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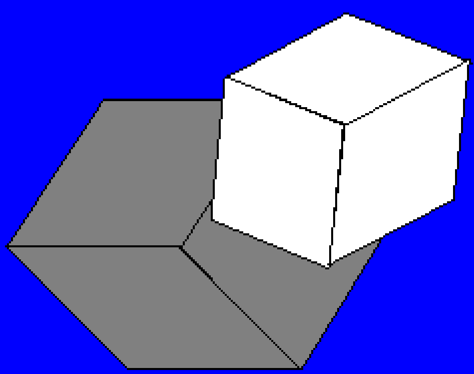

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Recursion formulas for Humbert's matrix functions

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Abstract. Special matrix functions have become a major area of study for mathematicians and physicists over the last two decades. The famous Humbert's matrix functions have received considerable attention by many authors from different points of view [5, 16, 24]. Inspired by the recent work by Abd-Elmageed *et.al.* [1], who established recursion formulas satisfied by the first Appell matrix function, namely F_1. In this paper, we find the recursion formulas for Humbert's matrix functions. This enriches the theory of special matrix functions. The obtained results are believed to be newly presented.

Keywords: Matrix functional calculus, Recursion formula, Humbert's matrix function.

1. Introduction

The theory of special functions and its generalisations appear frequently in physics, probability theory, engineering, and Lie theory, amongst other fields. Recursion formulas for the Appell functions have been studied in the literature, see [17, 28]. Recursion formulas forl multivariable hypergeometric functions were presented in [3, 19, 20, 21, 22]. Humbert's functions constitute a set of seven hy-

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pergeometric functions of two variables that are confluent cases of two variable Appell hypergeometric functions and generalize the Kummer's confluent hypergeometric function ${}_1F_1$ of one variable. The class of classical Humbert functions has been recently studied for reduction and summation formulas [4, 6, 25].

The matrix theory is used in orthogonal polynomials and special functions, and it is widely used in mathematics in general. Due to their applications in physics, engineering, probability theory, and Lie theory, special matrix functions have received a lot of attention [7, 10]. Special matrix functions connected to the matrix version of Laguerre, Hermite and Legendre differential equations and the corresponding polynomial families in [11, 12, 13]. Recently, Abd-Elmageed *et. al.* and Verma [1, 26] have obtained recursion formulas satisfied by the first Appell matrix function, namely F_1 and Srivastava's triple hypergeometric matrix functions. In [23, 27], recursion formulas for the Gauss hypergeometric matrix function and Lauricella matrix functions are presented. Motivated by this study, we obtain recursion formulas for Humbert's matrix functions.

The paper is organized as follows. In Section 2, we give a review of basic definitions that are needed in the sequel. In Section 3, we obtain the recursion formulas for Humbert's matrix function.

2. Preliminaries

Let $\mathbb{C}^{r\times r}$ be the vector space of r-square matrices with complex entries. For any matrix $A \in \mathbb{C}^{r\times r}$, its spectrum $\sigma(A)$ is the set of eigenvalues of A. The spectral abscissa of A is given by $\alpha(A)=$ max $[\Re(z)|z \in \sigma(A)]$, where $\Re(z)$ denotes the real part of a complex number z. If $\beta(A)=$ min $[\Re(z)|z \in \sigma(A)]$, then $\beta(A) = -\alpha(-A)$. A square matrix A in $\mathbb{C}^{r\times r}$ is said to be positive stable if $\beta(A) > 0$. The 2-norm of A is denoted by $||A||$ and defined by

$$||A|| = \max_{x\neq 0}\frac{||Ax||_2}{||x||_2} = \max[\sqrt{(}\lambda)|\lambda \in (A^\star A)], \tag{1}$$

where for any vector x in the r-dimensional complex space, $||x||_2 = (x^\star x)^{\frac{1}{2}}$ is the Euclidean norm of x and $A^\star$ denotes the transposed conjugate of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [8], it follows that

$$f(A)g(A) = g(A)f(A). \tag{2}$$

Furthermore, if $B \in \mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B) \subset \Omega$, and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{3}$$

If A is a positive stable matrix in $\mathbb{C}^{r\times r}$, then $\Gamma(A)$ can be expressed as [15]

$$\Gamma(A) = \int_0^\infty e^{-t}\, t^{A-I}\, dt, \tag{4}$$

where, $t^{A-I} = \exp((A-I)\ln t)$ and ln is the principal branch of the logarithmic function.

Furthermore, if $A+kI$ is invertible for all integers $k \geq 0$, then the reciprocal gamma matrix function is defined as [15]

$$\Gamma^{-1}(A) = A(A+I)\dots(A+(n-1)I)\Gamma^{-1}(A+nI), \ n \geq 1. \tag{5}$$

By application of the matrix functional calculus, the Pochhammer symbol for $A \in \mathbb{C}^{r\times r}$ is given by [15]

$$(A)_n = \begin{cases} I, & \text{if } n = 0, \\ A(A+I)\dots(A+(n-1)I), & \text{if } n \geq 1. \end{cases} \tag{6}$$

This gives

$$(A)_n = \Gamma^{-1}(A)\ \Gamma(A+nI), \qquad n \geq 1. \tag{7}$$

Humbert's matrix functions are defined as follows [2, 5, 18]:

$$\Phi_1(A,B;C;x,y) = \sum_{m,n=0}^{\infty} (A)_{m+n}(B)_n(C)^{-1}_{m+n}\,\frac{x^m\,y^n}{m!\,n!}, \tag{8}$$

$$\Phi_2(A,A';C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(A')_n(C)^{-1}_{m+n}\frac{x^m\,y^n}{m!\,n!}, \tag{9}$$

$$\Phi_3(A;C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(C)^{-1}_{m+n}\,\frac{x^m\,y^n}{m!\,n!}, \tag{10}$$

$$\Psi_1(A,B;C,C';x,y) = \sum_{m,n=0}^{\infty} (A)_{m+n}(B)_m(C)^{-1}_m(C')^{-1}_n\,\frac{x^m\,y^n}{m!\,n!}, \tag{11}$$

$$\Psi_2(A;C,C';x,y) = \sum_{m,n=0}^{\infty} (A)_{m+n}(C)^{-1}_m(C')^{-1}_n\,\frac{x^m\,y^n}{m!\,n!}, \tag{12}$$

$$\Xi_1(A,A',B;C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(A')_n(B)_m(C)^{-1}_{m+n}\frac{x^m\,y^n}{m!\,n!}, \tag{13}$$

$$\Xi_2(A,B;C;x,y) = \sum_{m,n=0}^{\infty} (A)_m(B)_m(C)^{-1}_{m+n}\frac{x^m\,y^n}{m!\,n!}, \tag{14}$$

where A, A', B, C and C' are matrices in $\mathbb{C}^{r\times r}$ such that $C+kI$ and $C'+kI$ are invertible for all integers $k \geq 0$.

3. Recursion formulas for Humbert's matrix functions

In this section, we obtain the recursion formulas for Humbert's matrix functions. Throughout the paper, I denotes the identity matrix and s denotes a non-negative integer.

Theorem 3.1. *Let $A+sI$ be an invertible matrix for all integers $s \geq 0$ and $BC = CB$. Then the following recursion formula holds true for Humbert's matrix function Φ_1:*

$$\begin{aligned}
&\Phi_1(A+sI,B;C;x,y)\\
&=\Phi_1(A,B;C;x,y)+x\left[\sum_{k=1}^{s}\Phi_1(A+kI,B;C+I;x,y)\right]C^{-1}\\
&+y\left[\sum_{k=1}^{s}\Phi_1(A+kI,B+I;C+I;x,y)\right]BC^{-1},
\end{aligned}\tag{15}$$

$$\begin{aligned}
&\Phi_1(A+sI,B;C;x,y)\\
&=\sum_{k_1+k_2\leq s}\binom{s}{k_1,k_2}x^{k_1}y^{k_2}\\
&\times\Big[\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)\Big](B)_{k_2}(C)^{-1}_{k_1+k_2}.
\end{aligned}\tag{16}$$

Furthermore, if $A-kI$ is invertible for integers $k \leq s$, then

$$\begin{aligned}
&\Phi_1(A-sI,B;C;x,y)\\
&=\Phi_1(A,B;C;x,y)-x\left[\sum_{k=0}^{s-1}\Phi_1(A-kI,B;C+I;x,y)\right]C^{-1}\\
&-y\left[\sum_{k=0}^{s-1}\Phi_1(A-kI,B+I;C+I;x,y)\right]BC^{-1},
\end{aligned}\tag{17}$$

$$\begin{aligned}
&\Phi_1(A-sI,B;C;x,y)\\
&=\sum_{k_1+k_2\leq s}\binom{s}{k_1,k_2}(-x)^{k_1}(-y)^{k_2}\\
&\times\Big[\Phi_1(A,B+k_2I;C+(k_1+k_2)I;x,y)\Big](B)_{k_2}(C)^{-1}_{k_1+k_2},
\end{aligned}\tag{18}$$

where $\binom{s}{k_1,k_2}=\frac{s!}{k_1!k_2!(s-k_1-k_2)!}$.

Proof. From the definition of Humbert's matrix function Φ_1 and the transformation

$$(A+I)_{m+n}=A^{-1}(A)_{m+n}\,(A+mI+nI)$$

we get the following contiguous matrix relation:

$$\begin{aligned}
&\Phi_1(A+I,B;C;x,y)\\
&=\Phi_1(A,B;C;x,y)+x\Big[\Phi_1(A+I,B;C+I;x,y)\Big]C^{-1}\\
&+y\Big[\Phi_1(A+I,B+I;C+I;x,y)\Big]BC^{-1}.
\end{aligned}\tag{19}$$

To calculate contiguous matrix relation for $\Phi_1(A+2I, B, B'; C; x, y)$, we replace A with $A+I$ in (19) and use in (19). This gives

$$\begin{aligned}&\Phi_1(A+2I, B; C; x, y) = \Phi_1(A, B; C; x, y)\\&+ x\Big[\Phi_1(A+I, B; C+I; x, y) + \Phi_1(A+2I, B; C+I; x, y)\Big]C^{-1}\\&+ y\Big[\Phi_1(A+I, B+I; C+I; x, y)+\Phi_1(A+2I, B+I; C+I; x, y)\Big]BC^{-1}.\end{aligned} \tag{20}$$

Iterating this process s times, we obtain (15). For the proof of (17), replace the matrix A with $A-I$ in (19). As $A-I$ is invertible, this gives

$$\begin{aligned}&\Phi_1(A-I, B; C; x, y) = \Phi_1(A, B; C; x, y) - x\Big[\Phi_1(A, B; C+I; x, y)\Big]C^{-1}\\&- y\Big[\Phi_1(A, B+I; C+I; x, y)\Big]BC^{-1}.\end{aligned} \tag{21}$$

Iteratively, we get (17).

The proof of (16) is based upon the principle of mathematical induction on $s \in \mathbb{N}$. For $s = 1$, the result (16) is true obviously. Suppose (16) is true for $s = t$, that is,

$$\begin{aligned}&\Phi_1(A+tI, B; C; x, y) = \sum_{k_1+k_2 \le t} \binom{t}{k_1, k_2} x^{k_1} y^{k_2}\\&\times \Phi_1(A+(k_1+k_2)I, B+k_2I; C+(k_1+k_2)I; x, y)(B)_{k_2}(C)^{-1}_{k_1+k_2},\end{aligned} \tag{22}$$

Replacing A with $A+I$ in (22) and using the contiguous matrix relation (19), we get

$$\begin{aligned}&\Phi_1(A+tI+I, B; C; x, y) = \sum_{k_1+k_2 \le t} \binom{t}{k_1, k_2} x^{k_1} y^{k_2}\\&\times \Big[\Phi_1(A+(k_1+k_2)I, B+k_2I; C+(k_1+k_2)I; x, y) + x\\&\times \Phi_1(A+(k_1+k_2)I+I, B+k_2I; C+(k_1+k_2)I+I; x, y)(C+(k_1+k_2)I)^{-1}\\&\times +y\Phi_1(A+(k_1+k_2)I+I, B+k_2I+I; C+(k_1+k_2)I+I; x, y)\\&\times (B+k_2I)(C+(k_1+k_2)I)^{-1}\Big](B)_{k_2}(C)^{-1}_{k_1+k_2}.\end{aligned} \tag{23}$$

Simplifying, (23) takes the form

$$\begin{aligned}&\Phi_1(A+tI+I, B; C; x, y)\\&= \sum_{k_1+k_2 \le t} \binom{t}{k_1, k_2} x^{k_1} y^{k_2}\\&\times \Phi_1(A+(k_1+k_2)I, B+k_2I; C+(k_1+k_2)I; x, y)(B)_{k_2}(C)^{-1}_{k_1+k_2}\\&+ \sum_{k_1+k_2 \le t+1} \binom{t}{k_1-1, k_2} x^{k_1} y^{k_2}\\&\times \Phi_1(A+(k_1+k_2)I, B+k_2I; C+(k_1+k_2)I; x, y)(B)_{k_2}(C)^{-1}_{k_1+k_2}\end{aligned}$$

$$+\sum_{k_1+k_2\leq t+1}\binom{t}{k_1,k_2-1}x^{k_1}y^{k_2}$$

$$\times\,\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)(B)_{k_2}(C)^{-1}_{k_1+k_2}. \tag{24}$$

Using Pascal's identity in (24), we have

$$\Phi_1(A+(t+1)I,B;C;x,y)=\sum_{k_1+k_2\leq t+1}\binom{t+1}{k_1,k_2}x^{k_1}y^{k_2}$$

$$\times\,\Phi_1(A+(k_1+k_2)I,B+k_2I;C+(k_1+k_2)I;x,y)(B)_{k_2}(C)^{-1}_{k_1+k_2}. \tag{25}$$

This establishes (16) for $s=t+1$. Hence by induction, result given in (16) is true for all values of s. The second recursion formula (18) can be proved in a similar manner. □

Now, we present the recursion formulas for the matrix B of the Humbert's matrix function Φ_1. We omit the proofs of the given below theorems.

Theorem 3.2. *Let $B+sI$ be invertible matrix for all integers $s\geq 0$. Then the following recursion formulas hold true for Humbert's matrix function Φ_1:*

$$\Phi_1(A,B+sI;C;x,y)$$

$$=\Phi_1(A,B;C;x,y)+yA\Big[\sum_{k=1}^{s}\Phi_1(A+I,B+kI;C+I;x,y)\Big]C^{-1}, \tag{26}$$

$$\Phi_1(A,B-sI;C;x,y)$$

$$=\Phi_1(A,B;C;x,y)-yA\Big[\sum_{k=0}^{s-1}\Phi_1(A+I,B-kI;C+I;x,y)\Big]C^{-1}. \tag{27}$$

Theorem 3.3. *Let $B+sI$ be invertible matrix for all integers $s\geq 0$ then the following recursion formulas hold true for Humbert's matrix function Φ_1:*

$$\Phi_1(A,B+sI;C;x,y)$$

$$=\sum_{k_1=0}^{s}\binom{s}{k_1}(A)_{k_1}\,y^{k_1}\Big[\ \Phi_1(A+k_1I,B+k_1I;C+k_1I;x,y)\Big](C)^{-1}_{k_1}. \tag{28}$$

Furthermore, if $B-kI$ are invertible for $k\leq s$, then

$$\Phi_1(A,B-sI;C;x,y)$$

$$=\sum_{k_1=0}^{s}\binom{s}{k_1}(A)_{k_1}\,(-y)^{k_1}\Big[\Phi_1(A+k_1I,B;C+k_1I;x,y)\Big](C)^{-1}_{k_1}. \tag{29}$$

Theorem 3.4. *Let $C - sI$ be an invertible matrix for all integers $s \geq 0$ and let $AB = BA$, then the following recursion formula holds true for Humbert's matrix function Φ_1:*

$$\begin{aligned}
&\Phi_1(A, B; C - sI; x, y) = \Phi_1(A, B; C; x, y)\\
&+ xA \sum_{k=1}^{s} \Big[\Phi_1(A + I, B; C + (2 - k)I; x, y)\Big]\\
&\times (C - kI)^{-1}(C - (k - 1)I)^{-1}\\
&+ yAB \sum_{k=1}^{s} \Big[\Phi_1(A + I, B + I; C + (2 - k)I; x, y)\Big]\\
&\times (C - kI)^{-1}(C - (k - 1)I)^{-1}. \qquad (30)
\end{aligned}$$

Proof. Applying the definition of Humbert's matrix function Φ_1 and the relation $(C - I)^{-1}_{m+n} = (C)^{-1}_{m+n}\left[1 + m(C - I)^{-1} + n(C - I)^{-1}\right]$, we obtain the following contiguous matrix relation:

$$\begin{aligned}
&\Phi_1(A, B, B'; C - I; x, y)\\
&= \Phi_1(A, B, B'; C; x, y) + xA\Big[\Phi_1(A + I, B; C + I; x, y)\Big]C^{-1}(C - I)^{-1}\\
&+ yAB\Big[\Phi_1(A + I, B + I; C + I; x, y)\Big]C^{-1}(C - I)^{-1}. \qquad (31)
\end{aligned}$$

We get (30) by using this contiguous matrix relation in Humbert's matrix function Φ_1 with the matrix $C - sI$ for s times. □

We state without proofs recursion formulas for remaining Humbert's matrix functions.

Theorem 3.5. *Let $A + sI$ and $A' + sI$ be an invertible matrix for all integers $s \geq 0$. Then the following recursion formula holds true for Humbert's matrix function Φ_2:*

$$\begin{aligned}
&\Phi_2(A + sI, A'; C; x, y)\\
&= \Phi_2(A, A'; C; x, y) + x\Big[\sum_{k=1}^{s} \Phi_2(A + kI, A'; C + I; x, y)\Big]C^{-1}, \qquad (32)\\
&\Phi_2(A + sI, A'; C; x, y)\\
&= \sum_{k_1 \leq s} \binom{s}{k_1} x^{k_1}\Big[\Phi_2(A + k_1 I, A'; C + k_1 I; x, y)\Big](C)^{-1}_{k_1}, \qquad (33)\\
&\Phi_2(A, A' + sI; C; x, y)\\
&= \Phi_2(A, A'; C; x, y) + y\Big[\sum_{k=1}^{s} \Phi_2(A, A' + kI; C + I; x, y)\Big]C^{-1}, \qquad (34)
\end{aligned}$$

$$\Phi_2(A, A'+sI; C; x, y) = \sum_{k_1 \le s} \binom{s}{k_1} y^{k_1} \Big[\Phi_2(A, A'+k_1I; C+k_1I; x, y)\Big] (C)_{k_1}^{-1}. \tag{35}$$

If $A-kI$ and $A'-kI$ are invertible for $k \le s$, then

$$\Phi_2(A-sI, A'; C; x, y) = \Phi_2(A, A'; C; x, y) - x\Big[\sum_{k=0}^{s-1} \Phi_2(A-kI, A'; C+I; x, y)\Big]C^{-1}, \tag{36}$$

$$\Phi_2(A-sI, A'; C; x, y) = \sum_{k_1 \le s} \binom{s}{k_1} (-x)^{k_1} \Big[\Phi_2(A, A'; C+k_1I; x, y)\Big] (C)_{k_1}^{-1}, \tag{37}$$

$$\Phi_2(A, A'-sI; C; x, y) = \Phi_2(A, A'; C; x, y) - y\Big[\sum_{k=0}^{s-1} \Phi_2(A, A'-kI; C+I; x, y)\Big]C^{-1}, \tag{38}$$

$$\Phi_2(A, A'-sI; C; x, y) = \sum_{k_1 \le s} \binom{s}{k_1} (-y)^{k_1} \Big[\Phi_2(A, A'; C+k_1I; x, y)\Big] (C)_{k_1}^{-1}. \tag{39}$$

Theorem 3.6. *Let $C-sI$ be invertible matrices for all integers $s \ge 0$ and $AA' = A'A$. Then the following recursion formulas hold true for Humbert's matrix function Φ_2:*

$$\begin{aligned}\Phi_2(A, A'; C-sI; x, y) &= \Phi_2(A, B, B'; C, C'; x, y) + xA\Big[\sum_{k=1}^{s} \Phi_2(A+I, A'; C+(2-k)I; x, y) \\ &\times (C-kI)^{-1}(C-(k-1)I)^{-1}\Big] + yA'\Big[\sum_{k=1}^{s} \Phi_2(A, A'+I; C+(2-k)I; x, y) \\ &\times (C-kI)^{-1}(C-(k-1)I)^{-1}\Big].\end{aligned} \tag{40}$$

Theorem 3.7. *Let $A+sI$ be invertible matrices for all integers $s \ge 0$. Then the following recursion formulas hold true for Humbert's matrix function Φ_3:*

$$\Phi_3(A+sI; C; x, y) = \Phi_3(A; C; x, y) + x\Big[\sum_{k=1}^{s} \Phi_3(A+kI; C+I; x, y)\Big]C^{-1}, \tag{41}$$

$$\Phi_3(A+sI;C;x,y) = \sum_{k_1=0}^{s} \binom{s}{k_1} x^{k_1} \Big[\Phi_3(A+k_1I;C+k_1I;x,y)\Big](C)_{k_1}^{-1}. \tag{42}$$

Furthermore, if $A-kI$ and $A'-kI$ are invertible for $k \le s$, then

$$\Phi_3(A-sI;C;x,y) = \Phi_3(A;C;x,y) - x\Big[\sum_{k=0}^{s-1} \Phi_3(A-kI;C+I;x,y)\Big]C^{-1}; \tag{43}$$

$$\Phi_3(A-sI;C;x,y) = \sum_{k_1=0}^{s} \binom{s}{k_1} (-x)^{k_1} \Big[\Phi_3(A;C+k_1I;x,y)\Big](C)_{k_1}^{-1}. \tag{44}$$

Theorem 3.8. *Let $C-sI$ be an invertible matrix for all integers $s \ge 0$. Then the following recursion formula holds true for Humbert's matrix function Φ_3:*

$$\begin{aligned}\Phi_3(A;C-sI;x,y) &= \Phi_3(A;C;x,y) + xA\Big[\sum_{k=1}^{s} \Phi_3(A+I;C+(2-k)I;x,y)\Big] \\ &\quad \times (C-kI)^{-1}(C-(k-1)I)^{-1} \\ &\quad + y\Big[\sum_{k=1}^{s} \Phi_3(A;C+(2-k)I;x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big].\end{aligned} \tag{45}$$

Theorem 3.9. *Let $A+sI$ be an invertible matrix for all integers $s \ge 0$ and let $AB = BA$; $CC' = C'C$. Then the following recursion formula holds true for Humbert's matrix function Ψ_1:*

$$\begin{aligned}\Psi_1(A+sI,B;C,C';x,y) &= \Psi_1(A,B;C,C';x,y) + xB\Big[\sum_{k=1}^{s} \Psi_1(A+kI,B+I;C+I,C';x,y)\Big]C^{-1} \\ &\quad + y\Big[\sum_{k=1}^{s} \Psi_1(A+kI,B;C,C'+I;x,y)\Big]C'^{-1},\end{aligned} \tag{46}$$

$$\begin{aligned}\Psi_1(A+sI,B;C,C';x,y) &= \sum_{k_1+k_2\le s} \binom{s}{k_1,k_2} (B)_{k_1} x^{k_1} y^{k_2} \\ &\quad \times \Big[\Psi_1(A+(k_1+k_2)I,B+k_1I;C+k_1I,C'+k_2I;x,y)\Big](C)_{k_1}^{-1}(C')_{k_2}^{-1}.\end{aligned} \tag{47}$$

Furthermore, if $A-kI$ is invertible for $k \leq s$, then

$$\begin{aligned}
&\Psi_1(A-sI,B;C,C';x,y)\\
&=\Psi_1(A,B;C,C';x,y)-xB\Big[\sum_{k=0}^{s-1}\Psi_1(A-kI,B+I;C+I,C';x,y)\Big]C^{-1}\\
&-y\Big[\sum_{k=0}^{s-1}\Psi_1(A-kI,B;C,C'+I;x,y)\Big]C'^{-1}, \qquad (48)\\
&\Psi_1(A-sI,B;C,C';x,y)\\
&=\sum_{k_1+k_2\leq s}\binom{s}{k_1,k_2}(B)_{k_1}(-x)^{k_1}(-y)^{k_2}\\
&\times\Big[\Psi_1(A,B+k_1I;C+k_1I,C'+k_2I;x,y)\Big](C)^{-1}_{k_1}(C')^{-1}_{k_2}. \qquad (49)
\end{aligned}$$

Theorem 3.10. *Let $B+sI$ be an invertible matrix for all integers $s \geq 0$ and let $CC' = C'C$. Then the following recursion formula holds true for Humbert's matrix function Ψ_1:*

$$\begin{aligned}
&\Psi_1(A,B+sI;C,C';x,y)=\Psi_1(A,B;C,C';x,y)\\
&+xA\Big[\sum_{k=1}^{s}\Psi_1(A+I,B+kI;C+I,C';x,y)\Big]C^{-1}, \qquad (50)\\
&\Psi_1(A,B+sI;C,C';x,y)\\
&=\sum_{k_1+k_2\leq s}\binom{s}{k_1}(A)_{k_1}x^{k_1}\\
&\times\Big[\Psi_1(A+k_1I,B+k_1I;C+k_1I,C';x,y)\Big](C)^{-1}_{k_1}. \qquad (51)
\end{aligned}$$

Furthermore, if $B-kI$ is invertible for $k \leq s$, then

$$\begin{aligned}
&\Psi_1(A,B-sI;C,C';x,y)=\Psi_1(A,B;C,C';x,y)\\
&-xA\Big[\sum_{k=0}^{s-1}\Psi_1(A+I,B-kI;C+I,C';x,y)\Big]C^{-1}, \qquad (52)\\
&\Psi_1(A,B-sI;C,C';x,y)\\
&=\sum_{k_1+k_2\leq s}\binom{s}{k_1}(A)_{k_1}(-x)^{k_1}\\
&\times\Big[\Psi_1(A+k_1I,B;C+k_1I,C';x,y)\Big](C)^{-1}_{k_1}. \qquad (53)
\end{aligned}$$

Theorem 3.11. *Let $C-sI$ and $C'-sI$ be invertible matrices for all integers $s \geq 0$ and let $AB = BA$; $CC' = C'C$. Then following recursion formulas hold* *true for Humbert's matrix function* Ψ_1*:*

$$\begin{aligned}&\Psi_1(A,B;C-sI,C';x,y)=\Psi_1(A,B;C,C';x,y)\\&+xAB\Big[\sum_{k=1}^{s}\Psi_1(A+I,B+I;C\\&+(2-k)I,C';x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big],\end{aligned}\tag{54}$$

$$\begin{aligned}&\Psi_1(A,B;C,C'-sI;x,y)=\Psi_1(A,B;C,C';x,y)\\&+yA\Big[\sum_{k=1}^{s}\Psi_1(A+I,B;\,C,C'\\&+(2-k)I;x,y)(C'-kI)^{-1}(C'-(k-1)I)^{-1}\Big].\end{aligned}\tag{55}$$

Theorem 3.12. *Let* $A+sI$ *be an invertible matrix for all integers* $s\geq 0$ *and let* $C'C=CC'$*. Then the following recursion formula holds true for Humbert's matrix function* Ψ_2*:*

$$\begin{aligned}&\Psi_2(A+sI;C,C';x,y)\\&=\Psi_2(A;C,C';x,y)+x\Big[\sum_{k=1}^{s}\Psi_2(A+kI;C+I,C';x,y)\Big]C^{-1}\\&+y\Big[\sum_{k=1}^{s}\Psi_2(A+kI;C,C'+I;x,y)\Big]C'^{-1},\end{aligned}\tag{56}$$

$$\begin{aligned}&\Psi_2(A+sI;C,C';x,y)\\&=\sum_{k_1+k_2\leq s}\binom{s}{k_1,k_2}x^{k_1}y^{k_2}\\&\times\Big[\Psi_2(A+(k_1+k_2)I;C+k_1I,C'+k_2I;x,y)\Big](C)_{k_1}^{-1}(C')_{k_2}^{-1}.\end{aligned}\tag{57}$$

Furthermore, if $A-kI$ *is invertible for* $k\leq s$*, then*

$$\begin{aligned}&\Psi_2(A-sI;C,C';x,y)\\&=\Psi_2(A;C,C';x,y)-x\Big[\sum_{k=0}^{s-1}\Psi_2(A-kI;C+I,C';x,y)\Big]C^{-1}\\&-y\Big[\sum_{k=0}^{s-1}\Psi_2(A-kI;C,C'+I;x,y)\Big]C'^{-1},\end{aligned}\tag{58}$$

$$\begin{aligned}&\Psi_2(A-sI;C,C';x,y)\\&=\sum_{k_1+k_2\leq s}\binom{s}{k_1,k_2}(-x)^{k_1}(-y)^{k_2}\\&\times\Big[\Psi_2(A;C+k_1I,C'+k_2I;x,y)\Big](C)_{k_1}^{-1}(C')_{k_2}^{-1}.\end{aligned}\tag{59}$$

Theorem 3.13. *Let $C - sI$ and $C' - sI$ be invertible matrices for all integers $s \geq 0$. Then the following recursion formulas hold true for Humbert's matrix function Ψ_2:*

$$\begin{aligned}
&\Psi_2(A; C - sI, C'; x, y) = \Psi_2(A; C, C'; x, y)\\
&+ xA\Big[\sum_{k=1}^{s} \Psi_2(A + I; C\\
&+ (2-k)I, C'; x, y)(C - kI)^{-1}(C - (k-1)I)^{-1}\Big], C'C = CC', \qquad (60)\\
&\Psi_2(A; C, C' - sI; x, y) = \Psi_2(A; C, C'; x, y)\\
&+ yA\Big[\sum_{k=1}^{s} \Psi_2(A + I;\, C, C'\\
&+ (2-k)I; x, y)(C' - kI)^{-1}(C' - (k-1)I)^{-1}\Big]. \qquad (61)
\end{aligned}$$

Theorem 3.14. *Let $A + sI$ be an invertible matrix for all integers $s \geq 0$ and let $BC = CB$. Then the following recursion formula holds true for Humbert's matrix function Ξ_1:*

$$\begin{aligned}
&\Xi_1(A + sI, A', B; C; x, y)\\
&= \Xi_1(A, A', B; C; x, y) + x\Big[\sum_{k=1}^{s} \Xi_1(A + kI, A', B + I; C + I; x, y)\Big] BC^{-1}, \qquad (62)\\
&\Xi_1(A + sI, A', B; C; x, y)\\
&= \sum_{k_1 \leq s} \binom{s}{k_1} x^{k_1} \Big[\Xi_1(A + k_1I, A', B + k_1I; C + k_1I; x, y)\Big] (B)_{k_1}(C)^{-1}_{k_1}. \qquad (63)
\end{aligned}$$

Furthermore, if $A - kI$ is invertible for $k \leq s$, then

$$\begin{aligned}
&\Xi_1(A - sI, A', B; C; x, y)\\
&= \Xi_1(A, A', B; C; x, y) - x\Big[\sum_{k=0}^{s-1} \Xi_1(A - kI, A', B + I; C + I; x, y)\Big] BC^{-1}, \qquad (64)\\
&\Xi_1(A - sI, A', B; C; x, y)\\
&= \sum_{k_1 \leq s} \binom{s}{k_1} (-x)^{k_1} \Big[\Xi_1(A, A', B + k_1I; C + k_1I; x, y)\Big] (B)_{k_1}(C)^{-1}_{k_1}. \qquad (65)
\end{aligned}$$

Theorem 3.15. *Let $A' + sI$ be an invertible matrix for all integers $s \geq 0$. Then the following recursion formula holds true for Humbert's matrix function Ξ_1:*

$$\begin{aligned}
&\Xi_1(A, A' + sI, B; C; x, y)\\
&= \Xi_1(A, A', B; C; x, y) + y\Big[\sum_{k=1}^{s} \Xi_1(A, A' + kI, B; C + I; x, y)\Big] C^{-1}, \qquad (66)
\end{aligned}$$

$$\begin{aligned}
&\Xi_1(A, A'+sI, B; C, C'; x, y)\\
&= \sum_{k_1 \le s} \binom{s}{k_1} y^{k_1} \Big[\Xi_1(A, A'+k_1 I, B; C+k_1 I; x, y)\Big] (C)_{k_1}^{-1}.
\end{aligned} \tag{67}$$

Furthermore, if $A' - kI$ is invertible for $k \le s$, then

$$\begin{aligned}
&\Xi_1(A, A'-sI, B; C; x, y)\\
&= \Xi_1(A, A', B; C; x, y) - y\Big[\sum_{k=0}^{s-1} \Xi_1(A, A'-kI, B; C+I; x, y)\Big] C^{-1},
\end{aligned} \tag{68}$$

$$\begin{aligned}
&\Xi_1(A, A'-sI, B; C, C'; x, y)\\
&= \sum_{k_1 \le s} \binom{s}{k_1} (-y)^{k_1} \Big[\Xi_1(A, A', B; C+k_1 I; x, y)\Big] (C)_{k_1}^{-1}.
\end{aligned} \tag{69}$$

Theorem 3.16. *Let $B+sI$ be an invertible matrix for all integers $s \ge 0$. Then the following recursion formula holds true for Humbert's matrix function Ξ_1:*

$$\begin{aligned}
&\Xi_1(A, A', B+sI; C; x, y)\\
&= \Xi_1(A, A', B; C; x, y) + xA\Big[\sum_{k=1}^{s} \Xi_1(A+I, A', B+kI; C+I; x, y)\Big] C^{-1},
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\Xi_1(A, A', B+sI; C, C'; x, y)\\
&= \sum_{k_1 \le s} \binom{s}{k_1} x^{k_1} (A)_{k_1} \Big[\Xi_1(A+k_1 I, A', B+k_1 I; C+k_1 I; x, y)\Big] (C)_{k_1}^{-1}.
\end{aligned} \tag{71}$$

Furthermore, if $B - kI$ is invertible for $k \le s$, then

$$\begin{aligned}
&\Xi_1(A, A', B-sI; C; x, y)\\
&= \Xi_1(A, A', B; C; x, y) - xA\Big[\sum_{k=0}^{s-1} \Xi_1(A+I, A', B+kI; C+I; x, y)\Big] C^{-1},
\end{aligned} \tag{72}$$

$$\begin{aligned}
&\Xi_1(A, A', B-sI; C, C'; x, y)\\
&= \sum_{k_1 \le s} \binom{s}{k_1} (-x)^{k_1} (A)_{k_1} \Big[\Xi_1(A+k_1 I, A', B; C+k_1 I; x, y)\Big] (C)_{k_1}^{-1}.
\end{aligned} \tag{73}$$

Theorem 3.17. *Let $C - sI$ be an invertible matrix for all integers $s \ge 0$ and let $AA' = A'A$; $BC = CB$. Then the following recursion formula holds true for*

Humbert's matrix function Ξ_1*:*

$$\begin{aligned}
&\Xi_1(A,A',B;C-sI;x,y)\\
&=\Xi_1(A,A',B;C;x,y)+xA\Big[\sum_{k=1}^{s}\Xi_1(A+I,A',B+I;C+(2-k)I;x,y)\Big]\\
&\times B(C-kI)^{-1}(C-(k-1)I)^{-1}\\
&+yA'\Big[\sum_{k=1}^{s}\Xi_1(A,A'+I,B;C+(2-k)I;x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big]. \qquad (74)
\end{aligned}$$

Theorem 3.18. *Let* $A+sI$ *and* $B+sI$ *be an invertible matrix for all integers* $s\geq 0$ *and let* $AB=BA$*. Then the following recursion formula holds true for Humbert's matrix function* Ξ_2*:*

$$\begin{aligned}
&\Xi_2(A+sI,B;C;x,y)\\
&=\Xi_2(A,B;C;x,y)+xB\Big[\sum_{k=1}^{s}\Xi_2(A+kI,B+I;C+I;x,y)\Big]C^{-1}, \qquad (75)\\
&\Xi_2(A,B+sI;C;x,y)\\
&=\Xi_2(A,B;C;x,y)+xA\Big[\sum_{k=1}^{s}\Xi_2(A+I,B+kI;C+I;x,y)\Big]C^{-1}, \qquad (76)\\
&\Xi_2(A+sI,B;C;x,y)\\
&=\sum_{k_1\leq s}\binom{s}{k_1}x^{k_1}(B)_{k_1}\left[\Xi_2(A+k_1I,B+k_1I;C+k_1I;x,y)\right](C)^{-1}_{k_1}, \qquad (77)\\
&\Xi_2(A,B+sI;C,C';x,y)\\
&=\sum_{k_1\leq s}\binom{s}{k_1}x^{k_1}(A)_{k_1}\left[\Xi_2(A+k_1I,B+k_1I;C+k_1I;x,y)\right](C)^{-1}_{k_1}. \qquad (78)
\end{aligned}$$

Furthermore, if $A-kI$ *and* $B-kI$ *is invertible for* $k\leq s$*, then*

$$\begin{aligned}
&\Xi_2(A-sI,B;C;x,y)\\
&=\Xi_2(A,B;C;x,y)-xB\Big[\sum_{k=0}^{s-1}\Xi_2(A-kI,B+I;C+I;x,y)\Big]C^{-1}, \qquad (79)\\
&\Xi_2(A,B-sI;C;x,y)\\
&=\Xi_2(A,B;C;x,y)-xA\Big[\sum_{k=0}^{s-1}\Xi_2(A+I,B-kI;C+I;x,y)\Big]C^{-1}, \qquad (80)
\end{aligned}$$

$$\begin{aligned}
&\Xi_2(A-sI,B;C;x,y)\\
&=\sum_{k_1\le s}\binom{s}{k_1}(-x)^{k_1}\,(B)_{k_1}\Big[\Xi_2(A,B+k_1I;C+k_1I;x,y)\Big](C)_{k_1}^{-1},
\end{aligned}\tag{81}$$

$$\begin{aligned}
&\Xi_2(A,B-sI;C;x,y)\\
&=\sum_{k_1\le s}\binom{s}{k_1}(-x)^{k_1}\,(A)_{k_1}\Big[\Xi_2(A+k_1I,B;C+k_1I;x,y)\Big](C)_{k_1}^{-1}.
\end{aligned}\tag{82}$$

Theorem 3.19. *Let $C-sI$ be an invertible matrix for all integers $s\ge 0$ and $AB=BA$. Then the following recursion formula holds true for Humbert's matrix function Ξ_2:*

$$\begin{aligned}
&\Xi_2(A,B;C-sI;x,y)\\
&=\Xi_2(A,B;C;x,y)+xAB\Big[\sum_{k=1}^{s}\Xi_2(A+I,B+I;C+(2-k)I;x,y)\\
&\times(C-kI)^{-1}(C-(k-1)I)^{-1}\Big]\\
&+y\Big[\sum_{k=1}^{s}\Xi_2(A,B;C+(2-k)I;x,y)(C-kI)^{-1}(C-(k-1)I)^{-1}\Big].
\end{aligned}\tag{83}$$

4. Conclusion

We have studied the recursion formulas for Humbert's matrix function. These matrix formulas will contribute to the literature on special function theory and have the potential to find new applications in mathematics and physics.

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Extended of generalized power series reversible rings

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Abstract. Let R be a ring and $(S, \leq)$ a strictly ordered monoid. This paper aims to introduce and study generalized power series *nil*-reversible rings. The researchers obtains various necessary or sufficient conditions for a generalized power series *nil*-reversible rings are 2-primal, *nil*-semicommutative and *nil*-symmetric. Examples are given to show that, a generalized power series *nil*-reversible which is neither generalized power series semicommutative nor generalized power series reversible. Also, we proved that a multiplicatively closed subset of R consisting of central non-zero divisors is generalized power series *nil*-reversible if and only if R is generalized power series *nil*-reversible. Moreover, other standard ring-theoretic properties are given.

Keywords: Armendariz ring, generalized power series reversible, ordered monoid $(S, \leq)$, semicommutative ring.
MSC 2020: 13M99, 13B10, 16S36

1. Introduction

Throughout this paper, any ring is associative and has an identity unless stated. We write $P(R)$, $nil(R)$, $Mat_n(R)$, $T_n(R,)$ $S_n(R)$ and $R[x]$ respectively for the prime radical, the set of all nilpotent elements of R, the ring of all $n \times n$ matrices, the ring of all $n \times n$ upper triangular matrices for a positive integer n with entries in R, the subring consisting of all upper triangular matrices over a ring R and the polynomial ring over a ring R.

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In [1], Cohn introduced the notion of a reversible ring. A ring R is said to be reversible, if whenever $a, b \in R$ satisfy $ab = 0$, then $ba = 0$. Anderson-Camillo [2] used the term ZC_2 for what is called reversible. While Krempa-Niewieczerzal [3] took the term C_0 for it.

In [4], a ring R is called semicommutative if for all $a, b \in R, ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator of R is an ideal of R. According to [5], semicommutativity of rings is generalized to nil-semicommutativity of rings. A ring R is called nil-semicommutative if $a, b \in R$ satisfy that ab is nilpotent, then $ahb \in nil(R)$, for any $h \in R$. Clearly, every semicommutative ring is nil-semicommutative. Reduced rings (*i.e.*, rings with no nonzero nilpotent elements in R) are symmetric by [6, P. 361], symmetric rings are clearly reversible, and reversible rings are semicommutative by Proposition 1.3 [6], but the converses are not true. Kim and Lee showed that polynomial rings over reversible rings need not be reversible Example 2.1 [7]. In [8], they consider these reversible rings over which polynomial rings are reversible and call them be strongly reversible, *i.e.*, a ring R is called strongly reversible, if whenever polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $g(x)f(x) = 0$. Reversible Armendariz rings are such rings Proposition 2.4 [7], so reduced rings are strongly reversible, but the converse is not true in general. A ring R is said to be 2- primal if $nil(R)$ coincides with $P(R)$. A ring R is called an NI-ring if the upper nilradical $Nil^*(R)$ coincides with the set of nilpotent elements $nil(R)$. Note that R is an NI-ring if and only if $nil(R)$ forms an ideal and 2-primal rings are NI.

The notion of Armendariz ring is introduced by Rege and Chhawchharia in [4]. A ring R to be an Armendariz if $f(x)g(x) = 0$ implies $a_i b_j = 0$, for all polynomials $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_mx^m$, $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n \in R[x]$.

This paper introduce and study generalized power series nil-reversible rings. Under some various necessary or sufficient conditions for a generalized power series nil-reversible rings to be nil-semicommutative and nil-symmetric. Also, we proved that, a multiplicatively closed subset of R consisting of central non-zero divisors is generalized power series nil-reversible if and only if R is generalized power series nil-reversible. Moreover, some results of generalized power series nil-reversible are given.

We will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $\ell_R(X)(r_R(X))$.

We use the following terminology. If A and B are non-empty subsets of a monoid S, then an element $u_0 \in AB = \{ab : a \in A, b \in B\}$ is said to be a unique product element (*u.p.* element) in the product of AB if it is uniquely presented in form $u = ab$ where $a \in A$ and $b \in B$. For a partially ordered set Y, we write $min(Y)$ to denote the set of minimal elements of Y.

Recall that a monoid S is called unique product monoid ($u.p.$- monoid) if for any two non-empty finite subsets $A, B \in S$ there exist $a \in A$ and $b \in B$ such that ab is $u.p.$ element in the product of AB.

We continue by recalling the structure of the generalized power series ring construction, introduced in [9]. Suppose that $(S, \leq)$ is an ordered set, then $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. Following definition is due to Ribenboim and Elliott [14].

Let $(S, \leq)$ is a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', d \in S$ and $s < s'$, then $s + d < s' + d$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \to R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let $X_s(f, g) = \{(u, v) \in S \times S | u + v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from Ribenboim [10, 4.1] that $X_s(f, g)$ is finite. This fact allows one to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v).$$

Clearly, $\mathrm{supp}(fg) \subseteq supp(f) + supp(g)$, thus by Ribenboim [9, 3.4] $\mathrm{supp}(fg)$ is artinian and narrow, hence $fg \in [[R^{S,\leq}]]$. With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes an associative ring, with identity element e, namely $e(0) = 1, e(s) = 0$ for every $0 \neq s \in S$. Which is called the ring of generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in ([11]$-$[13]), Elliott and Ribenboim [14] and Varadarajan ([15], [16]). For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] \cong R[S]$, the monoid ring of S over R. Further examples are given in Ribenboim [9]. To any $r \in R$ and $s \in S$, we associate the maps $c_r, e_s \in [[R^{S,\leq}]]$ defined by

$$c_r(x) = \begin{cases} r, & x = 0, \\ 0, & \text{otherwise,} \end{cases} \quad e_s(x) = \begin{cases} 1, & x = s, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $r \mapsto c_r$ is a ring embedding of R into $[[R^{S,\leq}]], s \mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}]]$, and $c_r e_s = e_s c_r$. Recall that a monoid S is torsion-free if the following property holds: If $s, t \in S$, k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$.

2. Generalized power series nil-reversible rings

In this section, we first give the following concept, so called generalized power series nil-reversible, that is a generalization of power series reversible rings and generalized power series reversible, we use this concept by studying the relations between nil generalized power series reversible and some certain classes of rings.

Definition 2.1. *Let $(S, \leq)$ be a strictly ordered monoid. A ring R is called generalized power series nil-reversible if whenever $f, g \in [[R^{S,\leq}]]$ satisfy $fg \in [[nil(R)^{S,\leq}]]$ implies $gf \in [[nil(R)^{S,\leq}]]$.*

Let $S = (\mathbb{N} \cup \{0\}, +)$, and $\leq$ is the usual order. Then, $[[R^{S,\leq}]] \cong R[[x]]$. So, the ring R is generalized power series nil-reversible if and only if R is power series nil-reversible. Hence, a generalized power series nil-reversible is a generalization of power series nil-reversible and power series reversible.

Remark 2.2. By definition, it is clear that generalized power series nil-reversible rings are closed under subrings.

Now, we can give example of nil-reversible rings of generalized power series which is neither generalized power series reversible nor generalized power series semicommutative. As we know, generalized power series reversible rings are both generalized power series semicommutative and generalized power series nil-reversible by definition. So, we may conjecture that generalized power series nil-reversible rings may be generalized power series semicommutative. But the following examples eliminate the possibility. We need the following Propositions.

Proposition 2.3. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. If R is a reduced ring with $nil(R)$ an ideal of R, then R is generalized power series nil-reversible ring.*

Proof. Assume that $f, g \in [[R^{S,\leq}]]$, satisfying that fg is nilpotent. So, there exists a positive integer n such that $(fg)^n = 0$, so $(f(u)g(v))^n = 0$, for any $u, v \in S$. Then, $f(u)g(v) \in nil(R)$. Hence, $g(v)f(u)$ is nilpotent by reducedness. Thus, gf is nilpotent. □

Proposition 2.4. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. A ring R is generalized power series nil-reversible ring if and only if, for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is generalized power series nil-reversible.*

Proof. Assume that $f, g \in [[T_n(R)^{S,\leq}]]$, such that $fg \in [[nil(T_n(R))^{S,\leq}]]$. So, by [17], $nil(T_n(R)) = \begin{pmatrix} nil(R) & R & R & \dots & R \\ 0 & nil(R) & R & \dots & R \\ 0 & 0 & nil(R) & \dots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & nil(R) \end{pmatrix}$.

Let R be a reduced ring. Then, $nil(R) = 0$ and so $nil(T_n(R))$ is an ideal. By Proposition 2.3, $T_n(R)$ is generalized power series nil-reversible. The if part follows Remark 2.2. □

Example 2.5. Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Let R be generalized power series nil-reversible ring. Then

$$T = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \mid a_{ij} \in R \right\}.$$

is a generalized power series nil-reversible ring by Proposition 2.4. Note that $fg = 0$, where $f = c_{E_{23}} + c_{E_{13}}e_s$ and $g = c_{E_{12}} + c_{E_{22}}e_s$, But we have $gf \neq 0$. So, T is not generalized power series reversible. In fact, T is not generalized power series semiccomutative by the same as argument from the last sentence of Example 3.17 [18] (with $n = 3$).

Also let S be a generalized power series nil-reversible ring. Then, the ring

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{23} & \dots & a_{2n} \\ 0 & 0 & a & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{pmatrix} \mid a, a_{ij} \in S; n \geq 3 \right\}.$$

is not generalized power series reversible by Example 1.5 [19]. But R_n is generalized power series nil-reversible by Proposition 2.4, since any subring of generalized power series nil-reversible ring is generalized power series nil-reversible. It is obvious that R_4 is not generalized power series semicommutative and it can be proved similarly that R_n is not generalized power series semicommutative for $n \geq 5$.

Proposition 2.6. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S, and R a generalized power series nil-reversible ring. If $f_1, f_2, \dots, f_n \in [[R^{S,\leq}]]$ such that $f_1 f_2 \dots f_n \in [[nil(R)^{S,\leq}]]$, then $f_1(u_1)f_2(u_2)\dots f_n(u_n) \in nil(R)$, for all $u_1, u_2, \dots, u_n \in S$.*

Proof. Suppose $f_1 f_2 \dots f_n \in [[nil(R)^{S,\leq}]]$. Then, for $f_1(f_2 \dots f_n) \in [[nil(R)^{S,\leq}]]$ it follows that $f_1(u_1)(f_2 \dots f_n)(v) \in nil(R)$, for all $u_1, v \in S$. Thus, $C_{f_1(u_1)}(f_2 \dots f_n)$ $(v) \in nil(R)$, for any $v \in S$, and so $C_{f_1(u_1)} f_2 \dots f_n \in [[nil(R)^{S,\leq}]]$. Now, from $(C_{f_1(u_1)} f_2) f_3 \dots f_n \in [[nil(R)^{S,\leq}]]$, it follows that $(C_{f_1(u_1)} f_2)(u_2)(f_3 \dots f_n)(w) \in nil(R)$, since $u_2, w \in S$. $(C_{f_1(u_1)} f_2)(u_2) = f_1(u_1)(f_2(u_2))$, for any $u_1, u_2 \in S$, we have $f_1(u_1)f_2(u_2)(f_3 \dots f_n)(w) \in nil(R)$, for all $u_1, u_2, w \in S$. Hence,

$$C_{f_1(u_1)} C_{f_2(u_2)}(f_3 \dots f_n) \in [[nil(R)^{S,\leq}]].$$

Continuing this manner, we see that $f_1(u_1)f_2(u_2)\dots f_n(u_n) \in nil(R)$, for any $u_1, u_2, \dots, u_n \in S$. As we are desired $f_1(u_1)f_2(u_2)\dots f_n(u_n) \in nil(R)$, for any $u_1, u_2, \dots, u_n \in S$. □

Corollary 2.7. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. If R is generalized power series nil-reversible, then $nil([[R^{S,\leq}]]) \subseteq [[nil(R)^{S,\leq}]]$.*

Proposition 2.8. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. If R is generalized power series nil-reversible rings, then:*
(1) R is abelian.
(2) R is 2-primal.

Proof. Let R be a generalized power series *nil*-reversible ring.
(1) Let e be an idempotent element of R. For any $f(u) \in R, u \in S, c_e f(u) - c_e f(u)c_e \in nil(R)$. Note that $(c_e f(u) - c_e f(u)c_e)c_e = 0$. By hypothesis, $c_e(c_e f(u) - c_e f(u)c_e) = 0$, so $c_e f(u) = c_e f(u)c_e$. Again, $f(u)c_e - c_e f(u)c_e \in nil(R)$ and $c_e(f(u)c_e - c_e f(u)c_e) = 0$. So by generalized power series *nil*-reversibility of R, we have $(f(u)c_e - c_e f(u)c_e)c_e = 0$, that is, $f(u)c_e = c_e f(u)c_e$. Hence, $c_e f(u) = f(u)c_e$.
(2) Note that $P(R) \subseteq nil(R)$. Suppose $g(v) \in nil(R)$. Then, there is a positive integer $m \geq 2$ such that $(g(v))^m = 0$. Thus, $R(g(v))^{m-1}g(v) = 0$. This implies that $g(v)R(g(v))^{m-1} = 0$ as R is generalized power series *nil*-reversible. This yields $(Rg(v))^m = 0$, so $g(v) \in P(R)$. □

Proposition 2.9. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Every generalized power series nil-reversible rings are generalized power series nil-Armendariz.*

Proof. Let $0 \neq f, g \in [[R^{S,\leq}]]$ be such that $fg \in [[nil(R)^{S,\leq}]]$. By Ribenboim [9], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$. We will use transfinite induction on the strictly totally ordered set $(S, \leq)$ to show that $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$. Let s and d denote the minimum elements of supp(f) and supp(g) in the $\leq'$ order, respectively. If $u \in supp(f)$ and $v \in supp(g)$ are such that $u + v = s + d$, then $s \leq' u$ and $d \leq' v$. If $s <' u$, then $s + d <' u + v = s + d$, a contradiction. Thus $u = s$. Similarly, $v = d$. Hence, $0 = (fg)(s + d) = \sum_{(u,v)\in X_{s+d}(f,g)} f(u)g(v) = f(s)g(d)$.

Now, suppose that $w \in S$ such that for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v <' w, f(u)g(v) = 0$. We will show that $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v = w$. We write $X_w(f, g) = \{(u, v) \mid u + v = w, u \in supp(f), v \in supp(g)\}$ as $\{(u_i, v_i) \mid i = 1, 2, \ldots, n\}$ such that

$$u_1 <' u_2 <' \ldots <' u_n.$$

Since S is cancellative, $u_1 = u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_1 = v_2$. Since $\leq'$ is a strict order, $u_1 <' u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_2 <' v_1$. Thus we have

$$v_n <' \ldots <' v_2 <' v_1.$$

Now,

$$0 = (fg)(w) = \sum_{(u,v)\in X_w(f,g)} f(u)g(v) = \sum_{i=1}^{n} f(u_i)g(v_i). \tag{2.1}$$

For any $i \geq 2, u_1 + v_i <' u_i + v_i = w$, and thus, by induction hypothesis, we have $f(u_1)g(v_i) \in nil(R)$. R is 2- primal by Proposition 2.8 this implies $f(u_1)g(v_i) \in nil(R)$. Hence, multiplying Eq. (2.1) on the right by $f(u_1)g(v_1)$, we obtain

$$\Big(\sum_{i=1}^{n} f(u_i)g(v_i)\Big) f(u_1)g(v_1) = f(u_1)g(v_1)f(u_1)g(v_1) = 0.$$

Then, $(f(u_1)rg(v_1))^2 = 0$ and so $f(u_1)g(v_1) \in nil(R)$. Now, Eq. (2.1) becomes

$$\sum_{i=2}^{n} f(u_i)g(v_i) = 0. \tag{2.2}$$

Multiplying $f(u_2)g(v_2)$ on Eq. (2.2) from the right-hand side, we obtain $f(u_2)g(v_2) = 0$ by the same way as the above. Continuing this process, we can prove $f(u_i)g(v_i) = 0$ for $i = 1, 2, \ldots, n$. Thus $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$ with $u+v = w$. Therefore, by transfinite induction, $f(u)g(v) \in nil(R)$, for any $u \in supp(f)$ and $v \in supp(g)$. □

Lemma 2.10. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. For a ring R, consider the following conditions.*
(1) R is generalized power series nil-reversible.
(2) If AB is a nil set, then so is BA, for any subsets A, B of R.
(3) If IJ is nil, then JI is nil for all right (or left) ideals I, J of R.
Then, (1) ⇒ (2) ⇒ (3).

Proof. (1) ⇒ (2) Assume that R is nil generalized power series reversible. Let A, B be two nonempty subsets of R with AB is a nil set. For any $f \in A$ and $g \in B$ is nilpotent. Then, gf is nilpotent. This implies that BA is nil.
(2) ⇒ (3) Let I and J be any right ideals of R such that IJ is nil. Since $IR \subseteq I, IJ$ is nil. By (2), JI is nil. Since $JI \subseteq JRI$, we get JI is nil. Assume that I and J be any left ideals of R such that IJ is nil. Since $RJ \subset J$ and then $IRJ \subseteq IJ, IJ$ is nil. By (2), JRI is nil. Since $JI \subseteq JRI$, we get JI is nil. □

Lemma 2.11. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then, every generalized power series nil-reversible rings are generalized power series nil-semicommutative.*

Proof. Let $f, g \in [[R^{S,\leq}]]$ with $fg \in [[nil(R)^{S,\leq}]]$. Then, $gf \in [[nil(R)^{S,\leq}]]$ and $g(v)h(w)f(u) \in nil(R)$, for any $u, v, w \in S$ and $h(w) \in R$, so $f(v)h(w)g(u) \in nil(R)$. Thus, $fhg \in [[nil(R)^{S,\leq}]]$ by [7, Lemma 1.1]. Therefore, R is generalized power series *nil*-semicommutative. □

Let I be an index set and R_i be a ring for each $i \in I$. Let $(S, \leq)$ be a strictly ordered monoid, if there is an injective homomorphism $f : R \to \prod_{i\in I} R_i$ such that, for each $j \in I, \pi_j f : R \to R_j$ is a surjective homomorphism, where $\pi_j : \prod_{i\in I} R_i \to R_j$ is the jth projection. We have the following.

Proposition 2.12. *Let R_i be a ring, $(S, \leq)$ a strictly totally ordered monoid, for each i in a finite index set I. If R_i is generalized power series nil-reversible ring. for each i, then $R = \prod_{i\in I} R_i$ is generalized power series nil-reversible ring.*

Proof. Let $R = \prod_{i\in I} R_i$ be the direct product of rings $(R_i)_{i\in I}$ and R_i is generalized power series nil-reversible, for each $i \in I$. Denote the projection $R \to R_i$ as Π_i. Suppose that $f, g \in [[R^{S,\leq}]]$ such that $fg \in [[nil(R)^{S,\leq}]]$. Set $f_i = \prod_i f$, $g_i = \prod_i g$. Then, $f_i, g_i \in [[R_i^{S,\leq}]]$. For any $u, v \in S$, assume $f(u) = (a_i^u)_{i\in I}$, $g(v) = (b_i^v)_{i\in I}$. Now, for any $s \in S$,

$$\begin{aligned}
(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v) = &\sum_{(u,v)\in X_s(f,g)} (a_i^u)_{i\in I}(b_i^v)_{i\in I} \\
= \sum_{(u,v)\in X_s(f,g)} ((a_i^u)(b_i^v))_{i\in I} = &\sum_{(u,v)\in X_s(f,g)} (f_i(u)g_i(v))_{i\in I} \\
= &\Big(\sum_{(u,v)\in X_s(f_i,g_i)} f_i(u)g_i(v) \Big)_{i\in I} \\
= &((f_i g_i)(s))_{i\in I}.
\end{aligned}$$

Since $(fg)(s) \in nil(R)$, we have $(f_i g_i)(s) \in nil(R_i)$. Thus, $f_i g_i \in [[nil(R_i)^{S,\leq}]]$. Now, it follows $f_i(u)g_i(v) \in nil(R_i)$, for any $u, v \in S$ and any $i \in I$, since R_i is generalized power series nil-reversible. Hence, for any $u, v \in S$,

$$f(u)g(v) = (f_i(u)g_i(v))_{i\in I} \in nil(R),$$

since I is finite. Thus, $f(u)g(v) \in nil(R)$. Then, by reversible ring, we have

$$(g_i(v)f_i(u))_{i\in I} = g(v)f(u) \in nil(R).$$

This means that $gf \in [[nil(R)^{S,\leq}]]$. The proof is done. □

Proposition 2.13. *Let $(S, \leq)$ be a strictly ordered monoid. If R is finite subdirect product of generalized power series nil-reversible rings, then R is generalized power series nil-reversible ring.*

Proof. Let $I_k(k = 1, \dots, l)$ be ideals of R such that R/I_k is generalized power series nil-reversible and $\bigcap_{k=1}^l I_k = 0$. Let f and g be in $[[R^{S,\leq}]]$ with $fg \in [[nil(R)^{S,\leq}]]$. Clearly $\bar{f}\bar{g} \in [[nil(R/I_k)^{S,\leq}]]$. Since R/I_k is generalized power series nil-reversible, we have $(f(u)g(v))^{r_{u,v,k}} \in I_k$, for each $u, v \in S$ and $k = 1, \dots, l$. Assume that $r_{u,v} = \max\{r_{u,v,k} | k = 1, \dots, l\}$. So, $(f(u)g(v))^{r_{u,v}} \in \bigcap_{k=1}^l I_k = 0$. Hence, $f(u)g(v) \in nil(R)$, for each $u, v \in S$, then $g(v)f(u) \in nil(R)$. Thus, $gf \in [[nil(R)^{S,\leq}]]$, and we are done. □

Proposition 2.14. *Let $(S,\leq)$ a strictly ordered monoid. Let R be a ring and $e^2=e\in R$. If R is generalized power series nil-reversible, then so is eRe.*

Proof. Let $c_efc_e, c_egc_e \in [[(eRe)^{S,\leq}]]$ with $(c_efc_e)(c_egc_e) \in [[nil(eRe)^{S,\leq}]]$. Let e be an idempotent of R. It is easy to see that c_e is an idempotent element of $[[(eRe)^{S,\leq}]]$ and $c_eg = gc_e$ for every $g \in [[(R)^{S,\leq}]]$. Then, $(c_ef)(c_eg) \in [[nil(eR)^{S,\leq}]]$. Since R is generalized power series nil-reversible, we have $fg \in [[nil(R)^{S,\leq}]]$, and so $gf \in [[nil(R)^{S,\leq}]]$. Then, there exists $m \in \mathbb{N}$ such that $((c_efc_e)(c_egc_e))^m = 0$. Hence, $(c_egc_e)(c_efc_e) \in [[nil(eRe)^{S,\leq}]]$. □

Corollary 2.15. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid. For a central idempotent e of a ring R, eR and $(1-e)R$ are generalized power series nil-reversible if and only if R is generalized power series nil-reversible.*

Proof. Assume that eR and $(1-e)R$ are generalized power series nil-reversible. Since the nil generalized power series reversibility property is closed under finite direct products, $R \cong eR \times (1-e)R$ is generalized power series nil-reversible. The converse is trivial by Proposition 2.14. □

In [20], A homomorphic image of a nil-reversible ring may not be nil-reversible, so as generalized power series nil-reversible by the next example.

Example 2.16. Let R be a ring, $(S,\leq)$ a strictly ordered monoid. Assume that $R = D[[S,\leq]]$, where D is a division ring and $I =< xy >$, where $xy \neq yx$. As R is a domain, R is generalized power series nil-reversible. Clearly $\overline{yx} \in nil(R/I)[[S,\leq]]$ and $\overline{x}(\overline{yx}) = \overline{xyx} = 0$. But, $(\overline{yx})\overline{x} = \overline{yx^2} \neq 0$. This implies R/I is not generalized power series nil-reversible.

Theorem 2.17. *Let R be a ring and $(S,\leq)$ a strictly ordered monoid. If R is a generalized power series nil-reversible and I an ideal consisting of nilpotent elements of bounded index $\leq n$ in R, then R/I is generalized power series nil-reversible.*

Proof. Let $\bar{f},\bar{g} \in [[(R/I)^{S,\leq}]]$ with $\bar{f}\bar{g} \in [[nil(R/I)^{S,\leq}]]$. By hypothesis, the order $(S,\leq)$ can be refined to a strict total order $\leq$ on S. We will use transfinite induction on the strictly totally ordered set $(S,\leq)$ to show that $\bar{g}\bar{f} \in [[nil(R/I)^{S,\leq}]]$. Firstly, by transfinite induction to show $g(t)f(s) \in nil(R)$, for any $s \in supp(f)$ and any $t \in supp(g)$. Since $supp(f)$ and $supp(g)$ are nonempty subsets of S, the set of minimal elements of $supp(f)$ and $supp(g)$, respectively, are finite and non-empty. Let s_0 and t_0 denote the minimum elements of $supp(f)$ and $supp(g)$ in the $\leq$ order, respectively. By analogy with the proof of Theorem 2.25 [21], we can show that $f(s_0)g(t_0) = 0$. Therefore, by transfinite induction, we can proof that $f(s)g(t) = 0$. Since $\bar{f}\bar{g} \in [[nil(R/I)^{S,\leq}]]$, then, there is a positive integer $n \in \mathbb{N}$ such that $(\bar{f}\bar{g})^n = \bar{0}$. So, $(f(s)g(t))^n \in I$, for any $s,t \in S$. Since $I \subseteq nil(R), (f(s)g(t))^n = 0$. Hence, $f(s)g(t) \in nil(R)$, so $g(t)f(s) \in nil(R)$, by R is generalized power series nil-reversible, $gf \in [[nil(R)^{S,\leq}]]$. Thus, $\bar{g}\bar{f} \in [[nil(R/I)^{S,\leq}]]$. Therefore, R/I is generalized power series nil-reversible. □

Now, we give some characterizations of nil generalized power series reversibility by using the prime radical of a ring.

Corollary 2.18. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. If a ring R is generalized power series nil-reversible, then $R/P(R)$ is generalized power series nil-reversible.*

Proof. Since every element of $P(R)$ is nilpotent, the proof follows from Theorem 2.17. $\square$

Proposition 2.19. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Let J be a reduced ideal of a ring R such that R/J is generalized power series nil-reversible. Then, R is generalized power series nil-reversible.*

Proof. Let $f, g \in [[R^{S,\leq}]]$ and suppose that $fg \in [[nil(R)^{S,\leq}]]$. Then, $\bar{f}\bar{g} \in [[nil(R/J)^{S,\leq}]]$ and so $\bar{g}\bar{f} \in [[nil(R/J)^{S,\leq}]]$, since R/J is nil generalized power series reversible. There exists $m \in \mathbb{N}$ such that $(\bar{f}\bar{g})^m = \bar{0}$. This shows that $(f(s)g(t))^m \in J$, for any $s, t \in S$. Since J is reduced, we have $f(s)g(t) = 0$ yields $g(t)f(s) = 0$. Thus, $gf \in [[nil(R)^{S,\leq}]]$. Therefore, R is generalized power series nil-reversible. $\square$

A ring is called semiperfect if every finitely generated R-right-module has a projective cover by [22]. For abelian semiperfect, here we have.

Theorem 2.20. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Consider the following statements.*
(1) *R is a finite direct sum of local generalized power series nil-reversible rings.*
(2) *R is a semiperfect generalized power series nil-reversible ring.*
Then, (1) $\Rightarrow$ (2) *and the converse is true when R is abelian.*

Proof. (1) $\Rightarrow$ (2) Assume that R is a finite direct sum of local generalized power series nil-reversible rings. Then, R is semiperfect because local rings are semiperfect and a finite direct sum of semiperfect rings is semiperfect, and moreover R is generalized power series nil-reversible by Proposition 2.12.
(2) $\Rightarrow$ (1) Suppose that R is an abelian semiperfect generalized power series nil-reversible ring. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by Theorem 27.6 [23], say $1 = e_1 + e_2 + \ldots + e_n$ such that each e_iRe_i is a local ring where $1 \leq i \leq n$. The ring R being abelian implies $e_iRe_i = e_iR$. Each e_iR is a generalized power series nil-reversible by Proposition 2.14. Hence, R is generalized power series nil-reversible by Proposition 2.12. $\square$

3. Weak annihilator of generalized power series reversible and some rings property

In [24], Ouyang introduced the notion of weak annihilators and investigated their properties. For a subset X of a ring R put $Nr_R(X) = \{a \in R | Xa \in nil(R)\}$

and $Nl_R(X) = \{b \in R | bX \in nil(R)\}$. By a simple computation, we can see that $Nr_R(X) = Nl_R(X)$. The set $Nr_R(X)$ is called the weak annihilator of X. It is also easy to see that, $Nr_R(X)$ is an ideal of R in case R is a NI-ring. Furthermore when R is reduced, then $r_R(X) = Nr_R(X) = l_R(X) = Nl_R(X)$. For more details and results on weak annihilators see [25].

Now, we investigate the relations between weak annihilators in a ring R and weak annihilators in a generalized power series ring $R[[S, \leq]]$. Given a ring R and let $\gamma = C(f)$ be the content of f, i.e., $C(f) = \{f(u) | u \in supp(f)\} \subseteq R$. Since, $R \simeq c_R$ we can identify, the content of f with

$$c_{C(f)} = \{c_{f(u_i)} | u_i \in supp(f)\} \subseteq [[R^{S,\leq}]].$$

Then, we have two maps $\phi : NrAnn_R(id(R)) \to NrAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ and $\psi : NlAnn_R(id(R)) \to NlAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ defined by $\phi(I) = I[[R^{S,\leq}]]$ and $\psi(J) = [[R^{S,\leq}]]J$ for every $I \in NrAnn_R(id(R)) = \{Nr_R(U) | U$ is an ideal of $R\}$ and $J \in NlAnn_R(id(R)) = \{Nl_R(U) | U$ is an ideal of $R\}$, respectively. Obviously, ϕ is injective. In the following theorem, we show that ϕ and ψ are bijective maps if and only if R is generalized power series *nil*-reversible.

Theorem 3.1. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. If R is reduced and $nil(R)$ is a nilpotent ideal of R, then the following are equivalent:*
(1) *R is generalized power series nil-reversible ring.*
(2) *The function $\phi : NrAnn_R(id(R)) \to NrAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\phi(I) = I[[R^{S,\leq}]]$ for every $I \in NrAnn_R(id(R))$.*
(3) *The function $\psi : NlAnn_R(id(R)) \to NlAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\psi(J) = [[R^{S,\leq}]]J$ for every $J \in NlAnn_R(id(R))$.*

Proof. (1)⇒(2) Let $Y \subseteq [[R^{S,\leq}]]$ and $\gamma = \bigcup_{f \in Y} C(f)$. From Proposition 2.6 it is sufficient to show that $Nr_{[[R^{S,\leq}]]}(f) = Nr_R C(f)[[R^{S,\leq}]]$, for all $f \in Y$. In fact, let $g \in Nr_{[[R^{S,\leq}]]}(f)$. Then, $fg \in [[nil(R)^{S,\leq}]]$ and by assumption $f(u_i)g(v_j) \in nil(R)$ for each $u_i \in supp(f)$ and each $v_j \in supp(g)$. Then, for a fixed $u_i \in supp(f)$ and each $v_j \in supp(g), 0 = f(u_i)g(v_j) = (c_{f(u_i)}g)(v_j)$, it follows that

$$g \in Nr_R \bigcup_{u_i \in supp(f)} c_{f(u_i)}[[R^{S,\leq}]] = Nr_R C(f)[[R^{S,\leq}]].$$

So,

$$Nr_{[[R^{S,\leq}]]}(f) \subseteq Nr_R C(f)[[R^{S,\leq}]].$$

Conversely, let $g \in Nr_R C(f)[[R^{S,\leq}]]$, then $c_{f(u_i)}g \in [[nil(R)^{S,\leq}]]$ for each $u_i \in supp(f)$. Hence, $(c_{f(u_i)}g)(v_j) = f(u_i)g(v_j) \in nil(R)$ and $v_j \in supp(g)$. Thus,

$$(fg)(s) = \sum_{(u_i,v_j) \in X_s(f,g)} f(u_i)g(v_j) = 0$$

and it follows that $g \in Nr_{[[R^{S,\leq}]]}(f)$. Hence, $Nr_RC(f)[[R^{S,\leq}]] \subseteq Nr_{[[R^{S,\leq}]]}(f)$ and it follows that $Nr_RC(f)[[R^{S,\leq}]] = Nr_{[[R^{S,\leq}]]}(f)$. So,

$$Nr_{[[R^{S,\leq}]]}(Y) = \bigcap_{f\in Y} Nr_{[[R^{S,\leq}]]}(f) = \bigcap_{f\in Y} C(f)[[R^{S,\leq}]] = Nr_R(\gamma)[[R^{S,\leq}]].$$

(2)⇒(1) Suppose that $f, g \in [[R^{S,\leq}]]$ be such that $fg \in [[nil(R)^{S,\leq}]]$. Then, $g \in Nr_{[[R^{S,\leq}]]}(f)$ and by assumption $Nr_{[[R^{S,\leq}]]}(f) = \gamma[[R^{S,\leq}]]$ for some right ideal γ of R. Consequently, $0 = fc_{g(v_j)}$ and for any $u_i \in supp(f), (fc_{g(v_j)})(u_i) = f(u_i)g(v_j) \in nil(R)$ for each $u_i \in supp(f)$ and $v_j \in supp(g)$. Thus by reduced ring, $g(v_j)f(u_i) \in nil(R)$, then $gf \in [[nil(R)^{S,\leq}]]$. Hence, R is generalized power series nil-reversible. The proof of (1)⇔(3) is similar to the proof of (1)⇔(2). □

According to Liu [26], the ring R is called S-Armendariz if whenever $f, g \in [[R^{S,\leq}]]$ satisfy $fg = 0$, then $f(u)g(v) = 0$ for each $u, v \in S$. Now, we given a strong condition under which $[[R^{S,\leq}]]$ is nil-reversible.

Theorem 3.2. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Assume that R is an S-Armendariz ring, then R is generalized power series nil-reversible if and only if $[[R^{S,\leq}]]$ is nil-reversible.*

Proof. Suppose R is generalized power series nil-reversible. Let $f, g \in [[R^{S,\leq}]]$ be such that $fg \in [[nil(R)^{S,\leq}]]$. By [27, Proposition 2.17], $[[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]])$. So, $f(u_i)g(v_j) \in nil(R)$, for any $u, v \in S$ and any i, j. Since R is S-Armendariz, $f(u_i)g(v_j) = 0$, for all i, j. By nil-reversibility, $g(v_j)f(u_i) \in nil(R)$, for all i, j. So, $gf = 0$. Therefore, $[[R^{S,\leq}]]$ is nil-reversible. The proof of the converse is trivial. □

Theorem 3.3. *Let R be a ring and $(S, \leq)$ a strictly ordered monoid. Let Δ denotes a multiplicatively closed subset of R consisting of central non-zero divisors. Then, R is generalized power series nil-reversible if and only if $\Delta^{-1}R$ is generalized power series nil-reversible.*

Proof. Suppose R is generalized power series nil-reversible and $p_i, d_j, u, v \in R$. Let $u^{-1}C_{p_i}, v^{-1}C_{d_j} \in \Delta^{-1}R[[S, \leq]]$, for all i, j satisfying that $u^{-1}C_{p_i}v^{-1}C_{d_j} \in nil(\Delta^{-1}R[[S, \leq]])$. Then, $(u^{-1}C_{p_i}v^{-1}C_{d_j})^n = 0$ for some positive integer n. This implies $(C_{p_i}C_{d_j})^n = 0$, so $p_id_j \in nil(R)$. For any $u^{-1}C_{p_i}, v^{-1}C_{d_j} \in \Delta^{-1}R[[S, \leq]]$ having the property that $(u^{-1}C_{p_i})(v^{-1}C_{d_j}) = 0$, we have $(uv)^{-1}C_{p_i}C_{d_j} = 0, C_{p_i}C_{d_j} = 0$, for all i, j. Since R is generalized power series nil-reversible, $d_jp_i \in nil(R)$, so $(v^{-1}u^{-1})C_{d_j}C_{p_i} = 0$ which further yields $(v^{-1}C_{d_j})(u^{-1}C_{p_i}) \in nil(\Delta^{-1}R[[S, \leq]])$. Hence, $\Delta^{-1}R$ is generalized power series nil-reversible. The converse part is trivial. □

Following Lambek [28], a ring R is called symmetric if $abc = 0$ implies $acb = 0$, for all $a, b, c \in R$. It is obvious that commutative rings are symmetric and symmetric rings are reversible ring.

