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The ranks of classes and nX-complementary generations of the Tits group $^2\mathrm{F}_4(2)'$

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Abstract. Let G be a finite non-abelian simple group. The *rank* of non-trivial conjugacy class X of G, denoted by $\mathrm{rank}(G{:}X)$, is defined to be the minimal number of elements of X generating G. Also, a group G is said to be nX-complementary generated if given an arbitrary non-identity element $x \in G$ then there exists an element $y \in nX$ such that $G = \langle x, y\rangle$. In this paper we establish the ranks of all the conjugacy classes of

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the Tits group ${}^2\mathrm{F}_4(2)'$ and also classify all the non-trivial conjugacy classes of ${}^2\mathrm{F}_4(2)'$ whether they are complementary generators of ${}^2\mathrm{F}_4(2)'$ or not.

Keywords: conjugacy classes, nX-complementary generation, rank, structure constant, Tits group.

MSC 2020: 20C15, 20D06.

1. Introduction

A finite group G can be generated in many different ways. For example the probabilistic generation, $\frac{3}{2}$-generation, (p, q, r)-generations, ranks of non-trivial classes of G, nX-complementary generation and many other methods. Generation of finite groups by suitable subsets has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [42] for details). Also Di Martino et al. [32] established a useful connection between generation of groups by conjugate elements and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [40], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions.

A finite group G is said to be (l, m, n)-*generated* if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of lX in G containing elements of order l. The same applies to $[y]$ and $[z]$. Since a group G can be generated by a number of elements in a given conjugacy class X, there is a considerable interest of finding the minimal number of elements of X generating G. This minimal number is given as the rank of X in G and is denoted by $\mathrm{rank}(G{:}X)$.

Moori in the articles [33, 34] and [36], computed the ranks of involution classes of the Fischer sporadic simple group Fi_{22}. Furthermore, Moori and Basheer in [8] computed the rank of the class $3A$ of A_n, $n \geq 5$. They proved by mathematical induction that $\mathrm{rank}(A_n{:}3A)$ is $\frac{n-1}{2}$ if n is odd and is $\frac{n}{2}$ if n is even. Ali and Ibrahim in [1, 2, 3] determined the ranks of Conway group Co_1, the Higman-Sims group HS, McLaughlin group McL, Conway's sporadic simple groups Co_2 and Co_3. In 2008, Ali and Moori [4] established the ranks of conjugacy classes of the Janko groups J_1, J_2, J_3 and J_4. Recently, Motalane [38] computed the ranks of the classes of the Mathieu group M_{23} and the Alternating group A_{11}.

For a non-trivial conjugacy class nX of a finite non-abelian group G, we say that G is nX-**complementary generated** if for any $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. We say y is a complementary. The motivation of studying this kind of generation comes from a conjecture by Brenner-Guralnick-Wiegold [20] that every finite simple group can be generated by an arbitrary non-trivial element together with another suitable element.

In a series of papers [5, 24, 26, 27, 28, 30, 35] and [37], the nX-complementary generations of the sporadic simple groups Th, Co_1, J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22} have been investigated.

The aim of this paper is two fold. Firstly, we intend to establish the ranks of all nontrivial conjugacy classes of an exceptional group of Lie type, namely, the Tits group ${}^2\mathrm{F}_4(2)'$. Secondly, we establish all the nX-complementary generations of ${}^2\mathrm{F}_4(2)'$, where nX is a non-trivial conjugacy class of elements of order n as in the Atlas [23]. We follow the methods used in the papers [6, 7, 9, 10, 11, 12, 13,14] and [15]. Note that, in general, if G is a $(2,2,n)$-generated group then G is a dihedral group and therefore G is not simple. Also by [22], if G is a non-abelian (l,m,n)-generated group then either $G \cong A_5$ or $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$. Thus, for our purpose of establishing the nX-complementary generations of $G = {}^2\mathrm{F}_4(2)'$, the only cases we need to consider are when $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$.

The main results in this paper can be summarized in Theorems 1.1 and 1.2.

Theorem 1.1. *Let nX be a non-trivial conjugacy class of the exceptional Tits group ${}^2\mathrm{F}_4(2)'$. Then*

1. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) = \mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) = 3$,
2. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}nX) = 2$ *for all* $nX \notin \{2A, 2B\}$.

Theorem 1.2. *The group ${}^2\mathrm{F}_4(2)'$ is nX-complementary generated if and only if $n \geq 3$.*

The group ${}^2\mathrm{F}_4(2)'$ as outlined in the Atlas [23] is a simple group of order $17971200 = 2^{11} \times 3^3 \times 5^2 \times 13$. It has exactly 22 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Table 1 depicts representatives of the maximal subgroups of ${}^2\mathrm{F}_4(2)'$ and their orders.

Table 1: Maximal subgroups of ${}^2\mathrm{F}_4(2)'$

Maximal Subgroup	Order
$L_3(3):2 = M_1$	$11232 = 2^5 \times 3^3 \times 13$
$L_3(3):2 = M_2$	$11232 = 2^5 \times 3^3 \times 13$
$2^{\cdot}(2^8){:}5{:}4 = M_3$	$10240 = 2^{11} \times 5$
$L_2(25) = M_4$	$7800 = 2^3 \times 3 \times 5^2 \times 13$
$2^{2\cdot}(2^8){:}S_3 = M_5$	$6144 = 2^{11} \times 3$
$A_6^{\cdot}2^2 = M_6$	$1440 = 2^5 \times 3^2 \times 5$
$A_6^{\cdot}2^2 = M_7$	$1440 = 2^5 \times 3^2 \times 5$
$5^2{:}4A_4 = M_8$	$1200 = 2^4 \times 3 \times 5^2$

Using Equation (4) on GAP [31], we calculated the values of $h(g, M_i)$, where g is a representative of a non-trivial conjugacy class of ${}^2\mathrm{F}_4(2)'$ and over all the maximal subgroups M_i of ${}^2\mathrm{F}_4(2)'$. We list these values in Table 2.

Table 2: The values $h(g, M_i),\ 1 \leq i \leq 8$, for non-identity classes and maximal subgroups of ${}^2\mathrm{F}_4(2)'$

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$2A$	0	0	91	0	45	256	256	256
$2B$	48	48	27	64	61	80	80	32
$3A$	7	7	0	9	9	6	6	18
$4A$	4	4	15	16	13	12	12	8
$4B$	0	0	3	0	5	0	0	0
$4C$	4	4	7	0	5	12	12	8
$5A$	0	0	5	4	0	5	5	1
$6A$	3	3	0	1	1	2	2	2
$8A$	0	0	3	0	1	0	0	0
$8B$	0	0	3	0	1	0	0	0
$8C$	2	2	1	0	1	2	2	0
$8D$	2	2	1	0	1	2	2	0
$10A$	0	0	1	0	0	1	1	1
$12A$	1	1	0	1	1	0	0	2
$12B$	1	1	0	1	1	0	0	2
$13A$	1	1	0	3	0	0	0	0
$13B$	1	1	0	3	0	0	0	0
$16A$	0	0	1	0	1	0	0	0
$16B$	0	0	1	0	1	0	0	0
$16C$	0	0	1	0	1	0	0	0
$16D$	0	0	1	0	1	0	0	0

2. Preliminaries

Let G be a finite group and $C_1, C_2, \ldots, C_k$ (not necessarily distinct) for $k \geq 3$ be conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i,\ 1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \ldots \times C_{k-1}$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant.* With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G using Equation (1),

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{1}$$

Also, for a fixed $g_k \in C_k$, we denote by $\Delta_G^*(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ satisfying

$$g_1 g_2 \ldots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \ldots, g_{k-1} \rangle \,. \tag{2}$$

Definition 2.1. *If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then the group G is said to be $(C_1, C_2, \ldots, C_k)$-**generated**.*

Furthermore, if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \ldots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \ldots, h_{k-1})$ which are in $C_1 \times C_2 \times \ldots \times C_{k-1}$ such that

$$h_1 h_2 \ldots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \ldots, h_{k-1}\rangle \leq H. \tag{3}$$

The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \cap C_i$.

Lastly, for conjugacy classes $c_1, c_2, \ldots, c_k$ of a proper subgroup H of G and a fixed $g_k \in c_k$, let $\Sigma^*_H(c_1, c_2, \ldots, c_k)$ represents the number of tuples $(h_1, h_2, \ldots, h_{k-1}) \in c_1 \times c_2 \times \ldots \times c_{k-1}$ such that $h_1 h_2 \ldots h_{k-1} = g_k$ and $\langle h_1, h_2, \ldots, h_{k-1}\rangle = H$.

When it is clear from the context which conjugacy classes of H are considered, we will use the notation $\Sigma(H)$ and $\Sigma^*(H)$ to denote $\Sigma_H(c_1, c_2, \ldots, c_k)$ and $\Sigma^*_H(c_1, c_2, \ldots, c_k)$, respectively.

Theorem 2.1. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular,*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|}, \tag{4}$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof of Theorem 2.1. See [27] and [29, Theorem 2.1].

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta^*_G(C_1, C_2, \ldots, C_k)$, namely $\Delta^*_G(C_1, C_2, \ldots, C_k) \geq \Theta_G(C_1, C_2, \ldots, C_k)$, where

$$\Theta_G(C_1, C_2, \ldots, C_k) = \Delta_G(C_1, C_2, \ldots, C_k) - \sum h(g_k, H)\Sigma^*_H(C_1, C_2, \ldots, C_k),$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \ldots, C_k$.

In the following, Theorem 2.2 and Lemma 2.2 are in many cases useful in determining whether G is nX-complementary generated, while Lemma 2.1 is in some cases useful in establishing non-generation for finite groups.

Theorem 2.2 ([8, Lemma 2.5])**.** *Let G be a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.*

Lemma 2.1 (e.g., see Ali and Moori [4] or Conder et al. [21])**.** *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$ and $g_k \in C_k$ then $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$ and therefore, G is not $(C_1, C_2, \ldots, C_k)$-generated.*

Lemma 2.2 ([25])**.** *If G is sY-complementary generated and $(rX)^n = sY$ then G is rX -complementary generated.*

Proof of Theorem 2.2. Let rX and sY be non-trivial conjugacy classes of G such that $(rX)^n = sY$ for some integer n. If G is not rX-complementary generated then there exits an element x of prime order such that $\langle x, y\rangle < G$ for all $y \in rX$. Since $x, y^n \in \langle x, y\rangle$, it follows that $\langle x, y^n\rangle \leq \langle x, y\rangle < G$ for all $y^n \in sY$. Thus the results follows by method of contrapositive.

Lemma 2.3 (Ali and Moori [4] or Conder et al. [21])**.** *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated.*

The following result is due to Scott ([21] and [39]).

Theorem 2.3 (Scott's Theorem)**.** *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G such that $g_1g_2\ldots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with* $\dim \mathbb{V} = n \geq 2$. *Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i\rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i\rangle}$ being the trivial character of the cyclic group $\langle g_i\rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([25]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^G_{\langle g_i\rangle}, \mathbf{1}_{\langle g_i\rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i\rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{5}$$

The following proposition gives a criterion for G to be nX-complementary generated or not.

Proposition 2.1 ([25])**.** *A finite non-abelian group G is nX-complementary generated if and only if for each conjugacy class pY of G, where p is prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that G is $(pY, nX, t_{pY}Z)$-generated. Moreover, if G is a finite simple group then G is not $2X$-complementary generated for any conjugacy class of involutions.*

3. The ranks of non-trivial conjugacy classes of ${}^2\mathrm{F}_4(2)'$

We start our investigation on the ranks of the non-trivial classes of ${}^2\mathrm{F}_4(2)'$ by looking at the two classes of involutions, namely $2A$ and $2B$. It is well-known that two involutions generate a dihedral group. Thus the lower bound for the rank of a class of involutions in a finite simple group G is 3.

Lemma 3.1. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 3A, 13A)$*-generated.*

Proof. Direct computations on GAP show that $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A) = 13$. We deduce from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ will be involved in the calculations of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A)$. Hence, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A) = 13$ and generation occurs immediately. □

Proposition 3.1. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) = 3$.

Proof. Since by Lemma 3.1, ${}^2\mathrm{F}_4(2)'$ is $(2A, 3A, 13A)$-generated group, it follows by applications of Lemma 2.3 that ${}^2\mathrm{F}_4(2)'$ is $(2A, 2A, 2A, (13A)^3)$-generated, i.e., $(2A, 2A, 2A, 13A)$-generated group. Thus, $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) \leq 3$. Since $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) \notin \{1, 2\}$, it follows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) = 3$. □

Lemma 3.2. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2B, 3A, 16A)$*-generated.*

Proof. Direct calculations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) = 112$. We deduce from Table 2 that only one maximal subgroup of ${}^2\mathrm{F}_4(2)'$, namely M_5, has a nonempty intersection with the triple $(2B, 3A, 16A)$. We further calculated $\Sigma(M_5) = 0 + 0 + 0 + 32 = 32$ and $h(g, M_5) = 1$ for $g \in 16A$. Thus,

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) \quad \geq \quad \Delta_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) - 1 \cdot \Sigma(M_5) = 112 - 32 = 80.$$

We find $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) > 0$ showing that ${}^2\mathrm{F}_4(2)'$ is $(2B, 3A, 16A)$-generated. □

Proposition 3.2. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) = 3$.

Proof. By Lemma 3.2, ${}^2\mathrm{F}_4(2)'$ is $(2B, 3A, 16A)$-generated. We see from the Atlas [41] that $(16A)^3 = 16A$. Now, application of Lemma 2.3 reveals that ${}^2\mathrm{F}_4(2)'$ is $(2B, 2B, 2B, (16A)^3)$-generated, i.e., $(2B, 2B, 2B, 16A)$- generated. Thus, $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) \leq 3$. Since $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) \notin \{1, 2\}$, it follows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) = 3$. □

Proposition 3.3. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}3A) = 2$.

Proof. Since by Lemma 3.1, ${}^2\mathrm{F}_4(2)'$ is $(2A, 3A, 13A)$-generated group, it follows by applications of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $(3A, 3A, (13A)^2)$-generated. Since $(13A)^2 = 13B$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(3A, 3A, 13B)$-generated. This shows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}3A) = 2$. □

Remark 3.1. The result of Proposition 3.3 can be obtained directly from Proposition 12 of [16].

Lemma 3.3. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 4A, 12A)$*-generated.*

Proof. We calculated, with GAP, the structure constant $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = 12$. We deduce from Table 2 that only the maximal subgroups M_5 and M_8 of ${}^2\mathrm{F}_4(2)'$ have nonempty intersection with the triple $(2A, 4A, 12A)$. Further calculations with GAP reveal that $\Sigma(M_5) = 0 = \Sigma(M_8)$. We see from Table 2 that $h(g, M_5) = 1$ and $h(g, M_8) = 2$ for $g \in 12A$. Therefore,

$$\begin{aligned}\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) &= \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) - 1 \cdot \Sigma^*(M_5) - 2 \cdot \Sigma^*(M_8)\\ &= 12 - 0 - 0 = 12\end{aligned}$$

showing that ${}^2\mathrm{F}_4(2)'$ is $(2A, 4A, 12A)$-generated. □

Proposition 3.4. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}4A) = 2$.

Proof. Applying Lemma 2.2 to the result in Lemma 3.3 we see that ${}^2\mathrm{F}_4(2)'$ is $(4A, 4A, (12A)^2)$-generated. Since $(12A)^2 = 12A$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(4A, 4A, 12A)$-generated and the result follows. □

Lemma 3.4. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 4C, 13B)$*-generated.*

Proof. Calculations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13B) = 26$. We see from Table 2 that all the maximal subgroups of ${}^2\mathrm{F}_4(2)'$ have an empty intersection with the triple $(2A, 4C, 13B)$. Thus, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13B) = \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13B)$ $= 26$ showing that ${}^2\mathrm{F}_4(2)'$ is $(2A, 4C, 13B)$-generated. □

Proposition 3.5. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}4C) = 2$.

Proof. Since by Lemma 3.4, ${}^2\mathrm{F}_4(2)'$ is $(2A, 4C, 13B)$-generated group, it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $(4C, 4C, (13B)^2)$-generated. Since $(13B)^2 = 13A$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(4C, 4C, 13A)$-generated. We thus conclude that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}4C) = 2$. □

Lemma 3.5. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 5A, 13B)$*-generated.*

Proof. See Proposition 9 of [16]. □

Proposition 3.6. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}5A) = 2$.

Proof. Since by Lemma 3.5, ${}^2\mathrm{F}_4(2)'$ is $(2A, 5A, 13B)$-generated group, it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $(5A, 5A, (13B)^2)$-generated. Since $(13B)^2 = 13A$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(5A, 5A, 13A)$-generated. This shows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}5A) = 2$. □

Remark 3.2. Alternatively, the result of Proposition 3.6 follows immediately from either Proposition 16 or 17 of [16].

Proposition 3.7. *Let $T = \{4B, 8A, 8B, 10A\}$. For all conjugacy classes $pX \in T$ we have* $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}pX) = 2$.

Proof. We achieve the result by showing that ${}^2\mathrm{F}_4(2)'$ is $(pX, pX, 13A)$-generated for all conjugacy classes $pX \in T = \{4B, 8A, 8B, 10A\}$. We observe from Table 2 that for all $pX \in T$, all the maximal subgroups of ${}^2\mathrm{F}_4(2)'$ have an empty intersection with the triple $(pX, pX, 13A)$. Thus, we will have $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, pX, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(pX, pX, 13A)$. Now, direct computations with GAP yield

- $\Delta_{{}^2\mathrm{F}_4(2)'}(4B, 4B, 13A) = 1222$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(8A, 8A, 13A) = 13624$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(8B, 8B, 13A) = 13624$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(10A, 10A, 13A) = 17000$.

Since all of the structure constants calculated above are greater than zero, it follows that $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, pX, 13A) > 0$ for all $pX \in T$. Therefore, ${}^2\mathrm{F}_4(2)'$ is $(pX, pX, 13A)$-generated for all conjugacy classes $pX \in T$ and thus

$$\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}pX) = 2. \qquad \square$$

Lemma 3.6. *The group ${}^2\mathrm{F}_4(2)'$ is $(2A, 13X, 13Y)$-generated for $X \in \{A, B\}$.*

Proof. The proof is given in Proposition 10 of [16]. $\square$

Proposition 3.8. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}13X) = 2$ *for $X \in \{A, B\}$.*

Proof. The proof follows similarly from the proofs of Propositions 3.3 and 3.6. Alternatively, the result follows immediately from Proposition 19 of [16]. $\square$

Proposition 3.9. *Let $S = \{6A, 12A, 12B\}$. For all conjugacy classes $qX \in S$ we have* $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}qX) = 2$.

Proof. We achieve the result by showing that ${}^2\mathrm{F}_4(2)'$ is $(qX, qX, 16A)$-generated for all conjugacy classes $qX \in S = \{6A, 12A, 12B\}$. Table 2 shows that only the maximal subgroup M_5 has a nonempty intersection with the triple $(qX, qX, 16A)$. Direct computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(qX, qX, 16A) = 124416$ and $\Sigma(M_5) = 0$ for all $qX \in S$. With $h(g, M_5) = 1$ for $g \in 16A$ we find that

$$\begin{aligned} \Delta^*_{{}^2\mathrm{F}_4(2)'}(qX, qX, 16A) &= \Delta_{{}^2\mathrm{F}_4(2)'}(qX, qX, 16A) - 1 \cdot \Sigma(M_5) \\ &= 124416 - 0 = 124416. \end{aligned}$$

Thus, the group ${}^2\mathrm{F}_4(2)'$ is $(qX, qX, 16A)$-generated, hence $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}qX) = 2$ for all $qX \in S$. $\square$

Proposition 3.10. *Let $R = \{8C, 8D, 16A, 16B, 16C, 16D\}$. For all conjugacy classes $rX \in R$ we have* $\text{rank}({}^2\text{F}_4(2)'{:}rX) = 2$.

Proof. We obtain the result by showing that ${}^2\text{F}_4(2)'$ is $(rX, rX, 16A)$-generated for all conjugacy classes $rX \in R = \{8C, 8D, 16A, 16B, 16C, 16D\}$. Table 2 shows that only the maximal subgroups M_3 and M_5 have a nonempty intersection with the triple $(rX, rX, 16A)$. Direct computations with GAP yield $\Delta_{2\,\text{F}4(2)'}(rX, rX, 16A) = 69888$ and $\Sigma(M_3) = \Sigma(M_5) = 0$ for all $rX \in R$. We see from Table 2 that $h(g, M_3) = h(g, M_5) = 1$ for $g \in 16A$. Therefore,

$$\begin{aligned}\Delta^*_{{}^2\text{F}_4(2)'}(rX, rX, 16A) &= \Delta_{{}^2\text{F}_4(2)'}(rX, rX, 16A) - 1\cdot\Sigma(M_3) - 1\cdot\Sigma(M_5)\\ &= 69888 - 0 - 0 = 69888.\end{aligned}$$

Thus, the group ${}^2\text{F}_4(2)'$ is $(rX, rX, 16A)$-generated, hence $\text{rank}({}^2\text{F}_4(2)'{:}rX) = 2$ all $rX \in R$. □

The proof of Theorem 1.1 follows directly from Propositions 3.1 to 3.10.

4. The nX-complementary generations of ${}^2\text{F}_4(2)'$

In this section we apply the results discussed in Section 2 to the group $G = {}^2\text{F}_4(2)'$. We determine the non-trivial conjugacy classes nX such that ${}^2\text{F}_4(2)'$ is nX-complementary generated.

If $2X$ is a class of involutions of a non-abelian finite simple group G then G is not $2X$-complementary generated as if it is, then G would be $(2Y, 2X, t_{2Y}Z)$-generated. But this would mean that G is a dihedral group, which would contradict the fact that G is a simple group. Thus, in the process of investigating whether a group G is nX-complementary generated or not, we consider classes nX of G with $n \geq 3$.

Let $T = \{2A, 2B, 3A, 5A, 13A, 13B\}$ be a set of all conjugacy classes of ${}^2\text{F}_4(2)'$ of elements of prime orders. This set T will be useful whenever we apply Proposition 2.1 to prove that the group ${}^2\text{F}_4(2)'$ is nX-complementary generated.

Proposition 4.1. *The group ${}^2\text{F}_4(2)'$ is $3A$-complementary generated.*

Proof. We achieve the result by showing that ${}^2\text{F}_4(2)'$ is $(pX, 3A, 16A)$-generated for all conjugacy classes $pX \in T$. We divide our proof into two parts.

The first part involves the conjugacy classes in $R = \{2A, 2B, 3A\} \subset T$. We observe from Table 2 that only the maximal subgroup M_5 has a nonempty intersection with the triple $(pX, 3A, 16A)$ for $pX \in R$. Computations with GAP yield the structure constants and the values of $\Sigma(M_5)$ in Table 3. We see from Table 3 that $\Delta^*_{{}^2\text{F}_4(2)'}(pX, 3A, 16A) > 0$ for all $pX \in R$.

The second part involves the conjugacy classes in $U = \{5A, 13A, 13B\} \subset T$. We observe from Table 2 that for $pX \in U$, all the maximal subgroups of ${}^2\text{F}_4(2)'$ have an empty intersection with the triple $(pX, 3A, 16A)$. Thus, we will have $\Delta^*_G(pX, 3A, 16A) = \Delta_G(pX, 3A, 16A)$. Now, direct computations with GAP yield

Table 3: Values of $\Delta({}^2\mathrm{F}_4(2)')$, $\Sigma(M_5)$, $\Delta^*({}^2\mathrm{F}_4(2)')$ for $(pX, 3A, 16A)$, $pX \in R$

	$\Delta({}^2\mathrm{F}_4(2)')$	$\Sigma(M_5)$	$\Delta^*({}^2\mathrm{F}_4(2)')$
$(2A, 3A, 16A)$	16	0	16
$(2B, 3A, 16A)$	112	32	80
$(3A, 3A, 16A)$	1536	0	1536

- $\Delta_{{}^2\mathrm{F}_4(2)'}(5A, 3A, 16A) = 3328$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(13X, 3A, 16A) = 12800,\ X \in \{A, B\}$.

Since the structure constants computed above are greater than zero it follows that $\Delta^*_G(pX, 5Y, 21A) > 0$ for all $pX \in U$.

We conclude from the two parts that the group ${}^2\mathrm{F}_4(2)'$ is $(pX, 3A, 16A)$-generated for $pX \in T = R \cup U$. It follows, by Proposition 2.1, that ${}^2\mathrm{F}_4(2)'$ is $3A$-complementary generated. □

Proposition 4.2. *The group ${}^2\mathrm{F}_4(2)'$ is $4A$-complementary generated.*

Proof. We show that ${}^2\mathrm{F}_4(2)'$ is $(pX, 4A, tZ)$-generated for all conjugacy classes $pX \in T$ and some class tZ depending on pX. Our proof consists of six cases.
Case $(2A, 4A, 12A)$**:** Computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = 12$. That is, for a fixed $z \in 12A$ there are 12 triples (x, y, z) with $x \in 2A$ and $y \in 4A$ such that $xy = z$ and $\langle x, y\rangle \leq {}^2\mathrm{F}_4(2)'$. We can see from Table 2 that only the maximal subgroups M_5 and M_8 have a nonempty intersection with the triple $(2A, 4A, 12A)$. Further computations with GAP reveal that $\Sigma(M_5) = 0 = \Sigma(M_8)$. Therefore, none of the 12 triples generate a proper subgroup of ${}^2\mathrm{F}_4(2)'$. Hence, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = 12$.
Case $(2B, 4A, 16A)$**:** We can see from Table 2 that only the maximal subgroups M_3 and M_5 have a nonempty intersection with the triple $(2B, 4A, 16A)$. Direct computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) = 56$, $\Sigma(M_3) = 16+8 = 24$ and $\Sigma(M_5) = 16$. By Table 2, $h(g, M_3) = h(g, M_5) = 1$. Thus,

$$\begin{aligned}\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) &\geq \Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) - 1 \cdot \Sigma(M_3) - 1 \cdot \Sigma(M_5)\\ &= 56 - 24 - 16 = 16.\end{aligned}$$

Since $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) \geq 16 > 0$ it follows by Definition 2.1 that ${}^2\mathrm{F}_4(2)'$ is $(2B, 4A, 16A)$-generated.
Case $(3A, 4A, 16A)$**:** For this case we find that only one maximal subgroup, namely M_5, will be involved in the computation of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A)$. Direct computations with GAP give $\Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A) = 864$ and $\Sigma(M_5) = 32$. So, with $h(g, M_3) = 1$ we get

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A) \geq \Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A) - 1 \cdot \Sigma(M_5) = 864 - 32 = 832$$

which is clearly greater than zero. Hence, ${}^2\mathrm{F}_4(2)'$ is $(3A, 4A, 16A)$-generated.
Case $(5A, 4A, 16A)$**:** Also in this case we find that only one maximal subgroup, namely M_3, will be involved in the computation of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A)$. Computations with GAP give $\Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A) = 1856$ and $\Sigma(M_5) = 128$. So,

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A) \geq \Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A) - 1 \cdot \Sigma(M_3) = 1856 - 128 = 1728$$

which is clearly greater than zero. Hence, ${}^2\mathrm{F}_4(2)'$ is $(5A, 4A, 16A)$-generated.
Case $(13X, 4A, 16A)$**,** $X \in \{A, B\}$**:** In this case we deal with two cases involving $13A$ and $13B$ at the same time. We see from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ has a nonempty intersection with the triple $(13X, 4A, 16A)$ for $X \in \{A, B\}$. Therefore,

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(13X, 4A, 16A) = \Delta_{{}^2\mathrm{F}_4(2)'}(13X, 4A, 16A) = 7168$$

showing generation by the triple $(13X, 4A, 16A)$ for $X \in \{A, B\}$.

We see from these cases that ${}^2\mathrm{F}_4(2)'$ is $(pX, 4A, tZ)$-generated for all $pX \in T$. Thus, by Proposition 2.1, the group ${}^2\mathrm{F}_4(2)'$ is $4A$-complementary generated. $\square$

Proposition 4.3. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $4B$*-complementary generated.*

Proof. Here, we show that ${}^2\mathrm{F}_4(2)'$ is $(pX, 4B, 13A)$-generated for all $pX \in T$. From GAP, we get the following structure constants:

- $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4B, 13A) = 13$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4B, 13A) = 65$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4B, 13A) = 1352$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4B, 13A) = 2392$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(13A, 4B, 13A) = 10192$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(13B, 4B, 13A) = 10192$.

From the calculations above we see that $\Delta_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A) > 0$ for all $pX \in T$. From Table 2, we can see that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ contains elements from $4B$ and $13A$. Therefore, maximal subgroups of ${}^2\mathrm{F}_4(2)'$ make no contribution in the calculations of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A)$. Hence, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A) > 0$. Therefore, by Definition 2.1, the group ${}^2\mathrm{F}_4(2)'$ is $(pX, 4B, 13A)$-generated for all the conjugacy classes $pX \in T$. It follows by Proposition 2.1 that ${}^2\mathrm{F}_4(2)'$ is $4B$-complementary generated. $\square$

Proposition 4.4. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $4C$*-complementary generated.*

Proof. We will treat cases involving classes in T separately.
Case $(2A, 4C, 13A)$**:** We observe from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ contains elements from $2A$ and $13A$ together. Thus, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13A)$ $= \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13A) = 26$ and generation by $(2A, 4C, 13A)$ follows since $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13A) > 0$.
Case $(2B, 4C, 16A)$**:** In this case the maximal subgroups M_3 and M_5 have the required fusions. Computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4C, 16A) = 168$, $\Sigma(M_3) = 16 + 8 = 24$ and $\Sigma(M_5) = 16$. Therefore, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 4C, 16A) \geq \Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4C, 16A) - 1 \cdot \Sigma(M_3) - 1 \cdot \Sigma(M_5) = 168 - 24 - 16 = 128$ and thus, ${}^2\mathrm{F}_4(2)'$ is $(2B, 4C, 16A)$-generated.
Case $(3A, 4C, 16A)$**:** We observe from Table 2 that only the maximal subgroup M_5 has a nonempty intersection with the triple $(3A, 4C, 16A)$. Computations with GAP reveal that $\Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4C, 16A) = 2592$ and $\Sigma^*(M_5) = \Sigma(M_5) = 32$. Therefore, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(3A, 4C, 16A) = \Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4C, 16A) - 1 \cdot \Sigma^*(M_5) = 2592 - 32 = 2560$ showing that ${}^2\mathrm{F}_4(2)'$ is $(3A, 4C, 16A)$-generated.
Case $(5A, 4C, 13A)$**:** We observe from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ has a nonempty intersection with the triple $(5A, 4C, 13A)$. Therefore, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(5A, 4C, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4C, 13A) = 4784$ showing that ${}^2\mathrm{F}_4(2)'$ is $(5A, 4C, 13A)$-generated.
Case $(13A, 4C, 16A)$**:** Again, from Table 2, we see that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ contains elements from $13A$ and $16A$ together. So, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(13A, 4C, 16A)$ $= \Delta_{{}^2\mathrm{F}_4(2)'}(13A, 4C, 13A) = 21504$, hence ${}^2\mathrm{F}_4(2)'$ is $(13A, 4C, 16A)$-generated.
Case $(13B, 4C, 16A)$**:** The proof in this case is similar to the proof of $(13A, 4C, 16A)$.

Thus, ${}^2\mathrm{F}_4(2)'$ is $(pX, 4C, tZ)$-generated, where $tZ = 13A$ when $pX \in \{2A, 5A\}$ and $tZ = 16A$ when $pX \in \{2B, 3A, 13A, 13B\}$, for all the conjugacy classes $pX \in T$. It follows by Proposition 2.1 that ${}^2\mathrm{F}_4(2)'$ is $4C$-complementary generated. $\square$

Proposition 4.5. *The group ${}^2\mathrm{F}_4(2)'$ is $5A$-complementary generated.*

Proof. By Propositions 8, 13 and 16 of [16], the group ${}^2\mathrm{F}_4(2)'$ is $(pX, 5A, 13A)$-generated for all $pX \in \{2A, 2B, 3A, 5A\} \subset T$. For the remaining classes $13A$ and $13B$ in T we have $\Delta_{{}^2\mathrm{F}_4(2)'}(13X, 5A, 16A) = 27648$. We can see from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ has a nonempty intersection with the triple $(13X, 5A, 16A)$ for $X \in \{A, B\}$. Hence, generation by the triple $(13X, 5A, 16A)$ occurs because $\Delta^*_{{}^2\mathrm{F}_4(2)'}(13X, 5A, 16A) = \Delta_{{}^2\mathrm{F}_4(2)'}(13X, 5A, 16A) = 27648$. Thus, ${}^2\mathrm{F}_4(2)'$ is $(pX, 5A, tZ)$-generated, where $tZ = 13A$ when $pX \in \{2A, 2B, 3A, 5A\}$ and $tZ = 16A$ when $pX \in \{13A, 13B\}$, for all the conjugacy classes $pX \in T$. We conclude, by Proposition 2.1, that ${}^2\mathrm{F}_4(2)'$ is $5A$-complementary generated. $\square$

Proposition 4.6. *The group ${}^2\mathrm{F}_4(2)'$ is $6A$-complementary generated.*

Proof. By Proposition 4.1, the group ${}^2\mathrm{F}_4(2)'$ is $3A$-complementary generated. Since $(6A)^2 = 3A$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $6A$-complementary generated. $\square$

Proposition 4.7. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $8X$*-complementary generated for* $X \in \{A, B\}$.

Proof. By Proposition 4.3, the group ${}^2\mathrm{F}_4(2)'$ is $4B$-complementary generated. Since $(8A)^2 = (8B)^2 = 4B$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{A, B\}$. $\square$

Proposition 4.8. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $8X$*-complementary generated for* $X \in \{C, D\}$.

Proof. By Proposition 4.4, the group ${}^2\mathrm{F}_4(2)'$ is $4C$-complementary generated. Since $(8C)^2 = (8D)^2 = 4C$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{C, D\}$. $\square$

Proposition 4.9. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $10A$*-complementary generated.*

Proof. By Proposition 4.5, the group ${}^2\mathrm{F}_4(2)'$ is $5A$-complementary generated. Since $(10A)^2 = 5A$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $10A$-complementary generated. $\square$

Proposition 4.10. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $12X$*-complementary generated for* $X \in \{A, B\}$.

Proof. By Proposition 4.2, the group ${}^2\mathrm{F}_4(2)'$ is $4A$-complementary generated. Since $(12A)^3 = (12B)^3 = 4A$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $12X$-complementary generated for $X \in \{A, B\}$. $\square$

Proposition 4.11. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $13Y$*-complementary generated for* $Y \in \{A, B\}$.

Proof. By Propositions 9, 14, 17 and 18 of [16] and for $X, Y \in \{A, B\}$, the group G is $(2X, 13Y, 13B)$-, $(3A, 13Y, 13B)$-, $(5A, 13Y, 13B)$- and $(13X, 13Y, 13B)$-generated, respectively. Thus ${}^2\mathrm{F}_4(2)'$ is $(pX, 13Y, 13B)$-generated for all $pX \in T$, hence, it is $13Y$-complementary generated for $Y \in \{A, B\}$. $\square$

Proposition 4.12. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $16X$*-complementary generated for* $X \in \{A, B, C, D\}$.

Proof. Since, by Proposition 4.7, the group ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{A, B\}$ and since $(16A)^2 = (16C)^2 = 8A$ and $(16B)^2 = (16D)^2 = 8B$, it follows by Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $16X$-complementary generated for $X \in \{A, B, C, D\}$. $\square$

The results established in the Propositions 4.1 to 4.12 show that the Tits group ${}^2\mathrm{F}_4(2)'$ is nX-complementary generated if and only if $n \geq 3$. Hence Theorem 1.2 is proved.

5. Conclusion

The ranks of the nontrivial conjugacy classes nX of the Tits group ${}^2\mathrm{F}_4(2)'$ are all equal to 2 except when nX is an involutory class. Furthermore, all these conjugacy classes of rank 2 are the nX-complementary generators of ${}^2\mathrm{F}_4(2)'$. However, $rank(G : nX) = 2$ does not necessarily imply nX-complementary generation. Since the Tits group is classified under the Twisted Chevalley groups, it would be interesting to find if all the conjugacy classes nX of rank 2 in the other Twisted Chevalley groups are the nX-complementary generators.

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Inequalities of DVT-type – the two-dimensional case continued

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Abstract. In this note, particular two-dimensional inequalities dealing with two n-tuples of integer numbers under relatively general assumptions are investigated. Moreover, systems of integers for which the equality holds are completely described.

Keywords: integer numbers, inequality

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1. Introduction

In [4], A. Drápal and V. Valent proved that in a finite quasigroup Q of order n the number of associative triples $a(Q) \geq 2n - i(Q) + (\delta_1 + \delta_2)$, where $i(Q)$ is the number of idempotents in Q, i.e., $i(Q) = |\{\, x \in Q \,|\, xx = x \,\}|$, $\delta_1 = |\{\, z \in Q \,|\, zx \neq x \text{ for all } x \in Q \,\}|$ and $\delta_2 = |\{\, z \in Q \,|\, xz \neq x \text{ for all } x \in Q \,\}|$. This important result is an easy consequence of the inequality

$$\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) - \sum_{i=1}^{k}(a_i + b_i) \geq 3n - 2k + (r + s),$$

where $n \ge k \ge 0$, $a_1, \dots, a_n, b_1 \dots, b_n$ are non-negative integers such that $\sum a_i = n = \sum b_i$, $a_i \ge 1$ and $b_i \ge 1$ for $1 \le i \le k$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$. It should be noted that quasigroups with small $a(Q)$ may have applications in cryptography [5]. The lengthy and complicated proof of this DVT-inequality (inequality of Drápal-Valent type) in [4] is based on highly semantically involved insight.

In [6], a very short elementary arithmetical proof of a more general inequality of this type was found under assumption that $\sum_{i=1}^n a_i \ge n$, $\sum_{i=1}^n b_i \ge n$. This inequality is two-dimensional in the sense that it works with two n-tuples of integers. The approach in [6] opens a road to investigation of similar DVT-inequalities which could be useful in further investigations of estimates in non-associative algebra and they are also of independent interest. Hence they deserve a thorough examination; however, the research is only at its beginning. In [1] and [2], the one-dimensional case working with one n-tuple of real numbers was investigated. In [3], the investigation of the two-dimensional case working with two n-tuples of integer numbers was begun. The main aim of this note is to show that

$$2\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) \ge 3\sum_{i=1}^{n}(a_i + b_i) + 2r + 2s,$$

where $a_1, \dots, a_n, b_1 \dots, b_n$ are integers such that $\sum_{i=1}^n |a_i| \ge n$, $\sum_{i=1}^n |b_i| \ge n$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$. Moreover, the case when the equality holds is completely described.

2. The inequalities

Throughout this section, let $n \ge 1$ and $a_1, \dots, a_n, b_1, \dots, b_n$ be integers. Put $\alpha = (a_1, \dots, a_n)$, $\beta = (b_1, \dots, b_n)$, $I = \{1, \dots, n\}$, $A = \{\, i \in I \mid a_i \ge 0, b_i \ge 0, a_i + b_i \ge 3\}$, $B_1 = \{\, i \in I \mid (a_i, b_i) = (2, 0)\}$, $B_2 = \{\, i \in I \mid (a_i, b_i) = (0, 2)\}$, $B_3 = \{\, i \in I \mid (a_i, b_i) = (1, 1)\}$, $B = B_1 \cup B_2 \cup B_3$, $C_1 = \{\, i \in I \mid (a_i, b_i) = (2, -1)\}$, $C_2 = \{\, i \in I \mid (a_i, b_i) = (-1, 2)\}$, $C = C_1 \cup C_2$, $D_1 = \{\, i \in I \mid (a_i, b_i) = (0, 1)\}$, $D_2 = \{\, i \in I \mid (a_i, b_i) = (1, 0)\}$, $D = D_1 \cup D_2$ and $E = \{\, i \in I \mid (a_i, b_i) = (0, 0)\}$. For $X = A, B_1, \dots, E$, denote $\mathrm{x} = |X|$. Further, for integers x, y put $z(0) = 1$, $z(x) = 0$ otherwise and $t(x, y) = 2x^2 + 2y^2 + 2xy - 3x - 3y - 2z(x) - 2z(y)$. Finally, put $z(\alpha) = |\{\, i \in I \mid a_i = 0\}| = \sum_{i=1}^n z(a_i)$ and $t(\alpha, \beta) = \sum_{i=1}^n t(a_i, b_i)$.

Lemma 2.1. *$a^2 - a - 2z(a) \ge 2|a| - 2$ for every integer a.*

Proof. Obviously, $a^2 - 3a + 2 = (a-1)(a-2) \ge 0$. If $a > 0$ then $z(a) = 0$ and $a^2 - a \ge 2a - 2 = 2|a| - 2$. If $a < 0$ then $z(a) = 0$ and $a^2 \ge |a| = -a > -a - 2$, and hence $a^2 - a - 2z(a) > -2a - 2 = 2|a| - 2$. Finally, if $a = 0$ then $z(a) = 1$ and $a^2 - a - 2z(a) = -2 = 2|a| - 2$. □

Lemma 2.2. *Let $a \ge 1$ and $b \ge 0$ be integers. Then:*

(i) $t(a + 1, b) > t(a, b)$.

(ii) *If c, d are integers such that $c \geq a$, $d \geq b$ and $c + d > a + b$ then $t(c, d) > t(a, b)$.*

(iii) *If $i \in A$ then $t(a_i, b_i) \geq t(2, 1) = 5$.*

(iv) $t(2, -1) = t(-1, 2) = 3$, $t(1, 0) = t(0, 1) = -3$, $t(2, 0) = t(0, 2) = t(1, 1) = 0$ *and* $t(0, 0) = -4$.

(v) *If $I = B \cup C \cup D \cup E$ and $3\mathrm{c} = 3\mathrm{d} + 4\mathrm{e}$ then $t(\alpha, \beta) = 0$.*

Proof. We have $t(a+1, b) - t(a, b) = 4a + 2b - 1 \geq 4a - 1 \geq 3$ and the rest is clear. $\square$

Theorem 2.3. *Let $\sum_{i=1}^{n} |a_i| \geq n$ and $\sum_{i=1}^{n} |b_i| \geq n$. Put $\alpha = (a_1, \dots, a_n)$, $\beta = (b_1, \dots, b_n)$. Then*

$$2\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) \geq 3\sum_{i=1}^{n}(a_i + b_i) + 2z(\alpha) + 2z(\beta),$$
$$2\sum_{i=1}^{n}(a_i + b_i)^2 \geq 2\sum_{i=1}^{n} a_i b_i + 3\sum_{i=1}^{n}(a_i + b_i) + 2z(\alpha) + 2z(\beta)\,.$$

The equalities hold if and only if the following conditions are satisfied:

1. $I = B \cup C \cup D$.
2. $\mathrm{d}_1 \leq \mathrm{c}_1 + \mathrm{c}_2$ *and* $k = 2\mathrm{c}_1 + 2\mathrm{c}_2 + |\mathrm{c}_1 - \mathrm{d}_1| \leq n$.
3. $\mathrm{d}_2 = \mathrm{c}_1 + \mathrm{c}_2 - \mathrm{d}_1$.
4. *If* $\mathrm{c}_1 \geq \mathrm{d}_1$ *then* $\mathrm{b}_2 = \mathrm{b}_1 + |\mathrm{c}_1 - \mathrm{d}_1|$.
5. *If* $\mathrm{c}_1 < \mathrm{d}_1$ *then* $\mathrm{b}_1 = \mathrm{b}_2 + |\mathrm{c}_1 - \mathrm{d}_1|$.
6. $2p \leq n - k$ *and* $\mathrm{b}_3 = n - k - 2p$, *where* $p = \min(\mathrm{b}_1, \mathrm{b}_2)$.

In this case, $\sum_{i=1}^{n} |a_i| = n = \sum_{i=1}^{n} |b_i|$.