Theorem 3.4. *Let R be a reversible ring and $(S, \leq)$ a strictly ordered monoid with $nil(R)$ is a nilpotent ideal of R. Then, R is generalized power series nil-symmetric if and only if $R[[S, \leq]]$ is nil-symmetric.*

Proof. Assume that R is generalized power series nil-symmetric and $f, g, h \in R[[S, \leq]]$ are such that $fgh \in nil(R[[S, \leq]])$. Hence, by Proposition 2.6, $f(u)g(v)h(t) \in nil(R)$, for all $u, v, t \in S$. Since R is nil-symmetric, we have $f(u)h(t)g(v) \in nil(R)$. Now, for all $s \in S$, we have

$$(fhg)(s) = \sum_{(u,t,v) \in X_s(f,h,g)} f(u)h(t)g(v).$$

So, the reversibility of R imply that $fhg \in nil(R[[S, \leq]])$, hence $R[[S, \leq]]$ is nil-symmetric. Conversely, if $R[[S, \leq]]$ is nil-symmetric, then R is generalized power series nil-symmetric, as subrings of generalized power series nil-symmetric rings are also generalized power series nil-symmetric. □

4. Conclusion

In this paper, we have introduced the notion of generalized power series nil-reversible rings. The researchers obtains various necessary or sufficient conditions for a generalized power series nil-reversible rings to be some rings related. We use this concept by studying the relations between generalized power series reversible and some certain classes of rings. One can extend this work to study different rings on this structure. Further one can identify some real life applications in a monoid homomorphism and ideal rings. In our future work we will introduce the concept of skew generalized power series nil-reversible, that is a generalization of power series nil-reversible, when R is S-compatible, $(S, \leq)$ a strictly ordered monoid and connected by annihilator rings.

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Some even-odd mean graphs in the context of arbitrary super subdivision

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Abstract. In this paper, we establish some new results on even vertex odd mean labeling of graph. We prove that the graphs obtained by arbitrary super subdivision of cycle, comb, crown, slanting ladder and planar grid are even-odd mean graphs.

Keywords: labeling, even-odd mean labeling, biclique, arbitrary super subdivision.
MSC 2020: 05C78

1. Introduction

Unless mentioned or otherwise, all graphs in this paper are simple, finite, connected and undirected. For all other standard terminology and notations in graph theory we follow Harary [7]. A (p, q)-graph G is a graph such that $| V(G) |= p$ and $| E(G) |= q$. A labeling (or valuation) of a graph is a function that carries graphs elements to numbers usually to non negative integers or positive. If the domain is the vertex set or edge set the labeling called vertex labeling or edge labeling respectively. If the domain is both vertices and edges then the labeling is called total labeling. According to Beineke and Hegde [1] graph labeling serves as frontier between number theory and structure of graphs. Labeled graph have variety of applications in coding theory, mathematical modeling, x-ray, crystallography and to determine optimal circuit layouts. For a dynamic survey of various graph labeling problems along with extensive bibliography we refer to Gallian [6]. The concept of even-odd mean labeling of the graph was introduced by Vasuki, Nagarajan and Arockiaraj [11]. They studied even-odd mean labeling of some standard graphs. The subject of even-odd mean labeling has been further studied in [2], [3], [4], [8], [9], [12]. The concept of super subdivision of graphs was introduced by Sethuraman and Selvaraju [10]. They proved that the arbitrary super subdivision of graphs admit graceful labeling. In [5] Basher et.al proved that the super subdivision of some families of graphs admit an even-odd mean labeling. Motivated by the work in [5], in this paper, we study the even-odd mean labeling of cycle, comb, crown, slanting ladder and planar grid. We will give a brief summary of definitions and terminology which are useful for our study.

Definition 1.1 ([11]). *A vertex labeling of G is an injective function $f : V(G) \to \{0, 2, 4, ..., 2q\}$. For a vertex labeling f, the induced edge labeling f^* is defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$ for any edge uv in G, then the vertex labeling f is called even-odd mean labeling of graph of G if its induced edge labeling $f^* : E(G) \to \{1, 3, 5, ..., 2q-1\}$ is a bijection, that is $f^*(E) = \{1, 3, 5, ..., 2q-1\}$.*

If a graph G has even-odd mean labeling , then we say that G is an even-odd mean graph.

Definition 1.2 ([10]). *Let G be a graph with p vertices and q edges. A graph $G^{'}$ is said to be an arbitrary super subdivision of G if $G^{'}$ is obtained from G by replacing each edge e_i by a complete bipartite graph (biclique) K_{2,t_i} where t_i is any positive integer and may vary for each edge arbitrary by identifying the ends of each edge e_i with the two vertices of 2-vertices part of K_{2,t_i} after removing the edge from G.*

In this work a cycle on n vertices denoted by C_n, a slanting ladder $SL_n, n \geq 2$ is a graph obtained from two parallel paths with vertices $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ respectively by joining each u_i with $v_{i+1}, 1 \leq i \leq n-1$. The corona $G \odot K_1$ of a graph G on p vertices $u_1, u_2, ..., u_p$ is the graph obtained from G by adding p new vertices $v_1, v_2, ..., v_p$ and joining each vertex u_i to a vertex v_i,$1 \leq i \leq n$. The graph $P_n \odot K_1$ is called a comb and the graph $C_n \odot K_1$ is called a crown. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices, respectively. The Cartesian product $G_1 \times G_2$ is the graph such that $V = p_1 p_2$ with vertices set $\{(u, v) : u \in G_1, v \in G_2\}$ and the two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and v_1, v_2 are adjacent in G_2 or $v_1 = v_2$ and u_1, u_2 are adjacent in G_1. The product $P_m \times P_n$ is called a planar grid and $P_2 \times P_n$ is called ladder, denoted by L_n. Let a and b be two positive numbers, we refer to $[a, b]$ an interval of numbers s, where $a \leq s \leq b$.

Notation. We denote the arbitrary super subdivision of G by $ASS(G)$.

2. Main results

Theorem 2.1. *$ASS(C_n)$ is an even-odd mean graph where the edges $u_i u_{i+1} (i \in [1, n-1])$, $u_n u_1$ of the cycle C_n are replaced by K_{2,t_i}, K_{2,t_n} respectively, such that $n \equiv 0$ (mod 4), $t_{\frac{n}{2}} = t_n$ and $\sum_{i=1}^{\frac{n}{2}} t_i = \sum_{i=\frac{n}{2}+1}^{n} t_i$.*

Proof. Let C_n be a cycle graph of length n, where $n \equiv 0$ (mod 4) whose vertex set is $V = \{u_1, u_2, ..., u_n\}$ and the edge set is $E = \{e_i = u_i u_{i+1}, e_n = u_n u_1 : i \in [1, n-1]\}$. Let $ASS(C_n)$ be an arbitrary super subdivision of a cycle graph C_n. That is, for $i \in [1, n]$ each edge e_i of the cycle C_n replaced by a biclique K_{t_i}, where t_i is any positive integer, $t_{\frac{n}{2}} = t_n$ and $\sum_{i=1}^{\frac{n}{2}} t_i = \sum_{i=\frac{n}{2}+1}^{n} t_i$. Let $u_{ij} (i \in [1, n], j \in [1, t_i])$ be the vertices which are used for arbitrary super subdivision. Therefore, the edge set is $E(ASS(C_n)) = \{u_i u_{ij}, u_{ij} u_{i+1}, u_{nj} u_1 : i \in [1, n], j \in$

$[1, t_i]\}$. Then, it is clear that $ASS(C_n)$ has $n + \sum_{i=1}^{n} t_i$ vertices and $2\sum_{i=1}^{n} t_i$ edges. Define labeling $f : V(ASS(C_n)) \to \{0, 2, 4, ..., 2q-2, 2q = 4\sum_{s=1}^{n} t_s\}$ as follows:

$$f(u_i) = \begin{cases} 0, & i = 1 \\ 4\sum_{s=1}^{i-1} t_s, & i \in [2, \frac{n}{2}] \\ 4\sum_{s=1}^{i-1} t_s + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n]. \end{cases}$$

For $j \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4j-2, & i = 1, \\ 4\sum_{s=1}^{i-1} t_s + 4j - 2, & i \in [2, n]. \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:

$$f^*(u_i u_{ij}) = \begin{cases} 2j-1, & i = 1 \\ 4\sum_{s=1}^{i-1} t_s + 2j - 1, & i \in [2, \frac{n}{2}] \\ 4\sum_{s=1}^{i-1} t_s + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2}+1, n]. \end{cases}$$

$$f^*(u_{ij} u_{(i+1)}) = \begin{cases} 2t_1 + 2j - 1, & i = 1 \\ 4\sum_{s=1}^{i-1} t_s + 2t_i + 2j - 1, & i \in [2, \frac{n}{2} - 1] \\ 4\sum_{s=1}^{\frac{n}{2}} t_s + 2j - 1, & i = \frac{n}{2} \\ 4\sum_{s=1}^{i-1} t_s + 2(t_{\frac{n}{2}} + t_i) + 2j - 1, & i \in [\frac{n}{2}+1, n-1]. \end{cases}$$

$$f^*(u_{nj} u_1) = 2\sum_{s=1}^{n-1} t_s + 2j - 1.$$

Hence, f is an even-odd mean labeling of $ASS(C_n)$. Thus, $ASS(C_n)$ is an even-odd mean graph. □

Illustration 2.1. Consider $ASS(C_8)$ where the edges $u_i u_{i+1}, i \in [1, 7]$, $u_8 u_1$ are replaced by $K_{2,3}, K_{2,2}, K_{2,5}, K_{2,4}, K_{2,3}, K_{2,4}, K_{2,3}$ and $K_{2,4}$ respectively. An even-odd mean labeling of $ASS(C_8)$ is shown in Figure 1.

Theorem 2.2. *$ASS(P_n \odot K_1)$ is an even-odd mean graph where the edges $u_i u_{i+1}, u_i v_i$ and $u_n v_n$ are replaced by $K_{2,t_i}, K_{2,t_i'}$ and $K_{2,t_n'}$ respectively, such that $t_i' = t_{i+1}'$ when i is even, $i \in [1, n-1]$.*

Proof. Let $P_n \odot K_1$ be a comb graph. Let $u_i (i \in [1, n])$ be the vertices of the path P_n and v_i be the pendant vertex adjacent to $u_i (i \in [1, n-1])$. Then, the vertex set is $V = \{u_i, v_i : i \in [1, n]\}$ and the edge set is $E = \{e_i = u_i u_{i+1}, e_i' = u_i v_i, e_n' = u_n v_n : i \in [1, n-1]\}$. Let $ASS(P_n \odot K_1)$ be an arbitrary super subdivision of a comb graph $P_n \odot K_1$. The edges e_i, e_i' and

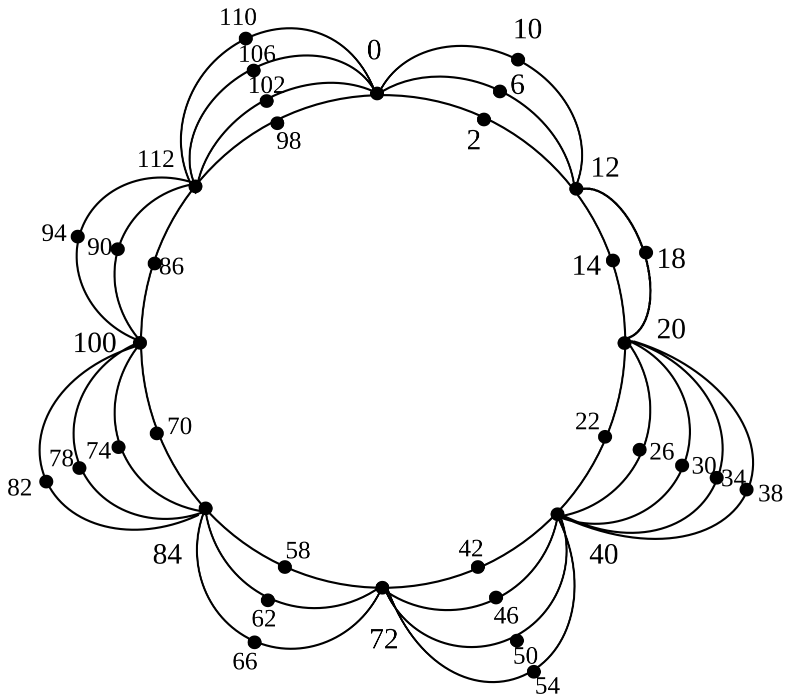

Figure 1: An even-odd mean graph of $ASS(C_8)$

$e_n^{'}(i \in [1, n-1])$ are replaced by bicliques K_{t_i},$K_{t_i^{'}}$ and $K_{t_n^{'}}(i \in [1, n-1])$ respectively, where $t_i, t_i^{'}$ are positive integer numbers, $t_i^{'} = t_{i+1}^{'}$ when i is even. Let $u_{ij}(i \in [1, n-1], j \in [1, t_i])$, $w_{ij}(i \in [1, n], j \in [1, t_i^{'}])$ be the vertices which are used for arbitrary super subdivision of $P_n \odot K_1$. Thus, the edge set is $E(ASS(P_n \odot K_1)) = \{\{u_i u_{ij}, u_{ij} u_{i+1} : i \in [1, n-1], j \in [1, t_i^{'}]\} \cup \{u_i w_{ij}, w_{ij} v_i : i \in [1, n], j \in [1, t_i^{'}]\}\}$. Here, we note that $ASS(P_n \odot K_1)$ has $2n + \sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n} t_i^{'}$ vertices and $2(\sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n} t_i^{'})$ edges. Define labeling $f : V(ASS(P_n \odot K_1)) \to \{0, 2, 4, ..., 2q-2, 2q = 4(\sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n} t_i^{'})\}$ as follows:

$$f(u_i) = \begin{cases} 4t_1^{'}, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4t_1^{'} + 4j - 2, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f(w_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

Thus, the induced edge labeling f^* is obtained as follows:
For $j \in [1, t_i]$.

$$f^*(u_i u_{ij}) =$$
$$\begin{cases} 4t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2(t_{(i+1)}^{'} + t_i^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(u_{ij} u_{(i+1)}) =$$
$$\begin{cases} 4t_1^{'} + 2t_1 + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2t_i + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 2t_i + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f^*(u_i w_{ij}) =$$
$$\begin{cases} 2t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2t_{(i-1)} + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(w_{ij} v_i) =$$
$$\begin{cases} 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{i-1} + t_i^{'}) + 2j - 1, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + t_s^{'}) + 2t_i^{'} + 2j - 1, & i \in [2, n], \text{ and } i \text{ is even.} \end{cases}$$

Hence, f is an even-odd mean labeling of $ASS(P_n \odot K_1)$. Then, $ASS(P_n \odot K_1)$ is an even-odd mean graph. $\square$

Illustration 2.2. Consider $ASS(P_7 \odot K_1)$ where the edges $u_iu_{i+1}, i \in [1,6]$ are replaced by $K_{2,3}, K_{2,5}, K_{2,4}, K_{2,3}, K_{2,4}$ and $K_{2,2}$ respectively and the edges $u_iv_i, i \in [1,7]$ are replaced by $K_{2,5}, K_{2,4}, K_{2,4}, K_{2,2}, K_{2,2}, K_{2,3}$ and $K_{2,3}$ respectively . An even-odd mean labeling of $ASS(P_7 \odot K_1)$ is shown in Figure 2.

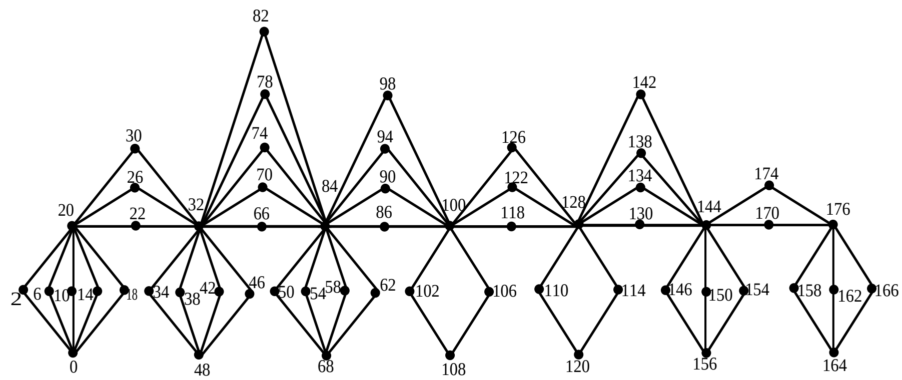

Figure 2: An even-odd mean graph of $ASS(P_7 \odot K_1)$

Theorem 2.3. *$ASS(C_n \odot K_1)$ is an even-odd mean graph where the edges $u_iu_{i+1}, u_iv_i, u_nu_1$ and u_nv_n of $C_n \odot K_1$ are replaced by $K_{2,t_i}, K_{2,t_i'}, K_{2,t_n}$ and $K_{2,t_n'}$ respectively, such that $n \equiv 0$ (mod 4), $\sum_{i=1}^{\frac{n}{2}}(t_i + t_i') = \sum_{i=\frac{n}{2}+1}^{n}(t_i + t_i')$, $t_n = t_{\frac{n}{2}}$, $t_n' = t_1'$ and $t_i' = t_{i+1}'$ when i is even, $i \in [1, n-1]$.*

Proof. Let $C_n \odot K_1$ be a crown graph. Let $u_i(i \in [1,n])$ be the vertices of the cycle C_n, $n \equiv 0$ (mod 4). Let v_i be the pendant vertices adjacent to $u_i(i \in [1,n])$. Then, the vertex set of the crown $C_n \odot K_1$ is $V = \{u_i, v_i : i \in [1,n]\}$ and the edge set is $E = \{\{e_i = u_iu_{i+1}, e_n = u_nu_1 : i \in [1, n-1]\} \cup \{e_i' = u_iv_i : i \in [1,n]\}\}$. Let $ASS(C_n \odot K_1)$ be an arbitrary super subdivision of the crown $C_n \odot K_1$. The edges $e_i, e_i'(i \in [1,n])$ are replaced by the bicliques K_{2,t_i}, $K_{2,t_i'}$ respectively where t_i, t_i' are positive integer numbers, $\sum_{i=1}^{\frac{n}{2}}(t_i + t_i') = \sum_{i=\frac{n}{2}+1}^{n}(t_i + t_i')$, $t_n = t_{\frac{n}{2}}$, $t_n' = t_1'$ and $t_i' = t_{i+1}'$ when i is even. Let $u_{ij}(i \in [1,n], j \in [1,t_i])$, $w_{ij}(i \in [1,n], j \in [1,t_i'])$ be the vertices which are used for arbitrary super subdivision of $C_n \odot K_1$. Therefore, the edge set of $ASS(C_n \odot K_1)$ is $E(ASS(C_n \odot K_1)) = \{\{u_iu_{ij}, u_{ij}u_{i+1}, u_{nt_n}u_1 : i \in [1,n], j \in [1,t_i]\} \cup \{u_iw_{ij}, w_{ij}v_i : i \in [1,n], j \in [1,t_i']\}\}$. We observe that $ASS(C_n \odot K_1)$ has $2n + \sum_{i=1}^{n}(t_i + t_i')$ vertices and $2\sum_{i=1}^{n}(t_i + t_i')$ edges. Define labeling $f : V(ASS(C_n \odot K_1)) \to \{0, 2, 4, ..., 2q - 2, 2q = 4\sum_{i=1}^{n}(t_i + t_i')\}$ as follows:

$$f(u_i) = \begin{cases} 4t_1^{'}, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, \frac{n}{2}], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, \frac{n}{2}], \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n], \text{ and } i \text{ is even.} \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}), & i \in [2, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}), & i \in [2, n], \text{ and } i \text{ is even} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4t_{\frac{n}{2}}, & i \in [\frac{n}{2}+1, n], \text{ and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4t_1^{'} + 4j - 2 & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 4j - 2, & i \in [2, n-1], \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 4j - 2, & i \in [2, n-1], \text{ and } i \text{ is even} \\ 4\sum_{s=1}^{n-1}(t_s + t_s^{'}) + 4j - 2, & i = n. \end{cases}$$

For $j \in [1, t_i^{'}]$.

$$f(w_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 4j - 2, & i \in [2, n], \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{n} t_s + \sum_{s=1}^{n-1} t_s^{'}) + 4j - 2, & i = n. \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:
For $j \in [1, t_i]$.

$$f^*(u_i u_{ij}) = \begin{cases} 4t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2j - 1, & i \in [2, \frac{n}{2}] \quad \text{and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2j - 1, & i \in [2, \frac{n}{2}] \quad \text{and } i \text{ is even} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2}+1, n-1] \quad \text{and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2}+1, n-1] \quad \text{and } i \text{ is even} \\ 4\sum_{s=1}^{n-1}(t_s + t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i = n. \end{cases}$$

$$f^*(u_{ij}u_{(i+1)}) = \begin{cases} 4t_1^{'} + 2t_1 + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + & \\ 2t_i + 2j - 1, & i \in [2, \frac{n}{2} - 1] \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + & \\ 2t_i + 2j - 1, & i \in [2, \frac{n}{2} - 1] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{\frac{n}{2}} t_s + \sum_{s=1}^{\frac{n}{2}+1} t_s^{'}) + & \\ 2j - 1, & i = \frac{n}{2} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i} t_s^{'}) + 2(t_{\frac{n}{2}} + t_i) + & \\ 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is odd} \\ 4(\sum_{s=1}^{i-1} t_s + \sum_{s=1}^{i+1} t_s^{'}) + 2(t_{\frac{n}{2}} + t_i) + & \\ 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is even .} \end{cases}$$

$f^*(u_{nj}u_1) = 2\sum_{s=1}^{n-1}(t_s + t_s^{'}) + 2t_1^{'} + 2j - 1$.
For $j \in [1, t_i^{'}]$.

$$f^*(u_i w_{ij}) = \begin{cases} 2t_1^{'} + 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{(i-1)} + t_i^{'}) + & \\ 2j - 1, & i \in [2, \frac{n}{2}] \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2j - 1, & i \in [2, \frac{n}{2}] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{\frac{n}{2}} + t_{(i-1)} + t_i^{'}) + & \\ 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2t_{\frac{n}{2}} + 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{n} t_s + \sum_{s=1}^{n-1} t_s^{'}) + 2j - 1, & i = n. \end{cases}$$

$$f^*(v_i w_{ij}) = \begin{cases} 2j - 1, & i = 1 \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2t_{(i-1)} + 2j - 1, & i \in [2, \frac{n}{2}] \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2t_i^{'} + 2j - 1, & i \in [2, \frac{n}{2}] \text{ and } i \text{ is even} \\ 4(\sum_{s=1}^{i-2} t_s + \sum_{s=1}^{i-1} t_s^{'}) + 2(t_{(i-1)} + t_{\frac{n}{2}}) + 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is odd} \\ 4\sum_{s=1}^{i-1}(t_s + t_s^{'}) + 2(t_i^{'} + t_{\frac{n}{2}}) + 2j - 1, & i \in [\frac{n}{2} + 1, n - 1] \text{ and } i \text{ is even.} \\ 4(\sum_{s=1}^{n} t_s + \sum_{s=1}^{n-1} t_s^{'}) + 2t_n^{'} + 2j - 1, & i = n \end{cases}$$

Thus, f is an even-odd mean labeling of $ASS(C_n \odot K_1)$. Hence $ASS(C_n \odot K_1)$ is an even-odd mean graph. $\square$

Illustration 2.3. Consider $ASS(C_8 \odot K_1)$ where the edges $u_iu_{i+1}, i \in [1, 7]$ and u_8u_1 are replaced by $K_{2,2}, K_{2,3}, K_{2,5}, K_{2,4}, K_{2,3}, K_{2,6}, K_{2,3}$ and $K_{2,4}$ respectively and the edges $u_iv_i, i \in [1, 8]$ are replaced by $K_{2,2}, K_{2,4}, K_{2,4}, K_{2,5}, K_{2,5}, K_{2,3}, K_{2,3}$ and $K_{2,2}$ respectively . An even-odd mean labeling of $ASS(C_8 \odot K_1)$ is shown in Figure 3.

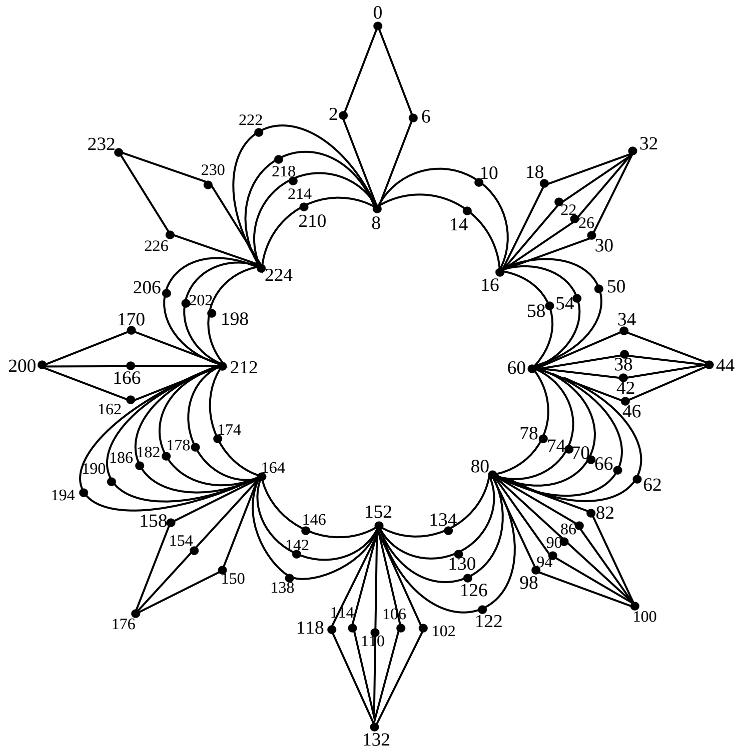

Figure 3: An even-odd mean graph of $ASS(C_8 \odot K_1)$

Theorem 2.4. *$ASS(SL_n)$ is an even-odd mean graph where the edges u_iu_{i+1}, v_iv_{i+1} and u_iv_{i+1} of SL_n are replaced by $K_{2,t_i}, K_{2,t_i'}$ and $K_{2,t_i''}(i \in [1, n-1])$ respectively, such that $t_i'' = t$ for all $(i \in [1, n-1])$, $t_{i+1}' = t_i$, $t_{i+1} \geq t_i$ for all $(i \in [1, n-2])$ and $t_1' = t_1 = t$ when n is odd.*

Proof. Let (SL_n) be a slanting ladder graph whose vertex set is $V = \{u_i, v_i : i \in [1, n]\}$ and edge set is $E = \{e_i = u_iu_{i+1}, e_i' = v_iv_{i+1}, e'' = u_iv_{i+1} : i \in [1, n-1]\}$. Let $ASS(SL_n)$ be an arbitrary super subdivision of SL_n. the edges e_i, e_i' and $e_i''(i \in [1, n-1])$ are replaced by the bicliques K_{2,t_i}, $K_{2,t_i'}$ and $K_{2,t_i''}$ respectively where t_i, t_i', and t_i'' are positive integer numbers, $t_i'' = t$ for some fixed $t \in N$, $t_{i+1}' = t_i$, $t_{i+1} \geq t_i$ for all $i \in [1, n-2]$ and $t_1' = t_1 = t$ when n is odd. Let $u_{ij}(i \in [1, n-1], i \in [1, t_i]), v_{ij}(i \in [1, n-1], j \in [1, t_i']),$ and $w_{ij}(i \in [1, n], j \in [1, t])$ be the vertices which are used for arbitrary super subdivision. Therefore, the edge set of $ASS(SL_n)$ is $E(ASS(SL_n) = \{\{u_iu_{ij}, u_{ij}u_{i+1} : i \in [1, n-1], j \in [1, t_i]\} \cup \{v_iv_{ij}, v_{ij}v_{i+1} : i \in [1, n-1], j \in [1, t_i^{'}]\} \cup \{u_iw_{ij}, w_{ij}v_{i+1} : i \in [1, n-1], j \in [1, t]\}\}$. Then, it obvious that $ASS(SL_n)$ has 2n+$2\sum_{i=1}^{n-2} t_i + t_{(n-1)} + (n-1)t + t_1$ vertices and $4\sum_{i=1}^{n-2} t_i + 2((n-1)t + t_{(n-1)} + t_1)$ edges. Define labeling $f : V(SS(C_n \odot K_1)) \to \{0, 2, 4, ..., 2q-2, 2q = 8\sum_{i=1}^{n-2} t_i + 4((n-1)t + t_{(n-1)} + t_1)\}$ as follows:

Case (i). n is odd.

$$f(u_i) = \begin{cases} 0, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1), & i \in [1, n-1] \\ 8\sum_{s=1}^{n-2} t_s + 4(nt + t_{(n-1)}), & i = n. \end{cases}$$

$$f(v_i) = \begin{cases} 4t, & i = 1 \\ 8t, & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4t(i-1), & i \in [3, n] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4ti, & i \in [3, n] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4t + 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 4j - 2, & i \in [2, n-2] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t(i+1) + t_i) + 4j - 2, & i \in [2, n-2] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4nt + 4j - 2, & i = n-1. \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f(v_{ij}) = \begin{cases} 8t + 4j - 2, & i = 1 \\ 12t + 4j - 2, & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4t(i-1) + 4j - 2, & i \in [3, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4(ti + t_{(i-1)}) + 4j - 2, & i \in [3, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f(w_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(ti + t_i) + 4j - 2, & i \in [2, n-2] \\ 8\sum_{s=1}^{n-2} t_s + 4t(n-1) + 4j - 2, & i = n-1. \end{cases}$$

Hence the induced edge labeling f^* is obtained as follows:

For $i \in [1, t_i]$.

$$f^*(u_iu_{ij}) = \begin{cases} 2t + 2j - 1, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 2j - 1, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1) + 2t_i + 2j - 1, & i \in [1, n-2] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-1} t_s + 4nt + 2j - 1, & i = n-1. \end{cases}$$

$$f^*(u_{ij}u_{(i+1)}) = \begin{cases} 12t+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1) + 4t_i + 2j - 1, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4t(i+1) + 6t_i + 2j - 1, & i \in [1, n-2] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4nt + 2t_{n-1} + 2j - 1, & i = n-1. \end{cases}$$

For $i \in [1, t_i^{'}]$.

$$f^*(v_iv_{ij}) = \begin{cases} 6t+2j-1, & i=1 \\ 10t+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s + 4t(i-1) + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 2t_{(i-1)} + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

$$f^*(v_{ij}v_{(i+1)}) = \begin{cases} 8t+2j-1, & i=1 \\ 14t+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 4t_{(i-1)} + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 6t_{(i-1)} + 2j - 1, & i \in [3, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f^*(u_iw_{ij}) = \begin{cases} 2j-1, & i=1 \\ 8\sum_{s=1}^{i-2} t_s + 4ti + 2t_i + 2j - 1, & i \in [2, n-2] \\ 8\sum_{s=1}^{n-2} t_s + 4nt - 2t + 2j - 1, & i = n-1. \end{cases}$$

$$f^*(w_{ij}v_{(i+1)}) = \begin{cases} 4t+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 2(t+t_i) + 2j - 1, & i \in [2, n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4ti + 2t_i + 2j - 1, & i \in [2, n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(n-1)t + 2j - 1, & i = n-1. \end{cases}$$

Then, f is an even-odd mean labeling of $ASS(SL_n)$. Thus, $ASS(SL_n)$ is an even-odd mean graph.
Case (ii). n is even.

$$f(u_i) = \begin{cases} 4(t_1^{'}+t), & i=1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + ti), & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1^{'} + t(i-1)), & i \in [2, n-1] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1^{'} + t(n-1) + t_{(n-1)}), & i = n. \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 4t_1', & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1' + t(i-1)), & i \in [1, n] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1' + t(i-2)), & i \in [1, n] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(u_{ij}) = \begin{cases} 4(t_1' + t_1 + t) + 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1' + ti + t_i) + 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1' + t(i-1)) + 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1' + t(n-1)) + 4j - 2, & i = n-1. \end{cases}$$

For $i \in [1, t_i']$.

$$f(v_{ij}) = \begin{cases} 4j - 2, & i = 1 \\ 4t_1' + 4j - 2, & i = 2 \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1' + t(i-1) + & \\ t_{(i-1)}) + 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s + 4(t_1' + t(i-2)) + & \\ 4j - 2, & i \in [2, n-1] \quad \text{and } i \text{ is even.} \end{cases}$$

For $i \in [1, t_i]$.

$$f(w_{ij}) = \begin{cases} 4(t_1' + t_1) + 4j - 2, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1' + t(i-1) + t_i) + 4j - 2, & i \in [2, n-2] \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1' + t(n-2)) + 4j - 2, & i = n-1. \end{cases}$$

Thus, the induced edge labeling f^* is obtained as follows:
For $i \in [1, t_i]$.

$$f^*(u_i u_{ij}) = \begin{cases} 4(t_1' + t) + 2t_1 + 2j - 1, & i = 1 \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1' + ti) + 2t_i + 2j - 1, & i \in [2, n-1], \quad \text{and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s + 4(t_1' + t(i-1)) + 2j - 1, & i \in [2, n-1], \quad \text{and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s + 4(t_1' + t(n-1)) + 2j - 1, & i = n-1. \end{cases}$$

$$f^*(u_{ij}u_{(i+1)}) =$$
$$\begin{cases} 4(t_1^{'}+t)+6t_1+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti)+6t_i+2j-1, & i\in[2,n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti+t_i)+2j-1, & i\in[2,n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s+4(t_1^{'}+t(n-1))+ & \\ 2t_{(n-1)}+2j-1 & i=n-1. \end{cases}$$

For $i\in[1,t_i^{'}]$.

$$f^*(v_iv_{ij}) =$$
$$\begin{cases} 2j-1, & i=1 \\ 4t_1^{'}+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-1))+ & \\ 2t_{(i-1)}+2j-1, & i\in[2,n-1], \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-2))+2j-1, & i\in[2,n-1], \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(v_{ij}v_{(i+1)}) =$$
$$\begin{cases} 2t_1^{'}+2j-1, & i=1 \\ 4(t_1^{'}+t+t_1)+2j-1, & i=2 \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-1))+ & \\ 6t_{(i-1)}+2j-1, & i\in[3,n-1], \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-2} t_s+4(t_1^{'}+t(i-1)+ & \\ t_{(i-1)})+2j-1, & i\in[3,n-1], \text{ and } i \text{ is even.} \end{cases}$$

For $i\in[1,t]$.

$$f^*(u_iw_{ij}) =$$
$$\begin{cases} 4t_1^{'}+2(t+t_1)+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti)+2(t_i-t)+2j-1, & i\in[2,n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+t(i-1))+2t_i+2j-1, & i\in[2,n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s+4(t_1^{'}+tn)-6t+2j-1, & i=n-1. \end{cases}$$

$$f^*(w_{ij}v_{(i+1)}) =$$
$$\begin{cases} 4t_1^{'}+2t_1+2j-1, & i=1 \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+t(i-1))+2t_i+2j-1, & i\in[2,n-2] \text{ and } i \text{ is odd} \\ 8\sum_{s=1}^{i-1} t_s+4(t_1^{'}+ti)+2(t_i-t)+2j-1, & i\in[2,n-2] \text{ and } i \text{ is even} \\ 8\sum_{s=1}^{n-2} t_s+4(t_1^{'}+t(n-2))+2j-1, & i=n-1. \end{cases}$$

Then, f is an even-odd mean labeling of $ASS(SL_n)$. Thus, $ASS(SL_n)$ is an even-odd mean graph. □

Illustration 2.4. Consider $ASS(SL_5)$ where the edges $u_iu_{i+1}, i \in [1,4]$, are replaced by $K_{2,2}, K_{2,3}, K_{2,4}$ and $K_{2,3}$ respectively and the edges $v_iv_{i+1}, i \in [1,4]$ are replaced by $K_{2,2}, K_{2,2}, K_{2,3}, K_{2,4}$ respectively and all the edges $u_ivi+1, i \in [1,4]$, are replaced $K_{2,2}$. An even-odd mean labeling of $ASS(SL_5)$ is shown in Figure 4.

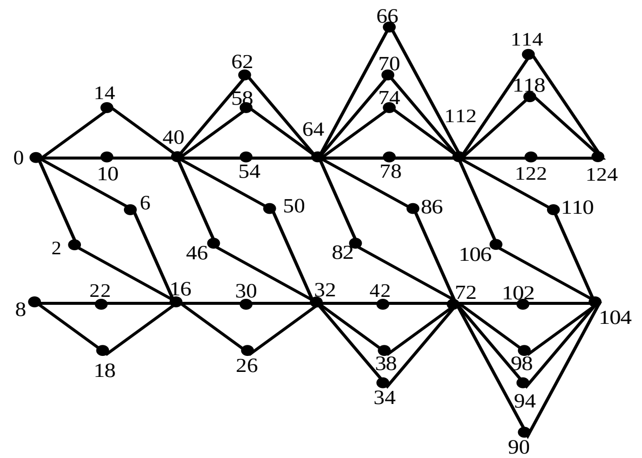

Figure 4: An even-odd mean graph of $ASS(SL_5)$

Illustration 2.5. Consider $ASS(SL_6)$ where the edges $u_iu_{i+1}, i \in [1,5]$, are replaced by $K_{2,2}, K_{2,3}, K_{2,4}, K_{2,5}, K_{2,3}$ respectively and the edges $v_iv_{i+1}, i \in [1,5]$ are replaced by $K_{2,4}, K_{2,2}, K_{2,3}, K_{2,4}, K_{2,5}$ respectively and all the edges $u_ivi+1, i \in [1,5]$, are replaced $K_{2,3}$. An even-odd mean labeling of $ASS(SL_6)$ is shown in Figure 5.

Theorem 2.5. *$ASS(P_m \times P_n)$ is an even-odd mean graph where the edges $u_{ij}u_{i(j+1)}$, $(i \in [1,m], j \in [1,n-1])$, $u_{ij}u_{(i+1)j}(i \in [1,m-1], j \in [1,n])$ of $P_m \times P_n$ are replaced by $K_{2,t_{ij}}$, $K_{2,t'_{ij}}$ respectively such that t_{ij} are equals for all j and t'_{ij} are equals for all i.*

Proof. Let the vertex set of planar grid $P_m \times P_n$ be $V = \{u_{ij} : i \in [1,m], j \in [1,n]\}$ and the edge set be $E = \{\{e_{ij} = u_{ij}u_{i(j+1)} : i \in [1,m], j \in [1,n-1]\} \cup \{e'_{ij} = u_{ij}u_{(i+1)j} : i \in [1,m-1], j \in [1,n]\}$. Let $ASS(P_m \times P_n)$ be an arbitrary super subdivision of the planar grid $P_m \times P_n$. The horizontal and vertical edges e_{ij}, e'_{ij} are replaced by the bicliques $K_{2,t_{ij}}, K_{2,t'_{ij}}$ respectively where t_{ij}, t'_{ij} are positive integer numbers, t_{ij} are equals for all j and t'_{ij} are equal for all i. Let $v_{ij,k}(i \in [1,m], j \in [1,n-1], k \in [1,t_{ij}])$ and $w_{ij,k}(i \in [1,m-1], j \in [1,n], k \in [1,t'_{ij}])$ be the vertices which are used for arbitrary super subdivision of the edges e_{ij} and e'_{ij} respectively. Thus, the edge set of $ASS(P_m \times P_n)$ is $E(ASS(P_m \times P_n)) = \{\{u_{ij}v_{ij,k}, v_{ij,k}u_{i(j+1)} : i \in [1,m], j \in [1,n-1], k \in$

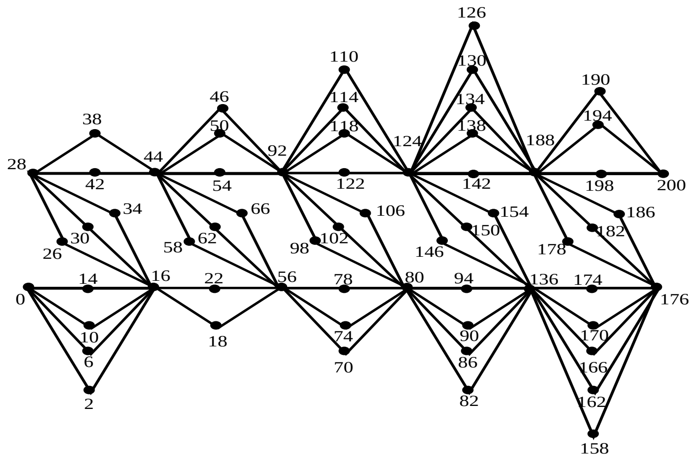

Figure 5: An even-odd mean graph of $ASS(SL_6)$

$[1,t_{ij}]\} \cup \{u_{ij}w_{ij,k}, w_{ij,k}u_{(i+1)j} : i \in [1,m-1], j \in [1,n], k \in [1,t'_{ij}]\}$. Therefore, it is clear that $ASS(P_m \times P_n)$ has $mn + m\sum_{s=1}^{n-1} t_{is} + n\sum_{s=1}^{m-1} t'_{sj}$ vertices and $2(m\sum_{s=1}^{n-1} t_{is} + n\sum_{s=1}^{m-1} t'_{sj})$ edges. Define labeling $f : V(ASS(P_m \times P_n)) \rightarrow \{0,2,4,...,2q-2,2q = 4(m\sum_{s=1}^{n-1} t_{is} + n\sum_{s=1}^{m-1} t'_{sj})\}$ as follows:

$$f(u_{1j}) = \begin{cases} 0, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s}, & j \in [2,n]. \end{cases}$$

$$f(u_{ij}) =$$
$$\begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1}, & j = 1, i \in [2,m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + 4\sum_{s=1}^{j-1} t_{is}, & j \in [2,n], i \in [2,m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1}, & j = 1, i \in [2,m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j-1} t_{is}, & j \in [2,n], i \in [2,m] \text{ and } i \text{ is even.} \end{cases}$$

For $k \in [1, t_{ij}]$.

$$f(v_{1j,k}) = \begin{cases} 4t'_{11} + 4k - 2, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 4jt'_{1j} + 4k - 2, & j \in [2,n-1]. \end{cases}$$

$$f(v_{2j,k}) = \begin{cases} 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{11} - 4t_{21} + 4k - 2, & j = 1 \\ 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{1j} - 4\sum_{s=1}^{j} t_{2s} + 4k - 2, & j \in [2,n-1]. \end{cases}$$

$f(v_{ij,k}) =$

$$\begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{s1} & \\ +4t'_{(i-1)1} + 4k - 2, & j = 1, i \in [3,m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{sj} & \\ +4jt'_{(i-1)j} + 4\sum_{s=1}^{j-1} t_{is} + 4k - 2, & j \in [2,n-1], i \in [3,m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} & \\ -4t'_{(i-1)1} - 4t_{i1} + 4k - 2, & j = 1, i \in [3,m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j} t_{is} & \\ -4jt'_{(i-1)j} + 4k - 2, & j \in [2,n-1], i \in [3,m] \text{ and } i \text{ is even.} \end{cases}$$

For $k \in [1, t'_{ij}]$.

$$f(w_{1j,k}) = \begin{cases} 4k-2, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 4(j-1)t'_{1j} + 4k - 2, & j \in [2, n-1]. \end{cases}$$

$f(w_{ij,k}) =$

$$\begin{cases} 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{s1} & \\ -4(t'_{i1}) + 4k - 2, & j = 1, i \in [2, m-1] \text{ and } i \text{ is odd} \\ 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{sj} & \\ -4\sum_{s=1}^{j-1} t_{is} - 4j(t'_{ij}) + 4k - 2, & j \in [2,n-1], i \in [2,m-1] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} & \\ +4k-2, & j = 1, i \in [2,m-1] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} & \\ +4(j-1)t'_{ij} + 4\sum_{s=1}^{j-1} t_{is} + 4k - 2, & j \in [2,n-1], i \in [2,m-1] \text{ and } i \text{ is even.} \end{cases}$$

Then, the induced edge labeling f^* is obtained as follows:
For $k \in [1, t_{ij}]$.

$$f^*(u_{1j}\ v_{1j,k}) = \begin{cases} 2t'_{11} + 2k - 1, & j = 1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 2jt'_{1j} + 2k - 1, & j \in [2, n-1]. \end{cases}$$

$f^*(u_{2j}\ v_{2j,k}) =$

$$\begin{cases} 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{11} - 2t_{21} + 2k - 1, & j = 1 \\ 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{1j} - 4\sum_{s=1}^{j-1} t_{2s} - 2t_{2j} + 2k - 1, & j \in [2, n-1]. \end{cases}$$

$$f^*(u_{ij}\ v_{ij,k}) = \begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{s1} & \\ +2(n+1)t'_{(i-1)1} + 2k-1, & j=1, i\in[3,m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{sj} & \\ +2(n+j)t'_{(i-1)j} + 4\sum_{s=1}^{j-1} t_{is} + 2k-1, & j\in[2,n-1], i\in[3,m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} - 2t'_{(i-1)1} & \\ -2t_{i1} + 2k-1, & j=1, j\in[3,m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j-1} t_{is} & \\ -2jt'_{(i-1)j} - 2t_{ij} + 2k-1, & j\in[2,n-1], j\in[3,m] \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(v_{1j,k}\ u_{1(j+1)}) = \begin{cases} 2t_{11} + 2t'_{11} + 2k-1, & j=1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 2t_{ij} + 2jt'_{1j} + 2k-1, & j\in[2,n-1]. \end{cases}$$

$$f^*(v_{2j,k}\ u_{2(j+1)}) = \begin{cases} 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{11} - 4t_{21} + 2k-1, & j=1 \\ 8\sum_{s=1}^{n-1} t_{2s} + 4nt'_{1j} - 4\sum_{s=1}^{j} t_{2s} + 2k-1, & j\in[2,n-1]. \end{cases}$$

$$f^*(v_{ij,k}\ u_{i(j+1)}) = \begin{cases} 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{s1} & \\ +2(n+1)t'_{(i-1)1} + 2t_{i1} + 2k-1, & j=1, i\in[3,m] \text{ and } i \text{ is odd} \\ 4(i-1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-2} t'_{sj} & \\ +4\sum_{s=1}^{j-1} t_{is} + 2(n+j)t'_{(i-1)j} & \\ +2t_{ij} + 2k-1, & j\in[2,n-1], i\in[3,m] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} - 2t'_{(i-1)1} & \\ -4t_{i1} + 2k-1, & j=1, j\in[3,m] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} - 4\sum_{s=1}^{j} t_{is} & \\ -2jt'_{(i-1)j} + 2k-1, & j\in[2,n-1], i\in[3,m] \text{ and } i \text{ is even.} \end{cases}$$

For $k \in [1, t'_{ij}]$.

$$f^*(u_{1j}\ w_{1j,k}) = \begin{cases} 2k-1, & j=1 \\ 4\sum_{s=1}^{j-1} t_{1s} + 2(j-1)t'_{1j} + 2k-1, & j\in[2,n] \end{cases}$$

$$f^*(u_{ij}\ w_{ij,k}) = \begin{cases} 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} + \\ 2(n-1)t'_{i1} + 2k - 1, & j=1, i\in[2,m-1] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + \\ 2nt'_{ij} - 2t'_{ij} + 2k - 1, & j\in[2,n], i\in[2,m-1] \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} + \\ 2k - 1, & j=1, i\in[2,m-1] \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + \\ 2(j-1)t'_{ij} + 2k - 1, & i\in[2,n], i\in[2,m-1] \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(w_{1j,k}\ u_{2j}) = \begin{cases} 4\sum_{s=1}^{n-1} t_{1s} + 2nt'_{11} + 2k - 1, & j=1 \\ 4\sum_{s=1}^{n-1} t_{1s} + 2(n+j-1)t'_{1j} + 2k - 1, & j\in[2,n]. \end{cases}$$

For $i \in [2, m-1]$.

$$f^*(w_{ij,k}\ u_{(i+1)j}) = \begin{cases} 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{s1} \\ -2t'_{i1} + 2k - 1, & j=1, \text{ and } i \text{ is odd} \\ 4(i+1)\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i} t'_{sj} \\ -4\sum_{s=1}^{j-1} t_{is} - 2jt'_{ij} + 2k - 1, & j\in[2,n], \text{ and } i \text{ is odd} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{s1} \\ +2nt'_{i1} + 2k - 1, & j=1, \text{ and } i \text{ is even} \\ 4i\sum_{s=1}^{n-1} t_{is} + 4n\sum_{s=1}^{i-1} t'_{sj} + 4\sum_{s=1}^{j-1} t_{is} \\ +2(n+j-1)t'_{ij} + 2k - 1, & j\in[2,n], \text{ and } i \text{ is even.} \end{cases}$$

Hence, f is an even-odd mean labeling of $ASS(P_m \times P_n)$. Thus, $ASS(P_m \times P_n)$ is an even-odd mean graph. □

Illustration 2.6. Consider $ASS(P_6 \times P_5)$ where the edges $u_{i1}u_{i2}, u_{i2}u_{i3}, u_{i3}u_{i4}$ and $u_{i4}u_{i5}$ are replaced by $K_{2,5}, K_{2,2}, K_{2,4}$ and $K_{2,5}$ respectively for all $i \in [1, 6]$ and the edges $u_{1j}u_{2j}, u_{2j}u_{3j}, u_{3j}u_{4j}, u_{4j}u_{5j}$ and $u_{5j}u_{6j}$ are replaced by $K_{2,2}, K_{2,4}, K_{2,3}, K_{2,5}$ and $K_{2,2}$ respectively for all $j \in [1, 5]$. An even-odd mean labeling of $ASS(P_6 \times P_5)$ is shown in Figure 6.

Corollary 2.1. *$ASS(L_n)$ is an even-odd mean graph for all n.*

Proof. From the definition of Ladder L_n and by Theorem 2.5, the arbitrary super subdivision of L_n is also an even-odd mean graph. □

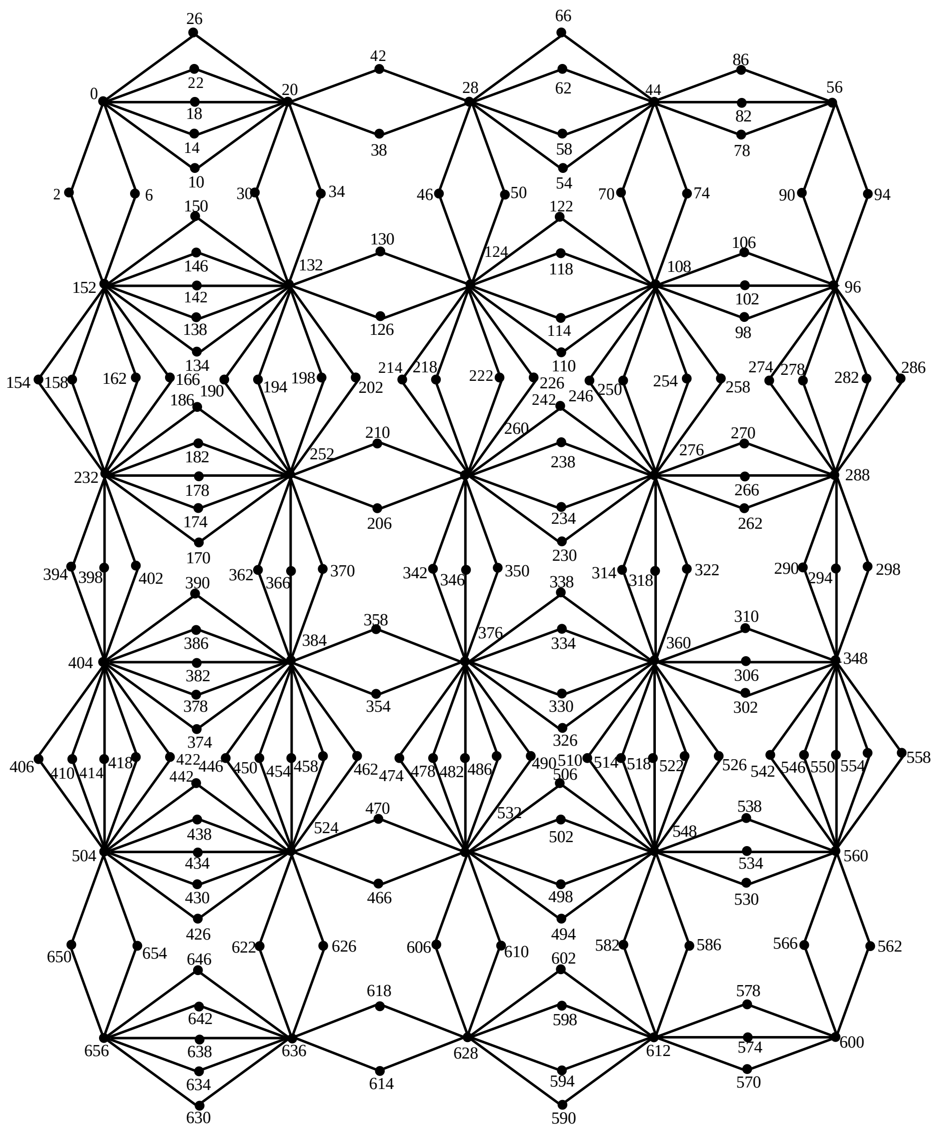

Figure 6: An even-odd mean graph of $ASS(P_6 \times P_5)$

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The separator of Green's classes of the full transformation semigroup

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Abstract. This paper investigates the separator of Green's classes of the full transformation semigroup. The separator of a subset A of a semigroup S is the set of all elements $x \in S$ satisfying the following conditions: $xA \subseteq A$, $Ax \subseteq A$, $x(S\backslash A) \subseteq S\backslash A$ and $(S\backslash A)x \subseteq S\backslash A$. We establish the relationship between the separator of Green's classes and the permutations preserving partition and/or permuting image.

Keywords: semigroup, full transformation semigroup, Green's Relations, symmetric group.
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1. Introduction

The separator of a subset A of a semigroup S is the set of all elements $x \in S$ satisfying the following conditions: $xA \subseteq A$, $Ax \subseteq A$, $x(S\backslash A) \subseteq S\backslash A$ and $(S\backslash A)x \subseteq S\backslash A$. Let π be an equivalence relation on a set X. We say that $\alpha : X \to X$ preserves π if, for all $x, y \in X$, $(x, y) \in \pi$ implies $(x\alpha, y\alpha) \in \pi$. Let T_n and S_n denote the full transformation semigroup and symmetric group, respectively, on $\underline{n} = \{1, \ldots, n\}$. Denote by $S_n(\pi)$ the set of all permutations on $\underline{n}$ that preserve π. For a nonempty subset Y of $\underline{n}$, denote by $S_n(Y)$ the set of all permutations on $\underline{n}$ that permute Y. Moreover, let $S_n(\pi, Y) = S_n(\pi) \cap S_n(Y)$. The Green's relations on a semigroup were first studied by J.A. Green [7] in 1951. Let a and b be elements of a semigroup S. We define $a\mathscr{L}b$ ($a\mathscr{R}b$) if a and b generate the same principal left (right) ideal of S. The join of $\mathscr{L}$ and $\mathscr{R}$ is

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denoted by $\mathscr{D}$ and their intersection by $\mathscr{H}$ (see [3]). In 2011, A. Nagy proved that the separator of a proper ideal of T_n is the symmetric group S_n. Guided by the result put forth by C.G. Doss [5], we will describe the separator of the Green's classes of T_n. Following the convention used in [3], by a partition π of a set X we mean the partition X/π determined by an equivalence relation π on X. First, we show that the separator of a $\mathscr{D}$-class of T_n is the symmetric group S_n. Then, we prove that $S_n(Y)$ is the separator of the $\mathscr{L}$-class consisting of all elements of T_n whose image is Y. Next, we show that $S_n(\pi)$ is the separator of the $\mathscr{R}$-class consisting of all elements of T_n with partition π. Finally, we show that $S_n(\pi, Y)$ is the separator of the $\mathscr{H}$-class consisting of all elements of T_n with partition π and image Y.

2. Preliminaries

The following definitions are found in [3]. A *transformation* of a set X is a single-valued mapping of X into itself. The image of an element x of X under a transformation or mapping α is denoted by $x\alpha$ (rather than αx or $\alpha(x)$). The *product* (or *iterate* or *composition*) of two transformations α and β of X is the transformation $\alpha\beta$ defined by $x(\alpha\beta) = (x\alpha)\beta$, for all $x \in X$ (that is, α followed by β). The set T_X of all transformations of X is a semigroup with respect to iteration. We call T_X the *full transformation semigroup on* X. A one-to-one mapping of a set X onto itself will be called a *permutation of* X, even when X is infinite. The *symmetric group* S_X *on* X consists of all permutations of X under the operation of iteration.

Definition 2.1 ([3]). *With each element α of T_X we associate two things: (1) the image $X\alpha$ of α, also denoted by $Im(\alpha)$, which is defined by $X\alpha = \{x\alpha \mid x \in X\}$ and (2) the partition $\pi_\alpha = \alpha \circ \alpha^{-1}$ of X corresponding to α, i.e., the equivalence relation on X defined by $(x, y) \in \pi_\alpha$ if $x\alpha = y\alpha$, where $x, y \in X$. Let $\pi_\alpha^\natural$ be the natural mapping of X upon the set X/π_α of equivalence classes of X mod π_α. Then, $x\pi_\alpha^\natural \mapsto x\alpha$ is a one-to-one mapping of X/π_α upon $X\alpha$. It follows that $|X/\pi_\alpha| = |X\alpha|$, and this cardinal number is called the rank of α.*

The following theorem characterizes Green's classes in terms of rank, partition, and image.

Theorem 2.1 ([3]). *Let T_X be the full transformation semigroup on a set X.*

i. In the semigroup T_X, we have $\mathscr{D} = \mathscr{J}$.

ii. There is a one-to-one correspondence between the set of all principal ideals of T_X and the set of all cardinal numbers $r \leq |X|$ such that the principal ideal corresponding to r consists of all elements of T_X of rank $\leq r$.

iii. There is a one-to-one correspondence between the set of all $\mathscr{D}$-classes of T_X and the set of all cardinal numbers $r \leq |X|$ such that the $\mathscr{D}$-class D_r corresponding to r consists of all elements of T_X of rank r.

iv. *Let r be a cardinal number $\leq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{L}$-classes in D_r and the set of all subsets Y of X of cardinal r such that the $\mathscr{L}$-class corresponding to Y consists of all elements of T_X having image Y.*

v. *Let r be a cardinal number $\leq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{R}$-classes contained in D_r and the set of all partitions π of X for which $|X/\pi| = r$ such that the $\mathscr{R}$-class corresponding to π consists of all elements of T_X having partition π.*

vi. *Let r be a cardinal number $\leq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{H}$-classes in D_r and the set of all pairs (π, Y) where π is a partition of X and Y is a subset of X such that $|X/\pi| = |Y| = r$, such that the $\mathscr{H}$-class corresponding to (π, Y) consists of all elements of T_X having partition π and image Y.*

Throughout this paper, we will only consider the finite full transformation semigroup. Let T_n and S_n denote the full transformation semigroup and symmetric group, respectively, on $\underline{n} = \{1, \ldots, n\}$.

Lemma 2.1 ([6])**.** *Let $\alpha \in T_n$. Then, the following conditions are equivalent:*

i) *α is surjective.*

ii) *α is injective.*

iii) *α is bijective.*

Lemma 2.2 ([4])**.** *Let $\alpha, \beta \in T_n$. Then, $rank(\alpha\beta) \leq \min\{rank(\alpha), rank(\beta)\}$.*

Lemma 2.3 ([2])**.** *If $\alpha \in S_n$ and $\beta \in T_n$, then $rank(\alpha\beta) = rank(\beta\alpha) = rank(\beta)$.*

Next, we introduce notations for the Green's classes of T_n. Let $k \leq n$. We denote by D_k the set of all $\alpha \in T_n$ whose rank is k. For a partition π of $\underline{n}$ and $Y \subseteq \underline{n}$ where $|\underline{n}/\pi| = |Y| = k$, let $L_k(Y)$ be the set of all $\alpha \in D_k$ with image Y. Moreover, let $R_k(\pi)$ be the set of all $\alpha \in D_k$ with $\pi_\alpha = \pi$. Finally, we denote by $H_k(\pi, Y)$ the set of all $\alpha \in D_k$ with $\pi_\alpha = \pi$ and $Im\alpha = Y$. Then, $H_k(\pi, Y) = R_k(\pi) \cap L_k(Y)$. By Theorem 2.1, D_k, $L_k(Y)$, $R_k(\pi)$, and $H_k(\pi, Y)$ are precisely the $\mathscr{D}$-, $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{H}$-classes of T_n.

Definition 2.2 ([8])**.** *Let S be a semigroup and let $A \subseteq S$. The separator of A, denoted by $Sep(A)$, is the set of all elements $x \in S$ satisfying the following conditions: $xA \subseteq A$, $Ax \subseteq A$, $x(S \backslash A) \subseteq S \backslash A$ and $(S \backslash A)x \subseteq S \backslash A$.*

2.1 Transformations preserving a partition

Definition 2.3 ([1]). *Let $\mathcal{P}$ be a partition of a set X. We say that $\alpha \in T_X$ preserves $\mathcal{P}$ if, for all $P \in \mathcal{P}, \exists Q \in \mathcal{P}$ such that $P\alpha \subseteq Q$.*

Let $T(X, \mathcal{P})$ denote the semigroup of all full transformations of X that preserve the partition $\mathcal{P}$. We now define a transformation preserving an equivalence relation π. It is straightforward to show that this definition is equivalent to the definition of a transformation preserving X/π.

Definition 2.4. *Let π be an equivalence relation on a set X. We say that $\alpha \in T_X$ preserves π if, for all $x, y \in X$, $(x, y) \in \pi$ implies $(x\alpha, y\alpha) \in \pi$.*

Definition 2.5 ([10]). *Let E be an equivalence relation on a set X. A selfmap $\alpha : X \to X$ is said to be E^*-preserving if α satisfies the following: $(x, y) \in E$ if and only if $(x\alpha, y\alpha) \in E$.*

Remark 2.1. In view of Definition 2.4, an E^*-preserving map preserves E and satisfies the condition that $(x\alpha, y\alpha) \in E$ implies $(x, y) \in E$.

Definition 2.6 ([10]). *Let $\mathcal{P} = \{X_i \mid i \in I\}$ be a partition of an arbitrary set X, and let $\alpha \in T(X, \mathcal{P})$. The character of α is a selfmap $\chi^{(\alpha)} : I \to I$ defined by $i\chi^{(\alpha)} = j$ whenever $X_i\alpha \subseteq X_j$.*

Denote by $\Sigma(X, \mathcal{P})$ the set of all $\alpha \in T(X, \mathcal{P})$ whose image intersects every block of $\mathcal{P}$. Sarkar and Singh [10] gave a characterization of elements in $\Sigma(X, \mathcal{P})$. It is useful in proving our result on the separator of an $\mathscr{R}$-class.

Corollary 2.1 ([10]). *Let $\mathcal{P} = \{X_1, \dots, X_m\}$ be an m-partition associated with an equivalence relation E on an arbitary set X, and let $\alpha \in T(X, \mathcal{P})$. Then, the following statements are equivalent:*

(i) $\alpha \in \Sigma(X, \mathcal{P})$.

(ii) $\chi^{(\alpha)}$ *is a bijective map on* $\{1, \dots, m\}$.

(iii) α *is an* E^**-preserving map.*

3. Main Results

In view of the definition of the separator of a subset of a semigroup [8], we have the following remark.

Remark 3.1. Let S be a semigroup. Let $A \subseteq S$ and $x \in S$. Then, $x \in Sep(A)$ if and only if x satisfies the following four conditions:

i) $xa \in A$, for all $a \in A$.

ii) $ax \in A$, for all $a \in A$.

iii) $xb \in S\backslash A$, for all $b \in S\backslash A$.

iv) $bx \in S\backslash A$, for all $b \in S\backslash A$.

Remark 3.2 ([8])**.** Let S be a semigroup. Then, $Sep(\emptyset) = Sep(S) = S$.

Using Theorem 2.2 (ii), Nagy proved the following result.

Theorem 3.1 ([8])**.** *If I is a proper ideal of T_n, then $Sep(I) = S_n$.*

3.1 The separator of $\mathscr{D}$-classes

Lemma 3.1. *If $k \geq 2$ and $\beta \in T_n\backslash S_n$, then $\exists \alpha \in D_k$ such that $rank(\alpha\beta) \leq k-1$.*

Proof. Suppose $k \geq 2$ and $\beta \in T_n\backslash S_n$. Then, $\exists x \neq y$ such that $x\beta = y\beta$. Choose an element $\alpha \in D_k$ such that $x, y \in Im\alpha$. Then, $|\underline{n}/\pi_\alpha| = |Im\alpha| = k$ so we may choose distinct elements $p_1, p_2, \ldots, p_k \in \underline{n}$ such that the equivalence classes $[p_s]_{\pi_\alpha}$ and $[p_t]_{\pi_\alpha}$ are disjoint for $s \neq t$. Let $m_i = p_i\alpha$ for $i = 1, 2, \ldots, k$. Then, $Im\alpha = \{m_1, m_2, \ldots, m_k\}$. Since $x, y \in Im\alpha$, we have $x = m_{i_1}$ and $y = m_{i_2}$, for some $1 \leq i_1, i_2 \leq k$ with $i_1 \neq i_2$; hence, $m_{i_1}\beta = x\beta = y\beta = m_{i_2}\beta$. Note that, $(Im\alpha)\beta = \{m_i\beta \mid i = i_1, i_2\} \cup \{m_i\beta \mid i \in \{1, 2, \ldots, k\}\backslash\{i_1, i_2\}\}$. Therefore, $|Im(\alpha\beta)| = |(Im\alpha)\beta| \leq 1 + (k-2) = k-1$. □

Applying Lemma 2.3, we have the following results.

Lemma 3.2. *If $\alpha \in S_n, \beta \in D_k$, and $\gamma \in T_n\backslash D_k$, then $\alpha\beta, \beta\alpha \in D_k$ and $\alpha\gamma, \gamma\alpha \in T_n\backslash D_k$.*

Lemma 3.3. *If $\alpha \in S_n$ and $\beta \in \bigcup_{i=1}^m D_{k_i}$, then $\alpha\beta, \beta\alpha \in \bigcup_{i=1}^m D_{k_i}$.*

Lemma 3.4. *Let $\alpha, \gamma \in T_n$. If $\alpha \in S_n$ and $\gamma \notin \bigcup_{i=1}^m D_{k_i}$, where $m < n$, then $\alpha\gamma, \gamma\alpha \notin \bigcup_{i=1}^m D_{k_i}$.*

Theorem 3.2. *$Sep(D_k) = S_n$*

Proof. If $n = 1$, then $D_1 = S_1 = T_1$. By Remark 3.2, $Sep(D_1) = Sep(T_1) = T_1 = S_1$. Suppose $n \geq 2$ and $k = 1$. Note that, D_1 is a proper ideal of T_n. By Theorem 3.1, $Sep(D_1) = S_n$. Suppose $k \geq 2$. By Lemma 3.2, $S_n \subseteq Sep(D_k)$. Suppose $\beta \notin S_n$. By Lemma 3.1, $\exists \alpha \in D_k$ such that $rank(\alpha\beta) \leq k-1$. Hence, $\alpha\beta \notin D_k$. Therefore, $\beta \notin Sep(D_k)$. □

Next, we investigate the separator of union of $\mathscr{D}$-classes. The following result is a generalization of Theorem 3.1.

Theorem 3.3. *If $1 \leq k_1 < \ldots < k_m \leq n$ where $m < n$, then $Sep(\bigcup_{i=1}^m D_{k_i}) = S_n$.*

Proof. If $m = 1$, apply Theorem 3.2. Suppose $m \geq 2$. By Lemmas 3.3 and 3.4, $S_n \subseteq Sep(\bigcup_{i=1}^{m} D_{k_i})$. Suppose $\alpha \notin S_n$.

Case 1. $k_1 \geq 2$. By Lemma 3.1, $\exists \beta \in D_{k_1}$ such that $rank(\beta\alpha) \leq k_1 - 1$. Hence, $\beta\alpha \notin \bigcup_{i=1}^{m} D_{k_i}$. Therefore, $\alpha \notin Sep(\bigcup_{i=1}^{m} D_{k_i})$.

Case 2. $k_1 = 1$. Suppose $k_1, k_2, \ldots, k_m$ are consecutive positive integers. Then, $\bigcup_{i=1}^{m} D_{k_i}$ is a proper ideal of T_n. Since $\alpha \notin S_n$, by Theorem 3.1, $\alpha \notin Sep(\bigcup_{i=1}^{m} D_{k_i})$. Suppose $k_{i+1} - k_i > 1$, for some $1 \leq i \leq m-1$. By the Well-ordering principle, $b = \min\{i \mid k_{i+1} - k_i > 1\}$ exists. Then, $k_1, \ldots, k_b$ are consecutive positive integers and $k_b < k_b+1 < k_{b+1}$. But Lemma 3.1 tells us that $\exists \beta$ with $rank(\beta) = k_b + 1$ such that $rank(\beta\alpha) \leq k_b$. Note that, $\beta \notin \bigcup_{i=1}^{m} D_{k_i}$ but $\beta\alpha \in \bigcup_{i=1}^{m} D_{k_i}$. Therefore, $\alpha \notin Sep(\bigcup_{i=1}^{m} D_{k_i})$. □

3.2 The separator of $\mathscr{L}$-classes

Given a subset Y of $\underline{n}$ with $|Y| = k$, let $S_n(Y) = \{\alpha \in S_n \mid Y\alpha = Y\}$ and $L_k(Y) = \{\alpha \in D_k \mid Im\alpha = Y\}$.

Remark 3.3. If $n = k = 1$, then $|Y| = 1$ so that $L_1(Y) = T_1 = S_1 = S_1(Y)$. Then, $Sep(L_1(Y)) = Sep(T_1) = T_1 = S_1(Y)$.

We will show that $S_n(Y)$ is the separator of the $\mathscr{L}$-class consisting of all elements of T_n whose image is Y. The next two lemmas follow immediately from the properties of S_n and $L_k(Y)$.

Lemma 3.5. *If $\alpha \in S_n(Y)$ and $\beta \in L_k(Y)$, then $\alpha\beta, \beta\alpha \in L_k(Y)$.*

Lemma 3.6. *If $\alpha \in S_n(Y)$ and $\beta \in T_n \setminus L_k(Y)$, then $\alpha\beta, \beta\alpha \in T_n \setminus L_k(Y)$.*

For $m = 1, \ldots, n$, let c_m denote the constant transformation on $\underline{n}$ defined by $x \mapsto m$.

Theorem 3.4 ([2]). *Let $n \geq 2$. If $A = \{c_{k_1}, \ldots, c_{k_r}\}$, then $Sep(A) = S_n(K)$, where $K = \{k_1, \ldots, k_r\}$.*

Lemma 3.7. *If $k \geq 2$ and $\alpha \in T_n \setminus S_n$ with $Y\alpha = Y$, then $\exists \gamma \in L_k(Y)$ such that $\alpha\gamma \notin L_k(Y)$.*

Proof. Suppose $k \geq 2$ and $\alpha \in T_n \setminus S_n$ with $Y\alpha = Y$. Since $\alpha \notin S_n$, it is not surjective. Let $s \in \underline{n} \setminus Im\alpha$, $Y = \{y_1, \ldots, y_k\}$, and $Z = \underline{n} \setminus (Y \cup \{s\})$. Then, $s \notin Y$ since $Y = Y\alpha \subseteq Im\alpha$. For $i = 1, 2, \ldots, k$, let

$$P_i = \begin{cases} \{s\}, & \text{if } i = 1 \\ \{y_1, y_2\} \cup Z, & \text{if } i = 2 \\ \{y_i\}, & \text{if } i \notin \{1, 2\}. \end{cases}$$

Consider $\gamma : \underline{n} \to \underline{n}$ where $\underline{n}/\pi_\gamma = \{P_1, \ldots, P_k\}$ and $P_i\gamma = \{y_i\}, \forall i = 1, 2, \ldots, k$. Then, $\gamma \in L_k(Y)$ since $Im\gamma = Y$. Note that, $P_1\gamma = \{s\}\gamma = \{y_1\}$.

Claim. $y_1 \notin Im\alpha\gamma$. Suppose $y_1 \in Im\alpha\gamma$. Then, $\exists x \in Im\alpha$ such that $x\gamma = y_1 = s\gamma$. Hence, $(x, s) \in \pi_\gamma$ which implies that $x \in [s]_{\pi_\gamma} = P_1$. Then, $x = s$, a contradiction, since $s \notin Im\alpha$. Hence, $y_1 \notin Im\alpha\gamma$ which implies that $Y \neq Im\alpha\gamma$. Therefore, $\alpha\gamma \notin L_k(Y)$. □

Theorem 3.5. $Sep(L_k(Y)) = S_n(Y)$

Proof. If $n = k = 1$, by Remark 3.3, $Sep(L_1(Y)) = S_1(Y)$. Suppose $n \geq 2$ and $k = 1$. Then, $|Y| = 1$. Let $Y = \{m\}$. Then, $L_1(Y) = \{c_m\}$. By Theorem 3.4, $Sep(L_1(Y)) = S_n(Y)$. Now, suppose $k \geq 2$. By Lemmas 3.5 and 3.6, $S_n(Y) \subseteq Sep(L_k(Y))$. Suppose $\alpha \notin S_n(Y)$.

Case 1. $Y\alpha \neq Y$. Let $\beta \in L_k(Y)$. Then, $Im\beta\alpha = (Im\beta)\alpha = Y\alpha \neq Y$ which implies that $\beta\alpha \notin L_k(Y)$. Therefore, $\alpha \notin Sep(L_k(Y))$.

Case 2. $\alpha \notin S_n$ with $Y\alpha = Y$. By Lemma 3.7, $\alpha \notin Sep(L_k(Y))$. □

3.3 The separator of $\mathscr{R}$-classes

The next two lemmas are immediate from the definitions.

Lemma 3.8. *Let $\alpha, \beta \in T_X$ and $x, y \in X$. Then, $(x\alpha, y\alpha) \in \pi_\beta$ if and only if $(x, y) \in \pi_{\alpha\beta}$.*

Lemma 3.9. *If $\alpha, \beta \in T_X$, then $\pi_\alpha \subseteq \pi_{\alpha\beta}$.*

Lemma 3.10. *If $\alpha \in S_X$ and $\beta \in T_X$, then $\pi_{\beta\alpha} = \pi_\beta$.*

Proof. Let $x, y \in X$. Since α is injective,

$$x(\beta\alpha) = y(\beta\alpha) \iff (x\beta)\alpha = (y\beta)\alpha \iff x\beta = y\beta.$$ □

Let π be an equivalence relation on $\underline{n}$. Then, $\underline{n}/\pi$ is a partition of $\underline{n}$. Denote by $T_n(\pi)$ the semigroup $T(\underline{n}, \underline{n}/\pi)$. Moreover, let $\Sigma_n(\pi) = \Sigma(\underline{n}, \underline{n}/\pi)$ and $S_n(\pi) = S(\underline{n}, \underline{n}/\pi)$. Since $S_n(\pi) = T_n(\pi) \cap S_n$ and $S_n(\pi) \subseteq \Sigma_n(\pi) \subseteq T_n(\pi)$, we have $S_n(\pi) \subseteq S_n \cap \Sigma_n(\pi) \subseteq S_n \cap T_n(\pi) = S_n(\pi)$. Thus, we have the following remark.

Remark 3.4. $S_n(\pi) = S_n \cap \Sigma_n(\pi) = S_n \cap T_n(\pi)$.

Let $R_k(\pi)$ denote the $\mathscr{R}$-class consisting of all $\alpha \in D_k$ with partition π.

Lemma 3.11. *If $\alpha \in S_n$ and $\beta \in R_k(\pi)$, then $\beta\alpha \in R_k(\pi)$.*

Proof. By Lemma 3.10, $\pi_{\beta\alpha} = \pi_\beta = \pi$. Therefore, $\beta\alpha \in R_k(\pi)$. □

Lemma 3.12. *If $\alpha \in \Sigma_n(\pi)$ and $\beta \in R_k(\pi)$, then $\alpha\beta \in R_k(\pi)$.*

Proof. Let $x, y \in \underline{n}$. By Corollary 2.1, α is π^*-preserving. Then, by Lemma 3.8, $(x, y) \in \pi_{\alpha\beta} \iff (x\alpha, y\alpha) \in \pi_\beta = \pi \iff (x, y) \in \pi$. Thus, $\pi_{\alpha\beta} = \pi$. Therefore, $\alpha\beta \in R_k(\pi)$. □

Lemma 3.13. *If $\alpha \in S_n(\pi)$ and $\gamma \in T_n \backslash R_k(\pi)$, then $\alpha\gamma, \gamma\alpha \in T_n \backslash R_k(\pi)$.*

Proof. By Remark 3.4, $S_n(\pi) = S_n \cap \Sigma_n(\pi)$. Suppose $\alpha \in S_n(\pi)$ and $\gamma \in T_n \backslash R_k(\pi)$. Since $S_n(\pi)$ is a group, $\alpha^{-1} \in S_n(\pi)$. Suppose $\gamma \notin D_k$. By Lemma 3.2, $\alpha\gamma, \gamma\alpha \notin D_k$ which implies that $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$. Suppose $\pi_\gamma \neq \pi$.

Case 1. $\pi \not\subseteq \pi_\gamma$. Then, $\exists (u, v) \in \pi$ such that $(u, v) \notin \pi_\gamma$. Then, $u\gamma \neq v\gamma$. Since α is injective, $u\gamma\alpha \neq v\gamma\alpha$. Then, $(u, v) \notin \pi_{\gamma\alpha}$. Thus, $\pi \neq \pi_{\gamma\alpha}$. Let $u' = u\alpha^{-1}$ and $v' = v\alpha^{-1}$. Then, $u'\alpha = u$ and $v'\alpha = v$. Since α^{-1} preserves π, we have that

$$(u, v) \in \pi \implies (u\alpha^{-1}, v\alpha^{-1}) \in \pi \implies (u', v') \in \pi.$$

However, since $u'\alpha\gamma = u\gamma \neq v\gamma = v'\alpha\gamma$, we have $(u', v') \notin \pi_{\alpha\gamma}$. Thus, $\pi \neq \pi_{\alpha\gamma}$. Therefore, $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$.

Case 2. $\pi_\gamma \not\subseteq \pi$. Then, $\exists (x, y) \in \pi_\gamma$ such that $(x, y) \notin \pi$. Then, $x\gamma = y\gamma$ and

$$x\gamma = y\gamma \implies x\gamma\alpha = y\gamma\alpha \implies (x, y) \in \pi_{\gamma\alpha}.$$

Thus, $\pi_{\gamma\alpha} \neq \pi$. Let $x' = x\alpha^{-1}$ and $y' = y\alpha^{-1}$. Then, $x'\alpha = x$ and $y'\alpha = y$. By Corollary 2.1, α^{-1} is π^*-preserving. Then

$$(x, y) \notin \pi \implies (x\alpha^{-1}, y\alpha^{-1}) \notin \pi \implies (x', y') \notin \pi.$$

However, since $x'\alpha\gamma = x\gamma = y\gamma = y'\alpha\gamma$, we have $(x', y') \in \pi_{\alpha\gamma}$. Thus, $\pi_{\alpha\gamma} \neq \pi$. Therefore, $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$. □

Note that, $|\underline{n}/\pi| = 1$ if and only if $\underline{n}/\pi = \{\underline{n}\}$. Clearly, $R_1(\pi) \subseteq D_1$. Let $\alpha \in D_1$. Then, α has rank 1 which means that it only has one equivalence class. Then, $\pi_\alpha = \pi$. Thus, we have the following remark

Remark 3.5. $R_1(\pi) = D_1$.

Theorem 3.6. *$Sep(R_k(\pi)) = S_n(\pi)$.*

Proof. Suppose $k = 1$. By Theorem 3.2, $Sep(R_1(\pi)) = Sep(D_1) = S_n = S_n(\pi)$. Suppose $k \geq 2$. Since $S_n(\pi) = S_n \cap \Sigma_n(\pi)$, by Lemmas 3.11, 3.12, and 3.13, $S_n(\pi) \subseteq Sep(R_k(\pi))$. Now, suppose $\alpha \notin S_n(\pi)$. Let $\beta \in R_k(\pi)$.

Case 1. $\alpha \notin T_n(\pi)$. Then, α does not preserve π; hence, $\exists (x, y) \in \pi$ such that $(x\alpha, y\alpha) \notin \pi = \pi_\beta$. By Lemma 3.8, $(x, y) \notin \pi_{\alpha\beta}$. Thus, $\pi \neq \pi_{\alpha\beta}$ which implies that $\alpha\beta \notin R_k(\pi)$. Therefore, $\alpha \notin Sep(R_k(\pi))$.

Case 2. $\alpha \notin S_n$. Then, $\exists x, y \in \underline{n}$ with $x \neq y$ such that $x\alpha = y\alpha$. Suppose $(x, y) \notin \pi$. Since $x\alpha\beta = y\alpha\beta$, we have $(x, y) \in \pi_{\alpha\beta}$. Thus, $\pi_{\alpha\beta} \neq \pi$ which implies that $\alpha\beta \notin R_k(\pi)$. Therefore, $\alpha \notin Sep(R_k(\pi))$.

Suppose $(x, y) \in \pi$. Since $k \geq 2$, we can choose $q \in \underline{n}$ such that $(x, q) \notin \pi$. Consider an element $\gamma \in R_k(\pi)$ such that $x\gamma = x$ and $q\gamma = y$. Then, $x\gamma\alpha = x\alpha = y\alpha = q\gamma\alpha$ which implies that $(x, q) \in \pi_{\gamma\alpha}$. Thus, $\pi_{\gamma\alpha} \neq \pi$. It follows that $\gamma\alpha \notin R_k(\pi)$. Therefore, $\alpha \notin Sep(R_k(\pi))$. □

3.4 The separator of $\mathscr{H}$-classes

For a partition π of $\underline{n}$ and $Y \subseteq \underline{n}$ with $|\underline{n}/\pi| = |Y|$, let $H_k(\pi, Y)$ denote the $\mathscr{H}$-class consisting of all $\alpha \in D_k$ with partition π and image Y. Clearly, $H_k(\pi, Y) = R_k(\pi) \cap L_k(Y)$. Moreover, denote by $S_n(\pi, Y)$ the intersection of $S_n(\pi)$ and $S_n(Y)$. We will show that $S_n(\pi, Y)$ is the separator of $H_k(\pi, Y)$.

Lemma 3.14. *$S_n(\pi, Y) \subseteq Sep(H_k(\pi, Y))$.*

Proof. Suppose $\alpha \in S_n(\pi, Y)$. Let $\beta \in H_k(\pi, Y)$. Applying Lemma 3.5, we have $\alpha\beta, \beta\alpha \in L_k(Y)$. Then, by Lemma 3.12, $\alpha\beta \in R_k(\pi)$. Moreover, by Lemma 3.10, $\pi_{\beta\alpha} = \pi_\beta = \pi$, which implies that $\beta\alpha \in R_k(\pi)$. Therefore, $\alpha\beta, \beta\alpha \in H_k(\pi, Y)$. Let $\gamma \in T_n \backslash H_k(\pi, Y)$. Suppose $\gamma \notin R_k(\pi)$. By Lemma 3.13, $\alpha\gamma, \gamma\alpha \notin R_k(\pi)$. Suppose $\gamma \notin L_k(Y)$. By Lemma 3.6, $\alpha\gamma, \gamma\alpha \notin L_k(Y)$. Then, $\alpha\gamma, \gamma\alpha \in T_n \backslash H_k(\pi, Y)$. Therefore, $\alpha \in Sep(H_k(\pi, Y))$. □

Lemma 3.15. *If $\alpha \in T_n \backslash S_n$ with $Y\alpha = Y$ such that $(\underline{n} \backslash Im\alpha)\alpha \cap Y \neq \emptyset$, then $\exists \beta \in T_n \backslash H_k(\pi, Y)$ such that $\beta\alpha \in H_k(\pi, Y)$.*

Proof. Let $\underline{n}/\pi = \{P_1, \dots, P_k\}$ and $Y = \{y_1, \dots, y_k\}$. Suppose $\alpha \in T_n \backslash S_n$ with $Y\alpha = Y$ such that $(\underline{n} \backslash Im\alpha)\alpha \cap Y \neq \emptyset$. Let $t \in (\underline{n} \backslash Im\alpha)\alpha \cap Y$. Then, $t = sa$, for some $s \in \underline{n} \backslash Im\alpha$. Since $Y = Y\alpha$, $\exists y_m \in Y$ such that $t = y_m\alpha$. Note that, $s \notin Y$ since $Y = Y\alpha \subseteq Im\alpha$. Let $Y' = Y \backslash \{y_m\} \cup \{s\}$ and consider $\beta \in T_n$ with $\pi_\beta = \pi$ and $Im\beta = Y'$, where $P_m\beta = \{s\}$ and $P_i\beta = \{y_i\}$, for all $i \neq m$. Since $Im\beta \neq Y$, we have $\beta \notin H_k(\pi, Y)$. By Lemma 3.9, $\pi_\beta \subseteq \pi_{\beta\alpha}$.

Claim. $\pi_{\beta\alpha} \subseteq \pi_\beta$. Suppose $(x, y) \notin \pi_\beta$. Then, $x\beta \neq y\beta$. Then, at least one of $x\beta$ or $y\beta$ must belong to Y; otherwise, $x\beta = s = y\beta$, a contradiction. Suppose both are elements of Y, that is, $x\beta, y\beta \in Y$. Since $Y\alpha = Y$, the map $\alpha|_Y : Y \to Y$ is surjective hence injective. Then, $x\beta\alpha \neq y\beta\alpha$ which implies that $(x, y) \notin \pi_{\beta\alpha}$. Suppose only one of them is an element of Y. Without loss of generality, assume $x\beta \in Y$ and $y\beta \notin Y$. Then, $x\beta = y_i$, for some $i \neq m$ and $y\beta = s$. Since $\alpha|_Y$ is injective, we have

$$x\beta\alpha = y_i\alpha \neq y_m\alpha = t = s\alpha = y\beta\alpha.$$

Hence, $(x, y) \notin \pi_{\beta\alpha}$. This proves our claim. We have shown that $\pi_{\beta\alpha} = \pi_\beta = \pi$. Moreover, $P_m\beta\alpha = \{s\}\alpha = \{t\} = \{y_m\alpha\}$ and $P_i\beta\alpha = \{y_i\}\alpha = \{y_i\alpha\}$, for all $i \neq m$. Hence, $Im\beta\alpha = Y\alpha = Y$. Therefore, $\beta\alpha \in H_k(\pi, Y)$. □

Lemma 3.16. *If $\alpha \in \Sigma_n(\pi) \backslash S_n$ with $Y\alpha = Y$, then $\alpha \notin Sep(H_k(\pi, Y))$.*

Proof. Let $\underline{n}/\pi = \{P_1, \dots, P_k\}$ and $Y = \{y_1, \dots, y_k\}$. Suppose $\alpha \in \Sigma_n(\pi) \backslash S_n$ with $Y\alpha = Y$. Since $\alpha \notin S_n$, α is not surjective; hence, $\underline{n} \backslash Im\alpha \neq \emptyset$. If $(\underline{n} \backslash Im\alpha)\alpha \cap Y \neq \emptyset$, by Lemma 3.15, $\alpha \notin Sep(H_k(\pi, Y))$. Suppose $(\underline{n} \backslash Im\alpha)\alpha \cap Y = \emptyset$. Let $s \in \underline{n} \backslash Im\alpha$. Then, $s\alpha \notin Y$. Since $\underline{n}/\pi$ is a partition of $\underline{n}$, $s \in P_j$, for some j with $1 \leq j \leq k$. Then, by Corollary 2.1, $\chi^{(\alpha)}$ is bijective; hence there exists m with $1 \leq m \leq k$ such that $m\chi^{(\alpha)} = j$, that is, $P_m\alpha \subseteq P_j$. Let $z \in P_m$.

Then, $z\alpha \in P_j$ and $z\alpha \neq s$, since $s \notin Im\alpha$. Thus, $P_j \backslash \{s\} \neq \emptyset$. Consider an element $\beta \in D_{k+1}$ with $\underline{n}/\pi_\beta = \{Q_1, \ldots, Q_{k+1}\}$, where

$$Q_i = \begin{cases} P_i, & \text{if } i \notin \{j, k+1\} \\ P_j \backslash \{s\}, & \text{if } i = j \\ \{s\}, & \text{if } i = k+1, \end{cases}$$

with $Q_i\beta = \{y_i\}$, for all $i = 1, \ldots, k$ and $Q_{k+1}\beta = \{s\alpha\}$.

Claim 1. $\pi_\beta \subseteq \pi$. Suppose $(x, y) \in \pi_\beta$. Then, $x, y \in Q_i$, for some $1 \leq i \leq k+1$. If $i \notin \{j, k+1\}$, then $x, y \in P_i$. If $i = j$, then $Q_i = P_j \backslash \{s\}$ so $x, y \in P_j$. If $i = k+1$, then $x = y = s$. Thus, $(x, y) \in \pi$. This proves Claim 1.

Claim 2. $(x\alpha, y\alpha) \in \pi$ implies $(x\alpha, y\alpha) \in \pi_\beta$. Suppose $(x\alpha, y\alpha) \in \pi$. Then $x\alpha, y\alpha \in P_i$, for some $1 \leq i \leq k$. If $i \neq j$, then $x\alpha, y\alpha \in Q_i$. We now consider the case where $i = j$. Then, $x\alpha, y\alpha \in P_j$. Note that, $x\alpha$ and $y\alpha$ are both not equal to s, since $s \notin Im\alpha$. Then

$$x\alpha, y\alpha \in P_j \implies x\alpha, y\alpha \in P_j \backslash \{s\} \implies x\alpha, y\alpha \in Q_j.$$

Thus, $(x\alpha, y\alpha) \in \pi_\beta$. This proves Claim 2. Note that, the converse of Claim 2 is true by Claim 1. By Corollary 2.1, α is π^*-preserving. By Lemma 3.8,

$$(x, y) \in \pi_{\alpha\beta} \iff (x\alpha, y\alpha) \in \pi_\beta \iff (x\alpha, y\alpha) \in \pi \iff (x, y) \in \pi.$$

Thus, $\pi_{\alpha\beta} = \pi$. Let $P_i \in \underline{n}/\pi$. Then, by Corollary 2.1, $\chi^{(\alpha)}$ is bijective; hence there exists i^* with $1 \leq i^* \leq k$ such that $i^*\chi^{(\alpha)} = i$, that is, $P_{i^*}\alpha \subseteq P_i$. For $i \neq j$, we have $P_i = Q_i$. Then $P_{i^*}\alpha\beta \subseteq P_i\beta = Q_i\beta = \{y_i\}$. Suppose $i = j$. Then, $Q_j = P_j \backslash \{s\}$. Since $s \notin Im\alpha$, we have $s \notin P_{j^*}\alpha$ which implies that $P_{j^*}\alpha \subseteq P_j \backslash \{s\}$. Then $P_{j^*}\alpha\beta \subseteq (P_j \backslash \{s\})\beta = Q_j\beta = \{y_j\}$. Hence, $Im\alpha\beta = Y$. Note that, $\beta \notin H_k(\pi, Y)$ but $\alpha\beta \in H_k(\pi, Y)$. Therefore, $\alpha \notin Sep(H_k(\pi, Y))$. $\square$

Theorem 3.7. $Sep(H_k(\pi, Y)) = S_n(\pi, Y)$.

Proof. By Lemma 3.14, $S_n(\pi, Y) \subseteq Sep(H_k(\pi, Y))$. Suppose $\alpha \notin S_n(\pi, Y)$. Let $T_n(Y) = \{\alpha \in T_n \mid Y\alpha = Y\}$. Note that,

$$S_n(\pi, Y) = S_n(\pi) \cap S_n(Y) = S_n \cap \Sigma_n(\pi) \cap S_n \cap T_n(Y) = \Sigma_n(\pi) \cap S_n \cap T_n(Y).$$

Let $\beta \in H_k(\pi, Y)$. Then, $\pi_\beta = \pi$ and $Im\beta = Y$.

Case 1. $\alpha \notin T_n(Y)$. Then, $Y\alpha \neq Y$ which implies that $Im\beta\alpha = (Im\beta)\alpha = Y\alpha \neq Y$. Thus, $\beta\alpha \notin H_k(\pi, Y)$. Therefore, $\alpha \notin Sep(H_k(\pi, Y))$.

Case 2. $\alpha \notin \Sigma_n(\pi)$. Suppose $\alpha \in T_n \backslash T_n(\pi)$. Then, α does not preserve π so $\exists (x, y) \in \pi$ such that $(x\alpha, y\alpha) \notin \pi = \pi_\beta$. By Lemma 3.8, $(x, y) \notin \pi_{\alpha\beta}$. Thus, $\pi \neq \pi_{\alpha\beta}$ which implies that $\alpha\beta \notin H_k(\pi, Y)$.

Suppose $\alpha \in T_n(\pi) \backslash \Sigma_n(\pi)$. Then, α preserves π but is not π^*-preserving. By Remark 2.1, $\exists (u\alpha, v\alpha) \in \pi$ such that $(u, v) \notin \pi$. Since $\pi = \pi_\beta$, by Lemma

3.8, $(u, v) \in \pi_{\alpha\beta}$. Thus, $\pi_{\alpha\beta} \neq \pi$ which implies that $\alpha\beta \notin H_k(\pi, Y)$. Therefore, $\alpha \notin Sep(H_k(\pi, Y))$.

Case 3. $\alpha \notin S_n$ but $\alpha \in \Sigma_n(\pi)$ with $Y\alpha = Y$. Then, by Lemma 3.16, $\alpha \notin Sep(H_k(\pi, Y))$. □

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Picture fuzzy multisets

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Abstract. In this paper, the notion of picture fuzzy multiset was introduced. Some standard operations on picture fuzzy multiset such as intersection, union, complement were defined and their properties were investigated. Also, cut set and Cartesian product of picture fuzzy multiset were defined and the connections of Cartesian product with intersection and union were obtained.

Keywords: multiset, fuzzy multiset, intuitionistic fuzzy multiset, picture fuzzy multiset.

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1. Introduction

In 1965, the theory of fuzzy sets was introduced by Zadeh [34] as a generalisation of classical set theory. The theory only takes into consideration membership degree of an element belonging to a particular set. Atanassov [1], extended the work of Zadeh by introducing the theory of intuitionistic fuzzy sets which deals

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with both the membership and non-membership degrees of an element belonging to a set.

Cuong and Kreinovich [8] generalised the works of Zadeh and Atanassov into the theory of *picture fuzzy sets* (PFSs). This theory is a new concept for computational intelligence which is a set of Nature-inspired computational methodologies and approaches based on mathematics, computer science, artificial intelligence, to address applications of the real world complex problems that can not be solved by traditional methodologies and approaches. Basically, picture fuzzy sets based models may be appropriate in situations involving more answers of type: yes, abstain, no, refusal. A good example of such a situation is voting system in which human voters may decide to: vote for, vote against, abstain and refusal to vote. Thus, according to Cuong and Kreinovich [8], a given set is represented by three membership degrees i.e; positive membership degree, neutral membership degree and negative membership degree.

Picture fuzzy set has been extensively studied such as; in 2014, Cuong [11] investigated some characteristics of PFSs, introduced distance measure and defined convex combination between two PFSs. Cuong and Hai [13] investigated main fuzzy logic operators: negations, conjunctions, disjunctions and implications on picture fuzzy sets. Son [31], introduced a generalised picture distance measure and applied it to establish an Hierarchical Picture Clustering.

The theory of picture fuzzy set has been widely applied in decision making problems in the area of medical diagnosis, building material and minerals field recognitions, Covid-19 medicine selection among others (see [19, 26, 27, 32] for more details).

Yagar in 1986 [33], put forward the notion of *fuzzy multiset* (FM). In 2012, Shinoj and John [29] initiated *intuitionistic fuzzy multiset* (IFMS) from the combination of the concepts of fuzzy multisets and intuitionistic fuzzy sets and this was applied in medicine to diagnosis diseases. In 2013, Shinoj and John [30] defined some operations on intuitionistic fuzzy multisets and established some of its properties. Some researchers have also studied this notion of intuitionistic fuzzy multisets and applied it to medical diagnosis, binomial distribution (see [16, 17, 21] for more details). Due to the fact that the idea of intuitionistic fuzzy multisets also lacks accuracy in handling imprecision and uncertainties because of not taking into account neutrality degree, it is important to study the concept of picture fuzzy multiset as a generalisation of intuitionistic fuzzy multiset.

In this paper, we introduce the concept of *picture fuzzy multisets* (PFMSs), standard operations such as intersection, union, complement are defined and their properties are obtained. Cartesian product of picture fuzzy multiset are also defined and the connections of Cartesian product with intersection and union are established. The paper is organised as follows. Section 2 defines basic terms. Section 3 introduces the notion of PFMS, and some of its properties are obtained.

2. Preliminaries

In this section, some basic definitions are stated. Throughout this paper, E denotes a nonempty set.

Definition 2.1 ([21]). *A fuzzy set (FS) P drawn from E is defined as*

$$P = \{\langle y, \sigma_P(y)\rangle | y \in E\},$$

where $\sigma_P : E \longrightarrow [0,1]$ is the membership function of the fuzzy set P.

Definition 2.2 ([20]). *A fuzzy multiset (FMS) P drawn from E is characterised by a count membership function CM_P such that $CM_P : E \to N$, where N is the set of all crisp multisets drawn from $[0,1]$. Then, for any $y \in E$, the value $CM_P(y)$ is a crisp multiset drawn from $[0,1]$. For any $y \in E$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_P(y)$. It is denoted by $(\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))$ where $\sigma_P^1(y) \geq \sigma_P^2(y) \geq \cdots \geq \sigma_P^d(y)$.*

Definition 2.3 ([1]). *An intuitionistic fuzzy set (IFS) P of E is defined as*

$$P = \{\langle y, \sigma_P(y), \tau_P(y)\rangle | y \in E\},$$

where the functions $\sigma_P : E \to [0,1]$ and $\tau_P : E \to [0,1]$ are called the membership and non-membership degrees of $y \in P$, respectively, and for every $y \in E$,

$$0 \leq \sigma_P(y) + \tau_P(y) \leq 1.$$

Definition 2.4 ([18]). *An intuitionistic fuzzy multiset (IFMS) P drawn from E is characterised by count membership function CM_P and count nonmembership function CN_P such that $CM_P : E \to N$ and $CN_P : E \to N$, respectively, where N is the set of all crisp multisets drawn from $[0,1]$, such that for any $y \in E$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_P(y)$, denoted by $(\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))$ where $\sigma_P^1(y) \geq \sigma_P^2(y) \geq \cdots \geq \sigma_P^d(y)$ and the nonmembership sequence is given as $(\tau_P^1(y), \tau_P^2(y) \cdots, \tau_P^d(y))$ such that $0 \leq \sigma_P^i(y) + \tau_P^i(y) \leq 1$ for any $y \in E$, $i = 1, 2, \cdots, d$.*
Thus, an IFMS is given as

$$P = \{\langle y, (\sigma_P^1(y), \sigma_P^2(y) \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y))\rangle | y \in E\}.$$

Definition 2.5 ([8]). *A picture fuzzy set P of E is defined as*

$$P = \{\langle y, \sigma_P(y), \tau_P(y), \gamma_P(y)\rangle | y \in E\},$$

where the functions

$$\sigma_P : E \to [0,1], \ \tau_P : E \to [0,1] \text{ and } \gamma_P : E \to [0,1]$$

are called the positive, neutral and negative membership degrees of $y \in P$, respectively, and $\sigma_P, \tau_P, \gamma_P$ satisfy

$$0 \leq \sigma_P(y) + \tau_P(y) + \gamma_P(y) \leq 1, \ \forall y \in E.$$

For each $y \in E$, $S_P(y) = 1 - (\sigma_P(y) + \tau_P(y) + \gamma_P(y))$ is called the refusal membership degree of $y \in P$.

Definition 2.6 ([16]). *The Cut set of PFS P, denoted by $C_{r,s,t}(P)$ is defined by*

$$C_{r,s,t}(P) = \{y \in E | \sigma_P(y) \geq r,\ \tau_P(y) \geq s, \gamma_P(y) \leq t\},$$

where $r, s, t \in [0,1]$ with the condition $0 \leq r+s+t \leq 1$.

Definition 2.7 ([8]). *Let P and Q be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:*

- *$P \subseteq Q$ if and only if for all $y \in E$, $\sigma_P(y) \leq \sigma_Q(y)$, $\tau_P(y) \leq \tau_Q(y)$ and $\eta_P(y) \geq \eta_Q(y)$.*
- *$P = Q$ if and only if $P \subseteq Q$ and $Q \subseteq P$.*
- *$P \cup Q = \{(y, \sigma_p(y) \vee \sigma_Q(y), \tau_P(y) \wedge \tau_Q(y)), \eta_P(y) \wedge \eta_Q(y))|\ y \in\ E\}$.*
- *$P \cap Q = \{(y, \sigma_P(y) \wedge \sigma_Q(y), \tau_P(y) \wedge \tau_Q(y)), \eta_P(y) \vee \eta_Q(y))|\ y \in\ E\}$.*
- *$\overline{P} = \{(y, \eta_P(y), \tau_P(y), \sigma_P(y))|\ y \in\ E\}$.*

3. Picture fuzzy multisets

Here, we define picture fuzzy multiset, some basic operations and investigate some properties related to the operations.

Definition 3.1. *A picture fuzzy multiset (PFMS) P drawn from E is characterised by count positive membership function C_pM_P, count neutral membership function $C_{ne}M_P$ and count negative membership function C_nM_P such that $C_pM_P : E \to N$, $C_{ne}M_P : E \to N$ and $C_nM_P : E \to N$, respectively, where N is the set of all crisp multisets drawn from $[0,1]$, such that for any $y \in E$, the positive membership sequence is defined as the decreasingly ordered sequence of elements in $C_pM_P(y)$, denoted by $(\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))$ where $\sigma_P^1(y) \geq \sigma_P^2(y) \geq \cdots \geq \sigma_P^d(y)$, the neutral membership sequence and negative membership sequence is $(\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y))$ and $(\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))$, respectively such that $0 \leq \sigma_P^i(y) + \tau_P^i(y) + \eta_P^i(y) \leq 1$ for any $y \in E$, $\ i = 1, 2, \cdots, d$.*

So, a PFMS is denoted by

$$\begin{aligned} P \ = \ & \{\langle y, (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ & (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))\rangle | y \in E\}. \end{aligned}$$

Remark 3.1. Notice that since the positive membership sequence is arranged in decreasing order, neutral or negative membership sequence may not be decreasing or increasing order.

Definition 3.2. *Let $P = \{\langle y, \sigma_P^i(y), \tau_P^i(y), \eta_P^i(y)\rangle|\ y \in\ E\}$ be a PFMS. Then, the (r,s,t)-cut of P denoted by $[P]_{r,s,t}$ is defined by*

$$[P]_{(r,s,t)} = \{a \in P|\ \sigma_P^i(a) \geq r,\ \tau_P^i(a) \leq s,\ \eta_P^i(a) \leq t\},\ i = 1, 2, \cdots, d,$$

where $r, s, t \in [0,1]$ such that $0 \leq r+s+t \leq 1$.

Definition 3.3. *Let* $P = \{\langle y, (\sigma_P^i(y)), (\tau_P^i(y)), (\eta_P^i(y))\rangle | y \in E\},\ i = 1, 2, \cdots, d$ *be a PFMS and* $y \in P$*. Then, the size of* $y \in P$*, denoted by* $S(y; P)$ *is defined as the cardinality of* $C_pM_P(y)$ *or* $C_{ne}M_P(y)$ *or* $C_nM_P(y)$*, for which* $0 \leq \sigma_P^1(y) + \tau_P^1(y) + \eta_P^1(y) \leq 1$*. That is*

$$S(y;\ P) = |C_pM_P(y)| = |C_{ne}M_P(y)| = |C_nM_P(y)|.$$

Definition 3.4. *Given two PFMSs* P *and* Q *drawn from* E*. Then, the size of* P *and* Q *is defined as*

$$S(y;\ P,\ Q) = S(y;\ P) \vee S(y;\ Q).$$

Example 3.1. Let $E = \{a, b, c\}$

$$\begin{aligned} P &= \{\langle a; (0.5,\ 0.2), (0.3,\ 0.1), (0.2, 0.4)\rangle, \\ &\quad \langle c; (0.0,\ 0.4,\ 0.1),\ (0.2,\ 0.1,\ 0.3), (0.5, 0.2, 0.6)\rangle\} \end{aligned}$$

and

$$\begin{aligned} Q &= \{\langle a; (0.1,\ 0.5), (0.2,\ 0.4), (0.0, 0.0)\rangle, \\ &\quad \langle b; (0.2,\ 0,\ 0.3), (0.1,\ 1.0,\ 0.2), (0.2, 0.0, 0.4)\rangle, \\ &\quad \langle c; (0.8,\ 0.1),\ (0.1,\ 0.3),\ (0.1, 0.5)\rangle\}. \end{aligned}$$

Then

$$\begin{aligned} &S(a;\ P) = 2,\ S(b;\ P) = 0,\ S(c;\ P) = 3, \\ &S(a;\ Q) = 2,\ S(b;\ Q) = 3,\ S(c;\ Q) = 2, \\ &S(a; P,\ Q) = 2,\ S(b;\ P,\ Q) = 3,\ S(c;\ P,\ Q) = 3. \end{aligned}$$

3.1 Standard operations on picture fuzzy multisets

Definition 3.5. *Let*

$$P = \{\langle y, \sigma_P^i(y), \tau_P^i(y), \eta_P^i(y)\rangle|\ y \in\ E\}$$

and

$$Q = \{\langle y, \sigma_Q^i(y), \tau_Q^i(y), \eta_Q^i(y)\rangle|\ y \in\ E\},$$

where $i = 1, 2, \cdots, d$*, be two PFMSs drawn from* E*. Then, the inclusion, equality, union, intersection and complement are defined as follow:*

- $P \subseteq Q$ *if and only if,* $\sigma_P^i(y) \leq \sigma_Q^i(y)$*,* $\tau_P^i(y) \leq \tau_Q^i(y)$ *and* $\eta_P^i(y) \geq \eta_Q^i(y)$*;* $i = 1, 2, \cdots, S(y;\ P,\ Q),\ y \in E$*.*
- $P = Q$ *if and only if* $P \subseteq Q$ *and* $Q \subseteq P$*.*
- $P \cup Q = \{(y, \max(\sigma_P^i(y), \sigma_Q^i(y)), \min(\tau_P^i(y), \tau_Q^i(y)), \min(\eta_P^i(y), \eta_Q^i(y)))|y \in E\}$*, where* $i = 1, 2, \cdots, S(y; P, Q)$*.*

- $P \cap Q = \{(y, \min(\sigma_P^i(y), \sigma_Q^i(y)), \min(\tau_P^i(y), \tau_Q^i(y)), \max(\eta_P^i(y), \eta_Q^i(y)))|y \in E\}$, *where* $i = 1, 2, \cdots, S(y;\ P,\ Q)$.
- $\overline{P} = \{(y, \eta_P^i(y), \tau_P^i(y), \sigma_P^i(y))|\ y \in\ E\},\ i = 1, 2, \cdots, S(y;\ P,\ Q)$.

Example 3.2. Let $E = \{a, b, c\}$

$$\begin{aligned} P \;=\; & \{\langle a; (0.1,\ 0.5), (0.2,\ 0.4), (0.0, 0.0)\rangle, \\ & \langle b; (0.1,\ 0.4,\ 0.7), (0.1,\ 0.6,\ 0.0), (0.5, 0.0, 0.3)\rangle, \\ & \langle c; \langle c; (0.4,\ 0.1), (0.1,\ 0.7), (0.0, 0.0)\rangle\} \end{aligned}$$

and

$$\begin{aligned} Q \;=\; & \{\langle a; (0.5,\ 0.2), (0.3,\ 0.1), (0.2, 0.4)\rangle, \\ & \langle b; (0.2,\ 0.0,\ 0.3), (0.1,\ 0.6,\ 0.4), (0.2, 0.3, 0.1)\rangle, \\ & \langle c; \langle c; (0.8,\ 0.1),\ (0.1,\ 0.3),\ (0.1, 0.5)\rangle\}. \end{aligned}$$

Then

$$\begin{aligned} P \cup Q \;=\; & \{\langle a; (0.5,\ 0.5), (0.2,\ 0.1), (0.0, 0.0)\rangle, \\ & \langle b; (0.2,\ 0.4,\ 0.7),\ (0.1,\ 0.6,\ 0.0), (0.2, 0.0, 0.1)\rangle, \\ & \langle c; \langle c; (0.8,\ 0.1),\ (0.1,\ 0.3), (0.0, 0.0)\rangle\}, \\ P \cap Q \;=\; & \{\langle a; (0.1,\ 0.2),\ (0.2,\ 0.1), (0.2, 0.4)\rangle, \\ & \langle b; (0.1,\ 0.0,\ 0.3),\ (0.1,\ 0.6,\ 0.0), (0.5, 0.3, 0.3)\rangle, \\ & \langle c; \langle c; (0.4,\ 0.1),\ (0.1,\ 0.3), (0.1, 0.5)\rangle\}, \\ \overline{P} \;=\; & \{\langle c; (0.4,\ 0.1), (0.1,\ 0.7), (0.0, 0.0)\rangle, \\ & \langle b; (0.1,\ 0.4,\ 0.7), (0.1,\ 0.6,\ 0.0), (0.5, 0.0, 0.3)\rangle, \\ & \langle c; \langle a; (0.1,\ 0.5), (0.2,\ 0.4), (0.0, 0.0)\rangle\}. \end{aligned}$$

3.2 Basic properties

Proposition 3.1. *For every PFMS P, Q, R.*

1. **Involution**

   $\overline{\overline{P}} = P$.

2. **Commutative rule**

   $P \cap Q = Q \cap P,\ \ P \cup Q = Q \cup P$.

3. **Associative rule**

   $P \cap (Q \cap R) = (P \cap Q) \cap R,\ \ P \cup (Q \cup R) = (P \cup Q) \cup R$.

4. **Distributive rule**

$$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R), \quad P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R).$$

5. **Idempotent rule**

$$P \cap P = P, \quad P \cup P = P.$$

6. **De Morgan's rule**

$$\overline{P \cap Q} = \overline{P} \cup \overline{Q}, \quad \overline{P \cup Q} = \overline{P} \cap \overline{Q}.$$

Proof. Let

$$\begin{aligned} P &= \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\ &\quad (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))\rangle |\ y \in\ E\}, \\ Q &= \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\ &\quad (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y))\rangle |\ y \in\ E\} \end{aligned}$$

and

$$\begin{aligned} R &= \{\langle y, (\sigma_R^1(y), \sigma_R^2(y), \cdots, \sigma_R^d(y)), (\tau_R^1(y), \tau_r^2(y), \cdots, \tau_R^d(y)), \\ &\quad (\eta_R^1(y), \eta_R^2(y), \cdots, \eta_R^d(y))\rangle |\ y \in\ E\}. \end{aligned}$$

Then

1.

$$\begin{aligned} \overline{P} &= \{\langle y, (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ &\quad (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y))\rangle |\ y \in\ E\}, \\ \overline{\overline{P}} &= \{\langle y, (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ &\quad (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y))\rangle |\ y \in\ E\} \\ &= P. \end{aligned}$$

2.

$$\begin{aligned} & P \cap Q \\ & = \{\langle y, (\sigma_P^1(y), \sigma_P^2(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \tau_P^2(y), \cdots, \tau_P^d(y)), \\ & (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y)) \\ & \cap (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \end{aligned}$$

$$(\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y)))\rangle|\ y \in\ E\rangle\}$$
$$= \{\langle y, (\sigma_P^1(y) \wedge \sigma_Q^1(y), \sigma_P^2(y) \wedge \sigma_Q^2(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y)),$$
$$(\tau_P^1(y) \wedge \tau_Q^1(y), \tau_P^2(y) \wedge \tau_Q^2(y)), \cdots (\tau_P^d(y) \wedge \tau_Q^d(y)),$$
$$(\eta_P^1(y) \vee \eta_Q^1(y), \eta_P^2(y) \vee \eta_Q^2(y)), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y))|\ y \in\ E\rangle\}$$
$$= \{\langle y, (\sigma_Q^1(y) \wedge \sigma_P^1(y)), \cdots, (\sigma_Q^d(y) \wedge \sigma_P^d(y)), (\tau_Q^1(y) \wedge \tau_P^1(y)),$$
$$\cdots, (\tau_Q^d(y) \wedge \tau_P^d(y)), (\eta_Q^1(y) \vee \eta_P^1(y)), \cdots, (\eta_Q^d(y) \vee \eta_P^d(y))|\ y \in\ E\rangle\}$$
$$= Q \cap P.$$

Similarly, we can prove $P \cup Q = Q \cup P$.

3.

$$P \cap (Q \cap R)$$
$$= \{\langle y, (\sigma_P^1(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \cdots, \tau_P^d(y)), (\eta_P^1(y), \cdots, \eta_P^d(y))\rangle|\ y \in\ E\}$$
$$\cap \{\langle y, ((\sigma_Q^1(y) \wedge \sigma_R^1(y)), \cdots, (\sigma_Q^d(y) \wedge \sigma_R^d(y)), (\tau_Q^1(y) \wedge \tau_R^1(y)),$$
$$\cdots, (\tau_Q^d(y) \wedge \tau_R^d(y)), (\eta_Q^1(y) \wedge \eta_R^1(y)), \cdots, (\eta_Q^d(y) \wedge \eta_R^d(y))))\rangle|\ y \in\ E\}$$
$$= \{\langle y, (\sigma_P^1(y) \wedge \sigma_Q^1(y)) \wedge \sigma_R^1(y), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y)) \wedge \sigma_R^d(y),$$
$$(\tau_P^1(y)\tau_Q^1(y)) \wedge \tau_R^1(y), \cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)) \wedge \tau_R^d(y),$$
$$(\eta_P^1(y) \vee \eta_Q^1(y)) \vee \eta_R^1(y), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y)) \vee \eta_R^d(y)\rangle|\ y \in\ E\}$$
$$= (P \cap Q) \cap R.$$

Similarly, we can prove $P \cup (Q \cup R) = (P \cup Q) \cup R$.

4.

$$P \cap (Q \cup R)$$
$$= \{\langle y, (\sigma_P^1(y) \wedge (\sigma_Q^1(y)) \vee \sigma_R^1(y)), \cdots, \sigma_P^d(y) \wedge (\sigma_Q^d(y)) \vee \sigma_R^d(y)),$$
$$\tau_P^1(y) \wedge (\tau_Q^1(y)) \vee \tau_R^1(y)), \cdots, \tau_P^d(y) \wedge (\tau_Q^d(y)) \vee \tau_R^d(y)),$$
$$\eta_P^1(y) \vee (\eta_Q^1(y)) \wedge \eta_R^1(y)), \cdots, \eta_P^d(y) \vee (\eta_Q^d(y)) \wedge \eta_R^d(y))))\rangle|\ y \in\ E\}$$
$$= \{\langle y, ((\sigma_P^1(y) \wedge \sigma_Q^1(y)) \vee (\sigma_P^1(y) \wedge \sigma_R^1(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y))$$
$$\vee (\sigma_P^d(y) \wedge \sigma_R^d(y))), ((\tau_P^1(y) \wedge \tau_Q^1(y)) \vee (\tau_P^1(y) \wedge \tau_R^1(y)),$$
$$\cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)) \vee (\tau_P^d(y) \wedge \tau_R^d(y))), ((\eta_P^1(y) \vee \eta_Q^1(y))$$
$$\wedge (\eta_P^1(y) \vee \eta_R^1(y)), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y)) \wedge (\eta_P^d(y) \vee \eta_R^d(y))))\rangle|\ y \in\ E\}$$
$$= (P \cap Q) \cup (P \cap R).$$

Similarly, we can prove $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$.

5.

$$
\begin{aligned}
&P \cap P \\
&= \{\langle y, (\sigma_P^1(y) \wedge \sigma_P^1(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_P^d(y)), (\tau_P^1(y) \wedge \tau_P^1(y)), \\
&\cdots, (\tau_P^d(y) \wedge \tau_P^d(y)), (\eta_P^1(y) \vee \eta_P^1(y)), (\eta_P^d(y) \vee \eta_P^d(y))|\ y \in\ E\rangle\} \\
&= \{\langle y, (\sigma_P^1(y), \cdots, \sigma_P^d(y)), (\tau_P^1(y), \cdots, \tau_P^d(y)), (\eta_P^1(y), \cdots, \eta_P^d(y))|\ y \in\ E\rangle\} \\
&= P.
\end{aligned}
$$

Similarly, we can prove $P \cup P = P$.

6.

$$
\begin{aligned}
&\overline{P \cap Q} \\
&= \{\langle y, ((\eta_P^1(y) \vee \eta_Q^1(y)), \cdots, (\eta_P^d(y) \vee \eta_Q^d(y), ((\tau_P^1(y) \wedge \tau_Q^1(y)), \\
&\cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)), ((\sigma_P^1(y) \wedge \sigma_Q^1(y)), \cdots, (\sigma_P^d(y) \wedge \sigma_Q^d(y))\rangle|\ y \in\ E\} \\
&= \{\langle y, ((\sigma_P^1(y) \vee \sigma_Q^1(y)), \cdots, (\sigma_P^d(y) \vee \sigma_Q^d(y)), ((\tau_P^1(y) \wedge \tau_Q^1(y)), \\
&\cdots, (\tau_P^d(y) \wedge \tau_Q^d(y)), ((\eta_P^1(y) \wedge \eta_Q^1(y)), \cdots, (\eta_P^d(y) \wedge \eta_Q^d(y)\rangle|\ y \in\ E\} \\
&= \{\langle y, (\eta_P^1(y), \cdots, \eta_P^d(y)), (\tau_P^1(y), \cdots, \tau_P^d(y)), (\sigma_P^1(y), \cdots, \sigma_P^d(y))\rangle|\ y \in\ E\} \\
&\cup \{\langle y, (\eta_Q^1(y), \cdots, \eta_Q^d(y)), (\tau_Q^1(y), \cdots, \tau_Q^i(y)), (\sigma_Q^1(y), \cdots, \sigma_Q^d(y))\rangle|\ y \in\ E\} \\
&= \overline{P} \cup \overline{Q}.
\end{aligned}
$$

Similarly, we can prove $\overline{P \cup Q} = \overline{P} \cap \overline{Q}$. □

Definition 3.6. *Let*

$$
\begin{aligned}
P \quad = \quad & \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\
& (\eta_P^1(y), \eta_P^2(y), \cdots, \eta_P^d(y))\rangle|\ y \in\ E\}
\end{aligned}
$$

and

$$
\begin{aligned}
Q \quad = \quad & \{\langle y, (\sigma_Q^1(y), \sigma_Q^2(y), \cdots, \sigma_Q^d(y)), (\tau_Q^1(y), \tau_Q^2(y), \cdots, \tau_Q^d(y)), \\
& (\eta_Q^1(y), \eta_Q^2(y), \cdots, \eta_Q^d(y))\rangle|\ y \in\ E\}
\end{aligned}
$$

be two PFMSs on E.

Then, the Cartesian product of P and Q, $P \times Q$ is defined as

$$
\begin{aligned}
P \times Q \quad = \quad & \{\langle (x, y), (\sigma_{P\times Q}^1(x, y), \sigma_{P\times Q}^2(x, y), \cdots, \sigma_{P\times Q}^d(x, y)), \\
& (\tau_{P\times Q}^1(x, y), \tau_{P\times Q}^2(x, y), \cdots, \tau_{P\times Q}^d(x, y)), \\
& (\eta_{P\times Q}^1(x, y), \eta_{P\times Q}^2(x, y), \cdots, \eta_{P\times Q}^d(x, y))\rangle|\ x, y \in\ E\}
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{P\times Q}^i(x, y) = \sigma_P^i(x) \wedge \sigma_Q^i(y), \\
\tau_{P\times Q}^i(x, y) = \tau_P^i(x) \wedge \tau_Q^i(y)
\end{aligned}
$$

and

$$\eta^i_{P\times Q}(x,y) = \eta^i_P(x) \vee \eta^i_Q(y)$$

with $i = 1, 2, \cdots, d$.

3.3 Basic properties

Proposition 3.2. *Let P, Q, R be PFMSs. Then*

1. $P \times Q = Q \times P$.
2. $(P \times Q) \times R = P \times (Q \times R)$.
3. $P \times (Q \cup R) = (P \times Q) \cup (P \times R)$.
4. $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$.

Proof. Let PFMSs P, Q, R be defined as

$$\begin{aligned} P &= \{\langle y, (\sigma^1_Q(y), \sigma^2_Q(y), \cdots, \sigma^d_Q(y)), (\tau^1_Q(y), \tau^2_Q(y), \cdots, \tau^d_Q(y)), \\ &\quad (\eta^1_P(y), \eta^2_P(y), \cdots, \eta^d_P(y))\rangle |\ y \in\ E\}, \end{aligned}$$

$$\begin{aligned} Q &= \{\langle y, (\sigma^1_Q(y), \sigma^2_Q(y), \cdots, \sigma^d_Q(y)), (\tau^1_Q(y), \tau^2_Q(y), \cdots, \tau^d_Q(y)), \\ &\quad (\eta^1_Q(y), \eta^2_Q(y), \cdots, \eta^d_Q(y))\rangle |\ y \in\ E\} \end{aligned}$$

and

$$\begin{aligned} R &= \{\langle y, (\sigma^1_R(y), \sigma^2_R(y), \cdots, \sigma^d_R(y)), (\tau^1_R(y), \tau^2_r(y), \cdots, \tau^d_R(y)), \\ &\quad (\eta^1_R(y), \eta^2_R(y), \cdots, \eta^d_R(y))\rangle |\ y \in\ E\}. \end{aligned}$$

(1) and (2) are obvious from the definition

3.
$$\begin{aligned} & P \times (Q \cup R) \\ = & \{\langle (x,y), (\sigma^1_P(y) \wedge (\sigma^1_Q(y) \vee \sigma^1_R(y)), \cdots, \sigma^d_P(y) \wedge (\sigma^d_Q(y) \vee \sigma^d_R(y))), \\ & (\tau^1_P(y) \wedge (\tau^1_Q(y) \wedge \tau^1_R(y)), \cdots, \tau^d_P(y) \wedge (\tau^d_Q(y) \wedge \tau^d_R(y))), \\ & (\eta^1_P(y) \vee (\eta^1_Q(y) \wedge \eta^1_R(y)), \cdots, \eta^d_P(y) \vee (\eta^d_Q(y) \wedge \eta^d_R(y)))\rangle |\ x, y \in\ E\} \\ = & \{\langle (x,y), (\sigma^1_P(y) \wedge \sigma^1_Q(y), \cdots, \sigma^d_P(y) \wedge \sigma^d_Q(y)), (\tau^1_P(y) \wedge \tau^1_Q(y), \\ & \cdots, \tau^d_P(y) \wedge \tau^d_Q(y)), (\eta^1_P(y) \vee \eta^1_Q(y), \cdots, \eta^d_P(y) \vee \eta^d_Q(y))\rangle |\ x, y \in\ E\} \\ & \cup \{\langle (x,y), (\sigma^1_P(y) \wedge \sigma^1_R(y), \cdots, \sigma^d_P(y) \wedge \sigma^d_R(y)), (\tau^1_P(y) \wedge \tau^1_R(y), \\ & \cdots, \tau^d_P(y) \wedge \tau^d_R(y)), (\eta^1_P(y) \vee \eta^1_R(y), \cdots, \eta^d_P(y) \vee \eta^d_R(y))\rangle |\ x, y \in\ E\} \\ = & (P \times Q) \cup (P \times R). \quad \square \end{aligned}$$

Property 4 can also be proved in the same way as property 3.

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The matrix inverse based on the EP-nilpotent decomposition of a complex matrix

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Abstract. A generalized inverse for matrices is introduced, which is called the MPEPN-inverse. Let A be a complex matrix, the MPEPN-inverse can be described

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by using the part A_1 in the EP-nilpotent decomposition of A and the Moore-Penrose inverse of A. Let $A = A_1 + A_2$ be the EP-nilpotent decomposition of A, $A^{E,\ddagger}$ be the MPEPN-inverse of A and $A^\dagger$ be the Moore-Penrose inverse of A, one can show that $A^{E,\ddagger}AA^{E,\ddagger} = A^{E,\ddagger}$ does not hold in general, moreover, necessary and sufficient conditions to make the MPEPN-inverse to be an outer inverse of A are given, that is $A^{E,\ddagger}AA^{E,\ddagger} = A^{E,\ddagger}$ hold if and only if one of the conditions $(A_1A^\dagger)^2 = A_1A^\dagger$ and $P_{\mathcal{R}(A_2)}A^{\textcircled{\dagger}} = 0$ holds, where $A^{\textcircled{\dagger}}$ is the Core-EP inverse of A and $P_{\mathcal{R}(A_2)}$ is the projection on $\mathcal{R}(A_2)$. If $A_1A^\dagger$ is an idempotent, then the MPEPN-inverse of A coincides with the $(A^\dagger A_1 P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)}A_1A^\dagger)$-inverse of A , i.e. coincides the inverse along $A^\dagger A_1 P_{\mathcal{R}(A^*)}$ and $P_{\mathcal{R}(A)}A_1A^\dagger$.
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1. Introduction

Let $\mathbb{C}$ be the complex field. The set $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices over the complex field $\mathbb{C}$. Let $A \in \mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. Notations $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0, x \in \mathbb{C}^n\}$ will be used in the sequel. An integer k is called the index of $A \in \mathbb{C}^{n\times n}$ if k is the smallest positive integer such that $\mathrm{rank}\,(A^k) = \mathrm{rank}(A^{k+1})$ holds and is denoted by $\mathrm{ind}(A)$.

Let $A \in \mathbb{C}^{m\times n}$. A matrix $X = A^\dagger \in \mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse of A [8, 12] if $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$ hold. Let $A, X \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}\,(A) = k$. Then algebraic definition of the Drazin inverse as follows: if

$$AXA = A,\ XA^{k+1} = A^k \text{ and } AX = XA,$$

then X is called a Drazin inverse of A. If such X exists, then it is unique and denoted by A^D [4]. More generalized inverses can be seen as follows:core inverse [2] by using $\Sigma - K - L$ decomposition [7], core-EP inverse [9] and DMP inverse [11].

Let $A, B, C \in \mathbb{C}^{n\times n}$. The (B, C)-inverse of A is unique (see [1, 5, 13]). Several kinds of generalized inverses are all special cases of the (B, C)-inverse of the matrix A: Moore-Penrose inverse [8, 12], Drazin inverse [4], core inverse [2], DMP-inverse [11] and core-EP inverse [9].

For a complex matrix with a given index, there are three important matrix decompositions: core-nilpotent decomposition [10], Core-EP decomposition [14] and EP-nilpotent decomposition [15]. The CMP inverse can be introduced by the core-nilpotent decomposition and the MPCEP-inverse can be introduced by the Core-EP decomposition. Motivated by the idea of the CMP inverse and the MPCEP-inverse of a complex matrix, in this paper, the MPEPN-inverse was introduced. Specifically, the CMP inverse of $A \in \mathbb{C}^{n\times n}$ was introduced by Mehdipour and Salemi in [10], this inverse using the core part in core-nilpotent decomposition of A and the Moore-Penrose inverse of A. The MPCEP-inverse can be described by using the core part in Core-EP decomposition of A and the Moore-Penrose inverse of A [3]. Motivated by the above method, we have a natural question as follows: Using the core part A_1 in EP-nilpotent decomposition of A and the Moore-Penrose inverse of A to introduce a matrix $X = A^\dagger A_1 A^\dagger$. Thus, the MPEPN-inverse can be described by using the core part in EP-nilpotent decomposition of A and the Moore-Penrose inverse of A [15].

2. Existence criteria and expressions of the MPEPN-inverse

The EP-nilpotent decomposition of A was introduced by Wang and Liu in [15]. That is A can be written as $A = A_1 + A_2$, where k is the index of A, A_1 is an EP matrix (i.e. $A_1A_1^\dagger = A_1^\dagger A_1$), $A_2^{k+1} = 0$ and $A_2A_1 = 0$. The following lemma holds by [15, Theorem 2.2].

Lemma 2.1 ([15, Theorem 2.1]). *Let $A \in \mathbb{C}^{n\times n}$ and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A_1 = U \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} U^* \text{ and } A_2 = U \begin{bmatrix} 0 & S \\ 0 & N \end{bmatrix} U^*, \tag{1}$$

where $\operatorname{ind}(A) = k$, *$T$ is nonsingular, S and N are matrices with some suitable sizes.*

The Core-EP decomposition in the following lemma is useful in the study of the Core-EP inverse. Ferreyra et al.[6] given the explicit expressions of the Moore-Penrose inverse by using the Core-EP decomposition, which can be seen in Lemma 2.3.

Lemma 2.2 ([14, Theorem 2.1]). *Let $A \in \mathbb{C}^{n\times n}$ and $A = A_1' + A_2'$ be the Core-EP decomposition of A. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A_1' = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* \text{ and } A_2' = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \tag{2}$$

where $\operatorname{ind}(A) = k$, *$T$ is nonsingular, S and N are matrices with some suitable sizes.*

Lemma 2.3 ([6, Theorem 3.9]). *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *If A has the Core-EP decomposition of A as* (2.2) *in Lemma* 2.2, *then*

$$A^\dagger = U \begin{bmatrix} T^*\Delta & -T^*\Delta S N^\dagger \\ (E_{n-t} - N^\dagger N)S^\dagger\Delta & N^\dagger - (E_{n-t} - N^\dagger N)S^*\Delta S N^\dagger \end{bmatrix} U^*,$$

where $t = \operatorname{rank}(A^k)$, $\Delta = [TT^* + S(E_{n-t} - N^\dagger N)S^*]^{-1}$ and E_{n-t} is the identity of size $n - t$.

Lemma 2.4 ([8])**.** *Let $A \in \mathbb{C}^{n\times n}$. Then*

(1) *$A^*B = A^*C$ if and only if $A^\dagger B = A^\dagger C$ for any $B, C \in \mathbb{C}^{n\times n}$;*

(2) *$BA^* = CA^*$ if and only if $BA^\dagger = CA^\dagger$ for any $B, C \in \mathbb{C}^{n\times n}$.*

The core part of the EP-nilpotent decomposition can be expressed by the Moore-Penrose inverse of A^k, where $ind(A) = k$. The core part of the EP-nilpotent decomposition is useful in our paper.

Lemma 2.5 ([15, Theorem 2.2])**.** *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as (2.1). Then $A_1 = AA^k(A^k)^\dagger$.*

Lemma 2.6 ([5, Theorem 2.1 and Proposition 6.1])**.** *Let $A \in \mathbb{C}^{n\times n}$. Then $Y \in \mathbb{C}^{n\times n}$ is a (B,C)-inverse of A if and only if $YAY = Y$, $\mathcal{R}(Y) = \mathcal{R}(B)$ and $\mathcal{N}(X) = \mathcal{N}(C)$.*

Motivated by the definition of the CMP inverse in [10], in the following definition we will introduced the MPEPN-inverse of a complex matrix by using the Moore-Penrose inverse of such matrix and the core part of the EP-nilpotent decomposition of this matrix, then one can prove that this inverse is unique.

Definition 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as (1). Then $X = A^\dagger A_1 A^\dagger$ is called the MPEPN-inverse of A.*

Example 2.1. The MPEPN-inverse $A^\dagger A_1 A^\dagger$ is different to $A^\dagger A^D A^\dagger$. Since by Lemma *2.5*, we have $A_1 = AA^k(A^k)^\dagger$ and by [5], we have $A^D = A^k(A^{2k+1})^\dagger A^k$, thus $A^\dagger A_1 A^\dagger = A^\dagger A^{k+1}(A^k)^\dagger A^\dagger$ and $A^\dagger A^D A^\dagger = A^\dagger A^k(A^{2k+1})^\dagger A^k A^\dagger$. Let $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$, one check that $A^\dagger A_1 A^\dagger = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $A^\dagger A^D A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k. The equality $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$ does not hold in general, a counterexample will be given in the following example.

Example 2.2. Let $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then it is easy to check that the index of A is $k = 2$, but $AA^k(A^k)^\dagger = AA^2(A^2)^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^k(A^k)^\dagger A = A^2(A^2)^\dagger A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, that is, $AA^k(A^k)^\dagger \neq A^k(A^k)^\dagger A$. Moreover, we have $A^D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $A^{E,\ddagger} = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k. The following Example 2.3 shows that the equality $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$ can hold for some matrices.

Example 2.3. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then it is easy to check that $\mathrm{ind}(A) = k = 1$ and $AA^k(A^k)^\dagger = AAA^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$, $A^k(A^k)^\dagger A = AA^\dagger A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$, that is, $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$. Moreover, we have $A^{E,\ddagger} = A^D = A^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{5} & \frac{4}{5} \end{bmatrix}$.

Example 2.2 and Example 2.3 show that the equality $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$ does not hold in general. One sufficient condition such that the equality holds can be seen in the following proposition.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *If $PA^*A^k = 0$, then $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$, where $P = E_n - A^k(A^k)^\dagger$ and E_n is the identity of size n.*

Proof. Since $P = E_n - A^k(A^k)^\dagger$, then $PA^*A^\dagger = 0$ is equivalent to $[E_n - A^k(A^k)^\dagger]A^*A^k = 0$, which is equivalent to

$$A^*A^k = A^k(A^k)^\dagger A^*A^k. \tag{3}$$

Taking $*$ on (3) gives $(A^*A^k)^* = [A^k(A^k)^\dagger A^*A^k]^*$, then

$$(A^k)^*A = (A^k)^*A[A^k(A^k)^\dagger]^* = (A^k)^*AA^k(A^k)^\dagger. \tag{4}$$

By (4) and Lemma 2.4, we have

$$(A^k)^\dagger A = (A^k)^\dagger AA^k(A^k)^\dagger. \tag{5}$$

Pre-multiplying by A^k on (5) gives

$$A^k(A^k)^\dagger A = A^k(A^k)^\dagger AA^k(A^k)^\dagger = A^k(A^k)^\dagger A^kA(A^k)^\dagger = A^kA(A^k)^\dagger = AA^k(A^k)^\dagger,$$

that is, $AA^k(A^k)^\dagger = A^k(A^k)^\dagger A$. □

By using the Moore-Penrose inverse of A and the core part in the EP-nilpotent decomposition of A, the formula of the MPEPN-inverse of A was given. Moreover, we can get the formula $A^\dagger A^{k+1}(A^k)^\dagger A^\dagger$ is the MPEPN-inverse of A.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and A_1 be the core part in the EP-nilpotent decomposition of A, then $A^\dagger A^{k+1}(A^k)^\dagger A^\dagger$ is the MPEPN-inverse of A.*

Proof. Let X be the MPEPN-inverse of A, we have $A_1 = AA^k(A^k)^\dagger$ by Lemma 2.5. By Definition 2.1, we have $X = A^\dagger A_1 A^\dagger$. Thus, the conditions $A_1 = AA^k(A^k)^\dagger$ and $X = A^\dagger A_1 A^\dagger$ give

$$X = A^\dagger A_1 A^\dagger = A^\dagger AA^k(A^k)^\dagger A^\dagger = A^\dagger A^{k+1}(A^k)^\dagger A^\dagger.$$ □

3. When the MPEPN-inverse of complex matrix is an outer inverse of this matrix

Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $X \in \mathbb{C}^{n\times n}$ be the MPEPN-inverse of A. In general, the MPEPN-inverse is an outer inverse of A? The answer is no, $X = XAX$ does not hold, a counterexample will be given in the following example.

Example 3.1. Let $A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then $\operatorname{ind}(A) = 2$, but $A^{E,\ddagger} = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^{E,\ddagger}AA^{E,\ddagger} = \begin{bmatrix} \frac{14}{27} & -\frac{13}{27} & \frac{1}{27} & 0 \\ -\frac{13}{27} & \frac{14}{27} & \frac{1}{27} & 0 \\ \frac{1}{27} & \frac{1}{27} & \frac{2}{27} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, that is, $A^{E,\ddagger} \neq A^{E,\ddagger}AA^{E,\ddagger}$. Moreover, $A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then

$$A_1A^\dagger = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (A_1A^\dagger)^2 = \begin{bmatrix} \frac{5}{9} & -\frac{4}{9} & \frac{1}{9} & 0 \\ -\frac{4}{9} & \frac{5}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $A_1A^\dagger$ is not an idempotent.

The above counterexample shows that $X \neq XAX$, where X is the MPEPN-inverse of A. A natural question is: when $A^{E,\ddagger}$ is an outer inverse of A. One can show that if the condition $(A_1A^\dagger)^2 = A_1A^\dagger$ holds, then the MPEPN-inverse of A is an outer inverse of A.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and A_1 be the core part in the EP-nilpotent decomposition of A. Then $XAX = X$ if and only if $(A_1A^\dagger)^2 = A_1A^\dagger$, where X is the MPEPN-inverse of A.*

Proof. Let X be the MPEPN-inverse of A, then by Definition 2.1 we have $X = A^\dagger A_1 A^\dagger$. We have the following conditions of equation $XAX = X$.

$$XAX = X \Longleftrightarrow A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A_1 A^\dagger,$$

that is,

$$XAX = X \Longleftrightarrow A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A_1 A^\dagger. \tag{6}$$

By Lemma 2.5, we know $A_1 = AA^k(A^k)^\dagger$, thus (6) gives

$$XAX = X \Longleftrightarrow A^\dagger AA^k(A^k)^\dagger A^\dagger = A^\dagger AA^k(A^k)^\dagger A^\dagger AA^k(A^k)^\dagger A^\dagger. \tag{7}$$

Pre-multiplying by A on the right of (7) implies

$$AA^\dagger AA^k(A^k)^\dagger A^\dagger = AA^\dagger AA^k(A^k)^\dagger A^\dagger AA^k(A^k)^\dagger A^\dagger.$$

Then,

$$AA^k(A^k)^\dagger A^\dagger = AA^k(A^k)^\dagger A^\dagger AA^k(A^k)^\dagger A^\dagger. \tag{8}$$

Thus, we have the equality in (8) is equivalent to $A_1A^\dagger = A_1A^\dagger A_1A^\dagger$ by $A_1 = AA^k(A^k)^\dagger$, that is, $(A_1A^\dagger)^2 = A_1A^\dagger$. □

In the following, we show that the MPEPN-inverse of A is an outer inverse under the condition $S(E_{n-t} - N^\dagger N)S^* = 0$, where E_{n-t} is the identity of size $n-t$ and reciprocally.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as (1). Then $XAX = X$ if and only if $S(E_{n-t} - N^\dagger N)S^* = 0$, where $t = rank(A^k)$ and X is the MPEPN-inverse of A.*

Proof. By Lemma 1, we have $A = A_1 + A_2$, where $A_1 = U\begin{bmatrix} T & 0 \\ 0 & 0\end{bmatrix}U^*$ and $A_2 = U\begin{bmatrix} 0 & S \\ 0 & N\end{bmatrix}U^*$, where t is the rank of A^k, the size of T and N are t and $n-t$, respectively. Then by Lemma 1 and Lemma 2.3, we have

$$\begin{aligned} A_1A^\dagger &= U\begin{bmatrix} T & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} T^*\Delta & -T^*\Delta SN^\dagger \\ (E_{n-t} - N^\dagger N)S^\dagger\Delta & N^\dagger - (E_{n-t} - N^\dagger N)S^*\Delta SN^\dagger\end{bmatrix}U^* \\ &= U\begin{bmatrix} TT^*\Delta & -TT^*\Delta SN^\dagger \\ 0 & 0\end{bmatrix}U^*. \end{aligned}$$

By $(A_1A^\dagger)^2 = A_1A^\dagger$, we have

$$\left(U\begin{bmatrix} TT^*\Delta & -TT^*\Delta SN^\dagger \\ 0 & 0\end{bmatrix}U^*\right)^2 = U\begin{bmatrix} TT^*\Delta & -TT^*\Delta SN^\dagger \\ 0 & 0\end{bmatrix}U^*,$$

which is equivalent to

$$\begin{bmatrix} (TT^*\Delta)^2 & -TT^* \triangle TT^*\Delta SN^\dagger \\ 0 & 0\end{bmatrix} = \begin{bmatrix} TT^*\triangle & -TT^* \triangle SN^\dagger \\ 0 & 0\end{bmatrix} \tag{9}$$

since U is nonsingular because U is unitary. The equality in (9) gives

$$\begin{cases} (TT^*\Delta)^2 = TT^*\Delta \\ TT^*\Delta TT^*\Delta SN^\dagger = TT^*\Delta SN^\dagger \end{cases} \tag{10}$$

By Lemma 2.4, we know that (10) is equivalent to

$$\begin{cases} (TT^*\Delta)^2 = TT^*\Delta \\ TT^*\Delta TT^*\Delta SN^* = TT^*\Delta SN^* \end{cases} \tag{11}$$

Since T is nonsingular, then TT^* is nonsingular, then (11) is equivalent to

$$\begin{cases} TT^*\Delta = E_t \\ TT^*\Delta SN^* = SN^* \end{cases} \tag{12}$$

which is equivalent to

$$TT^*\Delta = E_t. \tag{13}$$

Since Δ is invertible, (13) is equivalent to

$$TT^* = \Delta^{-1}. \tag{14}$$

By Lemma 2.3,

$$\Delta^{-1} = TT^* + S(E_{n-t} - N^\dagger N)S^*. \tag{15}$$

By (14) and (15), we have $TT^* = TT^* + S(E_{n-t} - N^\dagger N)S^*$, that is, $S(E_{n-t} - N^\dagger N)S^* = 0$. □

Remark 3.1. By the proof of Theorem *3.2*, we have $X = XAX$ if and only if $TT^* = \Delta^{-1}$, where X is the MPEPN-inverse of A and $\Delta = [TT^* + S(E_{n-t} - N^\dagger N)S^*]^{-1}$.

In the following, we show that the MPEPN-inverse of A is an outer inverse of A if and only if $A_2 A_2^\dagger A^{\textcircled{\dagger}} = 0$.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as* (1) *and $A = A_1' + A_2'$ be the Core-EP decomposition of A as* (2.2)*. Then $XAX = X$ if and only if $A^{\textcircled{\dagger}} A_1 A_2 A_2^* A_1 A^{\textcircled{\dagger}} = A_1'(A_2')^\dagger A_2'(A_1')^*$, where X is the MPEPN-inverse of A.*

Proof. Let X be the MPEPN-inverse of A. By Theorem 3.2, we have $XAX = X$ if and only if $S(E_{n-t} - N^\dagger N)S^* = 0$, that is,

$$SS^* = SN^\dagger NS^*. \tag{16}$$

We have

$$A_2A_2^* = U\begin{bmatrix}0 & S\\ 0 & N\end{bmatrix}U^*U\begin{bmatrix}0 & 0\\ S^* & N^*\end{bmatrix}U^* = U\begin{bmatrix}SS^* & SN^*\\ NS^* & NN^*\end{bmatrix}U^* \tag{17}$$

by $A_2 = U\begin{bmatrix}0 & S\\ 0 & N\end{bmatrix}U^*$ and $A_2^* = U^*\begin{bmatrix}0 & 0\\ S^* & N^*\end{bmatrix}U$. Moreover, by Lemma 1 we have

$$A_1 = U\begin{bmatrix}T & 0\\ 0 & 0\end{bmatrix}U^*. \tag{18}$$

By (17) and (18), we have

$$A_2A_2^*A_1 = U\begin{bmatrix}SS^*T & 0\\ NS^*T & 0\end{bmatrix}U^*. \tag{19}$$

By (19), we have

$$A_1A_2A_2^*A_1 = U\begin{bmatrix}T & 0\\ 0 & 0\end{bmatrix}U^*U\begin{bmatrix}SS^*T & 0\\ NS^*T & 0\end{bmatrix}U^* = U\begin{bmatrix}TSS^*T & 0\\ 0 & 0\end{bmatrix}U^*. \tag{20}$$

By [14, Theorem 3.2], we have

$$A^{\oplus} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{21}$$

By (20) and (21), we have

$$A^{\oplus} A_1 A_2 A_2^* A_1 A^{\oplus} = U \begin{bmatrix} SS^* & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{22}$$

By Lemma 2.2, we have $(A_2')^{\dagger} = U \begin{bmatrix} 0 & 0 \\ 0 & N^{\dagger} \end{bmatrix} U^*$, then

$$(A_2')^{\dagger} A_2' = U \begin{bmatrix} 0 & 0 \\ 0 & N^{\dagger} N \end{bmatrix} U^*. \tag{23}$$

Since $(A_1')^* = U \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} U^*$. Thus by (23), we have

$$\begin{aligned} A_1'(A_2')^{\dagger} A_2' (A_1')^* &= U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & N^{\dagger} N \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} 0 & SN^{\dagger} N \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} SN^{\dagger} NS^* & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned} \tag{24}$$

By (22) and (24), the equality in (16) can be written as

$$A^{\oplus} A_1 A_2 A_2^* A_1 A^{\oplus} = A_1'(A_2')^{\dagger} A_2' (A_1')^*.$$

□

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ with the index of A is k and $A = A_1 + A_2$ be the EP-nilpotent decomposition of A as (1). Then $XAX = X$ if and only if $A_2 A_2^{\dagger} A^{\oplus} = 0$, where X is the MPEPN-inverse of A.*

Proof. By Lemma 2.3, we have

$$A_2^{\dagger} = U \begin{bmatrix} 0 & 0 \\ (E_{n-t} - N^{\dagger} N) S^* \Delta & N^{\dagger} - (E_{n-t} - N^{\dagger} N) S^* \Delta S N^{\dagger} \end{bmatrix} U^*, \tag{25}$$

where $\Delta = [TT^* + S(E_{n-t} - N^{\dagger} N) S^*]^{-1}$. Then

$$\begin{aligned} A_2^{\dagger} A^{\oplus} &= U \begin{bmatrix} 0 & 0 \\ (E_{n-t} - N^{\dagger} N) S^* \Delta & N^{\dagger} - (E_{n-t} - N^{\dagger} N) S^* \Delta S N^{\dagger} \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} 0 & 0 \\ (E_{n-t} - N^{\dagger} N) S^* \Delta T^{-1} & 0 \end{bmatrix} U^*. \end{aligned} \tag{26}$$

By (26), we have

$$\begin{aligned} A_2 A_2^\dagger A^{\oplus} &= U \begin{bmatrix} 0 & S \\ 0 & N \end{bmatrix} U^* U \begin{bmatrix} 0 & 0 \\ (E_{n-t} - N^\dagger N) S^* \Delta T^{-1} & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} S(E_{n-t} - N^\dagger N) S^* \Delta T^{-1} & 0 \\ N(E_{n-t} - N^\dagger N) & 0 \end{bmatrix} U^*. \end{aligned}$$

Thus,

$$S(E_{n-t} - N^\dagger N) S^* = 0 \Longleftrightarrow S(E_{n-t} - N^\dagger N) S^* \Delta T^{-1} = 0 \Longleftrightarrow A_2 A_2^\dagger A^{\oplus} = 0. \ \square$$

Note that, the condition $A_2 A_2^\dagger A^{\oplus} = 0$ in Theorem 3.4 can be written as $P_{\mathcal{R}(A_2)} A^{\oplus} = 0$, where $P_{\mathcal{R}(A_2)}$ is the orthogonal projectors onto $\mathcal{R}(A_2)$.

4. The "distance" between the MPEPN-inverse and the inverse along two matrices

In 2012, Drazin [5] introduced a new kind of generalized inverse based on two elements. In 2017, Benítez et al. [1] investigated the (B, C)-inverse of a rectangle complex matrix. The "distance" between the MPEPN-inverse and the inverse along two matrices can be stated by $A^{E,\ddagger}$ is the $(A^\dagger A_1 P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)} A_1 A^\dagger)$-inverse of A under the condition $(A_1 A^\dagger)^2 = A_1 A^\dagger$.