Proof. Clearly, the inequalities are equivalent to $t(\alpha, \beta) \geq 0$. Denote $I_1 = \{\, i \in I \mid a_i \geq 0, b_i \geq 0\}$, $n_1 = |I_1|$, $I_2 = \{\, i \in I \mid a_i \leq 0, b_i \leq 0\} \setminus \{(0, 0)\}$, $n_2 = |I_2|$, $I_3 = \{\, i \in I \mid a_i b_i < 0\}$ and $n_3 = |I_3|$. For $j = 1, 2, 3$ put $z_j(\alpha) = |\{\, a_i \in I_j \mid a_i = 0\}| = \sum_{i \in I_j} z(a_i)$, $z_j(\beta) = |\{\, b_i \in I_j \mid b_i = 0\}| = \sum_{i \in I_j} z(b_i)$ and $t_j = 2\sum_{i \in I_j} a_i^2 + 2\sum_{i \in I_j} b_i^2 + 2\sum_{i \in I_j} a_i b_i - 3\sum_{i \in I_j} a_i - 3\sum_{i \in I_j} b_i - 2z_j(\alpha) - 2z_j(\beta)$. Then $I = I_1 \cup I_2 \cup I_3$, $n = n_1 + n_2 + n_3$, $z_3(\alpha) = 0 = z_3(\beta)$ and $t(\alpha, \beta) = t_1 + t_2 + t_3$. The proof is divided into nine parts:

(i) First, denote $t_1(\alpha) = \sum_{i \in I_1} a_i^2 - \sum_{i \in I_1} a_i - 2z_1(\alpha)$, $t_1(\beta) = \sum_{i \in I_1} b_i^2 - \sum_{i \in I_1} b_i - 2z_1(\beta)$ and $q_1 = \sum_{i \in I_1}(a_i + b_i)^2 - 2\sum_{i \in I_1} a_i - 2\sum_{i \in I_1} b_i$. By 2.1, we get $t_1(\alpha) \geq 2\sum_{i \in I_1} |a_i| - 2n_1$, $t_1(\beta) \geq 2\sum_{i \in I_1} |b_i| - 2n_1$ and $q_1 = \sum_{i \in I_1}(a_i + b_i - 1)^2 - n_1 \geq \sum_{i \in I_1}(a_i + b_i - 1) - n_1 = \sum_{i \in I_1} |a_i| + \sum_{i \in I_1} |b_i| - 2n_1$. Then $t_1 = t_1(\alpha) + t_1(\beta) + q_1 \geq 3\sum_{i \in I_1} |a_i| + 3\sum_{i \in I_1} |b_i| - 6n_1$.

(ii) Further, we have $t_2 = 2\sum_{i\in I_2} a_i^2 + 2\sum_{i\in I_2} b_i^2 + 2\sum_{i\in I_2} a_ib_i + 3\sum_{i\in I_2}|a_i| + 3\sum_{i\in I_2}|b_i| - 2z_2(\alpha) - 2z_2(\beta) \geq 3\sum_{i\in I_2}|a_i| + 3\sum_{i\in I_2}|b_i| - 2n_2$, since $a_i = 0$ if and only if $b_i \neq 0$ for every $i \in I_2$. If $I_2 \neq \emptyset$ then $t_2 > 3\sum_{i\in I_2}|a_i| + 3\sum_{i\in I_2}|b_i| - 6n_2$.

(iii) If $i \in I_3$ and $a_i > 0$, $b_i < 0$, then $t(a_i, b_i) - 3|a_i| - 3|b_i| + 6 = 2a_i^2 + 2b_i^2 + 2a_ib_i - 6a_i + 6 = 2(b_i^2 + a_ib_i + a_i^2 - 3a_i + 3) = 2((b_i + \frac{a_i}{2})^2 + \frac{3}{4}(a_i - 2))^2 \geq 0$. Thus $t(a_i, b_i) \geq 3|a_i| + 3|b_i| - 6$ and the equality holds if and only if $(a_i, b_i) = (2, -1)$. The case $a_i < 0$, $b_i > 0$ is symmetric. Hence $t_3 \geq 3\sum_{i\in I_3}|a_i| + 3\sum_{i\in I_3}|b_i| - 6n_3$ and the equality holds if and only if $I_3 = C$.

(iv) Finally, $t(\alpha, \beta) = t_1 + t_2 + t_3 \geq 3\sum_{i=1}^n |a_i| + 3\sum_{i=1}^n |b_i| - 6n \geq 0$.

(v) Now, assume that $t(\alpha, \beta) = 0$. Then $\sum_{i=1}^n |a_i| = n = \sum_{i=1}^n |b_i|$, $I_2 = \emptyset$ and $I_3 = C$. Thus $I = A \cup B \cup C \cup D \cup E$.

(vi) First, let $t(\alpha, \beta) = 0$ and $C = \emptyset$. Then $I = I_1$, $\sum_{i=1}^n a_i = n = \sum_{i=1}^n b_i$ and (see (i)) $0 = q_1 = \sum_{i=1}^n (a_i + b_i)^2 - 2\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n (a_i + b_i - 2)^2 + 2\sum_{i=1}^n a_i + 2\sum_{i=1}^n b_i - 4n = \sum_{i=1}^n (a_i + b_i - 2)^2 \geq 0$. Thus $a_i + b_i = 2$ for every $i = 1, \dots, n$, $I = B$ and $A = \emptyset = E$.

(vii) Further, let $t(\alpha, \beta) = 0$, $C \neq \emptyset$ and $E \neq \emptyset$. Take $j \in C$ and $k \in E$. If $j \in C_1$ (i.e., $(a_j, b_j) = (2, -1)$) , put $c_j = 1$, $d_j = 0$, $c_k = 1 = d_k$ and $c_i = a_i$, $d_i = b_i$ otherwise. Denote $\gamma = (c_1, \dots, c_n)$ and $\delta = (d_1, \dots, d_n)$. Then $\sum_{i=1}^n |c_i| = \sum_{i\neq j,k} |a_i| + 1 + 1 = \sum_{i\neq j,k} |a_i| + 2 + 0 = n = \sum_{i\neq j,k} |b_i| + 1 + 0 = \sum_{i\neq j,k} |b_i| + 0 + 1 = \sum_{i=1}^n |d_i|$, and hence $0 \leq t(\gamma, \delta) = \sum_{i\neq j,k} t(a_i, b_i) - 3 + 0 < \sum_{i\neq j,k} t(a_i, b_i) + 3 - 4 = t(\alpha, \beta) = 0$, a contradiction. The proof for $j \in C_2$ is similar. We have proved that if $t(\alpha, \beta) = 0$ and $C \neq \emptyset$ then $E = \emptyset$.

(viii) Now, let $t(\alpha, \beta) = 0$, $C \neq \emptyset$ and $A \neq \emptyset$. Then $E = \emptyset$ by (vii), $t(\alpha, \beta) = \sum_{i\in A} t(a_i, b_i) + 3\mathrm{c} - 3\mathrm{d}$ and $\sum_{i=1}^n |a_i| = \sum_{i\in A} a_i + 2\mathrm{b}_1 + \mathrm{b}_3 + 2\mathrm{c}_1 + \mathrm{c}_2 + \mathrm{d}_2 = n = \mathrm{a} + \mathrm{b}_1 + \mathrm{b}_2 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + \mathrm{d}_1 + \mathrm{d}_2 = \sum_{i=1}^n |b_i| = \sum_{i\in A} b_i + 2\mathrm{b}_2 + \mathrm{b}_3 + \mathrm{c}_1 + 2\mathrm{c}_1 + \mathrm{d}_1$.

In order to obtain $t(\alpha, \beta) = 0$, to each pair (a_i, b_i), $i \in C$, must correspond a pair (a_j, b_j), $j \in D$. Hence $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2 > \mathrm{c}_1 + \mathrm{c}_2$ and the remaining $\mathrm{d} - \mathrm{c}$ pairs (a_i, b_i), $i \in D$ (their increment to $t(\alpha, \beta)$ is $-3(\mathrm{d} - \mathrm{c})$) must compensate $\sum_{i\in A} t(a_i, b_i) \geq 5\mathrm{a}$.

Now, suppose that $\mathrm{d}_1 \geq \mathrm{c}_1$ and $\mathrm{d}_2 \geq \mathrm{c}_2$. Then for every $i \in C_1$ we can choose $j_i \in D_1$ and for every $i \in C_2$ we can choose $j_i \in D_2$. Put $K = I \setminus \{i, j_i \,|\, i \in C\}$. Then $K = \{\, i \in K \,|\, a_i \geq 0,\, b_i \geq 0 \,\}$, $|K| = n - 2\mathrm{c}$, $\sum_{i\in K} |a_i| = \sum_{i\in A} a_i + 2\mathrm{b}_1 + \mathrm{b}_3 + d_2 - \mathrm{c}_2 = |K| = \sum_{i\in A} a_i + 2\mathrm{b}_1 + \mathrm{b}_3 + 2\mathrm{c}_1 + \mathrm{c}_2 + \mathrm{d}_2 - 2\mathrm{c}_1 - 2\mathrm{c}_2 = n - 2\mathrm{c} = |K| = \sum_{i\in A} b_i + 2\mathrm{b}_2 + \mathrm{b}_3 + \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 - 2\mathrm{c}_1 - 2\mathrm{c}_2 = \sum_{i\in A} b_i + 2\mathrm{b}_2 + \mathrm{b}_3 + \mathrm{d}_1 - \mathrm{c}_1 = \sum_{i\in K} |b_i|$ and $0 = t(\alpha, \beta) = \sum_{i\in K} t(a_i, b_i) + \sum_{i\in C_1} (t(a_i, b_i) + t(a_{j_i}, b_{j_i})) + \sum_{i\in C_2} (t(a_i, b_i) + t(a_{j_i}, b_{j_i})) = \sum_{i\in K} t(a_i, b_i) + \mathrm{c}_1(t(2, -1) + t(0, 1)) + \mathrm{c}_2(t(-1, 2) + t(1, 0)) = \sum_{i\in K} t(a_i, b_i)$. By (vi) (for K instead of I) we obtain $K = \{\, i \in K \,|\, a_i + b_i = 2 \,\}$, a contradiction with $\emptyset \neq A \subseteq K$.

Further, suppose that $\mathrm{d}_1 > \mathrm{c}_1$ and $\mathrm{d}_2 < \mathrm{c}_2$. Again, for every $i \in C_1$ we can choose $j_i \in D_1$, and for every $i \in D_2$ we can choose $j_i \in C_2$. Taking into account that the increment of pairs $(2, -1), (0, 1)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 2 and $t(2, -1) + t(0, 1) = 3 - 3 = 0$, the increment of pairs $(1, 0), (-1, 2)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 2 and $t(1, 0) + t(-1, 2) = -3 + 3 = 0$, the increment of pairs $(2, 0), (0, 2)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 2 and $t(2, 0) = 0 = t(0, 2)$, and the increment of pair $(1,1)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 1 and $t(1,1) = 0$, we may assume without loss of generality that $\mathrm{b}_3 = 0$, $\min(\mathrm{b}_1, \mathrm{b}_2) = 0$, $\mathrm{c}_1 = 0$ and $\mathrm{d}_2 = 0$. Of course, $\mathrm{d}_1 > \mathrm{c}_2$ and $C = C_2$, $D = D_1$. Now, for each $i \in C_2$ we can choose $j_i \in D_1$. Put $L = \{ j_i \mid i \in C_2 \} \subseteq D_1$ and $K = I \setminus (C_2 \cup L)$. For every $i \in C_2$ put $c_i = 0$, $d_i = 2$, $c_{j_i} = 1$ and $d_{j_i} = 1$. Further, put $c_i = a_i$, $d_i = b_i$ for every $i \in K$ and $\gamma = (c_1, \ldots, c_n)$, $\delta = (d_1, \ldots, d_n)$. Then $(c_i, d_i) = (0,2)$ for every $i \in C_2$, $(c_i, d_i) = (1,1)$ for every $i \in L$ and $c_i \geq 0$, $d_i \geq 0$ for every $i \in I$, $\sum_{i=1}^n |c_i| = \sum_{i \in K} |a_i| + \mathrm{c}_2 = \sum_{i=1}^n |a_i| = n = \sum_{i=1}^n |b_i| = \sum_{i \in K} |b_i| + \mathrm{c}_2 + 2\mathrm{c}_2 = \sum_{i=1}^n |d_i|$ and $t(\gamma, \delta) = \sum_{i \in K} t(a_i, b_i) + \sum_{i \in C_2} t(c_i, d_i) + \sum_{i \in L} t(c_i, d_i)) = \sum_{i \in K} t(a_i, b_i) = \sum_{i \in K} t(a_i, b_i) + 3\mathrm{c}_2 - 3\mathrm{c}_2 = \sum_{i \in K} t(a_i, b_i) + \sum_{i \in C_2} t(a_i, b_i) + \sum_{i \in L} t(a_i, b_i) = \sum_{i=1}^n t(a_i, b_i) = 0$. By (vi) for γ, δ, we obtain $K = \{ i \in K \mid a_i + b_i = 2 \}$, a contradiction with $\emptyset \neq A \subseteq K$. The proof for $\mathrm{d}_1 < \mathrm{c}_1$, $\mathrm{d}_2 > \mathrm{c}_2$ is similar. We have proved that if $t(\alpha, \beta) = 0$ and $C \neq \emptyset$ then $A = \emptyset$.

(ix) Finally, let $t(\alpha, \beta) = 0$. By (vi), (vii) and (viii), we have $A = \emptyset = E$, $I = B \cup C \cup D$, $0 = t(\alpha, \beta) = 3\mathrm{c} - 3\mathrm{d}$ and $\mathrm{c} = \mathrm{c}_1 + \mathrm{c}_2 = \mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$. Hence $\mathrm{d}_1 \leq \mathrm{c}_1 + \mathrm{c}_2$ and $\mathrm{d}_2 = \mathrm{c}_1 + \mathrm{c}_2 - \mathrm{d}_1$. Further, $\sum_{i=1}^n |a_i| = 3\mathrm{c}_1 + 2\mathrm{c}_2 - \mathrm{d}_1 + 2\mathrm{b}_1 + \mathrm{b}_3 = n = 2\mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{b}_1 + \mathrm{b}_2 + \mathrm{b}_3 = \sum_{i=1}^n |b_i| = \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 + 2\mathrm{b}_2 + \mathrm{b}_3$, and hence $\mathrm{c}_1 + b_1 = \mathrm{d}_1 + \mathrm{b}_2$. If $\mathrm{c}_1 \geq \mathrm{d}_1$ then $\mathrm{b}_2 = b_1 + |\mathrm{c}_1 - \mathrm{d}_1|$, and if $\mathrm{c}_1 < d_1$ then $\mathrm{b}_1 = \mathrm{b}_2 + |\mathrm{c}_1 - \mathrm{d}_1|$. As $|C \cup D| = 2\mathrm{c}_1 + 2\mathrm{c}_2$, we obtain $k = 2\mathrm{c}_1 + 2\mathrm{c}_2 + |\mathrm{c}_1 - \mathrm{d}_1| \leq n$. Now, denote $p = \min(\mathrm{b}_1, \mathrm{b}_2)$. Then $2p \leq n - k$ and $\mathrm{b}_3 = n - k - 2p$. Indeed, if $\mathrm{c}_1 \geq \mathrm{d}_1$ then $n - k = n - 2\mathrm{c}_1 - 2\mathrm{c}_2 - \mathrm{c}_1 + \mathrm{d}_1 = \mathrm{b}_1 + \mathrm{b}_2 + \mathrm{b}_3 - \mathrm{c}_1 + \mathrm{d}_1 = \mathrm{b}_1 + \mathrm{b}_1 + \mathrm{b}_3 = 2p + \mathrm{b}_3$. Thus $2p \leq n - k$ and $\mathrm{b}_3 = n - k - 2p$. The proof in case $\mathrm{c}_1 < \mathrm{d}_1$ is similar.

Conversely, assume that the conditions (1) – (6) are satisfied. If $\mathrm{c}_1 \geq \mathrm{d}_1$ then $\sum_{i=1}^n |a_i| = 3\mathrm{c}_1 + 2\mathrm{c}_2 - \mathrm{d}_1 + 2\mathrm{b}_1 + \mathrm{b}_3 = 3\mathrm{c}_1 + 2\mathrm{c}_2 - \mathrm{d}_1 + 2p + n - k - 2p = 2\mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{c}_1 - \mathrm{d}_1 + n - k = n$ and $\sum_{i=1}^n |b_i| = \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 + 2\mathrm{b}_2 + \mathrm{b}_3 = \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 + 2p + 2(\mathrm{c}_1 - \mathrm{d}_1) + n - k - 2p = 2\mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{c}_1 - \mathrm{d}_1 + n - k = n$. The proof for $\mathrm{c}_1 < \mathrm{d}_1$ is similar. Finally, $t(\alpha, \beta) = 0$ by 2.2(v). □

Remark 2.4. If $\sum_{i=1}^n |a_i| + \sum_{i=1}^n |b_i| \geq 2n$ then the inequalities in Theorem 2.3 hold.

Theorem 2.5. *If $\sum_{i=1}^n a_i \geq n$ and $\sum_{i=1}^n b_i \geq n$ then the inequalities in Theorem 2.3 hold and the equalities hold if and only if $I = B$, $2\mathrm{b}_1 \leq n$, $\mathrm{b}_2 = \mathrm{b}_1$ and $\mathrm{b}_3 = n - 2\mathrm{b}_1$.*

Proof. The inequalities follow from Theorem 2.3. Now, suppose that $t(\alpha, \beta) = 0$. Then $\sum_{i=1}^n |a_i| = n = \sum_{i=1}^n |b_i|$. If $C \neq \emptyset$ then $\sum_{i=1}^n a_i < \sum_{i=1}^n |a_i| = n$ or $\sum_{i=1}^n b_i < \sum_{i=1}^n |b_i| = n$, a contradiction. Thus $C = \emptyset$ and the rest follows from Theorem 2.3 and its proof. □

Remark 2.6. (i) The situation $\sum_{i=1}^n |a_i| \geq n$, $\sum_{i=1}^n |b_i| \geq n$, $t(\alpha, \beta) = 0$ is completely described by conditions (1) – (6). In order to find all such pairs α, β for given n, choose non-negative integers $\mathrm{c}_1, \mathrm{c}_2, \mathrm{d}_1, p$ such that $\mathrm{d}_1 \leq \mathrm{c}_1 + \mathrm{c}_2$, $k = 2\mathrm{c}_1 + 2\mathrm{c}_2 + |\mathrm{c}_1 - \mathrm{d}_1| \leq n$ and $2p \leq n - k$, calculate $\mathrm{d}_2 = \mathrm{c}_1 + \mathrm{c}_2 - \mathrm{d}_1$, $\mathrm{b}_1 = p$ and $\mathrm{b}_2 = p + |\mathrm{c}_1 - \mathrm{d}_1|$ if $\mathrm{c}_1 \geq \mathrm{d}_1$, $\mathrm{b}_2 = p$ and $\mathrm{b}_1 = p + |\mathrm{c}_1 - \mathrm{d}_1|$ if $\mathrm{c}_1 < \mathrm{d}_1$,

$b_3 = n - k - 2p$ and take c_1 pairs (2,-1), c_2 pairs (-1,2), d_1 pairs (0,1), d_2 pairs (1,0), b_1 pairs (2,0), b_2 pairs (0,2) and b_3 pairs (1,1).

(ii) For instance, for $n = 17$ choose, e.g., $c_1 = 3$, $c_2 = 2$, $d_1 = 4$ and $p = 2$. Then $d_2 = 1$, $b_2 = 2$, $b_1 = 3$, $b_3 = 2$ and we obtain one type of α, β. In this way, to each choice of c_1, c_2, d_1, p satisfying $d_1 \leq c_1 + c_2$, $k = 2c_1 + 2c_2 + |c_1 - d_1| \leq n$ and $2p \leq n-k$ corresponds one type of α, β such that $\sum_{i=1}^{n} |a_i| \geq n$, $\sum_{i=1}^{n} |b_i| \geq n$ and $t(\alpha, \beta) = 0$. All $n!$ pairs α, β of this type can be obtained by permutations of I.

(iii) By Theorem 2.5, the situation $\sum_{i=1}^{n} a_i \geq n$, $\sum_{i=1}^{n} b_i \geq n$, $t(\alpha, \beta) = 0$ is completely described.

(iv) For instance, for $n = 5$ choose, e.g., $p = 2$. Then we obtain one type of α, β, namely 2 pairs (2,0), 2 pairs (0,2) and one pair (1,1). In this way, to each choice of p such that $2p \leq n$ corresponds one type of α, β such that $\sum_{i=1}^{n} a_i \geq n$, $\sum_{i=1}^{n} b_i \geq n$ and $t(\alpha, \beta) = 0$, namely p pairs (2,0), p pairs (0,2) and $n - 2p$ pairs (1,1). All $n!$ pairs of this type can be obtained by permutations of I.

3. Conclusions

In the paper, two relatively complicated inequalities concerning two n-tuples of integers are proved and the case when the equality holds is solved. Inequalities of similar type already proved useful in obtaining some estimates of the number of non-associative triples in quasigroups and hence the investigation of such inequalities can lead to further applications.

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Automorphisms and isomorphisms of heptavalent symmetric graphs of order $32p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we determine the automorphisms and isomorphisms of connected heptavalent symmetric graphs of order $32p$ for each prime p. As a result, we get the complete classification of such graphs, and there are two sporadic such graphs with $p = 2$ and 3.
Keywords: symmetric graph, s-transitive graph, Cayley graph, bi-Cayley graph.
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1. Introduction

Let X be a finite, simple graph with vertex set $V(X)$ and edge set $E(X)$. If the full automorphism group $\mathrm{Aut}(X)$ acts transitively on $V(X)$, then X is said to be *vertex-transitive*. Recall that an *arc* in a graph X is an ordered pair of adjacent vertices. Thus, a graph X is said to be *arc-transitive* or *symmetric* if X is vertex-transitive and $\mathrm{Aut}(X)$ acts transitively on the set of all arcs in X. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [27, 30] or [2, 3], respectively.

As we all known that the vertex stabilizers of symmetric graphs are the foundation of the study of such graphs, which have been definitely known for small valences, see for example [11, 15, 25, 26]. By using this structures, classifying symmetric graphs with small valencies has been received considerable attention and a lot of results have been achieved, see[9, 13, 20, 21, 22, 31, 32, 33, 34], and reference therein. In particular, the classification of pentavalent and heptavalent symmetric graphs of order $16p$ with p a prime was given in [13, 12]. Thus, as a

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natural continuation, we classify heptavalent symmetric graphs of order $32p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and two orbits adjacent if there is an edge in X between those two orbits. In view of [23, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has at least 3 orbits on $V(X)$, N acts semiregularly on $V(X)$, and the quotient graph X_N is a connected heptavalent G/N-symmetric graph.*

For a graph X and a positive integer s, an s-arc in X is a sequence of $s+1$ vertices of which any two consecutive vertices are adjacent and any three consecutive vertices are distinct. Let $G \leq \mathrm{Aut}(X)$. Then a graph X is said to be (G,s)*-arc-transitive* if G is transitive on the set of s-arcs in X. Furthermore, if X is (G,s)-arc-transitive but not $(G,s+1)$-arc-transitive, then X is said to be (G,s)*-transitive*. The following proposition characterizes the vertex stabilizers of connected heptavalent (G,s)-transitive graphs (see [15, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G,s)-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

(1) *For $s = 1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;*
(2) *For $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\mathrm{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ or $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$;*
(3) *For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\mathrm{PSL}(3,2) \times \mathrm{S}_4$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ or $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$.*

In particular, a Sylow 3-subgroup of G_v is elementary abelian.

To extract a classification of connected heptavalent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graph $G(2p,r)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0,1,\cdots,p-1\}$ and $V' = \{0',1',\cdots,(p-1)'\}$. Let r be a positive integer dividing $p-1$ and $H(p,r)$ the unique subgroup of $\mathbb{Z}_p^*$ of order r. Define the graph $G(2p,r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x-y \in H(p,r)\}$.

Proposition 2.3. *Let X be a connected heptavalent symmetric graph of order $2p$ with p a prime. Then X is isomorphic to $K_{7,7}$ or $G(2p,7)$ with $7 \mid (p-1)$. Furthermore,* $\mathrm{Aut}(G(2p,7)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.

In view of [16, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order $4p$ for a prime p.

Proposition 2.4. *Let X be a connected heptavalent symmetric graph of order $4p$ with p a prime. Then X is isomorphic to K_8.*

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G with respect to S is defined to have vertex set $V(\mathrm{Cay}(G,S)) = G$ and edge set $E(\mathrm{Cay}(G,S)) = \{\{g, sg\} \mid g \in G, s \in S\}$. Clearly, a Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if S generates G. Furthermore, $\mathrm{Aut}(G,S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G,S))$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. The Cayley graph is vertex-transitive because it admits the right regular representation $R(G)$ of G as a regular group of automorphisms of $\mathrm{Cay}(G,S)$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [28]). For two subsets S and T of G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$ then S and T are said to be *equivalent*, denoted by $S \equiv T$. We may easily show that if $S \equiv T$ then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ and $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Cay}(G,T)$ is normal. The next example is about connected heptavalent symmetric Cayley graphs of order 24.

Example 2.1. Let $\mathrm{S}_4 = \langle (1,2), (1,3), (1,4) \rangle$. Then define the Cayley graph on the symmetric group S_4:

$$\mathcal{G}_{24} = \mathrm{Cay}(\mathrm{S}_4, S).$$

where $S = \{(1,2,3,4), (1,4,3,2), (1,2,4), (1,4,2), (3,4), (2,4), (1,4)(2,3)\}$. By Magma [4], $\mathrm{Aut}(\mathcal{G}_{24}) = \mathrm{S}_4.D_{14} \cong \mathrm{PGL}(2,7)$ and $\mathcal{G}_{24}$ is a connected heptavalent 1-transitive graph.

Following this construction and the result in [14, Theorem 3.1], we have the classification of heptavalent symmetric graphs of order $8p$ with p a prime.

Proposition 2.5. *Let X be a connected heptavalent symmetric graph of order $8p$ with p a prime. Then $X \cong K_{8,8} - 8K_2$ or $\mathcal{G}_{24}$.*

Construction 2.1. Let $G = 2.\mathrm{PSL}(2,7).2i$. Then by Atlas [6], G has a representation of degree 32, its suborbits are: 1^4 and 7^4. These suborbits of length 7 can form orbital graphs of valency 7 or 14. By Magma [4], up to isomorphism, there is only one orbital graph of valency 7, denoted by $\mathcal{G}_{32}$. Furthermore, $\mathrm{Aut}(\mathcal{G}_{32}) \cong \mathbb{Z}_2.(\mathrm{PGL}(2,7) \times \mathbb{Z}_2)$. Conversely, any connected heptavalent symmetric graph of order 32 admitting $G = 2.\mathrm{PSL}(2,7).2i$ as an arc-transitive automorphism group is isomorphic to $\mathcal{G}_{32}$.

In order to construct some heptavalent symmetric graphs, we need to introduce the so called coset graph (see [25, 28]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(G,H,D)$ of G with respect to H and D is defined to have vertex set $[G:H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $\mathrm{Cos}(G,H,D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G. The action of G on $V(\mathrm{Cos}(G,H,D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G in H. Clearly, $\mathrm{Cos}(G,H,D) \cong \mathrm{Cos}(G,H^\alpha,D^\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$. For more details regarding coset graphs, see for example [8, 23, 28].

Construction 2.2. Let $G = \mathrm{PGL}(2,7) = \langle (1,2,6)(3,4,8), (3,8,7,6,5,4)\rangle$. Then G has a Sylow 7-subgroup $H = \langle (2,5,7,6,3,4,8)\rangle$. Let $g = (1,2)(3,5)(6,8)$. Define the coset graph:

$$\mathcal{G}_{48} = \mathrm{Cos}(G,H,HgH).$$

Moreover, $\mathrm{Aut}(\mathcal{G}_{48}) \cong \mathrm{PGL}(2,7) \times \mathrm{S}_3$.

Construction 2.3. Let $G_{(2^3,2p)} = \langle a,b,x,y,z \mid a^p = b^2 = x^2 = y^2 = z^2 = 1, a^b = a^{-1}, a^x = a^y = a^z = a, b^x = b^y = b^z = b, x^y = x^z = x, y^z = y\rangle \cong D_{2p} \times \mathbb{Z}_2^3$. Take two subsets in G:

$$S = \{b, abx, a^2bxy, a^3bxz, a^4byz, a^5by, a^6bxyz\},$$
$$T = \{abx, a^rby, a^{r^2}byz, a^{r^3}bxy, a^{r^4}bz, a^{r^5}bxz, a^{r^6}bxyz\}.$$

where $r \in \mathbb{Z}_p^*$ has order 7. Then define the Cayley graphs:

$$\mathcal{G}_{112} = \mathrm{Cay}(G_{(2^3,14)}, S), \qquad \mathcal{G}_{(2^3,2p)} = \mathrm{Cay}(G_{(2^3,2p)}, T).$$

Moreover, $\mathrm{Aut}(\mathcal{G}_{112}) \cong (\mathbb{Z}_2^3 \times D_{14}) \rtimes F_{21}$ and $\mathrm{Aut}(\mathcal{G}_{(2^3,2p)}) \cong (\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$.

The next result is about heptavalent symmetric graphs of order $16p$, which is from [12, Theorem 1.1].

Proposition 2.6. *Let X be a connected heptavalent graph of order $16p$. Then X is symmetric if and only if X is isomorphic to $\mathcal{G}_{32}$, $\mathcal{G}_{48}$, $\mathcal{G}_{112}$ or $\mathcal{G}_{(2^3,2p)}$.*

Remark. The graph of $\mathcal{G}_{32}$ is missing in [12, Theorem 1.1]. In fact, in the proof of [12, lemma 4.1], the groups $\mathbb{Z}_4.\mathrm{PSL}(2,7)$ and $2.\mathrm{PSL}(2,7).2i$ are not considered, and by Atlas [6], these two groups have representations of degree 32. With the calculation of Magma [4], these two groups give the same orbital graph $\mathcal{G}_{32}$, up to isomorphism.

From [6, pp.12-14], [29, Theorem 2] and [18, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

Proposition 2.7. *Let p be a prime, and let G be a non-abelian simple group of order $|G| \mid (2^{29}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p)$. Then G has 3-prime factor, 4-prime factor or 5-prime factor, and is isomorphic to one of the following groups:*

Table 1: **Non-abelian simple $\{2,3,5,7,p\}-$groups**

3-prime factor					
G	Order	G	Order	G	Order
A_5	$2^2{\cdot}3{\cdot}5$	PSL(2,8)	$2^3{\cdot}3^2{\cdot}7$	PSU(4,2)	$2^6{\cdot}3^4{\cdot}5$
A_6	$2^3{\cdot}3^2{\cdot}5$	PSL(2,17)	$2^4{\cdot}3^2{\cdot}17$	PSU(3,3)	$2^5{\cdot}3^3{\cdot}7$
PSL(2,7)	$2^3{\cdot}3{\cdot}7$	PSL(3,3)	$2^4{\cdot}3^3{\cdot}13$		
4-prime factor					
G	Order	G	Order	G	Order
A_7	$2^3{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2,27)	$2^2{\cdot}3^3{\cdot}7{\cdot}13$	PSp(4,4)	$2^8{\cdot}3^2{\cdot}5^2{\cdot}17$
A_8	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2,31)	$2^5{\cdot}3{\cdot}5{\cdot}31$	PSp(6,2)	$2^9{\cdot}3^4{\cdot}5{\cdot}7$
A_9	$2^6{\cdot}3^4{\cdot}5{\cdot}7$	PSL(2,49)	$2^4{\cdot}3{\cdot}5^2{\cdot}7^2$	M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$
A_{10}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7$	PSL(2,81)	$2^4{\cdot}3^4{\cdot}5{\cdot}41$	M_{12}	$2^6{\cdot}3^3{\cdot}5{\cdot}11$
PSL(2,11)	$2^2{\cdot}3{\cdot}5{\cdot}11$	PSL(2,127)	$2^7{\cdot}3^2{\cdot}7{\cdot}127$	J_2	$2^7{\cdot}3^3{\cdot}5^2{\cdot}7$
PSL(2,13)	$2^2{\cdot}3{\cdot}7{\cdot}13$	PSL(3,4)	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	$P\Omega^+(8,2)$	$2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$
PSL(2,16)	$2^4{\cdot}3{\cdot}5{\cdot}17$	PSU(3,4)	$2^6{\cdot}3{\cdot}5^2{\cdot}13$	Sz(8)	$2^6{\cdot}5{\cdot}7{\cdot}13$
PSL(2,19)	$2^2{\cdot}3^2{\cdot}5{\cdot}19$	PSU(3,8)	$2^9{\cdot}3^4{\cdot}7{\cdot}19$	${}^2F_4(2)'$	$2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$
PSL(2,25)	$2^3{\cdot}3{\cdot}5^2{\cdot}13$				
5-prime factor					
G	Order	G	Order	G	Order
A_{11}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	PSL(2,449)	$2^6{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}449$	M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$
PSL(2,29)	$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}29$	$PSL(2,2^6)$	$2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$	$P\Omega^-(8,2)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$
PSL(2,41)	$2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}41$	PSL(4,4)	$2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	$G_2(4)$	$2^{12}{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}13$
PSL(2,71)	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}71$	PSL(5,2)	$2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$		

3. Graph constructions

In this section, we discuss some heptavalent graphs of order 64 and construct some heptavalent symmetric graphs of order $32p$ with p a prime.

Construction 3.1. Let $a=(1,2)$, $b=(3,4)$, $c=(5,6)$, $d=(7,8)$, $e=(9,10)$, $f=(11,12)$. Then $G=\langle a,b,c,d,e,f\rangle \cong \mathbb{Z}_2^6$. Define the Cayley graph as follows:

$$\mathcal{G}_{64}=\mathrm{Cay}(G,\{a,acd,bde,df,d,bcf,e\}).$$

Then by Magma [4], $\mathcal{G}_{64}$ is symmetric and $\mathrm{Aut}(\mathcal{G}_{64})\cong \mathbb{Z}_2^6 \rtimes \mathrm{S}_7$.

With the same notation as above, we have the following lemma.

Lemma 3.1. *Let X be a connected heptavalent normal Cayley graph on $\mathbb{Z}_2^6$. Then X is symmetric if and only if $X\cong\mathcal{G}_{64}$.*

Proof. By Construction 3.1, $\mathcal{G}_{64}$ is symmetric. Let $X = \mathrm{Cay}(G, S)$ be a connected heptavalent normal Cayley graph on $G = \langle a, b, c, d, e, f\rangle \cong \mathbb{Z}_2^6$. Then $\langle S\rangle = G$, $|S| = 7$. Since X is normal, we have that $\mathrm{Aut}(X)_1 = \mathrm{Aut}(G, S)$.

Suppose that X is symmetric. Then $\mathrm{Aut}(G, S)$ is transitive on S and $7 \,\big|\, |\mathrm{Aut}(G, S)|$. Let P be a Sylow 7-subgroup of $\mathrm{Aut}(G, S)$. By Proposition 2.2, $|P| = 7$. Since $\mathrm{Aut}(G) \cong \mathrm{GL}(6, 2)$ is transitive on $G\backslash\{1\}$, we may assume that $a \in S$, and since $\mathrm{Aut}(G, S)$ is transitive on S, we have that $S = a^P$. Clearly, $\mathrm{GL}(6, 2) \cong \mathrm{SL}(6, 2) \cong \mathrm{PSL}(6, 2)$. By Magma [4], a Sylow 7-subgroup of $\mathrm{PSL}(6, 2)$ is isomorphic to $\mathbb{Z}_7^2$. Thus, we only need to find every element x of order 7 in $\mathrm{Aut}(G) \cong \mathrm{PSL}(6, 2)$ such that $\langle a^{\langle x\rangle}\rangle = G$. With the calculation of Magma [4], there are 18 such elements, which form three subgroups of order 7, and the corresponding Cayley sets are: $\{a, acd, bde, df, d, bcf, e\}$, $\{a, abcf, adf, b, ae, abcdef, abf\}$, $\{a, bcdef, ab, abde, bf, cd, ad\}$. Furthermore, the three corresponding Cayley graphs are isomorphic to each other. Thus, $X \cong \mathcal{G}_{64}$. □

For a finite group G and a subset S of G, a *bi-Cayley* graph $\mathrm{BCay}(G, S)$ of G with respect to S is the bipartite graph with the vertex set $G \times \{0, 1\}$ and edge set $\{\{(g, 0), (sg, 1)\} \mid g \in G, s \in S\}$. By [7, 35], $\mathrm{BCay}(G, S)$ is connected if and only if $\langle SS^{-1}\rangle = G$ and that a bipartite graph X is a bi-Cayley graph if and only if there exists a subgroup of $\mathrm{Aut}(X)$ which acts regularly on the bipartition sets. Let $g \in G$. Then g induces an automorphism of $\mathrm{BCay}(G, S)$ as follows: $R(g) : (x, 0) \mapsto (xg, 0), (x, 1) \mapsto (xg, 1), x \in G$. Set $R(G) = \{R(g) \mid g \in G\}$. Then $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{BCay}(G, S))$ which has the bipartition sets of $\mathrm{BCay}(G, S)$ as its orbits. By [24, Lemma 2.2], for each $\alpha \in \mathrm{Aut}(G)$ and $g \in G$, we have:

$$\mathrm{BCay}(G, S) \cong \mathrm{BCay}(G, gS) \cong \mathrm{BCay}(G, Sg) \cong \mathrm{BCay}(G, S^{\alpha}).$$

The next lemma is about the existence of connected heptavalent symmetric bi-Cayley graph on the group $\mathbb{Z}_2^5$ and of order 32.

Lemma 3.2. *Any connected heptavalent bi-Cayley graph on $\mathbb{Z}_2^5$ cannot be symmetric.*

Proof. Let $G = \langle a, b, c, d, e\rangle \cong \mathbb{Z}_2^5$ and $X = \mathrm{BCay}(G, S)$ be a connected heptavalent graph. Then $|S| = 7$ and $\langle SS^{-1}\rangle = G$. Suppose to the contrary that X is symmetric.

Since all non-identity elements in G has order 2, we have that $\langle S\rangle = G$. Note that $\mathrm{Aut}(G) \cong \mathrm{GL}(5, 2)$ and all the minimal sets of generators of G have 5 elements. Thus, $\mathrm{Aut}(G)$ is transitive on the minimal sets of generators of G. Without loss of generality, we may assume that $\{1, a, b, c, d, e\} \subset S$. Set $S = \{1, a, b, c, d, x\}$. Following the above argument, $X \cong \mathrm{BCay}(G, S^{\alpha})$ with $\alpha \in \mathrm{Aut}(G)$. Each element of $G\backslash\{1, a, b, c, d, e\}$ and $\{1, a, b, c, d, e\}$ forms a connected heptavalent bi-Cayley graph. Let H be the subgroup of $\mathrm{Aut}(G)$ fixing the set $\{1, a, b, c, d, e\}$ setwise. Then $H \cong \mathrm{S}_5$. By Magma [4], H acting on $G\backslash\{1, a, b, c, d, e\}$ has four orbits and their representatives are: $\{ab\}$, $\{abc\}$, $\{abcd\}$ and $\{abcde\}$. However, by Magma [4], all the four corresponding graphs are not symmetric, a contradiction. $\square$

Construction 3.2. Let $G = 2.\mathrm{PSL}(2, 7).2i$. Then by Atlas [6], G has a representation of degree 96, its suborbits are: 1^{12} and 7^{12}. These suborbits of length 7 can form orbital graphs of valency 7 or 14. By Magma [4], up to isomorphism, there is only one orbital graph of valency 7, denoted by $\mathcal{G}_{96}$. Furthermore, $\mathrm{Aut}(\mathcal{G}_{96}) \cong \mathbb{Z}_2.(\mathrm{PGL}(2, 7) \times \mathrm{S}_3)$. Conversely, any connected heptavalent symmetric graph of order 96 admitting $G = 2.\mathrm{PSL}(2, 7).2i$ as an arc-transitive automorphism group is isomorphic to $\mathcal{G}_{96}$.

4. Main result

This section is devoted to classifying the connected heptavalent symmetric graphs of order $32p$ for each prime p. In what follows, we always let X be a connected heptavalent graph of order $32p$, $A = \mathrm{Aut}(X)$ and A_v the vertex stabilizer of $v \in V(X)$ in A. By Proposition 2.2, $|A_v| \bigm| 2^{24}{\cdot}3^4{\cdot}5^2{\cdot}7$ and hence $|A| \bigm| 2^{29}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$. Let N be a minimal normal subgroup of A. Then $N = T^k$ with T a non-abelian simple group, or $N \cong \mathbb{Z}_2^i$ with $|\mathbb{Z}_2^i| \bigm| 32p$ or $\mathbb{Z}_p$.

Lemma 4.1. *Suppose that $p = 2$. Then $X \cong \mathcal{G}_{64}$.*

Proof. Since $p = 2$, we have that $|N| \bigm| 2^{30}{\cdot}3^4{\cdot}5^2{\cdot}7$. Assume that N is nonsolvable. Then $N \cong T^k$ with T a non-abelian simple $\{2, 3, 5, 7\}$-group. Clearly, $3 \bigm| |N|$ and hence $N_v \neq 1$. By Proposition 2.1, N has at most two orbits on $V(X)$. It follows that $2^5 \bigm| |N|$ and $|N_v| = |N|/32$ or $|N|/64$. If $k \geq 2$, then by Proposition 2.7, $N \cong \mathrm{A}_6^2$. However, by Magma [4], A_6^2 has no subgroup of order $2{\cdot}3^2{\cdot}5^2$ or $3^2{\cdot}5^2$, a contradiction. Thus, $k = 1$ and $N = T$ is a non-abelian simple group. By Proposition 2.7, N is isomorphic to

$$\mathrm{PSU}(3, 3),\ \mathrm{PSU}(4, 2),\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10},\ \mathrm{PSL}(3, 4),\ \mathrm{PSp}(6, 2),\ \mathrm{J}_2.$$

However, by Atlas [6] and Magma [4], N has no subgroup of order $|N_v| = |N|/32$ or $|N|/64$, a contradiction.

Thus, N is solvable and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_2^5$ or $\mathbb{Z}_2^6$. If $N \cong \mathbb{Z}_2^6$, then by Lemma 3.1, $X \cong \mathcal{G}_{64}$. If $N \cong \mathbb{Z}_2^5$, then X is a Bi-Cayley graph on N, by Lemma 3.2, this is impossible. For the remaining four cases, N has at least 4 obits on $V(X)$, by Proposition 2.1, X_N is a heptavalent symmetric graph with A/N as an arc-transitive automorphism group. Since the order of X_N is at least 8, we have that $N \ncong \mathbb{Z}_2^4$. Thus, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$.

Let $N \cong \mathbb{Z}_2$. Then by Proposition 2.6, $X_N \cong \mathcal{G}_{32}$ and by Magma [4], the minimal arc-transitive subgroup of $\mathrm{Aut}(\mathcal{G}_{32}) \cong \mathbb{Z}_2.(\mathrm{PGL}(2, 7) \times \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_4.\mathrm{PSL}(2, 7)$ or $2.\mathrm{PSL}(2, 7).2i$. It forces that A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_4.\mathrm{PSL}(2, 7)$ or $2.\mathrm{PSL}(2, 7).2i$. Furthermore, M/N has a normal subgroup $K/N \cong \mathrm{SL}(2, 7)$. The normality of the derived subgroup K' in K

implies that $K'N/N \trianglelefteq K/N$. Since K/K' is abelian, K' is non-solvable and so is $K'N/N$. It forces that $K'N/N = K/N \cong \mathrm{SL}(2,7)$ and $K'N = K$. Note that $N \cong \mathbb{Z}_2$. Thus, $K'N = K' = K$ or $K' \times N$. For the former, we have that K is a central extension of a group of order 4 by $\mathrm{PSL}(2,7)$, this is impossible because $\mathrm{Mult}(\mathrm{PSL}(2,7)) \cong \mathbb{Z}_2$. For the later, $K = \mathrm{SL}(2,7) \times \mathbb{Z}_2$. It follows that K has a characteristic subgroup $H \cong \mathrm{SL}(2,7)$. The normality of K in M forces that $H \trianglelefteq M$. Since $|H| = 2^4{\cdot}21$, we have that H acting on $V(X)$ has at least 4 orbits and hence H is semiregular, a contradiction.