Theorem 4.1. *Let $A \in \mathbb{C}^{n \times n}$ with the index of A is k and A_1 be the core part in the EP-nilpotent decomposition of A. If $A_1 A^\dagger$ is an idempotent, then X is the $(A^\dagger A_1 P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)} A_1 A^\dagger)$-inverse of A, where X is the MPEPN-inverse of A.*

Proof. By Theorem 3.1, when $A_1 A^\dagger$ is an idempotent, we have $XAX = X$, where $X = A^\dagger A_1 A^\dagger = A^\dagger A^{k+1} (A^k)^\dagger A^\dagger$. Let $B = A^\dagger A_1 P_{\mathcal{R}(A^*)}$ and $C = P_{\mathcal{R}(A)} A_1 A^\dagger$, then $X = XAX = A^\dagger A_1 A^\dagger A X = A^\dagger A_1 P_{\mathcal{R}(A^*)} X = BX$, which gives

$$\mathcal{R}(X) \subseteq \mathcal{R}(B). \tag{27}$$

Moreover, the condition $B = A^\dagger A_1 P_{\mathcal{R}(A^*)} = A^\dagger A_1 A^\dagger A = XA$ implies

$$\mathcal{R}(B) \subseteq \mathcal{R}(X). \tag{28}$$

By (27) and (28), we can get $\mathcal{R}(B) = \mathcal{R}(X)$. For any $u \in \mathcal{N}(P_{\mathcal{R}(A)} A_1 A^\dagger)$, that is, $P_{\mathcal{R}(A)} A_1 A^\dagger u = 0$, then $Xu = XAXu = XAA^\dagger A_1 A^\dagger u = XP_{\mathcal{R}(A)} A_1 A^\dagger u = 0$, which gives

$$\mathcal{N}(P_{\mathcal{R}(A)} A_1 A^\dagger) \subseteq \mathcal{N}(X). \tag{29}$$

For any $v \in \mathcal{N}(X)$, that is, $Xv = 0$, then the condition $P_{\mathcal{R}(A)} A_1 A^\dagger v = AA^\dagger A_1 A^\dagger v = AXv = 0$ implies

$$\mathcal{N}(X) \subseteq \mathcal{N}(P_{\mathcal{R}(A)} A_1 A^\dagger). \tag{30}$$

By (29) and (30), we have $\mathcal{N}(C) = \mathcal{N}(X)$. Thus, by Lemma 2.6, we have X is the $(A^\dagger A_1 P_{\mathcal{R}(A^*)}, P_{\mathcal{R}(A)} A_1 A^\dagger)$-inverse of A. $\square$

The MPEPN-inverse of A is different from the Moore-Penrose inverse, the DMP inverse $A^{D,\dagger}$ of A ([11]), the Core-EP inverse $A^{\textcircled{\dagger}}$ of A ([9]) and the MPCEP-inverse $A^{\dagger,\textcircled{\dagger}}$ of A ([3]). The example can been seen in the following example.

Example 4.1. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then it is easy to check that

$$A^{E,\ddagger} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{\dagger} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{D,\dagger} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{\textcircled{\dagger}} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{\dagger,\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the MPEPN-inverse is different from the above generalized inverses.

5. Conclusions

Let A be a given complex matrix with a given index, then one can get that the computation of the MPEPN inverse of A by using the EP-nilpotent decomposition of this matrix. There is a interesting fact about the EP-nilpotent decomposition of A, that is one can using the Core-EP decomposition of A to get the the EP-nilpotent decomposition of A. The future perspectives for research are proposed:

Part 1. The MPEPN inverse is one of the useful tools to investigate the matrix partial orders.

Part 2. The rank properties of a given matrix, such as rank $(AA^{E,\ddagger} - A^{E,\ddagger}A)$.

Part 3. The weighted generalized inverse of matrices related given range space and null space.

Author contributions

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Fantastic (weak) hyper filters in hyper BE-algebras

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Abstract. In this paper, fantastic (weak) hyper filters in hyper BE-algebras are introduced and investigated. The relationships between fantastic (weak) hyper filters and (weak) hyper filters are discussed and the related examples are delivered. Then, fantastic (weak) hyper filters are characterized respectively. Moreover, examples are given in which fantastic weak hyper filters and fantastic hyper filters may not be deduced from each other in hyper BE-algebras, meanwhile the conditions are found that fantastic weak hyper filters become fantastic weak hyper filters in hyper BE-algebras.

Keywords: hyper BE-algebra, $\circ$-reflexive subset, (weak) hyper filter, fantastic (weak) hyper filter.

1. Introduction

The hyper algebraic theory was introduced by Marty [15] at the 8th Congress of Scandinavian Mathematicians. Since then, hyper algebraic structure has been intensively researched such as hyper BCK-algebras [12, 13], hyper K-algebras [11, 18], hyper residuated lattices [2, 17], hyper EQ-algebras [4, 8] and hyper equality algebras [3, 7], etc. At present, hyper algebraic theory has been widely applied to many disciplines [9, 10]. Borzooei et al. investigated the filter theory of residuated lattices and hyper equality algebras in [2] and [3] respectively.

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Then, Borzooei and Aaly in [1] systematically summarized various of hyper algebraic structures and presented the relationships among these hyper algebraic structures. Radfar et al. [16] in 2014 introduced the notion of hyper BE-algebras as a generalization of BE-algebras [14]. Moreover, they proposed some special types of hyper BE-algebras and (weak) hyper filters in hyper BE-algebras. In fact, hyper BE-algebras are closely related to many hyper algebras and it is a generalization of dual hyper BCK-algebras, dual hyper K-algebras and hyper hoops [5]. Cheng and Xin in [6] focused on investigating (positive) implicative hyper filters in hyper BE-algebras and induced quotient hyper BE-algebras by use of implicative hyper filters. Based on the above, the present paper considers fantastic (weak) hyper filters in hyper BE-algebras so as to further explore the structure of hyper BE-algebras.

2. Preliminaries

In this section, we recollect some definitions and results about hyper BE-algebras which will be used in the following.

Definition 2.1 ([16]). *Let H be a nonempty set and $\circ : H \times H \to P^*(H)$ be a hyperoperation. Then, $(H, \circ, 1)$ is called a hyper BE-algebra provided it satisfies the following axioms:*

$(HBE1)$ $x \ll 1$ *and* $x \ll x$*;*

$(HBE2)$ $x \circ (y \circ z) = y \circ (x \circ z)$*;*

$(HBE3)$ $x \in 1 \circ x$*;*

$(HBE4)$ $1 \ll x$ *implies* $x = 1$*, for all* $x, y \in H$*, where the relation* $\ll$ *is defined by* $x \ll y \Leftrightarrow 1 \in x \circ y$*. For any two nonempty subsets* A *and* B *of* H*,* $A \ll B$ *means that there exist* $a \in A, b \in B$ *such that* $a \ll b$*.*

Notice that, in any hyper BE-algebra, $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$ and $A \leq B$ means for any $a \in A$, there exists $b \in B$ such that $a \ll b$.

In the following sequel, by H denote a hyper BE-algebra $(H, \circ, 1)$, unless otherwise specified.

Proposition 2.1 ([6, 16]). *In any hyper BE-algebra H, the following hold:*

(1) $A \circ (B \circ C) = B \circ (A \circ C)$*;*

(2) $A \subseteq 1 \circ A, 1 \in A \circ 1, 1 \in A \circ A$*;*

(3) $x \leq y \circ x, A \leq B \circ A$*;*

(4) $A \ll B$ *iff* $1 \in A \circ B$*;*

(5) $1 \in A$ *and* $A \leq B$ *imply* $1 \in B$*;*

(6) $1 \ll A$ *implies* $1 \in A$, *for all* $x, y \in H, A, B \subseteq H$.

Definition 2.2 ([16]). *We say that a hyper BE-algebra H is a*

(1) *C-hyper BE-algebra, if* $x \circ 1 = \{1\}$ *for all* $x \in H$;

(2) *R-hyper BE-algebra, if* $1 \circ x = \{x\}$ *for all* $x \in H$;

(3) *D-hyper BE-algebra, if* $x \circ x = \{1\}$ *for all* $x \in H$;

(4) *RD-hyper BE-algebra, if H is both a R-hyper BE-algebra and a D-hyper BE-algebra;*

(5) *RC-hyper BE-algebra, if H is both a R-hyper BE-algebra and a C-hyper BE-algebra.*

Definition 2.3 ([16]). *A nonempty subset F containing 1 of H is said to be a*

(1) *hyper filter if* $x \circ y \cap F \neq \emptyset$ *and* $x \in F$ *imply* $y \in F$, *for any* $x, y \in H$;

(2) *weak hyper filter if* $x \circ y \subseteq F$ *and* $x \in F$ *imply* $y \in F$, *for any* $x, y \in H$.

It is well known that every hyper filter is a weak hyper filter in a hyper BE-algebra, but the converse is not true. Moreover, every hyper filter satisfies the condition (F):

(F) $x \in F$ and $x \ll y$ imply $y \in F$ for all $x, y \in H$.

3. Fantastic (weak) hyper filters

In this section, we introduce fantastic (weak) hyper filters in hyper BE-algebras and deliver some related results of them.

Definition 3.1. *A nonempty subset F containing 1 of H is said to be a*

(1) *fantastic hyper filter, if* $z \circ (x \circ y) \cap F \neq \emptyset$ *and* $z \in F$ *imply* $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$, *for any* $x, y, z \in H$;

(2) *fantastic weak hyper filter, if* $z \circ (x \circ y) \subseteq F$ *and* $z \in F$ *imply* $((y \circ x) \circ x) \circ y \subseteq F$, *for any* $x, y, z \in H$.

Example 3.2. Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1, b\}$	$\{1\}$	$\{1\}$
b	$\{1, b\}$	$\{1\}$	$\{1\}$

Then, $(H, \circ, 1)$ is a hyper BE-algebras [16]. It is easy to verify that $F = \{1\}$ is a fantastic weak hyper filter and $G = \{1, a\}$ is a fantastic hyper filter of H.

Proposition 3.1. *Let H be a RC-hyper BE-algebra. If F is a fantastic (weak) hyper filter of H, then F is a (weak) hyper filter of H.*

Proof. (1) Let $x \circ y \cap F \neq \emptyset$ and $x \in F$, for any $x, y \in F$. Then, by $x \circ y \subseteq x \circ (1 \circ y)$ we have $x \circ (1 \circ y) \cap F \neq \emptyset$. Again since $x \in F$ and F is a fantastic hyper filter of H, we can obtain that $\{y\} \cap F = 1 \circ y \cap F = ((y \circ 1) \circ 1) \circ y \cap F \neq \emptyset$ and thus $y \in F$. Therefore, F is a hyper filter of H.

(2) Let $x \circ y \subseteq F$ and $x \in F$, for any $x, y \in F$. Since H is a R-hyper BE-algebra, we have $x \circ (1 \circ y) = x \circ y \subseteq F$. Again since $x \in F$ and F is a fantastic weak hyper filter of H, then $\{y\} = 1 \circ y = ((y \circ 1) \circ 1) \circ y \subseteq F$ and thus $y \in F$. Therefore, F is a weak hyper filter of H. □

Example 3.3. Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1\}$	$\{a, b\}$	$\{1, b\}$

Then, $(H, \circ, 1)$ is a RC-hyper BE-algebras [16]. One can calculate that $F = \{1, a\}$ is both a (weak) hyper filter and a fantastic (weak) hyper filter of H.

Notice that the condition of the RC-hyper from Proposition 3.1 is not necessary in general. In fact, in Example 3.2 H is not a RC-hyper BE-algebra, but it is easy to see that $F = \{1\}$ is both a (weak) hyper filter and a fantastic (weak) hyper filter of H.

The converse of Proposition 3.1 may not be true and see the following example.

Example 3.4. (1) Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1\}$	$\{1, a\}$
b	$\{1\}$	$\{1\}$	$\{1, a\}$

Then, $(H, \circ, 1)$ is a RC-hyper BE-algebras [16]. It is not difficult to check that $F = \{1\}$ is a weak hyper filter of H, but it is not a fantastic weak hyper filter of H since $1 \in F$ and $1 \circ (b \circ a) \subseteq F$ while $((a \circ b) \circ b) \circ a = \{1, a\} \nsubseteq F$.
(2) Let $H = \{1, a, b, c\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{1\}$	$\{a\}$	$\{1, b\}$	$\{c\}$
c	$\{1\}$	$\{a\}$	$\{1, b\}$	$\{1, b\}$

Then, $(H, \circ, 1)$ is a RC-hyper BE-algebra [11]. It is routine to verify that $F = \{1, a\}$ is a hyper filter of H, but it is not a fantastic hyper filter of H since $1 \in F$ and $1 \circ (a \circ b) \cap F \neq \emptyset$ while $((b \circ a) \circ a) \circ b = \{b\} \cap F = \emptyset$.

In what follows we deliver a characterization of the fantastic (weak) hyper filter of H, respectively.

Theorem 3.1. *Let F be a hyper filter of H. Then, the following are equivalent:*

(1) *F is a fantastic hyper filter of H;*

(2) *$x \circ y \cap F \neq \emptyset$ implies $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$, for any $x, y \in H$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds and $x \circ y \cap F \neq \emptyset$, for any $x, y \in H$. Since $x \circ y \subseteq 1 \circ (x \circ y)$ then $1 \circ (x \circ y) \cap F \neq \emptyset$. Since $1 \in F$ and F is a fantastic hyper filter of H, we have $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$.

(2) $\Rightarrow$ (1) Assume that (2) holds. Let $z \circ (x \circ y) \cap F \neq \emptyset$ and $z \in F$, for any $x, y, z \in H$. Since F is a hyper filter of H, then $x \circ y \cap F \neq \emptyset$ and so by hypothesis we can obtain $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$. It concludes that F is a fantastic hyper filter of H. □

Theorem 3.2. *Let F be a weak hyper filter of a R-hyper BE-algebra H. Then, the following are equivalent:*

(1) *F is a fantastic weak hyper filter of H;*

(2) *$x \circ y \subseteq F$ implies $((y \circ x) \circ x) \circ y \subseteq F$, for any $x, y \in H$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds and $x \circ y \subseteq F$, for any $x, y \in H$. Since $1 \in F, 1 \circ (x \circ y) = x \circ y \subseteq F$ and F is a fantastic weak hyper filter of H, we have $((y \circ x) \circ x) \circ y \subseteq F$.

(2) $\Rightarrow$ (1) Assume that (2) holds. Let $z \circ (x \circ y) \subseteq F$ and $z \in F$, for any $x, y, z \in H$. Since F is a weak hyper filter of H, then $x \circ y \subseteq F$ and so by hypothesis we can obtain $((y \circ x) \circ x) \circ y \subseteq F$. It concludes that F is a fantastic weak hyper filter of H. □

In general, a fantastic hyper filter of H may not be a fantastic weak hyper filter and vice versa.

Example 3.5. (1) In Example 3.2 one can check that the set $M = \{1, b\}$ is a fantastic hyper filter of H, but it is not a fantastic weak hyper filter since $b \in M$ and $b \circ (1 \circ a) = \{1\} \subseteq M$ while $((a \circ 1) \circ 1) \circ a = \{1, a, b\} \nsubseteq M$.
(2) Let $H = \{a, b, 1\}$. Define the operation $\circ$ on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a\}$	$\{1, b\}$
b	$\{1\}$	$\{1, a, b\}$	$\{1\}$

Then, $(H, \circ, 1)$ is a hyper BE-algebras [16]. It can be calculated that $F = \{1, a\}$ is a fantastic weak hyper filter of H, but it is not a fantastic hyper filter since $a \in F$ and $a \circ (1 \circ b) = \{1, b\} \cap F \neq \emptyset$ while $((b \circ 1) \circ 1) \circ b = \{b\} \cap F = \emptyset$.

In what follows, we provide the conditions that fantastic weak hyper filters become fantastic hyper filters in hyper BE-algebras.

Definition 3.6 ([6]). *A nonempty subset S of H is said to be $\circ$-reflexive if $x \circ y \cap S \neq \emptyset$ implies $x \circ y \subseteq S$ for all $x, y \in H$.*

Proposition 3.2. *Let H be a RC-hyper BE-algebra. If F is a $\circ$-reflexive fantastic weak hyper filters of H, then it is a fantastic hyper filter of H.*

Proof. As F is a $\circ$-reflexive weak hyper filters of H, we have that F is a hyper filter of H. Now, set $x \circ y \cap F \neq \emptyset$, for any $x, y \in H$. It follows from the $\circ$-reflexivity of F that $x \circ y \subseteq F$. Since F is a fantastic weak hyper filter of H, then by Theorem 3.2 we obtain $((y \circ x) \circ x) \circ y \subseteq F$ and so $((y \circ x) \circ x) \circ y \cap F \neq \emptyset$. Therefore, by Theorem 3.1 F is a fantastic hyper filter of H. □

Definition 3.7. *A hyper BE-algebra H is called right-ordered, if $x \ll y$ implies $y \circ z \ll x \circ z$ for all $x, y, z \in H$.*

Example 3.8. It is easy to verify that the hyper BE-algebra H from Example 3.5 (2) is right-ordered.

Theorem 3.3. *Let H be a right-ordered RD-hyper BE-algebra, and F, G be $\circ$-reflexive weak hyper filters of H. If $F \subseteq G$ and F is a fantastic weak hyper filter of H, then G is a fantastic hyper filter of H.*

Proof. Let $x \circ y \cap G \neq \emptyset$, for any $x, y \in H$. Denote $m = x \circ y$, since G is $\circ$-reflexive then $m \subseteq G$ Again since H is a D-hyper BE-algebra, we have $x \circ (m \circ y) = m \circ (x \circ y) = \{1\} \subseteq F$. Notice that H is a R-hyper BE-algebra and F is a fantastic weak hyper filter, it follows from Theorem 3.2 that $m \circ ((((m \circ y) \circ x) \circ x) \circ y) = (((m \circ y) \circ x) \circ x) \circ (m \circ y) \subseteq F$ and hence $m \circ ((((m \circ y) \circ x) \circ x) \circ y) \subseteq G$. Combing that $m \in G$ and G is a weak hyper filter, we can obtain $(((m \circ y) \circ x) \circ x) \circ y \subseteq G$. Again since $y \ll m \circ y$ and H is right-ordered, we get that $(((m \circ y) \circ x) \circ x) \circ y \ll ((y \circ x) \circ x) \circ y$. Considering $(((m \circ y) \circ x) \circ x) \circ y \subseteq G$ and the $\circ$-reflexivity of G, it can conclude that $((y \circ x) \circ x) \circ y \cap G \neq \emptyset$. Therefore, using Theorem 3.1 G is a fantastic hyper filter of H. □

4. Conclusions

Filters are an important tool in the research of algebraic structures. In this paper, fantastic (weak) hyper filters are proposed in hyper BE-algebras and also the relation between them is delivered. What is more, the characterizations of fantastic (weak) hyper filters are showed. In the further work, we shall explore some applications of fantastic (weak) hyper filters such as in quotient hyper BE-algebras and in the state theory of hyper BE-algebras.

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Flow-selfdual curves in a geometric surface

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Abstract. For a natural parametrization of a curve γ in an orientable two-dimensional Riemannian manifold, we compare two differential equations associated to γ. The main tool of our study is the geodesic curvature k of γ and when these equations coincide we call γ as being flow-selfdual since the second equation corresponds to the flow-curvature k_f of γ in the same manner as the first equation involves k. We obtain that these curves have a constant geodesic curvature and then we discuss four examples. Also, we generalize this type of differential equations to vector fields on Riemannian manifolds of arbitrary dimension.

Keywords: two-dimensional Riemannian manifold, geodesic curvature, flow-selfdual curve.

MSC 2020: 53A04, 53A45, 53A55

1. Flow-selfdual curves and tangential vector fields

The framework of this study is a *geometric surface* i.e. ([3]) a smooth, orientable two-dimensional Riemannian manifold (M^2, g). Being orientable M supports an almost complex structure J; in fact J is integrable and for an arbitrary point $p \in M$ we consider $J_p : T_pM \to T_pM$ as being the multiplication with the complex unit $i \in \mathbb{C}$. Let ∇ be the Levi-Civita connection of g.

Fix also a smooth curve $\gamma : I \subseteq \mathbb{R} \to M$ which we suppose to be *regular*: $\gamma'(t) \in T_{\gamma(t)}M \setminus \{0\}$. Let $\mathfrak{X}(\gamma)$ be the $C^\infty(I)$-module of vector fields along γ i.e. smooth maps $X : I \to TM$ with $X(t) \in T_{\gamma(t)}M$ for all $t \in I$. It follows the *unit tangent vector field* $T \in \mathfrak{X}(\gamma)$ with:

$$T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}, \tag{1.1}$$

where $\|\cdot\|$ denotes the norm induced by g on the tangent spaces. Therefore, the *Frenet frame* of γ is $\mathcal{F} := \begin{pmatrix} T \\ N := J(T) \end{pmatrix} \in \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma)$.

The Riemannian geometry of γ is described by its *geodesic curvature* $k : I \to \mathbb{R}$ provided by the *Frenet equations*:

$$\nabla_{T(t)}\mathcal{F}(t) = \begin{pmatrix} 0 & k(t) \\ -k(t) & 0 \end{pmatrix}\mathcal{F}(t) = k(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathcal{F}(t) \tag{1.2}$$

which means:

$$k(t) := \frac{g(\nabla_{\gamma'(t)}T(t), N(t))}{\|\gamma'(t)\|} = \frac{g(\nabla_{\gamma'(t)}\gamma'(t), J(\gamma'(t)))}{\|\gamma'(t)\|^3}. \tag{1.3}$$

Recall also the pair (g, J) yields *the symplectic form* $\Omega(\cdot,\cdot) := g(\cdot, J\cdot)$ and whence:

$$k(t) := \frac{\Omega(\nabla_{\gamma'(t)}\gamma'(t), \gamma'(t))}{\|\gamma'(t)\|^3}. \tag{1.4}$$

The starting point of this short note is the remark that under the hypothesis of γ being parametrized by arc-length the second covariant derivative applied to the Frenet equations yields the following differential equation:

$$\mathcal{E} : (\nabla_{\gamma'}\nabla_{\gamma'}\gamma')(t) - \frac{k'(t)}{k(t)}(\nabla_{\gamma'}\gamma')(t) + k^2(t)\gamma'(t) = 0. \tag{1.5}$$

The pair (g, J) being a Kähler structure (since $\dim M = 2$) it follows that ∇ commutes with N and then N satisfies the same differential equation. For curves parametrized by arc-length the vector field $\nabla_{\gamma'}\gamma'$ is called *the curvature vector field* of γ.

In the very recent paper [3] we introduce a modification of the curvature k called *flow-curvature* and denoted k^f. For a general parametrization of γ it holds:

$$k^f(t) := k(t) - \frac{1}{\|\gamma'(t)\|} < k(t). \tag{1.6}$$

Since k^f is obtained exactly in the same manner as k i.e. through the Frenet equation of the flow-frame:

$$\mathcal{F}^f(t) := \begin{pmatrix} E_1^f \\ E_2^f \end{pmatrix}(t) = Rotation(t)\mathcal{F}(t) = \begin{pmatrix} \cos tT(t) - \sin tN(t) \\ \sin tT(t) + \cos tN(t) \end{pmatrix} \tag{1.7}$$

it follows a second differential equation of third order satisfied by non-flow-flat curves:

$$\mathcal{E}^f : (\nabla_{\gamma'}\nabla_{\gamma'}E_1^f)(t) - \frac{(k^f)'(t)}{k^f(t)}(\nabla_{\gamma'}E_1^f)(t) + (k^f)^2(t)E_1^f(t) = 0. \tag{1.8}$$

It is natural to connect the differential equations $\mathcal{E}$ and $\mathcal{E}^f$ and this leads to our new type of curves:

Definition 1.1. The non-flow-flat curve γ, parametrized by arc-length, is called *flow-selfdual* it it satisfies also the differential equation:

$$\mathcal{E}_f : (\nabla_{\gamma'}\nabla_{\gamma'}\gamma')(t) - \frac{(k^f)'(t)}{k^f(t)}(\nabla_{\gamma'}\gamma')(t) + (k^f)^2(t)\gamma'(t) = 0. \tag{1.9}$$

Our main theoretical result is the following:

Proposition 1.2. *The non-flow-flat curve γ is a flow-selfdual one if and only if $k = \frac{1}{2} = -k^f$ which means that all four unit vector fields γ', N, E_1^f, E_2^f satisfy the same differential equation of Schrödinger type*:

$$\mathcal{E} = \mathcal{E}^f = \mathcal{E}_f : (\nabla_{\gamma'}\nabla_{\gamma'}U)(t) + \frac{1}{4}U(t) = 0, \quad U \in \mathfrak{X}(\gamma). \tag{1.10}$$

Proof. By comparing $\mathcal{E}$ and $\mathcal{E}_f$ it follows:

$$\frac{k'(t)}{k(t)(k(t)-1)}(\nabla_{\gamma'}\gamma')(t) + (2k(t)-1)\gamma'(t) = 0. \tag{1.11}$$

Due to the unit speed parametrization of γ the vector fields $\nabla_{\gamma'}\gamma'$ and γ' are orthogonal and then $2k - 1 = 0$. $\square$

Remarks 1.3. 1) Let (Γ_{ij}^k) denote the Christoffel symbols of the metric g in a local chart of M in which $\gamma(t) = (x^i(t))$, $1 \leq i \leq 2$. Then the differential equation (1.10) becomes a scalar third-order one for a fixed $k \in \{1, 2\}$:

$$\begin{aligned} &\frac{d}{dt}\left[\ddot{x}^k(t) + \Gamma_{ij}^k(\gamma(t))\dot{x}^i(t)\dot{x}^j(t)\right] \\ &+ \Gamma_{ij}^k(\gamma(t))\dot{x}^i(t)\left[\ddot{x}^j(t) + \Gamma_{ab}^j(\gamma(t))\dot{x}^a(t)\dot{x}^b(t)\right] + \frac{\dot{x}^k(t)}{4} = 0. \end{aligned} \tag{1.12}$$

2) In the same paper [3] the flow-frame is generalized with an arbitrary (smooth) angle function $\Omega = \Omega(t)$ obtaining the Ω-*curvature*:

$$k^\Omega(t) := k(t) - \frac{\Omega'(t)}{\|\gamma'(t)\|}. \tag{1.13}$$

Hence, an arc-length parametrized curve with $k^\Omega \neq 0$ will be called Ω-*flow-selfdual* if the differential equation (1.9) holds with k^f replaced by Ω. The characterization of the proposition 1.2 reads now:

$$k(t) = \frac{\Omega'(t)}{2} = -k^\Omega(t). \tag{1.14}$$

A second suitable generalization of our notion works at the level of vector fields $\xi \in \mathfrak{X}(M)$=the Lie algebra of vector fields on M. Fix a unit ξ; then we call

ξ as being *a tangential vector field* if there exists a strictly positive $k \in C^{\infty}(M)$ (which we call *the curvature* of ξ) such that:

$$\nabla_{\xi}\nabla_{\xi}\xi - \xi(\ln k)\nabla_{\xi}\xi + k^2\xi = 0. \tag{1.15}$$

Making the g-product of the left-hand-side term above with ξ gives, as is expected, that:

$$\|\nabla_{\xi}\xi\| = k > 0. \tag{1.16}$$

We remark that is not necessary to work in the initial dimension two. An example of tangential vector field is provided within the theory of *torse-forming vector fields*. Recall, after [2], that a fixed $V \in \mathfrak{X}(M)$ is called torse-forming if for all $X \in \mathfrak{X}(M)$ we have:

$$\nabla_X V = fX + \omega(X)V, \tag{1.17}$$

for a smooth function $f \in C^{\infty}(M)$ and a 1-form $\omega \in \Omega^1(M)$. Now, suppose that $\nabla_{\xi}\xi$ is a torse-forming vector field with the data:

$$f = -k^2, \quad \omega = d(\ln k), \tag{1.18}$$

for a given strictly positive function k. It follows:

$$\nabla_X\nabla_{\xi}\xi - X(\ln k)\nabla_{\xi}\xi + k^2X = 0 \tag{1.19}$$

and then for $X = \xi$ it results the definition (1.15).

Recall also that an important tool in dynamics on curves is the Fermi-Walker derivative, which is the map ([4]) $\nabla_{\gamma}^{FW} : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$:

$$\begin{aligned} \nabla_{\gamma}^{FW}(X) &:= \frac{d}{dt}X + \|r'(\cdot)\|k[\langle X, N\rangle T \\ &- \langle X, T\rangle N] \to \nabla_{\gamma}^{FW}(T) = \nabla_{\gamma}^{FW}(N) = 0. \end{aligned} \tag{1.20}$$

Hence, we generalize this derivative as follows: the Fermi-Walker derivative generated by a tangential vector field ξ is the map $\nabla^{\xi} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by:

$$\nabla^{\xi}(X) := \nabla_{\xi}X + g(X, \nabla_{\xi}\xi)\xi - g(X, \xi)\nabla_{\xi}\xi. \tag{1.21}$$

From the equation (1.14) it results that ξ and $\nabla_{\xi}\xi$ are eigenvector fields of ∇^{ξ}:

$$\nabla^{\xi}(\xi) = 0, \quad \nabla^{\xi}(\nabla_{\xi}\xi) = \xi(\ln k)\nabla_{\xi}\xi. \tag{1.22}$$

2. Examples of flow-selfdual curves

Example 2.1. As is expected the plane Euclidean geometry $\mathbb{E}^2 := (\mathbb{R}^2, \langle\cdot,\cdot\rangle_{can})$ is the simplest case. The circle $\mathcal{C}(O, R = 2)$ is the "generic" Euclidean flow-orthogonal curve; it has the arc-length parametrization and Frenet data:

$$\gamma(t) = 2\left(\cos\frac{t}{2}, \sin\frac{t}{2}\right), \gamma'(t) = \left(-\sin\frac{t}{2}, \cos\frac{t}{2}\right),$$
$$N(t) = -\exp\left(i\frac{t}{2}\right) = -\frac{1}{2}\gamma(t) \tag{2.1}$$

and then the flow-frame is:

$$E_1^f(t) = \left(\sin\frac{t}{2}, \cos\frac{t}{2}\right), \quad E_2^f(t) = \left(-\cos\frac{t}{2}, \sin\frac{t}{2}\right) = \overline{N(t)}, \tag{2.2}$$

where we use the complex conjugate; the ordinary differential equation (1.10) is: $u'' + \frac{1}{4}u = 0$. An associated interesting problem is if there exists a Riemannian metric on (an open subset of) $\mathbb{R}^2$ having as geodesics the Euclidean circles; for the case of Finslerian metric this problem is already solved in [1].

Example 2.2. Fix (M, g) a rotationally symmetric surface i.e., conform [6], M is the product $\mathbb{S}^1 \times I$ with $\mathbb{S}^1$ the Euclidean unit circle and $I \subseteq \mathbb{R}$, endowed with the warped product metric:

$$g = dr^2 + f(r)^2 d\varphi^2, \quad r \in I, \quad \varphi \in \mathbb{S}^1. \tag{2.3}$$

This surface is oriented by the 2-form $dr \wedge d\varphi$ and then:

$$J\left(\frac{\partial}{\partial r}\right) = \frac{1}{f(r)}\frac{\partial}{\partial\varphi}, \quad J\left(\frac{\partial}{\partial\varphi}\right) = -f(r)\frac{\partial}{\partial r}. \tag{2.4}$$

Fix now the curve $\gamma(t) := (r(t), \varphi(t))$ parameterized by the arc-length t. Let $\sigma = \sigma(t)$ be the *structural angle of* γ i.e. the oriented angle between $\frac{\partial}{\partial r}$ and T. It follows the Frenet frame:

(2.5)
$$T(t) = \cos\sigma(t)\frac{\partial}{\partial r}|_t + \frac{\sin\sigma(t)}{f(r(t))}\frac{\partial}{\partial\varphi}|_t, \quad N(t) = -\sin\sigma(t)\frac{\partial}{\partial r}|_t + \frac{\cos\sigma(t)}{f(r(t))}\frac{\partial}{\partial\varphi}|_t.$$

The first derivative of T is then:

$$(\nabla_{\gamma'}\gamma')(t) = \left(\sigma'(t) + \frac{f'(r)}{f(r)}(t)\sin\sigma(t)\right)N(t) \tag{2.6}$$

which provides the expression of the geodesic curvature for γ:

$$k(t) = \sigma'(t) + \frac{f'(r)}{f(r)}(t)\sin\sigma(t). \tag{2.7}$$

The Proposition 1.1 of the cited paper [6] (or [7, p. 89]) offers a conservation law along γ, which for our constant $k = \frac{1}{2}$ reads as follows:

Proposition 2.3. *The smooth function:*

$$t \in [0, L(\gamma)) \to \mathcal{F}(t) := f(r(t)) \sin \sigma(t) - \frac{1}{2} \int_{r(0)}^{r(t)} f(\xi) d\xi \tag{2.8}$$

is constant along a given flow-selfdual curve γ.

The Euclidean plane geometry means $f(r) = r$ and the circle $\mathcal{C}(O, R > 0)$ gives $r = constant = R$, $\varphi(t) = \frac{t}{R}, t \in [0, 2\pi R]$ and $\sigma = constant = \frac{\pi}{2}$.

Example 2.4. For the hyperbolic plane geometry we use the Poincaré model of [5, p. 103]: $\mathbb{H}^2 := (\mathbb{R}^2_{y>0}; g = \frac{1}{y^2}(dx^2 + dy^2))$. Fix a curve $\gamma : t \in [0, +\infty) \to (x(t), y(t)) \in \mathbb{H}^2$ parametrized by arc-length. With the computation of the geodesic curvature from the cited book it follows that γ is a flow-selfdual curve if and only if the following differential system is satisfied:

$$[x'(t)]^2 + [y'(t)]^2 = [y(t)]^2, \quad x'(t) \left(\frac{1}{y(t)} + \frac{y''(t)}{y(t)^2} \right) - x''(t) \frac{y'(t)}{y(t)^2} = \frac{1}{2}. \tag{2.9}$$

A straightforward computation gives a single second order differential equation, which is written in a more simple form as:

$$\ddot{y} - \frac{2}{y}\dot{y} + y = \frac{\sqrt{y^2 - \dot{y}^2}}{2} \tag{2.10}$$

Unfortunately, being a nonlinear differential equation we cannot solve explicitly. In fact, we know that the types of hyperbolic curves with constant geodesic curvature k are as follows: a) circles, for $k > 1$; b) horocycles, with $k = 1$; c) equidistant curves (i.e. curves of finite distance from a hyperbolic geodesic), for $k \in (0, 1)$. Hence, the hyperbolic flow-selfdual curves are a family of equidistant curves.

Example 2.5. Let γ be a arc-length parametrized curve in the unit sphere $\mathbb{S}^2 := (S^2 \subset \mathbb{R}^3, g = (\langle \cdot, \cdot \rangle_{can})|_{S^2})$. Its usual Frenet curvature and torsion as space curve are $k^F > 0$ and τ^F. In fact, from the relationship:

$$k^F = \sqrt{k^2 + 1} \geq 1 \tag{2.11}$$

it follows that a flow-selfdual curve on $\mathbb{S}^2$ have a constant Frenet curvature $k^F = \frac{\sqrt{5}}{2}$. As concrete example we have the horizontal circle:

$$\gamma(t) = \frac{2}{\sqrt{5}} \left(\cos\left(\frac{\sqrt{5}}{2} t \right), \sin\left(\frac{\sqrt{5}}{2} t \right), \frac{1}{2} \right), t \in \mathbb{R}, \quad \tau^F \equiv 0. \tag{2.12}$$

More generally, recall that for the given arc-length parametrized curve γ on the regular surface $S \subset \mathbb{R}^3$ its geodesic curvature satisfies:

$$k = k^F \sin\theta \tag{2.13}$$

with θ the oriented angle between the normal N_γ of the curve and the normal N_S of S. For a flow-selfdual curve on $S = S^2$ it results the angle θ provided by:

$$\sin\theta = \frac{1}{\sqrt{5}}, \quad \cos\theta = \pm\frac{2}{\sqrt{5}}. \tag{2.14}$$

3. Conclusions

This note concerns with a particular class of curves in an orientable geometric surface (M^2, g). The curves in this class have a constant geodesic curvature, and hence they are remarkable objects for the differential geometry of the given pair (M^2, g). Four examples illustrate the significance of these curves in some important geometries. From a dynamical point of view we generalise the Fermi-Walker derivative and we hope this operator to become more suitable in some future works.

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On the completion of symmetric metric spaces

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Abstract. In this work, we investigate particular properties on the completion of symmetric spaces. Symmetric spaces are metric spaces and, naturally, question arises as to whether their completions are also symmetric. In this work, we provide an affirmative response to this question. More precisely, we prove that every metric space is isometrically a subset of a symmetric space. In addition, we prove that the completion of a symmetric metric space is likewise symmetric. Some additional functorial properties are established along with some other results. Additionally, generic examples of symmetric spaces will be provided in this manuscript.

Keywords: symmetric metric spaces, point symmetric map, isometry map, completion, functorial properties.
MSC 2020: 54E35, 54E40, 54E50

1. Introduction

The purpose of this section is to recall some standard terminology and nomenclature related to metric spaces [1] and symmetric metric spaces [2]. To start with, recall that a **metric space** is a pair of the form (X, d_X), where X is a nonempty set and $d_X : X \times X \to \mathbb{R}$ is a function satisfying the following properties:

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(i) if $x, y \in X$, then $d_X(x,y) \geq 0$,

(ii) if $x, y \in X$, then $d_X(x,y) = 0$ if and only if $x = y$,

(iii) $d_X(x,y) = d_X(y,x)$, for any $x, y \in X$, and

(iv) $d_X(x,y) \leq d_X(x,z) + d_X(z,y)$, for any $x, y, z \in X$.

If (X, d_X) and (Y, d_Y) are metric spaces, then a function $f : X \to Y$ is an **isometry** if $d_Y(f(x), f(y)) = d_X(x,y)$, for all $x, y \in X$. Obviously, any isometry is an injective and continuous function. If f is a surjective isometry, then we say that (X, d_X) and (Y, d_Y) and **isometric** spaces. In such case, f^{-1} is likewise an isometry. Evidently, the relation of being isometric is an equivalence relation in the class of metric spaces.

Let (X, d_X) be a metric space, $x_0 \in X$ and $f : X \to X$ a surjective isometry. Then (X, d_X) is x_0**-symmetric with respect to** f if, for each $x \in X$,

$$d_X(x, x_0) = d_X(f(x), x_0) = \frac{1}{2} d_X(x, f(x)).$$

If there is no ambiguity, then (X, d_X) is simply called x_0**-symmetric**. As an example, if $X = [-1, 1]$ with the metric d of $\mathbb{R}$ and $f : X \to X$ is given by $f(x) = -x$, then (X, d) is 0-symmetric with respect to f. Also, if $(X, \|\cdot\|)$ is a real Banach space with the norm $\|\cdot\| : X \to \mathbb{R}$, d_X is the respective induced norm and $a \in X$, then (X, d_X) is a-symmetric with respect to $f(x) = 2a - x$.

The following are some properties satisfied by symmetric metric spaces.

Proposition 1. *Let (X, d_X) be a metric space, $x_0 \in X$ and $f : X \to X$ a surjective isometry. If (X, d_X) is x_0-symmetric with respect to f, then it is also x_0-symmetric with respect to f^{-1}.*

Proof. Beforehand, notice that f^{-1} is also a surjective isometry. Let $y \in X$, and take $x \in X$ such that $y = f(x)$. It follows that

$$\begin{aligned} d_X(y, x_0) &= d_X(f(x), x_0) = d_X(x, x_0) = d_X(f^{-1}(y), x_0) \\ &= \frac{1}{2} d_X(x, f(x)) = \frac{1}{2} d_X(f^{-1}(y), y). \end{aligned}$$

We conclude that (X, d_X) is x_0-symmetric with respect to f^{-1}. □

Proposition 2. *Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$ and suppose that $f : X \to X$ and $\phi : X \to Y$ are surjective isometries. If (X, d_X) is a x_0-symmetric metric space with respect to f, then (Y, d_Y) is $\phi(x_0)$-symmetric with respect to $g = \phi \circ f \circ \phi^{-1}$.*

Proof. Being the composition of surjective isometries, g itself is a surjective isometry. On the other hand, if $y \in Y$, then the x_0-symmetry of (X, d_X) with respect to f and isometry properties of f, ϕ and ϕ^{-1} assure that

$$d_Y(y, \phi(x_0)) = d_X(\phi^{-1}(y), x_0) = d_X(f(\phi^{-1}(y)), x_0) = d_Y(g(y), \phi(x_0))$$

and

$$\frac{1}{2}d_Y(y, g(y)) = \frac{1}{2}d_X(\phi^{-1}(y), f(\phi^{-1}(y))) = d_X(f(\phi^{-1}(y)), x_0)$$
$$= d_Y(g(y), \phi(x_0)).$$

These facts establish that (Y, d_Y) is $\phi(x_0)$-symmetric with respect to g. □

Let (X, d_X) be a metric space, $x_0 \in X$ and $f, g : X \to X$ to surjective isometries. In general, it is not true that (X, d_X) is a x_0-symmetric metric space with respect to $g \circ f$ when it is x_0-symmetric with respect to f and g. Indeed, let (X, d_X) be the real numbers with its usual distance, and let us define $f(x) = g(x) = -x$, for each $x \in X$. It is obvious that (X, d_X) is 0-symmetric with respect to f and g, but it is not 0-symmetric with respect to $g \circ f$. In fact, notice that $d_X(x, (g \circ f)(x)) = 0$, for each $x \in X$.

2. Main results

This section is devoted to providing additional properties and ways to construct symmetric metric spaces. In the remainder and unless we mention something different, we will assume that (X, d_X) is a metric space, $x_0 \in X$ and $f : X \to X$ will be a surjective isometry.

To start with, we recall some standard definitions. If (X, d_X) is a metric space, $x \in X$ and $A \subseteq X$ is nonempty, then we define

$$d_X(x, A) = \inf_{y \in A} d_X(x, y).$$

In addition, if $B \subseteq X$ is also nonempty, then we define the number $d_X(A, B)$ alternatively (and equivalently) in the following way:

$$d_X(A, B) = \inf_{\substack{x \in A \\ y \in B}} d_X(x, y) = \inf_{x \in A} d_X(x, B) = \inf_{y \in B} d_X(y, A).$$

Proposition 3. *Suppose that (X, d_X) is an x_0-symmetric metric space with respect to f, and let $A \subseteq X$ be nonempty. Then*

$$d_X(x_0, A) = d_X(x_0, f(A)) \geq \frac{1}{2}d_X(A, f(A)).$$

Proof. Observe that the following inequalities hold:

$$\begin{aligned} d_X(x_0, A) &= \inf_{x \in A} d_X(x_0, x) = \inf_{x \in A} d_X(x_0, f(x)) = \inf_{y \in f(A)} d_X(x_0, y) \\ &= d_X(x_0, f(A)) = \inf_{x \in A} d_X(x_0, f(x)) = \frac{1}{2} \inf_{x \in A} d_X(x, f(X)) \\ &\geq \frac{1}{2} \inf_{\substack{x \in A \\ y \in f(A)}} d_X(x, y) = \frac{1}{2} d_X(A, f(A)), \end{aligned}$$

which yields the conclusion of this result. □

The following result is motivated by the reduced cone CX defined in [3].

Theorem 1. *Every metric space is isometrically a subset of a symmetric space.*

Proof. Let (X, d_X) be any metric space, and fix $x_0 \in X$ arbitrarily. Throughout, we will let $Y = (X \times \{x_0\}) \cup (\{x_0\} \times X)$. Obviously, Y is a subset of $X \times X$. Define the function $d_Y : Y \times Y \to \mathbb{R}$ as

$$d_Y((x_1, x_2), (y_1, y_2)) = d_X(x_1, y_1) + d_X(x_2, y_2),$$

for each (x_1, x_2) and (y_1, y_2) in Y. It is easy to check then that (Y, d_Y) is a metric space. Let $\phi : X \to Y$ be given by $\phi(x) = (x, x_0)$, for each $x \in X$. Notice firstly that ϕ is an isometry by virtue of the fact that

$$d_Y(\phi(x), \phi(y)) = d_Y((x, x_0), (y, x_0)) = d_X(x, y), \quad \forall x, y \in X.$$

Let us define $f : Y \to Y$ by $f(x, x_0) = (x_0, x)$ and $f(x_0, x) = (x, x_0)$, for each $x \in X$. Evidently, f is a surjective function. Moreover, f is also an isometry. To check this fact, various cases need to be considered. Indeed, observe that

$$\begin{aligned}
&d_Y(f(x, x_0), f(y, x_0)) = d_X(x, y) = d_Y((x, x_0), (y, x_0)),\\
&d_Y(f(x_0, x), f(x_0, y)) = d_X(x, y) = d_Y((x_0, x), (x_0, y)),\\
&d_Y(f(x, x_0), f(x_0, y)) = d_X(y, x_0) + d_X(x_0, x) = d_Y((x, x_0), (y, x_0)),
\end{aligned}$$

for each $x, y \in X$. We claim now that Y is x^*-symmetric with respect to f, where $x^* = (x_0, x_0) \in Y$. To show that, notice firstly that, for each $x \in X$,

$$\begin{aligned}
d_Y(f(x, x_0), x^*) &= d_Y((x, x_0), x^*) = d_X(x, x_0)\\
&= \frac{1}{2}\left[d_X(x, x_0) + d_X(x, x_0)\right] = \frac{1}{2} d_Y((x, x_0), f(x, x_0)).
\end{aligned}$$

In similar fashion, we can prove also that

$$d_Y(f(x_0, x), x^*) = d_Y((x_0, x), x^*) = \frac{1}{2} d_Y((x_0, x), f(x_0, x)), \quad \forall x \in X.$$

We conclude that (Y, d_Y) is x^*-symmetric with respect to f, and that (X, d_X) is isometric to a subset of (Y, d_Y), as desired. $\square$

It is well known that every metric space (X, d_X) can be extended to be a complete metric space. Moreover, the metric space (X, d_X) is dense in its completion. Our last result establishes that the completion is symmetric if the space (X, d_X) is symmetric. Before proving the theorem, we recall some of the details in the construction of the proof for the completion of a metric space. Let $\mathcal{S}(X)$ be the set of all Cauchy sequences in (X, d_X), and define a relation of X as follows: if (x_n) and (y_n) are members of $\mathcal{S}(X)$, we say that they are equivalent if $\lim_{n\to\infty} d_X(x_n, y_n) = 0$. This is an equivalence relation on $\mathcal{S}(X)$, and the set of equivalence classes is denoted by $\mathcal{C}(X, d_X)$ or, simply, by $\mathcal{C}(X)$. For the sake of briefness, the equivalence class determined by the Cauchy sequence $(x_n) \in \mathcal{S}(X)$ will be denoted also by (x_n).

Define next the function $d_{\mathcal{C}(X)} : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}$ by

$$d_{\mathcal{C}(X)}((x_n),(y_n)) = \lim_{n\to\infty} d_X(x_n, y_n),$$

for any two equivalence classes (x_n) and (y_n) in $\mathcal{C}(X)$. This function is well defined on $\mathcal{C}(X)$ and, moreover, it is a metric. The space $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is a complete metric space. In addition, if $\iota_X : X \to \mathcal{C}(X)$ is the function that assigns to each $x \in X$ the constant sequence whose nth term is x, then ι_X is an isometry and $\iota_X(X)$ is dense in $\mathcal{C}(X)$. The space $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ constructed in this way is called the *completion* of the metric space (X, d_X).

Interestingly, if (X, d_X) is a metric space, $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is its completion, (Y, d_Y) and complete metric space and $f : X \to Y$ an isometry, then there exists a unique isometry $\overline{f} : \mathcal{C}(X) \to Y$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, "f"'] & \mathcal{C}(X) \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

The uniqueness of completions up to isometries is a consequence of this property. Moreover, if (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is an isometry, then there exists a unique isometry $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[d, "f"'] & \mathcal{C}(X) \arrow[d, "\mathcal{C}(f)"] \\ Y \arrow[r, "\iota_Y"'] & \mathcal{C}(Y) \end{tikzcd}$$

In addition, recall that $\mathcal{C}$ preserves compositions of isometries and identity mappings. This implies that $\mathcal{C}$ is a functor from the category of metric spaces with isometries, into the category of complete metric spaces. With these conventions, the following proposition shows that if (X, d_X) is an x_0-symmetric metric space with respect to the isometry $f : X \to X$, then $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is $\iota_X(x_0)$-symmetric with respect to $\mathcal{C}(f)$. The statement is summarized as follows.

Theorem 2. *The completion of a symmetric metric space is likewise symmetric.*

Proof. We will use the notation preceding the theorem. Since $f : X \to X$ is a surjective isometry, then $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(X)$ is likewise a surjective isometry. For the sake of convenience, let $\hat{f} = \mathcal{C}(f)$ and $x_0^* = \iota_X(x_0)$. To show that $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is x_0^*-symmetric with respect to $\hat{f}$, it remains to check that, for each $x^* \in \mathcal{C}(X)$, the following identities are satisfied:

$$d_{\mathcal{C}(X)}(x^*, x_0^*) = d_{\mathcal{C}(X)}(\hat{f}(x^*), x_0^*) = \frac{1}{2} d_{\mathcal{C}(X)}(x^*, \hat{f}(x^*)). \tag{1}$$

Let us assume firstly that $x^* \in \iota_X(X)$. So, there exists $x \in X$ with the property that $x^* = \iota_X(x)$. As a consequence of this, the fact that ι_X is an isometry, the functorial properties of the completion and the x_0-symmetry of (X, d_X) with respect to f, we obtain

$$\begin{aligned} d_{\mathcal{C}(X)}(x^*, x_0^*) &= d_{\mathcal{C}(X)}(\iota_X(x), \iota_X(x_0)) = d_X(x, x_0) = d_X(f(x), x_0) \\ &= d_{\mathcal{C}(X)}(\iota_X(f(x)), \iota_X(x_0)) = d_{\mathcal{C}(X)}(\hat{f}(\iota_X(x)), \iota_X(x_0)) \\ &= d_{\mathcal{C}(X)}(\hat{f}(x^*), x_0^*). \end{aligned}$$

Similarly, notice that

$$\begin{aligned} d_{\mathcal{C}(X)}(x^*, x_0^*) = d_X(x, x_0) &= \frac{1}{2} d_X(x, f(x)) = \frac{1}{2} d_{\mathcal{C}(X)}(\iota_X(x), \iota_X(f(x))) \\ &= \frac{1}{2} d_{\mathcal{C}(X)}(x^*, \hat{f}(\iota_X(x))) = \frac{1}{2} d_{\mathcal{C}(X)}(x^*, \hat{f}(x^*)). \end{aligned}$$

As a consequence, we have proved that (1) holds for each $x^* \in \iota_X(X)$. To show that the conclusion is also valid for all $x^* \in \mathcal{C}(X)$, recall that the closure of $\iota_X(X)$ is equal to $\mathcal{C}(X)$, and let (x_n^*) be any sequence in $\iota_X(X)$ which converges to x^*. Thus, if $n \in \mathbb{N}$, then

$$d_{\mathcal{C}(X)}(x_n^*, x_0^*) = d_{\mathcal{C}(X)}(\hat{f}(x_n^*), x_0^*) = \frac{1}{2} d_{\mathcal{C}(X)}(x_n^*, \hat{f}(x_n^*))$$

Taking now the limit when $n \to \infty$, using that the metric $d_{\mathcal{C}(X)}$ and $\hat{f}$ are both continuous functions, we prove that (1) is satisfied for all $x^* \in \mathcal{C}(X)$. We conclude that $(\mathcal{C}(X), d_{\mathcal{C}(X)})$ is x_0^*-symmetric with respect to $\hat{f}$. □

3. Examples

In this section, we provide some constructions of symmetric spaces. Various examples will be provided at this stage of our work. In the first of them, we will show that some products of symmetric spaces are likewise symmetric.

Example 1. Let (X_i, d_{X_i}) be metric spaces, $x_i^* \in X_i$ and $f_i : X_i \to X_i$ surjective isometries, and assume that (X_i, d_{X_i}) is x_i^*-symmetric with respect to f_i, for each $i = 1, 2$. Let $X = X_1 \times X_2$, fix $x^* = (x_1^*, x_2^*)$, and agree that $x = (x_1, x_2)$ and $y = (y_1, y_2)$, for each $x, y \in X$. Let us define $d_X : X \times X \to \mathbb{R}$ by means of the equation

$$d_X(x, y) = d_{X_1}(x_1, y_1) + d_{X_2}(x_2, y_2),$$

for each $x, y \in X$. It is obvious that (X, d_X) is a metric space. Let $f : X \to X$ be defined as $f(x) = (f_1(x_1), f_2(x_2))$, for each $x \in X$. Then f is surjective and, moreover, it is an isometry by virtue that

$$\begin{aligned} d_X(f(x), f(y)) &= d_X((f_1(x_1), f_2(x_2)), (f_1(y_1), f_2(y_2))) \\ &= d_{X_1}(f_1(x_1), f_1(y_1)) + d_{X_2}(f_2(x_2), f_2(y_2)) \\ &= d_{X_1}(x_1, y_1) + d_{X_2}(x_2, y_2) = d_X(x, y). \end{aligned}$$

Additionally,

$$\begin{aligned} d_X(x,x^*) &= d_{X_1}(x_1,x_1^*) + d_{X_2}(x_2,x_2^*) = d_{X_1}(f_1(x_1),x_1^*) + d_{X_2}(f_2(x_2),x_2^*) \\ &= d_X(f(x),x^*) = \frac{1}{2}\left[d_{X_1}(x_1,f_1(x_1)) + d_{X_2}(x_2,f_1(x_2))\right] \\ &= \frac{1}{2} d_X(x,f(x)). \end{aligned}$$

We conclude that (X,d_X) is x^*-symmetric with respect to f. □

It is worth pointing out that the last example can be generalized to the product of a finite number of symmetric metric spaces. Moreover, the example can be extended to account for different metrics, including the infinity metric and the Euclidean metric induced in d_{X_1} and d_{X_2}.

To state our next result, recall that if (X,d_X) is a metric space and $E \subseteq X$ is nonempty, we say that E is **bounded** if there exists $K \in \mathbb{R}$ such that $d_X(x,y) \leq K$, for each $x,y \in E$. If that is the case, then we let

$$\operatorname{diam} E = \sup\{d_X(x,y) : x,y \in E\}.$$

Theorem 3. *Let (X,d_X) be x_0-symmetric with respect to f, and let $E \neq \emptyset$. Let $B = \{g : E \to X : \operatorname{diam} g(E) < \infty\}$, and $d_B : B \times B \to \mathbb{R}$ be given by*

$$d_B(g,h) = \sup_{e\in E} d_X(g(e),h(e)), \quad \forall g,h \in B.$$

Let $\Phi : B \to B$ be given by $\Phi(g) = f \circ g$, for each $g \in B$. Then B is g_{x_0}-symmetric with respect to Φ, where $g_{x_0} : E \to X$ is the constant $g_{x_0} \equiv x_0$.

Proof. To start with, observe that (B,d_B) is indeed a metric space. To show that Φ is surjective, let $h : E \to X$ be such that $\operatorname{diam} h(E) < \infty$, and let $g = f^{-1} \circ h$. The fact that f is an isometry assures that $\operatorname{diam} g(E) = \operatorname{diam} h(E) < \infty$, which means that $g \in B$ and, moreover, $\Phi(g) = h$. The fact that Φ is an isometry is a consequence of the fact that f is an isometry, so

$$d_B(\Phi(g),\Phi(h)) = \sup_{e\in E} d_X(f(g(e)),f(h(e))) = \sup_{e\in E} d_X(g(e),h(e)) = d_B(g,h),$$

for each $g,h \in B$. Finally, observe that

$$\begin{aligned} d_B(g,g_{x_0}) &= \sup_{e\in E} d_X(g(e),x_0) = \sup_{e\in E} d_X(f(g(e)),x_0) \\ &= \sup_{e\in E} d_X((\Phi(g))(e),x_0) = d_B(\Phi(g),g_{x_0}) \\ &= \frac{1}{2}\sup_{e\in E} d_X(f(g(e)),g(e)) = \frac{1}{2} d_B(\Phi(g),g), \end{aligned}$$

for each $g \in B$. We conclude that B is g_{x_0}-symmetric with respect to Φ. □

Theorem 4. *Let* (X, d_X) *be a compact metric space, assume that* (Y, d_Y) *is* y_0*-symmetric with respect to* f*, and let* $C = \{g : X \to Y : g \text{ is continuous}\}$*. Let* $d_C : C \times C \to \mathbb{R}$ *be defined by*

$$d_C(g,h) = \sup_{x \in X} d_Y(g(x), h(x)), \quad \forall g, h \in C.$$

Then (C, d_C) *is* g_{y_0}*-symmetric with respect to* $\Phi(g) = f \circ g$*. Here,* $g_{y_0} : X \to Y$ *is the constant function* $g_{y_0} \equiv y_0$*.*

Proof. The proof is similar to that of the previous theorem. We just need to point out here that the function d_C is well defined in this case, in view of the compactness of the metric space (X, d_X). $\square$

It is worth pointing out that the compactness assumption on the metric space (X, d_X) can be omitted in the last theorem. To that end, we require to redefine the set C as

$$C = \{g : X \to Y : g \text{ is continuous and } \operatorname{diam} g(E) < \infty\}.$$

Using all the remaining assumptions in Theorem 4, we can readily reach the same conclusion.

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Tight partitions for packing circles in a circle

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Abstract. We develop a new strategy for proving optimal packing densities for N congruent circles in a circle. Specifically, we introduce tight partitions, which generalize filled rings of circles, and show that for the densest packing, the union of tight partitions forms a connected graph containing the center of every circle, except for possibly rattlers on the container boundary. We then apply this to the case of $N = 14$ to reduce the list of potentially optimal solutions to one basic shape, which in turn admits a one-parameter family of configurations with two local extrema, one of which is the global optimal.

Keywords: circle packing, rings of circles, tight partitions.

1. Introduction

Circle packing problems with various containers and radii arise in applications to factory layouts [2, 5], communications networks [1, 3, 8], circular cutting [16], cylinder packing [6], container loading [13], and social distancing [24], but in general are considered to be NP-hard [4, 7]. For packing N congruent circles of unit radius in a circle, minimum container radii (or equivalently maximum densities) have been proved only for $N \leq 14$ and $N = 19$ [9, 10, 11, 12, 19, 21]. For general N, only heuristic methods have been proposed to find approximate solutions [15, 17, 20]; the best known solutions up to $N = 2647$ can be found at [23]. Our goal in the current paper is to provide a new strategy for proving optimal density which we hypothesize can be systematically applied to increasing N. We demonstrate the utility of this new approach by providing an independent proof for the case of $N = 14$.

Specifically, we geometrically reduce the number of basic configurations for circles using a new tool that we refer to as *tight partitions*, which generalize filled rings of circles, and which characterize global ring structure that must exist for potentially optimal configurations. For the case of $N = 14$, we use tight

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partitions to geometrically reduce the problem to one basic shape. This basic shape admits a one-parameter family of geometric configurations that have as endpoints a symmetric arrangement and an extreme one, where no further deformation of the basic shape is possible. We then show that the container radius is monotone decreasing from the symmetric arrangement to the extreme one, which therefore yields the optimal solution. With a similar analysis, we believe it is possible to establish that for any N, and for any given feasible configuration, the distinct local minima occur either in a symmetric or extreme arrangement. A proof of such a conjecture, along with the identification of a finite configuration list using tight partitions, will lead to a tractable combinatorial optimization problem for increasing N.

The outline of the paper is as follows. In Section 2 we define tight partitions, and prove Theorem 2.1 that the union of tight partitions forms a connected graph containing the center of every circle, except for possibly outermost rattlers. In Section 3 we apply this to $N = 14$, determining the basic shape of the optimal solution in Theorem 3.1. In Section 4 we then state and prove Theorem 4.1 which establishes the densest packing.

2. Partitions and tight partitions

Consider a packing of circles $C_1, \ldots, C_N$ of unit radius into a circular container of radius r centered at O. As introduced in [8], there is a set of *rings*, $R_1, \ldots, R_n$, that are concentric circles with center O and corresponding radii $0 \leq r_1 < \ldots < r_n = r - 1$, such that each circle C_i has its center on some ring R_j. A *filled ring* is one for which consecutive circles along that ring are mutually tangent, so there are no gaps. Since filled rings cannot be assumed to be present, our goal in this section is to provide a more general notion of well-defined layers without gaps. The observations in this section are basic, yet will lead to useful conceptual organization of subsequent sections.

We will refer to the complex of rings as $\mathcal{R}$; we will assume in this section that $r_i > 0$, but observe at the outset that all results hold for $N > 13$ even if $r_1 = 0$, since in that case there are still at least two rings with $r_i > 0$, which will suffice for all proofs. Given two circles $C_i, C_j \in \mathcal{R}$, there is a central angle $\theta_{C_iC_j}$ formed by line segments joining the centers of C_i, C_j with O.

Definition 2.1 (Partition)**.** *A partition P is a piecewise linear simple closed loop whose segments connect centers of circles in $\mathcal{R}$, such that if there are m segments, then the corresponding central angles θ_i have measures $0 < \theta_i < \pi$, $i \in \{1, \ldots, m\}$, with $\sum_{i=1}^{m} \theta_i = 2\pi$.*

A partition is thus an edge-path which connects centers of circles, and proceeds strictly monotonically once around the center O of the ring complex. We use the word *partition* because the central angles partition 2π. We will assume our packing is optimal at minimum radius r, so that we may assume the following three conditions:

A. r_n is the minimum outer radius for ring complexes $\mathcal{R}$ with N circles.
B. Given Condition A, every other radius r_i for $1 \leq i < n$ is maximized.
C. Given Conditions A and B, the total number of rings n is maximized.

That Condition A holds is obvious. Conditions B and C then guarantee that with Condition A in place, no circle may be moved further outward from O; in particular *rattlers*, which have local freedom of movement, are pushed as far outward from O as possible. We prove three initial lemmas that show partitions exist through every circle.

Lemma 2.1. *Let R_i, R_{i+1} be successive rings with respective circles C_i, C_{i+1}. Then the centers of C_i, C_{i+1} are not on the same radial ray extending from O.*

Proof. Suppose, for contradiction, that C_i and C_{i+1} have centers on a common radial ray extending from O. Then C_i and C_{i+1} must be tangent, with $r_i + 2 = r_{i+1}$, and in fact any point where circles of R_i and R_{i+1} intersect must also be a point of tangency on some radial line. Thus, all R_j for $i+1 \leq j \leq n$ can be rotated simultaneously such that all circles of R_i are disjoint from all circles of R_{i+1}, and we can increase r_i, contradicting Condition B. $\square$

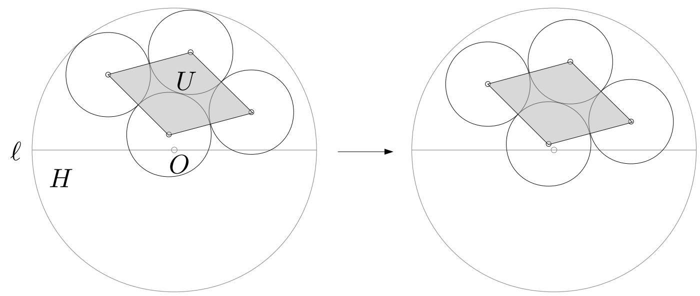
Lemma 2.2. *There exists a partition P for $\mathcal{R}$.*

Proof. Let U be the convex hull of all centers of circles in $\mathcal{R}$, and note that U is not a line segment due to Lemma 2.1 and the fact that $N > 2$. We observe that if $O \in \operatorname{Int} U$ we are done, for then $\operatorname{Bd} U$ is our desired partition P. If O is not initially contained in $\operatorname{Int} U$ we will show that the circles in $\mathcal{R}$ admit a perturbation within their circular container so that either $O \in \operatorname{Int} U$, or r_n can be reduced, contradicting Condition A.

To that end, if $O \notin \operatorname{Int} U$, since U is convex there is a diameter ℓ of the circular container for $\mathcal{R}$ that is disjoint from $\operatorname{Int} U$, so that $\operatorname{Int} U$ is entirely contained on one side of ℓ, as shown in part (a) of Figure 1. We call H the side disjoint from $\operatorname{Int} U$. We need to consider the cases when O is on $\operatorname{Bd} U$, or when O is disjoint from U altogether.

If O is on $\operatorname{Bd} U$, it is possible that O is a vertex of $\operatorname{Bd} U$, meaning a center of a circle is at O. If so, since U is convex we can translate that vertex and corresponding circle slightly into H so as to obtain $O \in \operatorname{Int} U$. The other possibility is that O lies on an edge of $\operatorname{Bd} U$. Then both endpoints of that edge are centers of circles that lie on ℓ on opposite sides of O, and we can rotate one of those circles slightly into H, so as to obtain $O \in \operatorname{Int} U$.

Finally, we consider the case where O is disjoint from U, as depicted in part (a) of Figure 1. Then all circles in $\mathcal{R}$ admit a translation within the circular container perpendicular to ℓ, eliminating all points of tangency between circles in $\mathcal{R}$ and the container as in part (b) of Figure 1. Thus, the container radius, and hence r_n, can be reduced, contradicting Condition A. $\square$



(a) (b)

Figure 1: Figure for Lemma 2.2.

Lemma 2.3. *If $C \in \mathcal{R}$, there is a partition P_C that contains the center of C.*

Proof. Consider the radial ray extending from O through the center p of C. Since we know there is at least one partition P, this ray must intersect P. If it intersects a vertex of P at a center p_1 of a circle C_1, we can replace p_1 with p to obtain a new partition P_C which has an edge-path now through p. If the ray intersects an edge of P joining two centers p_1 and p_2 of circles C_1 and C_2, then we can obtain a new partition P_C which has an edge-path going from p_1 to p then to p_2. □

We now present our primary definition.

Definition 2.2 (Tight partition)**.** *A tight partition P for $\mathcal{R}$ is a partition where all segments have length 2.*

A tight partition is an edge-path which connects successive centers of tangent circles strictly monotonically once around O, and generalizes filled rings. Note that every packing $\mathcal{R}$ comes equipped with a *tangency graph*, where centers of circles are vertices, and edges between two vertices indicate tangency between those two circles. Therefore, a tight partition is a particular kind of loop in the tangency graph which proceeds monotonically around O. We also note that every optimal packing must have edges in its tangency graph, since if no tangencies exist then all circles have freedom of movement, and we may reduce r_n.

Before proceeding to the existence of tight partitions, we need two definitions. Consider two circles C, C' with radii $r_C, r_{C'}$; if C, C' are tangent, then their central angle is $\theta_{CC'} = \cos^{-1}\left((r_C^2 + r_{C'}^2 - 4)/(2r_C r_{C'})\right)$, which may be acute, right or obtuse.

Definition 2.3 (Angular defect between C and C')**.** *The angular defect between C and C' is defined as*

$$\delta_{CC'} = \begin{cases} \theta_{CC'} - \cos^{-1}\left((r_C^2 + r_{C'}^2 - 4)/(2r_C r_{C'})\right), & \textit{if } |r_C - r_{C'}| < 2, \\ \theta_{CC'}, & \textit{otherwise.} \end{cases}$$

The angle $\delta_{CC'} \geq 0$, since it is the angle needed to rotate C along its ring until it is either tangent to C' (the first case) or along the same radial ray (the second case).

Definition 2.4 (Angular defect for $\mathcal{R}$)**.** *The angular defect for $\mathcal{R}$ is defined as*

$$\delta = \min\{\ \delta_{CC'} \ \mid \ C, C' \in \mathcal{R} \text{ and } \delta_{CC'} > 0\}.$$

We now can prove the existence of tight partitions.

Proposition 2.1. *There exists a tight partition P for $\mathcal{R}$.*

Proof. Suppose for contradiction that there does not exist a tight partition. By Lemma 2.2, let $\mathcal{P}$ be the non-empty set of all partitions for $\mathcal{R}$. For each $P \in \mathcal{P}$, there is at least one edge e of length greater than 2, from circle C to C', with an angular defect $\delta_e = \delta_{CC'} > 0$. We know that $\delta_e \geq \delta > 0$. Throughout the proof we will be rotating circles along their rings, and we consider the counterclockwise direction to be the forward direction of rotation around O.

We label the circles of $\mathcal{R}$ randomly as $C_1, \ldots, C_N$. For any circle C_i, by Lemma 2.1 any points of tangency with other circles will either occur before C_i's radial ray, or after. This will be seen in the tangency graph at the vertex C_i as adjacent edges which extend backward in the counterclockwise direction (which we term *backward edges*), or adjacent edges which extend forward (which we term *forward edges*). Note that if C_i had a backward edge to C_j, that edge acts as a forward edge for C_j.

With this in mind, we rotate circles forward along their rings in the following manner: First, we rotate C_1 by $\delta/2$. If C_1 has forward edges connecting it to adjacent circles, its rotation will force those circles to rotate by $\delta/2$, and this rotation may propagate forward via connections in the tangency graph. However, no new edges in the tangency graph, and hence no new tight partitions, will be created in $\mathcal{R}$, since $\delta/2 < \delta$. Moreover, any circles originally connected by backward edges to C_1 will stay fixed, since if they moved along with C_1, this would imply a monotonic loop around O in the tangency graph, and hence a tight partition. Thus, all backward edges connected to C_1 will be eliminated from the tangency graph. The new angular defect is at least $\delta/2$.

For $i > 1$ we then rotate each C_i forward, one at a time in orderly succession, by $\delta/2^i$, where prior to each rotation the angular defect is at least $\delta/2^{i-1}$. As above, this may force other circles to rotate forward by connections in the tangency graph, but no new edges in the tangency graph will be created since $\delta/2^i < \delta/2^{i-1}$. Moreover, since no tight partitions exist, each C_i's backward edges will be eliminated. After the final rotation of C_N, all edges in the tangency graph are eliminated, and thus r_n can be reduced, contradicting Condition A. Therefore, tight partitions must exist. □

We now show how tight partitions relate to the specific rings in $\mathcal{R}$.

Lemma 2.4. *For each ring R_i, there exists at least one tight partition P which contains centers of circles from R_i.*

Proof. Suppose for contradiction that no tight partition contains circles from R_i. We argue exactly as in Proposition 2.1, but now in the presence of existing tight partitions. Specifically, when we rotate each C_j, if it is in an existing tight partition, it will stay in that tight partition, since the whole partition will be forced to move via forward edges in the tangency graph. However, no new tight partitions will be created, and any backward edges not in a tight partition will be eliminated. After the N rotations, all edges in the tangency graph that were not originally in a tight partition will therefore be eliminated. As a result, since no circle in R_i was in a tight partition, the circles in R_i will have no adjacent edges in the tangency graph. If $i = n$, then r_n can be reduced, contradicting Condition A; if $i < n$ then r_i can be increased, contradicting Condition B. □

Recalling that a tight partition is a particular loop in the tangency graph, let $\mathcal{T}$ be the subgraph of the tangency graph obtained by letting $\mathcal{T}$ be the union of all tight partitions.

Theorem 2.1. *$\mathcal{T}$ forms a connected graph, and includes every circle in $\mathcal{R}$ except possibly a proper subset of the circles in R_n, which are rattlers.*

Proof. Suppose for contradiction that there are at least two distinct components of $\mathcal{T}$, which we call $\mathcal{T}_1$ and $\mathcal{T}_2$. We begin with some topological observations. First, because $\mathcal{T}_1$ and $\mathcal{T}_2$ are each connected unions of loops around O, they are each contained in topological annuli which we call $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, which are disjoint from O and which are separated by a topological circle $\mathcal{C}$ in the plane. $\mathcal{C}$ also separates the plane into a topological disc and its complement. Without loss of generality we may assume $\mathcal{A}_1$ is contained in the disc, and $\mathcal{A}_2$ in its complement. Since $\mathcal{A}_1$ contains at least one tight partition which is a closed loop around O, the center O must also be contained in the disc. Therefore, $\mathcal{C}$ must be a topological circle containing O with $\mathcal{T}_1$ and its annulus $\mathcal{A}_1$, with $\mathcal{T}_2$ and its annulus $\mathcal{A}_2$ enclosing all of these. A schematic for this basic topological configuration is shown in Figure 2. Moreover, we can assume that $\mathcal{T}_2$ is the outermost component of $\mathcal{T}$ from O, if there are more than two components. Thus, of all the components of $\mathcal{T}$, only $\mathcal{T}_1$ contains circles of R_1, and only $\mathcal{T}_2$ contains circles from R_n, for no other components of $\mathcal{T}$ can intersect the annuli in Figure 2.

We now turn our attention to the entire tangency graph. $\mathcal{T}_1$ may be connected by an edge in the tangency graph to another circle $C \notin \mathcal{T}_1$, meaning there is no tight partition in $\mathcal{T}_1$ that also contains C. The same holds true for $\mathcal{T}_2$. As in Proposition 2.1 and Lemma 2.4 we use the angular defect to rotate all circles in $\mathcal{R}$, and observe that because $\mathcal{T}_1$ and $\mathcal{T}_2$ are connected unions of tight partitions, they will remain connected after the rotation of all circles. However, the edges in the tangency graph between each of them and other circles are

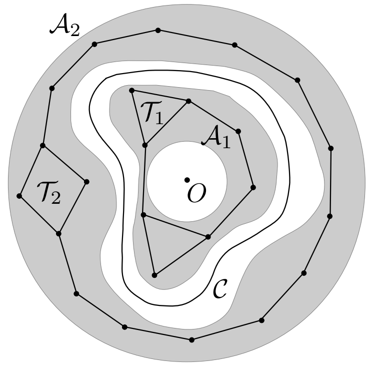


Figure 2: Figure for Theorem 2.1.

eliminated. Therefore, $\mathcal{T}_1$ and $\mathcal{T}_2$ are now disjoint components of the overall tangency graph, and any other circles are in components of the tangency graph disjoint from the $\mathcal{T}_i$. As a result, we can uniformly increase radii of all circles in $\mathcal{T}_1$, including all circles of R_1 and thus increasing r_1, without moving any circle in $\mathcal{T}_2$, and in particular not increasing r_n. This increase in r_1 contradicts Condition B, so that $\mathcal{T}$ must be a connected graph.

Finally, to see that $\mathcal{T}$ contains every circle in $\mathcal{R}$ except perhaps isolated rattlers in R_n, observe that the angular defect rotation ensures that any circles not in $\mathcal{T}$ are disconnected vertices in the tangency graph. Any such circles that are not in R_n can have their radii increased, contradicting either Conditions B or C, depending on whether an entire ring can increase, or just a subset of a ring. Thus, such disconnected circles must only be rattlers in R_n, and cannot include all of R_n, for otherwise r_n could be decreased. $\square$

We conclude with a useful corollary and definition.