Let $N \cong \mathbb{Z}_2^2$. Then by Proposition 2.5, $X_N \cong K_{8,8} - 8K_2$ and $A/N \lesssim \mathrm{S}_8 \times \mathbb{Z}_2$. By Magma [4], $\mathrm{S}_8 \times \mathbb{Z}_2$ has minimal arc-transitive subgroups isomorphic to $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_7$, $\mathrm{PGL}(2,7)$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$. Then A/N has a subgroup $M/N \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_7$, $\mathrm{PGL}(2,7)$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$.

Assume that $M/N \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_7$. Then M/N has a normal regular subgroup $K/N \cong \mathbb{Z}_2^4$ and $K \cong \mathbb{Z}_2^2.\mathbb{Z}_2^4$. Clearly, K is regular on $V(X)$. It follows that $X \cong \mathrm{Cay}(K,S)$ and $7 \,\big|\, |\mathrm{Aut}(K,S)| \,\big|\, |\mathrm{Aut}(K)|$. By GAP [10], $K \cong \mathbb{Z}_2^6$, $\mathbb{Z}_4^3$, $\mathbb{Z}_8 \times \mathbb{Z}_2^3$, $\mathbb{Z}_4 \times \mathbb{Z}_2^4$, `SmallGroup`(64,82), `SmallGroup`(64,261) or `SmallGroup`(64,262). With the similar argument as above, we can deduce that $K \cong \mathbb{Z}_2^6$ and $X \cong \mathcal{G}_{64}$.

Assume that $M/N \cong \mathrm{PGL}(2,7)$ or $M/N \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$. By Atlas [6], the Schur multiplier $\mathrm{Mult}(\mathrm{PSL}(2,7)) = \mathbb{Z}_2$ and $\mathrm{Aut}(N) \cong \mathrm{GL}(2,2)$ is solvable. Thus, $\mathrm{PSL}(2,7)$ commutes with N. It is easy to see that M has a normal subgroup $K \cong \mathbb{Z}_2$. It follows that X_K is a connected heptavalent symmetric graph of order 32, and by Proposition 2.6, $X_N \cong \mathcal{G}_{32}$. With the similar argument as above, we can also deduce that this is also impossible.

Let $N \cong \mathbb{Z}_2^3$. Then $X_N \cong K_8$ and $A/N \lesssim \mathrm{S}_8$. By Magma [4], $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$ and $\mathrm{PSL}(2,7)$ are minimal arc-transitive subgroups of S_8. Thus, we may assume that A/N has a subgroup $M/N \cong \mathrm{PSL}(2,7)$ or $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$. If $M/N \cong \mathrm{PSL}(2,7)$, then since $\mathrm{Aut}(N) \cong \mathrm{GL}(3,2) \cong \mathrm{SL}(3,2) \cong \mathrm{PSL}(2,7)$ and $\mathrm{Mult}(\mathrm{PSL}(2,7)) \cong \mathbb{Z}_2$, we have that $M \cong \mathrm{ASL}(3,2)$, $\mathbb{Z}_2^2 \times \mathrm{SL}(2,7)$ or $\mathbb{Z}_2^3 \times \mathrm{PSL}(2,7)$. However, by Magma [4], there is no connected heptavalent symmetric graph on these three groups, a contradiction. If $M/N \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, then M has normal regular subgroup $K \cong \mathbb{Z}_2^3.\mathbb{Z}_2^3$. It follows that X is a normal Cayley graph on K, that is, $X \cong \mathrm{Cay}(K,S)$. The normality of X implies that $7 \,\big|\, |\mathrm{Aut}(K)|$. By GAP [10], there are 7 such groups: $\mathbb{Z}_2^6$, $\mathbb{Z}_4^3$, $\mathbb{Z}_8 \times \mathbb{Z}_2^3$, $\mathbb{Z}_4 \times \mathbb{Z}_2^4$, `SmallGroup`(64,82), `SmallGroup`(64,261) and `SmallGroup`(64,262). Since X is normal, all elements in S have order 2 and hence K must be generated by elements of order 2. Thus, $K \ncong \mathbb{Z}_4^3$, $\mathbb{Z}_8 \times \mathbb{Z}_2^3$ or $\mathbb{Z}_4 \times \mathbb{Z}_2^4$ and by GAP [10], $K \ncong$`SmallGroup`(64,82) or `SmallGroup`(64,262). Assume that K is isomorphic to `SmallGroup`(64,261). Then by GAP [10], $|\mathrm{Aut}(K)| = 2^{15}{\cdot}3{\cdot}7$. Let p be a Sylow 7-subgroup of $\mathrm{Aut}(K,S) \leq \mathrm{Aut}(K)$. Take $s \in S$. The normality of X implies that the order $o(s) = 2$ and $S = s^P$. With the calculation of GAP [10], the orbits of P acting on the all the elements of K are: 1^5 and 7^6. It follows that S has six choices. On the other hand, the connectivity of X forces that $\langle S\rangle = K$. However, all these six choices cannot generate the group K, a contradiction. Thus, $K \cong \mathbb{Z}_2^6$ and $X \cong \mathcal{G}_{64}$. $\square$

Lemma 4.2. *Suppose that $p = 3$. Then $X \cong \mathcal{G}_{96}$.*

Proof. Since $p = 3$, we have that $|V(X)| = 64{\cdot}3$ and $|N| \mid 2^{29}{\cdot}3^5{\cdot}5^2{\cdot}7$. Assume that N is non-solvable, then $N = T^k$ with T a non-abelian simple $\{2, 3, 5, 7\}$-group. Since $|N|$ has at least 3 prime factors, we have that $N_v \neq 1$. By Proposition 2.1, N has at most two orbits on $V(X)$. It forces that $2^4{\cdot}3 \mid |N|$ and $|N_v| = |N|/(16{\cdot}3)$ or $|N|/(32{\cdot}3)$. If $k \geq 2$, then $|T|^2 \mid |N|$ and $7 \nmid |T|$. By Proposition 2.7, $N \cong \mathrm{A}_5^2$ or A_6^2. Thus, $|N_v| = 3{\cdot}5^2$ for $N \cong \mathrm{A}_5^2$ and $|N_v| = 2^2{\cdot}3^2{\cdot}5^2$ or $2{\cdot}3^2{\cdot}5^2$ for $N \cong \mathrm{A}_6^2$. However, by Magma [4], A_5^2 and A_6^2 has no subgroups of such orders, a contradiction. It follows that $k = 1$ and $N = T$ is a non-abelian simple group. Checking the orders of the simple groups in Proposition 2.7, N is isomorphic to

$$\mathrm{PSU}(3,3),\ \mathrm{PSU}(4,2),\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10},\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2).$$

However, by Atlas [6] and Magma [4], all the groups listed above have no subgroups of order $|N_v| = |N|/(16{\cdot}3)$ or $|N|/(32{\cdot}3)$, a contradiction.

Thus, N is solvable, and $N \cong \mathbb{Z}_3$, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^5$. Note that there is no heptavalent regular graph of order 3 or 6. Thus, $N \ncong \mathbb{Z}_2^5$ or $\mathbb{Z}_2^4$. By Proposition 2.4, there is no connected heptavalent symmetric graph of order 12, so we have $N \ncong \mathbb{Z}_2^3$. Thus, we have that $N \cong \mathbb{Z}_3$, $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$.

Let $N \cong \mathbb{Z}_3$. Then X_N is a heptavalent symmetric graph of order 32. By Proposition 2.6, $X_N \cong \mathcal{G}_{32}$. Since $\mathbb{Z}_4.\mathrm{PSL}(2,7)$ and $2.\mathrm{PSL}(2,7).2i$ are the minimal arc-transitive subgroup of $\mathrm{Aut}(\mathcal{G}_{32})$, we have that A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_4.\mathrm{PSL}(2,7)$ or $2.\mathrm{PSL}(2,7).2i$, and M/N has a normal subgroup $K/N \cong \mathrm{SL}(2,7)$. By "N/C-Theorem" (see [17, Chapter I, Theorem 4.5]), $K/C_K(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_2$. It is easy to see that $C_K(N) = K$ and hence $K \cong \mathrm{SL}(2,7) \times \mathbb{Z}_3$. It forces that K has a characteristic subgroup $H \cong \mathrm{SL}(2,7)$, which is normal in M. Since H has at least 3 orbits on $V(X)$, by Proposition 2.1, H is semiregular and $|H| \mid 32p$, a contradiction.

Let $N \cong \mathbb{Z}_2$. Then X_N is a heptavalent symmetric graph of order 48. By Proposition 2.6, $X_N \cong \mathcal{G}_{48}$ and $A/N \lesssim \mathrm{PGL}(2,7) \times \mathrm{S}_3$. By Magma [4], $\mathrm{PGL}(2,7) \times \mathrm{S}_3$ has minimal arc-transitive subgroups $\mathrm{PGL}(2,7)$ and $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$. It follows that A/N has an arc-transitive subgroup $M/N \cong \mathrm{PGL}(2,7)$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$. In both cases, M/N has a normal subgroup $K/N \cong \mathrm{PSL}(2,7)$. Note that $\mathrm{Mult}(\mathrm{PSL}(2,7)) \cong \mathbb{Z}_2$ by Atlas [6]. Thus, $K \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$ or $\mathrm{SL}(2,7)$. For $K \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$, we have that K has a characteristic subgroup $H \cong \mathrm{PSL}(2,7)$, which is normal in M. However, H acting on $V(X)$ has 4 orbits and hence is semiregular by Proposition 2.1. This is impossible because $H_v \cong \mathbb{Z}_7$. For $K \cong \mathrm{SL}(2,7)$, if $M/N \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$ then $M \cong \mathrm{SL}(2,7) \times \mathbb{Z}_2$ or $\mathbb{Z}_4.\mathrm{PSL}(2,7)$. By Magma [4], there is no such graph admitting $\mathrm{SL}(2,7) \times \mathbb{Z}_2$ as an arc-transitive subgroup, and there is only one heptavalent symmetric orbital graph on $\mathbb{Z}_4$.PSL$(2,7)$, which is isomorphic to $\mathcal{G}_{96}$. If $M/N \cong$ PGL$(2,7)$, then by Atlas [6], $M \cong 2$.PGL$(2,7)$ or 2.PSL$(2,7).2i$. By Magma [4], the only possibility is $M \cong 2$.PSL$(2,7).2i$, and by Construction 3.2, $X \cong \mathcal{G}_{96}$.

Let $N \cong \mathbb{Z}_2^2$. Then X_N is a heptavalent symmetric graph of order 24. By Proposition 2.5, $X_N \cong \mathcal{G}_{24}$ and $A/N \lesssim$ PGL$(2,7)$. Clearly, A/N has an arc-transitive subgroup $M/N \cong$ PSL$(2,7)$. Since Mult(PSL$(2,7)) \cong \mathbb{Z}_2$, we have that $M \cong$ SL$(2,7) \times \mathbb{Z}_2$ or PSL$(2,7) \times \mathbb{Z}_2^2$. Thus, M has a normal subgroup $K \cong \mathbb{Z}_2$. With a similar argument as above, we can easily deduce that this is impossible. □

Lemma 4.3. *Suppose that $p \geq 5$. Then there is no new graph.*

Proof. Since $p \geq 5$, we have that $|N| \,\big|\, 2^{29}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$. we separate the proof into two cases: A has a solvable minimal normal subgroup; A has no solvable normal subgroup.

Case 1: A has a solvable minimal normal subgroup.

Since A has a solvable minimal normal subgroup, we may assume that N is solvable, and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_2^5$ or $\mathbb{Z}_p$. Note that there is no heptavalent regular graph of odd order. Thus $N \ncong \mathbb{Z}_2^5$. By Propositions 2.4 and 2.5, there is no heptavalent symmetric graph of order $4p$ or $8p$ with $p \geq 5$. Thus, $N \ncong \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$ and hence $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^4$ or $\mathbb{Z}_p$.

Let $N \cong \mathbb{Z}_p$. Then X_N has order 32, and by Proposition 2.6, $X_N \cong \mathcal{G}_{32}$ and $A/N \leq \mathbb{Z}_2.(\text{PGL}(2,7) \times \mathbb{Z}_2)$. By Magma [4], $\mathbb{Z}_2.(\text{PGL}(2,7) \times \mathbb{Z}_2)$ has two minimal arc-transitive subgroups: $\mathbb{Z}_4$.PSL$(2,7)$ and 2.PSL$(2,7).2i$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_4$.PSL$(2,7)$ or 2.PSL$(2,7).2i$ and M/N has a normal subgroup $K/N \cong$ SL$(2,7)$. By "N/C-Theorem" (see [17, Chapter I, Theorem 4.5]), $K/C_K(N) \lesssim \text{Aut}(N) \cong \mathbb{Z}_{p-1}$. It is easy to see that $C_K(N) = K$ and hence $K \cong \text{SL}(2,7) \times \mathbb{Z}_p$. It forces that K has a characteristic subgroup $H \cong$ SL$(2,7)$, which is normal in M. Since H has at least p orbits on $V(X)$, by Proposition 2.1 H is semiregular and $|H| \,\big|\, 32p$, a contradiction.

Let $N \cong \mathbb{Z}_2^4$. Then X_N has order $2p$, and by Proposition 2.3, $X_N \cong K_{7,7}$ or $G(2p,7)$ and $A/N \lesssim (\text{S}_7 \times \text{S}_7) \rtimes \mathbb{Z}_2$ or $(\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.

Assume that $X_N \cong K_{7,7}$. Then $p = 7$. By Magma [4], $(\text{S}_7 \times \text{S}_7) \rtimes \mathbb{Z}_2$ has minimal arc-transitive subgroups $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ and $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. By "N/C-Theorem", $M/C_M(N) \lesssim \text{Aut}(N) \cong \text{GL}(4,2)$. Since $|\text{GL}(4,2)| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$, we have that $7 \,\big|\, |C_M(N)|$. Let H be a Sylow 7-subgroup of $C_M(N)$. Then $H = \mathbb{Z}_7^2$ or $\mathbb{Z}_7$ is normal in M. Considering the quotient graph X_H. Note that $p = 7$. Thus, X_H has order 32. By Propositions 2.1 and 2.6, H is semiregular, $H \cong \mathbb{Z}_7$ and $X_H \cong \mathcal{G}_{32}$. In particular, M/H is an arc-transitive subgroup of X_H. Clearly, M/H is solvable. However, by Magma [4], $\text{Aut}(\mathcal{G}_{32}) \cong \mathbb{Z}_2.(\text{PGL}(2,7) \times \mathbb{Z}_2)$ has no solvable arc-transitive subgroup, a contradiction.

Assume that $X_N \cong G(2p,7)$. Then $A/N \lesssim (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$ with $7 \,\big|\, (p-1)$. Since $G(2p,7)$ is 1-regular, we have $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. By "N/C-Theorem",

$A/C_A(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{GL}(4,2)$. Since $7 \mid (p-1)$ and $p \nmid |\mathrm{GL}(4,2)|$, we have $p \mid |C_A(N)|$. Let P be a Sylow p-subgroup of $C_A(N)$. Then $P \cong \mathbb{Z}_p$ and P is characteristic in $C_A(N)$. The normality of $C_A(N)$ implies that $P \trianglelefteq A$. Thus, X_P has order 32 and $X_P \cong \mathcal{G}_{32}$. By Proposition 2.1, $A/P \cong (\mathbb{Z}_2^4 \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$ is a solvable arc-transitive subgroup of X_P. However, by Magma [4], $\mathrm{Aut}(\mathcal{G}_{32}) \cong \mathbb{Z}_2.(\mathrm{PGL}(2,7) \times \mathbb{Z}_2)$ has no solvable arc-transitive subgroup, a contradiction.

Let $N \cong \mathbb{Z}_2$. Then since $p \geq 5$ and by Proposition 2.6, $X_N \cong \mathcal{G}_{112}$ or $\mathcal{G}_{(2^3,2p)}$ and $A/N \lesssim (\mathbb{Z}_2^3 \times D_{14}) \rtimes F_{21}$ or $(\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$. In both cases, we can easily deduce that $(\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$ is the unique minimal arc-transitive subgroup. Thus, we may assume that A/N has an arc-transitive subgroup $M/N \cong (\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$ with $p = 7$ or $7 \mid (p-1)$. Since $N \cong \mathbb{Z}_2$, we have that N lies in the center of M. It follows that a Sylow p-subgroup, say P, commutes with N. Thus, P is normal in M and by Proposition 2.1, X_P is heptavalent graph of order 32 with M/P as an arc-transitive subgroup. By Proposition 2.6, $X_P \cong \mathcal{G}_{32}$. Clearly, M/P is solvable. This is impossible because $\mathrm{Aut}(\mathcal{G}_{32})$ has no solvable arc-transitive subgroup by Magma [4].

Case 2: Suppose that A has no solvable normal subgroup.

Since every normal subgroup of A is non-solvable, N is non-solvable. Thus, $N = T^k$ with T a non-abelian simple group and $|T| \mid |N| \mid 2^{29} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p$. It follows that T is isomorphic to one of the groups listed in Proposition 2.7. Note that T has at least 3 prime factors. By checking the order of T, we have that $T_v \neq 1$ and hence $N_v \neq 1$. By Proposition 2.1, N has at most two orbits on $V(X)$, that is, $|N_v| = |N|/16p$ or $|N|/32p$. The normality of N in A implies that $N_v \trianglelefteq A_v$.

Assume that $k \geq 2$. Note that $p \geq 5$. Thus, $N \cong \mathrm{A}_5^2$, A_5^3 or A_6^2 for $p = 5$; $N \cong \mathrm{PSL}(2,7)^2$, $\mathrm{PSL}(2,8)^2$, A_7^2, A_8^2 or $\mathrm{PSL}(3,4)^2$ for $p = 7$. Let $N \cong \mathrm{A}_5^2$, A_6^2, A_5^3, $\mathrm{PSL}(2,8)$, A_7^2 or $\mathrm{PSL}(3,4)^2$. Then $|N_v| = |N|/16p$ or $|N|/32p$. By Magma [4], N has no subgroups of such orders, a contradiction. Let $N \cong \mathrm{PSL}(2,7)^2$. Then by Atlas [6], $N_v \cong \mathrm{A}_4 \times F_{21}$, $\mathrm{S}_3 \times F_{21}$. By Proposition 2.2, A_v has no such normal subgroups, a contradiction. Let $N \cong \mathrm{A}_8^2$. Then $|N_v| = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ or $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$. By Magma [4], $N_v \cong \mathrm{A}_8 \times ((A_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2)$ or $\mathrm{A}_7 \times \mathrm{S}_6$. However, A_v has no such normal subgroups, a contradiction.

Thus, $k = 1$ and N is a non-abelian simple group. Since N has at most two orbits, we have $16p \mid |N|$. It forces that N is isomorphic to the groups listed in Proposition 2.7 except for

$$\mathrm{A}_5,\ \mathrm{A}_6,\ \mathrm{PSL}(2,7),\ \mathrm{PSL}(2,8),\ \mathrm{A}_7,\ \mathrm{PSL}(2,11),\ \mathrm{PSL}(2,13),\ \mathrm{PSL}(2,19),$$
$$\mathrm{PSL}(2,25),\ \mathrm{PSL}(2,27),\ \mathrm{PSL}(2,29),\ \mathrm{PSL}(2,41) \text{ and } \mathrm{PSL}(2,71).$$

Note that $p \geq 5$. Since $|N_v| = |N|/16p$ or $|N|/32p$, we have that a Sylow 3-subgroup of N_v is also a Sylow 3-subgroup of N. By Proposition 2.2, a Sylow 3-subgroup of A_v is elementary abelian, and so is that of N_v. With the calculation of Magma [4], $N \ncong \mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(4,2)$, $\mathrm{PSU}(3,3)$, A_9, A_{10}, $\mathrm{PSU}(3,8)$, $\mathrm{PSp}(6,2)$, M_{12}, J_2, $\mathrm{P\Omega}^+(8,2)$, ${}^2F_4(2)'$, A_{11}, $\mathrm{PSL}(4,4)$, $\mathrm{P\Omega}^-(8,2)$ or $G_2(4)$.

Let $N \cong \mathrm{A}_8$. Then $|N_v| = 2^2{\cdot}3^2{\cdot}7$ or $2{\cdot}3^2{\cdot}7$ for $p = 5$ or $|N_v| = 2^2{\cdot}3^2{\cdot}5$ or $2{\cdot}3^2{\cdot}5$ for $p = 7$. By Magma [4], $|N_v| = 2^2{\cdot}3^2{\cdot}5$ and $N_v \cong \mathrm{A}_5 \times \mathbb{Z}_3$. The normality of N in A implies that N_v is normal in A_v. However, by Proposition 2.2, A_v has no normal subgroup isomorphic $\mathrm{A}_5 \times \mathbb{Z}_3$, a contradiction.

Let $N \cong \mathrm{PSL}(2,16)$ or $\mathrm{PSL}(2,31)$. Then $|N_v| = 15$ for $N \cong \mathrm{PSL}(2,16)$, $|N_v| = 15$ or 30 for $N \cong \mathrm{PSL}(2,31)$. By Magma [4], $N_v \cong \mathbb{Z}_{15}$ or D_{30}. However, A_v has no such normal subgroup, a contradiction.

Let $N \cong \mathrm{PSL}(2,49)$. Then $|N_v| = 3{\cdot}5^2{\cdot}7$. By Magma [4], $\mathrm{PSL}(2,49)$ has no subgroup of order $3{\cdot}5^2{\cdot}7$, a contradiction.

Let $N \cong \mathrm{PSL}(2,81)$. Then $|N_v| = 3^4{\cdot}5$. By Magma [4], $N_v \cong \mathbb{Z}_3^4 \rtimes \mathbb{Z}_5$. However, A_v has no such normal subgroup, a contradiction.

Let $N \cong \mathrm{PSL}(2,127)$, $\mathrm{PSL}(3,4)$, M_{11}, $\mathrm{Sz}(8)$, $\mathrm{PSL}(2,449)$, $\mathrm{PSL}(2,2^6)$,. Then $|N_v| = |N|/16p$ or $|N|/32p$. By Magma [4], N has no subgroups of such orders, a contradiction.

Let $N \cong \mathrm{PSU}(3,4)$. Then $|N_v| = 2^2{\cdot}3{\cdot}5^2$ or $2{\cdot}3{\cdot}5^2$. By Atlas [6], $N_v \cong \mathrm{A}_5 \times \mathbb{Z}_5$ or $\mathbb{Z}_5^2 \rtimes \mathrm{S}_3$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{PSp}(4,4)$. Then $|N_v| = 2^4{\cdot}3^2{\cdot}5^2$ or $2^3{\cdot}3^2{\cdot}5^2$. By Atlas [6], the only possibility is: $|N_v| = 2^4{\cdot}3^2{\cdot}5^2$ and $N_v \cong \mathrm{A}_5^2$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{PSL}(5,2)$. Then $|N_v| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$ or $2^5{\cdot}3^2{\cdot}5{\cdot}7$. By Magma [4], the only possibility is: $|N_v| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$ and $N_v \cong \mathrm{PSL}(4,2)$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{M}_{22}$. Then $|N_v| = 2^2{\cdot}3^2{\cdot}5{\cdot}7$ or $2^3{\cdot}3^2{\cdot}5{\cdot}7$. By Magma [4], the only possibility is: $|N_v| = 2^3{\cdot}3^2{\cdot}5{\cdot}7$ and $N_v \cong \mathrm{A}_7$. Clearly, N has two orbits on $V(X)$ and $p = 11$. By "N/C-Theorem", $A/C_A(N) \lesssim \mathrm{Aut}(N)$. If $C_A(N) \neq 1$, then $C_A(N) \cong C_A(N)N/N \leq A/N$. It forces that $|C_A(N)| \,\big|\, |A_v/N_v|$. By Proposition 2.2, $C_A(N)$ is a $\{2,3,5\}$-group. Thus, $C_A(N)$ acting on $V(X)$ has at least p orbits. By Proposition 2.1, $C_A(N)$ is semiregular and $|C_A(N)| \,\big|\, 32p$. It follows that $C_A(N)$ is solvable, which is contrary to our assumption. Thus, $C_A(N) = 1$ and $A \lesssim \mathrm{Aut}(N) \cong \mathrm{M}_{22} \rtimes \mathbb{Z}_2$. The intransitivity of N implies that $A \cong \mathrm{M}_{22} \rtimes \mathbb{Z}_2$. Note that $|V(X)| = 32{\cdot}11 = 352$. By Atlas [6], $\mathrm{M}_{22} \rtimes \mathbb{Z}_2$ has a representation of degree 352, its lengths of suborbits are: $1, 15, 35, 70, 105, 126$. Thus, there is no connected heptavalent orbital graph on $\mathrm{M}_{22} \rtimes \mathbb{Z}_2$, a contradiction. $\square$

Combining Lemmas 4.1, 4.2 and 4.3, we have the complete classification of connected heptavalent symmetric graphs of order $32p$ for each prime p.

Theorem 4.1. *Let p be a prime. Then up to isomorphism, there are only two connected heptavalent symmetric graphs of order $32p$, that is, $\mathcal{G}_{64}$ with $p = 2$ and $\mathcal{G}_{96}$ with $p = 3$.*

5. Conclusion

As is known to all, arc-transitive graphs have much higher symmetries and much larger full automorphism groups, and for prime valent arc-transitive graphs, the structure of their vertex stabilizers can be definitely controlled. Thus, the characterization and classification of such graphs can be achieved, and this reveals not only the local action but also global action of the full automorphism group acting on vertices and arcs. In the paper, we classify the arc-transitive graphs of order $32p$ and valency seven for each prime p. As a natural continuation, could we find and construct infinite families of arc-transitive graphs of order $32p$ with some more larger prime valency? Furthermore, we will classify arc-transitive graphs of order $32p$ and more general prime valencies.

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L-mosaics and orthomodular lattices

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Abstract. In this paper, we introduce a class of hypercompositional structures called dualizable L-mosaics. We prove that their category is equivalent to that formed by ortholattices and we formulate an algebraic property characterizing orthomodularity, suggesting possible applications to quantum logic. To achieve this, we establish an equivalence between the category of bounded join-semilattices and that of L-mosaics, thereby providing a categorical foundation for our framework.

Keywords: Hypercompositional Structure, Orthomodular Lattice, Mosaic, Polygroup, Effect algebra, Quantum logic.

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1. Introduction

We introduce the notion of *L-mosaic*, by extending a point of view initially pushed forward by T. Nakano in his studies on modular lattices [12] and by allowing non-associative multivalued operations as introduced in a recent work of S. Nakamura and M. L. Reyes [11]. We find a natural equivalence of categories between the category formed by ortholattices and the category of the multivalued algebraic structures we call *dualizable L-mosaics*. This study is a

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follow-up on a recent article by A. Jenčová and G. Jenča suggesting applications of multivalued algebraic structures in quantum logic [7].

The paper is structured as follows. In Section 2, we introduce the notion of mosaics, following recent developments on multivalued algebraic structures. Section 3 is devoted to the definition and analysis of Nakano mosaics, a class of L-mosaics naturally associated with bounded join-semilattices. In Section 4, we review the theory of orthomodular lattices, setting the stage for Section 5, where we establish an equivalence between the categories of ortholattices and dualizable L-mosaics. We also characterize orthomodularity in terms of a structural condition on Nakano mosaics. Finally, in Section 6, we discuss possible connections between these algebraic frameworks and quantum logic, and suggest directions for future research within quantum foundations.

2. Mosaics

Following Nakamura and M. L. Reyes [11], we introduce a multivalued algebraic structure called mosaic and review in detail some of their basic properties.

We consider two sets A and B. By a *multimap* $A \multimap B$ we mean a function $A \to \wp(B)$, with domain A and codomain the power set $\wp(B)$ of B. We call a multimap $f : A \multimap B$

1. *total* if $f(x) \neq \emptyset$, for all $x \in A$;
2. *partial* if it is not total;
3. *deterministic* if $|f(x)| \leq 1$, for all $x \in A$;
4. a *map* if it is total and deterministic.

The composition $g \circ f : A \multimap C$ of two multimaps $f : A \multimap B$ and $g : B \multimap C$ is defined on every $x \in A$ by the following formula:

$$(g \circ f)(x) := \bigcup_{y \in f(x)} g(y).$$

It is easily verified that this composition is associative and that the multimaps defined by the assignment $x \mapsto \{x\}$ serve as identities. Moreover, it is not difficult to verify that the obtained category formed by sets and multimaps as above is isomorphic to the familiar category `Rel` formed by sets and binary relations. The isomorphism is provided by the equivalence

$$y \in f(x) \iff (x, y) \in R,$$

where $f : A \multimap B$ is a multimap and $R \subseteq A \times B$ denotes the binary relation corresponding to f. By identifying these categories, we say that a multimap $f : A \multimap B$ *contains* a multimap $g : A \multimap B$ if the relation corresponding to f contains the relation corresponding to g.

A *(binary) multioperation* on a set A is a multimap $\boxdot : A^2 := A \times A \multimap A$. By a *magma* we mean a set A equipped with a multioperation. To any multioperation $\boxdot$ defined on a set A there correspond a multioperation *dual* of $\boxdot$, defined for $x, y \in A$ by the formula

$$x \boxdot^d y := y \boxdot x.$$

We call a magma $(A, \boxdot)$ *total/partial/deterministic* if so is the multimap $\boxdot$. The magma $(A, \boxdot)$ is *commutative* if the multioperations $\boxdot$ and $\boxdot^d$ coincide on A. If $(A, \boxdot)$ is total and deterministic, then we call it *classical*.

By a *morphism (of magmata)* from $(A, \boxdot_A)$ to $(B, \boxdot_B)$ we mean a function $f : A \to B$ such that, for all $x, y \in A$, the following inclusion holds:

$$f(x \boxdot_A y) \subseteq f(x) \boxdot_B f(y).$$

By a *strong morphism (of magmata)* from $(A, \boxdot_A)$ to $(B, \boxdot_B)$ we mean a function $f : A \to B$ such that, for all $x, y \in A$, the following inclusion holds:

$$f(x \boxdot_A y) = f(x) \boxdot_B f(y).$$

An injective morphism of magmata from $f : (A, \boxdot_A) \to (B, \boxdot_B)$ is called an *embedding* if

$$f(x \boxdot_A y) = (f(x) \boxdot_B f(y)) \cap f(A).$$

It is easily seen that there we have a category `Mag` formed by magmata and morphisms and that isomorphisms in `Mag` are precisely bijective strong morphisms or, equivalently, surjective embeddings. In order to lay the foundations for the notion of mosaic, we review and formalize a number of basic concepts related to multivalued operations and magmatic structures. The following definitions and lemmas will be essential in characterizing the algebraic behavior of mosaics and their morphisms.

Definition 2.1. *An element e in a magma $(A, \boxdot)$ is called* neutral *if $e \boxdot x = x \boxdot e = \{x\}$ holds, for all $x \in A$. A magma $(A, \boxdot)$ with a neutral element $e \in A$ for $\boxdot$ is called* unital magma*.*

Remark 2.1. If a neutral element e exists in a magma $(A, \boxdot)$, then it is unique.

Definition 2.2. *A morphism $f : A \to A'$ between two unital magmata A and A' with neutral elements e and e', respectively, is called* unitary *if $f(e) = e'$.*

Definition 2.3. *Let $(A, \boxdot)$ be a magma and $\rho : A \to A$ an endofunction. We say that $(A, \boxdot)$ is* reversible with respect to ρ *(briefly, ρ-*reversible*) if*

(RE) *$z \in x \boxdot y$ implies both $x \in z \boxdot \rho(y)$ and $y \in \rho(x) \boxdot z$, for all $x, y, z \in A$.*

Definition 2.4 ([11, Definition 2.3]). *A (commutative) unital magma $(A, \boxdot, e)$ which is ρ-reversible with respect to some endofunction $\rho : A \to A$ is called* (commutative) mosaic. *By a* (strong) morphism of mosaics *we mean a unitary (strong) morphism of the underlying unital magmata.*

The following is also observed in [11]. We make a slightly more precise statement and write down a short proof.

Lemma 2.1 ([11]). *Let $(A, \boxdot, e, \rho)$ be a mosaic. Then ρ is a unitary isomorphism of magmata*

$$\rho : (A, \boxdot, e) \xrightarrow{\sim} (A, \boxdot^d, e)$$

satisfying the following property:

(RINV) $e \in (x \boxdot \rho(x)) \cap (\rho(x) \boxdot x)$*, for all $x \in A$.*

In addition, the equivalences

$$z \in x \boxdot y \iff x \in z \boxdot \rho(y) \iff y \in \rho(x) \boxdot z$$

hold for all $x, y, z \in A$.

Proof. Indeed, $x \in (x \boxdot e) \cap (e \boxdot x)$ and ρ-reversibility imply $e \in (\rho(x) \boxdot x) \cap (x \boxdot \rho(x))$. Conversely, if $e \in (y \boxdot x) \cap (x \boxdot y)$ for some $y \in A$, then by ρ-reversibility we may deduce that $y \in e \boxdot \rho(x) = \{\rho(x)\}$ and hence $y = \rho(x)$. It follows that $\rho(e) = e$ and that ρ is an involution. In addition, the validity of the following equivalences is readily verified, for all $x, y, z \in A$:

$$\begin{aligned}
\rho(z) \in \rho(x) \boxdot^d \rho(y) &\iff \rho(z) \in \rho(y) \boxdot \rho(x) \\
&\iff \rho(y) \in \rho(z) \boxdot \rho(\rho(x)) = \rho(z) \boxdot x \\
&\iff x \in \rho(\rho(z)) \boxdot \rho(y) = z \boxdot \rho(y) \\
&\iff z \in x \boxdot \rho(\rho(y)) = x \boxdot y.
\end{aligned}$$

This shows that ρ is a strong morphism of magmata $(A, \boxdot) \to (A, \boxdot^d)$ and thus an isomorphism. The rest of the assertions follow as well. $\square$

Definition 2.5. *For an element x in a unital magma $(A, \boxdot, e)$, we call any $y \in A$ such that $e \in (x \boxdot y) \cap (y \boxdot x)$ an* inverse *of x. If in $(A, \boxdot, e)$ all elements have a unique inverse, then we denote by x^{-1} the inverse of any $x \in A$ and call $(A, \boxdot, e)$ an* invertible magma.

The following is an immediate consequence of Lemma 2.1.

Corollary 2.1. *Let $(A, \boxdot, e, \rho)$ be a mosaic. Then $\rho(x)$ is the unique inverse of x in A, for all $x \in A$.*

Lemma 2.2. *Let $(A, \boxdot, e)$ and $(A', \boxdot', e')$ be invertible magmata and $f : A \to A'$ a unitary morphism. Then $f(x^{-1}) = f(x)^{-1}$, for all $x \in A$.*

Proof. The standard argument as, e.g., in [9, Remark 2.6], applies. $\square$

Definition 2.6 ([4])**.** *A* (commutative) polygroup $(P, \boxdot, e)$ *is an invertible (commutative) magma, which is reversible with respect to the endofunction defined as* $x \mapsto x^{-1}$ *and where* $\boxdot$ *is* associative*, that is, the following property is valid:*

(ASC) $(x \boxdot y) \boxdot z = x \boxdot (y \boxdot z)$*, for all* $x, y, z \in P$

Remark 2.2. It follows from Corollary 2.1 that polygroups are precisely associative mosaics.

Definition 2.7. *By a* (strong) morphism of polygroups *we mean a (strong) morphism of the underlying mosaics.*

Definition 2.8. *Let* $(A, \boxdot, e)$ *be a mosaic. A subset* $B \subseteq A$ *is a* submosaic *if the inclusion map* $B \to A$ *is an embedding. A submosaic* B *of a mosaic* $(A, \boxdot, e)$ *is* strong *if the inclusion map* $B \to A$ *is a strong embedding.*

Examples of polygroups and mosaics can be found e.g. in [3, 11].

Definition 2.9. *A commutative mosaic* $(A, \boxplus, 0)$ *is called a* L-mosaic *if*

(Lms1) $0, x \in x \boxplus x$*, for all* $x \in A$*.*

(Lms2) $(x \boxplus x) \boxplus (x \boxplus x) = x \boxplus x$*, for all* $x \in A$*.*

(Lms3) $(x \boxplus (x \boxplus y)) \cap ((x \boxplus y) \boxplus y) \subseteq x \boxplus y$*, for all* $x, y \in A$*.*

(Lms4) *For all* $x, y \in A$ *there is a unique* $z \in x \boxplus y$ *such that* $x, y \in z \boxplus z$*.*

Definition 2.10. *Let* $(A, \boxplus, 0)$ *be an L-mosaic. A submosaic* B *of* A *is called* L-submosaic *if* $(B, \boxplus_B, 0)$ *satisfies properties* (Lms1)*,* (Lms2)*,* (Lms3) *and* (Lms4) *(property* (Lms1) *is automatic).*

Lemma 2.3. *Let* $(A, \boxplus, 0)$ *be an L-mosaic. Then*

$$y \leq x \overset{\text{def}}{\iff} y \in x \boxplus x.$$

is an order relation on A *with respect to which* 0 *is a bottom element.*

Proof. Reflexivity and the assertion on 0 can be deduced immediately from (Lms1). Regarding transitivity, note that if $x \in y \boxplus y$ and $y \in z \boxplus z$, then by (Lms2) implies

$$x \in y \boxplus y \subseteq (z \boxplus z) \boxplus (z \boxplus z) = z \boxplus z.$$

As for antisymmetry, if $y \in x \boxplus x$ and $x \in y \boxplus y$, then by reversibility we have $x, y \in x \boxplus y$. Thus, property (Lms4) now implies $x = y$. $\square$

Lemma 2.4. *Let* $(A, \boxplus, 0)$ *be a mosaic satisfying properties* (Lms1) *and* (Lms2). *If* $x \in A$, *then* $A_x := x \boxplus x$ *is a strong submosaic of* A. *In addition, for any strong submosaic* B *of* A *we have that*

$$B = \bigcup_{x \in B} A_x = \bigcup_{x \in B} B_x.$$

Proof. Let $x \in A$, then by (Lms1) we deduce that $0 \in A_x$. By property (Lms2), for all $y, z \in A_x$ we obtain that

$$y \boxplus z \subseteq (x \boxplus x) \boxplus (x \boxplus x) = A_x.$$

This suffices to prove that A_x is a strong submosaic of A. In addition, we obtain that $B_x = A_x$, for all strong submosaics B of A, whence

$$\bigcup_{x \in B} A_x = \bigcup_{x \in B} B_x \subseteq B$$

For the converse inclusion, note that by property (Lms1) it follows that $x \in B_x$ for all $x \in B$. $\square$

The following is an obvious corollary.

Corollary 2.2. *Let* $(A, \boxplus, 0)$ *be a mosaic satisfying properties* (Lms1) *and* (Lms2) *and* B *a submosaic of* A. *Then*

$$\bar{B} := \bigcup_{x \in B} A_x$$

is a strong submosaic of A, *which is contained in any strong submosaic of* A *containing* B.

Definition 2.11. *We call* $\bar{B}$ *the* strong closure *of a submosaic* B *in a mosaic* $(A, \boxplus, 0)$ *satisfying properties* (Lms1) *and* (Lms2).

The following notion will be useful in Section 5.

Definition 2.12. *A commutative mosaic* $(A, \boxplus, 0)$ *will be called* π-dualizable *if* $\pi : A \to A$ *is an involution such that* $(A, \boxplus_\pi, \pi(0))$ *is a commutative mosaic, where for* $x, y \in A$ *we have set*

$$x \boxplus_\pi y := \pi(x) \boxplus \pi(y).$$

Clearly, π becomes an isomorphism between the mosaic $(A, \boxplus, 0)$ and its π-*dual* $(A, \boxplus_\pi, \pi(0))$, whenever A is π-dualizable. Therefore, a π-dualizable mosaic is an L-mosaic if and only if its π-dual is.

3. Nakano mosaics

We introduce in this section the objects we shall call *Nakano mosaics*. These are inspired by the work of Nakano [12], where the modularity of a lattice L is shown to be equivalent to the associativity of some multioperations defined on L.

Let us start by fixing a *bounded join-semilattice* $(L, \vee, 0)$. Thus, $\vee$ is an associative, commutative and idempotent binary operation and $x \vee 0 = x$ for all $x \in L$. We highlight that the results of this section can be easily dualized to bounded-meet semilattices. For all $x, y \in L$ we define the following subset of L:

$$\mathrm{Nak}_\vee(x, y) := \{z \in L \mid x \vee y = x \vee z = z \vee y\}.$$

Much of the following proposition was already observed in [11, Example 2.20].

Proposition 3.1. *For all $x, y \in L$ set $x \boxplus y := \mathrm{Nak}_\vee(x, y)$. Then $(L, \boxplus, 0)$ is a commutative and total mosaic, where the inverse of each $x \in L$ is x itself. Moreover, for all $x, y \in L$ we have that*

$$x \in y \boxplus y \iff x \leq y,$$

where $x \leq y :\Leftrightarrow y = x \vee y$ is the canonical partial order associated to the join-semilattice $(L, \vee)$.

Proof. By definition $\mathrm{Nak}_\vee(x, 0) = \mathrm{Nak}_\vee(0, x) = \{x\}$ holds for all $x \in L$. Moreover, it is easily verified from the definitions that

$$0 \in \mathrm{Nak}_\vee(x, y) \implies y = x,$$

for all $x, y \in L$. Thus, $(L, \boxplus, 0)$ is an invertible magma where the inverse of each $x \in L$ is x itself. To show that it is a mosaic it suffices to note that for $x, y, z \in L$ we have that:

$$z \in x \boxplus y \iff x \vee y = z \vee x = z \vee y \iff x \in y \boxplus z.$$

Furthermore, we note that $x \vee y \in x \boxplus y$ holds for all $x, y \in L$, thus $\boxplus$ is total. For the last assertion, $x \in y \boxplus y$ if and only if the displayed equation holds,

$$y = y \vee y = x \vee y,$$

namely, if and only if $x \leq y$. □

Definition 3.1. *We call the mosaic $(L, \boxplus, 0)$ associated to a bounded join-semilattice $(L, \vee, 0)$ the* Nakano mosaic *associated to $(L, \vee, 0)$.*

Proposition 3.2. *The Nakano mosaic associated to a subsemilattice*[1] *L' of L such that $0 \in L'$ is a submosaic of the Nakano mosaic associated to L.*

1. Recall that a subset L' of a join-semilattice $(L, \vee)$ is a subsemilattice if it is closed with respect to the operation $\vee$.

Proof. Let L' be a sublattice of L such that $0 \in L'$. Consider the additive Nakano mosaic $(L', \boxplus', 0)$ associated to L'. We have to show that $x \boxplus' y = (x \boxplus y) \cap L'$, for all $x, y \in L'$. Fix $x, y \in L'$. If $z \in x \boxplus' y$, then $z \in L'$ and $x \vee y = x \vee z = y \vee z$ holds in L'. Thus, the same holds in L since L' is a subsemilattice of L and we obtain that $z \in (x \boxplus y) \cap L'$. Conversely, if $z \in (x \boxplus y) \cap L'$ then $x \vee y = x \vee z = y \vee z$ holds in L, but since L' is a subsemilattice and $x, y, z \in L'$ we obtain that $z \in x \boxplus' y$. □

Lemma 3.1. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$. The following equivalence*

$$x, y \in x \boxplus y \iff x = y.$$

holds, for all $x, y \in L$

Proof. We have $x, y \in x \boxplus y$ if and only if both $x \in y \boxplus y$ and $y \in x \boxplus x$ hold. The assertion thus follows by the antisymmetry of the canonical order $\leq$ associated to the join-semilattice L. □

Lemma 3.2. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$, then*

$$(x \boxplus x) \boxplus (x \boxplus x) = x \boxplus x$$

holds, for all $x \in L$.

Proof. If $y, z \leq x$ holds in L, then $t \leq t \vee z = t \vee y = z \vee y \leq x$ follows, for all $t \in y \boxplus z$, i.e., $t \in x \boxplus x$. The converse inclusion is immediately verified after noticing that $x \in x \boxplus x$ holds for all $x \in L$. □

Fact 3.1 ([5, Lemma 24, p. 132])**.** *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$ and take $x, y, z \in L$. Then*

$$x \boxplus (y \boxplus z) \subseteq \{t \in L \mid t \vee x \vee y = t \vee x \vee z = t \vee y \vee z = x \vee y \vee z\}.$$

Remark 3.1. Reference [5] actually contains the result for lattices, but the proof uses only the join-semilattice structure.

Corollary 3.1. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$, then*

$$(x \boxplus (x \boxplus y)) \cap ((x \boxplus y) \boxplus y) \subseteq x \boxplus y$$

holds, for all $x, y \in L$.

Proof. By Fact 3.1, for $z \in (x \boxplus (x \boxplus y)) \cap ((x \boxplus y) \boxplus y)$ we deduce

$$z \vee x = z \vee x \vee x = z \vee x \vee y = x \vee x \vee y = x \vee y$$

and

$$z \vee x \vee y = z \vee y \vee y = z \vee y.$$

It follows, that

$$x \vee y = z \vee x = z \vee x \vee y = z \vee y,$$

whence $z \in x \boxplus y$. $\square$

Lemma 3.3. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$. For all $x, y, z \in L$, we have $z \in x \boxplus y$ implies $z \leq x \vee y$. It follows that $x \boxplus y \subseteq (x \vee y) \boxplus (x \vee y)$.*

Proof. If $z \in x \boxplus y$, then

$$z \vee (x \vee y) = (z \vee x) \vee y = (x \vee y) \vee y = x \vee (y \vee y) = x \vee y,$$

hence $z \leq x \vee y$. $\square$

Lemma 3.4. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$. For $x, y, z \in L$ we have that $z = x \vee y$ if and only if*

$$x, y \in z \boxplus z \text{ and } z \in x \boxplus y \tag{1}$$

hold in $(L, \boxplus, 0)$.

Proof. If $z = x \vee y$, then $x, y \leq z$ and consequently $x \vee y = z = x \vee z = y \vee z$, which shows (1). Conversely, if (1) holds, then (e.g.) $x \leq z$ follows and thus $x \vee y = x \vee z = z$. $\square$

Corollary 3.2. *The Nakano mosaic $(L, \boxplus, 0)$ associated to a bounded join-semilattice $(L, \vee, 0)$ is an L-mosaic.*

Proof. Property (Lms1) is an immediate consequence of Proposition 3.1. Property (Lms2) follows from Lemma 3.2, while Property (Lms3) follows from Lemma 3.1.

It remains to verify Property (Lms4). For this observe that for $x, y \in L$ the element $z := x \vee y \in L$ satisfies $x, y \in z \boxplus z$ and $z \in x \boxplus y$ by Lemma 3.4. This proves the existence part. For uniqueness, note that if $z' \in L$ satisfies $x, y \in z' \boxplus z'$ and $z' \in x \boxplus y$, then by Lemma 3.4, we obtain that $z' = x \vee y = z$. $\square$

We now turn to a structural result that places the theory of Nakano mosaics within a categorical framework. By identifying the correspondence between bounded join-semilattices and L-mosaics, we are able to establish a categorical equivalence that highlights the algebraic robustness of these multivalued structures.

Theorem 3.1. *Consider the category* $\mathtt{JSem}^0$ *formed by bounded join-semilattices and their morphisms*[2] *and the category* $\mathtt{LMsc}$ *formed by L-mosaics and morphisms of mosaics. Then there is an equivalence of categories* $\mathtt{JSem}^0 \simeq \mathtt{LMsc}$.

Proof. We employ Corollary 3.2 to define the object assignment of a functor $\mathcal{E} : \mathtt{JSem}^0 \to \mathtt{LMsc}$ as

$$\mathcal{E}(L, \vee, 0) := (L, \boxplus, 0).$$

We start by claiming that $f : L \to L'$ is a morphism of join-semilattices if and only if the same map is a morphism of the corresponding mosaics. Indeed, if $z \in x \boxplus y$ holds in the mosaic $(L, \boxplus, 0)$ defined above, then we deduce

$$f(x) \vee f(y) = f(x \vee y) = f(x \vee z) = f(x) \vee f(z), \text{ and}$$
$$f(x) \vee f(y) = f(x \vee y) = f(z \vee y) = f(z) \vee f(y),$$

that is, $f(z) \in f(x) \boxplus f(y)$.

Conversely, if f is a morphism of mosaics, then since $x \vee y \in x \boxplus y$ holds for any $x, y \in L$, we obtain that $f(x \vee y) \in f(x) \boxplus' f(y)$, whence $f(x \vee y) \vee f(x) = f(x) \vee f(y)$. On the other hand, we have that $x \vee y \geq x$, i.e., $x \vee y \in x \boxplus x$. Therefore, $f(x \vee y) \in f(x) \boxplus' f(x)$, meaning that $f(x \vee y) \geq f(x)$ holds in L'. We conclude that $f(x \vee y) \vee f(x) = f(x \vee y) = f(x) \vee f(y)$.

Thus, $\mathcal{E}$ is fully faithful. It remains to prove that it is essentially surjective, i.e., if $(A, \boxplus, 0)$ is an L-mosaic, then we have to define a unique (up to isomorphism) join-semilattice structure on A such that $(A, \boxplus, 0)$ is the associated Nakano mosaic. Since $(A, \boxplus, 0)$ is an L-mosaic, by Lemma 2.3 we obtain the poset $(A, \leq)$ with bottom element 0.

We claim that for all $x, y \in A$ the unique z, whose existence is guaranteed by property (Lms4), is the least upper bound of x and y in $(A, \leq)$. Indeed, it is an upper bound since $x, y \in z \boxplus z$. If $z' \in A$ is any upper bound, i.e, $x, y \in z' \boxplus z'$, then $z \leq z'$ because

$$z \in x \boxplus y \subseteq (z' \boxplus z') \boxplus (z' \boxplus z') = z' \boxplus z'.$$

Now, we set $x \vee y := z$ and by the arbitrary choice of $x, y \in A$ we obtain a bounded join-semilattice $(A, \vee, 0)$. It remains to verify that $a \boxplus b = \mathrm{Nak}_\vee(a, b)$ for all a, b in the bounded join-semilattice $(A, \vee, 0)$.

If $c \in a \boxplus b$ holds in $(A, \boxplus, 0)$ and we set $x := a \vee b$, $y := a \vee c$ as well as $z := b \vee c$, then we have that $a, b \in x \boxplus x$ and

$$c \in a \boxplus b \subseteq (x \boxplus x) \boxplus (x \boxplus x) = x \boxplus x.$$

Thus, we deduce $a, c \leq x$ and $y = a \vee c \leq x$. On the other hand,

$$b \in a \boxplus c \subseteq (y \boxplus y) \boxplus (y \boxplus y) = y \boxplus y.$$

2. Recall that a *morphism* of join-semilattices $(L, \vee, 0)$ and $(L', \vee', 0')$ is a function $f : L \to L'$ such that $f(0) = 0'$ and $f(x \vee y) = f(x) \vee' f(y)$, for all $x, y \in L$.

Thus, we deduce $a, b \leq y$ and $x = a \vee b \leq y$. Therefore, $x = y$ follows by antisymmetry and similarly one proves that $z = x = y$. This proves that $a \boxplus b \subseteq \mathrm{Nak}_{\vee}(a, b)$.

For the converse inclusion, take $c \in \mathrm{Nak}_{\vee}(a, b)$, i.e., $a \vee b = a \vee c = b \vee c =: x$. By definition of $\vee$, we obtain that

$$x \in (a \boxplus b) \cap (a \boxplus c) \cap (b \boxplus c),$$

therefore, using reversibility,

$$c \in a \boxplus x \subseteq a \boxplus (a \boxplus b)$$

and

$$c \in x \boxplus b \subseteq (a \boxplus b) \boxplus b.$$

Now, $c \in a \boxplus b$ follows by property (Lms3) of L-mosaics. $\square$

Since fully faithful functors are conservative, we obtain a useful corollary.

Corollary 3.3. *Let $(L, \vee, 0), (L', \vee', 0')$ be bounded join-semilattices. The associated Nakano mosaics $(L, \boxplus, 0)$ and $(L', \boxplus', 0')$ are isomorphic if and only if $(L, \vee, 0)$ and $(L', \vee', 0')$ are.*

4. Orthomodular lattices

We now review the main concepts around the orthomodularity property in the framework of lattices. Despite their introduction in the 1930s is strictly linked with the arising of quantum mechanics and the studies of J. V. Neumann [13], orthomodular lattices have become an independent purely algebraic dimension beyond their physical significance. The reader interested in general details on orthomodular lattices is referred to [8].

Two elements x, y in a bounded lattice $(L, \wedge, \vee, 0, 1)$ are *complements*, written $x \bowtie y$, if and only if

$$x \vee y = 1 \quad \text{and} \quad x \wedge y = 0.$$

We define the multimap $\omega : L \multimap L$ by setting

$$\omega(x) := \{y \in L \mid x \bowtie y\}.$$

By the symmetry of the relation $\bowtie$, we obtain the following properties:

(O1) $\omega(x) \neq \emptyset \implies x \in (\omega \circ \omega)(x) =: \omega^2(x)$.

(O2) $\omega^3(x) = \omega(x)$, for all $x \in L$.

Definition 4.1. *A bounded lattice L is called* complemented *if ω is total. A lattice L is called an* ortholattice *if ω contains a map (called* orthocomplementation*).*

It follows from properties (O1) and (O2) above that all orthocomplementations on a bounded lattice are involutions. We shall denote by $\leq$ the canonical partial order associated to a lattice L.

Fact 4.1 ([8, Chpt. 1, Sect. 2])**.** *Let L be an ortholattice and $\pi : L \to L$ an orthocomplementation. Then the following statements hold:*

(OC1) $x \leq y$ *if and only if* $\pi(y) \leq \pi(x)$*, for all* $x, y \in L$*.*

(OC2) $\pi(x \vee y) = \pi(x) \wedge \pi(y)$*, for all* $x, y \in L$*.*

(OC3) $\pi(x \wedge y) = \pi(x) \vee \pi(y)$*, for all* $x, y \in L$*.*

Definition 4.2. *Let $(L, \wedge, \vee, 0, 1)$ be an ortholattice. If $\pi : L \to L$ is an orthocomplementation of L such that for all $x, y \in L$ we have that*

$$x \leq y \implies x \vee (\pi(x) \wedge y) = y.$$

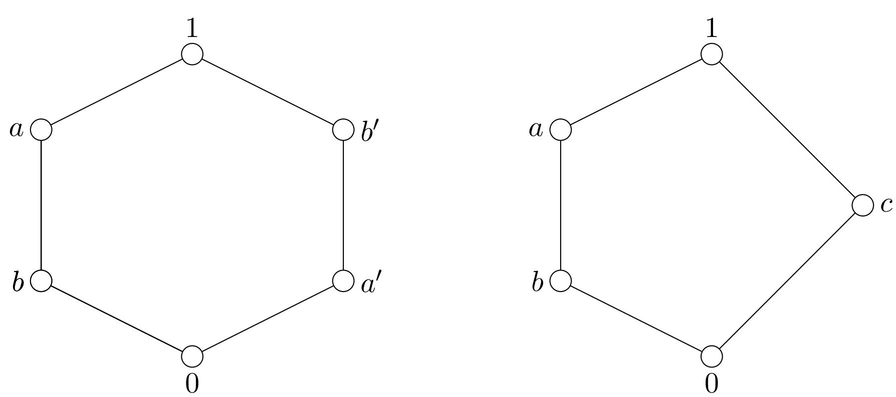
Then the lattice L is called orthomodular with respect to π *(*π-orthomodular*).*

Definition 4.3. *Two elements x, y in a π-orthomodular lattice are called* orthogonal*, written $x \perp y$, if $x \leq \pi(y)$.*

Definition 4.4. *A lattice $(L, \wedge, \vee)$ is called* modular *if for all $x, y, z \in L$*

$$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z).$$

Fact 4.2 ([8, Chpt. 1, Sect. 2])**.** *A lattice L is modular if and only if the pentagon lattice (Figure 1b) is not a sublattice of L.*



(a) The hexagon lattice H. (b) The pentagon lattice P.

Example 4.1. The ortholattice $H = \{0, a, a', b, b', 1\}$ in Figure 1a is not orthomodular. Indeed $b \leq a$ but

$$b \vee (b' \wedge a) = b \vee 0 = b \neq a.$$

The following characterization of orthomodularity is well-known.

Fact 4.3 ([8], Theorem 2). *Let L be a π-ortholattice. The following statements are equivalent:*

(OM1) *L is orthomodular.*

(OM2) *If $x \leq y$ and $\pi(x) \vee y = 1$, then $x = y$, for all $x, y \in L$.*

(OM3) *H is not a sublattice of L.*

(OM4) *If $x \leq y$, then the smallest sublattice L' of L containing x, y and closed under π is distributive.*

Since the pentagon lattice P of Figure 1b is a sublattice of the hexagon lattice H of Figure 1a, it follows from Fact 4.2 and Fact 4.3 (OM3) that modularity is a property stronger than orthomodularity. For an example of an orthomodular lattice which is not modular see [8, Section 3, p. 33].

The following is the main theorem of Nakano mentioned before.

Fact 4.4 ([12, Theorem 1]). *For a bounded lattice L, the following are equivalent statements:*

1. The Nakano mosaic $(L, \boxplus, 0)$ associated to $(L, \vee, 0)$ is a polygroup.

2. L is a modular lattice.

The abstract theory of orthomodularity admits concrete illustration through well-known lattice examples. We now examine two finite lattices, the pentagon and the hexagon, to show how modularity, associativity, and orthomodularity (or their failure) are reflected in the structure of their associated Nakano mosaics.

Example 4.2. The Pentagon lattice $P = \{0, a, b, c, 1\}$ in Figure 1b is not modular. Hence, the associated Nakano mosaic $\mathbf{P}_\vee$ is not associative. Table 1 fully describes the multioperation $\boxplus$ of $\mathbf{P}_\vee$. Note that $\mathbf{P}_\vee$ is not a polygroup as $a \boxplus (b \boxplus c) = \{c, 1\} \neq \{1\} = (a \boxplus b) \boxplus c$.

$\boxplus$	0	a	b	c	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a\}$	$\{1\}$	$\{c, 1\}$
b	$\{b\}$	$\{a\}$	$\{0, b\}$	$\{1\}$	$\{c, 1\}$
c	$\{c\}$	$\{1\}$	$\{1\}$	$\{0, c\}$	$\{a, b, 1\}$
1	$\{1\}$	$\{c, 1\}$	$\{c, 1\}$	$\{a, b, 1\}$	$\{0, a, b, c, 1\}$

Table 1: $\mathbf{P}_\vee$: The Nakano mosaic associated to the non-modular Pentagon lattice.

Corollary 4.1. *Let L be a bounded lattice. If $\mathbf{P}_\vee$ is not a submosaic of $(L, \boxplus, 0)$, then the Nakano mosaic $(L, \boxplus, 0)$ associated to $(L, \vee, 0)$ is a polygroup.*

Proof. If $(L, \boxplus, 0)$ is not a polygroup, then L is not modular by Fact 4.4, thus the Pentagon lattice is a sublattice of it. By Proposition 3.2, it follows that $\mathbf{P}_\vee$ is a submosaic of $(L, \boxplus, 0)$. □

Example 4.3. In Table 2 we describe the multioperation $\boxplus$ of the Nakano mosaic associated to the hexagon lattice H of Figure 1a.

$\boxplus$	0	a	b	a'	b'	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{a'\}$	$\{b'\}$	$\{1\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a\}$	$\{1\}$	$\{1\}$	$\{a', b', 1\}$
b	$\{b\}$	$\{a\}$	$\{0, b\}$	$\{1\}$	$\{1\}$	$\{a', b', 1\}$
a'	$\{a'\}$	$\{1\}$	$\{1\}$	$\{0, a'\}$	$\{b'\}$	$\{a, b, 1\}$
b'	$\{b'\}$	$\{1\}$	$\{1\}$	$\{a'\}$	$\{0, a', b'\}$	$\{a, b, 1\}$
1	$\{1\}$	$\{b, c\}$	$\{a, c\}$	$\{a, b\}$	$\{b'\}$	$\{0, a, b, c, 1\}$

Table 2: $\mathbf{H}_\vee$: The additive Nakano polygroup associated to the hexagon ortholattice.

5. Orthomodularity and Nakano mosaics

Let `Ort` be the category formed by pairs (L, π), where L is a π-ortholattice, as objects and lattice morphisms $f : L \to L'$ such that $f \circ \pi = \pi' \circ f$ as arrows $f : (L, \pi) \to (L', \pi')$. Similarly, let $\texttt{LMsc}^d$ the category formed by pairs (A, π), where A is a π-duaizable L-mosaic as objects and mosaic morphisms $f : A \to A'$ such that $f \circ \pi = \pi' \circ f$ as arrows $f : (A, \pi) \to (A', \pi')$.

We are now in a position to synthesize the preceding developments. Building on the equivalence established in Theorem 3.1, and taking into account the role of involutive dualities, we extend our framework to ortholattices and their categorical counterpart. This leads to the following key result.

Theorem 5.1. *The categories* `Ort` *and* $\texttt{LMsc}^d$ *are equivalent.*

Proof. By restricting the source of the functor $\mathcal{E}$ defined in the proof of Theorem 3.1 to `Ort` we obtain a functor $\texttt{Ort} \to \texttt{LMsc}$, setting

$$\mathcal{E}(L, \vee, \wedge, 0, 1) := (L, \boxplus, 0),$$

where $(L, \boxplus, 0)$ denotes the Nakano mosaic associated to $(L, \vee, 0)$. Since the orthocomplementation π satisfies $x \wedge y = \pi(\pi(x) \vee \pi(y))$ and $\pi(0) = 1$, we have that $(L, \boxplus_\pi, \pi(0))$ is the Nakano mosaic associated to the bounded semilattice $(L, \wedge, 1)$. In particular, $(L, \boxplus, 0)$ is π-dualizable. Hence, the restriction of the functor $\mathcal{E}$ to `Ort` has target $\texttt{LMsc}^d$. The same arguments as in Theorem 3.1 apply here to show that it is fully faithful and essentially surjective as a functor $\texttt{Ort} \to \texttt{LMsc}^d$. □

Proposition 5.1. *Let A be a π-dualizable L-mosaic. Then the π-ortholattice associated to A is π-orthomodular if and only if the implication*

$$x \in y \boxplus y \text{ and } 1 \in x \boxplus \pi(y) \implies x = y \tag{2}$$

holds, for all $x, y \in A$.

Proof. By Fact 4.3 (OM2), the π-ortholattice associated to A is π-orthomodular if and only if the conjunction of $x \leq y$ and $x \vee \pi(y) = 1$ implies $x = y$. By definition of $\vee$ in the π-ortholattice associated to A and since $1 \boxplus 1 = A$, we have that $x \vee \pi(y) = 1$ is equivalent to $1 \in x \boxplus \pi(y)$. On the other hand, we know that $x \leq y$ means that $x \in y \boxplus y$. The assertion follows. □

The following is a general result on orthomodular lattices.

Fact 5.1 ([8], Theorem 9)**.** *Let L be a π-orthomodular lattice and $x, y \in L$. Then the interstection of all sublattices L' of L such that $x, y \in L'$ and $\pi(L') = L'$ is a modular lattice.*

From the above result and Fact 4.4 we deduce:

Proposition 5.2. *Let A be a π-dualizable L-mosaic satisfying the implication* (2) *for all $x, y \in A$. Then for all $x, y \in A$ the smallest L-submosaic A' of A containing x, y and closed under π is a polygroup.*

Proof. Fix $x, y \in A$ and let A' be as in the statement. Since A' is an L-mosaic, we may consider its associated ortholattice structure. Call this lattice L'. Let further L be the ortholattice structure associated to the L-mosaic A. By Theorem 5.1, the lattice L' is the smallest of all sublattices of L such that $x, y \in L'$ and $\pi(L') = L'$, hence, by Fact 5.1, L' is a modular lattice. It now follows by Fact 4.4 that A' is a polygroup. □

6. Further research: the connection with the quantum world

Since the 1990s, *dagger compact categories* have played a central role in the development of topological quantum field theories, as introduced by John Baez and James Dolan [2]. Later, in the early 2000s, Samson Abramsky and Bob Coecke identified these categories as fundamental structures in their framework of categorical quantum mechanics [1].

Building on these foundational ideas, recent work has explored alternative algebraic approaches to quantum theories. In this context, we highlight the contribution of Jenčová and Jenča [7], which served as an initial inspiration for our study. Their research underscores the potential role of hypercompositional structures in the analysis of quantum phenomena. Specifically, their approach employs the concept of *effect algebras*—algebraic structures with a partial operation (see [6] for further details).

Both partial operations and multioperations can be naturally modeled by monoids in the category `Rel` (see [10] for further details). At this point, it should be noted that the category `Rel` possess a dagger compact structure. While partial algebraic structures, such as effect algebras, are already well-integrated into quantum theories, the study of algebraic structures with multioperations remains less developed. This work aims to encourage further research into this promising direction, which we also plan to explore in future investigations.

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Existence and approximate controllability for random functional differential equations with finite delay

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Abstract. This study examines second-order equations with delays, which frequently arise in various scientific and engineering applications. Within Banach spaces, these equations introduce unique challenges and opportunities for analysis and control. By exploring the existence and approximate controllability of solutions, the research enhances

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the understanding of dynamical systems with delayed feedback. Using mathematical tools such as cosine family theory and the Leray-Schauder theorem, it establishes rigorous conditions for solution existence, contributing to both theoretical and practical advancements. Additionally, the study incorporates empirical validation through a practical example, offering insights into the real-world behavior of these equations. This empirical analysis bridges the gap between theory and application, supporting the development of effective control strategies and engineering solutions. Ultimately, this research deepens the understanding of complex dynamical systems with delays and provides valuable contributions to both theoretical progress and practical implementation.

Keywords: differential equation, Lerray-Schauder fixed point, mild solution; finite delay, semigroup theory; approximate controllability.

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1. Introduction

In the area of mathematical analysis and its applications, various theories, methods, and models have been developed to study complex problems in science. This introduction explores key studies and recent research on functional differential equations, impulsive systems, controllability, and the links between mathematics and physics. The study of functional differential equations has been extensively explored, with foundational results established through semigroup theory and evolution operators [22, 1]. These methods have been widely applied to differential systems involving delays, as seen in the works of [19, 13, 27], which analyze the controllability and phase space properties of retarded systems. Further advancements in this field include research on integral inequalities [20] and stochastic fixed-point theorems [10], which provide crucial tools for investigating the existence of solutions in stochastic domains. The development of neutral functional differential equations has led to several significant contributions in the study of existence and controllability. Many studies focus on establishing uniqueness results in Fréchet spaces [3, 4] and on nonlinear impulsive systems [26, 9]. Recent works have extended these results to nonlocal conditions and fuzzy delay systems [12, 25], while others have examined integral equations on time scales [16]. Additionally, the application of fractional differential equations and numerical methods has gained attention, as demonstrated by [23, 14]. Further advancements include the study of abstract second-order neutral functional integrodifferential equations [6] and the investigation of controllability in Banach spaces [18], which have provided deeper insights into stability and control mechanisms in complex dynamical systems.

In recent years, controllability studies have expanded to include approximate controllability results for semilinear and impulsive systems [6, 7, 15]. Research on second-order neutral functional integro-differential equations with delay and random effects has been particularly relevant in this context [11, 17]. Several studies have investigated state-dependent delays and impulsive systems [2, 24], while others have analyzed random differential equations with nonlocal conditions [8, 5]. These studies highlight the increasing complexity of control problems involving randomness, delays, and impulsive effects, providing valuable insights into mathematical modeling and real-world applications. Random integral equations play a crucial role in modeling real-world problems in life sciences and engineering, providing a framework for studying systems influenced by randomness [28]. The concept of mild solutions for second-order semilinear impulsive differential inclusions in Banach spaces has been explored to address the complexities of impulsive systems and their applications in dynamical models [21]. These references help us understand how to control complex systems, especially those with impulses, delays, and nonlocal effects. They offer useful mathematical ideas and methods for studying and creating control strategies, which are important for science and engineering.

In this we demonstrate that the Second order functional differential equation with delay and random effects is of the form.

$$\begin{aligned} x''(t,\varpi) =& Ax(t,\varpi) + \Upsilon(t, x_t(.,\varpi),\varpi); \\ & t \in J = (0,\varrho], t \neq t_\xi, \xi = 1,2,3,...,m \\ x(0,\varpi) =& x_0(\varpi), \\ x'(0,\varpi) =& x_0'(\varpi), \\ \Delta x(t_\xi) =& I_\xi(x(t_\xi)), \\ \Delta' x(t_\xi) =& I_\xi'(x(t_\xi)). \end{aligned} \tag{1}$$

Consider a Banach space $\mathcal{S}$ equipped with the norm $\|.\|$, where A is the infinitesimal generator of a strongly continuous cosine family $\{T_1(t) : t \in \mathbb{R}\}$ consisting of bounded linear operators on $\mathcal{S}$. Additionally, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω represents the sample space and $\varpi \in \Omega$. Functions $\Upsilon : J \times \mathcal{D} \times \Omega \to \mathcal{S}$ where Ω is a random operator with stochastic domain and are continuous functions, $\mathcal{D}$ is a phase space.The term $x_t(\cdot,\varpi)$ represents the state history from time $-\infty$ to the present time t, with histories assumed to belong to the abstract phase space $\mathcal{S}$.The symbol $\Delta x(t_\xi) = x(t_\xi^+) - x(t_\xi^-)$ and $\Delta x'(t_\xi) = x'(t_\xi^+) - x'(t_\xi^-)$ where $x(t_\xi^+), x(t_\xi^-)$ and $x'(t_\xi^+), x'(t_\xi^-)$ represent right and left hand limit of $\Delta x(t_\xi)$ and $\Delta x'(t_\xi)$ respectively at $t = t_\xi$ and

$$\begin{aligned} x''(t,\varpi) =& Ax(t,\varpi) + \Upsilon(t, x_t(.,\varpi),\varpi) + By(t,\varpi); \quad t \in J \\ x(0,\varpi) =& x_0(\varpi), \\ x'(0,\varpi) =& x_0'(\varpi), \\ \Delta x(t_\xi) =& I_\xi(x(t_\xi)), \\ \Delta' x(t_\xi) =& I_\xi'(x(t_\xi)). \end{aligned} \tag{2}$$

Here, A is the operator and Υ is the continuous function, both defined previously. The control function $y(\cdot,\varpi)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, where U is a Banach space and $B : U \to \mathcal{S}$ is a bounded linear operator.

2. Preliminaries

In this section, we cover some key concepts and terms needed to understand our main results.

$$x''(t,\varpi) = Ax(t,\varpi)+\Upsilon(t,\varpi), \quad 0 \le t \le \varrho \quad x(0,\varpi) = x_0(\varpi), \quad x'(0,\varpi) = y_0(\varpi).$$

Here, $A : D(A) \subseteq \mathcal{S} \to \mathcal{S}$ is a closed operator that is densely defined, where $t \in J = [0,\varrho]$. Also, $\Upsilon : J \times \Omega \to \mathcal{S}$ is a suitable function. Many studies have explored equations of this type. In most cases, the existence of an evolution operator $T_2(t,j)$ for the homogeneous equation is essential for finding a solution to the problem.

$$x''(t,\varpi) = Ax(t,\varpi), \quad 0 \le j,t \le \varrho.$$

Definition 2.1. Let $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ be a seminormed linear space consisting of functions defined on $(-\delta,0]$ that take values in a Banach space $\mathcal{S}$. The space $\mathcal{D}$ is complete and satisfies the following axioms:

(A) For any continuous function $x : (-\delta,0] \to \mathcal{S}$ with $x_0 \in \mathcal{D}$ and $\rho > 0$, the following conditions hold for all $t \in J$:

1. The function x_t belongs to $\mathcal{D}$.

2. There exists a positive constant K such that

$$\|x(t,\varpi)\| \le K\|x_t(\cdot,\varpi)\|_{\mathcal{D}}.$$

   Moreover, there exist functions $U, \vartheta, \vartheta' : \mathbb{R}_+ \to \mathbb{R}_+$, where U is continuous and bounded, and ϑ, ϑ' are locally bounded and independent of x, such that

$$\|x_t(\cdot,\varpi)\|_{\mathcal{D}} \le U(t)\sup\{\|x(m,\varpi)\| : -\delta \le m \le 0\} + \vartheta\|x_0(\varpi)\|_{\mathcal{D}} + \vartheta'\|x_0'(\varpi)\|_{\mathcal{D}}.$$

(B) The function x_t is $\mathcal{D}$-valued and remains continuous on J for all functions x satisfying (A).
(C) The space $\mathcal{D}$ is complete.

Definition 2.2. A strongly continuous cosine family in the Banach space $\mathcal{S}$ is a collection of bounded linear operators $\{T_1(t) : t \in \mathcal{J}\}$ that satisfies the following conditions:

1. Identity Property: The operator at $t = 0$ is the identity operator, $T_1(0) = I$.

2. Continuity Condition: The operator $T_1(t)$ depends continuously on t for any fixed element $x \in \mathcal{S}$.

3. Addition Rule: For all $j, t \in \mathcal{J}$, the operators satisfy the equation:

$$T_1(j+t) + T_1(j-t) = 2T_1(j)T_1(t).$$

In this case, a set of bounded linear operators $\{T_1(t) : t \in \mathcal{J}\}$ is defined in the Banach space $\mathcal{S}$, where distances are measured using the norm $\|\cdot\|$. These operations change continuously over time and form what is known as a strongly continuous cosine family. The associated sine function, denoted as $\{T_2(t) : t \in \mathcal{J}\}$, corresponds to this family and is expressed as follows.

$$T_2(t)x = \int_0^t T_1(j)x\,dj \quad \text{for } x \in \mathcal{D} \text{ and } t \in \mathcal{J}.$$

For every $t \in J$, the bounds $\|T_1(t)\| \leq \vartheta$ and $\|T_2(t)\| \leq \vartheta_a$ hold, where ϑ and ϑ_a are positive constants that ensure these limits.

Definition 2.3. A system is considered approximately controllable on the interval $(0,T]$ if, for any initial state $x_0 \in \mathcal{S}$, desired final state $x_1 \in \mathcal{S}$, and $\epsilon > 0$, there exists a control $u \in L^2((0,T];U)$ such that the solution $x(t)$ to the equation

$$x'(t,\varpi) = Ax(t,\varpi) + \Upsilon(t, x_t(.,\varpi),\varpi), \quad t \in J = (0,\varrho],$$

with initial condition $x(0) = x_0$, satisfies the condition

$$\|x(\varrho) - x_1\| < \epsilon.$$

Lemma 2.1 (Leray-Schauder Nonlinear Alternative)**.** *Let $\mathcal{S}$ be a Banach space, and let Z be a closed and convex subset of $\mathcal{S}$. Suppose U is a relatively open subset of Z containing the point 0, and let $\Gamma : U \to Z$ be a compact mapping. Under these conditions, one of the following alternatives holds:*

1. *There exists a point $z \in \partial U$ such that $z \in \lambda\Gamma(z)$ for some $\lambda \in (0,1)$, or*
2. *The mapping Γ has a fixed point in U.*

Lemma 2.2. A set $\mathcal{D} \subset \mathcal{S}$ is relatively compact in $\mathcal{S}$ if and only if, for each $\xi = 0, 1, \ldots, m$, the set $\overline{\mathcal{D}}_\xi$ is relatively compact in $C[(t_\xi, t_{\xi+1}];\mathcal{S})$.

The analysis considers the case where impulses, delays, or nonlocal conditions are not included while examining the approximate controllability of the equations within their domain. It focuses on the following situation: given any function y from the L^2 space over the interval $(0,\varrho]$ with values in U and any initial point x_0 in the space $\mathcal{S}$, the initial-value problem is analyzed.

$$x'(t,\varpi) = Ax(t,\varpi) + By(t,\varpi), \quad x \in \mathcal{S}, \qquad x(0) = x_0, \tag{3}$$

A unique mild solution is obtained for the given problem, where the control function x is an element of $L^2(0,\varrho;U)$

$$x(t,\varpi) = T(t)x_0(\varpi) + \int_0^t T(t-j)By(j,\varpi)\,dj, \quad t \in (0,\varrho].$$

Definition 2.4. For $t > 0$, the controllability mapping $G : L^2((0, \varrho]; U) \to \mathcal{S}$ is defined within the given system

$$Gx = \int_0^t T(t-j)Bx(j)\, dj.$$

The adjoint operator $G^* : \mathcal{S} \to L^2((0, \varrho]; \mathcal{S})$ is defined by the following rule

$$(G^*x)(j) = B^*T^*(\varrho - j)x(\varpi) \quad \forall j \in [0, \varrho], \forall z \in \mathcal{S}.$$

Therefore, the Grammian operator $W : \mathcal{S} \to \mathcal{S}$ is given by $W = GG^*$

$$kx = GG^*x = \int_0^\varrho T(\varrho - j)BB^*T^*(\varrho - j)x\, dj.$$

3. Existence results

This section establishes the existence of solutions for the problem described by equations (1). Certain conditions must be considered to achieve this.

Definition 3.1. Let $\varrho > 0$ and define $PC_\delta = PC[(-\delta, \varrho], \mathcal{S}]$. Consider a function $x : J \times \Omega \to \mathcal{S}$ that is continuous and satisfies $x(0, \varpi) = x_0(t, \varpi)$. If x satisfies the integral equation, it is called a random mild solution of equation (1)

$$\begin{aligned} x(t,\varpi) =& T_1(t)x_0(\varpi) + T_2(t)x_0'(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, x_j(.,\varpi), \varpi)dj \\ &+ \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(x(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(x(t_\xi)). \end{aligned}$$

The following section will discuss the hypotheses that have been listed:

(G_1) There exist a continuous function $a_0, b_0 : J \times \Omega \to \mathbb{R}$ such that

$$\|\Upsilon(t, x, \varpi)\| \le a_0(\varpi)\|x, \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi),$$

for all $x \in \mathcal{S}, \varpi \in \Omega$.

(G_2) The function $\Upsilon(t, ., .) : \mathcal{D} \times \Omega \to \mathcal{S}$ is continuous for all $t, j \in J$. Additionally, for every $(x, \varpi) \in \mathcal{D} \times \Omega$, the function $\Upsilon(., x, \varpi) : J \to \mathcal{S}$ is strongly measurable.

(G_3) Let $I_\xi, I_\xi' \in C(\mathcal{S}, \mathcal{S})$ be compact operators for each $\xi = 1, 2, 3, \dots, m$

$$\begin{aligned} \|I_\xi(t, x)\| &\le a_\xi \|x\|_{\mathbb{R}}^{\alpha_\xi}, \\ \|I_\xi'(t, x)\| &\le a_\xi' \|x\|_{\mathbb{R}}^{\alpha_\xi}. \end{aligned}$$

(G_4) The function $\Upsilon : J \times \mathcal{D} \times \Omega \to \mathcal{S}$ is continuous, and there exists a constant $\mathcal{L}$ such that the following condition holds

$$\|\Upsilon(t, x_1, \varpi) - \Upsilon(t, x_2, \varpi)\| \le \mathcal{L}(\varpi)(\|(x_1, \varpi) - (x_2, \varpi)\|_{\mathcal{D}}^{\alpha_0}).$$

(G_5) A random function $R : \Omega \to \mathbb{R}^+$ exists such that the following condition holds

$$\vartheta\|z_0\| + \vartheta_a\|z_0'\| + \vartheta_a b_0(\varpi) + \vartheta \sum_{0<t_\xi<t} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}$$
$$+ \vartheta_a \sum_{0<t_\xi<t} a_\xi' \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \int_0^t \|[\sup_{j\in(0,\varrho]} \|z_j(.,\varpi)]\|dj \leq R(\varpi).$$

Theorem 3.1. If conditions (G_1)-(G_5) are satisfied, then the problem given in (1) has a mild random solution on the interval $(-\delta, \varrho]$.

Proof of Theorem 3.1. Consider a random operator $\Gamma(\varpi)$ defined as $\Gamma(\varpi) : \Omega \times PC_\delta \to PC_\delta$, where $PC_\delta = [(-\delta, \varrho), \mathcal{S}]$. This operator is given by $(\Gamma(\varpi))z(t)$ for $t \in (\delta, \varrho)$.

$$(\Gamma(\varpi))z(t,\varpi) = \begin{cases} \Upsilon(t,\varpi), & t \in (-\delta, \varrho], \\ T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) & \\ + \int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)dj & \\ + \sum\limits_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) & \\ + \sum\limits_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)), & t, j \in J \end{cases} \tag{4}$$

The proof is divided into several steps.

Step 1. The operator $(\Gamma(\varpi))$ maps bounded sets into bounded sets.

To prove this, it is enough to find a positive constant $r(\varpi)$ such that for every z in the bounded set $\mathcal{B}_r(\delta)$, where δ is defined as follows:

$$\mathcal{B}_r(\delta) := \{z \in PC_\delta : \sup_{\delta\leq t\leq\varrho} \|z(t,\varpi)\| \leq r(\varpi)\}$$

one has $\|(\Gamma(\varpi))z\|_{PC} \leq R(\varpi)$

$$\begin{aligned} \|(\Gamma(\varpi))z(t)\| \leq & \|T_1(t)z_0(\varpi)\| + \|T_2(t)z_0'(\varpi)\| \\ & + \|\int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)\|dj \\ & + \|\sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi))\| + \|\sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi))\|dj \\ & \leq \vartheta\|z_0(\varpi)\| + \vartheta_a\|z_0'(\varpi)\| + \vartheta_a \int_0^t \|\Upsilon(j, z_j(.,\varpi),\varpi)\|dj \\ & + \vartheta \sum_{0<t_\xi<t} \|I_\xi(z(t_\xi))\| + \vartheta_a \sum_{0<t_\xi<t} \|I_\xi'(z(t_\xi))\| \\ & \leq \vartheta\|z_0\| + \vartheta_a\|z_0'\| + \vartheta_a \int_0^t [\sup_{j\in(0.\varrho]} \|z_j(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj \end{aligned}$$

$$
\begin{aligned}
&+ \vartheta \sum_{0<t_\xi<t} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\leq \vartheta \|z_0\| + \vartheta_a \|z'_0\| + \vartheta_a b_0(\varpi) + \vartheta \sum_{0<t_\xi<t} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&+ \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \int_0^t \|[\sup_{j\in(0.\varrho]} \|z_j(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} dj \\
&\leq R(\varpi).
\end{aligned}
$$

Hence, $(\Gamma(\varpi))$ is bounded set in PC_δ.

Step 2. We now show that $(\Gamma(\varpi))$ is continuous on $\mathcal{B}_r(\delta)$.

Let us consider that for $z_1, z_2 \in \mathcal{B}_r(\delta)$, $t \in J$,

$$
\begin{aligned}
&\|(\Gamma(\varpi))z_1(t) - (\Gamma(\varpi))z_2(t)\| \\
&\quad \leq \| \int_0^t T_2(t-j)\Upsilon(j,(z_{1,j}(.,\varpi)),\varpi) - \Upsilon(j,(z_{2,j}(.,\varpi)),\varpi)\|dj \\
&\quad + \| \sum_{0<t_\xi<t} T_1(t-t_\xi)[I_\xi(z_1(t_\xi)) - I_\xi(z_2(t_\xi))]\| \\
&\quad + \| \sum_{0<t_\xi<t} T_2(t-t_\xi)[I'_\xi(z_1(t_\xi)) - I'_\xi(z_2(t_\xi))]\|dj \\
&\quad \leq \vartheta_a \int_0^t \|(\Upsilon(j,z_{1,j}(.,\varpi),\varpi) - \Upsilon(j,z_{2,j}(.,\varpi),\varpi)\|dj \\
&\quad + \vartheta \sum_{0<t_\xi<t} \|I_\xi(z_1(t_\xi)) - I_\xi(z_2(t_\xi))\| \\
&\quad + \vartheta_a \sum_{0<t_\xi<t} \|I'_\xi(z_1(t_\xi)) - I'_\xi(z_2(t_\xi))\| \\
&\quad \leq \vartheta_a \int_0^t \sup_{j\in(0.\varrho]} \|(z_{1,j}(.,\varpi),\varpi) - (z_{2,j}(.,\varpi),\varpi)\|_{\mathcal{D}}^{\alpha_0} dj \\
&\quad + \vartheta \sum_{0<t_\xi<t} a_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi},
\end{aligned}
$$

$$
\begin{aligned}
&\|(\Gamma(\varpi))z_1(t) - (\Gamma(\varpi))z_2(t)\| \\
&\quad \leq \vartheta \sum_{0<t_\xi<t} a_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \int_0^t \sup_{j\in(0.\varrho]} \|(z_{1,j}(.,\varpi),\varpi) - (z_{2,j}(.,\varpi),\varpi)\|_{\mathcal{D}}^{\alpha_0} dj.
\end{aligned}
$$

The operators are compact for $t > 0$, ensuring continuity in the uniform operator topology. For all $t \in (-\delta, \varrho]$, the right-hand side of the inequalities is independent when $z_1, z_2 \in \mathcal{B}_r(\delta)$.

As $(z_1 - z_2) \to 0$, the norm $\|((\Gamma(\varpi))z_1)(t) - ((\Gamma(\varpi))z_2)(t)\|$ also tends to zero. Thus, $(\Gamma(\varpi))$ is continuous.

Step 3. The operator $(\Gamma(\varpi))$ is compact. The operator $(\Gamma(\varpi))$ is expressed as the sum $(\Gamma_1(\varpi)) + (\Gamma_2(\varpi))$, where both are defined on $\mathcal{B}_r(\delta)$ as follows:

$$(\Gamma_1(\varpi))z(t) = T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi), \varpi)dj,$$

$$(\Gamma_2(\varpi))z(t) = \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)),$$

for all $t \in [\delta, \varrho]$.

Next, we establish that $(\Gamma_1(\varpi))$ is a compact operator.

(i) We show that $(\Gamma_1(\varpi))(\mathcal{B}_r(\delta))$ is equicontinuous.

Consider $\delta \le t_1 < t_2 \le \varrho$ and let $\epsilon > 0$ be small. Then:

$$\begin{aligned}
&\|(\Gamma_1(\varpi))z(t_2) - (\Gamma_1(\varpi))z(t_1)\| \\
&\quad \le \|[T_1(t_2) - T_1(t_2)]z_0(\varpi)\| + \|[T_2(t_2) - T_2(t_1)]z_0'(\varpi)\| \\
&\quad + \|\int_0^{t_1} T_2(t_2-j)\Upsilon(j, z_j(.,\varpi), \varpi) - \int_0^{t_2} T_2(t_1-j)\Upsilon(j, z_j(.,\varpi), \varpi)\|dj \\
&\quad \le \|[T_1(t_2) - T_1(t_2)]z_0(\varpi)\| + \|[T_2(t_2) - T_2(t_1)]z_0'(\varpi)\| \\
&\quad + \|\int_0^{t_1-\epsilon} (T_2(t_2-j) - T_2(t_1-j))\Upsilon(j, z_j(.,\varpi), \varpi)dj\| \\
&\quad + \|\int_{t_1-\epsilon}^{t_1} T_2(t_2-j) - T_2(t_1-j))\Upsilon(j, z_j(.,\varpi), \varpi)dj\| \\
&\quad + \|\int_{t_1}^{t_2} T_2(t_2-j) - T_2(t_1-j))\Upsilon(j, z_j(.,\varpi), \varpi)\|dj \\
&\quad \le \|[T_1(t_2) - T_1(t_2)]z_0\| + \|[T_2(t_2) - T_2(t_1)]z_0'\| \\
&\quad + \int_0^{t_1-\epsilon} \|T_2(t_2-j) - T_2(t_1-j)\|[\sup_{j\in(0.\varrho]} \|z_j(.,\varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj \\
&\quad + \int_{t_1-\epsilon}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)\|[\sup_{j\in(0.\varrho]} \|z_j(.,\varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj \\
&\quad + \int_{t_1}^{t_2} \|T_2(t_2-j) - T_2(t_1-j)\|[\sup_{j\in(0.\varrho]} \|z_j(.,\varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj.
\end{aligned}$$

As $t_2 - t_1$ approaches zero,

$$\left\|(\Gamma_1(\varpi))z(t_2) - (\Gamma_1(\varpi))z(t_1)\right\|.$$

Since $T_2(t)$ is compact for $t > 0$, the operator $(\Gamma_1(\varpi))$ maps $\mathcal{B}_r(\delta)$ into a set of uniformly continuous functions. For any $z \in \mathcal{B}_r(\delta)$, the above equation tends to zero.