Corollary 2.1. *For the connected graph $\mathcal{T}$ there is an outermost tight partition which contains every non-rattler circle in R_n, and an innermost tight partition which contains every circle in R_1.*

Proof. Since $\mathcal{T}$ is the union of tight partitions, which are loops in the tangency graph that proceed monotonically around O, there will be an innermost such loop closest to O, which is the innermost tight partition. Observe that this innermost tight partition bounds a disc containing O and no other vertex of $\mathcal{T}$. By Theorem 2.1, all circles in R_1 are vertices of $\mathcal{T}$, and hence must be in this innermost tight partition. Likewise there will be an outermost loop farthest from O, which is the outermost tight partition, and outside it can be no vertices of $\mathcal{T}$, so that by Theorem 2.1 it must contain every non-rattler circle of R_n. A priori these two tight partitions may be identical if $\mathcal{T}$ consists of only one tight partition, or they could possibly intersect along vertices or edges. $\square$

Definition 2.5 (P_{out}, P_{in}, gap chains)**.** $P_{\mathbf{out}}$ *is the outermost tight partition and* $P_{\mathbf{in}}$ *is the innermost tight partition. A gap chain* $C_1, \ldots, C_k$ *is a maximal sequence of consecutive circles in* P_{out} *from the inner rings* $R_1, \ldots, R_{n-1}$*.*

3. The basic shape of the optimal packing for $N = 14$

Our new proof for $N = 14$ leverages the fact that all minimum container radii for $1 \leq N \leq 13$ are known. More specifically, the case of $N = 13$ is known to have R_n a filled ring of 10 circles, yielding a radius of $A = 3.23606798$ [11], and the best known packing for $N = 14$ has r_n of $B = 3.32842855$, known since 1971 [14]. The inequality $A \leq r_n \leq B$ will be sufficient for us to hone in on the basic shape for $N = 14$ in this section, via the subsections below. We denote the number of circles in R_n by $|R_n|$.

3.1 $7 \leq |R_n| \leq 10$ and all circles in $R_1, \ldots, R_{n-1}$ touch R_n

We will assume that for minimum r_n, the number $|R_n|$ is maximized.

Lemma 3.1. *If a gap occurs between two consecutive circles* $C_1, C_2 \in R_n$*, then the central angle* $\theta_{C_1C_2} < 4\pi/10$*.*

Proof. The radius A is for a filled ring of 10 circles, so that any one circle in R_n has angular support at most $2\pi/10$, for $A \leq r_n \leq B$. If $\theta_{C_1C_2} \geq 4\pi/10$, there is enough angle in R_n for another circle; it remains to show that we can move another circle into R_n, contradicting the maximized $|R_n|$. We first assume both $C_1, C_2 \in P_{\text{out}}$; see part (a) of Figure 3, which shows C_1, C_2 connected by a gap chain $C_3, \ldots, C_k \in P_{\text{out}}$, where it is possible $k = 3$. Let p_i be the center of C_i. The polygon formed by the line segment $\overline{p_1p_2}$ and the portion of P_{out} from p_1 to p_2 must be convex, for otherwise at least one of the circles $C_3, \ldots, C_k$ could move freely out to R_n. We thus can assume that p_3 is closest to $\overline{p_1p_2}$ compared to $p_4, \ldots, p_k$, and as in part (a) of Figure 3 we can reflect C_3 through $\overline{p_1p_4}$ without obstruction, and rotate the resulting circle along C_1 to move it out to R_n. We conclude the proof by observing that if C_1 is a rattler, the only change will be that the line segment $\overline{p_1p_3}$ will have length greater than 2, but this does not affect the ability to reflect and rotate C_3 out to R_n. $\square$

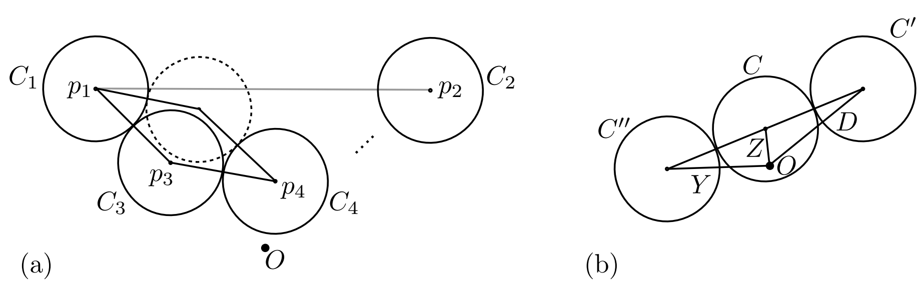


Figure 3: Figures for Lemmas 3.1, 3.4 and Proposition 3.1.

This then allows us to begin to narrow down possibilities for $|R_n|$.

Lemma 3.2. $6 \le |R_n| \le 10$, *and the sum of* $|R_n|$ *plus gaps in* R_n *is at least 11.*

Proof. A filled ring of 11 circles easily fits 4 inside, and so $|R_n| \le 10$. If we let j be the number of gaps in R_n then $j \le |R_n|$, and by Lemma 3.1 we require that the angular support around R_n is

$$2\pi < j \cdot \frac{4\pi}{10} + (|R_n| - j) \cdot \frac{2\pi}{10} = (|R_n| + j) \cdot \frac{2\pi}{10}$$

yielding $|R_n| + j > 10$. Since $j \le |R_n|$ this forces $|R_n| \ge 6$. □

Denote by D *the maximum value for* r_{n-1}, which occurs when a single circle C_3 between C_1, C_2 could be reflected out to $r_n = B$ as in Lemma 3.1. This is when the angular gap is $2\sin^{-1}(1/B) = \cos^{-1}\left((B^2 + D^2 - 4)/(2BD)\right)$ which yields $D = 2.126660$.

Let E be the distance from O for a circle that forms an equilateral triangle with two tangent circles from R_n, when $r_n = B$. Then $E = \sqrt{B^2 - 1} - \sqrt{3} = 1.442605$, and we call any circle C with distance greater than E a gap circle, since it forces a gap in R_n.

Lemma 3.3. $|R_n| \ge 7$.

Proof. If $|R_n| = 6$, then the other 8 circles fit in a container of radius $D + 1 = 3.126660$, but the minimum container radius for $N = 8$ is 3.304765 [21]. □

We now work toward showing that all circles in $R_1, \ldots, R_{n-1}$ touch R_n.

Lemma 3.4. *There is at most one circle* $C \in R_1, \ldots, R_{n-1}$ *which does not touch* R_n*, and the centers of all circles in* $R_1, \ldots, R_{n-1}$ *that touch* R_n *form a convex partition* P*.*

Proof. If C is disjoint from R_n, then C is prevented from moving out to R_n by two circles $C', C'' \in R_1, \ldots, R_{n-1}$, so that the maximum distance for C from O is if C'', C, C' have centers collinear with C', C'' at distance D from O; see part (b) of Figure 3 setting $Y = D$. This yields a maximum distance of $\sqrt{D^2 - 4}$ for C which we call $F = 0.722969$. Since $F < 1$, there is at most one such C. It also follows that the centers of all circles $C \in R_1, \ldots, R_{n-1}$ that touch R_n form a convex partition, since if not, one of them would likewise be forced to include O by a similar calculation. □

We now conclude this subsection, but first observe that the central angle between the centers of two circles in a ring R_i is at least $2\sin^{-1}(1/r_i)$, but if they have a circle of at least radius Y between them, the angle is at least $2\cos^{-1}\left((r_i^2 + Y^2 - 4)/(2r_iY)\right)$. We denote the angular support of P_{in} as Θ_{in}, and the angular support of P_{out} as Θ_{out}.

Proposition 3.1. *All circles in* $R_1, \ldots, R_{n-1}$ *touch* R_n*.*

Proof. If a circle C does not touch R_n, we seek a contradiction through two cases:

$|R_n| = 7, 8$: If $|R_n| = 7$, there are 7 circles not in R_n. Since the 6 that touch R_n form a convex partiton, their total angular contribution is minimized when their distances from O are maximized. Thus, the farthest C can be from O is when these other 6 circles in $R_1, \ldots, R_{n-1}$ are at a maximal distance D from O, and all 7 inner circles form P_{in}; this opens up the most room for C to move a distance Z away from O. Then Θ_{in} must be $2\pi = 10\sin^{-1}(1/D) + 2\cos^{-1}\big((D^2+Z^2-4)/(2DZ)\big)$ which yields $Z = .168539$. Since this is the maximum value for Z, the closest the remaining 6 circles in $R_1, \ldots, R_{n-1}$ can be to O is $Y = 2 - Z = 1.831461$, which is greater than E, so they must be gap circles. Since the minimum container radius for both $N = 6, 7$ is 3 [21], we know two of these gap circles have distance from O of at least 2. Since Θ_{out} is minimized when $r_n = B$, it must be at least

$$2\sin^{-1}(1/B) + 4\cos^{-1}(B/4) + 8\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \approx 7.312832 > 2\pi,$$

contradicting the fact that it must equal 2π. Thus, when $|R_n| = 7$, C must touch R_n.

For $|R_n| = 8$, there are 6 circles not in R_n, so C can be further from O. Θ_{in} is $2\pi = 8\sin^{-1}(1/D) + 2\cos^{-1}\big((D^2+Z^2-4)/(2DZ)\big)$ which yields $Z = .453080$. All 5 remaining circles in $R_1, \ldots, R_{n-1}$ have distance at least $Y = 2 - Z = 1.546920$ which is greater than E, so they are gap circles. Since the minimum container radius for $N = 5$ is $G = 1.701302$ [21], Θ_{out} must be at least

$$\begin{aligned} 6\sin^{-1}(1/B) &+ 2\cos^{-1}(B/4) + 2\cos^{-1}\left(\frac{B^2+G^2-4}{2BG}\right) \\ &+ 6\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \approx 6.414042 > 2\pi, \end{aligned}$$

contradicting the fact that it must equal 2π. This concludes the proof for $|R_n| = 7, 8$.

$|R_n| = 9, 10$: We use a different argument. In order for C not to touch R_n, it must be constrained by two circles $C', C'' \in R_1, \ldots, R_{n-1}$; we call Z, Y the distances of C, C'' from O, respectively. For a given Z, Y is minimized when the centers of C'', C, C' are collinear with C' at maximal distance D in part (b) of Figure 3. For $0.130750 \le Z \le F$, we have $Y = \sqrt{4 + Z^2 - 4Z\cos\left(\pi - \cos^{-1}(\frac{Z^2+4-D^2}{4Z})\right)}$, which is minimized when $Y = 1.873902$ at the left endpoint of its domain. Thus, Θ_{out} is at least

$$14\sin^{-1}(1/B) + 4\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \approx 6.499456 > 2\pi,$$

for $|R_n| = 9$, contradicting the fact that it must equal 2π. Since the Θ_{out} calculation for $|R_n| = 10$ adds a $2\sin^{-1}(1/B)$, it too is greater than 2π and the lemma is proved. □

3.2 $|R_n| = 10$ and all circles in $R_1, \ldots, R_{n-1}$ form P_{in}

We now define P to be the convex partition of all circles in $R_1, \ldots, R_{n-1}$, which follows from Lemma 3.4 and Proposition 3.1.

Lemma 3.5. *If two circles $C, C' \in P$ touch a circle $C_1 \in R_n$, then $|R_n| < 9$.*

Proof. Refer to part (a) of Figure 4, where we have $C, C' \in P$ touching a single circle C_1 from R_n between them; the distances x, y from O are for C, C', respectively. Observe that given x, then y is minimized when the centers of the circles form an equilateral triangle. When $r_n = B$, y is a function of x via $s = \cos^{-1}\left((B^2 + 4 - x^2)/(4B)\right)$ and $y = \sqrt{4 + B^2 - 4B\cos(\pi/3 - s)}$. Then for $|R_n| \geq 9$, we know Θ_{out} is at least

$$\begin{aligned} 14\sin^{-1}(1/B) + 2\cos^{-1}\left(\frac{B^2 + y^2 - 4}{2By}\right) \quad &+ \quad 2\cos^{-1}\left(\frac{B^2 + x^2 - 4}{2Bx}\right) \\ &\geq \quad 6.445686 > 2\pi \end{aligned}$$

where the minimum is when $x = D$ and $y \approx 1.665517$ is minimized. □

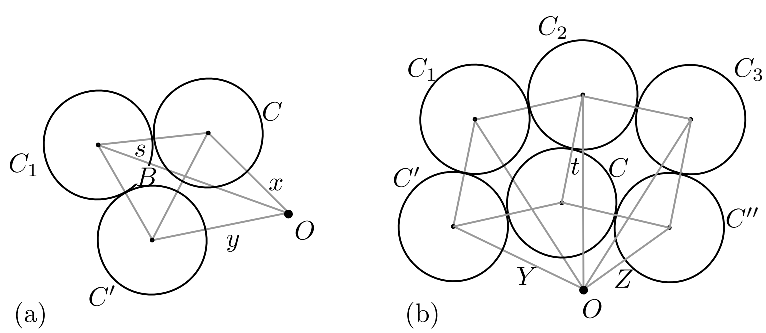


Figure 4: Figures for Lemmas 3.5 and 3.6.

If a sequence of circles $C_1, C_2, \ldots, C_k \in R_n$ proceeds from one gap circle to the next, we will call this sequence in R_n an overpass of length k. Note that we may assume that none of the C_i in an overpass are rattlers, since in maximizing the gaps for the two gap circles the $C_1, \ldots, C_k$ will rotate to form a path in the tangency graph.

Lemma 3.6. *Let $C \in P$ be a non-gap circle which touches an overpass of R_n. Then if $|R_n| \geq 9$, the overpass is at least length 4.*

Proof. If C touches an overpass of length 3, there are 3 circles $C_1, C_2, C_3 \in R_n$ between two gap circles $C', C'' \in P$ with distances Y, Z as in part (b) of Figure

4. The angle t can vary between $0 \le t \le \cos^{-1}(1/B) - \pi/3$, and given t then Y, Z are minimized when there are no gaps between C', C, C''. Then when $r_n = B$,

$$Y = \sqrt{B^2 + 4 - 4B\cos(\pi + t - 2\cos^{-1}(1/B))},$$
$$Z = \sqrt{B^2 + 4 - 4B\cos(\pi - t - 2\cos^{-1}(1/B))},$$

and we have that Θ_{out} is at least

$$\begin{aligned} 14\sin^{-1}(1/B) + 2\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \quad &+ \quad 2\cos^{-1}\left(\frac{B^2+Z^2-4}{2BZ}\right) \\ &\ge \quad 6.650365 > 2\pi, \end{aligned}$$

where the minimum value is achieved at $t = .04502$ when $Y = D$. We then observe that for gaps between C', C, C'', or shorter overpasses, Θ_{out} will be even greater, thus proving the lemma. □

We can now prove the main results of this subsection.

Proposition 3.2. *$|R_n| = 10$.*

Proof. If $|R_n| = 7$, the inner convex partition P has 7 circles, whose minimum container is a filled ring with radius $1 + 1/\sin(\pi/7) \approx 3.304765 > D + 1$.

If $|R_n| = 8$, suppose first that there is a non-gap circle $C \in P$. Then its maximum distance is E. Let $C' \in P$ with distance Y, where $C' \neq C$. Then Y is minimized when the other four circles in P are at maximal distance D, and the angular support of P is $2\pi = 2\cos^{-1}\left((D^2 + E^2 - 4)/(2DE)\right) + 2\cos^{-1}\left((D^2 + Y^2 - 4)/(2DY)\right) + 4\sin^{-1}(1/D)$ yielding $Y \approx 1.917185$. Thus, all circles in P besides C are gap circles, and as in Proposition 3.1 we have $\Theta_{\text{out}} > 2\pi$, since $Y > 1.546920$, the value used in that proposition. Thus, there are six gap circles when $|R_n| = 8$ and by counting gaps, at most one of these gap circles avoids the situation of Lemma 3.5, where consecutive gap circles touched a common circle from R_n. Therefore, $|R_n| > 8$ since Θ_{out} is at least

$$\begin{aligned} 6\sin^{-1}(1/B) \quad &+ \quad 2\cos^{-1}(B/4) + 2\cos^{-1}\left(\frac{B^2+G^2-4}{2BG}\right) \\ &+ \quad 6\cos^{-1}\left(\frac{B^2+y^2-4}{2By}\right) \approx 6.852784 > 2\pi, \end{aligned}$$

where $y \approx 1.665517$ is minimized from Lemma 3.5.

For $|R_n| = 9$, we have at most 3 gap circles, since with 5 gap circles we could not avoid the situation in Lemma 3.5, and with 4 gap circles we could not avoid an overpass of length 3, contradicting Lemma 3.6. We then note that the equation $2\pi = (18 - 2k)\sin^{-1}(1/B) + 2k\cos^{-1}\left((B^2 + y^2 - 4)/(2By)\right)$ has solutions $W = 1.721602$ for $k = 2$, and $V = 1.595722$ for $k = 3$, meaning there cannot be 2 gap circles of distance greater than W, nor 3 gap circles of distance greater than V, for otherwise $\Theta_{\text{out}} > 2\pi$. Then with 3 gap circles at distances V, W, D, we would have Θ_{in} is at least

$$\begin{aligned} &2\cos^{-1}\left(\frac{E^2+D^2-4}{2ED}\right) \quad + \quad \cos^{-1}\left(\frac{V^2+W^2-4}{2VW}\right) \\ &+\cos^{-1}\left(\frac{E^2+V^2-4}{2EV}\right) \quad + \quad \cos^{-1}\left(\frac{E^2+W^2-4}{2EW}\right) \approx 6.350338 > 2\pi, \end{aligned}$$

where no distances of circles can be increased in order to decrease Θ_{in}. This proves the proposition. □

Proposition 3.3. *All circles in $R_1, \ldots, R_{n-1}$ form P_{in}.*

Proof. We first show that P is tight. Suppose for contradiction that P has a gap, so that there are two circles $C \in P$ on either side of that gap. C is prevented from moving inward by a circle $C' \in P$ and a circle $C_1 \in R_n$. For a given r_n, the closest distance Y for C is when the centers of C_1, C, C' are collinear and C' is at minimal distance $r_n - 2$. Thus, $Y = \sqrt{r_n^2 - 2r_n - 2}$ using Laws of Cosines, and graphing $Y(r_n)$ yields $Y > \sqrt{r_n^2 - 1} - \sqrt{3}$ for $A \le r_n \le B$, so that the C's are gap circles. Then Θ_{out} is at least

$$16\sin^{-1}(1/r_n) + 4\cos^{-1}\left(\frac{r_n^2+Y^2-4}{2r_nY}\right) \ge 6.522939 > 2\pi,$$

where the angular support is minimized at $r_n = B$. Thus, P is tight.

Finally, if a subset of P formed a tight partition, then since all circles in P are at least distance $A-2$ from O, at least one circle in P would be at distance at least $\sqrt{(A-2)^2-1} + \sqrt{3} = 2.458593 > D$, which cannot happen. □

3.3 The basic optimal shape for $N = 14$

We begin with a definition and two lemmas.

Definition 3.1. *A minimal polygon is formed by joining centers of circles, so that all sides are length 2, and no subset of the sides forms a polygon.*

Lemma 3.7. *The only minimal rhombus is P_{in}.*

Proof. By Proposition 3.1 any minimal rhombus different from P_{in} must have two circles $C_1, C_2 \in R_n$ and two circles $C, C' \in P_{\text{in}}$; see part (a) of Figure 5. At $r_n = B$, for varying angle θ where $\pi/2 \le \theta \le 2\pi/3$, the distances x, y of C, C' are

$$\begin{aligned} x &= \sqrt{4 + B^2 - 4B\cos(\pi - (\theta + \cos^{-1}(1/B)))}, \\ y &= \sqrt{4 + B^2 - 4B\cos(\theta - \cos^{-1}(1/B))}. \end{aligned}$$

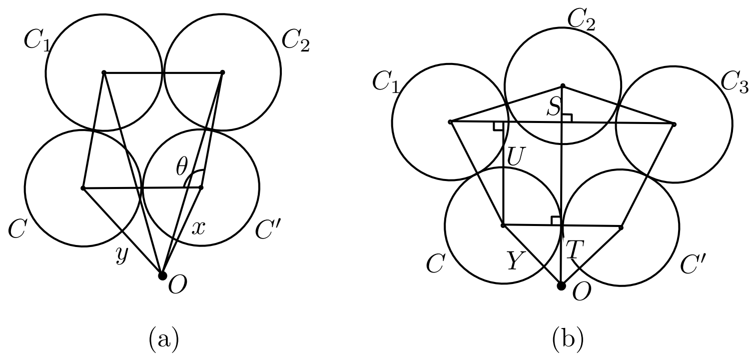


Figure 5: Figures for Lemmas 3.7 and 3.8.

Solving for $x = E$ yields two solutions $\theta = 1.657510, 2\pi/3$ with $y(1.657510) = Y \approx 1.665517$. Graphing Θ_{out} as a function of θ shows it is minimized at Y, namely

$$\begin{aligned} 16\sin^{-1}(1/B) + 2\cos^{-1}\left(\frac{B^2+E^2-4}{2BE}\right) &+ 2\cos^{-1}\left(\frac{B^2+Y^2-4}{2BY}\right) \\ &\approx 6.445686 > 2\pi. \end{aligned}$$

This proves the lemma. □

Lemma 3.8. *Any minimal pentagon has two circles from P_{in}, at most one of which is a gap circle.*

Proof. Since the distance between the centers of 4 consecutive circles on R_n is at least $5.236068 > 4$, we cannot have just one circle from P_{in} in a minimal pentagon. Thus, we have two circles $C, C' \in P_{\text{in}}$ and $C_1, C_2, C_3 \in R_n$, where the symmetric configuration is shown in part (b) of Figure 5. Now for general r_n, $|\overline{C_1C_3}| = \sqrt{2r_n^2 - 2r_n^2\cos(4\sin^{-1}(1/r_n))}$ using ΔOC_1C_3. Thus, $S = \sqrt{4 - (|\overline{C_1C_3}|/2)^2}$ and $U = \sqrt{4 - ((|\overline{C_1C_3}| - 2)/2)^2}$, with $T = r_n - (S+U)$ and $Y = \sqrt{1+T^2}$. Graphing Y for $A \le r_n \le B$ yields $Y(r_n) < \sqrt{r_n^2 - 1} - \sqrt{3}$ and thus neither of C, C' are gap circles in the symmetric configuration. Now in order for C to be pushed out to be a gap circle, C' would be pushed inward, so at most one of them is a gap circle. □

We have two lemmas whose proofs we defer until after our main theorem for this section.

Lemma 3.9. *If there are no rattlers on R_n, then a minimal polygon has at most 5 sides.*

Lemma 3.10. *No rattlers exist on R_n.*

We can now prove our theorem, which refers ahead to Figure 9.

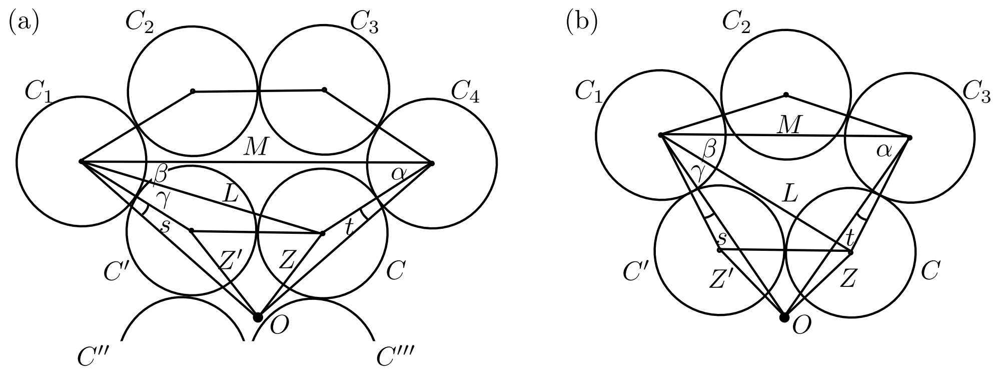


Figure 6: Hexagon and pentagon for Lemmas 3.9 and 3.10.

Theorem 3.1. *The basic shape of the optimal packing of 14 equal circles in a circle is in Figure 9, with the following features not mentioned previously:*

1. *Only the two circles in P_{in} with centers on ℓ touch two circles in R_n;*
2. *The packing has reflective symmetry across the vertical line ℓ;*
3. *The top or bottom triangles may be minimal.*

Proof. Since only minimal triangles or pentagons are possible along R_n, two circles in P_{in} are forced to touch R_n twice, and these cannot be consecutive on P_{in}; the scheme in part (a) of Figure 7, where points are circles and arcs are tangencies, is useful to verify this. The remainder of the packing must be 4 minimal pentagons. By symmetry of the rhombus and the 5 circles in R_n on either side of the rhombus, the theorem follows. $\square$

For the two remaining proofs we set notation that $C, C', C'', C''' \in P_{\text{in}}$ with respective distances Z, Z', Z'', Z''' from O.

Proof of Lemma 3.9. Since the distance between 5 consecutive circles on R_n is at least $6.155367 > 6$, we cannot have minimal polygons with more than 6 sides. Thus, we consider a hexagon with $C_1, C_2, C_3, C_4 \in R_n$ and $C, C' \in P_{\text{in}}$; see part (a) of Figure 6, where we indicate $C'', C''' \in P_{\text{in}}$ for context. We first describe dependencies for the next lemma. For a given value of r_n, the angle t is our variable, which may be positive or negative depending on whether it is to the right or left of the radial line through the center of C_4. Everything else is determined by t as follows in part (a) of Figure 6:

$$M = \sqrt{2r_n^2 - 2r_n^2 \cos(6 \sin^{-1}(1/r_n))}, \qquad Z = \sqrt{r_n^2 + 4 - 4r_n \cos t},$$
$$\alpha = \cos^{-1}(1/r_n) + t - \cos^{-1}((M-2)/4), \qquad L = \sqrt{M^2 + 4 - 4M \cos \alpha},$$
$$\beta = \cos^{-1}((M^2 + L^2 - 4)/(2LM)), \qquad \gamma = \cos^{-1}(L/4),$$
$$s = \cos^{-1}(1/r_n) - \cos^{-1}((M-2)/4) - \beta - \gamma, \qquad Z' = \sqrt{r_n^2 + 4 - 4r_n \cos s}.$$

We observe for the next lemma that similar equations hold for the pentagon in part (b) of Figure 6 provided $6\sin^{-1}(1/r_n)$ is replaced by $4\sin^{-1}(1/r_n)$ in the formula for M, and where $M-2$ is used in α and s, then M is used instead.

Referring back to part (a) of Figure 6, a priori C, C' could be gap circles, forcing gaps to the right of C_4 and left of C_1, respectively. To see that in fact neither of C, C' are gap circles, first observe that the farthest C' can be rolled along C_1 to the right is when $Z' = \sqrt{r_n^2-1}-\sqrt{3}$ and the hexagon becomes a pentagon. For $A \le r_n \le B$ we thus fix this Z', and graphing Z shows $Z < \sqrt{r_n^2-1}-\sqrt{3}$; thus C, C' are not gap circles. But since we are assuming no rattlers on R_n, we must have at least one gap circle by Lemma 3.2, which without loss of generality is C'''. We consider part (a) of Figure 7, where points of tangency between circles in our hexagon are indicated by black line segments, with the curvature of the segments giving the direction of tangency. In order to avoid rhombuses the three solid gray lines must be positioned exactly where they are. But then since all three of the pentagons are minimal, the positions of all circles are determined, meaning the dashed gray line from C' to C_2 must be present as well and C' must be a gap circle. Thus, in fact this is our optimal shape shown in Figure 9 and we conclude that no hexagons exist, provided there are no rattlers. $\square$

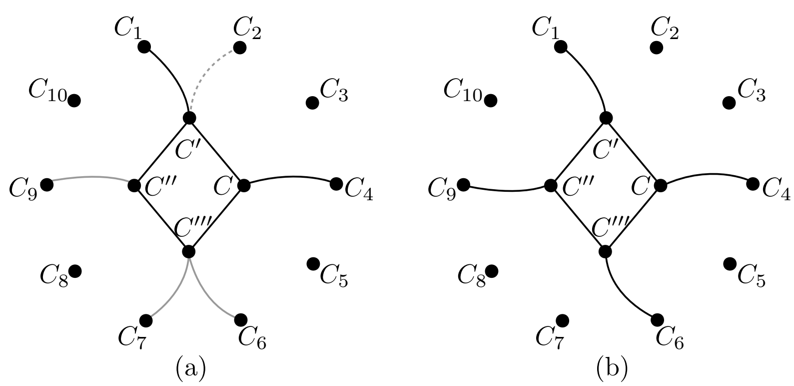


Figure 7: Tangencies for Lemmas 3.9 and 3.10.

Proof of Lemma 3.10. We now show there are no rattlers. Any consecutive rattlers $C_2, \ldots, C_{k-1}$ must occur between two circles $C_1, C_k \in R_n$ with a gap chain of exactly two circles $C, C' \in P_{\text{in}}$, where the centers of C_1, C', C, C_k are in clockwise order. The first sentence in the proof of Lemma 3.9 shows that $k = 3, 4$, meaning we have a non-minimal pentagon or hexagon. The maximum total angular gap on R_n is $2\pi - 20\sin^{-1}(1/B) \approx .180063 \equiv \Phi$. If we had a pentagon with a rattler, the minimum angle between C_1, C_3 is when the centers of C', C_1 and C, C_3 share radial rays, yielding an angle of $2\sin^{-1}(1/(B-2)) \approx 1.704517$. But subtracting $4\sin^{-1}(1/B)$ from this for C_1, C_2, C_3 yields $.483893 > \Phi$. Thus, we may assume only hexagons have rattlers.

If we have one hexagon with rattlers, we have two cases. First, if we have no minimal hexagons, then as in part (a) of Figure 7, to avoid rhombuses we may assume that C''' must be a gap circle. The pentagons are minimal, and the exact same argument holds as in Lemma 3.9, showing that we have the basic optimal shape with no rattlers. Second, if we have a minimal hexagon then we also have two minimal pentagons. These are all adjacent in some order and via parts (a) and (b) of Figure 6 with possible relabeling, all of Z, Z', Z'', Z''' are functions of one variable t for the minimal hexagon in part (a). For $r_n = B$ we have $-0.21844 \le t \le 0.05348$, where the left endpoint is when $Z = E$ and the right is when $Z' = E$. If the minimal hexagon has a pentagon on either side, graphing Θ_{in} in part (a) of Figure 8 shows it attains a minimum of 2π at either endpoint where the hexagon becomes a pentagon. But this is the optimal shape as in Lemma 3.9. Likewise, if the two pentagons are to the left of the minimal hexagon, graphing Z'', Z''' shows that one of C'', C''' is always a gap circle, and graphing Θ_{out} in part (b) of Figure 8 shows it has a minimum of 2π where $Z' = E$, again realizing the optimal shape. If the two pentagons are to the right, the minimum is 2π when $Z = E$. This eliminates the case of one hexagon with rattlers.

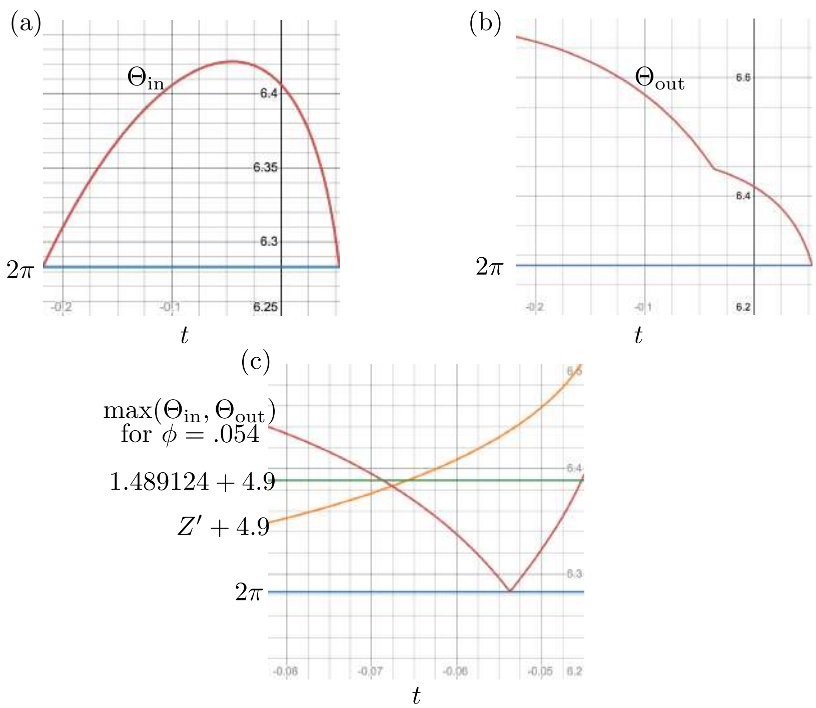


Figure 8: Graphs for Lemma 3.10 generated in Desmos.

If there are two hexagons with rattlers, then the tangencies are in part (b) of Figure 7, since a minimal pentagon must prevent C_1, C_4, C_6, C_9 from moving outward. Thus, each circle C, C', C'', C''' is in a minimal pentagon, and the direction of the tangencies require that $s \leq 0$ and $t \geq 0$ in part (b) of Figure 6. In particular the maximum value for Z (and so also Z', Z'', Z''') must be when $s = 0$, which yields 1.489124. Since the total angular gap Φ is shared by the two non-minimal hexagons, the hexagon with gap chain C, C' has total angular gap ϕ of at most $\Phi/2 \approx .0900315$. As ϕ increases, M in part (a) of Figure 6 can increase for this hexagon, but the dependent quantities change accordingly, so that we can still calculate $\max(\Theta_{\text{in}}, \Theta_{\text{out}})$ as a function of $t \leq 0$ for the hexagon, but now with graphs parametrized by $0 \leq \phi \leq \Phi/2$. Graphing these show that they attain a minimum near 2π at $t = -.05375$ when $\phi = .054$; this is shown in part (c) of Figure 8 where the red curve is tangential to the blue line at 2π. But at $t = -.05375$, we clearly have $Z' > 1.489124$, as indicated by the orange curve $(Z' + 4.9)$ being above the green line $(1.489124 + 4.9)$; this is true for a neighborhood of (t, ϕ) values and violates the constraint $Z' \leq 1.489124$. This proves the lemma. □

4. The optimal solution

We can now determine the optimal packing. We refer the reader to Figure 9 which shows the basic shape of the optimal packing, with the center O placed at the origin, and having reflective symmetry across the y-axis.

We may assume that $\beta_2 \leq \beta_1$, with Figure 9 showing the case $\beta_2 = \beta_1$ which has reflective symmetry over the x-axis. For convenience of notation we have used r to denote the radius of R_n. The tight partition P_{out} applied to the left side of the packing yields the equation

$$\gamma_1 + \gamma_2 + 8\sin^{-1}(1/r) = \pi \tag{1}$$

The quantities L_1 and L_2 denote the distance from O to the centers of C' and C'', respectively.

With this notation, we can now prove our main theorem.

Theorem 4.1. *The optimal packing for 14 circles occurs when* $\beta_2 = \pi/6$ *in Figure 9, meaning there is no gap between* C_5 *and* C_6 *in* P_{out}.

Proof. We show that if $\beta_2 > \pi/6$, then r can be reduced and is not optimal. The conclusion is then that the optimal solution occurs when $\beta_2 = \pi/6$ and there is no gap between C_5 and C_6 in P_{out}.

We therefore consider a value of β_2 satisfying $\pi/6 < \beta_2 \leq \beta_1 < \pi/2$, and for the moment fix the outer radius r associated with that packing. We will also for the moment assume that the rhombus formed by P_{in} is rigid, meaning that the quantity $L_1 + L_2$ is fixed. We will, however, be examining vertical translations of this rigid P_{in}, with the result that if L_2 is decreased, then L_1 must be increased by the same amount.

Since $\beta_2 > \pi/6$, we can rotate C_1 through C_5 counterclockwise along the container boundary by some positive angle $\epsilon > 0$, and likewise C_{10} through C_6

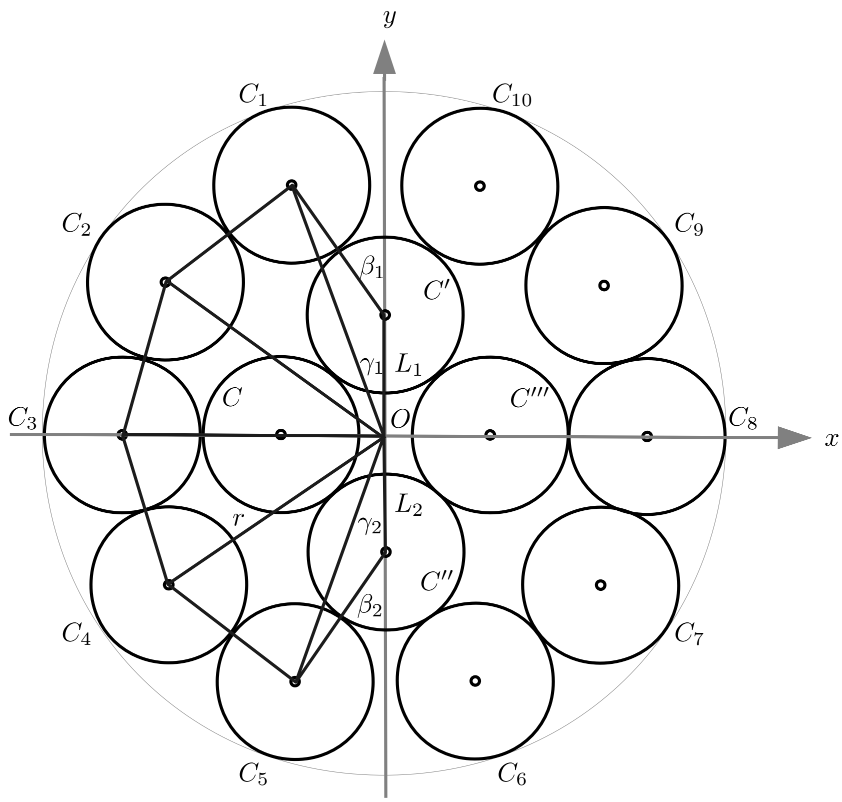


Figure 9: Quantities needed for the proof of Theorem 4.1.

clockwise by the same positive angle $\epsilon > 0$, to decrease β_2. This will force the rhombus P_{in} upward. Since $\beta_2 \leq \beta_1$, the points of tangency between C and C_3, and C''' and C_8, have nonnegative y-coordinate, so that these present no obstruction to the upward translation of the rhombus P_{in}.

It therefore only remains to show that the decreasing of the gap between C_5 and C_6 results in an increasing of the gap between C_1 and C_{10} that is large enough to accommodate the upward translation of C'. This can be formalized by considering γ_2 and γ_1, and first observing that by differentiating Equation 1, the rotation of circles in P_{out} results in

$$\frac{d\gamma_1}{d\gamma_2} = -1. \tag{2}$$

Now we need to compare this with the effect the upward translation of P_{in} has on γ_1. The Law of Cosines for the triangle having vertices O and the centers of C_1 and C' is

$$L_1^2 + r^2 - 2rL_1 \cos\gamma_1 = 4,$$

and implicitly differentiating this yields the positive derivative

$$\frac{d\gamma_1}{dL_1} = \frac{2r\cos\gamma_1 - 2L_1}{2rL_1\sin\gamma_1}.$$

Applying a similar Law of Cosines calculation for the triangle having vertices O and the centers of C_5 and C'', we obtain the positive derivative

$$\frac{dL_2}{d\gamma_2} = \frac{2rL_2\sin\gamma_2}{2r\cos\gamma_2 - 2L_2}.$$

Since $L_1 + L_2$ is constant we know $\frac{dL_1}{dL_2} = -1$ so that by the chain rule

$$\begin{aligned}
\frac{d\gamma_1}{d\gamma_2} &= \frac{d\gamma_1}{dL_1}\cdot\frac{dL_1}{dL_2}\cdot\frac{dL_2}{d\gamma_2} \\
&= \frac{2r\cos\gamma_1 - 2L_1}{2rL_1\sin\gamma_1}\cdot -1 \cdot \frac{2rL_2\sin\gamma_2}{2r\cos\gamma_2 - 2L_2} \\
&= \frac{2r\cos\gamma_1 - 2L_1}{2r\cos\gamma_2 - 2L_2}\cdot -1 \cdot \frac{2rL_2\sin\gamma_2}{2rL_1\sin\gamma_1}.
\end{aligned}$$

Since $\beta_2 \leq \beta_1$ we also have $\gamma_2 \leq \gamma_1$, and since $L_2 \leq L_1$ as well, we know that the first and third factors in the last expression are both positive values at most one. The result is that the upward translation of the rhombus P_{in} yields

$$\frac{d\gamma_1}{d\gamma_2} \leq -1. \tag{3}$$

Comparing Equation 2 with Inequality 3 shows that the rotation of circles in P_{out} will open up γ_1 enough to translate P_{in} upward. The result is that both C and C''' will no longer touch the outer ring, and thus have just two points of tangency with the circles in P_{in}. Both C and C''' can therefore be perturbed to be rattlers, and r can then be decreased. This establishes the theorem. □

We conclude the paper by observing that the optimal configuration established in Theorem 4.1 is indeed that conjectured by Pirl [21]. This is shown in Figure 10, where the global optimal, rotated clockwise by $\pi/2$, is obtained with container radius 4.328 (accurate up to four decimal places) using the trust-region Dogleg algorithm with the MATLAB *fsolve* function.

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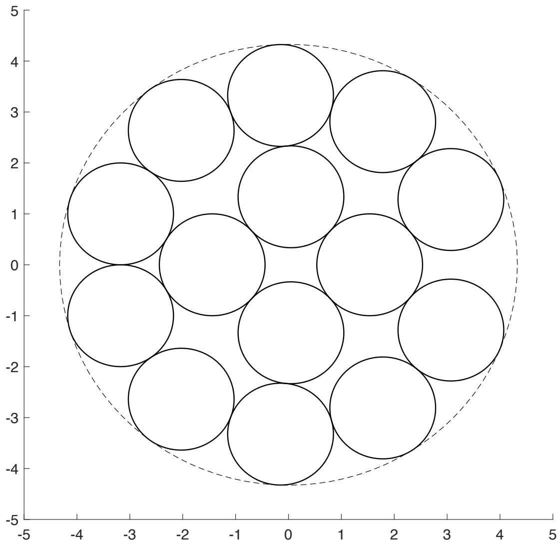

Figure 10: The optimal packing with container radius 4.328, plotted with MATLAB.

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On the characterization of regular ring lattices and their relation with the Dirichlet kernel

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Abstract. Regular ring lattices (RRLs) are defined as peculiar undirected circulant graphs constructed from a cycle graph, wherein each node is connected to pairs of neighbors that are spaced progressively in terms of vertex degree. This kind of network topology is extensively adopted in several graph-based distributed scalable protocols and their spectral properties often play a central role in the determination of convergence rates for such algorithms. In this work, basic properties of RRL graphs and the eigenvalues of the corresponding Laplacian and Randić matrices are investigated. A deep characterization for the spectra of these matrices is given and their relation with the Dirichlet kernel is illustrated. Consequently, the Fiedler value of such a network topology is found analytically. With regard to RRLs, properties on the bounds for the spectral radius of the Laplacian matrix and the essential spectral radius of the Randić matrix are also provided, proposing interesting conjectures on the latter quantities.

Keywords: regular ring lattices, circulant graphs, spectral graph theory.
MSC 2020: 05C50, 05C75, 93A16, 94C15.

1. Introduction

Regular Ring Lattices (RRLs) are often exploited in a wide range of research fields and they are also known in literature as *k-cycles* or "*pristine worlds*"[1, 2, 3, 4]. A RRL can be considered a peculiar undirected circulant network [5] constructed from a cycle graph, wherein each node is connected to pairs of neighbors spaced progressively in terms of vertex degree. Remarkably, RRLs are employed in many graph-based distributed scalable algorithms (see, e.g., [6, 7, 8, 9, 10, 11, 12]), as their symmetry can be exploited for design purposes. Possible applications for this class of networks may encompass intelligent surveillance of public spaces [13], tracking-by-detection [14], identification of sparse reciprocal graphical models [15], definition of *shift* in graph signal processing [16], modeling of quantum walks [17], video circulant sampling schemes [18], compressive three-dimensional sensing techniques [19] and sensor network monitoring algorithms [20]. The latter examples, in fact, represent only few state-of-the-art topics that motivate this study. Also, although being of straightforward derivation, a rigorous characterization for the basic and spectral properties of RRLs is lacking or, in some dissertations, incorrect information about their features is provided (see, e.g., the computation of the largest Laplacian eigenvalue λ_M associated to a RRL in the recently published [21]).

In light of this premise, RRLs are here examined in detail. In particular, the main contributions of this note consist in:

- the investigation of some of their basic properties;
- the spectral analysis of the associated Laplacian and Randić matrices.

Furthermore, an exact relationship for the spectra of these matrices is yielded through the Dirichlet kernel. A special effort is then directed towards the analytical computation of the Fiedler value [22, 23, 24], representing the *algebraic connectivity* of such graphs. With regard to RRLs, properties on the bounds for the spectral radius of the Laplacian matrix [25, 26] and the essential spectral radius of the Randić matrix [27, 28] are also provided. Lastly, conjectures on the latter quantities are also proposed.

The remainder of the note is organized as follows. The mathematical preliminaries in Sec. 2 offer an overview on RRLs. The main results of this work are then presented in Sec. 3, where basic and spectral properties of RRLs are widely explored. The study continues with the discussion in Sec. 4, in which two conjectures related to the spectral radius (for the Laplacian matrix) and the essential spectral radius (for the Randić matrix) of a RRL are given. To conclude, Sec. 5 summarizes all the reported findings and examines future research directions.

Notation The sets of integer, natural, real, complex numbers are indicated by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, respectively; whereas, the empty set and the imaginary unit are denoted by $\varnothing$ and $\boldsymbol{i}$, respectively. The cosine and sine functions of $\alpha \in \mathbb{R}$ are respectively denoted with $\cos(\alpha)$ and $\sin(\alpha)$, or abbreviated as c_α and s_α. The inverse sine and cosine function of $\alpha \in [-1,1]$ are denoted by $\arcsin(\alpha)$ and $\arccos(\alpha)$; while, the inverse tangent function of $\alpha \in \mathbb{R}$ is denoted by $\arctan(\alpha)$. The complex exponential, floor and ceiling functions are defined respectively as $e : z \in \mathbb{C} \mapsto e^z \in \mathbb{C} \setminus \{0\}$, $\lfloor\rfloor : x \in \mathbb{R} \mapsto \lfloor x \rfloor \in \mathbb{Z}$ and $\lceil\rceil : x \in \mathbb{R} \mapsto \lceil x \rceil \in \mathbb{Z}$. Given $N \in \mathbb{N} \setminus \{0\}$, the quantity $\theta = \pi/N$ is assigned and used throughout the note to shortly address the N-th part of a straight angle π; moreover, $n = \lfloor N/2 \rfloor$ is set. The modulo and transpose operations are denoted by mod and $\top$, respectively. Given an n-dimensional real-valued vector $\mathbf{w} = (w_k) \in \mathbb{R}^n$, the j-th *cyclic permutation* over $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix}^\top$, with $j \in \mathbb{Z}$, is defined as $\mathbf{w}^j = \begin{bmatrix} w_{1+(j \bmod N)} & w_{1+(j-1 \bmod N)} & \cdots & w_{1+(j-1+N \bmod N)} \end{bmatrix}^\top$ and it holds $\mathbf{w}^j = \mathbf{w}$ for all $j \in \mathbb{Z}$ such that $j \bmod N = 0$. Also, $\|\mathbf{w}\|_1$ denotes the 1-norm of $\mathbf{w}$. Given an $N \times N$-dimensional squared real-valued matrix $\mathbf{T} = (t_{h,k}) \in \mathbb{R}^{N \times N}$ its h-th row is denoted by $\mathrm{row}_h(\mathbf{T})$; furthermore, its j-th eigenvalue of is denoted by $\lambda_j^{\mathbf{T}}$, with $j \in \{0, \ldots, N-1\}$. The *spectrum* of $\mathbf{T}$ is defined as the set $\Lambda(\mathbf{T}) = \{\lambda_0^{\mathbf{T}}, \ldots, \lambda_{N-1}^{\mathbf{T}}\}$. Notably, it is assumed that eigenvalues $\lambda_j^{\mathbf{T}}$ are not necessarily ordered according to their index j. To conclude, $\mathbf{I}_N$ denotes the identity matrix of dimension N and the matrix $\mathrm{diag}(\delta_1, \ldots, \delta_N) \in \mathbb{R}^{N \times N}$ is equivalent to a squared diagonal matrix $\mathbf{\Delta} = (\delta_{h,k}) \in \mathbb{R}^{N\times N}$ such that $\delta_{k,k} = \delta_k$, for $k \in \{1,\dots,N\}$; $\delta_{h,k} = 0$, if $h \neq k$.

2. Preliminaries

This research begins by briefly illustrating some bases of graph theory and a few well-known mathematical preliminaries about circulant matrices, showing familiar algebraic relations. Also, the definition and a few properties of the Dirichlet kernel are reported.

2.1 Basic notions of graph theory

An *undirected graph* $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is a networked structure formed by a *vertex set* $\mathcal{V} = \{v_1,\dots,v_N\}$ and an *edge set* $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$, in which each edge $e_{h,k} = (v_h, v_k) = (v_k, v_h)$, with $h \neq k$, belongs to $\mathcal{E}$ if and only if there exists a connection between vertices v_h and v_k. The cardinality of the edge set is denoted respectively by $M(\mathcal{G}) = |\mathcal{E}|$. Equivalently, the whole structure of $\mathcal{G}$ can be described by the so-called *adjacency matrix* $\mathbf{A} = (a_{h,k}) \in \{0,1\}^{N\times N}$, where $a_{h,k} = 1$ if $e_{h,k} \in \mathcal{E}$; $a_{h,k} = 0$, otherwise. The k-th *neighborhood* of vertex v_k is then defined as $\mathcal{N}_k = \{v_k \in \mathcal{V} \mid e_{h,k} \in \mathcal{E}\}$ and its cardinality $d_k = |\mathcal{N}_k|$ is called *vertex degree*. The latter quantity also contributes to the definition of the *degree matrix* $\mathbf{D} = \mathrm{diag}(d_1,\dots,d_N)$. Graph $\mathcal{G}$ is said to be *regular* if all the vertex degrees are equal to some *common degree* $d(\mathcal{G}) \in \mathbb{N}$. The volume of $\mathcal{G}$ is defined as $\mathrm{vol}(\mathcal{G}) = \sum_{v_k\in\mathcal{V}} d_k$. Vertex v_k is said to be *isolated* if $d_k = 0$. From the above entities, three very relevant matrices associated to $\mathcal{G}$ can be finally defined: the *Laplacian* matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and, assuming that none of the vertices in $\mathcal{V}$ is isolated, the *normalized Laplacian* matrix $\boldsymbol{\mathcal{L}} = \mathbf{D}^{-\frac{1}{2}}\mathbf{L}\mathbf{D}^{-\frac{1}{2}}$ and the *Randić matrix* $\mathbf{R} = \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$ [29, 30, 31, 32, 33, 34]. Assuming that regularity holds for $\mathcal{G}$, the adjacency, Randić, normalized Laplacian and Laplacian matrices associated to $\mathcal{G}$ can be mutually computed through

$$\mathbf{L} = d(\mathcal{G})\mathbf{I}_N - \mathbf{A} = d(\mathcal{G})(\mathbf{I}_N - \mathbf{R}) = d(\mathcal{G})\boldsymbol{\mathcal{L}}. \tag{1}$$

In addition, a sequence of edges without repetition $\pi_{h,k} \subseteq \mathcal{E}$ that links vertices v_h and v_k, in which all traversed vertices are distinct, is called *path*. A *cycle* π_k passing through vertex v_k can be identified as a particular nondegenerate path for which $v_h = v_k$, i.e. $\pi_k = \pi_{k,k}$, with $\pi_{k,k} \neq \varnothing$. If it holds $\pi_{h,k} \neq \varnothing$ for all the couples of vertices v_h and v_k such that $v_h \neq v_k$ then $\mathcal{G}$ is said to be *connected*. The *length of a path* $\pi_{h,k}$ is identified with its cardinality $|\pi_{h,k}|$, the *distance* between v_h and v_k is yielded by $\mathrm{dist}(v_h, v_k) = \min\{|\pi_{h,k}| \mid \pi_{h,k} \subseteq \mathcal{E}\}$ (note that $\mathrm{dist}(v_k, v_k) = 0$) and the *eccentricity* of vertex v_k is computed as $\epsilon(v_k) = \max\{\mathrm{dist}(v_h, v_k) \mid v_h \in \mathcal{V}\}$. The *diameter* $\phi(\mathcal{G})$ and *radius* $r(\mathcal{G})$ of $\mathcal{G}$ are defined as $\phi(\mathcal{G}) = \max\{\epsilon(v_k) \mid v_k \in \mathcal{V}\}$ and $r(\mathcal{G}) = \min\{\epsilon(v_k) \mid v_k \in \mathcal{V}\}$. Also, the *periphery* $\mathcal{P}(\mathcal{G})$ and *center* $\mathcal{C}(\mathcal{G})$ of $\mathcal{G}$ are defined as the sets $\mathcal{P}(\mathcal{G}) = \{v_k \in \mathcal{V} \mid \epsilon(v_k) = \phi(\mathcal{G})\}$ and $\mathcal{C}(\mathcal{G}) = \{v_k \in \mathcal{V} \mid \epsilon(v_k) = r(\mathcal{G})\}$. Quantities

$g(\mathcal{G}) = \min\{|\pi_k| \mid v_k \in \mathcal{V}\}$ and $c(\mathcal{G}) = \max\{|\pi_k| \mid v_k \in \mathcal{V}\}$ are said respectively *girth* and *circumference* of $\mathcal{G}$.

Lastly, a *cycle graph* C_N is an undirected connected regular graph with N vertices such that $d(C_N) = 2$; a *complete graph* K_N is an undirected connected regular graph with N vertices such that $d(K_N) = N - 1$; an *edgeless graph* $\overline{K}_N$ is a nonconnected regular graph with N isolated vertices ($d(\overline{K}_N) = 0$). An undirected connected graph is *Eulerian* if and only if every vertex in it has even degree [35]. An undirected graph is said *Hamiltonian* if it has a cycle passing through each vertex in it. The smallest number of colors needed to color[1] a graph $\mathcal{G}$ is denoted by the *chromatic number* $\chi(\mathcal{G})$. A graph $\mathcal{G}$ with $\chi(\mathcal{G}) = 2$ is said *bipartite*. The following lemma concludes this paragraph.

Lemma 2.1 (Handshaking lemma [35]). *For an undirected graph $\mathcal{G}$, the sum of all its degrees equals twice the number of its edges, i.e.* $\mathrm{vol}(\mathcal{G}) = 2M(\mathcal{G})$.

2.2 Circulant matrices

In this paragraph, a few fundamental facts about circulant matrices are provided[2]. A circulant matrix is a matrix where each row in it is shifted one entry to the right relative to the previous row vector. The following lines provide its formal definition.

Definition 2.1 (Circulant matrix [5]). *Given an arbitrary vector* $\mathbf{w} = (w_k) \in \mathbb{R}^N$*, the matrix* $\mathbf{T} \in \mathbb{R}^{N\times N}$ *is circulant if its h-th rows satisfies* $\mathrm{row}_h(\mathbf{T}) = (\mathbf{w}^{h-1})^\top$*, for all* $h \in \{1, \dots, N\}$*. The vector* $\mathbf{w}$ *is called* generator *of* $\mathbf{T}$.

A circulant topology is thus a structure such that each element in it shares the same "local panorama" w.r.t. the other elements. Remarkably, a general expression for the spectrum of circulant matrices can be found. The latter is given in the next theorem.

Theorem 2.1 (Spectrum of circulant matrices [5]). *Let* $\mathbf{T} \in \mathbb{R}^{N\times N}$ *be a circulant matrix according to Def. 2.1. The spectrum* $\Lambda(\mathbf{T})$ *is composed by the eigenvalues* $\lambda_j^{\mathbf{T}}$ *such that*

$$\lambda_j^{\mathbf{T}} = \sum_{k=0}^{N-1} w_{k+1} e^{-\boldsymbol{i} jk\frac{2\pi}{N}}, \quad \forall j \in \{0, \dots, N-1\}. \tag{2}$$

2.3 Definition and properties of the Dirichlet kernel

According to [36], the definition and few fundamental properties of the Dirichlet kernel are provided in the sequel.

1. Coloring is intended as labeling each vertex with a nonnegative integer such that no two vertices sharing the same edge have the same label.
2. Only squared real-valued matrices are considered, as this investigation focuses on undirected (unweighted) RRLs.

Definition 2.2 (Dirichlet kernel [36]). *The Dirichlet kernel of order* $m \in \mathbb{N}$ *is defined as the function* $D_m : x \in \mathbb{R} \mapsto D_m(x) = \frac{1}{2}\sum_{k=-m}^{m} e^{\boldsymbol{i}kx}$.

Theorem 2.2 (Well-known properties of the Dirichlet kernel [36, 37, 38]). *The following properties for the Dirichlet kernel in Def. 2.2 hold.*

1. *Each* $D_m(x)$ *is a real-valued, continuous,* 2π*-periodic, even function and (for* $m > 0$*) assumes both positive and negative values.*
2. *For each* $m \in \mathbb{N}$*, the Dirichlet kernel can be rewritten as*

$$D_m(x) = \begin{cases} \dfrac{\sin\left(\left(m+\frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)}, & \text{if } x \neq 2\pi\ell, \text{ with } \ell \in \mathbb{Z}, \\ m + \frac{1}{2}, & \text{otherwise} \end{cases} \tag{3}$$

   or as

$$D_m(x) = \frac{1}{2} + \sum_{k=1}^{m} \cos(kx). \tag{4}$$

3. *For each* $m \in \mathbb{N}$ *it holds that* $|D_m(x)| \leq m + 1/2$, $\forall x \in \mathbb{R}$.
4. *For each* $m \in \mathbb{N} \setminus \{0\}$ *the Dirichlet kernel restricted to* $[0, 2\pi)$ *has* $2m$ *zeros at* $x_k^\star = 2k\pi/(2m+1)$, $\forall k \in \{1, \ldots, 2m\}$*. In particular, between each pair of consecutive zeros* $(x_k^\star, x_{k+1}^\star)$, $D_m(x)$ *has one local extremum: a minimum, if* k *is odd, or a maximum, if* k *is even.*
5. *For each* $m \in \mathbb{N} \setminus \{0\}$ *the Dirichlet kernel restricted to* $[0, 2\pi)$ *has one global maximum at* $\overline{x}_0 = 0$*, for which* $D_m(\overline{x}_0) = m + 1/2$*, and two global minima at* $\underline{x}_1 \in (x_1^\star, x_2^\star)$ *and* $\underline{x}_m = 2\pi - \underline{x}_1 \in (x_{2m-1}^\star, x_{2m}^\star)$*. The value of* $\underline{x}_1$ *is approximately given by* $\underline{x}_1 \approx \upsilon x_1^\star/\pi$*, with* $\upsilon = 4.493409457909064$.

3. Main results related to RRLs

In this section, the main results on the spectral properties of RRLs are given. In detail, the RRLs are firstly defined and some basic properties are presented. Then, a spectral analysis of the graph Laplacian matrix $\mathbf{L}$ via the Dirichlet kernel is carried out. This discussion will yield a characterization of its spectrum $\Lambda(\mathbf{L})$, with particular attention directed towards the *Fiedler value* (i.e. the smallest nonzero eigenvalue of $\mathbf{L}$) and its *spectral radius* (i.e. the largest eigenvalue of $\mathbf{L}$). Then, the investigation continues with a study on the so-called *essential spectral radius* of the Randić matrix $\mathbf{R}$ associated to a RRL.

3.1 Definition and basic properties

Hereafter, a particular kind of circulant graphs is addressed. The elements belonging to the class in question are referred to as *RRLs* and described in the following definition.

Definition 3.1 (RRL C_N^m)**.** *Let N and m be two natural numbers such that $N \geq 4$ and $1 \leq m < n = \lfloor N/2 \rfloor$. A RRL $C_N^m = C_N^m(\mathcal{V}, \mathcal{E})$ of order m is an undirected graph with N vertices having a circulant adjacency matrix $\mathbf{A}$ generated by a vector $\mathbf{w} \in \{0,1\}^N$ whose components are such that*

$$w_k = \begin{cases} 1, & \text{if } k \in \{2, \dots, m+1\} \cup \{N-m+1, \dots, N\}, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Remark 3.1. The order m of a RRL C_N^m can be interpreted as the identical local field-of-view width of each vertex. In other words, a RRL C_N^m can ba also said to be a k-cycle with N vertices, wherein $k = 2m$ neighbors are adjacent to each vertex as depicted in Fig. 1.

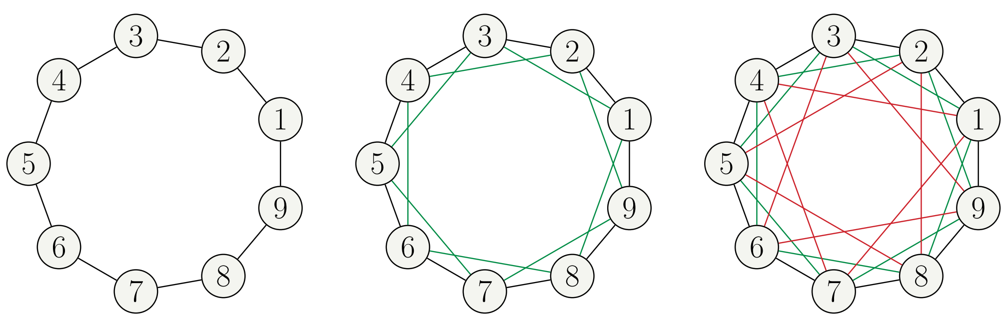
It is worth to notice that a RRL C_N^m is uniquely determined by its number of vertices N and order m only. The following propositions yield all the remaining derived quantities and properties introduced in Ssec. 2.1.

Proposition 3.1 (Regularity and common degree of RRLs)**.** *Any RRL $C_N^m(\mathcal{V}, \mathcal{E})$ is regular, with common degree*

$$d(C_N^m) = 2m. \tag{6}$$

Consequently, any C_N^m is Eulerian.

Proof of Proposition 3.1. The adjacency matrix $\mathbf{A}$ of C_N^m is circulant and generated by vector $\mathbf{w}$, thus the regularity is shown by observing that for all $v_k \in \mathcal{V}$ it holds that $d_k = |\mathcal{N}_k| = \|\mathrm{row}_k(\mathbf{A})\|_1 = \|\mathbf{w}\|_1 = d(C_N^m)$. From (5), the common degree $d(C_N^m)$ is given by the cardinality of $\{2, \dots, m+1\} \cup \{N-m+1, \dots, N\}$. Therefore, one has $d(C_N^m) = (m+1-2+1)+(N-N+m-1+1) = 2m$.



(a) C_9^1 (b) C_9^2 (c) C_9^3

Figure 1: All the three RRLs with $N = 9$ vertices. A layer of edges is added for each increasing value of $m \in \{1, 2, 3\}$: (a) first layer in black, (b) second layer in green, (c) third layer in red.

Proposition 3.2 (Connectivity of RRLs)**.** *Any RRL $C_N^m(\mathcal{V},\mathcal{E})$ is connected.*

Proof of Proposition 3.2. By definition, the adjacency matrix $\mathbf{A} = (a_{h,k})$ of C_N^m satisfies $a_{h,h+1} = 1$ for all $h \in \{1,\dots,N-1\}$. Hence, the path $\pi_{1,N} = \{e_{1,2}, e_{2,3}, \dots, e_{N-1,N}\}$ exists in C_N^m, implying its connectivity.

Remark 3.2. From Prop. 3.1 and Prop. 3.2 it follows that $C_N^1 = C_N$, since RRLs are connected and $d(C_N^m) = 2$ if $m = 1$. This implies that cycle graphs are a subclass of RRLs and represent a proper basic case in this setting. Moreover, one can also observe that $\lim_{m\to n} C_N^m = K_N$ follows directly from (5). Therefore, complete graphs represent a degenerate upper limit case for RRLs. One the other hand, one has that $\lim_{m\to 0} C_N^m = \overline{K}_N$ follows directly from (5). Hence, edgeless graphs represent a degenerate lower limit case for RRLs.

Corollary 3.1 (Volume and number of edges of a RRL)**.** *The volume* $\mathrm{vol}(C_N^m)$ *and number of edges $M(C_N^m)$ of a RRL $C_N^m(\mathcal{V},\mathcal{E})$ are yielded by*

$$\mathrm{vol}(C_N^m) = 2mN, \tag{7}$$

$$M(C_N^m) = mN. \tag{8}$$

Proof of Corollary 3.1. By leveraging the definition of volume of a graph and the regularity of RRLs shown in (6), relation (7) is verified. Whereas, exploiting Lem. 2.1 on C_N^m, the result in (8) follows.

Proposition 3.3 (Chromatic number of RRLs)**.** *A RRL $C_N^m(\mathcal{V},\mathcal{E})$ has chromatic number*

$$\chi(C_N^m) = m + 1 + (N \mod (m+1)). \tag{9}$$

Proof of Proposition 3.3. A RRL C_N^m can be minimally colored exploiting its circulant symmetry. Starting e.g. from vertex v_1, one can use a group of $m+1$ distinct colors to label subsequent subsets of $m+1$ vertices. In this way, vertices v_k share the same color $(k \mod (m+1)) \in \{0,\dots,m\}$ for all k such that $1 \le k \le N - (N \mod (m+1))$. Finally, the remaining $(N \mod (m+1))$ vertices need to be labeled with $(N \mod (m+1))$ additional distinct colors.

Corollary 3.2 (Bipartiteness of RRLs)**.** *A RRL $C_N^m(\mathcal{V},\mathcal{E})$ is bipartite if and only if $m = 1$ and N is even.*

Proof of Corollary 3.2. From Prop. 3.3, expression (9) yields $\chi(C_N^m) = 2$ if and only if $m = 1$ and $N \mod 2 = 0$.

Proposition 3.4 (Diameter and radius of a RRL)**.** *The diameter $\phi(C_N^m)$ and radius $r(C_N^m)$ of a RRL $C_N^m(\mathcal{V},\mathcal{E})$ are yielded by*

$$\phi(C_N^m) = r(C_N^m) = \lceil \lfloor N/2 \rfloor / m \rceil. \tag{10}$$

No. Vertices	No. Edges	Common degree
$N\colon\ N \geq 4$	$M(C_N^m) = mN$	$d(C_N^m) = 2m$
Order	Volume	Chromatic number
$m\colon\ 1 \leq m < \lfloor N/2 \rfloor$	$\mathrm{vol}(C_N^m) = 2mN$	$\chi(C_N^m) = m + 1 + (N \mod (m+1))$
Diameter	Periphery	Circumference
$\phi(C_N^m) = \lceil \lfloor N/2 \rfloor / m \rceil$	$\mathcal{P}(C_N^m) = \mathcal{V}$	$c(C_N^m) = N$
Radius	Center	Girth
$r(C_N^m) = \lceil \lfloor N/2 \rfloor / m \rceil$	$\mathcal{C}(C_N^m) = \mathcal{V}$	$g(C_N^m) = \lceil N/m \rceil$

Table 1: Basic topological quantities of a RRL $C_N^m(\mathcal{V}, \mathcal{E})$.

Proof of Proposition 3.4. As each vertex in C_N^m shares the same local perspective and any C_N^m is connected (see Prop. 3.2), the eccentricity of each $v_k \in \mathcal{V}$ is given by $\epsilon(v_k) = \epsilon_0(C_N^m)$, with constant $\epsilon_0(C_N^m) = \lceil n/m \rceil$.

Corollary 3.3 (Periphery and center of a RRL)**.** *The periphery $\mathcal{P}(C_N^m)$ and center $\mathcal{C}(C_N^m)$ of a RRL $C_N^m(\mathcal{V}, \mathcal{E})$ are yielded by*

$$\mathcal{P}(C_N^m) = \mathcal{C}(C_N^m) = \mathcal{V}. \tag{11}$$

Proof of Corollary 3.3. Relation (11) derives from (10) in Prop. 3.4.

Proposition 3.5 (Circumference and girth of a RRL)**.** *The circumference $c(C_N^m)$ and the girth $g(C_N^m)$ of a RRL $C_N^m(\mathcal{V}, \mathcal{E})$ are yielded by*

$$c(C_N^m) = N, \tag{12}$$

$$g(C_N^m) = \lceil N/m \rceil. \tag{13}$$

Consequently, any C_N^m is Hamiltonian.

Proof of Proposition 3.5. Relation (12) holds trivially, since C_N^m always encompasses the cycle graph C_N (see Rmk. 3.2). This implies that any C_N^m is Hamiltonian. Whereas, (13) is retrieved similarly to what done with eccentricity in Prop. 3.4.

In Tab. 1, all the discussed properties of RRLs are summarized.

3.2 Spectral analysis

The analysis starts by showing the key insight to examine the spectral properties of RRLs via the theoretical support provided by the properties of the Dirichlet kernel D_m. A characterization for the eigenvalues of the Laplacian matrix $\mathbf{L}$ associated to the RRLs in terms of D_m is given by the following theorem, explaining the reason why m is considered the order for this class of graphs. To avoid heavy notation, $d = d(C_N^m) = 2m$ is adopted henceforward.

Theorem 3.1 (Spectral characterization of RRLs)**.** *Let* $\mathbf{L}$ *be the graph Laplacian matrix associated to a RRL* C_N^m. *Setting* $\theta = \pi/N$, *the spectrum* $\Lambda(\mathbf{L})$ *can be expressed in function of the Dirichlet kernel* D_m *as*

$$\lambda_j^{\mathbf{L}} = 1 + 2\left(m - D_m(2\theta j)\right), \quad \forall N \geq 4, \forall m \in \{1, \ldots, n-1\}, \tag{14}$$

with $\lambda_{N-j}^{\mathbf{L}} = \lambda_j^{\mathbf{L}}$, $\forall j \in \{1, \ldots, n\}$. *Furthermore, the following properties hold for all* $N \geq 4$ *and* $m \in \{1, \ldots, n-1\}$.

1. *Each eigenvalue* $\lambda_j^{\mathbf{L}}$ *belongs to* $[0, 4m]$ *for all* $j \in \{0, \ldots, N-1\}$.
2. *Eigenvalue* $\lambda_0^{\mathbf{L}} = 0$ *is simple, i.e. it has algebraic multiplicity* 1.
3. *If* $\exists \lambda_{j^\star}^{\mathbf{L}} = 4m$ *for some* $j^\star \in \{1, \ldots, n\}$ *then eigenvalue* $\lambda_{j^\star}^{\mathbf{L}}$ *is simple.*

Proof of Theorem 3.1. Let $\mathbf{A}$ be the adjacency matrix of C_N^m generated by the vector $\mathbf{w}$, according to Def. 3.1. Recalling that given $\alpha \in \mathbb{R}$ and a matrix $\mathbf{T} \in \mathbb{R}^{N \times N}$ it holds that $\lambda_j^{\mathbf{I}_N + \alpha \mathbf{T}} = 1 + \alpha \lambda_j^{\mathbf{T}}$ for all $j \in \{0, \ldots, N-1\}$ (see [39]), the relations between the j-th eigenvalue of matrices in (1) are the following:

$$\lambda_j^{\mathbf{L}} = d - \lambda_j^{\mathbf{A}} = d(1 - \lambda_j^{\mathbf{R}}) = d\lambda_j^{\boldsymbol{\mathcal{L}}}. \tag{15}$$

Now, the j-th eigenvalue of the adjacency matrix $\mathbf{A}$ can be computed resorting to (2) in Thm. 2.1 and Def. 2.2 as follows:

$$\begin{aligned}
\lambda_j^{\mathbf{A}} &= \sum_{k=0}^{N-1} w_{k+1} e^{-2\boldsymbol{i}jk\theta} = \sum_{k=1}^{m} e^{-2\boldsymbol{i}jk\theta} + \sum_{k=N-m}^{N-1} e^{-2\boldsymbol{i}jk\theta} \\
&= \sum_{k=1}^{m} e^{-2\boldsymbol{i}jk\theta} + \sum_{k=1}^{m} e^{2\boldsymbol{i}jk\theta} = \left(\sum_{k=-m}^{m} e^{\boldsymbol{i}k(2\theta j)} \right) - 1 \\
&= 2(D_m(2\theta j) - 1/2), \quad \forall N \geq 4, \forall m \in \{1, \ldots, n-1\}.
\end{aligned} \tag{16}$$

Therefore, substituting (16) in (15) and leveraging Prop. 3.1 and Thm. 2.2, relation (14) can be found. In particular, $\lambda_{N-j}^{\mathbf{L}} = \lambda_j^{\mathbf{L}}$ holds $\forall j \in \{1, \ldots, n\}$ since $D_m(x)$ is 2π-periodic and even (see Thm. 2.2).

Lastly, regarding the rest of the statement, authors in [40] have already shown that matrix $\mathbf{R}$ has eigenvalues belonging to the interval $[-1, 1]$, where $\lambda_0^{\mathbf{R}} = 1$ and, possibly, $\lambda_{j^\star}^{\mathbf{R}} = -1$ for some $j^\star \in \{1, \ldots, n\}$ are both associated to a single eigenvector. Also, leveraging the connectivity of C_N^m shown in Prop. 3.2, it holds that $\lambda_0^{\boldsymbol{\mathcal{L}}} = 0$ and $0 < \lambda_j^{\boldsymbol{\mathcal{L}}} \leq 2$ for all $j \in \{1, \ldots, N-1\}$ (see [29]). Resorting to (15), one has

$$\lambda_j^{\boldsymbol{\mathcal{L}}} = 1 - m^{-1}(D_m(2\theta j) - 1/2), \quad \forall N \geq 4, \forall m \in \{1, \ldots, n-1\}$$

and the thesis easily follows.

The result provided by Theorem 3.1 contributes with equalities (14), yielding an interesting interconnection between the Dirichlet kernel and the eigenvalues of the graph Laplacian matrix $\mathbf{L}$ corresponding to a RRL. The analysis proceeds by focusing on the extremal (maximum and minimum) eigenvalues belonging to the restricted spectrum $\Lambda_0(\mathbf{L}) = \Lambda(\mathbf{L}) \setminus \{\lambda_0^{\mathbf{L}}\} \subseteq (0, 4m]$. In the following lines, some properties related to the *Fiedler value* $\nu(\mathbf{L}) = \min_{\lambda \in \Lambda_0(\mathbf{L})}\{\lambda\}$ and the *spectral radius* $\rho(\mathbf{L}) = \max_{\lambda \in \Lambda(\mathbf{L})}\{\lambda\}$ of a RRL Laplacian matrix are provided.