Next, we prove that $(\Gamma_1(\varpi))(\mathcal{B}_r(\delta))(t)$ is precompact in $\mathcal{S}$.

Let $\delta < t \leq j \leq \varrho$ be fixed, and take a real number ϵ such that $0 < \epsilon < t$. For $z \in \mathcal{B}_r(\delta)$, we define $((\Gamma_1(\varpi)), \epsilon)(t)$ as follows:

$$T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^{t-\epsilon} T_2(t-j)\Upsilon(j, z_j(.,\varpi), \varpi)dj.$$

For each $z \in B_r(\delta)$, we establish that the set $\{((\Gamma(\varpi))_{1,\epsilon}z)(t) : z \in B_r(\delta)\}$ is precompact for $0 < \epsilon < t$. This follows from the compactness property of $T_2(t)$ for $t > 0$. Additionally, we ensure that

$$\begin{aligned}
\|((\Gamma_1(\varpi))z)(t) - &((\Gamma_{1,\epsilon}(\varpi))z)(t)\| \\
&\leq \int_{t-\epsilon}^{t} \|T_2(t-j)\Upsilon(j, z_j(.,\varpi), \varpi)\| \\
&\leq \vartheta_a \int_{t-\epsilon}^{t} [\sup_{j\in(0.\varrho]} \|z_j(.,\varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj.
\end{aligned}$$

Thus, there exist precompact sets that are arbitrarily close to

$$\{(\Gamma_1(\varpi))\, z : z \in B_r(\delta)\}.$$

Since $(\Gamma_1(\varpi))(B_r(\delta))$ is uniformly bounded and consists of equicontinuous functions, the Arzelà–Ascoli theorem implies that proving $\Gamma_1(\varpi)$ maps $B_r(\delta)$ into a precompact set in $\mathcal{S}$ is sufficient. As a result, the set $\{\Gamma_1(\varpi)\, z : z \in B_r(\delta)\}$ is precompact in $\mathcal{S}$.

Next, we establish the compactness of $\Gamma_2(\varpi)$. By applying Lemma 2.1, we show that $\Gamma_2(\varpi)$ is completely continuous. Its continuity follows from the phase space axioms. Furthermore, for $r > 0$, $t \in (t_z, t_{z+1}] \cap (0, \varrho]$, and $z \geq 1$, where $z \in \mathcal{B}_r = \mathcal{B}_r(0, \mathcal{B}_r(\delta))$, we observe that

$$(\Gamma(\varpi))z(t) \in \begin{cases} \sum_{j=1}^{z} T(t-t_j)I_j(\mathcal{B}_{r^*}(0,\mathcal{S})), & t \in (t_z, t_{z+1}), \\ \sum_{j=0}^{z} T(t_{z+1}-t_j)I_j(\mathcal{B}_{r^*}(0,\mathcal{S})), & t = t_{z+1}, \\ \sum_{j=0}^{z} T(t_z-t_j)I_j(\mathcal{B}_{r^*}(0,\mathcal{S})) + I_z(\mathcal{B}_{r^*}(0,\mathcal{S})), & t = t_z. \end{cases} \tag{5}$$

Since the mappings I_j are completely continuous, the set $[(\Gamma(\varpi))_2(B_r)]_\xi(t)$ is relatively compact in $\mathcal{S}$ for every $t \in [t_\xi, t_{\xi+1}]$. Furthermore, the strong continuity of $(T(t))_{t_0}$ and the compactness of the operators I_ξ guarantee that $[(\Gamma(\varpi))_2(B_r)]_\xi$ remains equicontinuous at t for all $t \in [t_\xi, t_{\xi+1}]$ and for each

$\xi = 1, 2, \ldots, n$. Consequently, by Lemma 2.2, it follows that $(\Gamma(\varpi))_2$ is completely continuous.

Step 4. The goal is to determine an open set $U \subseteq PC_\delta$ such that for every $t \in (0, \varrho]$, any point z on its boundary is not part of $\lambda(\Gamma(\varpi))(z)$ for any $\lambda \in (0, 1)$

$$\begin{aligned}(\Gamma(\varpi))z(t) =& \lambda x(t, \varpi)\\ &= \lambda T_1(t)z_0(\varpi) + \lambda T_2(t)z_0'(\varpi) + \lambda \int_0^t T_2(t-j)\Upsilon(j, z_j(., \varpi), \varpi)dj\\ &+ \lambda \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \lambda \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)),\end{aligned}$$

for each $t \in (0, \varrho]$, we have $\|x(t, \varpi)\| \leq \|(\Gamma(\varpi))z(t)\|$ and

$$\begin{aligned}\|(\Gamma(\varpi))z(t)\| \leq& \|T_1(t)z_0(\varpi)\| + \|T_2(t)z_0'(\varpi)\|\\ &+ \int_0^t \|T_2(t-j)\Upsilon(j, z_j(., \varpi), \varpi)\|dj\\ &+ \|\sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi))\| + \|\sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi))\|.\end{aligned}$$

By step 1, $\|(\Gamma(\varpi))z(t)\| \leq R(\varpi)$. We can find a constant $R(\varpi)$ such that $\|z\|_{PC} \neq R(\varpi)$. Set

$$U = \{z \in PC([\delta, \varrho], \mathcal{S}) \mid \sup_{\delta \leq t \leq \varrho} \|z(t, \varpi)\| < R(\varpi)\}.$$

From Steps 1-3 in Theorem 3.1, it is sufficient to show that $(\Gamma(\varpi)) : U \to PC_\delta$ is a compact map. Since no $x \in \partial U$ satisfies $z \in \lambda(\Gamma(\varpi))(z)$ for any $\lambda \in (0, 1)$, Lemma 2.1 implies that $(\Gamma(\varpi))$ has a fixed point $z^* \in U$. Thus, we obtain the result

$$\begin{aligned}z^*(t, \varpi) =& T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, z_j^*(., \varpi), \varpi)dj\\ &+ \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z^*(t_\xi)).\end{aligned} \tag{6}$$

This shows that $z^*(t, \varpi)$ has a fixed point and satisfies the conditions of a mild solution to problem (1). Thus, the proof of the theorem is complete.

4. Approximate controllability of random functional differential equation

Definition 4.1. The problem (2) is considered controllable on the interval $(0, \varrho]$ if, for any given final state $z^1(\varpi)$, there exists a control function $y(t, \varpi)$ in $L^2(J, U)$ such that the solution $z(t, \varpi)$ satisfies $z(\varrho, \varpi) = z^1(\varpi)$.

Before presenting the main existence result for problem (2), we introduce the definition of a mild random solution

Definition 4.2. Let $\varrho > 0$ and define $PC_\delta = PC[(-\delta, \varrho], \mathcal{S}]$. Consider a function $z : J \times \Omega \to \mathcal{S}$ that is continuous and satisfies $z(0, \varpi) = z_0(\varpi)$. If z fulfills the given integral equation, it is referred to as a random mild solution of equation (2)

$$z(t,\varpi) = T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)[\Upsilon(j, z_j(.,\varpi),\varpi) + By(t,\varpi)]dj$$
$$+ \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)).$$

The next section will cover additional hypotheses, which have been listed here.

Let (G_6)The linear operator $k : L^2(J,U) \to \mathcal{S}$, defined by

$$ky = \int_0^\varrho T_2(\varrho - j)By(j,\varpi)dj,$$

admits a pseudo-inverse operator k^{-1} in the quotient space $L^2(J,U)/\ker k$.

(G_7) A random function $Q : \mathcal{S} \to \mathbb{R}_+$ exists, satisfying the following condition.

$$\vartheta_a Bk^{-1}\int_0^t [\|z^1(\varpi)\| + \vartheta\|z_0(\varpi)\| + \vartheta_a\|z_0'(\varpi)\|$$
$$+ \vartheta_a \int_0^\varrho [\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta$$
$$+ \vartheta \sum_{0<t_\xi<\varrho} a_\xi\|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \sum_{0<t_\xi<\varrho} a_\xi'\|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}]dj \le Q(\varpi).$$

Theorem 4.1. If (G_1) - (G_7) are satisfied, then the problem (2) is approximately controllable on J.

Proof of Theorem 4.1. Let us define the control:

$$\begin{aligned} z(t,\varpi) =& k^{-1}(z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\ &- \int_0^\varrho T_2(\varrho - j)\Upsilon(j, z_j(.,\varpi),\varpi)dj \\ &- \sum_{0<t_\xi<\varrho} T_1(\varrho - t_\xi)I_\xi(z(t_\xi)) - \sum_{0<t_\xi<\varrho} T_2(\varrho - t_\xi)I_\xi'(z(t_\xi))). \end{aligned}$$

The random operator $(\Gamma(\varpi))'$ is defined as $(\Gamma(\varpi))' : \Omega \times PC_\delta \to PC_\delta$

$$\begin{aligned} ((\Gamma(\varpi))'z)(t) &= T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)[\Upsilon(j, z_j(.,\varpi),\varpi)dj \\ &+ \int_0^t T_2(t-j)Bk^{-1}[(z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\ &- \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta - \sum_{0<t_\xi<\varrho} T_1(\varrho - t_\xi)I_\xi(z(t_\xi)))] \\ &- \sum_{0<t_\xi<\varrho} T_2(\varrho - t_\xi)I_\xi'(z(t_\xi))]dj + \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) \end{aligned} \tag{7}$$

$$+\sum_{0<t_\xi<t} T_2(t-t_\xi)I'_\xi(z(t_\xi)) \quad t\in(-\delta,\varrho].$$

$(\Gamma(\varpi))' = (\Gamma(\varpi))'_1 + (\Gamma(\varpi))'_2$

$$\begin{aligned}(\Gamma(\varpi))'_1 z(t) &= T_1(t)z_0(\varpi) + T_2(t)z'_0(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)dj \\ &+ \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I'_\xi(z(t_\xi)),\end{aligned} \tag{8}$$

$$\begin{aligned}(\Gamma(\varpi))'_2 z(t) = \int_0^t & T_2(t-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z'_0(\varpi) \\ &- \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta \\ &- \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)) - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))]dj.\end{aligned} \tag{9}$$

In Theorem (3.1), we have already considered four cases for $(\Gamma_1(\varpi))$. Therefore, it remains to verify the result for $(\Gamma(\varpi))_2$.

Step 1. The operator $(\Gamma(\varpi))_2$ maps bounded sets into bounded sets. To show this, we need to find a positive constant $q(\varpi)$ such that for every $z \in \mathcal{B}_q(\delta)$, the following holds:

$$\mathcal{B}_r(\delta) := \{z \in PC_\delta : \sup_{\delta\le t\le\varrho} \|z(t,\varpi)\| \le q(\varpi)\}$$

one has $\|(\Gamma(\varpi))_2 z\|_{PC} \le Q(\varpi)$.

$$\begin{aligned}&\|(\Gamma(\varpi))'_2 z(t)\| \\ &\le \int_0^t \|T_2(t-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z'_0(\varpi) \\ &- \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta \\ &- \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)) - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))]\|dj \\ &\le \int_0^t \|T_2(t-j)\|Bk^{-1}[\|z^1(\varpi)\| + \|T_1(\varrho)z_0(\varpi)\| + \|T_2(\varrho)z'_0(\varpi)\| \\ &+ \|\int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta\| \\ &+ \sum_{0<t_\xi<\varrho} \|T_1(\varrho-t_\xi)I_\xi(z(t_\xi))\| + \|\sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))\|]dj \\ &\le \vartheta_a Bk^{-1}\int_0^t [\|z^1(\varpi)\| + \vartheta\|z_0(\varpi)\| + \vartheta_a\|z'_0(\varpi)\| \\ &+ \vartheta_a \int_0^\varrho [\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|^{\alpha_0}_{\mathcal{D}} + b_0(\varpi)]d\eta\end{aligned}$$

$$+ \vartheta \sum_{0<t_\xi<\varrho} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \sum_{0<t_\xi<\varrho} a'_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}]dj$$
$$\leq Q(\varpi).$$

Hence, $(\Gamma(\varpi))'_2$ is bounded in PC_δ.

Step 2. We show that $(\Gamma(\varpi))'_2$ is continuous on $\mathcal{B}_r(\delta)$.

Let $z_1, z_2 \in \mathcal{B}_r(\delta)$ and $t \in J$.

$$\begin{aligned}
&\|(\Gamma(\varpi))'_2 z_1(t) - (\Gamma(\varpi))'_2 z_2(t)\| \\
&\quad \leq \int_0^t \|T_2(t-j)Bk^{-1}[\int_0^\varrho T_2(\varrho-\eta)[\Upsilon(\eta,(z_{1,\eta}(.,\varpi),\varpi),\varpi) \\
&\quad - \Upsilon(\eta,(z_{2,\eta}(.,\varpi),\varpi),\varpi)]d\eta - \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)[I_\xi(z_1(t_\xi)) - I_\xi(z_2(t_\xi))] \\
&\quad - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)[I'_\xi(z_1(t_\xi)) - I'_\xi(z_2(t_\xi))]\|]dj \\
&\quad \leq \int_0^t \vartheta_a Bk^{-1}[\int_0^\varrho \vartheta_a \|[\sup_{\eta\in(0.\varrho]} \|(z_{1,\eta}(.,\varpi),\varpi) - (z_{2,\eta}(.,\varpi),\varpi)\|_{\mathbb{R}}^{\alpha_0} d\eta \\
&\quad + \vartheta \sum_{0<t_\xi<\varrho} a_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \sum_{0<t_\xi<\varrho} a'_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}]dj.
\end{aligned}$$

For all $t \in (-\delta, \varrho]$, the uniform operator topology remains continuous due to compactness for $t > 0$. Since $z_1, z_2 \in \mathcal{B}_r(\delta)$, the right-hand side of the inequalities is independent. As $(z_1 - z_2) \to 0$, it follows that

$$\|((\Gamma(\varpi))'_2 z_1)(t) - ((\Gamma(\varpi))'_2 z_2)(t)\| \to 0.$$

Thus, $(\Gamma(\varpi))$ is continuous.

Step 3. $(\Gamma(\varpi))'_2$ is a compact operator. To show this, we write $(\Gamma(\varpi))'_2$ as

$$(\Gamma(\varpi))'_2 = (\Gamma_a(\varpi))'_2 + (\Gamma_b(\varpi))'_2.$$

Here, $(\Gamma_1(\varpi))$ and $(\Gamma(\varpi))_2$ are operators on $\mathcal{B}_r(\delta)$ and are defined as follows.

$$\begin{aligned}
(\Gamma_a(\varpi))'_2 &= \int_0^t \|T_2(t-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z'_0(\varpi) \\
&\quad - \int_0^\varrho T_2(\varrho-\eta)[\Upsilon(\eta, z_\eta(.,\varpi),\varpi)]]d\eta dj, \\
(\Gamma_b(\varpi))'_2 &= \int_0^t \|T_2(t-j)Bk^{-1}[\sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)) \\
&\quad - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))]dj.
\end{aligned}$$

First, we show that $(\Gamma(\varpi))_{2,a}(\mathcal{B}_r(\delta))$ is equicontinuous.

Let $\delta \leq t_1 < t_2 \leq \varrho$ and take a small $\epsilon > 0$. Then,

$$
\begin{aligned}
&\|(\Gamma_a(\varpi))_2' z(t_2) - (\Gamma_a(\varpi))_2' z(t_1)\| \\
&\qquad \leq \int_0^{t_1} \|T_2(t_2-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\
&\qquad - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]dj \\
&\qquad - \int_0^{t_2} T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\
&\qquad - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\qquad \leq \int_0^{t_1-\theta} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\qquad - T_2(\varrho)z_0'(\varpi) - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\qquad + \| \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\qquad - T_2(\varrho)z_0'(\varpi) - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\qquad + \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\qquad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\qquad \leq \int_0^{t_1-\theta} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\qquad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta]\|dj \\
&\qquad + \| \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\qquad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta]\|dj \\
&\qquad + \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\qquad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta]\|dj.
\end{aligned}
$$

As $t_2 - t_1$ approaches zero, the expression $\left\|(\Gamma_a(\varpi))_2' z(t_2) - (\Gamma_a(\varpi))_2' z(t_1)\right\|$. Additionally, for any $z \in \mathcal{B}_r(\delta)$, the value tends to zero. This occurs because the operator $T_2(t)$ is compact for $t > 0$, ensuring continuity in the uniform operator topology. Consequently, $(\Gamma_1(\varpi))$ maps $\mathcal{B}_r(\delta)$ into an equicontinuous family of functions.

We now show that the set $(\Gamma_a(\varpi))'_2(\mathcal{B}_r(\delta))(t)$ is precompact in $\mathcal{S}$. Consider $\delta < t \leq j \leq \varrho$ and a real number ϵ satisfying $0 < \epsilon < t$. For $z \in \mathcal{B}_r(\delta)$, we define $((\Gamma_{a,\epsilon}(\varpi))'_2 z)(t)$ as $\int_0^{t-\epsilon} T_2(t_2-j)Bk^{-1}[z^1(\varpi)-T_1(\varrho)z_0(\varpi)-T_2(\varrho)z'_0(\varpi)+\int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)]d\eta$.

Since $T_2(t)$ is compact for $t > 0$, it follows that the set $\{((\Gamma_{a,\epsilon}(\varpi))'_2 z)(t) : z \in \mathcal{B}_r(\delta)\}$ is precompact for $z \in \mathcal{B}_r(\delta)$ and $0 < \epsilon < t$.

Additionally, for every $z \in \mathcal{B}_r(\delta)$, we state that

$$\begin{aligned}
&\|((\Gamma_a(\varpi))_2 z)(t) - ((\Gamma_{a,\epsilon}(\varpi))_2 z)(t)\| \\
&\qquad \leq \int_{t-\epsilon}^{t} \|T_2(t_2-j)Bk^{-1}\int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)\|d\eta\, dj \\
&\qquad \leq \int_{t-\epsilon}^{t} \vartheta_a Bk^{-1}\int_0^{\varrho} \vartheta_a[\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)d\eta]dj.
\end{aligned}$$

Thus, we can find sets that are precompact and close to $\{(\Gamma_a(\varpi))'_2 z : z \in B_r(\delta)\}$. This shows that the set $\{(\Gamma_a(\varpi))'_2 z : z \in B_r(\delta)\}$ is precompact in $\mathcal{S}$. The set $(\Gamma_a(\varpi))'_2(B_r(\delta))$ is uniformly bounded. Since the functions in this set are equicontinuous, the Arzelà-Ascoli theorem implies that it is enough to show that $(\Gamma_a(\varpi))'_2$ maps $B_r(\delta)$ into a precompact set in $\mathcal{S}$.

Next, we need to confirm that $(\Gamma_b(\varpi))'_2$ is also a compact operator. Using Step 3 of Theorem 3.1, we establish that $(\Gamma_b(\varpi))'_2$ is compact.

Step 4. Next, we find an open set $U \subseteq PC_\delta$ where any z on its boundary satisfies $z \notin \lambda(\Gamma(\varpi))'(z)$ for $\lambda \in (0,1)$.

Let $\lambda \in (0,1)$ and assume $z \in PC_\delta$ is a solution of $z = \lambda(\Gamma(\varpi))'(z)$ for some $0 < \lambda < 1$. This means that for every $t \in (0,\varrho]$, the condition holds

$$\begin{aligned}
z(t,\varpi) =& \lambda T_1(t)z_0(\varpi) + \lambda T_2(t)z'_0(\varpi) + \lambda\int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)dj \\
&+ \lambda\int_0^t T_2(t-j)Bk^{-1}[(z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z'_0(\varpi) \\
&- \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta \\
&- \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)))] - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))]dj \\
&+ \lambda\sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \lambda\sum_{0<t_\xi<t} T_2(t-t_\xi)I'_\xi(z(t_\xi)).
\end{aligned}$$

By step 1 of Theorem 3.1 and 3.2, $\|(\Gamma(\varpi))'z(t)\| \leq R(\varpi) + Q(\varpi)$. We can find a constant $R(\varpi) + Q(\varpi)$ such that $\|z\|_{PC} \neq R(\varpi) + Q(\varpi)$. Set

$$U = \{z \in PC([\delta,\varrho],\mathcal{S}) \mid \sup_{\delta\leq t\leq\varrho} \|z(t)\| < R(\varpi) + Q(\varpi)\}.$$

From Steps 1-3 in Theorem 3.2, it is enough to show that $(\Gamma(\varpi))' : U \to PC_\delta$ is a compact operator.

With the chosen set U, no x on its boundary satisfies $z \in \lambda(\Gamma(\varpi))(z)$ for $\lambda \in (0,1)$.

Using Lemma 2.1, we conclude that the operator $(\Gamma(\varpi))$ has a fixed point $z^* \in U$. Hence, we obtain the result

$$
\begin{aligned}
z^*(t,\varpi) &= T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) \\
&\quad + \lambda \int_0^t T_2(t-j)[\Upsilon(j, z_j^*(.,\varpi),\varpi)dj \\
&\quad + \int_0^t T_2(t-j)Bk^{-1}[(z^{*,1}(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\
&\quad - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta^*(.,\varpi),\varpi)d\eta - \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z^*(t_\xi)))] \\
&\quad - \sum_{0<t_\xi<\varrho} T_2(k-t_\xi)I_\xi'(z^*(t_\xi))]dj + \lambda \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z^*(t_\xi)) \\
&\quad + \lambda \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z^*(t_\xi))].
\end{aligned}
\tag{10}
$$

Thus, $z^*(t,\varpi)$ is a mild solution of problem (2) and has a fixed point. This concludes the proof of the theorem.

5. Example

Before using our abstract results, we first establish some necessary details. This section provides an example to illustrate the findings.

Let $\mathcal{S} = L^2([0,\pi])$ and define $D(A)$ as the set of functions $x \in \mathcal{S}$ for which $x'' \in \mathcal{S}$ and $x(0) = x(\pi) = 0$. The linear operator $A : D(A) \subseteq \mathcal{S} \to \mathcal{S}$ is defined by $Ax = x''$. It is well known that A generates a strongly continuous cosine family $(T_1(t))_{t\in\mathbb{R}}$ on $\mathcal{S}$.

The operator A has a discrete spectrum, with eigenvalues $-n^2$ for $n \in \vartheta$. Each eigenvalue corresponds to an eigenvector given by $z_n(\varrho) = (\frac{2}{\pi})^{1/2}$. Now, we consider the following impulsive partial functional integro-differential equation.

$$
\frac{\partial^2}{\partial t^2} z(t,x,\varpi) = \frac{\partial^2}{\partial x^2} z(t,x,\varpi) + \Upsilon(t, z(\sin t, x, \varpi), \varpi), \quad \varpi \in (-\infty, 0],
\tag{11}
$$

$$
\begin{aligned}
\Delta z(t_\xi, x, \varpi) &= \int_0^\pi q_\xi(x,y) z(t_\xi, y, \varpi)dy \quad \text{and} \\
\Delta' z(t_\xi, x, \varpi) &= \int_0^\pi q_\xi'(x,y) z(t_\xi, y, \varpi)dy, \quad \xi = 1, \dots, m,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
z(t,0,\varpi) &= z(t,\pi,\varpi) = 0; \quad z(0,x,\varpi) = z_0(x,\varpi), \\
z_t(0,x,\varpi) &= z_1(x,\varpi), \quad t \in J = [0,1], \quad 0 \le x \le \pi,
\end{aligned}
\tag{13}
$$

$$z(0,x,\varpi)=z_0(x,\varpi), \quad \text{and} \quad z_t(0,x,\varpi)=z_1(x,\varpi), \quad 0\leq x\leq \pi, \tag{14}$$

where we assume the following conditions:

The functions $\Upsilon(\cdot,\varpi)$ and are continuous on $[0,1]$ with

$$n=\sup_{0\leq j\leq 1}|\Upsilon(j,\varpi)|<1.$$

The functions q_ξ, $q'_\xi : [0,\pi]\times[0,\pi]\to\mathbb{R}$, $k=1,1,\ldots,m$, are continuously differentiable, and

$$\psi_\xi=\left(\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial x}q_\xi(x,y)\right)^2dxdy\right)^{\frac{1}{2}}<\infty,$$
$$\psi'_\xi=\left(\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial x}q'_\xi(x,y)\right)^2dxdy\right)^{\frac{1}{2}}<\infty,$$

for every $\xi=1,2,\ldots,m$.

Now, we define the operators respectively $\Upsilon : J\times\mathcal{D}\times\Omega\to\mathcal{S}$,

$$\Upsilon(t,z_t(.,\varpi),\varpi)(x)=\Upsilon(t,z(\sin t,x,\varpi),\varpi),$$
$$I_\xi(z,\varpi)(x)=\int_0^\pi q_\xi(x,y)z(t_\xi,y,\varpi)dy, \quad \xi=1,2,...,m,$$
$$I'_\xi(z,\varpi)(x)=\int_0^\pi q'_\xi(x,y)z(t_\xi,y,\varpi)dy, \quad \xi=1,2,...,m.$$

The equations (5.13-5.16) can be rewritten in a more general form, represented as (1). Using the previously defined functions, we satisfy the conditions of Theorem 3.1. Therefore, by applying Theorem 3.1, we conclude that the nonlocal impulsive Cauchy problem (5.13-5.16) has a mild solution on the interval J.

The graphical illustration of the solution surface for the second-order impulsive random functional differential equation with finite delay, given in equation (1). This illustration provides a more comprehensive understanding of the spatio-temporal behavior of the system in question. The solution surface $z(t,x,\varpi)$ is shown over the spatial domain $x\in[0,\pi]$ and the temporal domain $t\in[0,1]$. The graph gives a visual representation of the dynamic behavior of the system, which includes the interaction between diffusion effects, delay factors, and impulsive influences. The exponential decay term, in conjunction with the sinusoidal components, captures the oscillatory character of the solution, which is heavily dominated by the boundary and initial conditions imposed. The plot also shows the influence of the impulses at the discrete time instances t_k. These impulses cause discontinuities in the system, which then manifest as jumps in the solution trajectory. As for the influence of the kernel functions $q_k(x,y)$ and $q'_k(x,y)$ of the integral formulas, it will be reflected through the solution's amplitudes in the space domain.

Inspecting the above graph, it is possible to make several useful observations. As the solution depends on the sines, one can see its oscillations along with the periodic variation within the above domain. In other words, the exponential decay term alone causes an amplitude to fall off gradually due to time going on, showing system stability under specific conditions. Impulsive effects are captured by the solution surface at certain instances of time, and these effects appear as sudden deviations or peaks in the surface. The boundary condition $z(t, 0, \varpi) = z(t, \pi, \varpi) = 0$ is used to enforce that the solution is discarded at the spatial boundaries, and this causes characteristic nodal structures. The solution surface is represented in below figure. The plot of the time development of the system with respect to the spatial variable is depicted here, and color gradient represents the magnitude of $z(t, x, \varpi)$. Light regions indicate high values, while the dark regions signify low values. The graphical interpretation of the solution

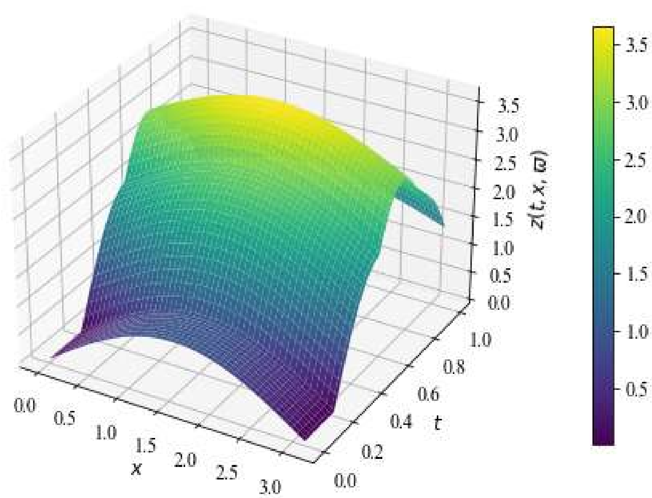

Figure 1: Graphical representation of the solution $z(y, t, \varpi)$ for the second-order impulsive random functional differential equation.

surface would carry an important behavior of the system with respect to different parametric conditions. The solutions profile could be discussed by adjusting the parameters such as kernel functions q_k and initial conditions $z_0(x, \varpi)$. The visualization more strongly supports the verification process of theoretical results through which it ensures the provided solutions are in accordance with expected physical and mathematical properties. In general, graphical representation is essential for the deeper understanding of such a complex dynamics system and helps in further research and analysis.

6. Conclusion

This study explores a class of second-order equations with delays, which are widely applicable in scientific and engineering fields. By analyzing these equations in Banach spaces, it reveals unique challenges and opportunities for control and analysis. The research rigorously examines the existence and approximate controllability of solutions, contributing to a deeper understanding of dynamical systems with delayed feedback. Using mathematical tools such as cosine family theory and the Leray-Schauder theorem, it establishes precise conditions for solution existence. A practical example provides empirical validation, offering valuable insights into real-world behavior. This investigation not only advances theoretical knowledge but also aids in developing effective control strategies and engineering solutions across various domains.

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On intuitionistic Q-fuzzy subalgebras/ideals/deductive systems in Hilbert algebras

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Abstract. This study introduces the concepts of intuitionistic Q-fuzzy subalgebras, ideals, and deductive systems in the setting of Hilbert algebras and investigates their fundamental properties and interrelations. The theoretical results are supported by concrete examples and are structured in a way that facilitates understanding of the algebraic-logical framework underlying fuzzy logic extensions. By presenting clear definitions, illustrative cases, and step-by-step reasoning, the paper serves not only as a contribution to the field of abstract algebra but also as a useful learning resource for upper secondary science students beginning to engage with research. The work encourages early exploration of mathematical structures, fosters critical and creative thinking,

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and promotes accessibility of advanced topics through collaboration across academic levels and institutions.

Keywords: Hilbert algebra, intuitionistic Q-fuzzy subalgebra, intuitionistic Q-fuzzy ideal, intuitionistic Q-fuzzy deductive system.

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1. Introduction and preliminaries

The concept of fuzzy sets was proposed by Zadeh [23]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in [1, 3, 6]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multi-criteria decision-making [12, 13, 14]. The concept of Hilbert algebra was introduced in the early 50-ties by Henkin [15] for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [9] from an algebraic point of view. Diego proved that Hilbert algebras form a locally finite variety. Hilbert algebras were treated by Busneag [4, 5] and Jun [16], and some of their filters forming deductive systems were recognized. Dudek [10] considered the fuzzification of subalgebras and deductive systems in Hilbert algebras. Zhan and Tan [24] studied intuitionistic fuzzifications of the concept of deductive systems in Hilbert algebras. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy deductive systems in Hilbert algebras.

The study of Q-fuzzy sets has attracted considerable interest due to its ability to model graded uncertainty with respect to an index set Q. Researchers have explored this framework in various algebraic structures, including semigroups, rings, and algebras. Notably, Kim [17, 18] investigated intuitionistic Q-fuzzy ideals and semiprime ideals in semigroups, establishing foundational results that extend classical fuzzy ideal theory and contribute to the development of generalized fuzzy algebraic systems. This line of research was further developed in the context of UP-algebras by Tanamoon et al. [21], who explored the general structure of Q-fuzzy sets and their algebraic behavior. In a subsequent study, Sripaeng et al. [20] introduced and analyzed anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras, offering dual perspectives that enriched the understanding of fuzziness in non-classical algebraic systems. Recent studies have extended the concept of intuitionistic Q-fuzzy ideals to various algebraic structures. Derseh et al. [8] examined intuitionistic Q-fuzzy PMS-ideals in PMS-algebras, offering structural properties and characterizations. Wang [22] investigated similar notions within BE-algebras, while Massa'deh [19] contributed to the development of intuitionistic Q-fuzzy KU-ideals. These works demonstrate the broad applicability of the intuitionistic Q-fuzzy framework and support its further exploration in diverse algebraic settings. Collectively, these works contribute to advancing generalized fuzzy structures in algebra and support further applications in logic and uncertainty modeling.

Given a set Q, this paper introduces the notions of intuitionistic Q-fuzzy subalgebras, ideals, and deductive systems within Hilbert algebras, and explores their fundamental properties and interrelationships. The study also examines the behavior of these structures under homomorphisms, with results presented in a clear and illustrative manner to enhance accessibility for learners and support further research development.

Before we begin the study, let's review the definition of Hilbert algebras, which was defined by Diego [9] in 1966.

Definition 1.1. *A Hilbert algebra is a triplet $H = (H, \cdot, 1)$, where H is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of H such that the following axioms hold:*

1. $(\forall x, y \in H)(x \cdot (y \cdot x) = 1)$,
2. $(\forall x, y, z \in H)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
3. $(\forall x, y \in H)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

The following results were proved in [10].

Lemma 1.1. *Let $H = (H, \cdot, 1)$ be a Hilbert algebra. Then*

1. $(\forall x \in H)(x \cdot x = 1)$,
2. $(\forall x \in H)(1 \cdot x = x)$,
3. $(\forall x \in H)(x \cdot 1 = 1)$,
4. $(\forall x, y, z \in H)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $H = (H, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in H)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on H with 1 as the largest element.

Definition 1.2 ([7])**.** *A nonempty subset I of a Hilbert algebra $H = (H, \cdot, 1)$ is called an ideal of H if*

1. $1 \in I$,
2. $(\forall x \in H, \forall y \in I)(x \cdot y \in I)$,
3. $(\forall x \in H, \forall y_1, y_2 \in I)((y_1 \cdot (y_2 \cdot x)) \cdot x \in I)$.

Let X and Q be two nonempty sets. An *Q-fuzzy set* in a nonempty set X is defined to be a function $\mu : X \times Q \to [0,1]$, where $[0,1]$ is the unit closed interval of real numbers.

Definition 1.3 ([11]). *An Q-fuzzy set μ in a Hilbert algebra $H = (H, \cdot, 1)$ is called a Q-fuzzy ideal of H if the following conditions hold:*

1. $(\forall x \in H, \forall q \in Q)(\mu(1,q) \geq \mu(x,q))$,
2. $(\forall x, y \in H, \forall q \in Q)(\mu(x \cdot y, q) \geq \mu(y,q))$,
3. $(\forall x, y_1, y_2 \in H, \forall q \in Q)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu(y_1,q), \mu(y_2,q)\})$.

Definition 1.4. *Let X and Q be two nonempty sets. An intuitionistic Q-fuzzy set in X is defined to be a structure*

$$A := \{(x, \mu_A(x,q), \gamma_A(x,q)) \mid x \in X, q \in Q\}, \tag{1.1}$$

where the functions $\mu_A : X \times Q \to [0,1]$ is the degree of membership of x and $\gamma_A : X \times Q \to [0,1]$ is the degree of non-membership of x such that

$$(\forall x \in X, \forall q \in Q)(0 \leq \mu_A(x,q) + \gamma_A(x,q) \leq 1),$$

and the intuitionistic Q-fuzzy set in (1.1) *is simply denoted by $A = (\mu_A, \gamma_A)$.*

2. Intuitionistic Q-fuzzy Hilbert algebras

In this section, we introduce the notions of intuitionistic Q-fuzzy subalgebras, intuitionistic Q-fuzzy ideals, and intuitionistic Q-fuzzy deductive systems of Hilbert algebras and investigate some related properties.

Definition 2.1. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra $H = (H, \cdot, 1)$ is called an intuitionistic Q-fuzzy subalgebra of H if*

$$(\forall x, y \in H, \forall q \in Q)\begin{pmatrix} \mu_A(x \cdot y, q) \geq \min\{\mu_A(x,q), \mu_A(y,q)\} \\ \gamma_A(x \cdot y, q) \leq \max\{\gamma_A(x,q), \gamma_A(y,q)\} \end{pmatrix}. \tag{2.1}$$

Example 2.1. Let $H = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then H is a Hilbert algebra. We define an intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in H as follows:

$$\mu_A(1,q) = 1, \mu_A(x,q) = 0.8, \mu_A(y,q) = 0.8, \mu_A(z,q) = 0.7, \mu_A(0,q) = 0.4,$$
$$\gamma_A(1,q) = 0.3, \gamma_A(x,q) = 0.5, \gamma_A(y,q) = 0.7, \gamma_A(z,q) = 0.3, \gamma_A(0,q) = 0.6,$$

for all $q \in Q$. Then A is an intuitionistic Q-fuzzy subalgebra of H.

Proposition 2.1. *Every intuitionistic Q-fuzzy subalgebra $A = (\mu_A, \gamma_A)$ of a Hilbert algebra H satisfies $\mu_A(1, q) \geq \mu_A(x, q)$ and $\gamma_A(1, q) \leq \gamma_A(x, q)$ for all $x \in H$ and $q \in Q$.*

Proof. For any $x \in H$ and $q \in Q$, we have

$$\mu_A(1, q) = \mu_A(x \cdot x, q) \geq \min\{\mu_A(x, q), \mu_A(x, q)\} = \mu_A(x, q)$$

and

$$\gamma_A(1, q) = \gamma_A(x \cdot x, q) \leq \max\{\gamma_A(x, q), \gamma_A(x, q)\} = \gamma_A(x, q). \qquad \square$$

Definition 2.2. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is said to be an intuitionistic Q-fuzzy ideal of H if the following conditions hold:*

$$(\forall x \in H, \forall q \in Q) \left(\begin{array}{c} \mu_A(1, q) \geq \mu_A(x, q) \\ \gamma_A(1, q) \leq \gamma_A(x, q) \end{array} \right), \tag{2.2}$$

$$(\forall x, y \in H, \forall q \in Q) \left(\begin{array}{c} \mu_A(x \cdot y, q) \geq \mu_A(y, q) \\ \gamma_A(x \cdot y, q) \leq \gamma_A(y, q) \end{array} \right), \tag{2.3}$$

$$\begin{array}{l} (\forall x, y_1, y_2 \in H, \forall q \in Q) \\ \left(\begin{array}{c} \mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_A(y_2, q)\} \\ \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\} \end{array} \right). \end{array} \tag{2.4}$$

Example 2.2. Let $H = \{1, x, y, z\}$ with the following Cayley table:

$\cdot$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	x	1	z
z	1	1	y	1

Then H is a Hilbert algebra. We define an intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ as follows:

$$\begin{array}{l} \mu_A(1, q) = 0.9, \mu_A(x, q) = 0.3, \mu_A(y, q) = 0.1, \mu_A(z, q) = 0.6, \\ \gamma_A(1, q) = 0.1, \gamma_A(x, q) = 0.2, \gamma_A(y, q) = 0.8, \gamma_A(z, q) = 0.3, \end{array}$$

for all $q \in Q$. Then A is an intuitionistic Q-fuzzy ideal of H.

Proposition 2.2. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then*

$$(\forall x, y \in H, \forall q \in Q) \left(\begin{array}{c} \mu_A((y \cdot x) \cdot x, q) \geq \mu_A(y, q) \\ \gamma_A((y \cdot x) \cdot x, q) \leq \gamma_A(y, q) \end{array} \right). \tag{2.5}$$

Proof. Let $x, y \in H$ and $q \in Q$. Putting $y_1 = y$ and $y_2 = 1$ in (2.4), we have

$$\mu_A((y \cdot x) \cdot x, q) \geq \min\{\mu_A(y, q), \mu_A(1, q)\} = \mu_A(y, q)$$

and

$$\gamma_A((y \cdot x) \cdot x, q) \leq \max\{\gamma_A(y, q), \gamma_A(1, q)\} = \gamma_A(y, q). \qquad \square$$

Lemma 2.1. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then we have the following*

$$(\forall x, y \in H, \forall q \in Q)\left(x \leq y \;\Rightarrow \left\{\begin{array}{l} \mu_A(x, q) \leq \mu_A(y, q) \\ \gamma_A(x, q) \geq \gamma_A(y, q) \end{array}\right.\right). \tag{2.6}$$

Proof. Let $x, y \in H$ be such that $x \leq y$ and $q \in Q$. Then $x \cdot y = 1$ and so

$$\begin{array}{rcl} \mu_A(y, q) & = & \mu_A(1 \cdot y, q) \\ & = & \mu_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \\ & \geq & \min\{\mu_A(x \cdot y, q), \mu_A(x, q)\} \\ & = & \min\{\mu_A(1, q), \mu_A(x, q)\} \\ & = & \mu_A(x, q) \end{array}$$

and

$$\begin{array}{rcl} \gamma_A(y, q) & = & \gamma_A(1 \cdot y, q) \\ & = & \gamma_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \\ & \leq & \max\{\gamma_A(x \cdot y, q), \gamma_A(x, q)\} \\ & = & \max\{\gamma_A(1, q), \gamma_A(x, q)\} \\ & = & \gamma_A(x, q). \end{array}$$

$\square$

Theorem 2.1. *Every intuitionistic Q-fuzzy ideal of a Hilbert algebra H is an intuitionistic Q-fuzzy subalgebra of H.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q-fuzzy ideal of H. Let $x, y \in H$ and $q \in Q$. Since $y \leq x \cdot y$, it follows from Lemma 2.1 that

$$\mu_A(y, q) \leq \mu_A(x \cdot y, q) \text{ and } \gamma_A(y, q) \geq \gamma_A(x \cdot y, q).$$

It follows from (2.3) that

$$\mu_A(x \cdot y, q) \geq \mu_A(y, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\}$$

and

$$\gamma_A(x \cdot y, q) \leq \gamma_A(y, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\}.$$

Hence, A is an intuitionistic Q-fuzzy subalgebra of H. $\square$

Proposition 2.3. *If $A_i = \{(\mu_{A_i}, \gamma_{A_i}) : i \in \Delta\}$ is a family of intuitionistic Q-fuzzy ideals of a Hilbert algebra H, then $\bigwedge_{i \in \Delta} A_i$ is an intuitionistic Q-fuzzy ideal of H.*

Proof. Let $A_i = \{(\mu_{A_i}, \gamma_{A_i}) : i \in \Delta\}$ be a family of intuitionistic Q-fuzzy ideals of H. Let $x \in H$ and $q \in Q$. Then

$$(\bigwedge_{i\in\Delta} \mu_{A_i})(1,q) = \inf_{i\in\Delta}\{\mu_{A_i}(1,q)\} \geq \inf_{i\in\Delta}\{\mu_{A_i}(x,q)\} = (\bigwedge_{i\in\Delta} \mu_{A_i})(x,q)$$

and

$$(\bigwedge_{i\in\Delta} \gamma_{A_i})(1,q) = \sup_{i\in\Delta}\{\gamma_{A_i}(1,q)\} \leq \sup_{i\in\Delta}\{\gamma_{A_i}(x,q)\} = (\bigwedge_{i\in\Delta} \gamma_{A_i})(x,q).$$

Let $x, y \in H$ and $q \in Q$. Then

$$(\bigwedge_{i\in\Delta} \mu_{A_i})(x\cdot y,q) = \inf_{i\in\Delta}\{\mu_{A_i}(x\cdot y,q)\} \geq \inf_{i\in\Delta}\{\mu_{A_i}(y,q)\} = (\bigwedge_{i\in\Delta} \mu_{A_i})(y,q)$$

and

$$(\bigwedge_{i\in\Delta} \gamma_{A_i})(x\cdot y,q) = \sup_{i\in\Delta}\{\gamma_{A_i}(x\cdot y,q)\} \leq \sup_{i\in\Delta}\{\gamma_{A_i}(y,q)\} = (\bigwedge_{i\in\Delta} \gamma_{A_i})(y,q).$$

Let $x, y_1, y_2 \in H$ and $q \in Q$. Then

$$\begin{aligned}(\bigwedge_{i\in\Delta} \mu_{A_i})((y_1\cdot(y_2\cdot x))\cdot x,q) &= \inf_{i\in\Delta}\{\mu_{A_i}((y_1\cdot(y_2\cdot x))\cdot x,q)\} \\ &\geq \inf_{i\in\Delta}\{\min\{\mu_{A_i}(y_1,q),\mu_{A_i}(y_2,q)\}\} \\ &= \min\{\inf_{i\in\Delta}\{\mu_{A_i}(y_1,q)\},\inf_{i\in\Delta}\{\mu_{A_i}(y_2,q)\}\} \\ &= \min\{(\bigwedge_{i\in\Delta} \mu_{A_i})(y_1,q),(\bigwedge_{i\in\Delta} \mu_{A_i})(y_2,q)\}\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in\Delta} \gamma_{A_i})((y_1\cdot(y_2\cdot x))\cdot x,q) &= \sup_{i\in\Delta}\{\gamma_{A_i}((y_1\cdot(y_2\cdot x))\cdot x,q)\} \\ &\leq \sup_{i\in\Delta}\{\max\{\gamma_{A_i}(y_1,q),\gamma_{A_i}(y_2,q)\}\} \\ &= \max\{\sup_{i\in\Delta}\{\gamma_{A_i}(y_1,q)\},\sup_{i\in\Delta}\{\gamma_{A_i}(y_2,q)\}\} \\ &= \max\{(\bigwedge_{i\in\Delta} \gamma_{A_i})(y_1,q),(\bigwedge_{i\in\Delta} \gamma_{A_i})(y_2,q)\}.\end{aligned}$$

Hence, $\bigwedge_{i\in\Delta} A_i$ is an intuitionistic Q-fuzzy ideal of H. □

Definition 2.3. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is said to be an intuitionistic Q-fuzzy deductive system of H if the following conditions hold:*

$$(\forall x \in H, \forall q \in Q)\begin{pmatrix} \mu_A(1,q) \geq \mu_A(x,q) \\ \gamma_A(1,q) \leq \gamma_A(x,q) \end{pmatrix}, \tag{2.7}$$

$$(\forall x, y \in H, \forall q \in Q)\begin{pmatrix} \mu_A(y,q) \geq \min\{\mu_A(x\cdot y,q),\mu_A(x,q)\} \\ \gamma_A(y,q) \leq \max\{\gamma_A(x\cdot y,q),\mu_A(x,q)\} \end{pmatrix}. \tag{2.8}$$

Proposition 2.4. *Every intuitionistic Q-fuzzy ideal of a Hilbert algebra H is an intuitionistic Q-fuzzy deductive system of H.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q-fuzzy ideal of H. Let $x, y \in H$ and $q \in Q$. If $y_1 = x \cdot y$ and $y_2 = x$, then by (1) and (2) of Lemma 1.1 and (2.4), we have

$$\mu_A(y, q) = \mu_A(1 \cdot y, q) = \mu_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \geq \min\{\mu_A(x \cdot y, q), \mu_A(x, q)\}$$

and

$$\gamma_A(y, q) = \gamma_A(1 \cdot y, q) = \gamma_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \leq \max\{\gamma_A(x \cdot y, q), \gamma_A(x, q)\}.$$

Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy deductive system of H. $\square$

Lemma 2.2. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H if and only if μ_A and $\overline{\gamma}_A$ are Q-fuzzy ideals of H.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H. Then obviously μ_A is a Q-fuzzy ideal of H. Consider for every $x, y \in H$ and $q \in Q$, we have $\overline{\gamma}_A(1, q) = 1 - \gamma_A(1, q) \geq 1 - \gamma_A(x, q) = \overline{\gamma}_A(x, q)$. Let $x, y \in H$ and $q \in Q$. Then $\overline{\gamma_A}(y, q) = 1 - \gamma_A(y, q) \leq 1 - \gamma_A(x \cdot y, q) = \overline{\gamma_A}(x \cdot y, q)$.
Let $x, y_1, y_2 \in H$ and $q \in Q$. Then

$$\begin{aligned} \overline{\gamma_A}((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= 1 - \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \\ &\geq 1 - \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\} \\ &= \min\{1 - \gamma_A(y_1, q), 1 - \gamma_A(y_2, q)\} \\ &= \min\{\overline{\gamma_A}(y_1, q), \overline{\gamma_A}(y_2, q)\}. \end{aligned}$$

Hence, $\overline{\gamma_A}$ is a Q-fuzzy ideal of H.

Conversely, let us take μ_A and $\overline{\gamma_A}$ are Q-fuzzy ideals of H. Then obviously for every $x \in H$ and $q \in Q$, we have $\mu_A(1, q) \geq \mu_A(x, q)$ and $1 - \gamma_A(1, q) = \overline{\gamma_A}(1, q) \geq \overline{\gamma_A}(x, q) = 1 - \gamma_A(x, q)$, that is, $\gamma_A(1, q) \leq \gamma_A(x, q)$. Let $x, y \in H$ and $q \in Q$. Then obviously, $\mu_A(x \cdot y, q) \geq \mu_A(y, q)$ and $1 - \gamma_A(x \cdot y, q) = \overline{\gamma_A}(x \cdot y, q) \geq \overline{\gamma_A}(y, q) = 1 - \gamma_A(y, q)$, that is, $\gamma_A(x \cdot y, q) \leq \gamma_A(y, q)$. Let $x, y_1, y_2 \in H$ and $q \in Q$. Then obviously $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_A(y_2, q)\}$ and

$$\begin{aligned} 1 - \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= \overline{\gamma_A}((y_1 \cdot (y_2 \cdot x, q), q \cdot x, q) \\ &\geq \min\{\overline{\gamma_A}(y_1, q), \overline{\gamma_A}(y_2, q)\} \\ &= \min\{1 - \gamma_A(y_1, q), 1 - \gamma_A(y_2, q)\} \\ &= 1 - \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\}, \end{aligned}$$

that is, $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\}$. Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H. $\square$

Theorem 2.2. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is an intuitionistic Q-fuzzy ideal of H if and only if $(\mu_A, \overline{\mu}_A)$ and $(\gamma_A, \overline{\gamma}_A)$ are intuitionistic Q-fuzzy ideals of H.*

Proof. If an intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H, then $\mu_A = \overline{\overline{\mu}}_A$ and γ_A are Q-fuzzy ideals of H from Lemma 2.2, hence $(\mu_A, \overline{\mu}_A)$ and $(\overline{\gamma}_A, \gamma_A)$ are intuitionistic Q-fuzzy ideals of H.

Conversely, if $(\mu_A, \overline{\mu}_A)$ and $(\gamma_A, \overline{\gamma}_A)$ are intuitionistic Q-fuzzy ideals of H, then μ_A and $\overline{\gamma}_A$ are Q-fuzzy ideals of H. Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H. $\square$

Definition 2.4. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q-fuzzy set of a Hilbert algebra H and $\alpha \in [0, 1]$. Then we define the subsets $U(\mu_A, \alpha) = \{x \in X : \mu_A(x, q) \geq \alpha \ \forall q \in Q\}$ and $L(\gamma_A, \alpha) = \{x \in X : \gamma_A(x, q) \leq \alpha \ \forall q \in Q\}$ of H.*

Theorem 2.3. *Let A be a nonempty subset of a Hilbert algebra H and (μ_A, γ_A) be an intuitionistic Q-fuzzy set in H defined by $\mu_A(x, q) = \begin{cases} \alpha_0, & \text{if } x \in A \\ \alpha_1, & \text{otherwise} \end{cases}$ $\gamma_A(x, q) = \begin{cases} \beta_0, & \text{if } x \in A \\ \beta_1, & \text{otherwise} \end{cases}$ for all $x \in H$, $q \in Q$ and $\alpha_i, \beta_i \in [0, 1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$. Then (μ_A, γ_A) is an intuitionistic Q-fuzzy ideal of H if and only if A is an ideal of H.*

Proof. Assume that (μ_A, γ_A) is an intuitionistic Q-fuzzy ideal of H. Since $\mu_A(1, q) \geq \mu_A(x, q)$ and $\gamma_A(1, q) \leq \gamma_A(x, q)$ for all $x \in H$ and $q \in Q$, we have $\mu_A(1, q) = \alpha_1$ and $\gamma_A(1, q) = \beta_1$ and so $1 \in A$. Let $x \in H$ and $y \in A$. Then $\mu_A(x \cdot y, q) \geq \mu_A(y, q) = \alpha_1$ for all $q \in Q$ and then $\mu_A(x \cdot y, q) = \alpha_1$. Also $\gamma_A(x \cdot y, q) \leq \gamma_A(y, q) = \beta_1$ and then $\gamma_A(x \cdot y, q) = \beta_1$. Hence, $x \cdot y \in A$. For any $y_1, y_2 \in A$ and $x \in H$, we get $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_A(y_2, q)\} = \alpha_1$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\} = \beta_1$ for all $q \in Q$, which implies that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \alpha_1$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \beta_1$. It follows that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in A$. Therefore, A is an ideal of H.

Conversely, assume that A is an ideal of H. Since $1 \in A$, it follows that $\mu_A(1, q) = \alpha_1 \geq \mu_A(x, q)$ for all $x \in H$ and $q \in Q$. Let $x, y \in H$ and $q \in Q$. If $y \in A$, then $x \cdot y \in A$ and so $\mu_A(x \cdot y, q) = \alpha_1 = \mu_A(y, q)$ and $\gamma_A(x \cdot y, q) = \beta_1 = \gamma_A(y, q)$. If $y \in H \backslash A$, then $\mu_A(y, q) = \alpha_2$ and $\gamma_A(y, q) = \beta_2$, and hence $\mu_A(x \cdot y, q) \geq \alpha_2 = \mu_A(y, q)$ and $\gamma_A(x \cdot y, q) \leq \beta_2 = \gamma_A(y, q)$. Finally, let $x, y_1, y_2 \in H$ and $q \in Q$. If $y_1 \in H \backslash A$ or $y_2 \in H \backslash A$, then $\mu_A(y_1, q) = \alpha_2$ or $\mu_A(y_2, q) = \alpha_2$. It follows that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \alpha_2 = \min\{\mu_A(y_1, q), \mu_A(y_2, q)\}$. Also if $y_1 \in H \backslash A$ or $y_2 \in H \backslash A$, then $\gamma_A(y_1, q) = \beta_2$ or $\gamma_A(y_2, q) = \beta_2$. It follows that $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \beta_2 = \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\}$. Assume that $y_1, y_2 \in A$. Then $(y_1 \cdot (y_2 \cdot x)) \cdot x \in A$ and thus $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \alpha_1 = \min\{\mu_A(y_1, q), \mu_A(y_2, q)\}$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \beta_1 = \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\}$. Hence, (μ_A, γ_A) is an intuitionistic Q-fuzzy ideal of H. $\square$

Theorem 2.4. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then the subsets $U(\mu_A, \alpha)$ and $L(\gamma_A, \alpha)$ of H are ideals of H for every $\alpha \in Im(f_A) \cap Im(g_A) \subset [0, 1]$.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H and let $\alpha \in Im(f_A) \cap Im(g_A) \subset [0,1]$. Let $x \in U(\mu_A, \alpha)$. Then $\mu_A(x,q) \geq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\mu_A(1,q) \geq \mu_A(x,q) \geq \alpha$ for all $q \in Q$. Hence, $1 \in U(\mu_A, \alpha)$. Let $x \in L(\gamma_A, \alpha)$. Then $\gamma_A(x,q) \leq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\gamma_A(1,q) \leq \gamma_A(x,q) \leq \alpha$ for all $q \in Q$. Hence, $1 \in L(\gamma_A, \alpha)$. Let $x \in H$ and $y \in U(\mu_A, \alpha)$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\mu_A(x \cdot y, q) \geq \mu_A(y,q) \geq \alpha$ for all $q \in Q$. Hence, $x \cdot y \in U(\mu_A, \alpha)$. Let $x_1 \in H$ and $y_1 \in L(\gamma_A, \alpha)$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\gamma_A(x_1 \cdot y_1, q) \leq \gamma_A(y_1, q) \leq \alpha$ for all $q \in Q$. Hence, $x_1 \cdot y_1 \in L(\gamma_A, \alpha)$. Let $x \in H$ and $y_1, y_2 \in U(\mu_A, \alpha)$. Then $\mu_A(y_1,q) \geq \alpha$ and $\mu_A(y_2,q) \geq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\mu_A(y_1 \cdot (y_2 \cdot x, q), q) \cdot x, q) \geq \min\{\mu_A(y_1,q), \mu_A(y_2,q)\} \geq \alpha$ for all $q \in Q$. Hence, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\mu_A, \alpha)$. Let $x' \in H$ and $y_1', y_2' \in L(\gamma_A, \alpha)$. Then $\gamma_A(y_1', q) \leq \alpha$ and $\gamma_A(y_2', q) \leq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\gamma_A((y_1' \cdot (y_2' \cdot x', q), q) \cdot x', q) \leq \max\{\gamma_A(y_1', q), \gamma_A(y_2', q)\} \leq \alpha$ for all $q \in Q$. Hence, $(y_1' \cdot (y_2' \cdot x')) \cdot x' \in L(\gamma_A, \alpha)$. Therefore, $U(\mu_A, \alpha)$ and $L(\gamma_A, \alpha)$ are ideals of H. □

Theorem 2.5. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then for all $s, t \in [0,1]$, the sets $U(\mu_A, t)$ and $L(\gamma_A, s)$ are either empty or ideals of H.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H and let $s, t \in [0,1]$ be such that $U(\mu_A, t)$ and $L(\gamma_A, s)$ are nonempty subsets of H. It is clear that $1 \in U(\mu_A, t) \cap L(\gamma_A, s)$ since $\mu_A(1,q) \geq t$ and $\gamma_A(1,q) \leq s$ for all $q \in Q$. Let $x \in H$ and $y \in U(\mu_A, t)$. Then $\mu_A(y,q) \geq t$ for all $q \in Q$. It follows that $\mu_A(x \cdot y, q) \geq \mu_A(y,q) \geq t$ so that $x \cdot y \in U(\mu_A, t)$. Let $x \in H$ and $y_1, y_2 \in U(\mu_A, t)$. Then $\mu_A(y_1,q) \geq t$ and $\mu_A(y_2,q) \geq t$ for all $q \in Q$. Hence, $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1,q), \mu_a(y_2,q)\} \geq t$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\mu_A, t)$. Hence, $U(\mu_A, t)$ is an ideal of H. Let $x \in H$ and $y \in L(\gamma_A, s)$. Then $\gamma_A(y,q) \leq s$ for all $q \in Q$. It follows that $\gamma_A(x \cdot y, q) \leq \gamma_A(y,q) \leq s$ so that $x \cdot y \in L(\gamma_A, s)$. Let $x \in H$ and $y_1, y_2 \in L(\gamma_A, s)$. Then $\gamma_A(y_1,q) \leq s$ and $\gamma_A(y_2,q) \leq s$ for all $q \in Q$. Hence, $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1,q), \gamma_A(y_2,q)\} \leq s$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(\gamma_A, s)$. Hence, $L(\gamma_A, s)$ is an ideal of H. □

A mapping $f : X \to Y$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras. For any intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in Y, we define a new intuitionistic Q-fuzzy set $f^{-1}(A) = (\mu_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$ in X by

$$(\forall x \in X)(\mu_{f^{-1}(A)}(x,q) = \mu_A(f(x), q))$$

and

$$(\forall x \in X)(\gamma_{f^{-1}(A)}(x,q) = \gamma_A(f(x), q)).$$

Proposition 2.5. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and $A = (\mu_A, \gamma_A)$ an intuitionistic Q-fuzzy subalgebra of Y. Then the inverse image $f^{-1}(A)$ of A is an intuitionistic Q-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $q \in Q$. Then

$$\begin{aligned}
\mu_{f^{-1}(A)}(x \cdot y, q) &= \mu_A(f(x \cdot y), q) \\
&= \mu_A(f(x) \cdot f(y), q) \\
&\geq \min\{\mu_A(f(x), q), \mu_A(f(y), q)\} \\
&= \min\{\mu_{f^{-1}(A)}(x, q), \mu_{f^{-1}(A)}(y, q)\}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{f^{-1}(A)}(x \cdot y, q) &= \gamma_A(f(x \cdot y), q) \\
&= \gamma_A(f(x) \cdot f(y), q) \\
&\leq \max\{\gamma_A(f(x), q), \gamma_A(f(y), q)\} \\
&= \max\{\gamma_{f^{-1}(A)}(x, q), \gamma_{f^{-1}(A)}(y, q)\}.
\end{aligned}$$

Hence, $f^{-1}(A)$ is an intuitionistic Q-fuzzy subalgebra of X. □

Theorem 2.6. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and $A = (\mu_A, \gamma_A)$ an intuitionistic Q-fuzzy ideal of Y. Then the inverse image $f^{-1}(A)$ of A is an intuitionistic Q-fuzzy ideal of X.*

Proof. Since f is a homomorphism of X into Y, we have $f(1) = 1 \in Y$ and, by the assumption, $\mu_A(f(1), q) = \mu_A(1, q) \geq \mu_A(y, q)$ for all $y \in Y$ and $q \in Q$. In particular, $\mu_A(f(1), q) \geq \mu_A(f(x), q)$ for all $x \in X$ and $q \in Q$. Hence, $\mu_{f^{-1}(A)}(1, q) \geq \mu_{f^{-1}(A)}(x, q)$ for all $x \in X$ and $q \in Q$. Also $\gamma_A(f(1), q) = \gamma_A(1, q) \leq \gamma_A(y, q)$ for all $y \in Y$ and $q \in Q$. In particular, $\gamma_A(f(1), q) \leq \gamma_A(f(x), q)$ for all $x \in X$ and $q \in Q$. Hence, $\gamma_{f^{-1}(A)}(1, q) \leq \gamma_{f^{-1}(A)}(x, q)$ for all $x \in X$ and $q \in Q$, which proves (2.2). Now, let $x, y \in X$ and $q \in Q$. Then, by the assumption, we have

$$\mu_{f^{-1}(A)}(x \cdot y, q) = \mu_A(f(x \cdot y), q) = \mu_A(f(x) \cdot f(y), q) \geq \mu_A(f(y), q) = \mu_{f^{-1}(A)}(y, q)$$

and

$$\gamma_{f^{-1}(A)}(x \cdot y, q) = \gamma_A(f(x \cdot y), q) = \gamma_A(f(x) \cdot f(y), q) \leq \gamma_A(f(y), q) = \gamma_{f^{-1}(A)}(y, q),$$

which proves (2.3). Let $x, y_1, y_2 \in X$ and $q \in Q$. Then, by the assumption, we have

$$\begin{aligned}
\mu_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= \mu_A(f((y_1 \cdot (y_2 \cdot x)) \cdot x), q) \\
&= \mu_A((f(y_1) \cdot (f(y_2) \cdot f(x))) \cdot f(x), q) \\
&\geq \min\{\mu_A(f(y_1), q), \mu_A(f(y_2), q)\} \\
&= \min\{\mu_{f^{-1}(A)}(y_1, q), \mu_{f^{-1}(A)}(y_2, q)\}
\end{aligned}$$

and

$$\begin{aligned}\gamma_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= \gamma_A(f((y_1 \cdot (y_2 \cdot x)) \cdot x), q) \\ &= \gamma_A((f(y_1) \cdot (f(y_2) \cdot f(x))) \cdot f(x), q) \\ &\leq \max\{\gamma_A(f(y_1), q), \gamma_A(f(y_2), q)\} \\ &= \max\{\gamma_{f^{-1}(A)}(y_1, q), \gamma_{f^{-1}(A)}(y_2, q)\},\end{aligned}$$

which proves (2.4). Hence, $f^{-1}(A)$ is an intuitionistic Q-fuzzy ideal of X. □

3. Conclusions

This work presented a detailed study of intuitionistic Q-fuzzy subalgebras, ideals, and deductive systems in Hilbert algebras. We established key relationships among these structures, including that every intuitionistic Q-fuzzy ideal is both a subalgebra and a deductive system. In addition, we examined how these structures behave under homomorphisms by analyzing their inverse images. The results not only contribute to the theoretical development of fuzzy algebraic systems but also provide a foundation that is accessible to new researchers and adaptable for educational purposes.

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Some properties of the zero-set intersection graph of $C(X)$ and its line graph

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Abstract. Let $C(X)$ be the ring of all continuous real valued functions defined on a completely regular Hausdorff topological space X. The zero-set intersection graph $\Gamma(C(X))$ of $C(X)$ is a simple graph with vertex set all non units of $C(X)$ and two vertices are adjacent if the intersection of the zero sets of the functions is non empty. In this paper, we study the zero-set intersection graph of $C(X)$ and its line graph. We show that if X has more than two points, then these graphs are connected with diameter and radius 2. We show that the girth of the graph is 3 and the graphs are both triangulated and hypertriangulated. We find the domination number of these graphs and finally we prove that $C(X)$ is a von Neumann regular ring if and only if $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$. Finally, we derive some properties of the line graph of $\Gamma(C(X))$.

Keywords: zero-set intersection graph, diameter, girth, cycles, dominating sets, line graph.

MSC 2020: 54C30, 54C40, 05C69, 05C76.

1. Introduction

The characterization of various algebraic structures by means of graph theory is an interesting area of research for mathematicians in the recent years. It was started by Beck in [12] by defining the graph $\Gamma(R)$ of a commutative ring R with vertices as elements of R in which two different vertices a and b are adjacent if and only if $ab = 0$. The author studied finitely colorable rings by associating this graph structure and this study was further continued by Anderson and Naseer in [14]. Sharma and Bhatwadekar first defined the comaximal graph of a commutative ring in [16] and further investigation was continued in [5], [6], [7], [8], [9], [13], [15], [17], [18], [20], [21], [22].

Throughout this paper, let X be a Tychonoff space and $C(X)$ the ring of all real valued continuous functions on X. For $f \in C(X)$, the set $Z(f) = \{x \in X | f(x) = 0\}$ is called the zero set of f. Let $Coz(f) = X \backslash Z(f)$. The family of all zero sets is denoted by $Z(C(X))$. A space X is said to be a P-space if every zero set of $f \in C(X)$ is an open set. If I is an ideal in $C(X)$, let $Z[I] = \{Z(f) | f \in I\}$. An ideal I in $C(X)$ is said to be a z-ideal if $g \in C(X)$ and $Z(g) \in Z[I]$ imply that $g \in I$. A space X is said to be an almost P-space if every non empty zero set in X has a non empty interior. For $x \in X$, let $O^x = \{f \in C(X) | x \in int_{\beta X} Z(f)\}$. O^x is a prime z-ideal in $C(X)$.

In recent years, many mathematicians have associated graphs to the ring of functions on topological spaces and derived graph theoretic characterizations of both the ring and the associated topological space. Badie [4] in the year 2016 studied the comaximal graph of $C(X)$. In the year 2022, Badie [26] associated the annihilating graph to $C(X)$ and derived various characterizations of $C(X)$ and X in terms of this graph. Acharyya et al. [24] studied the zero divisor graph of the rings $C_{\mathcal{P}}(X)$ and $C_{\infty}^{\mathcal{P}}(X)$ in the year 2022. Here, $C_{\mathcal{P}}(X)$ is the family of all functions whose support belong to $\mathcal{P}$ and $C_{\infty}^{\mathcal{P}}(X) = \{f \in C(X)|$ for each $\epsilon > 0, \{x \in X || f(x)| \geq \epsilon\} \in \mathcal{P}\}$. Hejazipour et al. [25] studied the zero divisor graph of the rings of real measurable functions with measures in the year 2022. In the year 2024, Bharati et al. [27] studied the zero divisor graph and comaximal graph of the ring of continuous functions with countable range.

Let R be a commutative ring with unity. The ring R is called an almost regular ring if each non-unit element of R is a zero-divisor element of R. Also, we know that $C(X)$ is an almost regular ring if X is an almost P-space. An ideal I in $C(X)$ is said to be a pure ideal if for each $f \in I$ there exists $g \in I$ such that $f = fg$ [23].

A graph is a pair G = (V, E), where V is a set whose elements are called vertices, and E is a set of pair of vertices, whose elements are called edges. Two elements u and v in V are said to be adjacent if $\{u, v\} \in E$. A graph G is said to be complete if every pair of vertices can be joined by an edge and G is said to be connected if for any pair of vertices u, $v \in V$, there exists a path joining u and v. The distance between two vertices u and v denoted by $d(u, v)$ is the length of the shortest path between them. The diameter is defined as diam(G) = sup $\{d(u, v) : u, v \in V(G)\}$, where $d(u, v)$ is the length of a shortest path from u to v. The eccentricity of a vertex $u \in G$ is denoted by $e(u)$ and is defined as $\max\{d(u, v) : v \in G\}$. The $min\{e(u) : u \in G\}$ is called the radius of G and it is denoted by $Rad(G)$. The length of the shortest cycle in a graph G is called girth of the graph and is denoted by gr(G). In a graph G, a dominating set is a set of vertices A such that every vertex outside A is adjacent to at least one vertex in A. The minimum cardinality of a dominating set in a graph G is called the domination number and it is denoted by $dt(G)$. A graph is said to be triangulated(hypertriangulated) if every vertex of the graph is a vertex(edge) of a triangle. If two distinct vertices u and v in a graph G are adjacent and there is no vertex $w \in G$ such that w is adjacent to both u and v, then we say that u and v are orthogonal and is denoted by $u \perp v$. A graph G is called complemented if for each vertex $u \in G$, there is a vertex $v \in G$ such that $u \perp v$. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a chord of that cycle. A graph is said to be chordal if every cycle of length greater than three has a chord. In a graph G, the length of the smallest cycle containing two vertices x, y is denoted by $c(x, y)$ and if there is no cycle containing x, y then $c(x, y) = \infty$. The line graph of G, denoted by $L(G)$, is a graph whose vertices are the edges of G and two vertices of $L(G)$ are said to be adjacent wherever the corresponding edges in G share a common vertex [19]. For any undefined term in graph theory, we refer the reader to [10].

Bose et al. [11] in the year 2020 introduced a graph structure called the zero-set intersection graph $\Gamma(C(X))$ on the ring of real valued continuous functions $C(X)$ as a graph whose set of vertices consists of all non-units in the ring $C(X)$ and there is an edge between distinct vertices f and g in $C(X)$ if $Z(f) \cap Z(g) \neq \emptyset$. The authors showed that the graph is connected, studied the cliques and maximal cliques of $\Gamma(C(X))$ and the inter-relationship of cliques of $\Gamma(C(X))$ and ideals in $C(X)$. Further they showed that two graphs are isomorphic if and only if the corresponding rings are isomorphic if and only if the corresponding topological spaces are homeomorphic either for first countable topological spaces or for realcompact topological spaces.

The comaximal graph $\Gamma(R)$ is defined as a graph with set of vertices as R and two vertices a, b are adjacent if and only if $Ra + Rb = R$. Let $\Gamma_2(R)$ denote the subgraph of $\Gamma(R)$ whose vertex set consists of all non-unit elements of R. If $J(R)$ is the Jacobson radical of R, then the graph with vertex set $V(\Gamma_2(R)) \setminus J(R)$ is denoted by $\Gamma_2'(R)$. The comaximal ideal graph of $C(X)$, $\Gamma_2'(C(X))$, was studied in [2] and [4] where they derived the ring properties of $C(X)$ and topological properties of X via $\Gamma_2'(C(X))$. For any two vertices f and g in $\Gamma_2'(C(X))$, f is adjacent to g if and only if $Z(f) \cap Z(g) = \emptyset$ from lemma 1.2 [4]. The complement graph $\overline{\Gamma_2'(C(X))}$ coincides with the zero-set intersection graph of $C(X)$. In this paper, we study the zero-set intersection graph of $C(X)$. Vertex set of $\Gamma(C(X))$ is denoted by $V(\Gamma(C(X)))$ which consists of all the non-units in $C(X)$ and two vertices f and g are adjacent if $Z(f) \cap Z(g) \neq \emptyset$. If X is singleton, then $\Gamma(C(X))$ is empty. Thus we assume that $|X| > 1$.

In Section 2 of this paper, we show that $\Gamma(C(X))$ is connected and find the diameter and radius of the graph. In Section 3, we find the girth of the graph and show that the graph is always triangulated and hypertriangulated. We also find the conditions when the graph is chordal and show that $\Gamma(C(X))$ is never complemented. In Section 4, we find the dominating number of $\Gamma(C(X))$ and show that $C(X)$ is a von Neumann regular ring if and only if $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$. Finally in Section 5, we study the line graph $L(\Gamma(C(X)))$ of $\Gamma(C(X))$ and derive similar properties as in $\Gamma(C(X))$. For all notations and undefined terms concerning the ring $C(X)$, the reader may consult [1].

2. Diameter, radius of $\Gamma(C(X))$

We first note that $f \in V(\Gamma(C(X)))$ if and only if $Z(f) \neq \emptyset$. We study the condition when $\Gamma(C(X))$ is connected. We also calculate the diameter and radius of $\Gamma(C(X))$.

Theorem 2.1. *Let X be any topological space then $\Gamma(C(X))$ is connected with $diam(\Gamma(C(X))) = 2$ if and only if $|X| > 1$.*

Proof. If $|X| = 1$, then $C(X) \cong \mathbb{R}$ and so $\Gamma(C(X))$ is the null graph. Suppose $X = \{a, b\}$, then f is a vertex in $\Gamma(C(X))$ if $f(a) = 0$ and $f(b) \neq 0$ or $f(a) \neq 0$ and $f(b) = 0$ or $f = 0$. Thus $\Gamma(C(X))$ is the disjoint union of two complete subgraphs $A = \{f \in C(X) : f(a) = 0 \textit{ and } f(b) \neq 0\}$, $B = \{f \in C(X) : f(a) \neq 0 \textit{ and } f(b) = 0\}$ and $f = 0$ is adjacent to each vertex in A and also to each vertex in B. Thus $diam(\Gamma(C(X))) = 2$. We now assume that $|X| > 2$. Let $f,\, g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$. Let $a \in Z(f)$, $b \in Z(g)$ and $c \notin \{a, b\}$. By regularity of X, there exists an open set U such that $c \in U \subseteq \overline{U} \subseteq X \setminus \{a, b\}$. Since X is completely regular there exists $h_1, h_2 \in C(X)$ such that $h_1(\overline{U}) = 1, h_1(a) = 0, h_2(\overline{U}) = 1$ and $h_2(b) = 0$. Let $h = h_1h_2$, then $h \in V(\Gamma(C(X))) \setminus \{f, g\}$ and $f - h - g$ is a path of length 2 joining f and g. Thus $\Gamma(C(X))$ is connected with diameter 2. $\square$

Corollary 2.1. *If $|X| > 2$, then for any two distinct vertices $f,\, g \in V(\Gamma(C(X)))$ there exists a vertex $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f and g.*

Proof. If $Z(f) \cap Z(g) = \emptyset$, then by Theorem 2.1 there exists $h \in V(\Gamma(C(X)))$ such that $f - h - g$ is a path in $\Gamma(C(X))$. If $Z(f) \cap Z(g) \neq \emptyset$ then let $h = 2f,\ if\ g \neq 2f\ and\ h = 3f$ otherwise. Then $h \in V(\Gamma(C(X))) \setminus \{f, g\}$ and $f - h - g$ is a path in $\Gamma(C(X))$. $\square$

Corollary 2.2. *Let $f, g \in \Gamma(C(X))$. Then:*
(1) $d(f, g) = 1\ if\ and\ only\ if\ Z(f) \cap Z(g) \neq \emptyset$.
(2) $d(f, g) = 2\ if\ and\ only\ if\ Z(f) \cap Z(g) = \emptyset$.

Theorem 2.2. *For any space X with $|X| > 2$, $Rad(\Gamma(C(X))) = 2$.*

Proof. For any vertex $f \in \Gamma(C(X))$, it is evident that $1 \leq e(f) \leq 2$, so $1 \leq Rad(\Gamma(C(X))) \leq 2$. For any vertex $f \in \Gamma(C(X))$, it is clear that $1 \leq e(f) \leq 2$, and hence $1 \leq Rad(\Gamma(C(X))) \leq 2$. Let $f \in V(\Gamma(C(X)))$ and choose a point $x \in X \backslash Z(f)$, which is possible because $Z(f) \neq X$. Since X is completely regular, there exists an open set $x \in U \subseteq Cl_X(U) \subseteq X \backslash Z(f)$. Again, by complete regularity, there exists a function $g \in C(X)$ such that $g(x) = 0$, $g(Z(f)) = 1$. Then $Z(g) \subseteq U$ and $U \subseteq X \backslash Z(f)$, so $Z(f) \cap Z(g) = \emptyset$. By Corollary 2.1 we have $d(f, g) = 2$. Hence, $e(f) = 2$, for every $f \in V(\Gamma(C(X)))$, and therefore $Rad(\Gamma(C(X))) = 2$. $\square$

3. Cycles in $\Gamma(C(X))$

In this section we explore the existence of cycles in $\Gamma(C(X))$.

Theorem 3.1. *For any topological space X with $|X| > 1$, $\Gamma(C(X))$ is both triangulated and hypertriangulated.*

Proof. Let $f \in V(\Gamma(C(X)))$. Then $f - 2f - 3f - f$ is a cycle of length 3 in $\Gamma(C(X))$. Hence, $\Gamma(C(X))$ is triangulated. Suppose f and g are adjacent vertices in $\Gamma(C(X))$. Let $h = 2f$, if $g \neq 2f$, otherwise let $h = 3f$. Then $h \in V(\Gamma(C(X)))\backslash\{f, g\}$ and $f - g - h - f$ is a triangle in $\Gamma(C(X))$. Hence, $f - g$ is an edge in a triangle. Thus, $\Gamma(C(X))$ is hypertriangulated. $\square$

Corollary 3.1. *If $|X| > 1$, then $gr(\Gamma(C(X))) = 3$.*

Theorem 3.2. *Let f and g be two distinct vertices in $\Gamma(C(X))$. Then:*
(1) $c(f, g) = 3$ if and only if $Z(f) \cap Z(g) \neq \emptyset$.
(2) $c(f, g) = 4$ if and only if $Z(f) \cap Z(g) = \emptyset$.

Proof. (1) Let $c(f, g) = 3$. Then f and g are adjacent in $\Gamma(C(X))$. Thus $Z(f) \cap Z(g) \neq \emptyset$.

Conversely, if $Z(f) \cap Z(g) \neq \emptyset$ then f and g are adjacent in $\Gamma(C(X))$ and by Corollary 2.1 there exists a vertex $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f and g. Thus $f - g - h - f$ is a cycle containing f and g. Hence, $c(f, g) = 3$.

(2) Let $c(f, g) = 4$. Then $Z(f) \cap Z(g) = \emptyset$ by (1).

Conversely, suppose $Z(f) \cap Z(g) = \emptyset$. Then by (1) there is no cycle of length 3 containing f and g. By Corollary 2.1, there exists $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f and g. So, $f - h - g - 2h - f$ is a cycle of length 4 and it is the smallest cycle containing f and g. Hence, $c(f, g) = 4$. $\square$

Theorem 3.3. *$\Gamma(C(X))$ is chordal if and only if $|X| \leq 3$.*

Proof. Suppose $\Gamma(C(X))$ is chordal and let $|X| \geq 4$. Let $x_1, x_2, x_3, x_4 \in X$. Since X is a Tychonoff space, there exist mutually disjoint open sets U_1, U_2, U_3 *and* U_4 such that $x_i \in U_i$, where $i \in \{1, 2, 3, 4\}$. Let $h_i \in V(\Gamma(C(X)))$ such that $h_i(x_i) = 0$, and $h_i(X \setminus U_i) = 1$ for each $i \in \{1, 2, 3, 4\}$. Then $Z(h_i) \cap Z(h_j) = \emptyset$, whenever $i \neq j$. Consider the functions $f_1 = h_1h_4$, $f_2 = h_1h_2$, $f_3 = h_2h_3$ and $f_4 = h_3h_4$. Clearly, $f_i \in V(\Gamma(C(X)))$ $(i = 1, 2, 3, 4)$ and $f_1 - f_2 - f_3 - f_4 - f_1$ is a chordless cycle since $Z(h_1) \cap Z(h_3) = \emptyset$ and $Z(h_2) \cap Z(h_4) = \emptyset$ which is a contradiction. Hence, if $\Gamma(C(X))$ is chordal then $|X| \leq 3$.

Conversely, if $|X| \leq 3$ then the following two cases arise.
Case I: If $X = \{a, b\}$, then f is a vertex in $\Gamma(C(X))$ if $f(a) = 0$ and $f(b) \neq 0$, $f(a) \neq 0$ and $f(b) = 0$ or $f(a) = 0$ and $f(b) = 0$. Thus, $\Gamma(C(X))$ has two complete subgraphs $A = \{f \in C(X) : f(a) = 0 \text{ and } f(b) \neq 0\}$ and $B = \{f \in C(X) : f(a) \neq 0 \text{ and } f(b) = 0\}$ and $f = 0$ is adjacent to each vertex of A as

well as each vertex of B. Hence, if C is an induced cycle of length greater than 3 in $\Gamma(C(X))$, then it is contained in the complete subgraph induced by A or B and so C has a chord. Therefore, $\Gamma(C(X))$ is chordal.
Case II: Let $X = \{a, b, c\}$ and C be an induced cycle in $\Gamma(C(X))$ of length greater than 3. Consider a path $f_1 - f_2 - f_3 - f_4$ in C such that f_1, f_2, f_3 and f_4 are distinct. If $Z(f_1) \cap Z(f_3) \neq \emptyset$ or $Z(f_2) \cap Z(f_4) \neq \emptyset$, then we have a chord joining f_1 and f_3 or a chord joining f_2 and f_4. So, we assume that $Z(f_1) \cap Z(f_3) = \emptyset$ and $Z(f_2) \cap Z(f_4) = \emptyset$. Then, we must have $|Z(f_2)| = |Z(f_3)| = 2$. If f_1 and f_4 are adjacent then, there is a chord joining f_2 and f_4. So, assume that f_1 and f_4 are not adjacent then, there is a vertex f_5 which is adjacent to f_4. If $|Z(f_5)| = 1$, then f_5 is adjacent to f_3 and if $|Z(f_5)| = 2$, then either f_5 is adjacent to f_2 and f_3 or f_5 is adjacent to f_1 and f_2. Thus in each case C has a chord and hence, $\Gamma(C(X))$ is chordal. $\square$

Example 3.1. When $X = \{a, b\}$, we classify the non-zero non-unit functions $f \in C(X)$ into two disjoint classes of functions $[f_1]$ and $[f_2]$ and let the zero function be denoted by f_3. Let the class $[f_1]$ represent the functions with $Z(f_1) = \{a\}$, the class $[f_2]$ represent the functions having $Z(f_2) = \{b\}$. We note that the classes $[f_1]$ and $[f_2]$ are complete graphs and each function in these classes are adjacent to f_3. The graph $\Gamma(C(X))$ may be depicted as shown in Figure 1.

Figure 1: Zero-set intersection graph of $C(X)$ for $|X| = 2$.

Example 3.2. When $|X| = 3$, let $X = \{a, b, c\}$. The non unit functions $f \in C(X)$ can be classified into six classes $[f_i]$ with $1 \leq i \leq 6$ such that $Z(f_1) = \{a\}$, $Z(f_2) = \{b\}$, $Z(f_3) = \{c\}$, $Z(f_4) = \{a, b\}$, $Z(f_5) = \{b, c\}$, $Z(f_6) = \{a, c\}$ and the function $f_7 = 0$. Each function in a class is adjacent to f_7 and if the classes $[f_i]$, $[f_j]$ are adjacent then they form complete bipartite subgraph of the graph. The graph $\Gamma(C(X))$ is shown in Figure 2.

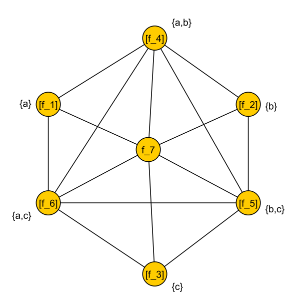

Figure 2: Zero-set intersection graph of $C(X)$ for $|X| = 3$.

Remark 3.1. We notice that the graphs in Figure 1 and Figure 2 are chordal graphs.

Theorem 3.4. *For* $|X| > 1$, $\Gamma(C(X))$ *is not complemented.*

Proof. Suppose $\Gamma(C(X))$ is complemented. Then, for every $f \in \overline{\Gamma_2'(C(X))}$ there exist g such that $f \perp g$, thus $c(f,g) = 4$. By Theorem 3.2, $Z(f) \cap Z(g) = \emptyset$, which is a contradiction. □

4. Dominating sets in $\Gamma(C(X))$

Theorem 4.1. *If* $|X| > 1$, *then* $dt(\Gamma(C(X))) = 2$.

Proof. It is evident that $dt(\Gamma(C(X))) \neq 1$. Let $f \in V(\Gamma(C(X)))$ and $a \in Int_X Z(f)$, then $f \in O^a$. Since O^a is a pure ideal, there exists $g \in O^a \subseteq Z(C(X))$ such that $f = fg$, that is $g = 1$ on $Supp(f)$ [23]. Hence, $g^2 \in V(\Gamma(C(X)))$ and $\{g, g^2\}$ dominates $V(\Gamma(C(X)))$. Therefore, $dt(\Gamma(C(X))) = 2$. □

Theorem 4.2. *If* $|X| > 1$, *then for any* $f, g \in V(\Gamma(C(X)))$, $\{f, g\}$ *dominates* $\Gamma(C(X))$ *if and only if* $Z(f) \cup Z(g) = X$.

Proof. Suppose $\{f, g\}$ dominates $\Gamma(C(X))$ and assume that $y \in X \setminus (Z(f) \cup Z(g))$. Let $h \in C(X)$ such that $h(y) = 0$ and $h(Z(f) \cup Z(g)) = 1$. Then, $h \in V(\Gamma(C(X)))$, $h \notin \{f, g\}$, $Z(f) \cap Z(h) = \emptyset$ and $Z(g) \cap Z(h) = \emptyset$. This is a contradiction since $\{f, g\}$ dominates $\Gamma(C(X))$.

Conversely, suppose $Z(f) \cup Z(g) = X$. Let $h \in C(X)$ such that $Z(f) \cap Z(h) = \emptyset$ and $Z(g) \cap Z(h) = \emptyset$. Then, $Z(h) \cap (Z(f) \cup Z(g)) = \emptyset$ which implies that $Z(h) = \emptyset$, i.e., $h \notin V(\Gamma(C(X)))$. Thus, $\{f, g\}$ dominates $\Gamma(C(X))$. □

Theorem 4.3. $C(X)$ *is a von Neumann regular ring if and only if* $C(X)$ *is an almost regular ring and for all* $f \in V(\Gamma(C(X)))$ *there exists* $g \in V(\Gamma(C(X)))$ *such that* $Z(f) \cap Z(g) = \emptyset$ *and* $\{f, g\}$ *dominates* $\Gamma(C(X))$.

Proof. Suppose $C(X)$ is a von Neumann regular ring. Then, it is clear that $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$, $Z(f)$ is open. Consider the function $g \in C(X)$ such that

$$g(x) = \begin{cases} 1, & x \in Z(f) \\ 0, & x \notin Z(f). \end{cases}$$

Then, $g \in V(\Gamma(C(X)))$ and $Z(f) \cup Z(g) = Z(f) \cup (X \setminus Z(f)) = X$. Hence, by Theorem 4.2 $\{f, g\}$ dominates $\Gamma(C(X))$.