Theorem 3.2 (Algebraic connectivity of the RRLs)**.** *Let C_N^m be a RRL and $\mathbf{L}$ be the corresponding Laplacian matrix with eigenvalues $\lambda_j^{\mathbf{L}}$ given by* (14)*. Then the algebraic connectivity of a RRL C_N^m is yielded by the Fiedler value $\nu(\mathbf{L})$ of $\mathbf{L}$, whose expression is*

$$\nu(\mathbf{L}) = \lambda_1^{\mathbf{L}} = \lambda_{N-1}^{\mathbf{L}}, \quad \forall N \geq 4, \ \forall m \in \{1, \dots, n-1\}. \tag{17}$$

Moreover, one has $\nu(\mathbf{L}) \in (0, 2m]$ and $\nu(\mathbf{L}) = 2m$ if and only if $2m = N - 2$.

Proof of Theorem 3.2. Exploiting the symmetry of $\Lambda(\mathbf{L})$ discussed in Thm. 3.1, let us restrict w.l.o.g. this analysis to eigenvalues in $\Lambda_0(\mathbf{L})$ indexed by $j \in \{1, \dots, n\}$. It can be noticed that relations (3) and (15) lead to

$$\lambda_j^{\mathbf{R}} = m^{-1}(D_m(2\theta j) - 1/2), \quad \forall N \geq 4, \forall m \in \{1, \dots, n-1\}, \tag{18}$$

which can be leveraged to prove that $\lambda_1^{\mathbf{L}} < \lambda_j^{\mathbf{L}}$ holds for all $j \in \{2, \dots, n\}$ by verifying the following chain of inequalities:

$$\lambda_1^{\mathbf{R}} > \lambda_j^{\mathbf{R}} \iff D_m(2\theta) > D_m(2\theta j) \iff \frac{\mathrm{s}_{(2m+1)\theta}}{\mathrm{s}_\theta} > \frac{\mathrm{s}_{(2m+1)\theta j}}{\mathrm{s}_{\theta j}}. \tag{19}$$

Considering that $\mathrm{s}_z = z \prod_{k=1}^{+\infty}(1 - \frac{z^2}{k^2\pi^2})$, $\forall z \in \mathbb{C}$ (see formula 4.3.89 in [41]), the following inequality can be derived from the rightmost expression in (19):

$$\prod_{k=1}^{+\infty} \frac{k^2N^2 - (2\mathrm{m}+1)^2}{k^2N^2 - 1} > \prod_{k=1}^{+\infty} \frac{k^2N^2 - (2\mathrm{m}+1)^2 j^2}{k^2N^2 - j^2}. \tag{20}$$

For relation (20) to be satisfied, it is sufficient to prove that:
(i) the k-th factor on the l.h.s. is strictly positive for all $k \in \mathbb{N} \setminus \{0\}$,
(ii) the k-th factor on the l.h.s. is strictly greater than the k-th factor on the r.h.s. for all $k \in \mathbb{N} \setminus \{0\}$.
Property (i) is verified, since this requirement boils down to the identity $2m < N - 1 \leq kN - 1$ for all $k \in \mathbb{N} \setminus \{0\}$; while, property (ii) is also satisfied, as this leads to the identities $m > 0$ and $j > 1$ for all $k \in \mathbb{N} \setminus \{0\}$. Hence, relation (17) is now proven.

To conclude, it is worth to show that $\lambda_1^{\mathbf{R}}$ is nonnegative for any given C_N^m. By (3) and (18) one has the relation

$$\lambda_1^{\mathbf{R}} = m^{-1}(D_m(2\theta) - 1/2) \geq 0 \iff \mathrm{s}_{(2m+1)\theta} \geq \mathrm{s}_\theta. \tag{21}$$

Since $\theta \in (0, \pi/4]$ and $m \geq 1$, the last inequality in (21) holds true for any admissible (N, m). Also, strict equality in (21) is satisfied for $m = n - 1$ and even N. Therefore, $\lambda_1^{\mathbf{R}}$ belongs to the interval $[0, 1)$ and, by (15), one has $\lambda_1^{\mathbf{L}} \in (0, 2m]$ and $\lambda_1^{\mathbf{L}} = 2m$ if and only if $2m = N - 2$.

Theorem 3.3 (Spectral radius properties of RRLs)**.** *Let C_N^m be a RRL and $\mathbf{L}$ be the corresponding Laplacian matrix with eigenvalues $\lambda_j^{\mathbf{L}}$ given by (14). Also, let $j^\star$ be an index for which the spectral radius of $\mathbf{L}$ can be expressed as $\rho(\mathbf{L}) = \lambda_{j^\star}^{\mathbf{L}} = \lambda_{N-j^\star}^{\mathbf{L}}$. Then the following properties are satisfied for all $N \geq 4$.*

1. *For all $m \in \{1, \ldots, n-1\}$ index $j^\star$ is yielded by*[3]

$$j^\star = \underset{j \in \{2, \ldots, n\}}{\arg\min} \{D_m(2\theta j)\} \in \{2, \ldots, n\}. \tag{22}$$

   In particular, the below partial characterization for $j^\star$ can be given.

   (a) If $m = 1$ then $j^\star = n$.

   (b) Let $b_2 = \arccos(-1/4)/(2\theta)$. If $m = 2$ then $j^\star \in \{\lfloor b_2 \rfloor, \lceil b_2 \rceil\}$.

   (c) Let $b_3 = \arccos\left((\sqrt{7} - 1)/6\right)/(2\theta)$. If $m = 3$ then $j^\star \in \{\lfloor b_3 \rfloor, \lceil b_3 \rceil\}$.

   (d) Let $b_4^- = \arccos\left((6\cos((4\arctan(1/\sqrt{5}) - \pi)/3) - 1)/8\right)/(2\theta)$ and $b_4^+ = \arccos\left((-6\cos(4\arctan(1/\sqrt{5})/3) - 1)/8\right)/(2\theta)$, where $b_4^- < b_4^+$. If $m = 4$ then $j^\star \in \{\lfloor b_4^- \rfloor, \lceil b_4^- \rceil, \lfloor b_4^+ \rfloor, \lceil b_4^+ \rceil\}$.

   (e) Let us assign

$$b_{5,1} = \sqrt{\sqrt{11} - 5\,\cos((\arctan(\sqrt{55}/11) + \pi)/3)},$$
$$b_{5,2} = \sqrt{\sqrt{11} - 5\,\cos((\arctan(\sqrt{55}/11) - \pi)/3)},$$
$$b_{5,3} = \sqrt{\sqrt{11} + 5\,\cos(\arctan(\sqrt{55}/11)/3)},$$

   $b_5^- = \arccos((\sqrt[4]{11}(b_{5,1} + b_{5,2} + b_{5,3}) - 1)/10)/(2\theta)$ and $b_5^+ = \arccos((\sqrt[4]{11}(b_{5,1} - b_{5,2} - b_{5,3}) - 1)/10)/(2\theta)$, where $b_5^- < b_5^+$. If $m = 5$ then $j^\star \in \{\lfloor b_5^- \rfloor, \lceil b_5^- \rceil, \lfloor b_5^+ \rfloor, \lceil b_5^+ \rceil\}$.

   (f) If $m = n - 1$ then $j^\star = 2$.

2. *For all $m \in \{1, \ldots, n-1\}$ it holds that $\rho(\mathbf{L}) \in (2m+1, 4m]$, with $\rho(\mathbf{L}) = 4m$ if and only if N is even and $m = 1$.*

3. *For all $m \in \{1, \ldots, n-1\}$ there exists $\underline{j} \in \{2, \ldots, n\}$ such that $\underline{j} \leq j^\star \leq n$ is satisfied. Moreover, the expression of $\underline{j}$ is given by*

$$\underline{j} = 1 + \lfloor N/(2m+1) \rfloor. \tag{23}$$

3. If there exist multiple distinct values $j_1, j_2, \ldots$ of j minimizing (22) then $j^\star = \min\{j_1, j_2, \ldots\}$ is assumed to be the principal minimizer.

Proof of Theorem 3.3. Let us restrict w.l.o.g. the analysis to $j \in \{1, \dots, n\}$ by exploiting the symmetry shown in Thm. 3.1. Each property of the statement is proven in the sequel.

1 Expression (22) holds as it is equivalent to

$$j^\star = \underset{j\in\{2,\dots,n\}}{\arg\max}\{\lambda_j^{\mathbf{L}}\} = \underset{j\in\{2,\dots,n\}}{\arg\max}\{1 + 2\,(m - D_m(2\theta j))\}, \tag{24}$$

as it directly descends from (14). Remarkably, in (24), $j = 0$ and $j = 1$ are excluded, as $\lambda_0^{\mathbf{L}} = 0$ and $\lambda_1^{\mathbf{L}} = \nu(\mathbf{L})$ are proven to be the smallest eigenvalues of $\mathbf{L}$ (see Thm. 3.1 and Thm. 3.2).

1a. Setting $m = 1$, equality $\lambda_j^{\mathbf{L}} = 4\mathrm{s}_{\theta j}^2$ follows by resorting to the triple angle identity $\mathrm{s}_{3z} = 3\mathrm{s}_z - 4\mathrm{s}_z^3$, $\forall z \in \mathbb{C}$. Hence, for $m = 1$, the j-th eigenvalue $\lambda_j^{\mathbf{L}}$ is trivially maximized by selecting $j^\star = \lfloor N/2 \rfloor = n$. Also, note that if N is even then $\rho(\mathbf{L}) = 4\mathrm{s}_{\theta n}^2 = 4$ holds in accordance to property 2.

1b. For $m = 2$, the global minimum of the Dirichlet kernel $D_m(x)$ is obtained for $x = \underline{x}_1 = \arccos(-1/4)$ by solving the trigonometric first-degree equation descending from $D_m'(x) = 0$, where $D_m'(x) = -\sum_{k=1}^m k\sin(kx)$ is the derivative w.r.t. x of $D_m(x)$ (see (4)), and verifying that $2\pi/5 = x_1^\star < \underline{x}_1 < x_2^\star = 4\pi/5$. Imposing $2\theta j \approx \underline{x}_1$ leads to the thesis.

1c. For $m = 3$, the global minimum of the Dirichlet kernel $D_m(x)$ is obtained for $x = \underline{x}_1 = \arccos\big((\sqrt{7} - 1)/6\big)$ by solving the trigonometric second-degree equation descending from $D_m'(x) = 0$ and verifying that $2\pi/7 = x_1^\star < \underline{x}_1 < x_2^\star = 4\pi/7$. Imposing $2\theta j \approx \underline{x}_1$ leads to the thesis. However, differently from the previous point, an additional check is here needed. In particular, because of the presence of a second local minimum[4] $\underline{x}_2 = \pi$ with ordinate $D_m(\underline{x}_2) = -1/2$, it is sufficient to show that $j_3^\star \in \{\lfloor b_3 \rfloor, \lceil b_3 \rceil\}$ satisfies $D_m(2\theta j_3^\star) \le -1/2$ in order. In this direction, one can find all the values of $x \in (0, \pi]$ such that $D_m(x) = -1/2$. These solutions are yielded by $\tilde{x}_1 = \pi/3$, $\tilde{x}_2 = \pi/2$ and, obviously, $\tilde{x}_3 = \underline{x}_2 = \pi$. To conclude the proof, it is sufficient to demonstrate that $\tilde{x}_2 - \tilde{x}_1 \ge 2\theta$. This inequality is however verified only if $N \ge 12$. Checking all the instances characterized by $4 \le N \le 11$ and $m = 3$, one has $j^\star \ne n$ for $N \ne 8$ and $j^\star = j_2^\star = 2$ or $j^\star = 4$, for $N = 8$. Thus, the thesis follows.

1d. This statement is obtained by solving the trigonometric third-degree equation descending from $D_m'(x) = 0$, similarly to what shown in point 1b.

1e. This statement is obtained by solving the trigonometric fourth-degree equation descending from $D_m'(x) = 0$, similarly to what shown in point 1c.

1f. It can be easily shown that, for all $j \in \{1, \dots, n\}$, one has $D_{n-1}(2\theta j) = (-1)^{j+1}/2$, if N is even $D_{n-1}(2\theta j) = (-1)^{j+1}\mathrm{c}_{\theta j}$, if N is odd. Therefore, $j = 2$ minimizes $D_{n-1}(2\theta j)$.

2. By (22), the maximum value for $\lambda_j^{\mathbf{L}}$ is attained when $D_m(2\theta j)$ is minimized in j. So, let us consider $D_m(2\theta y)$, with $y \in \mathbb{R}$. According to Thm. 2.2, the zeros of $D_m(2\theta y)$ can be expressed as $y_k^\star = kN/(2m + 1)$ for all

4. This is actually attained for $j = n$ when N is even, as $2\theta j = \underline{x}_2$ holds for $j = N/2$.

$k \in \{1, \dots, 2m\}$. Remarkably, each consecutive interval $(y_k^\star, y_{k+1}^\star)$ has uniform length $N/(2m+1) > 1$. Since the Dirichlet kernel is negative over intervals $(y_k^\star, y_{k+1}^\star)$ with odd k and $y_{k+1}^\star - y_k^\star > 1$, there exists an integer $j^\star$ for which $D_m(2\theta j^\star)$ is negative. As a consequence, it holds that $(1 + 2m - \lambda_{j^\star}^{\mathbf{L}})/2 = D_m(2\theta j^\star) < 0$, implying that $\lambda_j^{\mathbf{L}} > 2m + 1$. Moreover, $\rho(\mathbf{L}) = 2d = 4m$ holds if and only if C_N^m is bipartite [42], namely when N is even and $m = 1$, as shown in Cor. 3.2.

3. Since $\rho(\mathbf{L}) > 2m + 1$ follows from $D_m(2j^\star\theta) < 0$, a lower bound $\underline{j}$ for $j^\star$ can be computed by solving $D_m(2\theta j) < 0$ for $j \in \{2, \dots, n\}$. Via (3), this leads to the following system of inequalities

$$\begin{cases} j < 2\ell N/(2m+1), \\ j > (2\ell - 1)N/(2m+1), \end{cases} \tag{25}$$

where $\ell \in \mathbb{Z}$. Clearly, the first inequality in (25) requires that $\ell \geq 1$, as j is a positive index. Therefore, to find $\overline{j}$, it is imposed $\ell = 1$. Consequently, since $1 < N/(2m+1) < n$ holds true for any admissible values of (N, m), the second inequality in (25) evaluated at $\ell = 1$ provides the lower bound (23).

Remark 3.3. It is worth to note that index $j^\star$ can be easily computed in closed-form solutions through $D_m'(x) = -\sum_{k=1}^m k \sin(kx) = 0$ for $m \in \{1, 2, 3, 4, 5, n - 1\}$. However, for m such that $6 \leq m \leq n - 2$ this kind of expressions cannot be obtained in such a way, since $D_m'(x) = 0$ leads to trigonometric equations having degree five or higher.

3.2.1 Essential spectral radius analysis

According to [43], the *essential spectral radius* of a row-stochastic[5] Randić matrix $\mathbf{R}$ can be defined as

$$\sigma(\mathbf{R}) = \max_{\lambda \in \Lambda_0(\mathbf{R})} \{|\lambda|\}, \tag{26}$$

where $\lambda_0^{\mathbf{R}} = 1$ holds and $\Lambda_0(\mathbf{R}) = \Lambda(\mathbf{R}) \setminus \{\lambda_0^{\mathbf{R}}\}$ is assigned. Remarkably, the essential spectral radius of a Randić matrix $\mathbf{R}$ associated to a RRL C_N^m complies with definition in (26) for all admissible (N, m), since $\mathbf{R} = d^{-1}\mathbf{A}$ is a row-stochastic matrix with eigenvalues $|\lambda_j^{\mathbf{R}}| \leq 1$, $\forall j \in \{0, \dots, N-1\}$, and $\lambda_0^{\mathbf{R}} = 1$. A study on $\sigma(\mathbf{R})$ for each C_N^m is thus reported by starting from the next lemma.

Lemma 3.1. *Let $\mathbf{R}$ be the Randić matrix of a RRL C_N^m and $\theta = \pi/N \in (0, \pi/4]$. There exists a real number $m^\star \in (0, n)$ such that if $m \geq m^\star$ then $\lambda_1^{\mathbf{R}} + \lambda_2^{\mathbf{R}} \leq 0$, with the equality holding if and only if $m = m^\star$. Moreover, the value of $m^\star$ is yielded by*

$$m^\star = \theta^{-1} \arcsin\left(\sqrt{x^\star}\right), \tag{27}$$

5. The matrix $\mathbf{T} = (t_{h,k}) \in \mathbb{R}^{N \times N}$ is said *row-stochastic* if all its entries $t_{h,k}$ belong to interval $[0, 1]$ for all $h, k = 1, \dots, N$ and $\|\mathrm{row}_h(\mathbf{T})\|_1 = 1$ for all $h = 1, \dots, N$.

where $x^\star$ is the unique solution belonging to $(0,1)$ of the cubic equation

$$\mathrm{p}_\theta(x) = x^3 + a_2 x^2 + a_1 x + a_0 = 0, \tag{28}$$

in which $a_2 = -(\mathrm{c}_{2\theta}+5)/2$, $a_1 = (4\mathrm{c}_{2\theta}^2 + 7\mathrm{c}_{2\theta} + 13)/8$, $a_0 = -(3\mathrm{c}_{2\theta}+1)^2/16$.

Proof of Lemma 3.1. From (18), the eigenvalues of the Randić matrix $\mathbf{R}$ can be rewritten using the prosthaphaeresis formula for the difference of two sines as

$$\lambda_j^{\mathbf{R}} = \begin{cases} \dfrac{\sin(m\theta j)\cos((m+1)\theta j)}{m\sin(\theta j)}, & \text{if } j \in \{1,\dots,N-1\}, \\ 1, & \text{if } j = 0. \end{cases} \tag{29}$$

Thus, inequality $\lambda_1^{\mathbf{R}} + \lambda_2^{\mathbf{R}} \leq 0$ can be written as follows by means of the triple angle identities $\mathrm{c}_{3z} = 4\mathrm{c}_z^3 - 3\mathrm{c}_z$, $\mathrm{s}_{3z} = 3\mathrm{s}_z - 4\mathrm{s}_z^3$, $\forall z \in \mathbb{C}$, the Werner's formula for the product of two cosines and the basic trigonometric rules:

$$(1-\mathrm{c}_{2\theta}^2)(5-4\mathrm{s}_{m\theta}^2)^2\mathrm{s}_{m\theta}^2 \geq (1-\mathrm{s}_{m\theta}^2)(4\mathrm{c}_{2\theta}(1-\mathrm{s}_{m\theta}^2)+1-\mathrm{c}_{2\theta})^2. \tag{30}$$

Now, assigning $x = \mathrm{s}_{m\theta}^2 \in (0,1)$, inequality (30) can be solved in m by resorting to equation (28) and determining the solutions of $\mathrm{p}_\theta(x) \geq 0$. The application of the Routh-Hurwitz criterion to $\mathrm{p}_\theta(x)$, as illustrated in Table 2, ensures that there exists a solution $x^\star$ of $\mathrm{p}_\theta(x)$ having a strictly positive real part for any value of θ, since each pair of subsequent terms in the second column exhibits an alternating sign.

x^3	1	$(4\mathrm{c}_{2\theta}^2 + 7\mathrm{c}_{2\theta} + 13)/8$
x^2	$-(\mathrm{c}_{2\theta}+5)/2$	$-(3\mathrm{c}_{2\theta}+1)^2/16$
x^1	$(2\mathrm{c}_{2\theta}^3 + 9\mathrm{c}_{2\theta}^2 + 21\mathrm{c}_{2\theta} + 32)/(4(\mathrm{c}_{2\theta}+5))$	0
x^0	$-(3\mathrm{c}_{2\theta}+1)^2/16$	0

Table 2: Routh array for polynomial $\mathrm{p}_\theta(x)$.

Analogously, in order to show that $x^\star$ has real part smaller than 1 for all θ, the Routh-Hurwitz criterion can be also applied to $-\mathrm{p}_\theta(y)$, setting $y = 1 - x$. This leads to the analysis reported in Table 3: the fact that each pair of subsequent terms in the second column exhibits an alternating sign finally ensures that $x^\star \in (0,1)$, provided that $x^\star \in \mathbb{R}$.

y^3	1	$(4\mathrm{c}_{2\theta}^2 - \mathrm{c}_{2\theta} - 3)/8$
y^2	$-(1-\mathrm{c}_{2\theta})/2$	$-(1-\mathrm{c}_{2\theta}^2)/16$
y^1	$(2\mathrm{c}_{2\theta}^2 - \mathrm{c}_{2\theta} - 2)/4$	0
y^0	$-(1-\mathrm{c}_{2\theta}^2)/16$	0

Table 3: Routh array for polynomial $-\mathrm{p}_\theta(y)$.

According to method 3.8.2 in [41], equation (28) can be solved by setting

$$q_\theta = a_1/3 - a_2^2/9, \quad r_\theta = (a_1 a_2 - 3a_0)/6 - a_2^3/27, \tag{31}$$

through the computation and observation of the discriminant

$$\Delta_\theta = q_\theta^3 + r_\theta^2 = \frac{7\,(1-\mathrm{c}_{2\theta})\left(1-\mathrm{c}_{2\theta}^2\right)(\mathrm{c}_{2\theta}+13/14)}{1728}\left(\mathrm{c}_{2\theta}-\frac{1}{2}\right)^2 \geq 0. \tag{32}$$

Expression in (32) is strictly positive if and only if factor $(\mathrm{c}_{2\theta} - 1/2)^2$ is grater than zero: this occurs for values of $\mathrm{c}_{2\theta} \neq 1/2$, i.e. for $N \neq 6$. In this case, the presence of only one real solution is guaranteed and it is yielded via (31), (32) by

$$x^\star = -\frac{a_2}{3} + \sqrt[3]{r_\theta + \sqrt{\Delta_\theta}} + \sqrt[3]{r_\theta - \sqrt{\Delta_\theta}}. \tag{33}$$

Otherwise, for $N = 6$, the discriminant Δ_θ vanishes and the solutions for (28) are given by $\{1/4, 5/4, 5/4\}$. In fact, for $N = 6$, expression (33) boils down to $x^\star = 1/4 \in (0, 1)$.

Finally, the thesis in (27) is proven by inverting relation $x^\star = \mathrm{s}^2_{m^\star\theta}$.

In conclusion, some theoretical results on the essential spectral radius of $\mathbf{R}$ for RRLs are stated in the next theorem.

Theorem 3.4 (Essential spectral radius properties of RRLs). *Let C_N^m be a RRL and $\mathbf{R}$ the corresponding Randić matrix with eigenvalues $\lambda_j^{\mathbf{R}}$ given by (18). Also, according to Thm. 3.3, let $j^\star \in \{2, \dots, n\}$ be computed as in (22). Then, for the essential spectral radius $\sigma(\mathbf{R})$, the following properties are satisfied for all $N \geq 4$.*

1. *For all $m \in \{1, \dots, n-1\}$, it holds that $\sigma(\mathbf{R}) = \max\{\lambda_1^{\mathbf{R}}, -\lambda_{j^\star}^{\mathbf{R}}\}$ or, equivalently, $\sigma(\mathbf{R}) = \max\{\lambda_{N-1}^{\mathbf{R}}, -\lambda_{N-j^\star}^{\mathbf{R}}\}$, with $\sigma(\mathbf{R}) \in ((2m)^{-1}, 1] \subseteq (1/2, 1]$. In particular, it holds $\sigma(\mathbf{R}) = |\lambda_\gamma^{\mathbf{R}}| = |\lambda_{N-\gamma}^{\mathbf{R}}|$, with γ such that*

$$\gamma = \underset{j \in \{1,\dots,n\}}{\arg\min} \left\{ \left| D_m(2\theta j) - \frac{1}{2} \right| \right\} \in \{1, j^\star\}. \tag{34}$$

2. *If $m = 1$ then $\sigma(\mathbf{R}) = -\lambda_n^{\mathbf{R}} = -\lambda_{N-n}^{\mathbf{R}}$.*
3. *It holds that $\sigma(\mathbf{R}) = 1$ if and only if N is even and $m = 1$.*
4. *If $m \geq m^\star$, with $m^\star$ defined as in Lem. 3.1, then it holds that $\sigma(\mathbf{R}) = -\lambda_{j^\star}^{\mathbf{R}} = -\lambda_{N-j^\star}^{\mathbf{R}} \leq -\lambda_2^{\mathbf{R}} = -\lambda_{N-2}^{\mathbf{R}}$.*
5. *If $m = n-1$ then $\sigma(\mathbf{R}) = -\lambda_2^{\mathbf{R}} = -\lambda_{N-2}^{\mathbf{R}}$.*

Proof of Theorem 3.4. By the symmetry of the Dirichlet kernel, eigenvalues of $\mathbf{R}$ in (18) also exhibit the property $\lambda_j^{\mathbf{R}} = \lambda_{N-j}^{\mathbf{R}}$, for all $j \in \{1, \ldots, n\}$. At the light of this observation, the following analysis is restricted to indexes $j \in \{1, \ldots, n\}$.

1. Exploiting relation (15) and the fact that $\lambda_1^{\mathbf{L}} \leq 2m$ (see Thm. 3.2) and $\lambda_{j^\star}^{\mathbf{L}} > 2m+1$ (see Thm. 3.3), it follows that $\lambda_1^{\mathbf{R}} \geq 0$ and $\lambda_{j^\star}^{\mathbf{R}} < -(2m)^{-1} \leq -1/2$ are the largest eigenvalues of $\mathbf{R}$ in absolute value. In particular, (34) is directly derived from (18) applied to (26).

2. Applying (29) with $m = 1$, it holds that $\lambda_j^{\mathbf{R}} = \mathrm{c}_{2\theta j}$. If N is even then $j = n$ is trivially selected to provide the essential spectral radius $\sigma(\mathbf{R}) = -\lambda_n^{\mathbf{R}} = 1$. Otherwise, for odd N, $j = 1$ or $j = n$ can be both selected, since $\sigma(\mathbf{R}) = \lambda_1^{\mathbf{R}} = \mathrm{c}_{2\theta}$ or, equivalently, $\sigma(\mathbf{R}) = -\lambda_n^{\mathbf{R}} = -\mathrm{c}_{2\theta n} = \mathrm{c}_{2\theta}$.

3. In the previous point it is already shown that $\sigma(\mathbf{R}) = 1$ if $m = 1$ and N is even. To prove that $\sigma(\mathbf{R}) = 1$ also implies that $m = 1$ and N is even, property 2 of Thm. 3.3 is invoked. Indeed, recall that $\rho(\mathbf{L}) = 4m$ holds if and only if C_N^m is bipartite, namely it has even N and $m = 1$. Relation (15) is then used to conclude.

4. Lem. 3.1 shows that if $m \geq m^\star$ then $\lambda_1^{\mathbf{R}} + \lambda_2^{\mathbf{R}} \leq 0$. Since, in general, it holds that $\lambda_1^{\mathbf{R}} \geq 0$, then, if $m \geq m^\star$, one has $\lambda_2^{\mathbf{R}} \leq -\lambda_1^{\mathbf{R}} \leq 0$. In particular, if $m > m^\star$ then $|\lambda_2^{\mathbf{R}}| > \lambda_1^{\mathbf{R}}$ holds. Therefore, $j = 1$ cannot be a valid index for an eigenvalue $\lambda_j^{\mathbf{R}}$ selected to compute $\sigma(\mathbf{R})$ in this case. As a consequence, if $m \geq m^\star$ then $\sigma(\mathbf{R}) = -\lambda_{j^\star}^{\mathbf{R}} \leq -\lambda_2^{\mathbf{R}}$.

5. Again, for all $j \in \{1, \ldots, n\}$, one has $D_{n-1}(2\theta j) = (-1)^{j+1}/2$, if N is even; $D_{n-1}(2\theta j) = (-1)^{j+1}\mathrm{c}_{\theta j}$, if N is odd. Thus, to prove this statement, it is just required to check that $-\lambda_2^{\mathbf{R}} > \lambda_1^{\mathbf{R}}$ holds true for all odd $N \geq 5$. The latter inequality leads to an identity. Hence, $\lambda_2^{\mathbf{R}}$ is the eigenvalue that satisfies (26) if $m = n - 1$.

4. Further discussions and numerical examples

This section reports a discussion on a couple of conjectures about the spectral radius $\rho(\mathbf{L})$ of the Laplacian matrix $\mathbf{L}$ and on the essential spectral radius $\sigma(\mathbf{R})$ of the Randić matrix $\mathbf{R}$ associated to a RRL C_N^m. Meaningful numerical examples are also brought as evidence for these ideas.

4.1 Conjecture on a potential upper bound for $j^\star$

Let us consider the statement of Thm. 3.3. Finding analytically an upper bound $\bar{j}$ for $j^\star$, similarly to what done in (23), may not be trivial. Nonetheless, an interesting conjecture on this particular bound is here given.

Conjecture 1 (An upper bound for $j^\star$)**.** *Under the same assumptions of Thm. 3.3, there exists $\bar{j} \in \{2, \ldots, n\}$ such that $j^\star \leq \bar{j}$ and its expression is yielded by*

$$\bar{j} = \lceil 3N/(4m+2) - 1/2 \rceil, \quad \forall N \geq 4, \forall m \in \{1, \ldots, n-1\}. \tag{35}$$

Remark 4.1. Considering $\underline{j}$ and $\overline{j}$ computed respectively as in (23) and (35), the following properties holding for $N \geq 4$ can be easily proven to support the fact that $\overline{j}$ may represent a good candidate upper bound for $j^\star$.

1. If $m \geq \widetilde{m}$, where

   $$\widetilde{m} = 3N/10 - 1/2, \tag{36}$$

   then one has $\overline{j} = 2$.

2. One has $\overline{j} = n$ if and only if $m = 1$. This also implies that for $m = 1$ expression in (35) is, in fact, a valid upper bound for $j^\star$. Moreover, if $m \geq 2$ then $\overline{j} < 2N/(2m+1) = x_2^\star/(2\theta) < n$ (see Thm. 2.2)

3. If $m = 2$ (and $N \geq 6$) then $\overline{j}$ is, in fact, a valid upper bound for $j^\star$, since $\overline{j} = \lceil (3N-5)/2 \rceil \leq 2N/5 = x_m^\star/(2\theta)$ (see Thm. 2.2).

4. One has $2 \leq \underline{j} \leq \overline{j} \leq n$, in which $\underline{j} = \overline{j}$ holds if and only if at least one of the following three cases is verified: (i) $3N/14 - 1/2 \leq m \leq N/4 - 1/2$; (ii) $m \geq \widetilde{m}$; (iii) $N \mod (2m+1) = 0$ and $m \geq N/6 - 1/2$.

The upper bound in (35) is figured out after the attempt to minimize $D_m(2\theta j)$ w.r.t. j. Observing that $\mathrm{s}_{\theta j}$ is strictly increasing for $j \in \{1, \ldots, n\}$, relation (35) is derived by choosing the smallest $j \in \{2, \ldots, n\}$ such that $|(2m+1)\theta j - (3\pi/2 + 2\ell\pi)|$, $\ell \in \mathbb{Z}$, be minimum and, to make treatable the latter expression, $\ell = 0$ is forced. The aim of this careful selection is twofold: on one hand, we want to obtain a small positive value for the denominator of $D_m(2\theta j)$ and, on the other hand, a large (in modulus) negative value for the numerator of $D_m(2\theta j)$, see (3). However, in general, there may exist values of $j > \overline{j}$ that render the numerator of $D_m(2\theta j)$ even more negative! This consideration is crucial. Indeed, the reasoning shown for the derivation of formula (36) in [21] can be trivially disproved taking for instance $(N, m) = (67, 2)$, for which it holds that $j^\star = 19$ (there, $j^\star = 20$ is wrongly claimed).
Nevertheless, one has $\overline{j} \geq \lceil \upsilon N/(\pi(2m+1)) \rceil \approx \lceil \underline{x}_1/(2\theta) \rceil$, as $3/2 > \upsilon/\pi$ (see Thm. 2.2). Also, expression (35) has been tested in simulation for all N such that $4 \leq N \leq 10000$ and any relative admissible value of m. Remarkably, no counterexample has been found in any of the tested instances. Hence, this fact suggests that $\overline{j}$ in (35) might represent a suitable upper bound for $j^\star$.

The following remark illustrates the potential implications of Conj. 1.

Remark 4.2. Let $m^\star$ and $\widetilde{m}$ be defined as in (27) and (36), respectively. If Conj. (1) verifies then one would have these further implications.

1. $\rho(\mathbf{L}) = \lambda_n^{\mathbf{L}} = \lambda_{N-n}^{\mathbf{L}}$ holds for all $N \geq 4$ if and only if $m = 1$. Thus, property 1a in Thm. 3.3 would be reinforced.

2. With reference to the essential spectral radius $\sigma(\mathbf{R})$, one has, $\forall N \geq 4$, $\sigma(\mathbf{R}) = -\lambda_n^{\mathbf{R}} = -\lambda_{N-n}^{\mathbf{R}}$ if and only if $m = 1$. Thus, property 2 in Thm. 3.4 would be reinforced.

3. If $m \geq \widetilde{m}$ then $\rho(\mathbf{L}) = \lambda_2^{\mathbf{L}} = \lambda_{N-2}^{\mathbf{L}}$ holds for all $N \geq 4$.

4. Considering again $\sigma(\mathbf{R})$, if $m \geq \max\{m^\star, \widetilde{m}\}$ then it holds that $\sigma(\mathbf{R}) = -\lambda_2^{\mathbf{R}} = -\lambda_{N-2}^{\mathbf{R}}$ for all $N \geq 4$. Thus, property 4 in Thm. 3.4 would be reinforced.

5. The search space of minimization in 1c and 1d of Thm. 3.3 would be reduced into $j \in \{\lfloor b_4^- \rfloor, \lceil b_4^- \rceil\}$ and $j \in \{\lfloor b_5^- \rfloor, \lceil b_5^- \rceil\}$, respectively.

6. The spectral radius $\rho(\mathbf{L})$ could be computed efficiently through binary search algorithm, as it can be shown that $D_m(2\theta y)$ restricted to $y \in [\underline{j}, \overline{j}]$ has one global minimum given by $y = \underline{x}_1/(2\theta) \approx \upsilon N/(\pi(2m+1))$ (see Thm. 2.2). Consequently, the computation of $\sigma(\mathbf{R}) = \max\{1 - \nu(\mathbf{L})/(2m), -1 + \rho(\mathbf{L})/(2m)\}$ would also result more efficient.

7. A direct estimate $\widehat{j}^\star \in [\underline{j}, \overline{j}]$ for $j^\star$ could be provided by averaging $\underline{j}$ and $\overline{j}$ through convex combinations. For instance, given $\alpha \in [0, 1]$, one can choose[6]

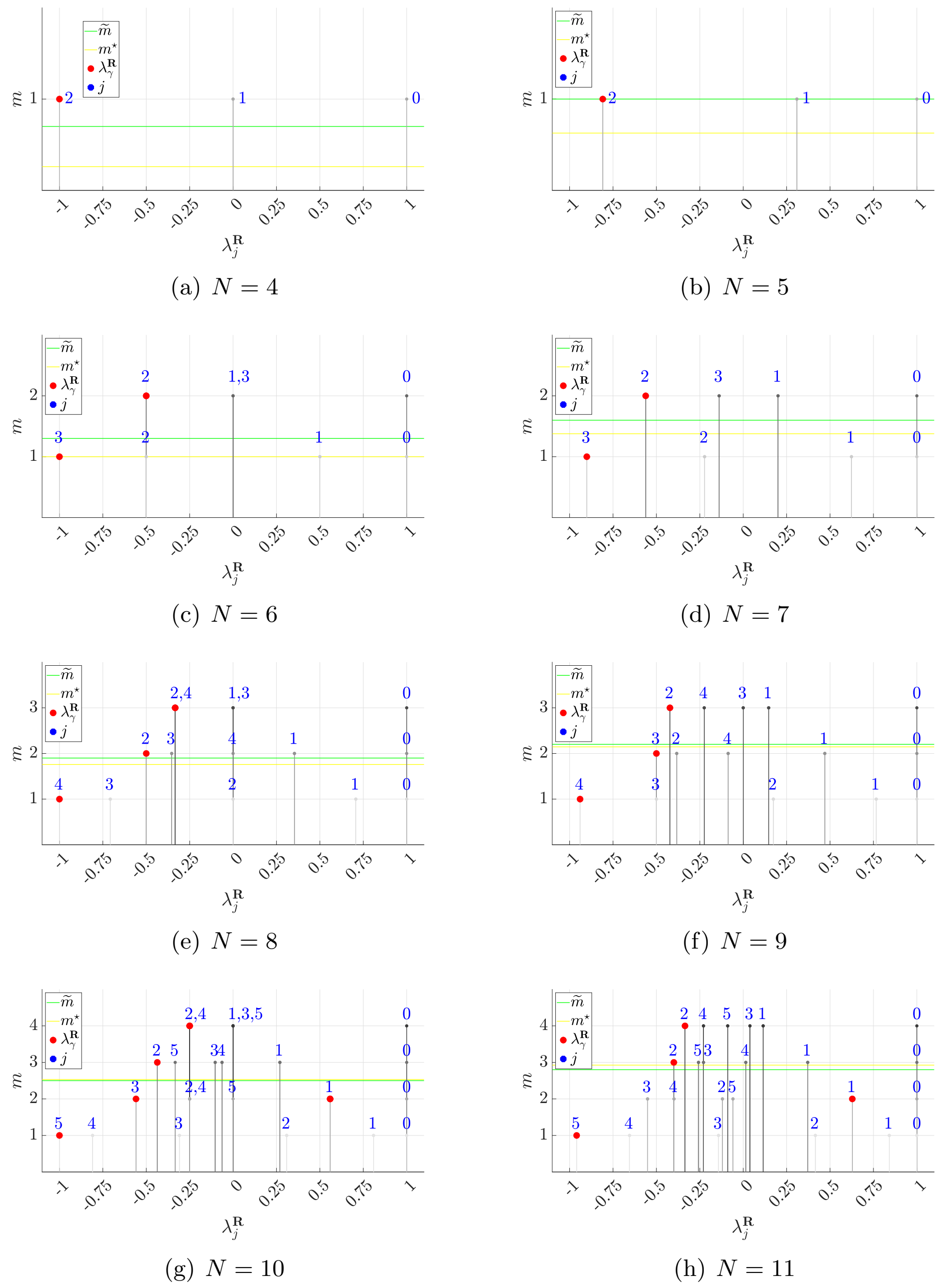
$$\widehat{j}^\star = \begin{cases} n, & \text{if } m = 1, \\ \lceil b_2^- - 1/2 \rceil, & \text{if } m = 2, \\ \lceil b_3^- - 1/2 \rceil, & \text{if } m = 3, \\ \lceil b_4^- - 1/2 \rceil, & \text{if } m = 4, \\ \lceil b_5^- - 1/2 \rceil, & \text{if } m = 5, \\ 2, & \text{if } m = n-1, \\ \lfloor \alpha \underline{j} + (1-\alpha)\overline{j} + 1/2 \rfloor, & \text{otherwise.} \end{cases} \tag{37}$$

4.2 Numerical examples for $4 \leq N \leq 11$

A few observations made on the pattern of values taken by $\sigma(\mathbf{R})$ are here provided. In this direction, examples in Fig. 2 grant to cover some of the most important aspects of this research, depicting a graphical representation of the spectrum $\Lambda(\mathbf{R})$. Specifically, each diagram in Fig. 2 shows how the eigenvalues $\lambda_j^{\mathbf{R}}$ spread over the interval $[-1, 1]$, as the order m changes for a fixed size N, with $4 \leq N \leq 11$. Plots 2(a)-2(h) also illustrate in blue all indexes $j = 0, \dots, n$ for relation (18), thresholds $m^\star$ and $\widetilde{m}$ (see point 4 in Rmk. 4.2) with a yellow and a green line respectively, and the eigenvalue $\lambda_\gamma^{\mathbf{R}}$ with a red dot (where γ is defined as in (34)).

With regard to Fig. 2, it is possible to observe the following facts descending from all the previous statements presented in Sec. 3.

6. For all N and m such that $4 \leq N \leq 2000$ and $1 \leq m < n$, coefficient $\alpha = 0.1313$ seems a good value to reduce the estimation error $|j^\star - \widehat{j}^\star|$, with $\widehat{j}^\star$ computed as in (37).



(a) $N = 4$ (b) $N = 5$

(c) $N = 6$ (d) $N = 7$

(e) $N = 8$ (f) $N = 9$

(g) $N = 10$ (h) $N = 11$

Figure 2: General eigenvalue distribution of the Randić matrix spectrum $\Lambda(\mathbf{R})$ for the RRLs C_N^m with $N = 4, \ldots, 11$ and $m = 1, \ldots, n-1 = \lfloor N/2 \rfloor - 1$.

- $\lambda_j^{\mathbf{R}} \in [-1, 1]$ holds $\forall j \in \{0, \dots, n\}$, with -1 and 1 simple eigenvalues.
- $\lambda_1^{\mathbf{R}} = \lambda_{N-1}^{\mathbf{R}} > \lambda_j^{\mathbf{R}}$ holds for all $j \in \{2, \dots, n\}$.
- For $m = 1$, one has $\lambda_\gamma^{\mathbf{R}}$ with $\gamma = n$ and if N is even then $\lambda_\gamma^{\mathbf{R}} = -1$.
- If $m \geq m^\star$ then $\lambda_{j^\star}^{\mathbf{R}} = \lambda_2^{\mathbf{R}} = \lambda_{N-2}^{\mathbf{R}}$ and if $m \geq \max\{m^\star, \widetilde{m}\}$ then $\lambda_\gamma^{\mathbf{R}} = \lambda_2^{\mathbf{R}} = \lambda_{N-2}^{\mathbf{R}}$, thus supporting property 4 in Rmk. 4.2.

To provide further evidences to the speculations made in Rmk. 4.2, some peculiarities and patterns can be also found for the following values of N.

- For $N = 5$ one has $\widetilde{m} = 1$ and, consequently, property 3 in Rmk. 4.2 holds tightly.
- For $N = 6$ one has $m^\star = 1$. Hence, if $m = 2 > m^\star$, the information about $\widetilde{m}$ becomes necessary in order to satisfy property 4 in Rmk. 4.2.
- For $N = 10$ and $m = 2$ one has $\sigma(\mathbf{R}) = \sqrt{5}/4 = \lambda_1^{\mathbf{R}} = \lambda_9^{\mathbf{R}} = -\lambda_3^{\mathbf{R}} = -\lambda_7^{\mathbf{R}}$, i.e. γ takes both the values in $\{1, n\}$. Moreover, in this case, it holds that $m^\star \approx 2.5330 > 2.5 = \widetilde{m}$, conversely to the previous cases with $N = 5$ and $N = 6$.

To sum up, each debated example in Fig. 2 gravitates, to some extent, around the key relation in (18), describing the spectrum $\Lambda(\mathbf{R})$ of the Randić matrix. It is important to recall that this investigation completely leverages the fundamental idea of studying the spectral properties of RRLs via the Dirichlet kernel redefined as in (3). Further clues are also given to support claims in Ssec. 4.1.

4.3 Conjecture on the values taken by $\sigma(\mathbf{R})$

All the previous discussions suggest few clues about the possibility of computing exactly $\sigma(\mathbf{R})$ by understanding the behavior of index γ defined in (34). The exact knowledge of the essential spectral radius of $\mathbf{R}$ is also motivated by various research areas, such as the convergence analysis of Page Rank and random walk processes [44].

Remarkably, from the numerical examples given in Ssec. 4.2, it is possible to observe the following facts. Graph C_9^2 in Fig. 2(f) is the unique example leading to $\gamma = 3$ only (if $m \geq 2$), as $\sigma(\mathbf{R}) = -\lambda_3^{\mathbf{R}} = -1/2 > \lambda_1^{\mathbf{R}} \approx 0.4698$. Graph C_{10}^2 in Fig. 2(g) is the unique example leading to both $\gamma = 1$ and $\gamma = 3$, as $\sigma(\mathbf{R}) = \lambda_1^{\mathbf{R}} = -\lambda_3^{\mathbf{R}} = \sqrt{5}/4$. In each diagram of Fig. 2 it holds that $\gamma = n$, if and only if $m = 1$, or $\gamma = 2$, if and only if $m \geq \max\{m^\star, \widetilde{m}\}$. In the remaining cases, it holds that $\gamma = 1$. Therefore, the following conjecture is drawn after having run some numerical tests[7].

7. These are performed for all N and m such that $4 \leq N \leq 10000$ and $1 \leq m < n$.

Conjecture 2 (Characterization of the essential spectral radius index γ). *Let $m^\star$ and $\widetilde{m}$ be defined as in* (27) *and* (36)*, respectively. For all $N \geq 4$, the essential spectral radius $\sigma(\mathbf{R}) = |\lambda_\gamma^{\mathbf{R}}| = |\lambda_{N-\gamma}^{\mathbf{R}}|$ associated to the Randić matrix $\mathbf{R}$ of a RRL C_N^m can be computed through index*

$$\gamma = \begin{cases} n, & \text{if } N \geq 8 \text{ and } m = 1, \\ 3, & \text{if } N = 6,7 \text{ and } m = 1 \text{ or if } N = 9,10 \text{ and } m = 2, \\ 2, & \text{if } N \geq 4 \text{ and } m \geq \min\{n-1, \max\{m^\star, \widetilde{m}\}\}, \\ 1, & \text{otherwise}. \end{cases} \tag{38}$$

Furthermore, a complete characterization of γ is given by taking into account (38) *along with the fact that $\gamma = 1$ also holds in the following four cases: (i) for all odd $N \geq 5$ and $m = 1$; (ii) for all $N \geq 4$ and $m = \max\{m^\star, \widetilde{m}\}$; (iii) for $N = 10$ and $m = 2$; (iv) for all even $N \geq 4$ and $m = n - 1$.*

5. Conclusions and future directions

In this work, a peculiar class of circulant graphs, referred to as *regular ring lattices*, is described highlighting the relationship between the spectrum of their characteristic matrices and the well-known Dirichlet kernel. Several properties related to the eigenvalues are described extensively, with a particular focus on the Fiedler value, the spectral radius of the Laplacian and the essential spectral radius of the Randić matrix associated to these graphs. Part of the proven results is also discussed in details with auxiliary diagrams depicting the related spectral distributions. Finally, the debated conjectures on the computation of the aforementioned spectral quantities represent an open problem to be solved in order to improve the latest analysis techniques for networked dynamic systems.

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On pseudo picture fuzzy cosets

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Abstract. In this paper, the concepts of pseudo picture fuzzy cosets, pseudo picture fuzzy double cosets and pseudo picture fuzzy middle cosets were introduced and some of their characteristics were established. In addition, we investigated the connections between pseudo picture fuzzy double cosets and picture fuzzy normal subgroup, also between pseudo picture fuzzy middle cosets and picture fuzzy normal subgroup.

Keywords: Picture fuzzy set, Picture fuzzy subgroup, Pseudo picture fuzzy cosets, Pseudo picture fuzzy double cosets, Pseudo picture fuzzy middle cosets.

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1. Introduction

The generalisation of theory of crisp sets into the theory of fuzzy sets was introduced by Zadeh [27]. This theory has become a vast and sprawling area of research in topology, algebra, engineering, convexity etc. The fuzzy sets only deal with the membership degree of an element in belonging to a set. In [1], Atanassov extended the theory of fuzzy sets into intuitionistic fuzzy sets which took care of both the membership and non-membership degrees of an element belonging to a set. Cuong and Kreinovich [14] generalised the notions of both fuzzy sets and intuitionistic fuzzy sets into picture fuzzy sets. In their work, one of the items needed to still determine the membership of an element in a set was added, and it is called the neutral membership degree. Thus, picture fuzzy sets theory comprises of positive membership, neutral membership and negative membership degrees.

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Rosenfeld [23] put forward the notion of fuzzy group. As an extension of fuzzy group, Biswas [6] initiated the idea of intuitionistic fuzzy subgroup of a group. Zhan and Tan [28] also studied intuitionistic fuzzy subgroup. Sharma [24] established some properties of intuitionistic fuzzy subgroup of a group through cut set of intuitionistic fuzzy sets. In [25], Sharma introduced t-intuitionistic fuzzy sets and obtained some properties. Dogra and Pal [15] initiated the concept of picture fuzzy subring of a crisp ring and studied some related basic results. They also investigated some properties of picture fuzzy subring under classical ring homomorphism. Dogra and Pal [16] put forward the notion of picture fuzzy subspace of a crisp vector space and obtained some basic results related to it on the basis of some basic operations on picture fuzzy sets. Furthermore, direct sum of two picture fuzzy subspaces, isomorphism between two picture fuzzy subspaces, picture fuzzy linear transformation and picture fuzzy linearly independent set of vectors and some properties connected to these were established.

In [21], Mukherjee and Bhattacharya initiated fuzzy cosets. The extension to pseudo fuzzy cosets was studied by Nagarajan and Solaraiju [20] and some properties were established. This concept was later studied by Onasanya and Ilori [20] to obtain some independent proofs of the properties established in [20]. Sharma [25] introduced t-intuitionistic fuzzy left (right) cosets and investigated some of its properties. Dogra and Pal [17] introduced picture fuzzy subgroup of a crisp group, picture fuzzy left (right, middle) cosets, and some of their properties were obtained. In [26], Sharma and Sandhu initiated pseudo intuitionistic fuzzy cosets, pseudo intuitionistic fuzzy double cosets and pseudo intuitionistic fuzzy middle cosets of a group and established some of their properties. Since the notion of picture fuzzy set was a generalisation of both fuzzy sets and intuitionistic fuzzy sets [14], the idea of fuzzy cosets was extended to intuitionistic fuzzy cosets [25] and the pseudo fuzzy cosets was also extended to pseudo intuitionistic fuzzy cosets [26]. Thus, the concept of pseudo picture fuzzy cosets which is a generalisation of pseudo intuitionistic fuzzy cosets can be a research focus.

In this paper, the concepts of pseudo intuitionistic fuzzy cosets was generalised to pseudo picture fuzzy cosets. We have put forward the *pseudo picture fuzzy cosets* (PPFCs), *pseudo picture fuzzy double cosets* (PPFDs) and *pseudo picture fuzzy middle cosets* (PPFMs), and some of their characterisations were established. It was established that, this concept is a generalisation of the notion introduced by Sharma and Sandhu in [26]. The paper is organised as follows. In Section 2, we give some definitions, basic operations and preliminary results. In Section 3, we introduce PPFCs, PPFDs and PPFMs and establish some of their characterisations.

2. Preliminaries

In this section, some basic definitions, operations and preliminary results are stated.

Definition 2.1 ([27]). *Let Y be a nonempty set. A fuzzy set (FS) Q of Y is defined as*

$$Q = \{\langle y, \sigma_Q(y)\rangle | y \in Y\}$$

with a membership function

$$\sigma_Q : Y \longrightarrow [0, 1],$$

where the function $\sigma_Q(y)$ denotes the degree of membership of $y \in Q$.

Definition 2.2 ([1]). *Let a nonempty set Y be fixed. An intuitionistic fuzzy set (IFS) Q of Y is defined as*

$$Q = \{\langle y, \sigma_Q(y), \tau_Q(y)\rangle | y \in Y\},$$

where the functions

$$\sigma_Q : Y \to [0, 1] \text{ and } \tau_Q : Y \to [0, 1]$$

are called the membership and non-membership degrees, respectively, and for every $y \in Y$,

$$0 \leq \sigma_Q(y) + \tau_Q(y) \leq 1.$$

Definition 2.3 ([14]). *A picture fuzzy set Q of Y is defined as*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y), \gamma_Q(y)) | y \in Y\},$$

where the functions

$$\sigma_Q : Y \to [0, 1],\ \ \tau_Q : Y \to [0, 1] \text{ and } \gamma_Q : Y \to [0, 1]$$

are called the positive, neutral and negative membership degrees, respectively, and $\sigma_Q, \tau_Q, \gamma_Q$ satisfy for any $y \in Y$,

$$0 \leq \sigma_Q(y) + \tau_Q(y) + \gamma_Q(y) \leq 1.$$

Then, $S_Q(y) = 1 - (\sigma_Q(y) + \tau_Q(y) + \gamma_Q(y))$ is called the refusal membership degree of $y \in Q$.

Definition 2.4 ([14]). *Let Q and R be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:*

- *$Q \subseteq R$ if and only if for all $y \in Y$, $\sigma_Q(y) \leq \sigma_R(y)$, $\tau_Q(y) \leq \tau_R(y)$ and $\gamma_Q(y) \geq \gamma_R(y)$.*

- $Q = R$ *if and only if* $Q \subseteq R$ *and* $R \subseteq Q$.
- $Q \cup R = \{(y, \sigma_Q(y) \vee \sigma_R(y), \tau_Q(y) \wedge \tau_R(y)), \gamma_Q(y) \wedge \gamma_R(y))|y \in Y\}$.
- $Q \cap R = \{(y, \sigma_Q(y) \wedge \sigma_R(y), \tau_Q(y) \wedge \tau_R(y)), \gamma_Q(y) \vee \gamma_R(y))|y \in Y\}$.
- $coQ = \overline{Q} = \{(y, \gamma_Q(y), \tau_Q(y), \sigma_Q(y))|y \in Y\}$.

Definition 2.5 ([23]). *Let* $(G, *)$ *be a group and* $Q = \{(y, \sigma_Q(y)) \mid y \in G\}$ *be an FS in* G. *Then,* Q *is called a fuzzy subgroup (FSG) of* G *if* $\sigma_Q(a * b) \geq \sigma_Q(a) \wedge \sigma_Q(b)$ *and* $\sigma_Q(a^{-1}) \geq \sigma_Q(a)$ *for all* $a, b \in G$, *where* a^{-1} *is the inverse of* $a \in G$.

Definition 2.6 ([6, 24, 28]). *Let* $(G, *)$ *be a crisp group and*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y)) \mid y \in G\}$$

be an IFS in G. *Then,* Q *is called intuitionistic fuzzy subgroup (IFSG) of* G *if*
(*i*) $\sigma_Q(a * b) \geq \sigma_Q(a) \wedge \sigma_Q(b),\ \tau_Q(a * b) \leq \tau_Q(a) \vee \tau_Q(b)$,
(*ii*) $\sigma_Q(a^{-1}) \geq \sigma_Q(a),\ \tau_Q(a^{-1}) \leq \tau_Q(a)$,
for all $a, b \in G$, *where* a^{-1} *is the inverse of* $a \in G$.

Definition 2.7 ([17]). *Let* $(G, *)$ *be a crisp group and*

$$Q = \{(y, \sigma_Q(y), \tau_Q(y), \eta_Q(y)) \mid y \in G\}$$

be a PFS in G. *Then,* Q *is called picture fuzzy subgroup (PFSG) of* G *if*
(*i*) $\sigma_Q(a * b) \geq \sigma_Q(a) \wedge \sigma_Q(b),\ \tau_Q(a * b) \geq \tau_Q(a) \wedge \tau_Q(b),\ \eta_Q(a * b) \leq \eta_Q(a) \vee \eta_Q(b)$,
(*ii*) $\sigma_Q(a^{-1}) \geq \sigma_Q(a),\ \tau_Q(a^{-1}) \geq \tau_Q(a),\ \eta_Q(a^{-1}) \leq \eta_Q(a)$,
for all $a, b \in G$, *where* a^{-1} *is the inverse of* $a \in G$.

Definition 2.8 ([17]). *Let* $(G, *)$ *be a crisp group and* $Q = (\sigma_Q, \tau_Q, \eta_Q)$ *be a PFSG of* G. *Then, for* $a \in G$, *the picture fuzzy left cosets (PFLCs) of* $Q \in G$ *is the PFS* $aQ = (\sigma_{aQ},\ \tau_{aQ},\ \eta_{aQ})$ *defined by*

$$\sigma_{aQ}(y) = \sigma_Q(a^{-1} * y),\ \tau_{aQ}(y) = \tau_Q(a^{-1} * y) \text{ and } \eta_{aQ}(y) = \eta_Q(a^{-1} * y),$$

for all $y \in G$.

Definition 2.9 ([17]). *Let* $(G, *)$ *be a crisp group and* $Q = (\sigma_Q, \tau_Q, \eta_Q)$ *be a PFSG of* G. *Then,* Q *is called a picture fuzzy normal subgroup (PFNSG) of* G *if*

$$\sigma_{Qa}(y) = \sigma_{aQ}(y),\ \tau_{Qa}(y) = \tau_{aQ}(y) \text{ and } \eta_{Qa}(y) = \eta_{aQ}(y),$$

for all $a,\ y \in G$.

Definition 2.10 ([17]). *Let $(G,*)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for $a \in G$, the picture fuzzy right cosets (PFRCs) of $Q \in G$ is the PFS $Qa = (\sigma_{Qa},\ \tau_{Qa},\ \eta_{Qa})$ defined by*

$$\sigma_{Qa}(y) = \sigma_Q(y * a^{-1}),\ \tau_{Qa}(y) = \tau_Q(y * a^{-1}) \text{ and } \eta_{Qa}(y) = \eta_Q(y * a^{-1}),$$

for all $y \in G$.

Definition 2.11 ([17]). *Let $(G,*)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for $a \in G$, the picture fuzzy middle cosets (PFMCs) of $Q \in G$ is the PFS $aQa^{-1} = (\sigma_{aQa^{-1}},\ \tau_{aQa^{-1}},\ \eta_{aQa^{-1}})$ defined by*

$$\sigma_{aQa^{-1}}(y) = \sigma_Q(a^{-1} * y * a),\ \tau_{aQa^{-1}}(y) = \tau_Q(a^{-1} * y * a)$$

and

$$\eta_{aQa^{-1}}(y) = \eta_Q(ya^{-1} * y * a),$$

for all $y \in G$.

3. Pseudo picture fuzzy sets

This section defines pseudo picture fuzzy cosets, pseudo picture fuzzy double cosets and pseudo picture fuzzy middle cosets were introduced and some of their charactristics are established.

Definition 3.1. *Let $(G,*)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for any $a \in G$ the pseudo picture fuzzy left cosets (PPFLCs) of Q with respect to some fixed PFS y of G is a PFS*

$$(aQ)^y = (\sigma_{(aQ)^y}(x),\ \tau_{(aQ)^y}(x),\ \eta_{(aQ)^y}(x))$$

defined by

$$\sigma_{(aQ)^y}(x) = \sigma_y(a)\sigma_Q(x),$$
$$\tau_{(aQ)^y}(x) = \tau_y(a)\tau_Q(x)$$

and

$$\eta_{(aQ)^y}(x) = \eta_y(a)\eta_Q(x),$$

for all $x \in G$.

Definition 3.2. *Let $(G,*)$ be a crisp group and $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, for any $a \in G$ the pseudo picture fuzzy right cosets (PPFRCs) of Q with respect to some fixed PFS y of G is a PFS*

$$(Qa)^y = (\sigma_{(Qa)^y}(x),\ \tau_{(Qa)^y}(x),\ \eta_{(Qa)^y}(x))$$

defined by

$$\sigma_{(Qa)^y}(x) = \sigma_Q(x)\sigma_y(a),$$
$$\tau_{(Qa)^y}(x) = \tau_Q(x)\tau_y(a)$$

and

$$\eta_{(Qa)^y}(x) = \eta_Q(x)\eta_y(a),$$

for all $x \in G$.

Example 3.1. Let $G = \{1, w, w^2\}$ be a group. Let

$$Q = \{(1, 0.1, 0.15, 0.7), (w, 0.2, 0.3, 0.4), (w^2, 0.3, 0.4, 0.1)\}$$

be a PFSG of G. Let y be a PFS of G defined as

$$\sigma_y(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0.4, & \text{if } x = w \\ 0.2, & \text{if } x = w^2 \end{cases}$$

$$\tau_y(x) = \begin{cases} 0, & \text{if } x = 1 \\ 0.3, & \text{if } x = w \\ 0.35, & \text{if } x = w^2 \end{cases}$$

and

$$\eta_y(x) = \begin{cases} 0, & \text{if } x = 1 \\ 0.2, & \text{if } x = w \\ 0.4, & \text{if } x = w^2 \end{cases}$$

Thus, PPFLC of Q determined by an element w is $(wQ)^y = (\sigma_{(wQ)^y}, \tau_{(wQ)^y}, \eta_{(wQ)^y})$. Now,

$$\sigma_{(wQ)^y}(x) = \sigma_y(w)\sigma_Q(x),$$
$$\tau_{(wQ)^y}(x) = \tau_y(w)\tau_Q(x)$$

and

$$\eta_{(wQ)^y}(x) = \eta_y(w)\eta_Q(x).$$

Hence,

$$\sigma_{(wQ)^y}(x) = \begin{cases} 0.04, & \text{if } x = 1 \\ 0.08, & \text{if } x = w \\ 0.12, & \text{if } x = w^2 \end{cases}$$

$$\tau_{(wQ)^y}(x) = \begin{cases} 0.045, & \text{if } x = 1 \\ 0.09, & \text{if } x = w \\ 0.12, & \text{if } x = w^2 \end{cases}$$

and

$$\eta_{(wQ)^y}(x) = \begin{cases} 0.14, & \text{if } x = 1 \\ 0.08, & \text{if } x = w \\ 0.02, & \text{if } x = w^2 \end{cases}.$$

Proposition 3.1. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, PPFLC $(aQ)^y$ is a PFSG of crisp group G for any $a \in G$.*

Proof. Let

$$Q = (\sigma_Q, \tau_Q, \eta_Q)$$

be a PFSG of G and

$$(aQ)^y = (\sigma_{(aQ)^y}(x),\ \tau_{(aQ)^y}(x),\ \eta_{(aQ)^y}(x))$$

be a PPFLC of $Q \in G$ for $a, x \in G$.

Now, for every $g,\ h \in G$, we have

$$\begin{aligned} \sigma_{(aQ)^y}(g * h) &= \sigma_y(a)\sigma_Q(g * h) \\ &\geq \sigma_y(a)\left(\sigma_Q(g) \wedge \sigma_Q(h)\right) \\ &= (\sigma_y(a)\sigma_Q(g)) \wedge (\sigma_y(a)\sigma_Q(h)) \\ &= \sigma_{(aQ)^y}(g) \wedge \sigma_{(aQ)^y}(h), \end{aligned}$$

$$\begin{aligned} \tau_{(aQ)^y}(g * h) &= \tau_y(a)\tau_Q(g * h) \\ &\geq \tau_y(a)\left(\tau_Q(g) \wedge \tau_Q(h)\right) \\ &= (\tau_y(a)\tau_Q(g)) \wedge (\tau_y(a)\tau_Q(h)) \\ &= \tau_{(aQ)^y}(g) \wedge \tau_{(aQ)^y}(h) \end{aligned}$$

and

$$\begin{aligned} \eta_{(aQ)^y}(g * h) &= \eta_y(a)\eta_Q(g * h) \\ &\leq \eta_y(a)\left(\eta_Q(g) \vee \eta_Q(h)\right) \\ &= (\eta_y(a)\eta_Q(g)) \vee (\eta_y(a)\eta_Q(h)) \\ &= \eta_{(aQ)^y}(g) \vee \eta_{(aQ)^y}(h). \end{aligned}$$

Therefore, the PPFLC $(aQ)^y$ is a PFSG of G. □

Proposition 3.2. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Then, PPFRC $(aQ)^y$ is a PFSG of crisp group G for any $a \in G$.*

Proof. This is similar to the proof of Proposition 3.1. □

Proposition 3.3. *Any two pseudo picture fuzzy cosets of PFSG are either disjoint or identical.*

Proof. Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFSG of G. Let

$$(aQ)^y = (\sigma_{(aQ)^y}(x),\ \tau_{(aQ)^y}(x),\ \eta_{(aQ)^y}(x))$$

and

$$(bQ)^y = (\sigma_{(bQ)^y}(x),\ \tau_{(bQ)^y}(x),\ \eta_{(bQ)^y}(x))$$

be any two identical PPFCs for $a,\ b \in G$, then for all $g \in G$,

$$\sigma_{(aQ)^y}(g) = \sigma_{(bQ)^y}(g),\ \tau_{(aQ)^y}(g) = \tau_{(bQ)^y}(g) \text{ and } \eta_{(aQ)^y}(g) = \eta_{(bQ)^y}(g).$$

Suppose on the contrary that the PPFCs $(aQ)^y$ and $(bQ)^y$ are disjoint. Then, there is no such $y \in G$ such that

$$\sigma_{(aQ)^y}(h) \neq \sigma_{(bQ)^y}(h),\ \tau_{(aQ)^y}(h) \neq \tau_{(bQ)^y}(h) \text{ and } \eta_{(aQ)^y}(h) \neq \eta_{(bQ)^y}(h),$$

which means that $\sigma_y(a)\sigma_Q(h) \neq \sigma_y(b)\sigma_Q(h)$, $\tau_y(a)\tau_Q(h) \neq \tau_y(b)\tau_Q(h)$ and $\eta_y(a)\eta_Q(h) \neq \eta_y(b)\eta_Q(h)$ and we get

$$\sigma_y(a) \neq \sigma_y(b),\ \tau_y(a) \neq \tau_y(b) \text{ and } \eta_y(a) \neq \eta_y(b).$$

So, the assumption that

$$\sigma_{(aQ)^y}(g) = \sigma_{(bQ)^y}(g),\ \tau_{(aQ)^y}(g) = \tau_{(bQ)^y}(g),\ \eta_{(aQ)^y}(g) = \eta_{(bQ)^y}(g),\ \forall g \in G$$

is not true.

Conversely, let

$$(aQ)^y = (\sigma_{(aQ)^y},\ \tau_{(aQ)^y},\ \eta_{(aQ)^y})$$

and

$$(bQ)^y = (\sigma_{(bQ)^y},\ \tau_{(bQ)^y},\ \eta_{(bQ)^y})$$

be two disjoint PPFCs for every $a,\ b,\ g \in G$. Then,

$$\sigma_{(aQ)^y}(g) \neq \sigma_{(bQ)^y}(g),$$
$$\tau_{(aQ)^y}(g) \neq \tau_{(bQ)^y}(g)$$

and

$$\eta_{(aQ)^y}(g) \neq \eta_{(bQ)^y}(g),$$

which implies that

$$\sigma_y(a)\sigma_Q(g) \neq \sigma_y(b)\sigma_Q(g),$$
$$\tau_y(a)\tau_Q(g) \neq \tau_y(b)\tau_Q(g)$$

and

$$\eta_y(a)\eta_Q(g) \neq \eta_y(b)\eta_Q(g),$$

but if they are assumed to be identical, then

$$\sigma_y(a)\sigma_Q(g) = \sigma_y(b)\sigma_Q(g),$$
$$\tau_y(a)\tau_Q(g) = \tau_y(b)\tau_Q(g)$$

and

$$\eta_y(a)\eta_Q(g) = \eta_y(b)\eta_Q(g).$$

So,

$$\sigma_y(a) = \sigma_y(b),$$
$$\tau_y(a) = \tau_y(b)$$

and

$$\eta_y(a) = \eta_y(b).$$

Thus, this makes the assumption that

$$\sigma_y(a)\sigma_Q(g) \neq \sigma_y(b)\sigma_Q(g),$$
$$\tau_y(a)\tau_Q(g) \neq \tau_y(b)\tau_Q(g)$$

and

$$\eta_y(a)\eta_Q(g) \neq \eta_y(b)\eta_Q(g),$$

i.e., $\sigma_{(aQ)^y}(g) \neq \sigma_{(bQ)^y}(g)$, $\tau_{(aQ)^y}(g) \neq \tau_{(bQ)^y}(g)$ and $\eta_{(aQ)^y}(g) \neq \eta_{(bQ)^y}(g)$ are false. $\square$

Proposition 3.4. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ and $R = (\sigma_R, \tau_R, \eta_R)$ be two PFSGs of G. Then $(aQ)^y \subseteq (aR)^y$ if and only if $Q \subseteq R$, for all $a \in G$ and $y \in Y$.*

Proof. Suppose that $(aQ)^y \subseteq (aR)^y$ and we get

$$\sigma_{(aQ)^y}(g) \leq \sigma_{(aR)^y}(g),$$
$$\tau_{(aQ)^y}(g) \leq \tau_{(aR)^y}(g)$$

and

$$\eta_{(aQ)^y}(g) \geq \eta_{(aR)^y}(g),$$

for all $g \in G$, which implies that

$$\sigma_y(a)\sigma_Q(g) \leq \sigma_y(a)\sigma_R(g),$$
$$\tau_y(a)\tau_Q(g) \leq \tau_y(a)\tau_R(g)$$

and

$$\eta_y(a)\eta_Q(g) \geq \eta_y(a)\eta_R(g),$$

for all $g \in G$. And we obtain

$$\sigma_Q(g) \leq \sigma_R(g),\ \tau_Q(g) \leq \tau_R(g) \text{ and } \eta_Q(g) \geq \eta_R(g),\ \forall\ g \in G.$$

Thus, $Q \subseteq R$.

Conversely, suppose that $Q \subseteq R$, and we get $\sigma_Q(g) \leq \sigma_R(g)$, $\tau_Q(g) \leq \tau_R(g)$ and $\eta_Q(g) \geq \eta_R(g)$, $\forall\ g \in G$. So,

$$\sigma_y(a)\sigma_Q(g) \leq \sigma_y(a)\sigma_R(g),$$
$$\tau_y(a)\tau_Q(g) \leq \tau_y(a)\tau_R(g)$$

and

$$\eta_y(a)\eta_Q(g) \geq \eta_y(a)\eta_R(g),$$

for all g$\in G$. And we obtain

$$\sigma_{(aQ)^y}(g) \leq \sigma_{(aR)^y}(g),\ \tau_{(aQ)^y}(g) \leq \tau_{(aR)^y}(g) \text{ and } \eta_{(aQ)^y}(g) \geq \eta_{(aR)^y}(g),$$

for all $g \in G$. □

Definition 3.3. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ and $R = (\sigma_R, \tau_R, \eta_R)$ be two PFSG. Then, for any $a \in G$ the pseudo picture fuzzy double cosets (PPFDCs) of Q and R with respect to some fixed PFS y of G is the PFS*

$$(QaR)^y = (\sigma_{(QaR)^y},\ \tau_{(QaR)^y},\ \eta_{(QaR)^y})$$

of G, which is defined as

$$\sigma_{(QaR)^y}(g) = \sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g),$$
$$\tau_{(QaR)^y}(g) = \tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)$$

and

$$\eta_{(QaR)^y}(g) = \eta_{(Qa)^y}(g) \vee \eta_{(aR)^y}(g)G,$$

for every $g \in G$.