Conversely, suppose $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$. Since X is an almost P-space, every non-unit element in $C(X)$ has a zero-set with non-empty interior. That is every element in $C(X)$ is either a unit or a zero-divisor. If $f \in C(X)$ is a unit then, $f = f^2 f^{-1}$. Suppose now

$f \in V(\Gamma(C(X)))$. By hypothesis $Z(f) \cap Z(g) = \emptyset$, so $Coz(f) \cup Coz(g) = X$ which implies that $Z(f)$ is open since $Z(f) = Coz(g)$. Therefore, $C(X)$ is a von Neumann regular ring. □

Example 4.1. Consider $X = \{a, b, c\}$ equipped with discrete topology. Then, every non-empty zero set $Z(f)$ is open. Hence, X is an almost P-space. Also, $C(X)$ is a von Neumann regular ring. Consider the functions

$$f_1(x) = \begin{cases} 0, & \text{when } x = a \\ 1, & \text{when } x = b \text{ or } x = c, \end{cases}, f_2(x) = \begin{cases} 1, & \text{when } x = a \text{ or } x = c \\ 0, & \text{when } x = b, \end{cases},$$
$$f_3(x) = \begin{cases} 1, & \text{when } x = a \text{ or } x = b \\ 0, & \text{when } x = c. \end{cases}$$

Let $g = f_2 f_3$. Then, $g \in V(\Gamma(C(X)))$ such that $Z(f_1) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$.

The following theorem is a direct consequence of Theorem 5.3 [4] and Theorem 4.2.

Theorem 4.4. *The graph $\Gamma(C(X))$ is complemented if and only if for every $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$.*

5. Line graph of $\Gamma(C(X))$

In this section, we study the properties of the line graph $L(\Gamma(C(X)))$ of $\Gamma(C(X))$. The study of the line graph of $\Gamma(C(X))$ is interesting as the results on some of the properties of $\Gamma(C(X))$ and $L(\Gamma(C(X)))$ differ from each other.

Suppose $f, g \in V(\Gamma(C(X)))$, then $[f, g]$ is a vertex in $L(\Gamma(C(X)))$ if $Z(f) \cap Z(g) \neq \emptyset$. In $L(\Gamma(C(X)))$, $[f, g] = [g, f]$ as $\Gamma(C(X))$ is an undirected graph and for two distinct vertices $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\Gamma(C(X)))$, $[f_1, f_2]$ is adjacent to $[g_1, g_2]$ if $f_i = g_j$, for some $i, j \in \{1, 2\}$.

We first investigate when is $L(\Gamma(C(X)))$ connected and then compute diameter and radius of $L(\Gamma(C(X)))$.

Lemma 5.1. *Let $[f_1, f_2]$ and $[g_1, g_2]$ be distinct vertices in $L(\Gamma(C(X)))$. Then, there is a vertex $[h_1, h_2]$ which is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\Gamma(C(X)))$ if and only if $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i, j \in \{1, 2\}$.*

Proof. Suppose there exists a vertex $[h_1, h_2]$ which is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\Gamma(C(X)))$. If $f_i = g_j$ for some $i, j \in \{1, 2\}$, then $Z(f_i) \cap Z(g_j) = Z(f_i) \neq \emptyset$. So, assume that $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. Consider a path $[f_1, f_2] - [h_1, h_2] - [g_1, g_2]$ in $L(\Gamma(C(X)))$, then $h_1 = f_i$ for some $i \in \{1, 2\}$ and $h_2 = g_j$ for some $j \in \{1, 2\}$. Thus $Z(f_i) \cap Z(g_j) = Z(h_1) \cap Z(h_2) \neq \emptyset$.

Conversely, suppose $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i, j \in \{1, 2\}$. Without loss of generality let $f_1 \neq g_1$, then $[f_1, g_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in

$L(\Gamma(C(X)))$. If $f_1 = g_1$ then there exists $r \in \mathbb{R} \setminus \{0, 1\}$ such that $g_2 \neq rg_1$ and $f_2 \neq rf_1$. Thus $[f_1, rf_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$. □

Theorem 5.1. *Let* $|X|{>}2$ *and* $[f_1, f_2]$, $[g_1, g_2]$ *be distinct vertices in* $L(\Gamma(C(X)))$. *Then:*
(1) $d([f_1, f_2], [g_1, g_2]) = 1$ *if and only if* $f_i = g_j$ *for some* $i, j \in \{1, 2\}$.
(2) $d([f_1, f_2], [g_1, g_2]) = 2$ *if and only if* $f_i \neq g_j$ *for all* $i, j \in \{1, 2\}$ *and* $Z(f_i) \cap Z(g_j) \neq \emptyset$ *for some* $i, j \in \{1, 2\}$.
(3) $d([f_1, f_2], [g_1, g_2]) = 3$ *if and only if* $f_i \neq g_j$ *for all* $i, j \in \{1, 2\}$ *and* $Z(f_i) \cap Z(g_j) = \emptyset$ *for all* $i, j \in \{1, 2\}$.

Proof. (1) Clearly holds.
(2) Suppose $d([f_1, f_2], [g_1, g_2]) = 2$. Then, $[f_1, f_2] - [f_i, g_j] - [g_1, g_2]$ is a path of length 2 for any $i, j \in \{1, 2\}$ implies that $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i, j \in \{1, 2\}$.
The converse follows clearly.
(3) From (1) and (2) we have $d([f_1, f_2], [g_1, g_2]) > 2$. By Corollary 2.1, there exists $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f_1 and g_1 in $\Gamma(C(X))$. Clearly, we can choose h such that $h \notin \{f_2, g_2\}$. Thus $[f_1, f_2] - [f_1, h] - [h, g_1] - [g_1, g_2]$ is a path in $L(\Gamma(C(X)))$ and hence $d([f_1, f_2], [g_1, g_2]) = 3$ □

The following corollary is immediate from Theorem 2.1 and Theorem 5.1.

Corollary 5.1. *If* $|X| > 2$, *then* $L(\Gamma(C(X)))$ *is a connected graph with*

$$diam(L(\Gamma(C(X))) \leq 3.$$

Theorem 5.2. *Let* $|X| > 2$ *and* $[f_1, f_2]$ *be a vertex in* $L(\Gamma(C(X)))$. *Then*

$$e([f_1, f_2]) = \begin{cases} 2, & if\ Z(f_1) \cup Z(f_2) = X \\ 3, & otherwise. \end{cases}$$

Proof. It is clear that $1 \leq e([f_1, f_2]) \leq 3$. Let $[g_1, g_2]$ be a vertex in $L(\Gamma(C(X)))$ such that $[g_1, g_2]$ is not adjacent to $[f_1, f_2]$. Then, for all $i, j \in \{1, 2\}$, $g_i \neq f_j$. But $\emptyset \neq Z(g_1) = Z(g_1) \cap X = Z(g_1) \cap (Z(f_1) \cup Z(f_2))$. Suppose, $Z(f_1) \cap Z(g_1) \neq \emptyset$ then, $[f_1, f_2] - [f_1, g_1] - [g_1, g_2]$ is a path in $L(\Gamma(C(X)))$ and so $d([f_1, f_2], [g_1, g_2]) = 2$. Hence, $e([f_1, f_2]) = 2$. Now, suppose $y \in X \setminus Z(f_1) \cup Z(f_2)$ and V be an open set in X such that $y \in V \subseteq Cl_X V \subseteq X \setminus Z(f_1) \cup Z(f_2)$. Consider $g_1, g_2 \in C(X)$, $g_1 \neq g_2$ such that $g_i(y) = 0$ and $g_i(Z(f_1) \cup Z(f_2)) = 1$ for $i \in \{1, 2\}$. Then, $f_i \neq g_j$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$. So, by Theorem 2.1, $d([f_1, f_2], [g_1, g_2]) = 3$ and hence $e([f_1, f_2]) = 3$. □

An immediate conclusion from Corollary 2.1 and Theorem 2.2 is the following corollary.

Corollary 5.2. *If* $|X| > 2$, *then* $2 \leq Rad(L(\Gamma(C(X)))) \leq 3$.

We now find the girth of $L(\Gamma(C(X)))$ and show that $L(\Gamma(C(X)))$ is always triangulated and hypertriangulated. We also show that $L(\Gamma(C(X)))$ is never chordal.

Theorem 5.3. *If $|X| > 1$, then $gr(L(\Gamma(C(X)))) = 3$.*

Proof. Let $[f, g]$ be a vertex in $L(\Gamma(C(X)))$, then $[f, g] - [g, 2f] - [2f, f] - [f, g]$ is a cycle of length 3. Hence, $gr(L(\Gamma(C(X)))) = 3$. □

Theorem 5.4. *For any space X with $|X| > 1$, $L(\Gamma(C(X)))$ is both triangulated and hypertriangulated .*

Proof. By Theorem 5.3, it follows that $L(\Gamma(C(X)))$ is triangulated. Let $[f_1, f_2] - [f_1, g]$ be an edge in $L(\Gamma(C(X)))$. Then, $f_1 \neq 2f_1$ and $f_2 \neq 2f_1$ and $[f_1, f_2] - [f_1, g] - [f_1, 2f_1] - [f_1, f_2]$ is a cycle in $L(\Gamma(C(X)))$. Hence, $L(\Gamma(C(X)))$ is hypertriangulated. □

Theorem 5.5. *If $[f_1, f_2]$ and $[g_1, g_2]$ are distinct vertices in $L(\Gamma(C(X)))$. Then:*
(1) $c([f_1, f_2], [g_1, g_2]) = 3$ if and only if $f_i = g_j$ for some $i, j \in \{1, 2\}$.
(2) $c([f_1, f_2], [g_1, g_2]) = 4$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i \in \{1, 2\}$ and $for\ all$ $j \in \{1, 2\}$ or ($Z(f_1) \cap Z(g_i) \neq \emptyset$ and $Z(f_2) \cap Z(g_j) \neq \emptyset$, where $i, j \in \{1, 2\}$).
(3) $c([f_1, f_2], [g_1, g_2]) = 5$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$ and only one $j \in \{1, 2\}$, $Z(f_i) \cap Z(g_j) \neq \emptyset$.
(4) $c([f_1, f_2], [g_1, g_2]) = 6$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$.

Proof. (1) If $c([f_1, f_2], [g_1, g_2]) = 3$, then it is clear that $f_i = g_j$ for some $i, j \in \{1, 2\}$.
Conversely, suppose $f_1 = g_1$. Then, there exists $r \in \mathbb{R} \setminus \{0\}$ such that $rg_1 \notin \{f_1, f_2, g_1\}$ and $[f_1, f_2] - [f_1, g_1] - [f_1, rg_1] - [f_1, f_2]$ is a cycle of length 3 in $L(\Gamma(C(X)))$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Hence, $c([f_1, f_2], [g_1, g_2]) = 3$.

(2) Suppose $c([f_1, f_2], [g_1, g_2]) = 4$ then by (1), $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. So, we have the cycle $[f_1, f_2] - [a, b] - [g_1, g_2] - [c', d] - [f_1, f_2]$, where $a, c' \in \{f_1, f_2\}$ and $b, d \in \{g_1, g_2\}$.
Case I. If $a = f_1 = c'$, $b = g_1$ and $d = g_2$ then $Z(f_1) \cap Z(g_1) \neq \emptyset$ and $Z(f_1) \cap Z(g_2) \neq \emptyset$.
Case II. If $a = f_1$, $c' = f_2$, $b = g_1$ and $d = g_2$, then $Z(f_1) \cap Z(g_1) \neq \emptyset$ and $Z(f_2) \cap Z(g_2) \neq \emptyset$.
Case III. If $a = f_1 = c'$ and $b = g_2 = d$, then $[a, b] = [c, d]$, which is a contradiction.
Conversely, if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ then there is no triangle containing $[f_1, f_2]$ and $[g_1, g_2]$. Suppose $Z(f_2) \cap Z(g_j) \neq \emptyset$ for all $j \in \{1, 2\}$ then $[f_1, f_2] - [f_2, g_2] - [g_1, g_2] - [g_1, f_2] - [f_1, f_2]$ is a cycle of length 4 in $L(\Gamma(C(X)))$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Suppose $Z(f_1) \cap Z(g_2) \neq \emptyset$ and $Z(f_2) \cap Z(g_2) \neq \emptyset$

then we get a cycle $[f_1, f_2] - [f_1, g_2] - [g_1, g_2] - [g_2, f_2] - [f_1, f_2]$ of length 4 in $L(\Gamma(C(X)))$. Hence, $c([f_1, f_2], [g_1, g_2]) = 4$.

(3) Suppose $c([f_1, f_2], [g_1, g_2]) = 5$. Then, by (1), $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. Suppose $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$ then, $[f_1, f_2] - [k_1, l_1] - [g_1, g_2] - [k_2, l_2] - [k_3, l_3] - [f_1, f_2]$ is a cycle of length 5 in $L(\Gamma(C(X)))$, where $k_1 \in \{f_1, f_2\}$ and $l_1 \in \{g_1, g_2\}$. But $Z(k_1) \cap Z(l_1) \neq \emptyset$ which is a contradiction to our assumption. Similarly, if we consider the cycle $[f_1, f_2] - [k_1, l_1] - [k_2, l_2] - [g_1, g_2] - [k_3, l_3] - [f_1, f_2]$, we get a contradiction. Hence, for only one $i \in \{1, 2\}$ and only one $j \in \{1, 2\}$, $Z(f_i) \cap Z(g_j) \neq \emptyset$.
Conversely, suppose $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$ and only one $j \in \{1, 2\}$, $Z(f_i) \cap Z(g_j) \neq \emptyset$. Let $Z(f_2) \cap Z(g_2) \neq \emptyset$. Then, by (1) and (2) it follows that there is no cycle of length 3 or 4 containing $[f_1, f_2]$ and $[g_1, g_2]$. Now there exists $r \in \mathbb{R} \setminus \{0\}$ such that $rg_2 \notin \{f_1, g_1, g_2\}$ and so $[f_1, f_2] - [f_2, g_2] - [g_1, g_2] - [g_2, rg_2] - [rg_2, f_2] - [f_1, f_2]$ is a cycle of length 5 in $L(\Gamma(C(X)))$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Hence, $c([f_1, f_2], [g_1, g_2]) = 5$.

(4) Suppose $c([f_1, f_2], [g_1, g_2]) = 6$. Then, by (1), (2) and (3), $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$.
Conversely, suppose $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$, then by (1), (2) and (3), $c([f_1, f_2], [g_1, g_2]) > 5$. By Corollary 2.1, there exists a vertex $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f_1 and g_1 in $\Gamma(C(X))$. Consider, $r \in \mathbb{R} \setminus \{0\}$ then we get a cycle $[f_1, f_2] - [f_1, h] - [h, g_1] - [g_1, g_2] - [g_1, rh] - [rh, f_1] - [f_1, f_2]$ of length 6 containing $[f_1, f_2]$ and $[g_1, g_2]$. Therefore, $c([f_1, f_2], [g_1, g_2]) = 6$. $\square$

Theorem 5.6. *The graph $L(\Gamma(C(X)))$ is never chordal.*

Proof. Let $f \in V(\Gamma(C(X)))$. Then, $[f, 3f] - [3f, 5f] - [5f, 7f] - [7f, f] - [f, 3f]$ is a chordless cycle of length 4 in $L(\Gamma(C(X)))$. Hence, $L(\Gamma(C(X)))$ is never chordal. $\square$

Example 5.1. Consider the graph in Figure 2. In the line graph of this graph, the cycle $[f, f_1] - [f, f_2] - [f_2, f_4] - [f_4, f_1] - [f_1, f]$ is not chordal.

6. Conclusion

Bose et al. [11] studied cliques and maximal cliques of $\Gamma(C(X))$ and graph isomorphism of the graph $\Gamma(C(X))$. In this paper, we determined diameter, girth, radius, domination number of $\Gamma(C(X))$. We connected the von Neumann regularity property of the ring $C(X)$ with the domination property of $\Gamma(C(X))$. We have studied the line graph $L(\Gamma(C(X)))$ of $\Gamma(C(X))$. The independence number of the graph $\Gamma(C(X))$, the vertex cover, matching and many other properties of the graph are yet to be determined. The problem of finding the genus of the graph $\Gamma(C(X))$ is still an open problem.

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On algebraically closed Krasner hyperfields

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Abstract. In this short paper, we prove two negative results: the first order theory of algebraically closed Krasner hyperfields is neither complete nor substructure complete, the latter meaning that the theory does not admit quantifier elimination.

Keywords: Krasner hyperfield, model theory, quantifier elimination, multioperation.

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1. Introduction

It is a classical result by A. Robinson ([19]) that the first order theory ACF_p of algebraically closed fields of fixed characteristic p (prime or 0) is complete. In addition, it is known that the theory ACF of algebraically closed fields of unspecified characteristic admits quantifier elimination.

Krasner hyperfields may be quickly described as field-like structures, where the additive operation is multivalued, that is, the result of an addition in a hyperfield F is allowed to be an arbitrary subset of F. Hyperfields have been first described in [8] and subsequently attract the interest of several mathematicians [16, 15, 5, 6, 1, 11, 3, 2, 10, 7, 13] because of a variety of reasons.

It is tempting to define and consider algebraically closed Krasner hyperfields (this has been done for instance in [4]) and to analyse if model theoretic results analogous to the case of algebraically closed fields hold for the theories of algebraically closed Krasner hyperfields (of fixed or unspecified characteristic, respectively).

By considering some simple but fundamental examples we prove in this paper that the first order theory (over a natural first order language) of algebraically closed Krasner hyperfields of fixed characteristic 0 is not complete. We prove in addition that the first order theory of algebraically closed Krasner hyperfields of unspecified characteristic does not admit quantifier elimination.

We also compare the notion of algebraically closed hyperfields with the notion of algebraically closed structures over the language we have selected. In the case of fields these two notions coincide, while we note that in the case of hyperfields they do not.

2. Hyperfields

Definition 2.1. *A* multioperation *on a set F is a map $\boxplus : F \times F \to \mathcal{P}(F)$, where $\mathcal{P}(F)$ denotes the power set of F.*

If $X, Y \subseteq F$, then we set

$$X \boxplus Y := \bigcup_{(x,y)\in X\times Y} x \boxplus y.$$

If $X = \{x\}$ (resp. $Y = \{y\}$), then we write $x \boxplus Y$ (resp. $X \boxplus y$) in place of $X \boxplus Y$.

Definition 2.2. *A* (Krasner) hyperfield *is a tuple $(F, \boxplus, \cdot, 0, 1)$, where $\boxplus$ is a multioperation on F and $\cdot : F \times F \to F$ is an operation on F while* 0 *and* 1 *are elements of F, subject to the following conditions:*

(KH1) The multioperation $\boxplus$ is associative on F, i.e., $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$ (as subsets of F), for all $x, y, z \in F$.

(KH2) The multioperation $\boxplus$ is commutative on F, i.e., $x \boxplus y = y \boxplus x$ (as subsets of F), for all $x, y \in F$.

(KH3) For all $x \in F$, there exists a unique $x^- \in F$ such that $0 \in x \boxplus x^-$.

(KH4) $(F \setminus \{0\}, \cdot, 1)$ is an abelian group and $0 \cdot x = x \cdot 0 = 0$, for all $x \in F$.

(KH5) The operation $\cdot$ is distributive over $\boxplus$, i.e.,

$$xy \boxplus xz = x(y \boxplus z) := \{xt \mid t \in y \boxplus z\}$$

(as subsets of F).

Remark 2.1. The expert reader may have noticed that the *reversibility* axiom is absent in our definition, while it explicitly appears in the original definition of hyperfields given by Krasner (cf. [8] or [9]):

- The multioperation $\boxplus$ is reversible on F, i.e., $z \in x \boxplus y$ implies $x \in z \boxplus y^-$, for all $x, y, z \in F$.

Let us briefly explain the reason for this absence. By subsequent applications of the axiom one notices that the multioperation $\boxplus$ of a hyperfield F is reversible if and only if

$$z^- \in x^- \boxplus y^-, \text{ for all } x, y, z \in F. \tag{1}$$

On the other hand, by axioms (KH4) and (KH5) we obtain, for all $x \in F$ that

$$0 \in x \cdot (1 \boxplus 1^-) = x \boxplus x \cdot 1^-.$$

By the uniqueness postulated in axiom (KH3), it follows that $x \cdot 1^- = x^-$, for all $x \in F$. Therefore, (1) and hence the reversibility axiom follow logically from the other axioms.[1]

1. The author is in debt with Ch. Massouros for presenting this observation.

Remark 2.2. By axiom (KH1), summations of finite length are well-defined in any hyperfield, with no need to specify where the parentheses lie.

Remark 2.3. Note that for a hyperfield $(F, \boxplus, \cdot, 0, 1)$ the multioperation $\boxplus$ only has non-empty values. Moreover, it follows from the uniqueness postulated in axiom (KH3) and Remark 2.1 that for all $x \in F$ one has $x \boxplus 0 = \{x\}$.

Definition 2.3. *A hyperfield F has* characteristic $n \in \mathbb{N}$ *if*

$$0 \in \underbrace{1 \boxplus \ldots \boxplus 1}_{n \text{ times}}$$

and n is the minimal such positive integer. If no such n exists, then F is said to have characteristic 0.

Example 2.1 (Sign hyperfield)**.** Consider the set $\mathbb{S} = \{1^-, 0, 1\}$ endowed with the multioperation $\boxplus$ defined by $1 \boxplus 0 = 1 \boxplus 1 := \{1\}$, $1^- \boxplus 0 = 1^- \boxplus 1^- := \{1^-\}$ and $1 \boxplus 1^- := \mathbb{S}$. With the obvious multiplication $(\mathbb{S}, \boxplus, \cdot, 0, 1)$ is a hyperfield. This hypefield has characteristic 0.

Example 2.2 (Phase hyperfield)**.** Consider the set of complex numbers $\mathbb{P} := S^1 \cup \{0\}$, where S^1 denotes the circle of units. Endow $\mathbb{P}$ with a commutative multioperation defined by $z \boxplus 0 := \{z\}$ and for $0 \le \varphi_1 \le \varphi_2 < 2\pi$

$$e^{i\varphi_1} \boxplus e^{i\varphi_2} := \begin{cases} \{e^{i\varphi_1}\}, & \text{if } \varphi_1 = \varphi_2 \\ \{e^{i\varphi} \mid \varphi_1 < \varphi < \varphi_2\}, & \text{if } \varphi_1 < \varphi_2 \text{ and } \varphi_2 - \varphi_1 < \pi \\ \{e^{i\varphi} \mid \varphi_2 < \varphi < \varphi_1 + 2\pi\}, & \text{if } \varphi_1 < \varphi_2 \text{ and } \varphi_2 - \varphi_1 > \pi \\ \{e^{i\varphi_1}, 0, e^{i\varphi_2}\}, & \text{if } \varphi_2 - \varphi_1 = \pi \end{cases}$$

With the multiplication of complex numbers $(\mathbb{P}, \boxplus, \cdot, 0, 1)$ is a hyperfield (see e.g. [17, p. 6] and also [16]). This hyperfield is nothing but the quotient hyperfield $\mathbb{C}/\mathbb{R}^+$ (cf. [9]), where $\mathbb{R}^+$ denotes the group of positive real numbers. The phase hyperfield has characteristic 0.

Definition 2.4 ([16])**.** *A hyperfield F is* closed *if $x, y \in x \boxplus y$ for all $x, y \in F$.*

Lemma 2.1 ([16, Construction II])**.** *Let $(F, \boxplus, \cdot, 0, 1)$ be a hyperfield. If we define on F a new multioperation $\dot{\boxplus}$ as follows: $x \dot{\boxplus} 0 = 0 \dot{\boxplus} x = \{x\}$ for all $x \in F$ and, for $x, y \neq 0$,*

$$x \dot{\boxplus} y := \begin{cases} x \boxplus y \cup \{x, y\}, & \text{if } y \neq x^- \\ F, & \text{if } y = x^-, \end{cases}$$

then $(F, \dot{\boxplus}, \cdot, 0, 1)$ is a hyperfield.

Definition 2.5. *We call the hyperfield $\dot{F} := (F, \dot{\boxplus}, \cdot, 0, 1)$ constructed as in the previous lemma the* closure *of the given hyperfield $(F, \boxplus, \cdot, 0, 1)$.*

We leave to the reader the straightforward verification of the following lemma.

Lemma 2.2. *The closure $\dot{\mathbb{P}}$ of the phase hyperfield has characteristic* 0.

Next, we define algebraically closed hyperfields by considering polynomials over hyperfields. Polynomials over hyperfields have been considered also e.g. in [2] from where we took the terminology analogous to that standardly adopted for fields.

Definition 2.6. *A hyperfield F is called* algebraically closed *if every non-constant polynomial with coefficients in F has a root in F, i.e., for all $a_0, \dots, a_n \in F$ with $n > 0$, there exists $z \in F$ such that*

$$0 \in a_0 \boxplus a_1 z \dots \boxplus a_n z^n.$$

Lemma 2.3. *Let $(F, \boxplus, \cdot, 0, 1)$ denote the phase hyperfield $\mathbb{P}$ or its closure $\dot{\mathbb{P}}$ and take $a, b, c \in F$. Then the following inclusion*

$$a \boxplus b \subseteq a \boxplus b \boxplus c$$

holds.

Proof. By definition, we have that

$$a \boxplus b \boxplus c = \bigcup_{d \in a \boxplus b} d \boxplus c.$$

There are two possibilities:

1. If $c \in a \boxplus b$, then the right hand side is equal to the arc $a \boxplus b$.
2. If $c \notin a \boxplus b$, then the right hand side is an arc connecting c and either a or b, which by the assumption contains the shortest arc connecting a and b, i.e., $a \boxplus b$. □

Theorem 2.1. *Let $(F, \boxplus, \cdot, 0, 1)$ denote the phase hyperfield $\mathbb{P}$ or its closure $\dot{\mathbb{P}}$ and take a non-constant polynomial $a_0 \boxplus a_1 z \boxplus \dots \boxplus a_n z^n$ such that $a_0 \neq 0$. Consider the minimal $m > 0$ such that $a_m \neq 0$, then*

$$a_0 \boxplus a_m z^m \subseteq a_0 \boxplus a_1 z \boxplus \dots \boxplus a_n z^n.$$

In particular, F is algebraically closed.

Proof. The inclusion is straightforward to verify by induction on n, using Lemma 2.3. The last assertion follows since the multiplicative group of F is divisible. □

3. Model theory of hyperfields

We consider hyperfields as structures over the language $\mathcal{L} = \{\boxplus, (\_)^{-}, \cdot, 0, 1\}$, where $\boxplus$ is a ternary relation symbol, which is interpreted as $z \in x \boxplus y$, $(\_)^{-}$ is a unary function symbol, $\cdot$ is a binary function symbol and $0, 1$ are constant symbols. This language is a natural choice and is the one employed in [12] in relation to the model theory of valued fields.

The following proposition is quickly verified by writing down the (countably many!) existential sentences needed.

Proposition 3.1. *Over the language $\mathcal{L}$ the theory of algebraically closed hyperfields and the theory of algebraically closed hyperfields of fixed characteristic are first order theories.*

Over the language $\mathcal{L}$, a *substructure* of a hyperfield $(F, \boxplus, \cdot, 0, 1)$ is formed by a tuple $(F', \boxplus', \cdot, 0, 1)$, where $F' \subseteq F$ is multiplicative closed, contains 0 and 1, and

$$x \boxplus' y := (x \boxplus y) \cap F' \quad (x, y \in F').$$

Definition 3.1. *Two hyperfields F_1, F_2 are* elementarily equivalent *(we write $F_1 \equiv F_2$) if F_1 and F_2 satisfy the same first order sentences over the language $\mathcal{L}$.*

If F is a common substructure of F_1 and F_2, then we say that F_1 and F_2 are elementarily equivalent over F *(and write $F_1 \equiv_F F_2$) if F_1 and F_2 satisfy the same first order sentences, with parameters from F, over the language $\mathcal{L}$.*

Definition 3.2. *A first order theory is* substructure complete *if every two models with a common substructure are elementarily equivalent over that substructure.*

It is well-known that a first order theory is *complete* if and only if any two of its models are elementarily equivalent. For more details we refer the reader to [18]. Perhaps the following fact is less known.

Fact 3.1 ([12, Theorem 1.3.1])**.** *A first order theory admits quantifier elimination if and only if it is substructure complete.*

4. Main results

Theorem 4.1. *The theory of algebraically closed hyperfields with characteristic* 0 *is not complete.*

Proof. In fact, both the phase hyperfield $\mathbb{P}$ and its closure $\dot{\mathbb{P}}$ are models of the theory of algebraically closed hyperfields by Theorem 2.1. We have already verified that they both have characteristic 0. On the other hand, the first order sentence $\forall x \forall y (x \in x \boxplus y)$ does not hold in $\mathbb{P}$ while it does hold in $\dot{\mathbb{P}}$. $\square$

Theorem 4.2. *The theory of algebraically closed hyperfields is not substructure complete.*

Proof. It is a quick verification that the sign hyperfield $\mathbb{S}$ is a common substructure of both $\mathbb{P}$ and $\dot{\mathbb{P}}$. The result follows by considering the first order sentence $\forall x \forall y (x \in x \boxplus y)$ as in the proof of Theorem 4.1. $\square$

5. Algebraic formulæ

An *algebraically closed structure* in model theory is an abstraction of the usual notion of an algebraically closed field, defined purely in terms of definability and finiteness. For more details we refer to e.g. [14, Chapter 5].

Definition 5.1. *Let $\mathcal{L}$ be any first-order language and $\mathfrak{A}$ an $\mathcal{L}$-structure.*

A formula $\varphi(x, a)$ with parameters $a \in \mathfrak{A}$ is called algebraic over $\mathfrak{A}$ *if, in every $\mathcal{L}$-structure $\mathfrak{B} \supseteq \mathfrak{A}$, the solution set*

$$\varphi(\mathfrak{B}, a) = \{b \in \mathfrak{B} : \mathfrak{B} \models \varphi(b, a)\}$$

is finite.

Example 5.1. The formula $0 \in x \boxplus 1$ is algebraic over any hyperfield F. Indeed, the polynomial $x \boxplus 1$ has only one root in any hyperfield H which extends F, that is, the unique additive inverse of 1.

Example 5.2. The formula $0 \in x^2 \boxplus x \boxplus 1$ is not algebraic over $\mathbb{P}$ (and neither over $\mathbb{S}$!). Indeed, it is a simple excercise to check that the polynomial $x^2 \boxplus x \boxplus 1$ has infinitely many roots in $\mathbb{P}$. These roots are indeed

$$\left\{ e^{i\varphi} \mid \frac{\pi}{2} < \varphi < \pi \right\}.$$

Definition 5.2. *An $\mathcal{L}$-structure $\mathfrak{A}$ is* algebraically closed *if whenever $\varphi(x, a)$ is algebraic over $\mathfrak{A}$ and there is some extension $\mathfrak{B} \supseteq \mathfrak{A}$ with*

$$\mathfrak{B} \models \exists x\, \varphi(x, a),$$

then already

$$\mathfrak{A} \models \exists x\, \varphi(x, a).$$

Example 5.2 shows that the notion of algebraically closed hyperfields that we have considered so far does not coincide with the notion of algebraically closed structure over the language of hyperfields we have selected.

6. Further research

It is known that hyperfields are allowed to have any $n \in \mathbb{N}$ as characteristic: not only prime numbers, as it happens in the case of fields (cf. [7]).

On the other hand, the structures associated to non-archimedean local fields originally considered by Krasner and which motivated the introduction of hyperfields in [8], all have finite prime characteristic.

It remains an open problem whether the theory of algebraically closed hyperfields with fixed characteristic $n > 0$ is complete or admits quantifier elimination.

The validity of model-completeness is an entirely open question.

Furthermore, the hyperfields associated to valued fields as in [12, 13] are themselves valued in a way which makes their structure considerably simpler than the general case. By an application of [13, Theorem 4.21] one quickly checks that the hyperfields considered in this paper are not of that form, namely, they are not Krasner valued hyperfields ([13, Definition 4.6]).

We shall consider the case of algebraically closed Krasner valued hyperfields in a subsequent research paper.

In future research, also the notion of algebraically closed structures over our language of hyperfields could unravel intersting facts on the peculiar behavior of hyperfields (in comparison with that of fields).

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Distance measures between picture fuzzy multisets and their application to medical diagnosis

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Abstract. In this paper, the distance measures between picture fuzzy multisets are proposed as a generalisation of the existing distance measures between picture fuzzy sets. The validity of the transformation from distance measures between PFS to PFMS is carried out using numerical example. Also, an application to medical diagnosis of the proposed distance measures between picture fuzzy multisets is carried out using hypothetical medical database.
Keywords: multiset, fuzzy multiset, picture fuzzy set, picture fuzzy medical diagnosis.
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1. Introduction

Zadeh [12] was the pioneer of fuzzy sets (FSs) which was the generalisation of classical sets. The concept of fuzzy set is a vast and sprawling area which has applications in many areas such as Engineering, Economics, etc. Fuzzy set has a membership function which describes the membership degree of an element with respect to a particular class. Atanassov [1] introduced the concept of intuitionistic fuzzy sets (IFSs) as a generalisation of Zadeh's work.

Picture fuzzy sets (PFSs) was introduced and studied by Cuong and Kreinovich [3] to generalise IFSs and FSs. While it is well-known that intuitionistic fuzzy set generalises fuzzy set in dealing with imprecisions and vagueness, the theory still lacks a very crucial parameter which is the degree of neutrality.

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This degree of neutrality is relevant in variety of situations like voting system, medical diagnosis, personal selection etc. For voting system, a voter has four choices, to vote for, to vote against, to abstain from voting and refuse to vote. In medical diagnosis, symptoms like malaria and typhoid have neutral effect on disorders such as chest pain and stomach pain; headache and temperature have neutral effect on stomach problem and chest problem. As a result of these situations, Cuong and Kreinovich [3] in 2013 introduced and studied PFSs as a generalisation of IFSs and FSs. Thus, PFS is made up of degrees of positive, neutral and negative memberships.

Yagar [11] introduced fuzzy multisets (FMs) to extend fuzzy sets. Shinoj and Sunil [9] extended IFSs and FSs by initiating intuitionistic fuzzy multisets (IFMSs). Cao et al [2] generalised IFMS and FMS by introducing picture fuzzy multisets (PFMS).

In fuzzy mathematics, there is as important concept called distance measure between IFSs. The concept is important because of its wide applications in real life problems. Some researchers have proposed some distance measures between IFSs. Dutta [4] introduced some distance measures between PFSs and extended the work of Wang and Xin [11] and established some properties of them.

In this paper, generalised definition of distance measures between PFMS is put forward, and distance measures between PFSs proposed in [4] are adapted. The validity of the transformation from distance measures between PFS to PFMS is carried out using numerical example. Finally, an application to medical diagnosis of proposed distance measures between picture fuzzy multisets is carried out using hypothetical medical database.

2. Preliminaries

In this section, we recall some basic definitions.

Definition 2.1 ([3]). *Given a nonempty set $\mathcal{C}$. A PFS $\mathcal{D}$ of $\mathcal{C}$ is written as*

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}(z), \tau_{\mathcal{D}}(z), \gamma_{\mathcal{D}}(z)}{z} \rangle | z \in \mathcal{C}\},$$

where the functions $\sigma_{\mathcal{D}}(z), \tau_{\mathcal{D}}(z), \gamma_{\mathcal{D}}(z)$ from $\mathcal{C}$ to $[0,1]$ are called the positive, neutral and negative membership degrees of $z \in \mathcal{C}$ to $\mathcal{D}$, respectively, and for all element $z \in \mathcal{C}$, $0 \leq \sigma_{\mathcal{D}}(z) + \tau_{\mathcal{D}}(z) + \gamma_{\mathcal{D}}(z) \leq 1$. For each PFS $\mathcal{D}$ of $\mathcal{C}$, $\pi_{\mathcal{D}}(z) = 1 - (\sigma_{\mathcal{D}}(z) + \tau_{\mathcal{D}}(z) + \gamma_{\mathcal{D}}(z))$ is the refusal membership degrees of $z \in \mathcal{D}$.

Definition 2.2 ([2]). *Given a nonempty set $\mathcal{C}$. The PFMS $\mathcal{D}$ in $\mathcal{C}$ is characterised by three functions, namely, positive membership count function pmc, neutral membership count function $n_e mc$ and negative membership count function nmc such that pmc, $n_e mc$ and nmc are functions from $\mathcal{C}$ to $\mathcal{W}$, where $\mathcal{W}$ refers to collection of crisp multisets taken from $[0,1]$.*

Thus, every element $r \in \mathcal{C}$, pmc is the crisp multiset from $[0,1]$ whose positive membership sequence is defined by $(\sigma^1_{\mathcal{D}}(r), \sigma^2_{\mathcal{D}}(r), \cdots, \sigma^n_{\mathcal{D}}(r))$ such that

$\sigma^1_{\mathcal{D}}(r) \geq \sigma^2_{\mathcal{D}}(r) \geq \cdots \geq \sigma^n_{\mathcal{D}}(r)$, $n_e mc$ is the crisp multiset from $[0,1]$ whose neutral membership sequence is defined by $(\tau^1_{\mathcal{D}}(r), \tau^2_{\mathcal{D}}(r), \cdots, \tau^n_{\mathcal{D}}(r))$ and nmc is the crisp multiset from $[0,1]$ whose negative membership sequence is defined by $(\eta^1_{\mathcal{D}}(r), \eta^2_{\mathcal{D}}(r), \cdots, \eta^n_{\mathcal{D}}(r))$, these can be either decreasing or increasing functions satisfying $0 \leq \sigma^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{D}}(r) \leq 1$, $\forall r \in \mathcal{C}$, $k = 1, 2, \cdots, n$. Thus, $\mathcal{D}$ is represented by $\mathcal{D} = \{\langle r, \sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{D}}(r)\rangle | r \in \mathcal{C}\}$, where $k = 1, 2, \cdots, n$.

Example 2.1. Let $\mathcal{C} = \{a, b, c\}$

$$\begin{aligned} \mathcal{D} = \quad & \{< a, (0.7, 0.2, 0.1), (0.55, 0.25, 0.10), (0.5, 0.3, 0.2) >, \\ & < b, (0.6, 0.2, 0.2), (0.4, 0.3, 0.2), (0.65, 0, 20, 0.15) >, \\ & < c, (0.8, 0.1, 0.1), (0.4, 0.4, 0.2), (0.9, 0.05, 0.05) >\} \end{aligned}$$

Definition 2.3 ([2]). *Let $\mathcal{D} = \{\langle z, \sigma^k_{\mathcal{D}}(z), \tau^k_{\mathcal{D}}(z), \eta^k_{\mathcal{D}}(z)\rangle \mid z \in \mathcal{C}\}$, $k = 1, 2, \cdots, n$ be a PFMS, $z \in \mathcal{D}$. Then, the order β of $z \in \mathcal{D}$ is defined as the cardinality of $C_p M_{\mathcal{D}}(z)$ or $C_{ne} M_{\mathcal{D}}(z)$ or $C_n M_{\mathcal{D}}(z)$ for $0 \leq \sigma^k_{\mathcal{D}}(z) + \tau^k_{\mathcal{D}}(z) + \eta^k_{\mathcal{D}}(z) \leq 1$. That is*

$$\beta = |C_p M_{\mathcal{D}}(z)| = |C_{ne} M_{\mathcal{D}}(z)| = |C_n M_{\mathcal{D}}(z)|.$$

Definition 2.4 ([2]). *Given two PFMSs $\mathcal{D}_1$ and $\mathcal{D}_2$ drawn from $\mathcal{C}$. Then, the size of $\mathcal{D}_1$ and $\mathcal{D}_2$ is defined as*

$$\beta = \beta(\mathcal{D}_1) \vee (\beta(\mathcal{D}_2).$$

Definition 2.5 ([8]). *Given that PFMS $\mathcal{D}$ is not empty. Then, a PFMR $\mathcal{U}$ on $\mathcal{D}$, defined as $\mathcal{U} = \{\langle (x, y), \sigma^k_{\mathcal{D}}(x, y), \tau^k_{\mathcal{D}}(x, y), \eta^k_{\mathcal{D}}(x, y)\rangle | (x, y) \in \mathcal{D} \times \mathcal{D}\}$ is also a PFMS, where $k = 1, 2, \cdots, \beta$, where β is the cardinality of the PFMS $\mathcal{D}$ and $\sigma^k_{\mathcal{D}}, \tau^k_{\mathcal{D}}, \eta^k_{\mathcal{D}}$ are functions from $\mathcal{C}$ to $\mathcal{W}$, where $\mathcal{W}$ is the set of all crisp multisets drawn from $[0, 1]$.*

2.1 Distance measures between picture fuzzy sets

Dutta [4], defined distance measures between PFSs as follow.

Let

$$\mathcal{D}_1 = \{< x, \sigma_{\mathcal{D}_1}(x), \tau_{\mathcal{D}_1}(x), \gamma_{\mathcal{D}_1}(x) > \ | \ x \in X\}$$

and

$$\mathcal{D}_2 = \{< x, \sigma_{\mathcal{D}_2}(x), \tau_{\mathcal{D}_2}(x), \gamma_{\mathcal{D}_2}(x) > \ | \ x \in X\}$$

defined on X. Then,

(1) the Hamming Distance is defined as

$$\begin{aligned} d(\mathcal{D}_1, \mathcal{D}_2) = \frac{1}{2} \sum_{j=1}^{n} \{ & |\sigma_{\mathcal{D}_1}(x_j) - \sigma_{\mathcal{D}_2}(x_j)| + |\tau_{\mathcal{D}_1}(x_j) - \tau_{\mathcal{D}_2}(x_j)| \\ & + |\eta_{\mathcal{D}_1}(x_j) - \eta_{\mathcal{D}_2}(x_j)| + |\pi_{\mathcal{D}_1}(x_j) - \pi_{\mathcal{D}_2}(x_j)|\}; \end{aligned}$$

(2) the Normalised Hamming Distance is defined as

$$l(\mathcal{D}_1,\mathcal{D}_2)=\frac{1}{2n}\sum_{j=1}^{n}\{|\sigma_{\mathcal{D}_1}(x_j)-\sigma_{\mathcal{D}_2}(x_j)|+|\tau_{\mathcal{D}_1}(x_j)-\tau_{\mathcal{D}_2}(x_j)|$$
$$+|\eta_{\mathcal{D}_1}(x_j)-\eta_{\mathcal{D}_2}(x_j)|+|\pi_{\mathcal{D}_1}(x_j)-\pi_{\mathcal{D}_2}(x_j)|\};$$

(3) the Euclidean Distance is defined as

$$e(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2}\sum_{j=1}^{n}[(\sigma_{\mathcal{D}_1}(x_j)-\sigma_{\mathcal{D}_2}(x_j))^2+(\tau_{\mathcal{D}_1}(x_j)-\tau_{\mathcal{D}_2}(x_j))^2$$
$$+(\eta_{\mathcal{D}_1}(x_j)-\eta_{\mathcal{D}_2}(x_j))^2+(\pi_{\mathcal{D}_1}(x_j)-\pi_{\mathcal{D}_2}(x_j))^2]\}^{\frac{1}{2}};$$

(4) the Normalised Euclidean Distance is defined as

$$q(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2n}\sum_{j=1}^{n}[(\sigma_{\mathcal{D}_1}(x_j)-\sigma_{\mathcal{D}_2}(x_j))^2+(\tau_{\mathcal{D}_1}(x_j)-\tau_{\mathcal{D}_2}(x_j))^2$$
$$+(\eta_{\mathcal{D}_1}(x_j)-\eta_{\mathcal{D}_2}(x_j))^2+(\pi_{\mathcal{D}_1}(x_j)-\pi_{\mathcal{D}_2}(x_j))^2)]\}^{\frac{1}{2}}.$$

3. Distance measure between picture fuzzy multisets

This section extends distance measures between PFSs to PFMSs. Throughout this section, $\mathbb{I}$ denotes closed interval $[0,1]$.

Definition 3.1. *Let $\mathcal{C}$ be a nonempty set such that PFMSs $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3\in\mathcal{C}$. Then, the distance measure is a mapping $d:\mathcal{C}\times\mathcal{C}\rightarrow\mathbb{I}$, if it satisfies the following properties;*

(i) $d(\mathcal{D}_1,\mathcal{D}_2)\in\mathbb{I}$.

(ii) $d(\mathcal{D}_1,\mathcal{D})_2=0$ *if and only if* $\mathcal{D}_1=\mathcal{D}_2$ *(faithful condition).*

(iii) $d(\mathcal{D}_1,\mathcal{D}_2)=d(\mathcal{D}_2,\mathcal{D}_1)$ *(Symmetric property).*

(iv) $d(\mathcal{D}_1,\mathcal{D}_3)\leq d(\mathcal{D}_1,\mathcal{D}_2)+d(\mathcal{D}_2,\mathcal{D}_3)$ *(Triangular inequality).*

(v) If $\mathcal{D}_1\subseteq\mathcal{D}_2\subseteq\mathcal{D}_3$, *then* $d(\mathcal{D}_1,\mathcal{D}_3)\geq d(\mathcal{D}_1,\mathcal{D}_2)$ *and* $d(\mathcal{D}_1,\mathcal{D}_3)\geq d(\mathcal{D}_2,\mathcal{D}_3)$.

Then, $d(\mathcal{D}_1,\mathcal{D}_2)$ measures the distance between PFMSs $\mathcal{D}_1$ and $\mathcal{D}_2$.

Distance measure can be considered as a dual concept of similarity measure. Distance measures between PFSs have been introduced and studied recently by Dutta [4]. These distance measures are extended to PFMSs since they satisfied conditions of metric distance.

Definition 3.2. *Let $\mathcal{D}_1=\{<z,\sigma^k_{\mathcal{D}_1}(z),\tau^k_{\mathcal{D}_1}(z),\eta^k_{\mathcal{D}_1}(z)>\ |\ z\in\mathcal{C}\}$ and $\mathcal{D}_2=\{<z,\sigma^k_{\mathcal{D}_2}(z),\tau^k_{\mathcal{D}_2}(z),\eta^k_{\mathcal{D}_2}(z)>\ |\ z\in\mathcal{C}\}$ defined on $\mathcal{C}$. Then,*

(1) the Hamming Distance is defined as

$$hd(\mathcal{D}_1,\mathcal{D}_2)=\frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j)|+|\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j)|$$
$$+|\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j)|+|\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j)|\};$$

(2) the Normalised Hamming Distance is defined as

$$nhd(\mathcal{D}_1,\mathcal{D}_2)=\frac{1}{2n}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j)|+|\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j)|$$
$$+|\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j)|+|\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j)|\};$$

(3) the Euclidean Distance is defined as

$$ed(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j))^2+(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j))^2$$
$$+(\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j))^2+(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}};$$

(4) the Normalised Euclidean Distance is defined as

$$ned(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2n}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j))^2+(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j))^2$$
$$+(\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j))^2+(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}}.$$

Theorem 3.1. *Given*

$$\mathcal{D}_1=\{<z,\sigma^k_{\mathcal{D}_1}(z),\tau^k_{\mathcal{D}_1}(z),\eta^k_{\mathcal{D}_1}(z)> \mid z\in\mathcal{C}\},$$
$$\mathcal{D}_2=\{<z,\sigma^k_{\mathcal{D}_2}(z),\tau^k_{\mathcal{D}_2}(z),\eta^k_{\mathcal{D}_2}(z)> \mid z\in\mathcal{C}\}$$

and

$$\mathcal{D}_3=\{<z,\sigma^k_{\mathcal{D}_3}(z),\tau^k_{\mathcal{D}_3}(z),\eta^k_{\mathcal{D}_3}(z)> \mid z\in\mathcal{C}\}$$

defined on $\mathcal{C}$. Then, the distance measures defined in Definition 3.2 are metric.

Proof. (1) Hamming Distance

Non-negativity: $hd(\mathcal{D}_1,\mathcal{D}_2)\geq 0$, because by definition, $hd(\mathcal{D}_1,\mathcal{D}_2)$ is always either 0 or 1, since the Hamming distance is the sum of non-negative terms. Faithful condition: Suppose that $\mathcal{D}_1=\mathcal{D}_2$, then every corresponding element is the same. That is

$$\sigma^k_{\mathcal{D}_1}(z_j)=\sigma^k_{\mathcal{D}_2}(z_j),\ \tau^k_{\mathcal{D}_1}(z_j)=\tau^k_{\mathcal{D}_2}(z_j),\ \eta^k_{\mathcal{D}_1}(z_j)=\eta^k_{\mathcal{D}_2}(z_j)$$

and

$$\pi^k_{\mathcal{D}_1}(z_j)=\pi^k_{\mathcal{D}_2}(z_j).$$

Hence, $hd(\mathcal{D}_1, \mathcal{D}_2) = 0$.

Conversely, suppose that $hd(\mathcal{D}_1, \mathcal{D}_2) = 0$, then all the terms in the summation must be zero, which means that $|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)| = 0$, $|\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)| = 0$, $|\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)| = 0$ and $|\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)| = 0$.

Hence, $\mathcal{D}_1 = \mathcal{D}_2$.

Symmetric:

$$\begin{aligned}
hd(\mathcal{D}_1, \mathcal{D}_2) &= \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)| + |\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)| \\
&\quad + |\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)| + |\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)|\} \\
&= \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_1}(z_j)| + |\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_1}(z_j)| \\
&\quad + |\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_1}(z_j)| + |\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_1}(z_j)|\} \\
&= hd(\mathcal{D}_2, \mathcal{D}_1).
\end{aligned}$$

Triangle inequality:

$$\begin{aligned}
hd(\mathcal{D}_1, \mathcal{D}_3) &= \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j)| + |\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_3}(z_j)| \\
&\quad + |\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_3}(z_j)| + |\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_3}(z_j)|\} \\
&= \frac{1}{2}\sum_{k,j=1}^{n}\{|(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)) + (\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j))| \\
&\quad + |(\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)) + (\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_3}(z_j))| \\
&\quad + |(\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)) + (\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_3}(z_j))| \\
&\quad + |(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)) + (\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_3}(z_j))|\} \\
&\leq \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)| + |\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)| \\
&\quad + |\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)| + |\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)|\} \\
&\quad + \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j)| + |\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_3}(z_j)| \\
&\quad + |\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_3}(z_j)| + |\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_3}(z_j)|\} \\
&= hd(\mathcal{D}_1, \mathcal{D}_2) + hd(\mathcal{D}_2, \mathcal{D}_3).
\end{aligned}$$

(2) Normalised Hamming Distance can be proved in a similar way.

(3) Euclidean Distance

Non-negativity: $ed(\mathcal{D}_1, \mathcal{D}_2) \geq 0$ since each squared sum and difference are always non-negative.

Faithful condition: Suppose that $\mathcal{D}_1 = \mathcal{D}_2$, then

$\sigma^k_{\mathcal{D}_1}(z_j) = \sigma^k_{\mathcal{D}_2}(z_j)$, $\tau^k_{\mathcal{D}_1}(z_j) = \tau^k_{\mathcal{D}_2}(z_j)$, $\eta^k_{\mathcal{D}_1}(z_j) = \eta^k_{\mathcal{D}_2}(z_j)$ and $\pi^k_{\mathcal{D}_1}(z_j) = \pi^k_{\mathcal{D}_2}(z_j)$.

Thus $ed(\mathcal{D}_1, \mathcal{D}_2) = \sqrt{0} = 0$.

Conversely, suppose that $ed(\mathcal{D}_1, \mathcal{D}_2) = 0$, this implies that

$$\begin{aligned}(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\ (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\ (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 &= 0\end{aligned}$$

and

$$(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2 = 0,$$

i.e.

$$\begin{aligned}&[\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2\\ &+ (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 + (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2]]^{\frac{1}{2}} = 0.\end{aligned}$$

Since the square root function is only zero when its argument is zero, thus

$$\begin{aligned}&\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 + (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2\\ &+ (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2] = 0.\end{aligned}$$

Since each of these

$$\begin{aligned}&(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2,\\ &(\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2,\\ &(\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2\end{aligned}$$

and

$$(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2$$

is non-negative, their sum can only be zero if

$$\begin{aligned}(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\ (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\ (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 &= 0\end{aligned}$$

and

$$(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2 = 0.$$

Thus $\sigma^k_{\mathcal{D}_1}(z_j) = \sigma^k_{\mathcal{D}_2}(z_j)$, $\tau^k_{\mathcal{D}_1}(z_j) = \tau^k_{\mathcal{D}_2}(z_j)$, $\eta^k_{\mathcal{D}_1}(z_j) = \eta^k_{\mathcal{D}_2}(z_j)$ and $\pi^k_{\mathcal{D}_1}(z_j) = \pi^k_{\mathcal{D}_2}(z_j)$.

Hence, $\mathcal{D}_1 = \mathcal{D}_2$.

Symmetric:

$$\begin{aligned}
ed(\mathcal{D}_1, \mathcal{D}_2) &= \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 + (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 \\
&\quad + (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}} \\
&= \{\frac{1}{2}\sum_{j=1}^{n}[(\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_1}(z_j))^2 + (\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_1}(z_j))^2 \\
&\quad + (\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_1}(z_j))^2 + (\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_1}(z_j))^2]\}^{\frac{1}{2}} \\
&= ed(\mathcal{D}_2, \mathcal{D}_1).
\end{aligned}$$

Triangle inequality:

$$\begin{aligned}
ed(\mathcal{D}_1, \mathcal{D}_3) &= \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j))^2 + (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_3}(z_j))^2 \\
&\quad + (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_3}(z_j))^2 + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_3}(z_j))^2]\}^{\frac{1}{2}} \\
&= \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j) + \sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j))^2 \\
&\quad + (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j) + \tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_3}(z_j))^2 \\
&\quad + (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j) + \eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_3}(z_j))^2 \\
&\quad + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j) + \pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_3}(z_j))^2]\}^{\frac{1}{2}} \\
&\leq \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 + (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 \\
&\quad + (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}} \\
&+ \{\frac{1}{2}\sum_{j=1}^{n}[(\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j))^2 + (\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_3}(z_j))^2 \\
&\quad + (\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_3}(z_j))^2 + (\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_3}(z_j))^2]\}^{\frac{1}{2}} \\
&\leq ed(\mathcal{D}_1, \mathcal{D}_2) + ed(\mathcal{D}_2, \mathcal{D}_3).
\end{aligned}$$

(4) Normalised Euclidean distance can be proved in a similar way. □

4. Application of picture fuzzy multisets in medical diagnosis via distance measures

This section presents an application of PFMSs to medical diagnosis via distance measures. Given that, the following parameters M, L, and K represent sets of symptoms, diseases and patients. Define the PFMR, F from M to L, $M \times L$ which reveals the degrees of association, non-association and nuetrality between symptoms and diseases. Also, define another PFMR, G from K to M, $K \times M$ which reveals the association degree, non-association degree and neutrality degree between patients and symptoms.

The steps to determine which disease explains patience's symptoms and signs better is what is called medical decision making. We employ the same methodology for the picture fuzzy medical diagnosis stated in [4] for PFMR. These are stated below:

i. Determine $M_i,\ i = 1, 2, 3, 4$.

ii. Formulate medical knowledge PFMR, F, $M \times L$ and PFMR, G, $K \times M$.

iii. Evaluate π in the data set.

iv. Computation of the distance between K_i and L_i using Definition 3.2.

v. Identification of the shortest distance between K_i and L_i reflects that the patient K_i is suffering from disease $L_i,\ i = 1, 2, 3, 4$.

4.1 Case study

Let

$$K = \{\text{Rabat, Roy, Ranic, Raja}\}$$

be a set of patients,

$$L = \{\text{Viral fever, malaria, Typhoid, Stomach problem}\}$$

be a set of diseases and

$$M = \{\text{Temperature, Headache, Stomach pain, Cough, Chest pain}\}$$

be a set of symptoms. Here, each patient was examined more than once because there is a possibility that a particular patient has symptoms of different diseases. This will enable the doctor to draw a conclusion that a particular patient is suffering from a particular disease based on the different examinations carried out on each patient.

In Table 1, each symptom M_i is described by four parameters: positive membership degree $\sigma(x)$, neutral membership degree $\tau(x)$, negative membership degree $\eta(x)$ and refusal degree $\pi(x)$.

In Table 2, the data were recorded at different intervals, and the distance of each patient K_i from the set of symptoms M_i for each diagnosis $L_i,\ i = 1, 2, 3, 4$. The first set represents the positive membership values, the second represents the neutral membership values, the third represents the negative membership values and the fourth represents the refusal membership values. Thus, using Definition 2.2 Table 2 is constructed.

In Table 3, conversion of PFMS to PFS was carried out by finding the mean values for each set in Table 2.

Table 1: SYMPTOMS VS DISEASES

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Temperature	(0.6,0.2,0.1,0.1)	(0.1,0.5,0.1,0.3)	(0.3,0.5,0.1,0.1)	(0.3,0.1,0.5,0.1)
Cough	(0.1,0.5,0.1,0.3)	(0.7,0.1,0.0,0.2)	(0.2,0.4,0.2,0.2)	(0.4,0.2,0.0,0.4)
Chest pain	(0.2,0.4,0.2,0.2)	(0.5,0.1,0.3,0.1)	(0.1,0.5,0.1,0.3)	(0.6,0.0,0.1,0.3)
Headache	(0.3,0.5,0.1,0.1)	(0.6,0.0,0.1,0.3)	(0.1,0.4,0.2,0.3)	(0.4,0.4,0.1,0.1)
Stomach pain	(0.4,0.3,0.2,0.1)	(0.6,0.0,0.1,0.3)	(0.3,0.3,0.2,0.2)	(0.5,0.2,0.0,0.3)

Table 2: PATIENTS VS SYMPTOMS

	Temperature	Cough	Chest pain	Headache	Stomach pain
Rabat	(0.5,0.2,0.1,0.2)	(0.3,0.4,0.2,0.1)	(0.4,0.3,0.1,0.2)	(0.2,0.3,0.2,0.3)	(0.6,0.0,0.3,0.0)
	(0.6,0.2,0.1,0.1)	(0.3,0.1,0.3,0.3)	(0.2,0.3,0.2,0.2)	(0.1,0.0,0.2,0.7)	(0.0,0.4,0.2,0.4)
	(0.4,0.3,0.2,0.1)	(0.4,0.4,0.1,0.1)	(0.3,0.2,0.3,0.2)	(0.0,0.4,0.3,0.3)	(0.2,0.3,0.2,0.3)
Roy	(0.4,0.3,0.1,0.2)	(0.6,0.0,0.2,0.2)	(0.5,0.1,0.1,0.3)	(0.1,0.3,0.3,0.3)	(0.4,0.1,0.2,0.3)
	(0.3,0.3,0.2,0.2)	(0.4,0.1,0.2,0.3)	(0.4,0.2,0.0,0.4)	(0.3,0.2,0.1,0.4)	(0.3,0.5,0.1,0.1)
	(0.4,0.2,0.1,0.3)	(0.3,0.3,0.0,0.4)	(0.1,0.3,0.1,0.5)	(0.6,0.1,0.2,0.1)	(0.2,0.3,0.3,0.2)
Ranic	(0.6,0.1,0.1,0.2)	(0.2,0.3,0.1,0.4)	(0.3,0.2,0.2,0.3)	(0.3,0.4,0.1,0.2)	(0.5,0.2,0.1,0.2)
	(0.2,0.4,0.2,0.2)	(0.1,0.4,0.0,0.5)	(0.2,0.5,0.1,0.2)	(0.4,0.2,0.1,0.3)	(0.4,0.3,0.2,0.1)
	(0.3,0.3,0.1,0.3)	(0.6,0.0,0.2,0.2)	(0.1,0.3,0.5,0.1)	(0.2,0.3,0.2,0.3)	(0.0,0.3,0.4,0.3)
Raja	(0.3,0.0,0.3,0.4)	(0.5,0.2,0.1,0.2)	(0.4,0.3,0.1,0.2)	(0.6,0.2,0.1,0.1)	(0.2,0.4,0.2,0.2)
	(0.4,0.2,0.2,0.2)	(0.3,0.3,0.2,0.2)	(0.2,0.5,0.2,0.1)	(0.3,0.2,0.3,0.2)	(0.3,0.1,0.3,0.3)
	(0.1,0.5,0.1,0.3)	(0.2,0.4,0.1,0.3)	(0.3,0.4,0.1,0.2)	(0.1,0.5,0.1,0.3)	(0.6,0.2,0.1,0.1)

Next, we convert PFMS to PFS by finding the mean values for each set in Table 2 to get Table 3.

Table 3: PATIENTS VS SYMPTOMS

	Temperature	Cough	Chest pain	Headache	Stomach pain
Rabat	(0.50,0.23, 0.13,0.13)	(0.33,0.30, 0.20,0.17)	(0.30,0.27, 0.20,0.23)	(0.10,0.23, 0.23,0.43)	(0.27,0.23, 0.23,0.27)
Roy	(0.37,0.27, 0.13,0.23)	(0.43,0.13, 0.13,0.30)	(0.33,0.20, 0.07,0.40)	(0.33,0.20, 0.20,0.27)	(0.30,0.30, 0.20,0.20)
Ranic	(0.37,0.27, 0.13,0.23)	(0.30,0.23, 0.10,0.37)	(0.20,0.33, 0.27,0.20)	(0.30,0.30, 0.13,0.27)	(0.30,0.27, 0.23,0.20)
Raja	(0.27,0.23, 0.20,0.30)	(0.33,0.30, 0.13,0.23)	(0.30,0.40, 0.13,0.17)	(0.33,0.30, 0.17,0.20)	(0.37,0.23, 0.20,0.20)

Table 4: HAMMING DISTANCE BETWEEN PATIENTS AND DISEASES

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Rabat	1.220	2.060	**0.945**	1.790
Roy	1.325	1.765	**1.125**	1.505
Ranic	**0.865**	1.860	0.980	1.570
Raja	0.995	1.705	**0.975**	1.395

Table 5: N-HAMMING DISTANCE BETWEEN PATIENTS AND DISEASES

	Viral ever	Tuberculosis	Typhoid	Stomach problem
Rabat	0.244	0.412	**0.189**	0.358
Roy	0.265	0.353	**0.225**	0.301
Ranic	**0.173**	0.372	0.196	0.314
Raja	0.199	0.341	**0.195**	0.279

Table 6: Euclidean distance between patients and diseases

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Rabat	0.475	0.755	**0.393**	0.645
Roy	0.541	0.581	**0.482**	0.514
Ranic	**0.362**	0.635	0.377	0.581
Raja	0.416	0.808	**0.390**	0.541

Table 7: N-Euclidean distance between patients and diseases

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Rabat	0.212	0.338	**0.176**	0.289
Roy	0.242	0.260	**0.215**	0.230
Ranic	**0.162**	0.284	0.169	0.260
Raja	0.186	0.361	**0.174**	0.242

4.2 Results and discussion

With respect to Tables 4 - 7, the decision making is presented. Decisions are made based on the smallest distance between the patients and diseases. From Tables 4 - 7, it was established that Rabat is suffering from Typhoid, Roy is suffering from Typhoid, Rank is suffering from Typhoid and Ranic is suffering from Viral fever.

5. Conclusion

In this paper, it has been established that the distance measures; Hamming Distance, Euclidean Distance together with their normalised versions transformed from PFSs to PFMSs are valid simply because the properties of the definition of distance measures proposed in [4] are satisfied. Also, an application to medical diagnosis of distance measures between picture fuzzy multisets is carried out using hypothetical medical database.

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A note on hyperrings and hypermodules-Corrigendum

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Abstract. In this corrigendum to the paper, "A note on hyperrings and hypermodules" [5], we present revised Theorem 3.3 and Example 3.3 and correct some typographical errors.

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1. Corrigendum

Unfortunately, we have found some errors in [5, Theorem 3.3] and [5, Example 3.3].

Let $m > 1$. Define the relation "$\equiv$" on $\mathbb{N}$ by for all x, $y \in \mathbb{N}$

$$\text{“}x \equiv y \Longleftrightarrow m|k, \text{ where } \min\{x, y\} + k = \max\{x, y\}\text{”}.$$

It can be seen that "$\equiv$" is an equivalence relation on $\mathbb{N}$. Let $\mathbb{N}_m = \{\overline{x} \mid x \in \mathbb{N}\}$, where $\overline{x} = \{0 + x, m + x, 2m + x, ...\} = \{nm + x \mid n \in \mathbb{N}\}$. Let $0 \leq x < y < m$. Suppose that $\overline{x} = \overline{y}$. Then, $y \in \overline{x}$ and so $m|k$, $x + k = y$ for some $k \in \mathbb{N}$. This is a contradiction since $0 < k < m$. Hence, the equivalence classes $\overline{0}$, $\overline{1}$, ..., $\overline{m-1}$

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are distinct. Let $\overline{x}$ be any element of $\mathbb{N}_m$. By the division algorithm, $x = mq+r$ for some elements q and r such that $0 \leq r < m$. Since $m|mq$, we obtain that $\overline{r} = \overline{x}$. Hence, $\mathbb{N}_m = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{m-1}\}$.

The definition of the hyperoperation given [5, Theorem 3.3] is incorrect and does not satisfy the condition of the uniqueness of an element, which is among the axioms for being a canonical hypergroup. We can observe this situation in [5, Example 3.3]. In this example, the inverse of the element $\overline{2}$ is both $\overline{2}$ and $\overline{4}$. This violates the definition of a canonical hypergroup. Therefore, the hyperoperation should be defined as we provide below.

The following Theorem are corrected versions of these results.

Theorem 1.1. *Let $m > 1$. Define the hyperoperation "$\oplus_m$" on $\mathbb{N}_m$ by*

$$\overline{x} \oplus_m \overline{y} = \begin{cases} \{\overline{x+y}\}, & \text{if } \overline{x} = \overline{y}; \\ \{\overline{x+y}, \overline{k}\}, & \text{if } \overline{x} \neq \overline{y}, \text{ where } \min\{x, y\} + k = \max\{x, y\}. \end{cases}$$

for all $\overline{x}, \overline{y} \in \mathbb{N}_m$. Then

(1) $(\mathbb{N}_m, \oplus_m)$ *is a canonical hypergroup with scalar identity* $\overline{0}$.

(2) $(\mathbb{N}_m, \oplus_m, .)$ *is a commutative and unitary hyperring, where "." is the usual multiplication.*

(3) $(\mathbb{N}^*_m, .)$ *is a group, where* $\mathbb{N}^*_m = \{\overline{x} \in \mathbb{N}_m \,|\, (x, m) = 1\}$.

(4) $(\mathbb{N}_m, \oplus_m, .)$ *is a hyperfield if and only if m is prime.*

(5) *The canonical hypergroup* $(\mathbb{N}_m, \oplus_m)$ *is a* $\mathbb{N}$*-hypermodule.*

Proof. (1), (2) and (3) are straightforward.

(4) $(\Rightarrow)$ Let $m = ab$, where $1 \leq a < b < m$. Then $\overline{a}, \overline{b} \in \mathbb{N}_m$ and so $\overline{a}.\overline{b} = \overline{ab} = \overline{0}$, a contradiction.

$(\Leftarrow)$ Let $\overline{a} \in \mathbb{N}^*_m$. Then $(a, m) = 1$ and so we get $1 = ax + my$ for some $x, y \in \mathbb{N}$. It follows that $\overline{1} = \overline{ax+my} = \overline{ax} = \overline{a}.\overline{x}$. Hence, $\mathbb{N}^*_m = \mathbb{N}_m \backslash \{\overline{0}\}$. By (3), $(\mathbb{N}_m, \oplus_m, .)$ is a hyperfield.

(5) Define the map $\cdot : \mathbb{N} \times \mathbb{N}_m \longrightarrow \mathbb{N}_m$ via $n \cdot \overline{x} = \overline{nx}$ for all $n \in \mathbb{N}$ and for all $\overline{x} \in \mathbb{N}_m$. According to the map, it can be checked that N_m is a $\mathbb{N}$-hypermodule. □

Note that the condition "prime positive integer" in the [5, Proposition 3.3] is necessary. Let's take the following example to see this.

Example 1.1. Given the the hyperring $\mathbb{N}_6$. Using Theorem 1.1, we obtain the following tables:

$\oplus_6$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$
$\overline{1}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{0}, \overline{4}\}$
$\overline{2}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{3}$	$\{\overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$
$\overline{4}$	$\{\overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{5}$	$\{\overline{5}\}$	$\{\overline{0}, \overline{4}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{4}\}$

and

.	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{0}$	$\overline{2}$	$\overline{4}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{2}$	$\overline{0}$	$\overline{4}$	$\overline{2}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Thus, the only maximal hyperideals of the hyperring $\mathbb{N}_6$ are $I_1 = \{\overline{0}, \overline{3}\}$ and $I_2 = \{\overline{0}, \overline{2}, \overline{4}\}$. Also, we have $\mathbb{N}_6 = I_1 \oplus_6 I_2$. It follows that every hyperideal of $\mathbb{N}_6$ is a direct summand of $\mathbb{N}_6$. Hence, the hyperring $\mathbb{N}_6$ is not local.

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Finite groups whose sums of irreducible character degrees of all proper subgroups are large

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Abstract. Some scholars investigated the influence of the sum of all character degrees of a finite group on group structure. In this paper, we will study the influence of sums of all character degrees of all proper subgroups of a finite group on its structure. We will show that a finite group G is solvable when all proper subgroups H of G satisfy $\mathrm{T}(H) \geq \frac{2}{3}|H|$, where $\mathrm{T}(H)$ is the sum of all character degrees of a finite group H.

Keywords: simple group, character degree sum, proper subgroup.

MSC 2020: 20C15, 20D06.

1. Introduction

Only finite groups are considered in this paper. Let G be a finite group and $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G, say $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_s\}$. Let $\mathrm{cd}(G) = \{\chi_i(1) : \chi_i \in \mathrm{Irr}(G)\}$. Then, $\mathrm{T}(G) = \sum_{i=1}^{s} \chi_i(1)$ is the sum of all character degrees of G. We see that $\mathrm{T}(G) = |G|$ if and only if G is abelian. Let p be a prime. The lower bound of $\mathrm{T}(G)$ is determined when $|G|$ is divisible by p^n with $n \leq 6$ (see [7]). Many scholars studied the relations between group structures and $\mathrm{T}(G)$. For instance, G is solvable when $\mathrm{T}(G) > \frac{4}{15}|G|$ (see [14]) or $\mathrm{T}(G) > \sqrt{3/8}|G|$ [11, Theorem 1.3]; G is p-solvable when $\mathrm{T}(G) \geq (\sqrt{3}/p)|G|$ [11, Theorem 1.1] or $\mathrm{T}(G) > |G|/f(p)$ with

$$f(p) = \begin{cases} p(p^2-1)/(p^2+p+2), & \text{if } p \equiv 1 \pmod 4, \\ p-1, & \text{if } p \geq 7 \text{ and } p \equiv -1 \pmod 4, \\ 15/4, & \text{if } p = 2, 3; \end{cases}$$

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(see [12, Corollary B]). Let $\tau = \sum_{i=1}^{s} \chi_i$ for all $\chi_i \in \mathrm{Irr}(G)$ and let $a(\phi) =< \tau_H, \phi >$ for $\phi \in \mathrm{Irr}(H)$. Let

$$\delta_0(G, H) = \sum_{\phi \in \mathrm{Irr}(H)} a(\phi)\phi(1) - (a - 1)$$

with $H < G$ and $a = a(1_H)$. Berkovich and Mann in [1] showed the structures of finite groups when $\delta_0(G, H)$ is small.

In this paper, we consider the influence of sums of all characters degrees of all proper subgroups of a finite group on its structure. For the sake of brevity, we give a definition.

Definition 1.1. *A group G is called an* SD-*group if* $\mathrm{T}(G) \geq \frac{2}{3}|G|$.

Now, the result of [11] shows that an SD-group is solvable. Let $\mathbf{prop}(G)$ be the set of all proper subgroups of G. In order to argue in short, a concept is introduced.

Definition 1.2. *A group G is called a* PSD-*group if every proper subgroup of G is an* SD-*group.*

In this paper, we mainly prove the following result which is corresponding to those results of [11] and [14].

Theorem 1.1. *A finite* PSD-*group is solvable.*

By [5, p. 2], A_5 has A_4, D_{10} and S_3 as its maximal subgroups. By [4],

$$\begin{aligned} \mathrm{T}(A_4) &= 1 + 1 + 1 + 3 = 6, |A_4| = 12. \\ \mathrm{T}(D_{10}) &= 1 + 1 + 2 + 2 = 6, |D_{10}| = 10. \\ \mathrm{T}(S_3) &= 1 + 1 + 2 = 4, \qquad |S_3| = 6. \end{aligned}$$

Hence

$$\frac{\mathrm{T}(A_4)}{|A_4|} = \frac{1}{2}, \frac{\mathrm{T}(D_{10})}{|D_{10}|} = \frac{3}{5} \text{ and } \frac{\mathrm{T}(S_3)}{|S_3|} = \frac{2}{3}.$$

It follows that the condition of Theorem 1.1 "for all $H \in \mathbf{prop}(G)$, $\frac{\mathrm{T}(H)}{|H|} \geq \frac{2}{3}$" is the best possible.

This paper is formed as follows. In Section 2, some properties of SD-groups are given and also the structures of minimal non-abelian simple groups are introduced. In Section 3, the result "a PSD-group is solvable" is proved. Let $\max G$ or $\max(G)$ be the set of all maximal subgroups of G with respect to subgroup order divisibility. All other symbols are standard, please see [5] and [9] for instance.

2. Basic results

In this section, some basic results are collected. First, in general, an SD-group must be a PSD-group. Let G be an SD-group. For any $H \in \mathbf{prop}(G)$, from [2, Chapter 11], $\frac{|H|}{\mathrm{T}(H)} \leq \frac{|G|}{\mathrm{T}(G)}$. Then, $\frac{\mathrm{T}(H)}{|H|} \geq \frac{\mathrm{T}(G)}{|G|} \geq \frac{2}{3}$. It follows that H is an SD-group and G is a PSD-group. But the converse is not true.

Example 2.1. Let $G = S_3 \times S_3$ where S_n is the symmetric group on n symbols. We see from [4] that

$$\mathrm{T}(G) = 4 \cdot 1 + 4 \cdot 2 + 1 \cdot 4 = 16 \text{ and } |G| = 36,$$

so

$$\frac{\mathrm{T}(G)}{|G|} = \frac{16}{36} = \left(\frac{2}{3}\right)^2 \ngeq \frac{2}{3}.$$

It follows that G is a non-SD-group. But its maximal subgroups $H \cong C_2 \times S_3$ and $K \cong C_3 \times S_3$ are SD-groups where C_n is a cyclic group of order n. In fact, $\mathrm{T}(H) = 4 \cdot 1 + 2 \cdot 2 = 8$, $|H| = 12$ and $\mathrm{T}(K) = 6 \cdot 1 + 3 \cdot 2 = 12$, $|K| = 18$, so

$$\frac{\mathrm{T}(H)}{|H|} = \frac{2}{3} = \frac{\mathrm{T}(K)}{|K|}.$$

It follows that $S_3 \times S_3$ is a PSD-group but not an SD-group.

Example 2.2. Let $G = A_4$. Then, G has subgroups K_4, C_3. We see that

$$\frac{\mathrm{T}(A_4)}{|A_4|} = \frac{3 \cdot 1 + 1 \cdot 3}{12} = \frac{1}{2}, \frac{\mathrm{T}(K_4)}{|K_4|} = \frac{\mathrm{T}(C_n)}{|C_n|} = 1.$$

Thus, G is a PSD-group but not an SD-group.

Lemma 2.1. *An* SD-*group is solvable.*

Proof. Let G be an SD-group. By the definition of SD-groups, we see that

$$\mathrm{T}(G) \geq \frac{2}{3}|G| = \frac{10}{15}|G| > \frac{4}{15}|G|.$$

By Theorem 1.1 of [11], G is solvable. □

The following conclusion, which is needed for proving our main result, is due to Thompson [13].

Lemma 2.2 (Corollary 1 of [13]). *Every minimal simple group, that is, non-abelian simple groups whose all proper subgroups are solvable, is isomorphic to one of the following simple groups:*

(1) $\mathrm{PSL}_2(2^p)$ *for p a prime;*

(2) $\mathrm{PSL}_2(3^p)$ *for p an odd prime;*

(3) $\mathrm{PSL}_2(p)$, *for p any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod 5$;*

(4) $Sz(2^p)$ *for p an odd prime;*

(5) $\mathrm{PSL}_3(3)$.

3. Solvability of PSD-groups

In this section, the key is to give a proof of Theorem 1.1. Let $k(G)$ be the number of conjugate classes of G. We first give some results which are needed in the proof.

Lemma 3.1. *Let G be a dihedral group of the form D_{2n}. Then*

(1) if n is odd, then $\mathrm{T}(G) = n+1$ and $k(G) = \frac{n+3}{2}$, in particular, G is an SD*-group if $n = 3$;*

(2) if n is even, then $\mathrm{T}(G) = n+2$ and $k(G) = \frac{n+6}{2}$, in particular, G is an SD*-group if $n \in \{2, 4, 6\}$.*

Proof. (1) Now, G is a Frobenius group $C_n : C_2$ with kernel C_n and complement C_2 respectively. For any $\chi \in \mathrm{Irr}(G)$, $\chi(1) \big| |G : G'| = 2$, where G' is the derived subgroup of G, and so $\chi(1) = 1$ or 2. Observe that $G' \cong C_n$, and let m be the number of irreducible characters of G of degree 2, so $2 \cdot 1^2 + m \cdot 2^2 = 2n$ and $m = \frac{n-1}{2}$. Now,

$$\mathrm{T}(G) = 2 \cdot 1 + \frac{n-1}{2} \cdot 2 = n+1, \;\; k(G) = 2 + \frac{n-1}{2}.$$

We see $|G| = 2n$, so $\frac{\mathrm{T}(G)}{|G|} = \frac{n+1}{2n} \geq \frac{2}{3}$ if $n = 3$. Thus, D_{2n} is an SD-group if $n = 3$.

(2) $G' \cong C_{n/2}$ and $G/G' \cong C_2 \times C_2$. Now, $\chi(1)^2 \big| |G/G'| = 4$ and so $\chi(1) = 1$ or 2. Denote by m the number of irreducible characters of G of degree 2. We obtain that $4 \cdot 1^2 + m \cdot 2^2 = 2n$ and so $m = \frac{n}{2} - 1$. Thus,

$$\mathrm{T}(G) = 4 \cdot 1 + (\frac{n}{2} - 1) \cdot 2 = 4 + n - 2 = n + 2, k(G) = 4 + \frac{n}{2} - 1 = 3 + \frac{n}{2}.$$

It follows from $|G| = 2n$ that $\frac{\mathrm{T}(G)}{|G|} = \frac{n+2}{2n} \geq \frac{2}{3}$ if $n = 2$, 4 or 6. $\square$

Let $\pi(n)$ denote the set of all prime divisors of n.

Lemma 3.2. *Let G be a Frobenius group $E_p : C_q$, where p is a prime power, with kernel E_p, the elementary abelian $\pi(p)$-group of order p and complement C_q respectively. Then, $\mathrm{T}(G) = p + q - 1$ and $k(G) = q + \frac{p-1}{q}$, in particular, G is an* SD*-group if $p = 2$ and $q = 3$.*

Proof. By [6, Theorem 13.8],

$$k(G) = k(C_q) + \frac{k(E_p) - 1}{|C_q|} = q + \frac{p-1}{q}$$

and

$$\mathrm{T}(G) = k(C_q) \cdot 1 + \frac{k(E_p) - 1}{|C_q|} \cdot |C_q| = q + (p-1).$$

Now, $\frac{\mathrm{T}(G)}{|G|} = \frac{1}{p} + \frac{1}{q} - \frac{1}{pq} \geq \frac{2}{3}$ if G is an SD-group. If $p > 2$ and $q > 2$, then as $(p, q) = 1$, we have that

$$\frac{6}{12} = \frac{1}{3} + \frac{1}{4} - \frac{1}{3 \cdot 4} > \frac{1}{p} + \frac{1}{q} - \frac{1}{pq} \geq \frac{2}{3},$$

a contradiction. This forces one of p and q to be 2. Since G is a Frobenius group, $p \neq 2$ and so $q = 2$. Let p be an odd number ≥ 5. Similarly, we obtain that

$$\frac{6}{10} = \frac{1}{2} + \frac{1}{5} - \frac{1}{2 \cdot 5} > \frac{1}{p} + \frac{1}{q} - \frac{1}{pq} \geq \frac{2}{3},$$

a contradiction. This forces $p = 3$ and $q = 2$. □

Let H^n denote the product of n times H and $f(G) = \frac{\mathrm{T}(G)}{|G|}$.

Lemma 3.3. *Let H be a group. Then, $f(H^2) = f(H)^2$.*

Proof. As $f(G) = \frac{\mathrm{T}(G)}{|G|}$, from Theorem 4.21 of [9] we have

$$\begin{aligned}
f(H^2) = \frac{\mathrm{T}(H^2)}{|H^2|} &= \frac{\sum_{i,j}(\chi_i \times \chi_j)(1)}{|H^2|} \\
&= \frac{\sum_{i,j} \chi_i(1)\chi_j(1)}{|H|^2} \\
&= \frac{\sum_i \chi_i(1) \left(\chi_1(1) + \chi_2(1) + \cdots + \chi_{k(H)}(1)\right)}{|H|^2} \\
&= \frac{\sum_i \chi_i(1)\mathrm{T}(H)}{|H|^2} \\
&= \frac{\mathrm{T}(H) \sum_i \chi_i(1)}{|H|^2} \\
&= \frac{\mathrm{T}(H)^2}{|H|^2} = f(H)^2.
\end{aligned}$$

The proof is complete. □

Let $A \curlyvee B$ be central product of two groups A and B (see [15, p. 75]). We rewrite Theorem 1.1 here.

Theorem 3.1. *A finite* PSD*-group is solvable.*

Proof. By way of contradiction, assume that G is a non-solvable PSD-group. Then, G has two normal subgroups H, N such that H/N is a product of isomorphic non-abelian simple groups. By Lemma 2.1, all proper subgroups of G are solvable, since G is a PSD-group. It follows that $G/N = H/N$, and G/N is a non-abelian simple group. Hence, G/N is a minimal simple group.

By Lemma 2.2, there are five possibilities for G/N. Note that (1), (2) and (3) of Lemma 2.2 can be treated together as follows

Table 1: $\mathrm{PSL}_2(q)$, $q \geq 5$ [8, Chap II, Theorem 8.27]

	$\max \mathrm{PSL}_2(q)$	Condition
$\mathcal{C}_1$	$E_q : C_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$\mathcal{C}_2$	$D_{2(q-1)/k}$	$q \notin \{5, 7, 9, 11\}$
C_3	$D_{2(q+1)/k}$	$q \notin \{7, 9\}$
$\mathcal{C}_5$	$\mathrm{PSL}_2(q_0).(k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
$\mathcal{C}_6$	S_4	$q = p \equiv \pm 1 \pmod 8$
	A_4	$q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$
$\mathcal{S}$	A_5	$q \equiv \pm 1 \pmod{10}, F_q = F_p[\sqrt{5}]$

Case 1. $G/N \cong \mathrm{PSL}_2(q)$ for q being a prime power and $q > 3$.

In this case, $G'/N \cong \mathrm{PSL}_2(q)$ and the Schur multiplier of $\mathrm{PSL}_2(q)$ has order at most two. So, $[G', N] \leqslant C_2$. Consequently, $G \cong N \times \mathrm{PSL}_2(q)$ or $N \curlyvee \mathrm{SL}_2(q)$. Since all proper subgroups of G are solvable, $G \cong \mathrm{PSL}_2(q)$ or $\mathrm{SL}_2(q)$.

Let $G \cong \mathrm{PSL}_2(q)$. First, suppose that $G \cong \mathrm{PSL}_2(2^2) \cong A_5$. There exists $H \in \max G$ such that $H \cong A_4$. Note that H is a Frobenius group $E_4 : C_3$. By Lemma 3.2, H is not an SD-group, a contradiction. This implies that $G \cong \mathrm{PSL}_2(q)$ with $q \geq 5$. From Table 1, $E_q : C_{(q-1)/k} \in \max(G)$ with $k = 1$ or 2. Hence, $E_q : C_{(q-1)/k}$ is an SD-group. By Lemma 3.2, $q = 3$, a contradiction.

Therefore, $G \cong \mathrm{SL}_2(q)$. Note that, in this case, we can assume that q is odd since $\mathrm{SL}_2(2^n) \cong \mathrm{PSL}_2(2^n)$. From [3, p. 377], $E_q : C_{q-1} \in \mathbf{prop}(G)$. This implies that $E_q : C_{q-1}$ is an SD-group. By Lemma 3.2, $q = 3$, a contradiction.

Case 2. $G/N \cong Sz(2^p)$ for p an odd prime.

From [5, p. xvi], the Schur multiplier of $Sz(2^p)$ has trivial order except for $p = 3$ and its order is 4 if $p = 3$. If $p > 3$, then $G \cong N \times Sz(2^p)$. From [3, p. 385], $D_{2(2^p-1)} \in \max Sz(2^p)$. Hence, $D_{2(2^p-1)}$ is an SD-group. By Lemma 3.1, $2^p - 1 = 3$, a contradiction, since p is odd. Therefore, $p = 3$. From [5, p. 28], $G \cong N \times Sz(8)$, $N \curlyvee 2.Sz(8)$ or $N \curlyvee 4.Sz(8)$. Since all proper subgroups of G are solvable, $G \cong Sz(8)$, $2.Sz(8)$ or $4.Sz(8)$.

Assume that $G \cong Sz(8)$. From [3, p. 385], $D_{2(2^3-1)} \in \max Sz(8)$. $D_{2(2^3-1)}$ is an SD-group. By Lemma 3.1, $2^3 - 1 = 3$, a contradiction. Therefore, $G \cong 2.Sz(8)$ or $4.Sz(8)$. Let $G \cong 2.Sz(8)$. Then, it follows from [5, p. 28] that $E_5 : C_4 \in \mathbf{prop}(2.Sz(8))$. From Lemma 3.2, $E_5 : C_4$ is a non-SD-group, a contradiction. Similarly, we can exclude the case $G \cong 4.Sz(8)$.

Case 3. $G/N \cong \mathrm{PSL}_3(3)$.

From [5, p. 13], the Schur multiplier of $\mathrm{PSL}_3(3)$ has trivial order. Then, we have that $G \cong N \times \mathrm{PSL}_3(3)$. It follows from [5, p. 13] that $S_4 \in \max \mathrm{PSL}_3(3)$. Hence, S_4 is an SD-group. However, $\mathrm{T}(S_4) = 1 + 1 + 2 + 3 + 3 = 10$, and so $\frac{\mathrm{T}(S_4)}{|S_4|} = \frac{10}{24} = \frac{5}{12} \not> \frac{8}{12} = \frac{2}{3}$. This implies that S_4 is a non-SD-group, a contradiction.

Hence, G is solvable. □

4. Conclusion

In this paper, we investigate the influence of sums of all character degrees of all proper subgroups of a finite group on its group structure, with a focus on the solvability of groups satisfying specific conditions. By defining SD-groups and PSD-groups, we prove that a finite PSD-group must be solvable. This work extends the approach of characterizing group structures through the sum of character degrees by incorporating the influence of character degree sums of proper subgroups. Future work may explore the implications of different thresholds, such as other values of c in $\mathrm{T}(H) \leq c|H|$, on group structures, or analyze broader classes of groups, such as nilpotent and supersolvable groups.

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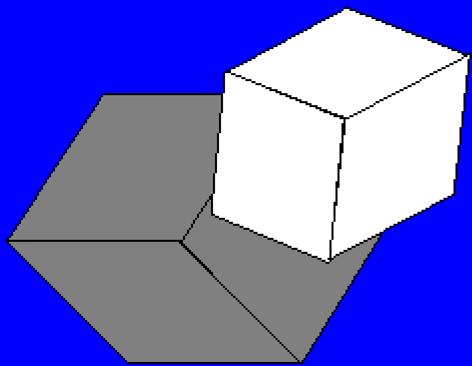

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