Proposition 3.5. *Every PPFDC is a PFSG of G.*

Proof. Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ and $R = (\sigma_R, \tau_R, \eta_R)$ be two PFSGs of G. Let

$$(QaR)^y = (\sigma_{(QaR)^y},\ \tau_{(QaR)^y},\ \eta_{(QaR)^y})$$

where

$$\sigma_{(QaR)^y}(g) = \sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g),$$
$$\tau_{(QaR)^y}(g) = \tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)$$

and

$$\eta_{(QaR)^y}(g) = \eta_{(Qa)^y}(g) \wedge \eta_{(aR)^y}(g)$$

$g \in G$ be PPFDC. Let $g, h \in G$ be any elements, then

$$\begin{aligned}
&\sigma_{(QaR)^y}(g*h)\\
&\quad= \sigma_{(Qa)^y}(g*h) \wedge \sigma_{(aR)^y}(g*h)\\
&\quad= \sigma_y(a)\sigma_{(Q)}(g*h) \wedge \sigma_y(a)\sigma_{(R)}(g*h)\\
&\quad\geq \sigma_y(a)(\sigma_Q(g) \wedge \sigma_Q(h)) \wedge \sigma_y(a)(\sigma_R(g) \wedge \sigma_R(h))\\
&\quad= [(\sigma_y(a)\sigma_Q(g)) \wedge (\sigma_y(a)\sigma_Q(h))] \wedge [(\sigma_y(a)\sigma_R(g)) \wedge (\sigma_y(a)\sigma_R(h))]\\
&\quad= [(\sigma_y(a)\sigma_Q(g)) \wedge (\sigma_y(a)\sigma_R(g))] \wedge [(\sigma_y(a)\sigma_Q(h)) \wedge (\sigma_y(a)\sigma_R(h))]\\
&\quad= \left[\sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g)\right] \wedge \left[\sigma_{(Qa)^y}(h) \wedge \sigma_{(aR)^y}(h)\right]\\
&\quad= \sigma_{(QaR)^y}(g) \wedge \sigma_{(QaR)^y}(h),\\
&\tau_{(QaR)^y}(g*h)\\
&\quad= \tau_{(Qa)^y}(g*h) \wedge \tau_{(aR)^y}(g*h)\\
&\quad= \tau_y(a)\tau_{(Q)}(g*h) \wedge \tau_y(a)\tau_{(R)}(g*h)\\
&\quad\geq \tau_y(a)(\tau_Q(g) \wedge \tau_Q(h)) \wedge \tau_y(a)(\tau_R(g) \wedge \tau_R(h))\\
&\quad= [(\tau_y(a)\tau_Q(g)) \wedge (\tau_y(a)\tau_Q(h))] \wedge [(\tau_y(a)\tau_R(g)) \wedge (\tau_y(a)\tau_R(h))]\\
&\quad= [(\tau_y(a)\tau_Q(g)) \wedge (\tau_y(a)\tau_R(g))] \wedge [(\tau_y(a)\tau_Q(h)) \wedge (\tau_y(a)\tau_R(h))]\\
&\quad= \left[\tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)\right] \wedge \left[\tau_{(Qa)^y}(h) \wedge \tau_{(aR)^y}(h)\right]\\
&\quad= \tau_{(QaR)^y}(g) \wedge \tau_{(QaR)^y}(h)
\end{aligned}$$

and

$$\begin{aligned}
&\eta(QaR)^y(g*h)\\
&\quad= \eta(Qa)^y(g*h) \vee \eta(aR)^y(g*h)\\
&\quad= \eta y(a)\eta(Q)(g*h) \vee \eta_y(a)\eta(R)(g*h)\\
&\quad\leq \eta_y(a)(\eta_Q(g) \vee \eta_Q(h)) \vee \eta_y(a)(\eta_R(g) \vee \eta_R(h))\\
&\quad= [(\eta_y(a)\eta_Q(g)) \vee (\eta_y(a)\eta_Q(h))] \vee [(\eta_y(a)\eta_R(g)) \vee (\eta_y(a)\eta_R(h))]\\
&\quad= [(\eta_y(a)\eta_Q(g)) \vee (\eta_y(a)\eta_R(g))] \vee [(\eta_y(a)\eta_Q(h)) \vee (\eta_y(a)\eta_R(h))]\\
&\quad= \left[\eta_{(Qa)^y}(g) \vee \eta_{(aR)^y}(g)\right] \vee \left[\eta_{(Qa)^y}(h) \vee \eta_{(aR)^y}(h)\right]\\
&\quad= \eta_{(QaR)^y}(g) \vee \eta_{(QaR)^y}(h).
\end{aligned}$$

Therefore, $(QaR)^y$ is a PFSG of G. □

Proposition 3.6. *Let Q and R be two PFSGs. If Q and R are PFNSGs, then the PPFDC $(QaR)^y$ is a PFNSG.*

Proof. Let $(QaR)^y = (\sigma_{(QaR)^y},\ \tau_{(QaR)^y},\ \eta_{(QaR)^y})$, where

$$\sigma_{(QaR)^y}(g) = \sigma_{(Qa)^y}(g) \wedge \sigma_{(aR)^y}(g),$$
$$\tau_{(QaR)^y}(g) = \tau_{(Qa)^y}(g) \wedge \tau_{(aR)^y}(g)$$

and

$$\eta_{(QaR)^y}(g) = \eta_{(Qa)^y}(g) \wedge \eta_{(aR)^y}(g),$$

$g \in G$ be PPFDC where Q and R are PFNSGs of G. By Proposition 3.5, $(QaR)^y$ is PFSG of G. Let $g, h \in G$, then

$$\begin{aligned}
\sigma_{(QaR)^y}(g*h*g^{-1}) &= \left[\sigma_{(Qa)^y}(g*h*g^{-1})\right] \wedge \left[\sigma_{(aR)^y}(g*h*g^{-1})\right] \\
&= \left[\sigma_y(a)\sigma_Q(g*h*g^{-1})\right] \wedge \left[\sigma_y(a)\sigma_R(g*h*g^{-1})\right] \\
&= \left[\sigma_y(a)\sigma_Q((g*h)*g^{-1})\right] \wedge \left[\sigma_y(a)\sigma_R((g*h)*g^{-1})\right] \\
&= \left[\sigma_y(a)\sigma_Q((g*h)*g^{-1})\right] \wedge \left[\sigma_y(a)\sigma_R((g*h)*g^{-1})\right] \\
&= \left[\sigma_y(a)\sigma_Q(g^{-1}*(g*h))\right] \wedge \left[\sigma_y(a)\sigma_R(g^{-1}*(g*h))\right] \\
&= \left[\sigma_y(a)\sigma_Q(g^{-1}*g)*h)\right] \wedge \left[\sigma_y(a)\sigma_R(g^{-1}*g)*h)\right] \\
&= [\sigma_y(a)\sigma_Q(h))] \wedge [\sigma_y(a)\sigma_R(h)] \\
&= (\sigma_{(Qa)^y}(h)) \wedge (\sigma_{(aR)^y}(h)) \\
&= \sigma_{(QaR)^y}(h), \\
\tau_{(QaR)^y}(g*h*g^{-1}) &= \left[\tau_{(Qa)^y}(g*h*g^{-1})\right] \wedge \left[\tau_{(aR)^y}(g*h*g^{-1})\right] \\
&= \left[\tau_y(a)\tau_Q(g*h*g^{-1})\right] \wedge \left[\tau_y(a)\tau_R(g*h*g^{-1})\right] \\
&= \left[\tau_y(a)\tau_Q((g*h)*g^{-1})\right] \wedge \left[\tau_y(a)\tau_R((g*h)*g^{-1})\right] \\
&= \left[\tau_y(a)\tau_Q((g*h)*g^{-1})\right] \wedge \left[\tau_y(a)\tau_R((g*h)*g^{-1})\right] \\
&= \left[\tau_y(a)\tau_Q(g^{-1}*(g*h))\right] \wedge \left[\tau_y(a)\tau_R(g^{-1}*(g*h))\right] \\
&= \left[\tau_y(a)\tau_Q(g^{-1}*g)*h)\right] \wedge \left[\tau_y(a)\tau_R(g^{-1}*g)*h)\right] \\
&= [\tau_y(a)\tau_Q(h))] \wedge [\tau_y(a)\tau_R(h)] \\
&= (\tau_{(Qa)^y}(h)) \wedge (\tau_{(aR)^y}(h)) \\
&= \tau_{(QaR)^y}(h)
\end{aligned}$$

and

$$\begin{aligned}
\eta_{(QaR)^y}(g*h*g^{-1}) &= \left[\eta_{(Qa)^y}(g*h*g^{-1})\right] \wedge \left[\eta_{(aR)^y}(g*h*g^{-1})\right] \\
&= \left[\eta_y(a)\eta_Q(g*h*g^{-1})\right] \wedge \left[\eta_y(a)\eta_R(g*h*g^{-1})\right] \\
&= \left[\eta_y(a)\eta_Q((g*h)*g^{-1})\right] \wedge \left[\eta_y(a)\eta_R((g*h)*g^{-1})\right] \\
&= \left[\eta_y(a)\eta_Q((g*h)*g^{-1})\right] \wedge \left[\eta_y(a)\eta_R((g*h)*g^{-1})\right] \\
&= \left[\eta_y(a)\eta_Q(g^{-1}*(g*h))\right] \wedge \left[\eta_y(a)\eta_R(g^{-1}*(g*h))\right] \\
&= \left[\eta_y(a)\eta_Q(g^{-1}*g)*h)\right] \wedge \left[\eta_y(a)\eta_R(g^{-1}*g)*h)\right] \\
&= [\eta_y(a)\eta_Q(h))] \wedge [\eta_y(a)\eta_R(h)] \\
&= (\eta_{(Qa)^y}(h)) \wedge (\eta_{(aR)^y}(h)) \\
&= \eta_{(QaR)^y}(h).
\end{aligned}$$

Hence, $(QaR)^y$ is a PFNSG of G. □

Definition 3.4. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be PFSG of G. Then, for any $a \in G$, pseudo picture fuzzy middle cosets (PPFMC) of Q is a PFS $(aQa^{-1})^y =$*

$(\sigma_{(aQa^{-1})^y},\ \tau_{(aQa^{-1})^y},\ \eta_{(aQa^{-1})^y})$ of G defined by

$$\sigma_{(aQa^{-1})^y}(g) = \sigma_y(a)\sigma_Q(a^{-1} * g * a)\ \sigma_y(a^{-1}),$$
$$\tau_{(aQa^{-1})^y}(g) = \tau_y(a)\tau_Q(a^{-1} * g * a)\ \tau_y(a^{-1})$$

and

$$\eta_{(aQa^{-1})^y}(g) = \eta_y(a)\eta_Q(a^{-1} * g * a)\ \eta_y(a^{-1}),$$

for all $g \in G$.

Proposition 3.7. *Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFNSG of G. Then for every $a \in G$, PPFMC $(aQa^{-1})^y$ is a PFNSG of G.*

Proof. Let $Q = (\sigma_Q, \tau_Q, \eta_Q)$ be a PFNSG of G and $a \in G$, let

$$(aQa^{-1})^y = (\sigma_{(aQa^{-1})^y},\ \tau_{(aQa^{-1})^y},\ \eta_{(aQa^{-1})^y}),$$

where $\sigma_{(aQa^{-1})^y}(g)$, $\tau_{(aQa^{-1})^y}(g)$, and $\eta_{(aQa^{-1})^y}(g)$ are as defined in Definition 3.4 for all $g \in G$. Let $g, h \in G$, then

$$\begin{aligned}
\sigma_{(aQa^{-1})^y}(g * h) &= \sigma_y(a)\sigma_Q(a^{-1} * (g * h) * a)\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q(a^{-1} * (g * h * a))\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q((g * h * a) * a^{-1})\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q((g * h) * (a * a^{-1}))\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q(g * h)\ \sigma_y(a^{-1}) \\
&= \sigma_y(a)\sigma_Q(h * g)\ \sigma_y(a^{-1}) \\
&= \sigma_{(aQa^{-1})^y}(h * g),
\end{aligned}$$

$$\begin{aligned}
\tau_{(aQa^{-1})^y}(g * h) &= \tau_y(a)\tau_Q(a^{-1} * (g * h) * a)\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q(a^{-1} * (g * h * a))\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q((g * h * a) * a^{-1})\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q((g * h) * (a * a^{-1}))\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q(g * h)\ \tau_y(a^{-1}) \\
&= \tau_y(a)\tau_Q(h * g)\ \tau_y(a^{-1}) \\
&= \tau_{(aQa^{-1})^y}(h * g)
\end{aligned}$$

and

$$\begin{aligned}
\eta_{(aQa^{-1})^y}(g * h) &= \eta_y(a)\eta_Q(a^{-1} * (g * h) * a)\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q(a^{-1} * (g * h * a))\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q((g * h * a) * a^{-1})\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q((g * h) * (a * a^{-1}))\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q(g * h)\ \eta_y(a^{-1}) \\
&= \eta_y(a)\eta_Q(h * g)\ \eta_y(a^{-1}) \\
&= \eta_{(aQa^{-1})^y}(h * g).
\end{aligned}$$

Hence, $(aQa^{-1})^y$ is a PFNSG of G. □

Conclusion and future scopes

In this paper, we have extended the concepts of pseudo fuzzy cosets and pseudo intuitionistc fuzzy cosets to pseudo picture fuzzy cosets (PPFCs), and established some of the properties related to pseudo picture fuzzy cosets, pseudo picture fuzzy double cosets (PPFDCs) and pseudo picture fuzzy middle cosets (PPFMCs). Furthermore, the connections between PPFDCs and PFNSG, and PPFMCs and PFNSG were obtained, respectively. In further research, it will be of interest to study the pseudo picture fuzzy cosets in more complicated uncertain environments like spherical fuzzy environment and establish the generalisation of these results.

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Area integral characterizations and Φ-Carleson measures for harmonic Bergman-Orlicz spaces

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Abstract. Let Φ be a growth function. In this paper, we define a harmonic Bergman-Orlicz space $\mathcal{B}_\alpha^\Phi$ and characterize it in terms of area integral functions. Furthermore, we define Φ-Carleson measures and then discuss Φ-Carleson measures for harmonic Bergman-Orlicz spaces.

Keywords: growth function, area integral, Bergman-Orlicz space, Carleson measure.
MSC 2020: 31B05, 31C05, 31C25

1. Introduction

Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$ be two vectors in the n-dimensional real vector space $\mathbb{R}^n$. We write

$$\langle x, y\rangle = x_1y_1 + ... + x_ny_n \quad \text{and} \quad |x| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + ... + x_n^2}.$$

For $a \in \mathbb{R}^n$, let $\mathbb{B}(a, r) = \{x : |x - a| < r\}$, $\mathbb{S}(a, r) = \partial\mathbb{B}(a, r)$ and $\overline{\mathbb{B}(a, r)} = \mathbb{B}(a, r) \cup \mathbb{S}(a, r)$. In particular, let $\mathbb{B} = \mathbb{B}(0, 1)$, $\mathbb{S} = \partial\mathbb{B}(0, 1)$ and $\overline{\mathbb{B}} = \mathbb{B} \cup \mathbb{S}$ the closure of $\mathbb{B}$. We denote by dv the normalized volume measure on $\mathbb{B}$ and $h(\mathbb{B})$ the class of all harmonic functions on $\mathbb{B}$. For each $\alpha > -1$, the weighted normalized volume measure $dv_\alpha(x) = c_\alpha(1 - |x|^2)^\alpha dv(x)$ and c_α is a positive constant so that $v_\alpha(\mathbb{B}) = 1$.

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A function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ is called a *growth function* if it is continuous and non-decreasing. The growth function Φ satisfies the Δ_2-condition if there exists a constant $K > 1$ such that

$$\Phi(2t) \le K\Phi(t), \quad t \in [0,\infty).$$

For $\alpha > -1$ and a growth function Φ satisfying Δ_2-condition, the *Orlicz space* $L^\Phi(\mathbb{B}, dv_\alpha)$ is the set of all measurable functions f such that

$$\|f\|_{\alpha,\Phi} = \int_{\mathbb{B}} \Phi(|f(x)|)dv_\alpha(x) < \infty.$$

The *harmonic Bergman-Orlicz* space $\mathcal{B}^\Phi_\alpha$ is the subspace of $L^\Phi(\mathbb{B}, dv_\alpha)$ consisting of all $f \in h(\mathbb{B})$. The Luxembourg gauge on $\mathcal{B}^\Phi_\alpha$ is defined by

$$\|f\|^{lux}_{\alpha,\Phi} = \inf\{\lambda > 0 : \int_{\mathbb{B}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dv_\alpha(x) \le 1\}.$$

We observe that $\Phi(t) = t^p$, the associated harmonic Bergman-Orlicz space is the classical weighted harmonic Bergman space $\mathcal{B}^p_\alpha$ (cf. [1, 9]).

For $f \in h(\mathbb{B})$, recall that the radial derivative $\mathcal{R}$ of f is given by

$$\mathcal{R}f(x) = x \cdot \nabla f(x) = \frac{\partial}{\partial t}(f(tx))_{t=1} = \sum_{m=1}^{\infty} m f_m(x),$$

where ∇ is the usual gradient and the last form is the homogeneous expansion of f. The fundamental theorem of calculus shows that

$$f(x) - f(0) = \int_0^1 (\mathcal{R}f)(tx)\frac{dt}{t}.$$

For $a \in \mathbb{B}$, we denote by φ_a the Möbius transformation in $\mathbb{B}$. It's an involution of $\mathbb{B}$ such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, which is of the form

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{[x,a]^2}, \ x \in \mathbb{B},$$

where $[x,a] = \sqrt{1 - 2\langle x,a\rangle + |x|^2|a|^2}$.

Let $a \in \mathbb{B}$ and $r \in (0,1)$, the *pseudo-hyperbolic ball* with center a and radius r is denoted by

$$E(a,r) = \{x \in \mathbb{B} : |\varphi_a(x)| < r\}.$$

Indeed, $E(a,r)$ is a Euclidean ball with center c_a and radius r_a given by

$$c_a = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad r_a = \frac{r(1-|a|^2)}{1-|a|^2r^2}, \tag{1}$$

respectively (cf. [16]). It is well known that for $\alpha > -1$ and any $x \in E(a,r)$,

$$1-|a|^2 \approx 1-|x|^2 \approx [a,x] \quad \text{and} \quad v_\alpha(E(a,r)) \approx (1-|a|^2)^{n+\alpha}. \tag{2}$$

For fixed $0 < s < \infty$ and $0 < r < \frac{1}{2}$, we consider the following area integral functions which were introduced by Chen and Ouyang (see [3, 4])

- $A^s_{\mathcal{R}}(f)(x) = \Big(\int_{E(x,r)} |(1-|y|^2)\mathcal{R}f(y)|^s d\tau(y) \Big)^{1/s}$,
- $A^s_{\nabla}(f)(x) = \Big(\int_{E(x,r)} |(1-|y|^2)\nabla f(y)|^s d\tau(y) \Big)^{1/s}$,
- $A^s(f)(x) = \Big(\int_{E(x,r)} |f(y)|^s d\tau(y) \Big)^{1/s}$,

where $d\tau(x) = (1-|x|^2)^{-n} dv(x)$ is the *invariant measure* on $\mathbb{B}$.

Let $\mathbf{B}_n$ be the unit ball of the n-dimensional complex vector space $\mathbb{C}^n$. For $0 < p < \infty$ and $\alpha > -1$, the standard weighted Bergman space $\mathcal{A}^p_\alpha(\mathbf{B}_n)$ consists of all holomorphic functions g on $\mathbf{B}_n$ such that

$$\int_{\mathbf{B}_n} |g(z)|^p dv_\alpha(z) < \infty.$$

It is well known that a holomorphic function $g \in \mathcal{A}^p_\alpha(\mathbf{B}_n)$ if and only if $(1-|z|^2)\nabla g(z) \in L^p(\mathbf{B}_n, dv_\alpha)$. In [18], B. Sehba extended this characterization to the holomorphic Bergman-Orlicz space. By adding the restriction $s > 1$, Chen and Ouyang [3, 4] proved that $g \in \mathcal{A}^p_\alpha(\mathbf{B}_n)$ is equivalent to one (and hence all) of the conditions $A^s_{\mathcal{R}}(g) \in L^p(\mathbf{B}_n, dv_\alpha)$, $A^s_{\nabla}(g) \in L^p(\mathbf{B}_n, dv_\alpha)$, $A^s(g) \in L^p(\mathbf{B}_n, dv_\alpha)$. As a consequence, they obtained some new maximal and area integral characterizations for Besov spaces. For the further discussions on this topic, we refer to [12].

Motivated by [3, 4, 18], our first aim in this paper is to extend Chen and Ouyang's result to the setting of harmonic Bergman-Orlicz space $\mathcal{B}^\Phi_\alpha$. In order to state our results, we need some more definitions on the growth function Φ.

We say that a growth function Φ is of upper type $q \geq 1$ if there exists $C > 0$ such that, for $s > 0$ and $t \geq 1$,

$$\Phi(st) \leq C t^q \Phi(s). \tag{3}$$

Denote by $\mathcal{U}^q$ the set of growth functions Φ of upper type q, (for some $q \geq 1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-decreasing.

We say that Φ is of lower type $p > 0$ if there exists $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$\Phi(st) \leq C t^p \Phi(s). \tag{4}$$

Denote by $\mathcal{L}_p$ the set of growth functions Φ of lower type p, (for some $p \leq 1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-increasing.

Let

$$\mathcal{U} = \bigcup_{q\geq 1} \mathcal{U}^q \qquad and \qquad \mathcal{L} = \bigcup_{0<p\leq 1} \mathcal{L}_p.$$

From the above definitions on Φ, we may always suppose that any $\Phi \in \mathcal{U}$ (resp. $\mathcal{L}$), is convex (resp. concave) and that Φ is a $\mathcal{C}^1$ function with derivative $\Phi'(t) \approx \frac{\Phi(t)}{t}$ (cf. [17, 18]).

Recall that the complementary function Ψ of a convex growth function Φ, is the function defined from $\mathbb{R}_+$ onto itself by

$$\Psi(s) = \sup_{t\in\mathbb{R}_+} \{ts - \Phi(t)\}.$$

A growth function Φ is said to satisfy the ∇_2-*condition* whenever both Φ and its complementary function Ψ satisfy the Δ_2-condition. See [15, 18] for more details on the complementary function Ψ.

Theorem 1.1. *Let $\alpha > -1$, $f \in h(\mathbb{B})$. Assume that Φ is a growth function satisfying one of the following conditions:*

(i) $\Phi \in \mathcal{U}^q$ *and satisfies the* ∇_2*-condition;*

(ii) $\Phi \in \mathcal{L}_p$ *and the function* $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ *satisfies the* ∇_2*-condition.*

Then the following statements are equivalent.

(a) $f \in \mathcal{B}^{\Phi}_{\alpha}$*;*

(b) $A^s_{\mathcal{R}}(f) \in L^{\Phi}(\mathbb{B}, dv_\alpha)$*;*

(c) $A^s_{\nabla}(f) \in L^{\Phi}(\mathbb{B}, dv_\alpha)$*;*

(d) $A^s(f) \in L^{\Phi}(\mathbb{B}, dv_\alpha)$.

For $a \in \mathbb{B} \setminus \{0\}$ and $\delta > 0$, the Carleson cone is defined as

$$\mathcal{C}_\delta(a) = \Big\{x \in \mathbb{B} : \Big|x - \frac{a}{|a|}\Big| < \delta\Big\}.$$

Let μ be a positive Borel measure on $\mathbb{B}$ and $s > 0$. We say that μ is an s-Carleson measure on $\mathbb{B}$ if there exists a constant C such that for any $a \in \mathbb{B} \setminus \{0\}$ and any $0 < \delta < 2$ such that

$$\mu(\mathcal{C}_\delta(a)) \leq C\delta^{(n-1)s}.$$

When $s = 1$, the above measure is called a Carleson measure. Carleson measures were first introduced in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ by Carleson [2]. These measures are pretty adapted to the studies of various questions on function spaces.

Given $0 < p, q < \infty$, the question of the characterization of the positive measures μ on $\mathbf{B}_n$ such that the embedding $I_\mu : \mathcal{A}^p_\alpha(\mathbf{B}_n) \to L^q(\mathbf{B}_n, d\mu)$ is continuous has attracted much attention. In the setting of Bergman spaces of the unit disk $\mathbb{D}$, this question was answered due to Hastings and Luecking [10, 13] by using Carleson measures. For the extensions of these results to the unit ball $\mathbf{B}_n$, see [5, 13, 14]. In [19], Ueki established the boundedness and compactness of composition operators between weighted Bergman spaces in $\mathbf{B}_n$ in terms of s-Carleson measures.

Our second aim of this paper is to investigate the Φ-Carleson measure in the real unit ball $\mathbb{B}$ whose definition is given as follows.

Definition 1.1. *Let Φ be a growth function. A positive Borel measure μ on $\mathbb{B}$ is called a Φ-Carleson measure if there exists a constant $C > 0$ such that for any $a \in \mathbb{B} \setminus \{0\}$ and any $0 < \delta < 2$,*

$$\mu(\mathcal{C}_\delta(a)) \leq \frac{C}{\Phi(\frac{1}{\delta^{n-1}})}.$$

Obviously, when $\Phi(t) = t^s$, the Φ-Carleson measure is the usual s-Carleson measure on $\mathbb{B}$.

The following result provides an equivalent definition of the Φ-Carleson measure.

Theorem 1.2. *Let $\tau > 0$, $\Phi \in \mathcal{U} \cup \mathcal{L}$ and μ be a positive measure on $\mathbb{B}$. Then μ is a Φ-Carleson measure if and only if*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \Phi\Big(\frac{(1-|a|^2)^\tau}{[a,x]^{(n-1)+\tau}}\Big) d\mu(x) < \infty. \tag{5}$$

Let Φ_1, Φ_2 be two growth functions. A positive measure μ on $\mathbb{B}$ is called a Φ_2-*Carleson measure for* $\mathcal{B}^{\Phi_1}_\alpha$ if there is a constant C such that

$$\int_{\mathbb{B}} \Phi_2\Big(\frac{|f(x)|}{C\|f\|^{lux}_{\alpha,\Phi_1}}\Big) d\mu(x) \leq 1,$$

for all $f \in \mathcal{B}^{\Phi_1}_\alpha$ with $\|f\|^{lux}_{\alpha,\Phi_1} \neq 0$.

In our final result, we discuss the Φ-Carleson measure for harmonic Bergman-Orlicz spaces.

Theorem 1.3. *Let $\alpha > -1$, $\Phi_1, \Phi_2 \in \mathcal{U} \cup \mathcal{L}_{(\frac{1}{2})}$ ($\mathcal{L}_{(\frac{1}{2})} = \cup_{\frac{1}{2}<p\leq 1}\mathcal{L}_p$) and μ be a positive measure on $\mathbb{B}$. If Φ_2/Φ_1 is non-decreasing, then the following statements are equivalent.*

(*a*) *There exists a constant $C_1 > 0$ such that for any $a \in \mathbb{B} \setminus \{0\}$ and any $0 < \delta < 1$,*

$$\mu(\mathcal{C}_\delta(a)) \leq \frac{C_1}{\Phi_2 \circ \Phi_1^{-1}(\frac{1}{\delta^{n+\alpha}})}; \tag{6}$$

(*b*) *μ is a Φ_2-Carleson measure for $\mathcal{B}_\alpha^{\Phi_1}$;*

(*c*) *There exists a constant $C_3 > 0$ such that*

$$(7)\quad \sup_{a\in\mathbb{B}}\int_{\mathbb{B}}\Phi_2\Big(\Phi_1^{-1}\big(\frac{1}{(1-|a|^2)^{n+\alpha}}\big)\frac{(1-|a|^2)^{2(n+\alpha)}}{[a,x]^{2(n+\alpha)}}\Big)d\mu(x)\le C_3.$$

The organization of this paper is as follows. In Section 2, some necessary terminologies are introduced and several known results are recalled. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 $\sim$ 1.3. Throughout this paper, we always assume without loss of generality that our growth functions Φ are satisfying $\Phi(1) = 1$. The constants are denoted by C, they are positive and may differ from one occurrence to the other. For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$.

2. Preliminaries

In this section, we introduce notations and collect some preliminary results that we will need later.

2.1 Operators on Orlicz spaces

Let Φ be a $\mathcal{C}^1$ growth function. Recall that the lower and the upper indices of Φ are respectively defined by

$$a_\Phi=\inf_{t>0}\frac{t\Phi'(t)}{\Phi(t)}\quad\text{and}\quad b_\Phi=\sup_{t>0}\frac{t\Phi'(t)}{\Phi(t)}.$$

It is known that when Φ is convex, then $1 \le a_\Phi \le b_\Phi < \infty$ and, if Φ is concave, then $0 \le a_\Phi \le b_\Phi \le 1$. Note that a convex growth function satisfies the ∇_2-condition if and only if $1 < a_\Phi \le b_\Phi < \infty$ (cf. [6], Lemma 2.1).

Definition 2.1. *Let Φ be a growth function. A linear operator T defined on $L^\Phi(\mathbb{B}, dv_\alpha)$ is said to be of mean strong type $(\Phi,\Phi)_\alpha$ if*

$$\int_{\mathbb{B}}\Phi(|Tf|)dv_\alpha(x)\le C\int_{\mathbb{B}}\Phi(|f|)dv_\alpha(x),$$

for any $f \in L^\Phi(\mathbb{B}, dv_\alpha)$, and T is said to be mean weak type $(\Phi,\Phi)_\alpha$ if

$$\sup_{t>0}\Phi(t)v_\alpha(\{x\in\mathbb{B}:|Tf(x)|>t\})\le C\int_{\mathbb{B}}\Phi(|f|)dv_\alpha(x),$$

for any $f \in L^\Phi(\mathbb{B}, dv_\alpha)$, where C is independent of f.

We remark that if $\Phi(t) = t^p$, then the mean strong type $(t^p, t^p)_\alpha$ is the usual strong type (p, p). The following interpolation result comes from [7, Theorem 4.3].

Lemma 2.1. *Let Φ_0, Φ_1 and Φ_2 be three convex growth functions. Sup- pose that their upper and lower indices satisfy the following condition*

$$1 \leq a_{\Phi_0} \leq b_{\Phi_0} < a_{\Phi_2} \leq b_{\Phi_2} < a_{\Phi_1} \leq b_{\Phi_1} < \infty.$$

If T is of mean weak types $(\Phi_0, \Phi_0)_\alpha$ and $(\Phi_1, \Phi_1)_\alpha$, then it is of mean strong type $(\Phi_2, \Phi_2)_\alpha$.

Let $\beta \in \mathbb{R}$ and consider the operator E_β defined for functions f on $\mathbb{B}$ by

$$E_\beta f(x) = \int_{\mathbb{B}} f(y) \frac{(1-|y|^2)^\beta}{[x,y]^{n+\beta}} dv(y).$$

For a proof of the following lemma, see [9, Theorem 1.6].

Lemma 2.2. *Let $1 \leq p < \infty$ and $\alpha, \beta > -1$. The operator $E_\beta : L^p(\mathbb{B}, dv_\alpha) \to L^p(\mathbb{B}, dv_\alpha)$ is bounded if and only if $\alpha + 1 < p(\beta + 1)$.*

Combining Lemmas 2.1 and 2.2, the following result can be easily derived, see [18, Theorem 2.5].

Lemma 2.3. *Let $\alpha, \beta > -1$ and Φ be a $\mathcal{C}^1$ convex growth function with its lower indice a_Φ. If $1 < p < a_\Phi$ and $\alpha + 1 < p(\beta + 1)$, then E_β is of mean strong type $(\Phi, \Phi)_\alpha$.*

2.2 Harmonic functions

It is well-known that the weighted harmonic Bergman spaces $\mathcal{B}^2_\alpha$ for $\alpha > -1$ is a reproducing kernel Hilbert space with reproducing kernel $R_\alpha(x, y)$:

$$f(x) = \int_{\mathbb{B}} f(y) R_\alpha(x, y) dv_\alpha(y), \ f \in \mathcal{B}^2_\alpha. \tag{8}$$

From [7], we know that (8) is also true for all $f \in \mathcal{B}^1_\alpha$.

The reproducing kernels $R_\alpha(x, y)$ can be expressed in terms of zonal harmonics as

$$R_\alpha(x, y) = \sum_{k=0}^{\infty} \frac{(1 + \frac{n}{2} + \alpha)_k}{(\frac{n}{2})_k} Z_k(x, y) = \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x, y),$$

where the series absolutely and uniformly converges on $K \times \mathbb{B}$ for any compact subset K of $\mathbb{B}$ and $(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$. A straightforward computation gives that

$$|R_\alpha(x, y)| \lesssim \frac{1}{[x, y]^{n+\alpha}}. \tag{9}$$

Note that $R_\alpha(x, y)$ is real-valued, symmetric in the variables x and y and harmonic with respect to each variable since the same is true for all $Z_k(x, y)$. For the extension of reproducing kernels $R_\alpha(x, y)$ to all $\alpha \in \mathbb{R}$, see [7, 9].

We recall some useful inequalities concerning harmonic functions which are useful for our investigations.

Lemma 2.4 ([7, 16])**.** *Let $0 < p < \infty$, $0 < r < 1$ and $f, g \in h(\mathbb{B})$. Then there exists some positive constant C such that*

(1) $|f(x)|^p \le C \int_{E(x,r)} |f(y)|^p d\tau(y)$;

(2) $|\nabla f(x)|^p \le \frac{C}{(1-|x|^2)^p} \int_{E(x,r)} |f(y)|^p d\tau(y)$.

Moreover, if $0 < p \le 1$ and $\alpha > -1$, then there exists a positive constant C such that

(3)

$$\int_{\mathbb{B}} |f(x)g(x)|(1-|x|^2)^{(n+\alpha)/p-n} dv(x) \le C\Big(\int_{\mathbb{B}} |f(x)g(x)|^p dv_\alpha(x)\Big)^{1/p}.$$

The following standard estimate will be needed in the sequel.

Lemma 2.5 ([16])**.** *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then for any $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\alpha}{[x,y]^{n+\alpha+\beta}} dv(y) \approx \begin{cases} (1-|x|^2)^{-\beta}, & \beta > 0, \\ \log \dfrac{1}{1-|x|^2}, & \beta = 0, \\ 1, & \beta < 0. \end{cases}$$

3. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. Before the proof, we need some preparation.

Lemma 3.1 ([8])**.** *Let $\Phi \in \mathcal{L}_p$. Then the growth function Φ_p, defined by $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$ for some $q \ge 1$. Moreover, for $s > 0$ and $t \ge 1$,*

$$\Phi_p(ts) \le t^{\frac{1}{p}} \Phi_p(s).$$

By Lemmas 2.4 and Lemma 3.1, we can obtain the following useful integral estimates.

Lemma 3.2. *Let $f \in h(\mathbb{B})$ and $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$. Then for $0 < r < 1$ and $x \in \mathbb{B}$,*

(1) $\Phi\big((1-|x|^2)|\nabla f(x)|\big) \lesssim \int_{E(x,r)} \Phi(|f(y))|d\tau(y)$;

(2) $\Phi(|f(x)|) \lesssim \int_{E(x,r)} \Phi(|f(y)|)d\tau(y)$.

Proof. Let

$$p_\Phi = \begin{cases} 1, & \text{if} \quad \Phi \in \mathcal{U}^q, \\ p, & \text{if} \quad \Phi \in \mathcal{L}_p. \end{cases}$$

By Lemma 2.4, for each $x \in \mathbb{B}$,

$$\big((1-|x|^2)|\nabla f(x)|\big)^{p_\Phi} \lesssim \int_{E(x,r)} |f(y)|^{p_\Phi} d\tau(y).$$

Set

$$\Phi_p(t) = \begin{cases} \Phi(t), & \text{if } \Phi \in \mathcal{U}^q, \\ \Phi(t^{\frac{1}{p}}), & \text{if } \Phi \in \mathcal{L}_p. \end{cases}$$

It follows from Lemma 3.1 and the convexity of $\Phi_p(t)$ that

$$\Phi((1-|x|^2)|\nabla f(x)|) \lesssim \int_{E(x,r)} \Phi(|f(y)|) d\tau(y).$$

This proves (1).

By Lemma 2.4 and an argument similar to the above, the assertion of (2) follows. □

Lemma 3.3. *Assume that Φ is a growth function satisfying one of the following conditions:*

(*i*) $\Phi \in \mathcal{U}^q$ *and satisfies the ∇_2-condition;*

(*ii*) $\Phi \in \mathcal{L}_p$ *and the function $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ satisfies the ∇_2-condition.*

If $\alpha > -1$ and $f \in h(\mathbb{B})$, then

$$\int_{\mathbb{B}} \Phi(|f(x)-f(0)|) dv_\alpha(x) \lesssim \int_{\mathbb{B}} \Phi((1-|x|^2)|\mathcal{R}f(x)|) dv_\alpha(x); \tag{10}$$

and

$$\int_{\mathbb{B}} \Phi((1-|x|^2)|\nabla f(x)|) dv_\alpha(x) \lesssim \int_{\mathbb{B}} \Phi(|f(x)|) dv_\alpha(x). \tag{11}$$

Proof. We first prove (10). Let $f \in h(\mathbb{B})$. Then for $s > -1$,

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y) R_s(x,y) dv_s(y).$$

Since $\int_{\mathbb{B}} \mathcal{R}f(y) dv_s(y) = 0$, subtracting this from the previous equation yields

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(x,y)-1) dv_s(y).$$

Consequently,

$$\begin{aligned} |f(x)-f(0)| &= \Big| \int_0^1 \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(tx,y)-1) dv_s(y) \frac{dt}{t} \Big| \\ &= \Big| \int_{\mathbb{B}} \mathcal{R}f(y) \int_0^1 \frac{R_s(tx,y)-1}{t} dt dv_s(y) \Big|. \end{aligned}$$

Set

$$G(x,y)=\int_0^1 \frac{R_s(tx,y)-1}{t}dt.$$

From the proof of [9, Lemma12.1], it deduces that

$$|G(x,y)|\le\int_0^1\Big|\frac{R_s(tx,y)-1}{t}\Big|dt\lesssim\int_0^1\frac{dt}{[tx,y]^{n+s}}\lesssim\frac{1}{[x,y]^{n+s-1}}.$$

Therefore,

$$|f(x)-f(0)|\quad\lesssim\quad\int_{\mathbb{B}}(1-|y|^2)|\mathcal{R}f(y)|\frac{1}{[x,y]^{n+s-1}}dv_{s-1}(y).$$

We first consider the case Φ satisfies the condition (i) of the lemma. Fix p so that $1<p<a_\Phi$. By taking s large enough so that $\alpha+1<ps$, we conclude from Lemma 2.3 that

$$\int_{\mathbb{B}}\Phi(|f(x)-f(0)|)dv_\alpha(x)\lesssim\int_{\mathbb{B}}\Phi((1-|x|^2)|\mathcal{R}f(x)|)dv_\alpha(x).$$

We next consider the case of $\Phi\in\mathcal{L}_p$ and $\Phi_p(t)=\Phi(t^{\frac{1}{p}})$ satisfies the ∇_2-condition. Set $s=(n+\alpha')/p-n$ and $\alpha'>\alpha+p$. By Lemma 2.4, it deduces that

$$\begin{aligned}|f(x)-f(0)|^p\quad&\lesssim\quad\int_{\mathbb{B}}|\mathcal{R}f(y)|^p|G(x,y)|^p dv_{\alpha'}(y)\\&\lesssim\quad\int_{\mathbb{B}}\frac{|\mathcal{R}f(y)|^p}{[x,y]^{p(n+s-1)}}dv_{\alpha'}(y)\\&\lesssim\quad\int_{\mathbb{B}}\frac{|(1-|y|^2)\mathcal{R}f(y)|^p}{[x,y]^{n+\alpha'-p}}dv_{\alpha'-p}(y).\end{aligned}$$

As the growth function $t\to\Phi_p(t)=\Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$ and satisfies the ∇_2-condition, proceeding as in the first part of this proof yields that

$$\begin{aligned}\int_{\mathbb{B}}\Phi(|f(x)-f(0)|)dv_\alpha(x)\quad&=\quad\int_{\mathbb{B}}\Phi_p(|f(x)-f(0)|^p)dv_\alpha(x)\\&\lesssim\quad\int_{\mathbb{B}}\Phi_p((1-|x|^2)|\mathcal{R}f(x)|)^p)dv_\alpha(x)\\&=\quad\int_{\mathbb{B}}\Phi((1-|x|^2)|\mathcal{R}f(x)|)dv_\alpha(x).\end{aligned}$$

We now come to prove (11). By Lemma 3.2, we have

$$\Phi((1-|x|^2)|\nabla f(x)|)\lesssim\int_{E(x,r)}\Phi(|f(y)|)d\tau(y),\quad x\in\mathbb{B}.$$

Integrating both sides of the above inequality over $\mathbb{B}$ with respect to $dv_\alpha(x)$ and applying Fubini's theorem, we get

$$\int_{\mathbb{B}} \Phi((1-|x|^2)|\nabla f(x)|)dv_\alpha(x) \lesssim \int_{\mathbb{B}} \Phi(|f(x)|)dv_\alpha(x).$$

This completes the proof. □

Proof of Theorem 1.1. We only prove $(a) \Leftrightarrow (b)$. Similar discussions can be applied to prove $(a) \Leftrightarrow (c)$ and $(a) \Leftrightarrow (d)$.

We first assume that $A^s_{\mathcal{R}}(f) \in L^\Phi(\mathbb{B}, dv_\alpha)$. By Lemma 2.4, for each $x \in \mathbb{B}$, we have

$$|(1-|x|^2)\mathcal{R}f(x)| \lesssim A^s_{\mathcal{R}}(f)(x).$$

Then $(b) \Rightarrow (a)$ follows from Lemma 3.3.

For the converse, we assume that $f \in \mathcal{B}^\Phi_\alpha$. For each fixed $x \in \mathbb{B}$, let

$$h(x) = \sup\{(1-|\zeta|^2)|\mathcal{R}f(\zeta)| : \zeta \in E(x, \frac{1}{2})\}.$$

From (1), we can find r' such that $0 < \frac{1}{2} < r' < 1$ and $E(\xi, \frac{1}{2}) \subset E(x, r')$ for every $\xi \in E(x, \frac{1}{2})$. It follows from Lemma 3.2 that

$$\Phi\big(|A^s_{\mathcal{R}}(f)(x)\big)| \lesssim \Phi(h(x)) \lesssim \int_{E(x,r')} \Phi(|f(y)|)d\tau(y)$$

Hence by Fubini's theorem and (2),

$$\begin{aligned}\int_{\mathbb{B}} \Phi\big(|A^s_{\mathcal{R}}(f)(x)\big)|dv_\alpha(x) &\lesssim \int_{\mathbb{B}} (1-|x|^2)^\alpha \int_{E(x,r')} \Phi(|f(y)|)d\tau(y)dv(x)\\ &\lesssim \int_{\mathbb{B}} \Phi(|f(y)|)d\tau(y) \int_{E(y,r')} (1-|x|^2)^\alpha dv(x)\\ &\lesssim \int_{\mathbb{B}} \Phi(|f(y)|)dv_\alpha(y).\end{aligned}$$

This completes the proof. □

4. Proofs of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Assume first that (5) holds. For each $a \in \mathbb{B} \setminus \{0\}$, set $\delta = 1 - |a|$. A simple computation gives that

$$[a, x] \leq 1 - |a|^2 \leq 2\delta,$$

for $x \in \mathcal{C}_\delta(a)$. Therefore

$$\begin{aligned} \mu(\mathcal{C}_\delta(a))\Phi(\frac{1}{\delta^{n-1}}) &= \int_{\mathcal{C}_\delta(a)} \Phi(\frac{1}{\delta^{n-1}})d\mu(x) \\ &\lesssim \int_{\mathcal{C}_\delta(a)} \Phi\Big(\frac{2^{n-1}}{[a,x]^{n-1}}\Big)d\mu(x) \\ &\lesssim \int_{\mathcal{C}_\delta(a)} \Phi\Big(\frac{2^{n-1}(1-|a|^2)^\tau}{[a,x]^{n-1+\tau}}\Big)d\mu(x) \\ &\lesssim \int_{\mathbb{B}} \Phi\Big(\frac{(1-|a|^2)^\tau}{[a,x]^{(n-1)+\tau}}\Big)d\mu(x), \end{aligned}$$

where the last inequality follows from the monotonicity of Φ or $\frac{\Phi(t)}{t}$.

Conversely, assume that μ is a Φ-Carleson measure. The proof is based on a standard slicing trick, see [11, Lemma 2.2]. Without loss of generality, let $\frac{1}{2} < |a| < 1$. Denote $Q_0(a) = \emptyset$ and

$$Q_k(a) = \Big\{x \in \mathbb{B} : \Big|x - \frac{a}{|a|}\Big| < 2^{k-1}(1-|a|)\Big\}, \ k = 1, 2, ..., N,$$

where N is the smallest integer such that $2^{N-1}(1-|a|) \geq 2$.

Since for each $x \in Q_k(a)\backslash Q_{k-1}(a)$, $[a,x] \geq |a|2^{(k-2)}(1-|a|)$, we have

$$\begin{aligned} &\int_{\mathbb{B}} \Phi\Big(\frac{(1-|a|^2)^\tau}{[a,x]^{(n-1)+\tau}}\Big)d\mu(x) \\ &\lesssim \sum_{k=1}^{N} \int_{Q_k(a)\backslash Q_{k-1}(a)} \Phi\Big(\frac{(1-|a|^2)^\tau}{2^{(k-2)(n-1+\tau)}(1-|a|)^{(n-1)+\tau}}\Big)d\mu(x) \\ &\lesssim \sum_{k=1}^{N} \frac{\Phi\Big(\frac{1}{2^{(k-2)(n-1+\tau)}(1-|a|)^{n-1}}\Big)}{\Phi\Big(\frac{1}{2^{(k-1)(n-1)}(1-|a|)^{n-1}}\Big)} \\ &\lesssim \sum_{k=1}^{N} \frac{1}{2^{k\tau\varsigma}} < \infty, \end{aligned}$$

where $\varsigma = 1$ if $\Phi \in \mathcal{U}$ and $\varsigma = p$ if $\Phi \in \mathcal{L}$ is of lower type $0 < p \leq 1$. The proof is complete.

In order to prove Theorem 1.3, we need the following two lemmas.

Lemma 4.1. *Let $\alpha > -1$, $\Phi \in \mathcal{U} \cup \mathcal{L}$ and $f \in \mathcal{B}_\alpha^\Phi$. Then there exists a positive constant C such that for each $a \in \mathbb{B}$,*

$$|f(a)| \leq C\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)\|f\|_{\alpha,\Phi}^{lux}. \tag{12}$$

Proof. If $\|f\|_{\alpha,\Phi}^{lux} = 0$, then $f = 0$ a.e. on $\mathbb{B}$ so that (12) obviously holds. Suppose that $\|f\|_{\alpha,\Phi}^{lux} \neq 0$. In view of (2) and Lemma 2.4, we see that for $a \in \mathbb{B}$

and $0 < p < \infty$,

$$|f(a)|^p \lesssim \int_{E(a,r)} |f(x)|^p \Big(\frac{(1-|a|^2)}{[x,a]^2}\Big)^{n+\alpha} dv_\alpha(x).$$

It follows a similar discussion in the proof of Lemma 3.2,

$$\begin{aligned}\Phi\Big(\frac{|f(a)|}{\|f\|^{lux}_{\alpha,\Phi}}\Big) &\lesssim \int_{E(a,r)} \Phi\Big(\frac{|f(x)|}{\|f\|^{lux}_{\alpha,\Phi}}\Big)\Big(\frac{(1-|a|^2)}{[x,a]^2}\Big)^{n+\alpha} dv_\alpha(x)\\ &\lesssim \frac{1}{(1-|a|^2)^{n+\alpha}},\end{aligned}$$

which gives (12). □

Lemma 4.2. *Let $\alpha > -1$, $\frac{1}{2} < p \le 1$ and $\Phi \in \mathcal{U} \cup \mathcal{L}_p$. Then each $a \in \mathbb{B}$, the following function*

$$f_a(x) = \Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) R_{n+2\alpha}(x,a)(1-|a|^2)^{2(n+\alpha)}$$

belongs to $\mathcal{B}^\Phi_\alpha$.

Proof. Let

$$h_a(x) = \frac{(1-|a|^2)^{2(n+\alpha)}}{[x,a]^{2(n+\alpha)}}.$$

Since $\alpha > -1$, from (8),

$$\begin{aligned}&\int_{\mathbb{B}} \Phi(|f_a(x)|) dv_\alpha(x)\\ &= \int_{\mathbb{B}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)|R_{n+2\alpha}(x,a)|(1-|a|^2)^{2(n+\alpha)}\Big) dv_\alpha(x)\\ &\lesssim \int_{\mathbb{B}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) h_a(x)\Big) dv_\alpha(x)\\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \int_{\{x\in\mathbb{B}: h_a(x)\le 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) h_a(x)\Big) dv_\alpha(x)$$

and

$$I_2 = \int_{\{x\in\mathbb{B}: h_a(x)\ge 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) h_a(x)\Big) dv_\alpha(x).$$

We now divide the remainder of the proof into the following two cases.

Case I. $\Phi \in \mathcal{U}$. By the monotonicity of $\frac{\Phi(t)}{t}$ and Lemma 2.5,

$$\begin{aligned} I_1 &\lesssim \int_{\{x\in\mathbb{B}:h_a(x)\le 1\}} h_a(x)\Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)\Big)dv_\alpha(x) \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(n+\alpha)}}{[x,a]^{2(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

Using (3), there exists some $q \ge 1$ such that

$$\begin{aligned} I_2 &= \int_{\{x\in\mathbb{B}:h_a(x)\ge 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)h_a(x)\Big)dv_\alpha(x). \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(2q-1)(n+\alpha)}}{[x,a]^{2q(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

Case II. $\Phi \in \mathcal{L}_p$ with $p > \frac{1}{2}$. Using (4) and Lemma 2.5, we have

$$\begin{aligned} I_1 &\lesssim \int_{\{x\in\mathbb{B}:h_a(x)\le 1\}} h_a(x)^p\Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)\Big)dv_\alpha(x) \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(2p-1)(n+\alpha)}}{[x,a]^{2p(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

By the monotonicity of $\frac{\Phi(t)}{t}$ and Lemma 2.5 again,

$$\begin{aligned} I_2 &= \int_{\{x\in\mathbb{B}:h_a(x)\ge 1\}} \Phi\Big(\Phi^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big)h_a(x)\Big)dv_\alpha(x) \\ &\lesssim \int_{\mathbb{B}} \frac{(1-|a|^2)^{(n+\alpha)}}{[x,a]^{2(n+\alpha)}}dv_\alpha(x) \lesssim 1. \end{aligned}$$

Combining the above two cases, the assertion of this lemma follows. □

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. The proof will follow by the routes $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

We first prove $(a) \Rightarrow (b)$. For $y \in \mathbb{B}\setminus\{0\}$ and $\frac{1}{4} < r < 1$. By (1) and (2), we see that for large enough k, $E(y,r) \subset Q_k(y)$ and

$$\mu(E(y,r)) \le \mu(Q_k(y)) \lesssim \frac{1}{\Phi_2\circ\Phi_1^{-1}\big(\frac{1}{2^{(k-1)(n+\alpha)}(1-|y|)^{n+\alpha}}\big)}. \tag{13}$$

Let $f \in \mathcal{B}_\alpha^{\Phi_1}$ with $\|f\|_{\alpha,\Phi_1}^{lux} \neq 0$. Note that $\Phi_2 \in \mathcal{U}\cup\mathcal{L}_{(\frac{1}{2})}$, then

$$\Phi_2\Big(\frac{|f(x)|}{\|f\|_{\alpha,\Phi_1}^{lux}}\Big) \lesssim \int_{E(x,\frac{1}{4})} \Phi_2\Big(\frac{|f(y)|}{\|f\|_{\alpha,\Phi_1}^{lux}}\Big)(1-|y|^2)^{-(n+\alpha)}dv_\alpha(y)$$

by Lemma 3.2. Thus

$$
\begin{aligned}
L &= \int_{\mathbb{B}} \Phi_2\Big(\frac{|f(x)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big) d\mu(x) \\
&\lesssim \int_{\mathbb{B}} d\mu(x) \int_{E(x,\frac{1}{4})} \Phi_2\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big)(1-|y|^2)^{-(n+\alpha)} dv_\alpha(y) \\
&\lesssim \int_{\mathbb{B}} \Big(\int_{\mathbb{B}} \chi_{E(y,\frac{1}{4})}(x) d\mu(x) \Big) \Phi_2\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big)(1-|y|^2)^{-n} dv(y).
\end{aligned}
$$

From (1), we can find an integer k such that and $E(x,\frac{1}{4}) \subset Q_k(y)$ for every $x \in E(y,\frac{1}{4})$. It follows from Lemma 3.2 and (13) that

$$
L \lesssim \int_{\mathbb{B}} \Phi_2\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big)\mu(Q_k(y))(1-|y|^2)^{-n} dv(y).
$$

By the assumption Φ_2/Φ_1 is non-decreasing and (12),

$$
\begin{aligned}
L &\lesssim \int_{\mathbb{B}} \Phi_1\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big) \frac{\Phi_2 \circ \Phi_1^{-1}\big(\frac{1}{(1-|y|^2)^{n+\alpha}}\big)}{\Phi_1 \circ \Phi_1^{-1}\big(\frac{1}{(1-|y|^2)^{n+\alpha}}\big)} (1-|y|^2)^{-n} \mu(Q_k(y)) dv(y) \\
&\lesssim \int_{\mathbb{B}} \Phi_1\Big(\frac{|f(y)|}{\|f\|^{lux}_{\alpha,\Phi_1}}\Big) dv_\alpha(y) \le 1.
\end{aligned}
$$

This implies that we can find a constant $C_2 > 0$ such that

$$
\int_{\mathbb{B}} \Phi_2\Big(\frac{|f(x)|}{C_2\|f\|^{lux}_{\alpha,\Phi_1}}\Big) d\mu(x) \le 1.
$$

$(b) \Rightarrow (c)$. For $a \in \mathbb{B}$, recall that

$$
f_a(x) = \Phi_1^{-1}\Big(\frac{1}{(1-|a|^2)^{n+\alpha}}\Big) R_{n+2\alpha}(x,a)(1-|a|^2)^{2(n+\alpha)} \in \mathcal{B}^{\Phi_1}_\alpha
$$

from Lemma 4.2. Thus, the implication easily follows by testing f_a and using the monotonicity of Φ_2 or the monotonicity of the function $\frac{\Phi_2(t)}{t}$.
$(c) \Rightarrow (a)$. The implication $(c) \Rightarrow (a)$ follows the same way as in the proof of Theorem 1.2. We omit the details here. □

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Mycielskian of signed graphs

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Abstract. In this paper, we define the Mycielskian of a signed graph and discuss the properties of balance and switching in the Mycielskian of a given signed graph. We provide a condition for ensuring the Mycielskian of a balanced signed graph remains balanced, leading to the construction of a balanced Mycielskian. We establish a relation between the chromatic numbers of a signed graph and its Mycielskian. We also study the structure of different matrices related to the Mycielskian of a signed graph.

Keywords: signed graph, signed graph coloring, Mycielskian of a signed graph.
MSC 2020: 05C15, 05C22

1. Introduction

A signed graph $\Sigma = (G, \sigma)$ consists of an underlying graph $G = (V, E)$, together with a function $\sigma : E \to \{-1, 1\}$, called the signature or sign function. The sign of a cycle C in Σ, denoted by $\sigma(C)$, is defined as the product of the signs of its edges, and the cycle C is said to be positive if $\sigma(C) = +1$. A signed graph Σ is said to be balanced if every cycle in it is positive, otherwise, Σ is unbalanced. A signed graph is called all-positive (all-negative) if all the edges are positive (negative).

A switching function for Σ is a function $\zeta : V(\Sigma) \to \{-1, 1\}$. For an edge $e = uv$ in Σ, the switched signature σ^ζ is defined as $\sigma^\zeta(e) = \zeta(u)\sigma(e)\zeta(v)$, and the switched signed graph is $\Sigma^\zeta = (G, \sigma^\zeta)$. The signs of cycles are unchanged by switching, and any balanced signed graph can be switched to an all-positive signed graph. If one signed graph can be switched from the other, they are said to be switching equivalent (see, [8, Section 3]).

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The net-degree of a vertex v in a signed graph Σ, denoted by $d^{\pm}_{\Sigma}(v)$ is defined as $d^{\pm}_{\Sigma}(v) = d^{+}_{\Sigma}(v) - d^{-}_{\Sigma}(v)$, where $d^{+}_{\Sigma}(v)$ and $d^{-}_{\Sigma}(v)$ respectively denotes the number of positive and negative edges incident with v in Σ. The total number of edges incident with v in Σ is denoted by $d_{\Sigma}(v)$ and $d_{\Sigma}(v) = d^{+}_{\Sigma}(v) + d^{-}_{\Sigma}(v)$.

Throughout this paper, we consider only finite, simple, connected, and undirected graphs and signed graphs. For the standard notation and terminology in graphs and signed graphs not given here, the reader may refer to [3, 9, 12].

The Mycielski construction of a simple graph was introduced by J. Mycielski [7] in his search for triangle-free graphs with arbitrarily large chromatic number. The Mycielskian for a finite, simple, connected graph $G = (V, E)$ is defined as follows.

Definition 1.1 ([1]). *The Mycielskian $M(G)$ of G is a graph whose vertex set is the disjoint union $V \cup V' \cup \{w\}$, where $V' = \{v' : v \in V\}$, and whose edge set is $E \cup \{u'v : uv \in E\} \cup \{v'w : v' \in V'\}$. The vertex w is called the root of $M(G)$ and $v' \in V'$ is called the twin of v in $M(G)$.*

The Mycielski construction is useful in various applications, including the study of planar graphs and coloring problems, as triangle-free graphs have unique properties and often behave differently from graphs with triangles. When it comes to signed graphs, triangle-free signed graphs are even more important, as recent studies indicate that the negative triangles affects the balance of a signed graph more than other negative cycles.

1.1 Mycielskian of signed graphs

Motivated from the Definition 1.1, we define the Mycielskian $M(\Sigma)$ of the signed graph Σ as follows.

Definition 1.2 (Mycielskian). *The Mycielskian of Σ is the signed graph $M(\Sigma) = (M(G), \sigma_M)$, where $M(G)$ is the Mycielskian of the underlying graph G of Σ, and the signature function σ_M is defined as $\sigma_M(uv) = \sigma_M(u'v) = \sigma(uv)$ and $\sigma_M(v'w) = 1$*

The following are some immediate observations.

Observation 1.1. Let Σ be a signed graph with p vertices and q edges and let $M(\Sigma)$ be its Mycielskian. Then, we have the following.

(i) $M(\Sigma)$ has $2p + 1$ vertices and $3q + p$ edges.

(ii) If Σ contains r positive edges and $q-r$ negative edges, then $M(\Sigma)$ contains $3r + p$ positive edges and $3(q - r)$ negative edges.

(iii) If Σ is triangle-free, then $M(\Sigma)$ is also triangle-free.

(iv) For each vertex $v \in V$, $d^{\pm}_{M(\Sigma)}(v) = 2d^{\pm}_{\Sigma}(v)$ and $d_{M(\Sigma)}(v) = 2d_{\Sigma}(v)$.

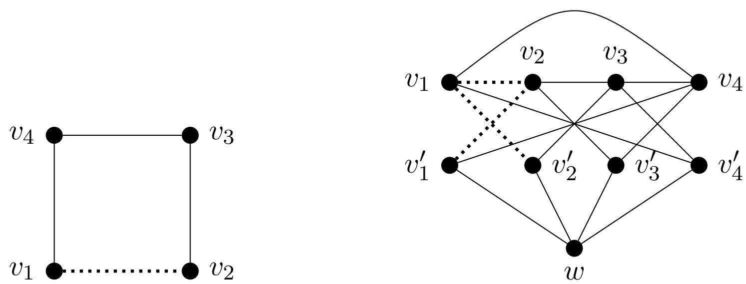
(v) For each vertex $v' \in V'$, $d^{\pm}_{M(\Sigma)}(v') = d^{\pm}_{\Sigma}(v) + 1$ and $d_{M(\Sigma)}(v') = d_{\Sigma}(v) + 1$.

(vi) $d^{\pm}_{M(\Sigma)}(w) = d_{M(\Sigma)}(w) = p$.

Note that, one can define the signature function for the Mycielskian of a signed graph in other ways. In this paper, we initiate a study on Mycielskian of a signed graph using this particular definition.

This particular construction of Mycielskian of a signed graph is illustrated in Example 1.1.

Example 1.1. Let Σ be the negative cycle C_4^-. The Mycielskian of C_4^- is constructed in Figure 1b.



(a) Σ (b) $M(\Sigma)$

Figure 1: A signed graph and its Mycielskian.

2. Balance and switching in Mycielskian of signed graphs

Balance and switching are two important concepts in signed graph theory.

In this section, we establish how the signed graph and its Mycielskian are related with respect to balance and switching. One may note that if Σ is unbalanced, then $M(\Sigma)$ is unbalanced. Also, in general, for a balanced signed graph Σ, the Mycielskian $M(\Sigma)$ need not be balanced.

The following is a characterization for $M(\Sigma)$ to be balanced.

Proposition 2.1. *The Mycielskian $M(\Sigma)$ is balanced if and only if Σ is all-positive.*

Proof. If Σ is all-positive, then so is $M(\Sigma)$, and hence is balanced. Conversely, If Σ has at least one negative edge, say v_iv_j, then $v_iv_jv_i'wv_j'v_i$ forms a negative 5 - cycle in $M(\Sigma)$, making it unbalanced. □

Consider any balanced signed graph Σ which is not all-positive. Then, Σ can be switched to an all-positive signed graph, say Σ'. By Proposition 2.1, $M(\Sigma)$ is not balanced, but $M(\Sigma')$ is balanced. Hence, the Mycielskians of two switching equivalent signed graphs need not to be switching equivalent.

The Mycielskian of an unbalanced signed graph is always unbalanced. However, for a balanced signed graph Σ, the Mycielskian $M(\Sigma) = (M(G), \sigma_M)$ can be made balanced by modifying the signature function σ_M. Though there are several ways to do so, to remain consistent with our original definition, we only look for changes that can be made in the signature of the edges incident to the root vertex w which makes the Mycielskian balanced, and leave the signatures of the other edges unchanged.

We need the following theorem [4].

Theorem 2.1 (Harary's bipartition theorem [4])**.** *A signed graph Σ is balanced if and only if there is a bipartition of its vertex set, $V = V_1 \cup V_2$, such that every positive edge is induced by V_1 or V_2 while every negative edge has one endpoint in V_1 and one in V_2. The bipartition $V = V_1 \cup V_2$ is called a Harary bipartition for Σ.*

Note that, if $V = V_1 \cup V_2$ is a Harary bipartition for Σ, then every path in Σ joining vertices in V_1 (similarly V_2) is positive, and every path between V_1 and V_2 is negative.

Theorem 2.2 provides a method to construct a balanced Mycielskian signed graph from a balanced signed graph.

Theorem 2.2. *Let Σ be a balanced signed graph and $M(\Sigma) = (M(G), \sigma_M)$ be its Mycielskian. If σ'_M is a signature function satisfying $\sigma'_M = \sigma_M$ on $M(G)\backslash\{w\}$ and satisfies the relation $\sigma'_M(v'_i w)\sigma'_M(v'_j w) = \sigma(v_i v_j)$ for every edge $v_i v_j$ in Σ, then the signed graph $M'(\Sigma) = (M(G), \sigma'_M)$ is balanced.*

Proof. Since Σ is balanced, by Harary bipartition theorem, there exists a bipartition $V = V_1 \cup V_2$ of V such that every negative edge in Σ has its one end vertex in V_1 and the other in V_2. We construct a Harary bipartition for $M'(\Sigma)$ as follows.

Let $V'_1 = \{v'_i : v_i \in V_1\}$ and $V'_2 = \{v'_i : v_i \in V_2\}$ be the subsets of V' corresponding to the subsets V_1 and V_2 of V. Since $V = V_1 \cup V_2$, we have $V' = V'_1 \cup V'_2$. Now, every edge with both its end vertices in V_1 is positive and no vertices in V'_1 are adjacent. Also, for edges of the form $v_i v'_j$, where $v_i \in V_1$ and $v'_j \in V'_1$, we have, $\sigma'_M(v_i v'_j) = \sigma_M(v_i v'_j) = \sigma(v_i v_j) = +1$. Thus, every edge with both its end vertices in $V_1 \cup V'_1$ is positive. Similarly, every edge with both its end vertices in $V_2 \cup V'_2$ is positive. Finally, consider any edge e having one end vertex in $V_1 \cup V'_1$ and the other in $V_2 \cup V'_2$. Without loss of generality, we can assume that $e = v_i v_j$, where $v_i \in V_1$ and $v_j \in V_2$. Then, $\sigma'_M(e) = \sigma_M(e) = \sigma_M(v_i v_j) = \sigma(v_i v_j) = -1$. Hence, every edge joining $V_1 \cup V'_1$ and $V_2 \cup V'_2$ is negative.

We now claim that *if $\sigma'_M(v'_k w)$ is positive for some $v_k \in V_1$, then $\sigma'_M(v'_i w)$ is positive for all $v_i \in V_1$ and $\sigma'_M(v'_j w)$ is negative for all $v_j \in V_2$*. To prove the claim, we first observe that if the relation $\sigma'_M(v'_iw)\sigma'_M(v'_jw) = \sigma(v_iv_j)$ holds for every edge v_iv_j in Σ, then for any path $P_{v_iv_j}$ joining the vertices v_i and v_j in Σ, the sign $\sigma(P_{v_iv_j})$ satisfies the relation $\sigma'_M(v'_iw)\sigma'_M(v'_jw) = \sigma(P_{v_iv_j})$. To prove this, consider a $v_i - v_j$ path, say $P_{v_iv_j} = v_iv_{i+1}v_{i+2}\cdots v_{j-1}v_j$, in Σ. Then, we have

$$\begin{aligned}\sigma(P_{v_iv_j}) &= \sigma(v_iv_{i+1}v_{i+2}\cdots v_{j-1}v_j)\\ &= \sigma(v_iv_{i+1})\sigma(v_{i+1}v_{i+2})\cdots\sigma(v_{j-1}v_j)\\ &= (\sigma'_M(v'_iw)\sigma'_M(v'_{i+1}w))(\sigma'_M(v'_{i+1}w)\sigma'_M(v'_{i+2}w))\cdots(\sigma'_M(v'_{j-1}w)\sigma'_M(v'_jw))\\ &= \sigma'_M(v'_iw)(\sigma'_M(v'_{i+1}w)\sigma'_M(v'_{i+2}w)\cdots\sigma'_M(v'_{j-1}w))^2\sigma'_M(v'_jw))\\ &= \sigma'_M(v'_iw)\sigma'_M(v'_jw).\end{aligned}$$

Now, consider $v_k \in V_1$ and let $v_i \in V_1$ and $v_j \in V_2$ be arbitrary. Then, every $v_i - v_k$ path is positive (i.e., $\sigma(P_{v_iv_k}) = +1$) and every $v_j - v_k$ path is negative (i.e., $\sigma(P_{v_jv_k}) = -1$). The connectedness of Σ guarantees the existence of such paths. Now, $\sigma'_M(v'_iw)\sigma'_M(v'_kw) = \sigma(P_{v_iv_k}) = +1$. Thus, $\sigma'_M(v'_iw)$ and $\sigma'_M(v'_kw)$ must have the same sign. Similarly, since $\sigma'_M(v'_jw)\sigma'_M(v'_kw) = \sigma(P_{v_jv_k}) = -1$, $\sigma'_M(v'_jw)$ and $\sigma'_M(v'_kw)$ are of the opposite sign. Thus, if $\sigma'_M(v'_kw)$ is positive for some $v_k \in V_1$, then $\sigma'_M(v'_iw)$ is positive for all $v_i \in V_1$ and $\sigma'_M(v'_jw)$ is negative for all $v_j \in V_2$. Hence, the claim is proved.

Now, consider the edges v'_iw, where $v'_i \in V'_1 \cup V'_2$. Because of the claim, if $\sigma'_M(v'_kw)$ is positive for some $v_k \in V_1$, then $\sigma'_M(v'_iw)$ is positive for all $v_i \in V_1$ and $\sigma'_M(v'_jw)$ is negative for all $v_j \in V_2$. In this case, let $(V_M)_1 = V_1 \cup V'_1 \cup \{w\}$ and $(V_M)_2 = V_2 \cup V'_2$. Similarly, if $\sigma'_M(v'_kw)$ is negative for some $v_k \in V_1$, then $\sigma'_M(v'_iw)$ is negative for all $v_i \in V_1$ and $\sigma'_M(v'_jw)$ is positive for all $v_j \in V_2$. In this case, let $(V_M)_1 = V_1 \cup V'_1$ and $(V_M)_2 = V_2 \cup V'_2 \cup \{w\}$.

Thus, in either case, $V_M = (V_M)_1 \cup (V_M)_2$ forms a Harary bipartition for $M'(\Sigma)$, and hence $M'(\Sigma)$ is balanced. □

Remark 2.1. One may note that σ'_M is a different signature on $M(G)$ that coincides with σ_M on $M(G)\backslash\{w\}$. The signature function σ'_M for the remaining edges v'_iw of $M(G)$ has to be defined using the relation stated in Theorem 2.2. One such construction is discussed in Section 2.1.

It is also worth noting that if $\sigma'_M = \sigma_M$ on $M(G)$, then Theorem 2.2 reduces to Proposition 2.1.

2.1 A balance-preserving construction

Given any balanced signed graph $\Sigma = (G, \sigma)$, there exist a switching function $\zeta : V(\Sigma) \to \{-1, +1\}$ that switches Σ to all-positive. Define $M_B(\Sigma)$ as the signed graph with underlying graph $M(G)$ and having the signature function

σ_B defined as

$$\begin{aligned}\sigma_B(v_iv_j) &= \sigma(v_iv_j),\\ \sigma_B(v_i'v_j) &= \sigma_B(v_iv_j') = \sigma(v_iv_j),\\ \sigma_B(v_i'w) &= \zeta(v_i).\end{aligned}$$

Define a switching function $\zeta_B : V(M_B(\Sigma)) \to \{-1,+1\}$ by

$$\begin{aligned}\zeta_B(v_i) &= \zeta(v_i),\\ \zeta_B(v_i') &= \zeta(v_i),\\ \zeta_B(w) &= 1.\end{aligned}$$

Since ζ switches Σ to all-positive, for edges v_iv_j,

$$\begin{aligned}\sigma_B^{\zeta_B}(v_iv_j) &= \zeta_B(v_i)\sigma_B(v_iv_j)\zeta_B(v_j)\\ &= \zeta(v_i)\sigma(v_iv_j)\zeta(v_j)\\ &= \sigma^{\zeta}(v_iv_j)\\ &= +1.\end{aligned}$$

Similarly, for edges $v_i'v_j$,

$$\begin{aligned}\sigma_B^{\zeta_B}(v_i'v_j) &= \zeta_B(v_i')\sigma_B(v_i'v_j)\zeta_B(v_j)\\ &= \zeta(v_i)\sigma(v_iv_j)\zeta(v_j)\\ &= \sigma^{\zeta}(v_iv_j)\\ &= +1.\end{aligned}$$

Also, for edges $v_i'w$,

$$\begin{aligned}\sigma_B^{\zeta_B}(v_i'w) &= \zeta_B(v_i')\sigma_B(v_i'w)\zeta_B(w)\\ &= \zeta(v_i)\zeta(v_i)(+1)\\ &= (\zeta(v_i))^2\\ &= +1.\end{aligned}$$

Hence, ζ_B switches $M_B(\Sigma)$ to all-positive. Thus, $M_B(\Sigma) = (M(G), \sigma_B)$ is balanced, and we call it as the balanced Mycielskian of Σ.

Definition 2.1 (Balanced Mycielskian)**.** *Let $\Sigma = (G, \sigma)$ be a balanced signed graph, where the underlying graph $G = (V, E)$, is a finite simple connected graph. The signed graph $M_B(\Sigma) = (M(G), \sigma_B)$ is called the balanced Mycielskian of Σ.*

One can observe that under this construction, if two balanced signed graphs Σ_1 and Σ_2 are switching equivalent, then their corresponding balanced Mycielskians $M_B(\Sigma_1)$ and $M_B(\Sigma_2)$ are also switching equivalent.

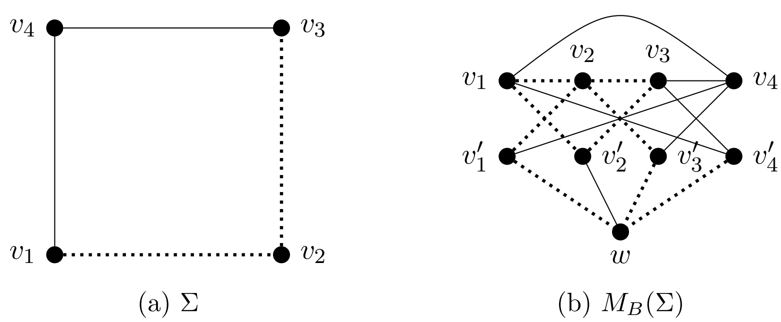


Figure 2: A balanced signed graph Σ and its balanced Mycielskian $M_B(\Sigma)$.

Example 2.1. Let Σ be the balanced 4-cycle shown in Figure 2a. The switching function $\zeta : V(\Sigma) \to \{-1, 1\}$ defined by $\zeta(v_1) = \zeta(v_3) = \zeta(v_4) = -1$ and $\zeta(v_2) = 1$ switches Σ to all-positive. The corresponding balanced Mycielskian is constructed in Figure 2b.

Remark 2.2. Note that, since $\sigma^{\zeta}(v_iv_j) = +1$, for every edge v_iv_j in Σ, we have $\zeta(v_i)\zeta(v_j) = \sigma(v_iv_j)$. Thus,

$$\sigma_B(v_i'w)\sigma_B(v_i'w) = \zeta(v_i)\zeta(v_j) = \sigma(v_iv_j).$$

Hence, the signature function defined for the balanced Mycielskian satisfies the condition given in Theorem 2.2.

3. The chromatic number of Mycielskian of signed graphs

In 1981, Zaslavsky [10] introduced the concept of coloring a signed graph. For a signed graph Σ, he defined the signed coloring of Σ in μ colors, or in $2\mu+1$ signed colors as a mapping $c : V(\Sigma) \to \{-\mu, -\mu+1, \ldots, 0, \ldots, \mu-1, \mu\}$. Whenever a coloring never assumes the value 0, it is referred to as a zero-free coloring. A coloring c is said to be proper if $c(u) \neq \sigma(e)c(v)$ for every edge $e = uv$ of Σ (see, [10, Section 1]).

Máčajová *et al.* in [5] defined the chromatic number of a signed graph as follows.

Definition 3.1 ([5]). *An n - coloring of a signed graph Σ is a proper coloring that uses colors from the set M_n, which is defined for each $n \geq 1$ as*

$$M_n = \begin{cases} \{\pm 1, \pm 2, \ldots \pm k\}, & \text{if } n = 2k \\ \{0, \pm 1, \pm 2, \ldots \pm k\}, & \text{if } n = 2k+1. \end{cases}$$

The smallest n such that Σ admits an n - coloring is called the chromatic number of Σ and is denoted by $\chi(\Sigma)$.

The chromatic number of a balanced signed graph coincides with the chromatic number of its underlying unsigned graph.

Proposition 3.1. *Let $M(\Sigma)\backslash\{w\}$ be the signed graph obtained by removing the root vertex w (and the corresponding edges) from $M(\Sigma)$. Then, $\chi(M(\Sigma)\backslash\{w\}) = \chi(\Sigma)$.*

Proof. Let $\chi(\Sigma) = n$ and let $c : V(\Sigma) \to M_n$ be an n - coloring for Σ. Define $c' : V((M(\Sigma)\backslash\{w\}) \to M_n$ by $c'(v_i') = c'(v_i) = c(v_i)$ for all i. Since $c(v_i) \neq \sigma(v_iv_j)c(v_j)$, it follows that $c'(v_i) \neq \sigma_M(v_iv_j)c'(v_j)$ and $c'(v_i') \neq \sigma_M(v_i'v_j)c'(v_j)$. Hence, c' is an n - coloring for $M(\Sigma)\backslash\{w\}$. □

For any given signed graph Σ, there exist a signed graph $-\Sigma$ obtained by reversing the signs of all edges of Σ. We say Σ is antibalanced when $-\Sigma$ is balanced. Note that, Σ is antibalanced if and only if it can be switched to all-negative.

We restate the Lemma 2.4 from [11] as follows.

Lemma 3.1 ([11])**.** *A signed graph Σ is antibalanced if and only if $\chi(\Sigma) \leq 2$.*

Theorem 3.1. *Let Σ be a signed graph and $M(\Sigma)$ be its Mycielskian. Then, $\chi(M(\Sigma)) \leq 2$ if and only if Σ is all-negative.*

Proof. If Σ is an all-negative signed graph with vertex set $\{v_1, v_2, \dots v_p\}$, then the only positive edges of $M(\Sigma)$ are $v_i'w$, $1 \leq i \leq p$. Now, the switching function $\zeta_M' : V(M(\Sigma)) \to \{-1, 1\}$ defined by $\zeta_M'(v_i) = \zeta_M'(v_i') = 1$ for all $1 \leq i \leq p$ and $\zeta_M'(w) = -1$ switches $M(\Sigma)$ to all-negative. Therefore, $M(\Sigma)$ is antibalanced and hence $\chi(M(\Sigma)) \leq 2$, by Lemma 3.1. Conversely, if Σ is not all-negative, it contains at least one positive edge, say v_iv_j. Then, $v_iv_jv_i'wv_j'v_i$ forms a negative 5 - cycle in $-M(\Sigma)$, making it unbalanced. Thus, $M(\Sigma)$ is not antibalanced and therefore, by Lemma 3.1, $\chi(M(\Sigma)) > 2$. □

We have the following theorem in [1].

Theorem 3.2 ([7])**.** *Let $\chi(G)$ and $\chi(M(G))$ be the chromatic numbers of a graph G and its Mycielskian $M(G)$ respectively. Then $\chi(M(G)) = \chi(G) + 1$.*

Theorem 3.3. *Let $M(\Sigma)$ be the Mycielskian of a signed graph (Σ). Then, $\chi(\Sigma) \leq \chi(M(\Sigma)) \leq \chi(\Sigma) + 1$. Furthermore, $\chi(M(\Sigma)) = \chi(\Sigma)$ if Σ is all-negative and $\chi(M(\Sigma)) = \chi(\Sigma) + 1$ if Σ is all-positive.*

Proof. Let $\chi(\Sigma) = n$ and let $c : V \to M_n$ be an n - coloring for Σ. We extend c to an $(n+1)$ - coloring of $M(\Sigma)$. If $n = 2k$, we extend c to an $(n+1)$ - coloring of $M(\Sigma)$ by setting $c(v_i') = c(v_i)$ for all i and $c(w) = 0$. If $n = 2k+1$, we extend c to an $(n+1)$ - coloring of $M(\Sigma)$ as follows. Let v_t be any vertex in V with $c(v_t) = 0$. Then, for all $v_i \neq v_t$, set $c(v_i') = c(v_i)$, $c(v_t') = c(v_t) = k+1$ and $c(w) = -(k+1)$. Hence, $\chi(M(\Sigma)) \leq \chi(\Sigma) + 1$.

Now, if Σ is all-negative, it can be colored using just one color, namely -1. Let $c : V(\Sigma) \to \{\pm 1\}$ be the proper 2 - coloring for Σ. This can be extended to a proper 2 - coloring for $M(\Sigma)$ by setting $c(v_i') = c(v_i) = -1$ for all i and $c(w) = +1$. If Σ is all-positive, then $M(\Sigma)$ is all-positive. Thus, $\chi(M(\Sigma)) = \chi(M(G)) = \chi(G) + 1 = \chi(\Sigma) + 1$. □

Remark 3.1. Let Σ be a signed graph with $\chi(\Sigma) = n$ and let $c : V(\Sigma) \to M_n$ be an n - coloring of Σ. The *deficiency* of the coloring c is the number of unused colors from M_n (see, [6]). The existence of signed graphs satisfying $\chi(M(\Sigma)) = \chi(\Sigma)$ is a consequence of the deficiency of the coloring of Σ. Specifically, if the coloring of Σ has a deficiency of at least 1, then an unused color can be assigned to w, making the chromatic number of $M(\Sigma)$ and Σ equal. As an example, consider Σ as the balanced 3 - cycle shown in Figure 3a. Note that, $\chi(\Sigma) = 3$ and the color -1 in the color set $\{0, \pm 1\}$ is unused.

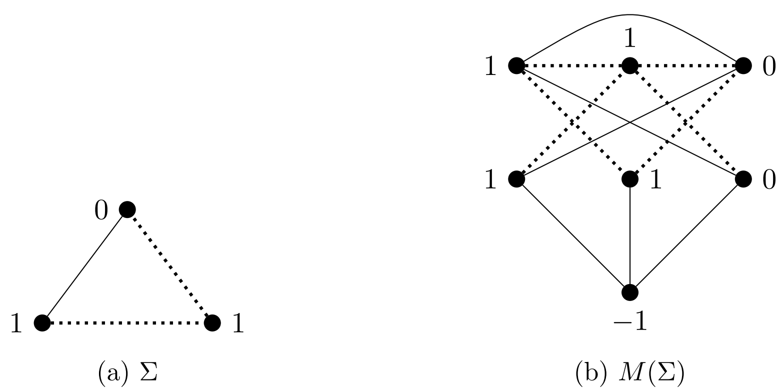


Figure 3: A signed graph Σ satisfying $\chi(M(\Sigma)) = \chi(\Sigma)$

We now establish some results on the balanced Mycielskian of signed graphs.

Proposition 3.2. *Let $\Sigma = (G, \sigma)$ be a balanced signed graph and $M_B(\Sigma) = (M(G), \sigma_B)$ be its balanced Mycielskian. Then, $\chi(M_B(\Sigma)) = \chi(\Sigma) + 1$.*

Proof. Since Σ and $M_B(\Sigma)$ are both balanced, $\chi(M_B(\Sigma)) = \chi(M(G))$ and $\chi(\Sigma) = \chi(G)$. The result then follows from Theorem 3.2. □

The following theorem was put forward by Mycielski in [7]

Theorem 3.4 ([7])**.** *For any positive integer n, there exists a triangle-free graph with chromatic number n.*

The next theorem is an analogous result for balanced signed graphs.

Theorem 3.5. *For any positive integer n, there exists a balanced triangle-free signed graph that is not all-positive, and having chromatic number n.*

Proof. The proof is based on mathematical induction. For $n = 1$ and $n = 2$, the signed graphs $\Sigma_1 = K_1$ and $\Sigma_2 = K_2^-$, where K_2^- is the all-negative signed complete graph on two vertices have the required property. Suppose that for $k > 2$, such a signed graph Σ_k satisfying the induction hypothesis exists. Then, $M_B(\Sigma_k)$ is a balanced signed graph that is not all-positive. Also, by Proposition 3.2, we have, $\chi(\Sigma_{k+1}) = \chi(\Sigma_k) + 1 = k + 1$. □

4. Matrices of the Mycielskian of signed graphs

Given a signed graph $\Sigma = (V, E, \sigma)$ where $V = \{v_1, v_2, \ldots, v_p\}$ is the vertex set, $E = \{e_1, e_2, \ldots, e_q\}$ is the edge set and $\sigma : E \to \{-1, 1\}$ is the sign function. Let $M(\Sigma)$ be the Mycielskian of Σ. In this section, we introduce the adjacency matrix, the incidence matrix and the Laplacian matrix of the Mycielskian $M(\Sigma)$ of Σ.

4.1 The adjacency matrix

The adjacency matrix of Σ, denoted by $\mathbf{A} = \mathbf{A}(\Sigma)$, is a $p \times p$ matrix (a_{ij}) in which $a_{ij} = \sigma(v_i v_j)$ if v_i and v_j are adjacent and 0 otherwise (see, [9, Section 3]).

Since v_i is adjacent to v'_j and v'_i is adjacent to v_j in $M(\Sigma)$ whenever v_i and v_j are adjacent in Σ, **the adjacency matrix $\mathbf{A_M}$** of the Mycielskian $M(\Sigma)$ takes the block form

$$\mathbf{A_M} = \mathbf{A}(M(\Sigma)) = \begin{bmatrix} \mathbf{A}(\Sigma) & \mathbf{A}(\Sigma) & \mathbf{0}_{p\times 1} \\ \mathbf{A}(\Sigma) & \mathbf{0}_{p\times p} & \mathbf{j}_{p\times 1} \\ \mathbf{0}^t_{1\times p} & \mathbf{j}^t_{1\times p} & 0 \end{bmatrix},$$

where $\mathbf{0}$ is a matrix of zeros and $\mathbf{j}$ is a matrix of ones of the specified order. $\mathbf{A_M}$ is a symmetric matrix of order $2p + 1$.

Given a graph G with adjacency matrix $A(G)$, the connection between the ranks of $A(G)$ and $A(M(G))$, the connection between the number of positive, negative and zero eigenvalues $A(G)$ and $A(M(G))$ were studied by Fisher *et al.* in [2]. We initiate a similar study in the case of signed graphs.

Let $\Sigma = (V, E, \sigma)$ be a given signed graph and let $t \notin V$. We denote the signed graph obtained by joining all the vertices of Σ to t with negative edges by Σ_{t^-}. That is, Σ_{t^-} is the negative join $\Sigma \vee_- K_1$. The adjacency matrix of Σ_t takes the block form

$$\mathbf{A}_{t^-} = \mathbf{A}(\Sigma_{t^-}) = \begin{bmatrix} \mathbf{A} & \textbf{-j} \\ \textbf{-j}^t & 0 \end{bmatrix}.$$

We now have the following theorem.

Theorem 4.1. *Let Σ be a signed graph and $\boldsymbol{A}(\Sigma)$ be the adjacency matrix of Σ. Let $r(\boldsymbol{A})$ denote the rank and $n_+(\boldsymbol{A})$, $n_-(\boldsymbol{A})$ and $n_0(\boldsymbol{A})$ respectively denote the number of positive, negative and zero eigenvalues of a symmetric matrix $\boldsymbol{A}$, then we have the following.*

(i) $r(\boldsymbol{A_M}) = r(\boldsymbol{A}) + r(\boldsymbol{A}_{t^-})$

(ii) $n_+(\boldsymbol{A_M}) = n_+(\boldsymbol{A}) + n_+(\boldsymbol{A}_{t^-})$

(iii) $n_-(\boldsymbol{A_M}) = n_-(\boldsymbol{A}) + n_-(\boldsymbol{A}_{t^-})$

(iv) $n_0(\boldsymbol{A_M}) = n_0(\boldsymbol{A}) + n_0(\boldsymbol{A}_{t^-})$

Proof. The adjacency matrix $\mathbf{A_M}$ can be factorized as

$$\mathbf{A_M} = \begin{bmatrix} \mathbf{A} & \mathbf{A} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{j} \\ \mathbf{0}^t & \mathbf{j}^t & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \textbf{-I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}^t & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \textbf{-A} & \textbf{-j} \\ \mathbf{0}^t & \textbf{-j}^t & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \textbf{-I} & \mathbf{0} \\ \mathbf{0}^t & \mathbf{0}^t & 1 \end{bmatrix} = \mathbf{P}\,\mathbf{B}\,\mathbf{P}^t$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \textbf{-I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}^t & 1 \end{bmatrix}$ is an invertible matrix and $\mathbf{B} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{t^-} \end{bmatrix}$.

Thus, the matrices $\mathbf{A_M}$ and $\mathbf{B}$ are congruent, and hence by Sylvester's law of inertia, they have the same rank and the same number of positive, negative and zero eigenvalues. □

4.2 The incidence matrix

The incidence matrix of Σ, denoted by $\mathbf{H} = \mathbf{H}(\Sigma)$, is the $p \times q$ matrix

$$\mathbf{H}(\Sigma) = \begin{bmatrix} \mathbf{x}(e_1) & \mathbf{x}(e_2) & \cdots & \mathbf{x}(e_q) \end{bmatrix},$$

where for each edge $e_k = v_i v_j$, $1 \le k \le q$, the vector $\mathbf{x}(e_k) = \begin{pmatrix} x_{1k} \\ \vdots \\ x_{pk} \end{pmatrix} \in \mathbb{R}^{p\times 1}$ has its i^{th} and j^{th} entries as $x_{ik} = \pm 1$ and $x_{jk} = \mp\sigma(e_k)$ respectively and all other entries as 0 (see, [9, Section 3]).

Let us denote the vertex set V_M and the edge set E_M of $M(\Sigma)$ as

$$V_M = \{v_1, v_2, \ldots, v_p, v'_1, v'_2, \ldots, v'_p, w\},$$
$$E_M = \{e_1, e_2, \ldots, e_q, e'_1, e''_1, e'_2, e''_2, \ldots, e'_q, e''_q, f_1, f_2, \cdots, f_p\}$$

respectively, where, for each $1 \le k \le q$, the edges e'_k and e''_k of $M(\Sigma)$ are defined by $e'_k = v_i v'_j$ and $e''_k = v'_i v_j$ whenever $e_k = v_i v_j$ is an edge of Σ with $1 \le i < j \le q$ and f_i is defined by $f_i = v'_i w$ for $1 \le i \le p$. Then, the **incidence matrix** $\mathbf{H_M} = \mathbf{H}(M(\Sigma))$ takes the block form

$$\mathbf{H_M} = \mathbf{H}(M(\Sigma)) = \left[\begin{array}{c|c|c|c|c|c|c|c|c} \mathbf{H}(\Sigma)_{p\times q} & \mathbf{x_1} & \mathbf{y_1} & \mathbf{x_2} & \mathbf{y_2} & \cdots & \mathbf{x_p} & \mathbf{y_p} & \mathbf{0}_{p\times p} \\ \hline \mathbf{0}_{p\times q} & \mathbf{y_1} & \mathbf{x_1} & \mathbf{y_2} & \mathbf{x_2} & \cdots & \mathbf{y_p} & \mathbf{x_p} & \mathbf{I}_{p\times p} \\ \hline \multicolumn{8}{c|}{\mathbf{0}_{1\times 3q}} & \textbf{-j}_{1\times p} \end{array}\right].$$

Here, $\mathbf{H}(\Sigma)$ is the incidence matrix of Σ, $\mathbf{I}$ is the identity matrix, $\mathbf{0}$ is the zero matrix and **-j** is the matrix with all entries -1 of the specified order. $\mathbf{x_i}$'s and $\mathbf{y_i}$'s are matrices of order $p \times 1$ and satisfies the condition $\mathbf{x_i} + \mathbf{y_i} = \mathbf{x}(e_i)$ for all $1 \le i \le q$.

4.3 The Laplacian matrix

The Laplacian matrix of Σ, denoted by $\mathbf{L} = \mathbf{L}(\Sigma)$ is the $p \times p$ matrix

$$\mathbf{L}(\Sigma) = \mathbf{D}(|\Sigma|) - \mathbf{A}(\Sigma),$$

where $\mathbf{A}(\Sigma)$ is the adjacency matrix of Σ and $\mathbf{D}(|\Sigma|)$ is the degree matrix of the underlying graph $|\Sigma|$ (see, [9, Section 3]).

Accordingly, we define the **Laplacian matrix for the Mycielskian** of Σ as

$$\mathbf{L_M} = \mathbf{L}(M(\Sigma)) = \mathbf{D}(|M(\Sigma)|) - \mathbf{A}(M(\Sigma)) = \mathbf{D_M} - \mathbf{A_M},$$

where $\mathbf{A_M}$ is the adjacency matrix and $\mathbf{D_M}$ is the diagonal degree matrix of the Mycielskian of Σ. Now, $\mathbf{D_M}$ takes the block form

$$\mathbf{D_M} = \begin{bmatrix} 2\mathbf{D}(|\Sigma|)_{p\times p} & \mathbf{0}_{p\times p} & \mathbf{0}_{p\times 1} \\ \mathbf{0}_{p\times p} & (\mathbf{D}(|\Sigma|)+\mathbf{I})_{p\times p} & \mathbf{0}_{p\times 1} \\ \mathbf{0}^t_{1\times p} & \mathbf{0}^t_{1\times p} & p \end{bmatrix},$$

where $p = |V|$, $\mathbf{D}(\Sigma)$ is the diagonal degree matrix of Σ, $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of the specified order.

Consequently, the Laplacian matrix $\mathbf{L_M} = \mathbf{L}(M(\Sigma))$ takes the block form

$$\mathbf{L_M} = \begin{bmatrix} (2\mathbf{D}(|\Sigma|) - \mathbf{A}(\Sigma))_{p\times p} & \textbf{-}\mathbf{A}(\Sigma)_{p\times p} & \mathbf{0}_{p\times 1} \\ \textbf{-}\mathbf{A}(\Sigma)_{p\times p} & (\mathbf{D}(|\Sigma|)+\mathbf{I})_{p\times p} & \textbf{-}\mathbf{j}_{p\times 1} \\ \mathbf{0}^t_{1\times p} & \textbf{-}\mathbf{j}^t_{1\times p} & p \end{bmatrix}.$$

5. Conclusion and scope

In this paper, we have defined the Mycielskian of a signed graph and discussed some of its properties. We have seen that the Mycielskian of a balanced signed graph need not be balanced and hence we provide an alternate construction in which the Mycielskian of Σ is balanced whenever Σ is balanced, This paper also discusses the chromatic number of the Mycielskian of a signed graph and established that the chromatic number of a signed graph and its Mycielskian are related. We also established the block forms of various matrices of the Mycielskian of a signed graph such as the adjacency matrix, the incidence matrix, and the Laplacian matrix.

This work finds its application in many areas, especially in sociology, where social systems can be represented by signed graphs. Triangle-free signed graphs are important for balanced social systems, and our construction creates larger triangle-free signed graphs from a given triangle-free signed graph. The balanced Mycielskian construction provides a method to extend a balanced system to a much larger system without affecting balance. Developing another balance preserving, switching preserving constructions for the Mycielskian of signed graphs, and computing the spectra of various matrices of the Mycielskian of signed graphs are some exciting areas for further investigation.

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Topological factor groups relative to normal soft int-groups

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Abstract. Given a group $\mathcal{G}$, let $\alpha_{\mathcal{G}}$ be a normal soft int-group in $\mathcal{G}$. We construct the factor group $\mathcal{G}/\alpha$ relative to $\alpha_{\mathcal{G}}$ by defining a congruence relation on $\mathcal{G}$. Using this construction, we establish soft Isomorphism Theorems which generalize the classical group Isomorphism Theorems. Finally, we give some topological structures on $\mathcal{G}$ and $\mathcal{G}/\alpha$.

Keywords: topological groups, soft int-groups, normal soft int-groups, soft isomorphism theorems.
MSC 2020: 40H05, 46A45

1. Introduction

In 1965, the concept of fuzzy set theory has been introduced by Zadeh [18]. The application of fuzzy sets can be found in many branches of mathematics and engineering sciences. Molodtsov in [11] introduced the soft set as a generalization of the fuzzy set to deal with uncertainty. A soft set (fuzzy soft sets, see[4]) is a set-valued function from a set of parameters to the power set(all fuzzy sets) of a universe set. The concept of soft groups (semigroups) is defined in [1, 2] as a collection of subgroups (subsemigroups) of a group (semigroup). In this direction, new types of soft ideals over semigroups are presented in recent works [6, 12, 13]. Cagman et al. [3], based on intersection and inclusion relation of sets, defined the soft int-group which are unlike that in [1, 14]. Some properties of soft int-groups and normal soft int-groups are introduced in [8, 9, 15]. Ideal theory in semigroups and ordered semigroups based on soft int- (uni-)semigroup is investigated in [5, 7, 17]. In this paper, we introduce a method to construct

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factor groups related to normal soft int-groups. We apply this construction to establish soft Isomorphism Theorems which generalize the classical group Isomorphism Theorems. Topological structures on $\mathcal{G}$ and the factor group $\mathcal{G}/\alpha$ are introduced.

2. Preliminaries

In this Section, we recall some definitions and results of soft set. Throughout our discussion, $\mathcal{U}$ refers to a universal set, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$ and $\mathcal{E}$ the set of parameters where $A, B, C, ... \subseteq \mathcal{E}$.

Definition 2.1 ([3])**.** *A soft set (α, A) over $\mathcal{U}$ is a set of ordered pairs*

$$(\alpha, A) := \{(x, \alpha(x)) : x \in \mathcal{E}, \alpha(x) \in \mathcal{P}(\mathcal{U})\},$$

where $\alpha : \mathcal{E} \longrightarrow \mathcal{P}(\mathcal{U})$ such that $\alpha(x) = \phi$ if $x \notin A$.

From now on, we write α_A instead of (α, A).

Definition 2.2 ([3])**.** *Let α_A and α_B be soft sets over $\mathcal{U}$. Then, union $\alpha_A \sqcup \alpha_B$ and intersection $\alpha_A \sqcap \alpha_B$ of α_A and α_B are defined by*

$$(\alpha_A \sqcup \alpha_B)(x) = \alpha_A(x) \cup \alpha_B(x), \qquad (\alpha_A \sqcap \alpha_B)(x) = \alpha_A(x) \cap \alpha_B(x)$$

respectively, for all $x \in \mathcal{E}$.

Definition 2.3 ([3])**.** *Let $\mathcal{G}$ be a group and $\alpha_{\mathcal{G}}$ be a soft set over $\mathcal{U}$. Then, $\alpha_{\mathcal{G}}$ is called a soft intersection group (**soft int-group**) over $\mathcal{U}$ if*

1. *$\alpha_{\mathcal{G}}(xy) \supseteq \alpha_{\mathcal{G}}(x) \cap \alpha_{\mathcal{G}}(y)$ for all $x, y \in \mathcal{G}$, and*
2. *$\alpha_{\mathcal{G}}(x^{-1}) = \alpha_{\mathcal{G}}(x)$ for all $x \in \mathcal{G}$.*

Or, equivalenty, if $\alpha_{\mathcal{G}}(xy^{-1}) \supseteq \alpha_{\mathcal{G}}(x) \cap \alpha_{\mathcal{G}}(y)$ for all $x, y \in \mathcal{G}$.

Theorem 2.1 ([8])**.** *Let $\alpha_{\mathcal{G}}$ be a soft int-group and $x, y \in \mathcal{G}$. Then*

1. *$\alpha_{\mathcal{G}}(e) \supseteq \alpha_{\mathcal{G}}(x)$,*
2. *$\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e) \Rightarrow \alpha_{\mathcal{G}}(x) = \alpha_{\mathcal{G}}(y)$.*

Definition 2.4 ([3])**.** *A soft int-group $\alpha_{\mathcal{G}}$ over $\mathcal{U}$ is called **normal**, if for all $x, y \in \mathcal{G}$, it satisfies one of the following equivalent conditions:*

1. *$\alpha_{\mathcal{G}}(xyx^{-1}) \supseteq \alpha_{\mathcal{G}}(y)$,*
2. *$\alpha_{\mathcal{G}}(xyx^{-1}) \subseteq \alpha_{\mathcal{G}}(y)$,*
3. *$\alpha_{\mathcal{G}}(xy) = \alpha_{\mathcal{G}}(yx)$.*

Definition 2.5 ([9]). *Let α_A be a soft set over $\mathcal{U}$ and $V \in \mathcal{P}(\mathcal{U})$). Then, V-inclusion of the soft set α_A, denoted by α^V is defined as*

$$\alpha^V = \{x \in A : \alpha(x) \supseteq V\}.$$

It is proved in [9] that "A soft set $\alpha_\mathcal{G}$ is a (normal) soft int-group over $\mathcal{U}$ iff for all $V \in \mathcal{P}(\mathcal{U})$, α^V is either empty or a (normal) subgroup of $\mathcal{G}$".

Definition 2.6 ([15]). *Let $f : \mathcal{G} \to \mathcal{H}$ be a function between groups. Then, the soft image $f(\alpha_\mathcal{G})$ of a soft set $\alpha_\mathcal{G}$ and the soft preimage $f^{-1}(\beta_\mathcal{H})$ of a soft set $\beta_\mathcal{H}$ under f are defined as*

$$f(\alpha_\mathcal{G})(y) = \begin{cases} \bigcup\{\alpha_\mathcal{G}(x) : x \in \mathcal{G}, f(x) = y\}, & \text{for } y \in f(\mathcal{G}), \\ \phi, & \text{otherwise,} \end{cases}$$

and

$$f^{-1}(\beta_\mathcal{H})(x) = \beta_\mathcal{H}(f(x)), \quad \forall x \in \mathcal{G}.$$

Theorem 2.2 ([15]). *If $f : \mathcal{G} \to \mathcal{H}$ is an epimorphism of groups, and $\alpha_\mathcal{G}$ is a normal soft int-group, then $f(\alpha_\mathcal{G})$ is a normal soft int-group.*

3. Construction of the factor group

In this Section, we represent our main findings. Given a group $\mathcal{G}$ we denote the identity element of $\mathcal{G}$ by $e_\mathcal{G}$, and the set of all soft int-groups over $\mathcal{U}$ with $\mathcal{G}$ as a set of parameters by $\mathcal{S}(\mathcal{G},\mathcal{U})$.

Recall that an equivalence relation δ on $\mathcal{G}$ is called a **congruence relation** if

$$x\delta y \Rightarrow xz\delta yz,\ zx\delta zy$$

for all $x, y, z \in \mathcal{G}$.

Let $\alpha_\mathcal{G} \in \mathcal{S}(\mathcal{G},\mathcal{U})$ be a normal soft int- group. For any $x, y \in \mathcal{G}$, we define the relation R on $\mathcal{G}$ by

$$xRy \Leftrightarrow \alpha_G(xy^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G}).$$

Lemma 3.1. *R is a congruence relation on $\mathcal{G}$.*

Proof. Clearly, R is reflexive and symmetric. Also, R is transitive. Indeed, let xRy and yRz, then $\alpha_\mathcal{G}(xy^{-1}) = \alpha_\mathcal{G}(yz^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G})$. Then, $\alpha_\mathcal{G}(xz^{-1}) = \alpha_\mathcal{G}(xy^{-1}yz^{-1}) \supseteq \alpha_\mathcal{G}(xy^{-1}) \cap \alpha_\mathcal{G}(yz^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G})$. Hence, $\alpha_\mathcal{G}(xz^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G})$, which proves that xRz and so R is an equivalence relation. If xRy, then $\alpha_\mathcal{G}(xy^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G})$. Thus, for all $z \in \mathcal{G}$ we have

$$\alpha_\mathcal{G}((xz)(yz)^{-1}) = \alpha_\mathcal{G}(xzz^{-1}y^{-1}) = \alpha_\mathcal{G}(xy^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G}).$$

Hence, $xzRyz$. Since $\alpha_\mathcal{G}$ is normal, we get $\alpha_\mathcal{G}((zx)(zy)^{-1}) = \alpha_\mathcal{G}(zxy^{-1}z^{-1}) = \alpha_\mathcal{G}(xy^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G})$. This gives $zxRzy$, and we conclude that R is a congruence relation on $\mathcal{G}$. $\square$

By $[x]_\alpha$, we denote the equivalence class containing $x \in \mathcal{G}$ and by $\mathcal{G}/\alpha$ the corresponding factor set relative to $\alpha_\mathcal{G}$.

Theorem 3.1. *$\mathcal{G}/\alpha$ is a group with the operation $[x]_\alpha[y]_\alpha = [xy]_\alpha$.*

Proof. Straightforward. □

Example 3.1. Assume that $\mathcal{U} = S_3$ is the set of permutations on $\{1, 2, 3\}$. Let $\mathcal{G} = Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ be the set of parameters. We define a soft set $\alpha_\mathcal{G}$ over U by

$$\begin{aligned}
\alpha_\mathcal{G}(\bar{0}) &= U,\\
\alpha_\mathcal{G}(\bar{1}) &= \{(12), (13), (132)\},\\
\alpha_\mathcal{G}(\bar{2}) &= \{(12), (13), (23), (123), (132)\},\\
\alpha_\mathcal{G}(\bar{3}) &= \{(1), (12), (13), (132)\},\\
\alpha_\mathcal{G}(\bar{4}) &= \{(12), (13), (23), (123), (132)\},\\
\alpha_\mathcal{G}(\bar{5}) &= \{(12), (13), (132)\}.
\end{aligned}$$

Clearly, $\alpha_\mathcal{G}$ is a normal soft int-group over $\mathcal{U}$ and

$$\mathcal{G}/\alpha = \{[\bar{0}]_\alpha, [\bar{1}]_\alpha, [\bar{2}]_\alpha, [\bar{3}]_\alpha, [\bar{4}]_\alpha, [\bar{5}]_\alpha\}.$$

By Definition 2.5, the set $K_{\alpha_\mathcal{G}} = \{x \in \mathcal{G} : \alpha_\mathcal{G}(x) = \alpha_\mathcal{G}(e_\mathcal{G})\}$ is a (normal) subgroup of $\mathcal{G}$ iff $\alpha_\mathcal{G}$ is a (normal)soft int-group over $\mathcal{U}$.

Proposition 3.1. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be a homomorphism of groups and $\alpha_\mathcal{G} \in \mathcal{S}(\mathcal{G},\mathcal{U})$, then*

(i) $f(K_{\alpha_\mathcal{G}}) \subseteq K_{f(\alpha_\mathcal{G})}$,

(ii) If $\alpha_\mathcal{G}$ is constant on $Kerf$, then $f(\alpha_\mathcal{G})(f(x)) = \alpha_\mathcal{G}(x)$ for all $x \in \mathcal{G}$.

Proof. (i) Let $y \in f(K_{\alpha_\mathcal{G}})$, then $y = f(x)$ for some $x \in K_{\alpha_\mathcal{G}}$. Since $\alpha_\mathcal{G}(x) = \alpha_\mathcal{G}(e_\mathcal{G})$, then

$$f(\alpha_\mathcal{G})(y) = \bigcup_{x \in f^{-1}(y)} \{\alpha_\mathcal{G}(x)\} = \alpha_\mathcal{G}(e_\mathcal{G}) = f(\alpha_\mathcal{G})(e_\mathcal{H}).$$

Therefore, $y \in K_{f(\alpha_\mathcal{G})}$.

(ii) Let $y = f(x)$, then $f(\alpha_\mathcal{G})(y) = \bigcup_{z \in f^{-1}(y)}\{\alpha_\mathcal{G}(z)\}$. But $f(zx^{-1}) = e_\mathcal{H}$ for all $z \in f^{-1}(y)$. Hence, $\alpha_\mathcal{G}(zx^{-1}) = \alpha_\mathcal{G}(e_\mathcal{G})$ because $\alpha_\mathcal{G}$ is constant on $Kerf$. By Theorem 2.1, we have $\alpha_\mathcal{G}(z) = \alpha_\mathcal{G}(x)$ for all $z \in f^{-1}(y)$. Therefore, $f(\alpha_\mathcal{G})(f(x)) = \alpha_\mathcal{G}(x)$. □

Theorem 3.2. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an epimorphism of groups and $\alpha_\mathcal{G} \in \mathcal{S}(\mathcal{G},\mathcal{U})$ be normal with $kerf \subseteq K_{\alpha_\mathcal{G}}$, then $\mathcal{G}/\alpha \cong \mathcal{H}/f(\alpha_\mathcal{G})$.*

Proof. From Theorem 2.2, $f(\alpha_{\mathcal{G}})$ is a normal soft int-group and hence $\mathcal{H}/f(\alpha_{\mathcal{G}})$ is a group. We define $\theta : \mathcal{G}/\alpha \longrightarrow \mathcal{H}/f(\alpha_{\mathcal{G}})$, such that $\theta([x]_\alpha) = [f(x)]_{f(\alpha_{\mathcal{G}})}$. Firstly, θ is well defined since $[x]_\alpha = [y]_\alpha$ implies $\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. Since $kerf \subseteq K_{\alpha_{\mathcal{G}}}$, then $\alpha_{\mathcal{G}}$ is constant on $kerf$, and by Proposition 3.1, we have

$$f(\alpha_{\mathcal{G}})(f(xy^{-1})) = f(\alpha_{\mathcal{G}})(f(e_{\mathcal{G}})).$$

Then, $f(\alpha_{\mathcal{G}})(f(x)f(y)^{-1}) = f(\alpha_{\mathcal{G}})(e_{\mathcal{H}})$, and so $[f(x)]_{f(\alpha_{\mathcal{G}})} = [f(y)]_{f(\alpha_{\mathcal{G}})}$. Therefore, θ is well defined.

Secondly, θ is a homomorphism because:

$$\begin{aligned}\theta([x]_\alpha[y]_\alpha) = \theta([xy]_\alpha) &= [f(xy)]_{f(\alpha_{\mathcal{G}})} = [f(x)f(y)]_{f(\alpha_{\mathcal{G}})}\\ &= [f(x)]_{f(\alpha_{\mathcal{G}})}[f(y)]_{f(\alpha_{\mathcal{G}})} = \theta([x]_\alpha)\theta([y]_\alpha).\end{aligned}$$

Now, we show that θ is an epimorphism. For any $[y]_{f(\alpha_{\mathcal{G}})} \in \mathcal{H}/f(\alpha_{\mathcal{G}})$, there exists $x \in \mathcal{G}$ such that $f(x) = y$ (since f is onto). So $\theta([x]_\alpha) = [f(x)]_{f(\alpha_{\mathcal{G}})} = [y]_{f(\alpha_{\mathcal{G}})}$, which means that θ is an epimorphism. Finally, θ is a 1-1 homomorphism since

$$\begin{aligned}[f(x)]_{f(\alpha_{\mathcal{G}})} = [f(y)]_{f(\alpha_{\mathcal{G}})}&\\ &\Longrightarrow f(\alpha_{\mathcal{G}})(f(x)f(y)^{-1}) = f(\alpha_{\mathcal{G}})(e_{\mathcal{H}})\\ &\Longrightarrow f(\alpha_{\mathcal{G}})(f(xy^{-1})) = f(\alpha_{\mathcal{G}})(f(e_{\mathcal{G}}))\\ &\Longrightarrow \alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})\\ &\Longrightarrow [x]_\alpha = [y]_\alpha,\end{aligned}$$

which proves that θ is injective. We conclude that θ is an isomorphism. □

Corollary 3.1. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an onto homomorphism of groups and $\beta_{\mathcal{H}} \in \mathcal{S}(\mathcal{H},\mathcal{U})$ be normal , then $\mathcal{G}/f^{-1}(\beta_{\mathcal{H}}) \cong \mathcal{H}/\beta$.*

Proof. It is known that $f^{-1}(\beta_{\mathcal{H}})$ is a normal soft int-group over $\mathcal{U}$ (see, [15]). Consequently, $\mathcal{G}/f^{-1}(\beta_{\mathcal{H}})$ and $\mathcal{H}/\beta$ are groups. Since f is onto, then $\beta_{\mathcal{H}} = f(f^{-1}(\beta_{\mathcal{H}}))$ [9]. Let x be an element in $kerf$, then $f(x) = f(e_{\mathcal{G}})$, and so $\beta_{\mathcal{H}}(f(x)) = \beta_{\mathcal{H}}(f(e_{\mathcal{G}}))$, that is $f^{-1}(\beta_{\mathcal{H}})(x) = f^{-1}(\beta_{\mathcal{H}})(e_{\mathcal{G}})$. Hence, $x \in K_{f^{-1}(\beta_{\mathcal{H}})}$, which means that $kerf \subseteq K_{f^{-1}(\beta_{\mathcal{H}})}$. By applying Theorem 3.2, we get the desired result. □

For a nonempty subset $\mathcal{A}$ of $\mathcal{G}$, define a map $\chi_{\mathcal{A}} : \mathcal{G} \longrightarrow \mathcal{P}(\mathcal{U})$ as follows:

$$\chi_{\mathcal{A}}(x) = \begin{cases}\mathcal{U}, & \text{if } x \in \mathcal{A},\\ \phi, & \text{otherwise.}\end{cases}$$

Then, $\chi_{\mathcal{A}}$ is a soft set over $\mathcal{U}$, which is called the characteristic soft set (see, [17]).

Theorem 3.3. *$\mathcal{A}$ is a (normal) subgroup of $\mathcal{G}$ if and only if $\chi_{\mathcal{A}}$ is a (normal) soft int-group over $\mathcal{U}$.*

Proof. Assume that $\chi_{\mathcal{A}}$ is a normal soft int-group over $\mathcal{U}$. For any $x, y \in \mathcal{A}$ we have $\chi_{\mathcal{A}}(xy^{-1}) \supseteq \chi_{\mathcal{A}}(x) \cap \chi_{\mathcal{A}}(y) = \mathcal{U}$. Thus, $\chi_{A}(xy^{-1}) = U$ and $xy^{-1} \in A$. Therefore $\mathcal{A}$ is a subgroup of $\mathcal{G}$. Similarly, for any $y \in \mathcal{A}, x \in \mathcal{G}$ we have $\chi_{\mathcal{A}}(xyx^{-1}) \supseteq \chi_{\mathcal{A}}(y) = \mathcal{U}$. Hence, $\chi_{\mathcal{A}}(xyx^{-1}) = \mathcal{U}$ and $xyx^{-1} \in \mathcal{A}$. This proves that $\mathcal{A}$ is a normal subgroup of $\mathcal{G}$. Conversely, suppose that $\mathcal{A}$ is a normal subgroup of $\mathcal{G}$. If $x, y \in \mathcal{A}$, then $\chi_{\mathcal{A}}(xy^{-1}) = \chi_{\mathcal{A}}(x) = \chi_{\mathcal{A}}(y) = \mathcal{U}$. Hence, $\chi_{\mathcal{A}}(xy^{-1}) = \chi_{\mathcal{A}}(x) \cap \chi_{\mathcal{A}}(y)$. If at least one of x and y is not in $\mathcal{A}$, then at least one of $\chi_{\mathcal{A}}(x)$ and $\chi_{\mathcal{A}}(y)$ is ϕ. Therefore $\chi_{\mathcal{A}}(xy^{-1}) \supseteq \chi_{\mathcal{A}}(x) \cap \chi_{\mathcal{A}}(y)$. Hence, $\chi_{\mathcal{A}}$ is a soft int-group over $\mathcal{U}$. Moreover, for any $x, y \in \mathcal{G}$, if $y \in \mathcal{A}$, then $xyx^{-1} \in \mathcal{A}$ and $\chi_{\mathcal{A}}(xyx^{-1}) = \mathcal{U} = \chi_{\mathcal{A}}(y)$. If $y \notin \mathcal{A}$, then $\chi_{\mathcal{A}}(xyx^{-1}) \supseteq \chi_{\mathcal{A}}(y) = \phi$. Hence, $\chi_{\mathcal{A}}$ is normal. □

Corollary 3.2. *Let $f : \mathcal{G} \longrightarrow \mathcal{H}$ be an onto homomorphism. Then, $\mathcal{G}/\chi_{kerf} \cong \mathcal{H}$.*

Proof. By Theorem 3.3, the characteristic soft set $\chi_{\{e_{\mathcal{H}}\}} \in \mathcal{S}(\mathcal{H},\mathcal{U})$ is normal. It is easy to see that the soft preimage $f^{-1}(\chi_{\{e_{\mathcal{H}}\}})$ is the soft set χ_{kerf}. Hence, the factor group $\mathcal{H}/\chi_{\{e_{\mathcal{H}}\}}$ is isomorphic to $\mathcal{H}$. By applying Corollary 3.1, we get $\mathcal{G}/\chi_{kerf} \cong \mathcal{H}/\chi_{\{e_{\mathcal{H}}\}} \cong \mathcal{H}$. □

In group theory, on the factor group $\mathcal{G}/kerf$ we can define an equivalence relation by $x \sim y \Leftrightarrow xy^{-1} \in kerf$. Easily, one shows that $x \sim y$ iff xRy relative to the normal soft int-group χ_{kerf}. Therefore, we have $\mathcal{G}/\chi_{kerf} \cong \mathcal{G}/kerf$ and Corollary 3.2 becomes the First Group Isomorphism Theorem.

Lemma 3.2. *Let $\mathcal{A}$ be a normal subgroup of $\mathcal{G}$ and $\alpha_{\mathcal{G}}$ a normal soft int-group over $\mathcal{U}$. Then, the restriction $\alpha_{\mathcal{G}} \mid \mathcal{A}$ is a normal soft int-group over $\mathcal{U}$ and $\mathcal{A}/\alpha$ is a normal subgroup of $\mathcal{G}/\alpha$.*

Proof. It is obvious from [9, Theorem 2.13] that $\alpha_{\mathcal{G}} \mid \mathcal{A}$ is a soft int-group. Since $\mathcal{A}$ is normal, $(\alpha_{\mathcal{G}} \mid \mathcal{A})(xy) = (\alpha_{\mathcal{G}} \mid \mathcal{A})(yx)$ for any $x, y \in \mathcal{A}$. Hence, $\alpha_{\mathcal{G}} \mid A$ is a normal soft int-group. If $[a]_\alpha, [b]_\alpha \in A/\alpha$, where $a, b \in \mathcal{A}$, then $([a]_\alpha)([b]_\alpha)^{-1} = ([a]_\alpha)([b^{-1}]_\alpha) = [ab^{-1}]_\alpha \in \mathcal{A}/\alpha$. Hence, $\mathcal{A}/\alpha$ is a subgroup of $\mathcal{G}/\alpha$. If $[a]_\alpha \in \mathcal{A}/\alpha, [x]_\alpha \in \mathcal{G}/\alpha$, where $a \in \mathcal{A}$ and $x \in \mathcal{G}$, then $xax^{-1} \in \mathcal{A}$ and

$$([x]_\alpha)([a]_\alpha)([x]_\alpha)^{-1} = ([x]_\alpha)([a]_\alpha)([x^{-1}]_\alpha) = [xax^{-1}]_\alpha \in \mathcal{A}/\alpha.$$

Thus, $\mathcal{A}/\alpha$ is a normal subgroup of $\mathcal{G}/\alpha$. □

Notation. For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$, we set $\mathcal{A} \cdot \mathcal{B} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$.

Theorem 3.4. *If $\alpha_{\mathcal{G}}$ and $\beta_{\mathcal{G}}$ are normal soft int-groups over $\mathcal{U}$ such that $\alpha_{\mathcal{G}}(e_{\mathcal{G}}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$, then $(K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})/\beta_{\mathcal{G}} \cong K_{\alpha_{\mathcal{G}}}/(\alpha_{\mathcal{G}} \sqcap \beta_{\mathcal{G}})$.*

Proof. Before we proceed and for simplicity, put $\gamma_{\mathcal{G}} = \alpha_{\mathcal{G}} \sqcap \beta_{\mathcal{G}}$. Since $\gamma_{\mathcal{G}}$ is a normal soft int-group over $\mathcal{U}$ (see, [9]) and by Lemma 3.2, the restrictions $\beta_{\mathcal{G}} \mid (K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})$ and $\gamma_{\mathcal{G}} \mid K_{\alpha_{\mathcal{G}}}$ are a normal soft int-groups over $\mathcal{U}$. Then, the factor sets $(K_{\alpha_{\mathcal{G}}} \cdot K_{\beta})/\beta$ and $K_{\alpha_{\mathcal{G}}}/\gamma$ are groups by Lemma 3.1. For any $x \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}, x = ab$, where $a \in K_{\alpha_{\mathcal{G}}}$ and $b \in K_{\beta_{\mathcal{G}}}$, we define $\Omega : (K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})/\beta \longrightarrow K_{\alpha_{\mathcal{G}}}/\gamma$ such that $f([x]_\beta) = [a]_\gamma$. The mapping Ω is well-defined. Indeed, if $[x]_\beta = [y]_\beta$, where $y = wz \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}$, then

$$\begin{aligned}\beta_{\mathcal{G}}(xy^{-1}) = \beta_{\mathcal{G}}(ab(wz)^{-1}) = \beta_{\mathcal{G}}(abz^{-1}w^{-1}) &= \beta_{\mathcal{G}}(w^{-1}abz^{-1})\\ &= \beta_{\mathcal{G}}(w^{-1}a(zb^{-1})^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}}).\end{aligned}$$

Hence, $\beta_{\mathcal{G}}(w^{-1}a) = \beta_{\mathcal{G}}(zb^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$. Thus,

$$\begin{aligned}\gamma_{\mathcal{G}}(aw^{-1}) = \alpha_{\mathcal{G}}(aw^{-1}) \cap \beta_{\mathcal{G}}(aw^{-1}) &= \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(w^{-1}a)\\ &= \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(e_{\mathcal{G}}) = \gamma_{\mathcal{G}}(e_{\mathcal{G}}),\end{aligned}$$

that is $[a]_\gamma = [w]_\gamma$.

Now, we prove that Ω is a homomorphism. Let $[x]_\beta, [y]_\beta \in (K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}})/\beta$, where $x = ab, y = wz, a, w \in K_{\alpha_{\mathcal{G}}}$ and $b, z \in K_{\beta_{\mathcal{G}}}$, then $xy = abwz$. Since $K_{\alpha_{\mathcal{G}}}$ is normal, $bwz \in K_{\alpha_{\mathcal{G}}}$. Hence,

$$\Omega([x]_\beta[y]_\beta) = \Omega([xy]_\beta) = [a(bwz)]_\gamma = [a]_\gamma[bwz]_\gamma$$

and

$$\begin{aligned}\gamma_{\mathcal{G}}((bwz)w^{-1}) &= \alpha_{\mathcal{G}}((bwz)w^{-1}) \cap \beta_{\mathcal{G}}((b(wzw^{-1}))\\ &= \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(e_{\mathcal{G}}) = \gamma_{\mathcal{G}}(e_{\mathcal{G}}).\end{aligned}$$

Hence, $[w]_\gamma = [bwz]_\gamma$, i.e.

$$\Omega([x]_\beta[y]_\beta) = [a]_{\gamma)}[w]_\gamma = \Omega([x]_\beta)\Omega([y]_\beta),$$

which implies that Ω is a homomorphism. It is also onto, since for any $[a]_\gamma \in K_{\alpha_{\mathcal{G}}}/\gamma$ and $b \in K_{\beta_{\mathcal{G}}}$, we have $x = ab \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}$ and $\Omega([x]_\beta) = [a]_{\gamma)}$. Finally, we show that Ω is injective. Let $x, y \in K_{\alpha_{\mathcal{G}}} \cdot K_{\beta_{\mathcal{G}}}$, where $x = ab, y = wz$. Assume that $[a]_\gamma = [w]_\gamma$, then $\gamma_{\mathcal{G}}(aw^{-1}) = \gamma_{\mathcal{G}}(e_{\mathcal{G}})$, that is

$$\alpha_{\mathcal{G}}(aw^{-1}) \cap \beta_{\mathcal{G}}(aw^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}}) \cap \beta_{\mathcal{G}}(e_{\mathcal{G}}).$$

But $\alpha_{\mathcal{G}}(e_{\mathcal{G}}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$ and $\alpha_{\mathcal{G}}(aw^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$ imply that $\beta_{\mathcal{G}}(aw^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$. Therefore,

$$\begin{aligned}\beta_{\mathcal{G}}(xy^{-1}) &= \beta_{\mathcal{G}}(ab(wz)^{-1}) = \beta_{\mathcal{G}}(abz^{-1}w^{-1}) = \beta_{\mathcal{G}}(w^{-1}abz^{-1})\\ &\supseteq \beta_{\mathcal{G}}(w^{-1}a) \cap \beta_{\mathcal{G}}(bz^{-1}) = \beta_{\mathcal{G}}(aw^{-1}) \cap \beta_{G}(bz^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}}).\end{aligned}$$

Hence, $[x]_\beta = [y]_\beta$. Therefore, Ω is an isomorphism. $\square$

In case $\mathcal{A}, \mathcal{B} \subseteq \mathcal{G}$ are normal subgroups, the result $(\mathcal{A} \cdot \mathcal{B})/\chi_{\mathcal{B}} \cong \mathcal{B}/\chi_{\mathcal{A}\cap\mathcal{B}}$ comes as a corollary of Theorem 3.4. and then we get the Second Group Isomorphism Theorem. Finally, the Third Group Isomorphism Theorem is outcome of the following result.

Theorem 3.5. *Let $\alpha_{\mathcal{G}}, \beta_{\mathcal{G}} \in \mathcal{S}(\mathcal{G},\mathcal{U})$ be normal such that $\beta_{\mathcal{G}} \sqsubseteq \alpha_{\mathcal{G}}$ and $\alpha_{\mathcal{G}}(e_{\mathcal{G}}) = \beta_{\mathcal{G}}(e_{\mathcal{G}})$. Then, $(\mathcal{G}/\beta)/(K_{\alpha_{\mathcal{G}}}/\beta) \cong \mathcal{G}/\alpha$*

Proof. For all $x \in \mathcal{G}$, we define $\theta : \mathcal{G}/\beta \longrightarrow \mathcal{G}/\alpha$ by $\theta([x]_\beta) = [x]_\alpha$. The mapping is well defined since $[x]_\beta = [y]_\beta$ implies $\beta_{\mathcal{G}}(xy^{-1}) = \beta_{\mathcal{G}}(e_{\mathcal{G}}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$. By assumption, we get $\alpha_{\mathcal{G}}(xy^{-1}) \supseteq \beta_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$ and hence $\alpha_{\mathcal{G}}(xy^{-1}) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})$, that is $[x]_\alpha) = [y]_\alpha$. By definition, θ is an onto homomorphism. We have $k_{\alpha_{\mathcal{G}}}/\beta = \{[z]_\beta : z \in k_{\alpha_{\mathcal{G}}}\} = \{[z]_\beta : \alpha_{\mathcal{G}}(z) = \alpha_{\mathcal{G}}(e_{\mathcal{G}})\} = \{[z]_\beta : [z]_\alpha = [e_{\mathcal{G}}]_\alpha\} = \{[z]_\beta \in \mathcal{G}/\beta : \theta([z]_\alpha) = [e_{\mathcal{G}}]_\alpha\} = ker\theta$. Therefore, it follows that $(\mathcal{G}/\beta)/(K_{\alpha_{\mathcal{G}}}/\beta) \cong \mathcal{G}/\alpha$. □

4. Topological structures on $\mathcal{G}/\alpha$

Group $\mathcal{G}$ with the congruence relation R construct an approximation space ([16]). The lower and upper approximations of $\mathcal{H} \subseteq \mathcal{G}$ are defined respectively as

$$R_\star(\mathcal{H}) = \{x \in \mathcal{G} : [x]_\alpha \subseteq \mathcal{H}\},$$
$$R^\star(\mathcal{H}) = \{x \in \mathcal{G} : [x]_\alpha \cap \mathcal{H} \neq \phi\}.$$

The lower approximation induces a topology on $\mathcal{G}$.

Proposition 4.1 ([10])**.** *$T_R = \{\mathcal{H} \subseteq \mathcal{G} : R_\star(\mathcal{H}) = \mathcal{H}\}$ is a topology on $\mathcal{G}$.*

Furthermore, we have the following result.

Theorem 4.1. *$(\mathcal{G}, T_R)$ is a topological group.*

Proof. Let x and y be elements in $\mathcal{G}$. Every open set $U \in T_R$ containing the element xy satisfies the condition $R_\star(U) = U$. This implies $[xy]_\alpha \subseteq U$. Since R is a congruence relation on $\mathcal{G}$, we have $[x]_\alpha[y]_\alpha \subseteq [xy]_\alpha \subseteq U$. Notice that, $[x]_\alpha$ and $[y]_\alpha$ are open sets containing x, y respectively such that $[x][y] \subseteq U$. Hence, the group operation : $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a continuous mapping. To complete the proof, we have to verify continuity of the inversion mapping $x \to x^{-1}$. Let x be an element in $\mathcal{G}$ and $V \in T_R$ an open set containing the element x^{-1}, then $[x^{-1}]_\alpha \subseteq V$. Let $y^{-1} \in [x]^{-1} = \{y^{-1} : y \in [x]\}$ then

$$\alpha_{\mathcal{G}}(x^{-1}(y^{-1})^{-1}) = \alpha_{\mathcal{G}}(x^{-1}y) = \alpha_{\mathcal{G}}(yx^{-1}) = \alpha_{\mathcal{G}}(e).$$

That is, $y^{-1} \in [x^{-1}]$. Since $[x]$ is an open set containing x such that $[x]^{-1} \subseteq [x^{-1}] \subseteq V$, then the inverse operation on $\mathcal{G}$ is continuous. Therefore, $(\mathcal{G}, T_R)$ is a topological group. □

Example 4.1. Assume that $\mathcal{G} = S_3$ is the set of permutations on $\{1, 2, 3\}$ and $\mathcal{U} = \mathbb{Z}$ is the set of parameters. We define a soft set $\alpha_{\mathcal{G}}$ over U by

$$\begin{aligned} \alpha_{\mathcal{G}}(e) &= \mathbb{Z}, \\ \alpha_{\mathcal{G}}((12)) = \alpha_{\mathcal{G}}((13)) = \alpha_{\mathcal{G}}((23)) &= \{-2, -1, 0, 1, 2\}, \\ \alpha_{\mathcal{G}}((123)) = \alpha_{\mathcal{G}}((132)) &= \{-3, -2, -1, 0, 1, 2, 3\}. \end{aligned}$$

$\alpha_{\mathcal{G}}$ is a soft int-group ([3]). Easily, one can verify that $\alpha_{\mathcal{G}}$ is a normal soft int-group over $\mathcal{U}$.

Obviously, the equivalence class $[p]_\alpha$ contains only the element p, for every $p \in \mathcal{G}$. This implies that the topology T_R is the discrete topology, that is $T_R = \mathcal{P}(\mathcal{G})$. Then, group $\mathcal{G}$ endowed with the topology T_R is a topological group.

Consider the quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\alpha$ defined by $x \to [x]_\alpha$, for all $x \in \mathcal{G}$. We equip the factor group $\mathcal{G}/\alpha$ with the quotient topology $\tau = \{K \subseteq \mathcal{G}/\alpha : \pi^{-1}(K) \in T_R\}$. In general topology, not every quotient map is open.

Proposition 4.2. *The quotient map $\pi : (\mathcal{G}, T_R) \longrightarrow (\mathcal{G}/\alpha, \tau)$ is open.*

Proof. For any open set $V \in T_R$, we show that $\pi(V) \in \tau$,

$$\pi^{-1}(\pi(V)) = \pi^{-1}(\bigcup_{x \in V} [x]_\alpha) = \bigcup_{x \in V} \pi^{-1}([x]_\alpha) = V.$$

So $\pi^{-1}(\pi(V))$ is open set and hence, by definition of quotient topology, $\pi(V)$ is open □

Theorem 4.2. *$(\mathcal{G}/\alpha, \tau)$ is a topological group.*

Proof. For $x, y \in \mathcal{G}$, let $[x]_\alpha, [y]_\alpha$ be elements in $\mathcal{G}/\alpha$ such that $[x]_\alpha[y]_\alpha = [xy]_\alpha \in W \in \tau$. Since $\pi(xy) = \pi(x)\pi(y) = [xy]_\alpha$ then $xy \in \pi^{-1}(W)$. Being $(\mathcal{G}, T_R)$ a topological group and $xy \in \pi^{-1}(W)$, there exists $V_x, V_y \in T_R$ containing x, y respectively and $V_xV_y \subseteq \pi^{-1}(W)$. Notice that $\pi(V_x)\pi(V_y) = \pi(V_xV_y) \in \pi(\pi^{-1}(W)) = W$. Since $\pi(x) = [x]_\alpha \in \pi(V_x), \pi(y) = [y]_\alpha \in \pi(V_y)$ and by Proposition 4.2, we verified that the product operation on $\mathcal{G}/\alpha$ is continuous. Now, we have to show that the inverse operation is also continuous. Let $[x]_\alpha$ be an element in $\mathcal{G}/\alpha$ and $V \in \tau$ an open set containing the element $[x]_\alpha^{-1} = [x^{-1}]_\alpha$, then $\pi(x^{-1}) = [x^{-1}]_\alpha \in V$ which implies $x^{-1} \in \pi^{-1}(V)$. Since $(\mathcal{G}, T_R)$ is a topological group, there exists $U \in T_R$ containing $x^{-1} \in \mathcal{G}$ such that $U^{-1} = \{z^{-1} \in \mathcal{G} : z \in U\} \subseteq \pi^{-1}(V)$. Since $\pi(x) = [x]_\alpha \in \pi(U) \in \tau$ and $\pi(U^{-1}) = \pi(U)^{-1}$ then we have $\pi(U)^{-1} \subseteq \pi(\pi^{-1}(V)) = V$. Therefore the mapping $[x]_\alpha \to [x^{-1}]_\alpha$ is continuous and hence $(\mathcal{G}/\alpha, \tau)$ is a topological group. □

5. Conclusion

In this paper, we constructed factor groups caused by normal soft int-groups. With the help of this construction, we established the group Isomorphism theorems. Further research can examine the factor groups caused by normal soft uni-groups.

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$\mathcal{J}$-ω^*-open sets and $\mathcal{J}$-ω^*-topology in ideal topological spaces

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Abstract. The aim of this study is to introduce $\mathcal{J}$-ω^*-open sets as a new set in ideal space which form topology on $\mathcal{X}$ known as $\mathcal{T}_{\mathcal{J}\omega^*}$ (or $\mathcal{J}$-ω^*-topology) which is strictly placed between $\mathcal{T}_{\omega^*}$ and $\mathcal{T}_{\omega}$. Additionally, we investigate the relationships of $\mathcal{J}$-ω^*-open sets with some other classes of sets.

Keywords: Ideal topological spaces, ω-topology, ω^*-topology, $\mathcal{J}$-ω^*-topology, ω-open sets, ω^*-open sets and $\mathcal{J}$-ω^*-open sets.
MSC 2020: 54A05, 54A10, 54A15, 54A20

1. Introduction

Kuratowski [17] and Vaidyanathaswamy [22] introduced the ideal topological space. Jankovic and Hamlett [4] presented the $\mathcal{J}$-open set in 1990. Dontchev [15] introduced *pre*-$\mathcal{J}$-open set in 1996. The concept of α-$\mathcal{J}$-open (resp., semi-$\mathcal{J}$-open, β-$\mathcal{J}$-open) set introduced by Hatir and Noiri [5]. $(\mathcal{X}, \mathcal{T})$ will be used to denote topological space in this paper, without losing any of separation qualities. The set of all real (resp., rational, irrational, and natural) numbers is denoted by R (resp., Q, Irr, N). Also, $P(\mathcal{X})$ mean the collection of all subsets of $\mathcal{X}$. We will denote the closure of any $\mathcal{H} \subseteq \mathcal{X}$ (resp., ω-closure, interior, ω-interior, θ-interior and δ-interior) of $\mathcal{H}$ by $cl\mathcal{H}$ (resp., $cl_\omega\mathcal{H}$, $int\mathcal{H}$,

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$int_\omega\mathcal{H}$, $int_\theta\mathcal{H}$ and $int_\delta\mathcal{H}$). An ideal $\mathcal{J}$ on $(\mathcal{X},\mathcal{T})$ is a nonempty collection of subsets of $\mathcal{X}$ which satisfies the following conditions:

1. If $\mathcal{H}\in\mathcal{J}$, $\mathcal{L}\subseteq\mathcal{H}$ implies $\mathcal{L}\in\mathcal{J}$.

2. If $\mathcal{H}\in\mathcal{J}$, $\mathcal{L}\in\mathcal{J}$ implies $\mathcal{H}\cup\mathcal{L}\in\mathcal{J}$.

If $\mathcal{J}$ is an ideal on $\mathcal{X}$, then $(\mathcal{X},\mathcal{T},\mathcal{J})$ is called an ideal topological space or ideal space. A set operator $(.)^*: P(\mathcal{X})\to P(\mathcal{X})$, called a local function of $\mathcal{H}$ with respect to $\mathcal{T}$ and $\mathcal{J}$ is defined as follows for: $\mathcal{H}\subseteq\mathcal{X}$, $\mathcal{H}^*=\{x\in\mathcal{X}: \mathcal{L}\cap\mathcal{H}\notin\mathcal{J}$ for each $x\in\mathcal{L}\subseteq\mathcal{T}\}$ [22]. Furthermore, in [17], [4] Kuratowski introduced $cl^*(.)$ defined by $cl^*(\mathcal{H})=\mathcal{H}\cup\mathcal{H}^*$ which construct a new topology on $\mathcal{X}$ finer than $\mathcal{T}$, it denoted by $\mathcal{T}^*$ called $*$-topology on $\mathcal{X}$, the members of $\mathcal{T}^*$ are called $\mathcal{T}^*$-open ($*$-open) sets. We will write the interior of $\mathcal{H}$ by $int^*(\mathcal{H})$ in $(\mathcal{X},\mathcal{T}^*)$ for every subset $\mathcal{H}$ of an ideal space$(\mathcal{X},\mathcal{T},\mathcal{J})$. The notion of ω-open set defined by Hdeib [13] several types of ω-open sets are introduced such as (ω^o-open, ω_θ-*open*, ω_δ-open, ω_p-open and ω^*-open) by (Al-Hamary et. al. [24], Ekici et. al. [8], Darwesh [11], Darwesh [10] and Darwesh and Shareef [12]). $(\mathcal{T}_\omega, \mathcal{T}_\theta, \mathcal{T}_{\omega^*})$ denote the families of (ω-open, θ-open, ω^*-open) which they are forms a topology on $\mathcal{X}$. Besides, O. Ravi, P. Sekar and K. Vidhyalakshmi [21] defined the notion of α-$\mathcal{J}_\omega$-open (resp., *pre*-$\mathcal{J}_\omega$-open, *b*-$\mathcal{J}_\omega$-open, β-$\mathcal{J}_\omega$-open) set in ideal space, which is weaker than the ω-*open* set.

In this study, by using a new notion $\mathcal{J}$-ω^*-open sets we construct a new topology $\mathcal{T}_{\mathcal{J}\omega^*}$ on $(\mathcal{X},\mathcal{T})$. Then, we show that $\mathcal{T}_{\mathcal{J}\omega^*}$ is strictly stronger than $\mathcal{T}_{\omega^*}$ (ω^*-topology) and weaker than $\mathcal{T}_\omega$ (ω-topology). Finally, we discussed several basic properties

2. Preliminaries

Definition 2.1. *A subset $\mathcal{H}$ of a space $(\mathcal{X},\mathcal{T})$ is said to be θ-open [20] (resp., θ_ω-open [23]), if for any $x\in\mathcal{H}$, there is an open set $\mathcal{F}$ containing x such that $x\in\mathcal{F}\subseteq cl\mathcal{F}\subseteq\mathcal{H}$ (resp., $x\in\mathcal{F}\subseteq cl_\omega\mathcal{F}\subseteq\mathcal{H}$).*

Definition 2.2. *A subset $\mathcal{H}$ of a space $(\mathcal{X},\mathcal{T})$ is said to be ω-open [13] (resp., ω^*-open [12], ω^o-open [24], ω_θ-open [8], ω_δ-open [11]), if for each $x\in\mathcal{H}$, there is an open $\mathcal{F}$ set containing x such that $\mathcal{F}\backslash\mathcal{H}$ (resp., $cl\mathcal{F}\backslash\mathcal{H}$, $\mathcal{F}\backslash int\mathcal{H}$, $\mathcal{F}\backslash int_\theta\mathcal{H}$, $\mathcal{F}\backslash int_\delta\mathcal{H}$) is a countable subset of $\mathcal{X}$.*

Definition 2.3. *A subset $\mathcal{H}$ of a space $(\mathcal{X},\mathcal{T})$ is said to be ω_p-open [10], if $\mathcal{H}\subseteq intcl_\omega(\mathcal{H})$.*

Definition 2.4. *A subset $\mathcal{H}$ of an ideal space $(\mathcal{X},\mathcal{T},\mathcal{J})$ is said to be α-$\mathcal{J}$-open [5] (resp., semi-$\mathcal{J}$-open [5], pre-$\mathcal{J}$-open [15], b-$\mathcal{J}$-open [3], strongly-β-$\mathcal{J}$-open[6])set, if $\mathcal{H}\subseteq int(cl^*(int\mathcal{H}))$ (resp., $\mathcal{H}\subseteq cl^*(int\mathcal{H}), \mathcal{H}\subseteq int(cl^*\mathcal{H})$, $\mathcal{H}\subseteq int(cl^*\mathcal{H})\cup cl^*(int\mathcal{H})$, $\mathcal{H}\subseteq cl^*(int(cl^*\mathcal{H}))$).*

The next definitions and result from [21]:

Definition 2.5. *A subset $\mathcal{H}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be α-$\mathcal{J}_\omega$-open (resp., pre-$\mathcal{J}_\omega$-open, b-$\mathcal{J}_\omega$-open, β-$\mathcal{J}_\omega$-open), if $\mathcal{H} \subseteq int_\omega (cl^* (int_\omega \mathcal{H}))$ (resp., $\mathcal{H} \subseteq int_\omega (cl^* \mathcal{H})$, $\mathcal{H} \subseteq int_\omega (cl^* \mathcal{H}) \cup cl^* (int_\omega \mathcal{H})$, $\mathcal{H} \subseteq cl^* (int_\omega (cl^* \mathcal{H}))$.*

Theorem 2.1. *For a subset of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$, the following properties hold:*

1. *Every ω-open set is α-$\mathcal{J}_\omega$ -open.*
2. *Every α-$\mathcal{J}_\omega$- open set is pre-$\mathcal{J}_\omega$ -open.*
3. *Every pre-$\mathcal{J}_\omega$- open set is b-$\mathcal{J}_\omega$ -open.*
4. *Every b-$\mathcal{J}_\omega$- open set is β-$\mathcal{J}_\omega$ -open.*

Proposition 2.1 ([1])**.** *Let $\mathcal{H}$ be a subset of $(\mathcal{X}, \mathcal{T}, \mathcal{J})$. If $\mathcal{J} = \{\emptyset\}$ (resp., $\mathcal{J} = P(\mathcal{X})$), then $\mathcal{H}^* = cl\mathcal{H}$ (resp., $\mathcal{H}^* = \emptyset$) and $cl^*\mathcal{H} = cl\mathcal{H}$ (resp., $cl^*\mathcal{H} = \mathcal{H}$).*

The next definition and result from [25]:

Definition 2.6. *Let $\mathcal{H} \subseteq \mathcal{X}$, is said to be α-ω-open (resp., pre-ω-open, b-ω-open, β-ω-open), if $\mathcal{H} \subseteq int_\omega (cl (int_\omega \mathcal{H}))$ (resp., $\mathcal{H} \subseteq int_\omega (cl\mathcal{H})$, $\mathcal{H} \subseteq int_\omega (cl\mathcal{H}) \cup cl (int_\omega \mathcal{H})$, $\mathcal{H} \subseteq cl (int_\omega (cl\mathcal{H}))$.*

Lemma 2.1. *For a subset of a topological space $(\mathcal{X}, \mathcal{T})$, the following properties hold:*

1. *Every ω-open set is α-ω-open.*
2. *Every α-ω-open set is pre-ω-open.*
3. *Every pre-ω-open set is b-ω-open.*
4. *Every b-ω-open set is β-ω-open.*

Definition 2.7. *A space $(\mathcal{X}, \mathcal{T})$ is defined as:*

1. *Locally countable [18] if every point of $\mathcal{X}$ has a countable open neighbourhood.*
2. *Hyperconnected [14] if each nonempty open subsets of $\mathcal{X}$ is dense in $\mathcal{X}$.*

Definition 2.8. *A subset $\mathcal{H}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is called:*

1. *$*$-dense in itself [16] if and only if $\mathcal{H} \subseteq \mathcal{H}^*$.*
2. *$*$-dense [7] if $cl^* (\mathcal{H}) = \mathcal{X}$.*
3. *$\mathcal{J}$-open set [19] if $\mathcal{H} \subseteq int\mathcal{H}^*$.*

Definition 2.9 ([7])**.** *An ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be $*$-hyperconnected. If $\mathcal{H}$ is $*$-dense, for any nonempty open subset $\mathcal{H}$ of $\mathcal{X}$.*

Lemma 2.2 ([27]). *If $\mathcal{H}$ is $*$- dense in itself, then $\mathcal{H}^* = cl\,(\mathcal{H}) = cl^*(\mathcal{H})$.*

Definition 2.10 ([2]). *An ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is a $R\mathcal{J}$-space if for each $x \in \mathcal{X}$ and every open set $\mathcal{F}$ containing x, there exists an open set $\mathcal{L}$ such that $x \in \mathcal{L} \subseteq cl^*\mathcal{L} \subseteq \mathcal{F}$.*

Lemma 2.3 ([9]). *Let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{X}$. Then, $cl^*_{\mathcal{M}}\mathcal{N} = cl^*(\mathcal{N}) \cap \mathcal{M}$.*

3. $\mathcal{J}$-ω^*-open sets with their relations with some other types of sets

This section establishes a new topology in the ideal space and introduces a new set called $\mathcal{J}$-ω^*- open sets. We also investigated their connections to other types of sets.

Definition 3.1. *A subset $\mathcal{H}$ of an ideal topological space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be an $\mathcal{J}$-ω^*- open set, if for each $x \in \mathcal{H}$,* there is an *open* set *$\mathcal{F}$ containing x such that $cl^*\mathcal{F} \backslash \mathcal{H}$ is a countable set. Also, $\mathcal{H}$ is said to be $\mathcal{J}$-ω^*-closed, if $\mathcal{X} \backslash \mathcal{H}$ is $\mathcal{J}$-ω^*-open.*

Remark 3.1. *In any ideal space $(X, \tau, \mathcal{J})$, it is clear that X and $\emptyset$ are always $\mathcal{J}$-ω^*-open sets.*

Theorem 3.1. *A subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is $\mathcal{J}$-ω^*-open if and only if for every $x \in \mathcal{M}$, there is an open set $\mathcal{F}_x$ containing x and a countable set $\mathcal{C}_x$ which does not containing x such that $cl^*\mathcal{F}_x \backslash \mathcal{C}_x \subseteq \mathcal{M}$.*

Proof. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-open subset of $\mathcal{X}$ and $x \in \mathcal{M}$, there exists $\mathcal{F}_x \in \mathcal{T}$ such that $x \in \mathcal{F}_x$ and $cl^*\mathcal{F}_x \backslash \mathcal{M}$ is a countable set. Then, $\mathcal{C}_x = cl^*\mathcal{F}_x \backslash \mathcal{M}$ is a countable set and $x \notin \mathcal{C}_x$. It remains to show that $cl^*\mathcal{F}_x \backslash \mathcal{C}_x \subseteq \mathcal{M}$. Then, $cl^*\mathcal{F}_x \backslash \mathcal{C}_x = cl^*\mathcal{F}_x \backslash (cl^*\mathcal{F}_x \backslash \mathcal{M}) = cl^*\mathcal{F}_x \backslash (cl^*\mathcal{F}_x \cap \mathcal{X} \backslash \mathcal{M}) = (cl^*\mathcal{F}_x \backslash cl^*\mathcal{F}_x) \cup (cl^*\mathcal{F}_x \cap \mathcal{M})$. Hence, $cl^*\mathcal{F}_x \backslash \mathcal{C}_x \subseteq \mathcal{M}$.

Conversely, let $x \in \mathcal{M}$, consequently by our hypothesis, there are open set $\mathcal{F}_x$ containing x and countable set $\mathcal{C}_x$ such that $x \notin \mathcal{C}_x$ and $cl^*\mathcal{F}_x \backslash C_x \subseteq \mathcal{M}$. This implies, $cl^*\mathcal{F}_x \backslash \mathcal{M} \subseteq \mathcal{C}_x$. Therefore, $\mathcal{M}$ is an $\mathcal{J}$-ω^*-open subset of $\mathcal{X}$. $\square$

Theorem 3.2. *If $\mathcal{M}$ is an $\mathcal{J}$-ω^*-closed subset of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$, then $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$ for a countable set $\mathcal{C}$ and a closed set $\mathcal{H}$.*

Proof. If $\mathcal{M}$ is equal to $\mathcal{X}$. Putting $\mathcal{H} = \mathcal{M}$ and $\mathcal{C} = \emptyset$, we get $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$. Otherwise, let x be an arbitrary point in $\mathcal{X}$ such that $x \notin \mathcal{M}$. Since $\mathcal{X} \backslash \mathcal{M}$ is $\mathcal{J}$-ω^*-open, consequently by Theorem 3.1, there exists $\mathcal{F} \in \mathcal{T}$ containing x and a countable set $\mathcal{C}_x$ which does not contains x such that $cl^*\mathcal{F}_x \backslash \mathcal{C}_x \subseteq \mathcal{X} \backslash \mathcal{M}$. Then, $\mathcal{H} = \mathcal{X} \backslash \mathcal{F}$ and $\mathcal{C}$ are the requisite closed set and a countable set. $\square$

Theorem 3.3. *A subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is $\mathcal{J}$-ω^*-closed if and only if $\mathcal{M} = \mathcal{X}$ or for any x not belong to $\mathcal{M}$, there is a closed set $\mathcal{H}$ and a countable set $\mathcal{C}$ such that $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$.*

Proof. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-*closed* subset of $\mathcal{X}$. Then, either $\mathcal{M} = \mathcal{X}$ or $\mathcal{M} \subset \mathcal{X}$. If $\mathcal{M} = \mathcal{X}$, then there is nothing to prove, otherwise $\mathcal{M}$ is a proper $\mathcal{J}$ -ω^*-*closed* subset of $\mathcal{X}$, then by Theorem 3.2, a closed set $\mathcal{H}$ and a countable set $\mathcal{C}$ exist such that $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$.

Conversely, if $\mathcal{M} = \mathcal{X}$, then it is $\mathcal{J}$-ω^*-*closed.* Otherwise, let for each $x \in \mathcal{X}\backslash\mathcal{M}$, there is a closed set $\mathcal{H}$ and a countable set $\mathcal{C}$ such that $\mathcal{M} \subseteq int^*\mathcal{H} \cup \mathcal{C}$. Then, $\mathcal{F} = \mathcal{X}\backslash\mathcal{H}$ is an *open* subset of $\mathcal{X}$ which contains x and $cl^*\mathcal{F}\backslash\mathcal{C} = cl^*(\mathcal{X}\backslash\mathcal{H})\backslash\mathcal{C}$. But, $cl^*(\mathcal{X}\backslash\mathcal{H}) = (\mathcal{X}\backslash\, int^*\mathcal{H})$ [26] thus, $cl^*\mathcal{F}\backslash\mathcal{C} = cl^*(\mathcal{X}\backslash\mathcal{H})\backslash\mathcal{C} = (\mathcal{X}\backslash int^*\mathcal{H})\backslash\mathcal{C} = (\mathcal{X}\backslash int^*\mathcal{H}) \cap (\mathcal{X}\backslash\mathcal{C}) = \mathcal{X}\backslash(int^*\mathcal{H} \cup \mathcal{C}) \subseteq \mathcal{X}\backslash\mathcal{M}$. This by Theorem 3.1 implies, $\mathcal{X}\backslash\mathcal{M}$ is $\mathcal{J}$-ω^*-open. Thus, $\mathcal{M}$ is $\mathcal{J}$-ω^*-*closed.* □

Theorem 3.4. *The intersection of two $\mathcal{J}$-ω^*-open sets is $\mathcal{J}$-ω^*-open.*

Proof. Let $\mathcal{M}$ and $\mathcal{P}$ be two $\mathcal{J}$-ω^*- *open* sets. If $\mathcal{M} \cap \mathcal{P} = \emptyset$, then there is nothing to prove. Otherwise, for $x \in \mathcal{M} \cap \mathcal{P}$, there are two open sets $\mathcal{G}$ and $\mathcal{L}$ containing x such that $cl^*\mathcal{G}\backslash\mathcal{M}$ and $cl^*\mathcal{L}\backslash\mathcal{P}$ are countable sets. Since $cl^*(\mathcal{G} \cap \mathcal{L}) \backslash (\mathcal{M} \cap \mathcal{P}) \subseteq (cl^*\mathcal{G} \cap cl^*\mathcal{L}) \cap (\mathcal{X}\backslash (\mathcal{M} \cap \mathcal{P}) = (cl^*\mathcal{G} \cap cl^*\mathcal{L}) \cap ((\mathcal{X}\backslash\mathcal{M}) \cup (\mathcal{X}\backslash\mathcal{P})) \subseteq (cl^*\mathcal{G} \cap (\mathcal{X}\backslash\mathcal{M})) \cup (cl^*\mathcal{L} \cap (\mathcal{X}\backslash\mathcal{P})) = (cl^*\mathcal{G}\backslash\mathcal{M}) \cup (cl^*\mathcal{L}\backslash\mathcal{P})$. Which means that, $cl^*(\mathcal{G} \cap \mathcal{L}) \backslash (\mathcal{M} \cap \mathcal{P})$ is countable. Hence, $\mathcal{M} \cap \mathcal{P}$ is $\mathcal{J}$-ω^*-open. □

Corollary 3.1. *The union of two $\mathcal{J}$-ω^*-closed sets is $\mathcal{J}$-ω^*-closed.*

Proof. It follows Theorem 3.4. □

Theorem 3.5. *The union (resp., intersection) of each family of $\mathcal{J}$-ω^*-open (resp., $\mathcal{J}$ -ω^*- closed) sets in any ideal topological space is $\mathcal{J}$ -ω^*-open (resp., $\mathcal{J}$ -ω^*-closed).*

Proof. Let $\{\mathcal{M}_\gamma\colon \gamma \in \Delta\}$ be any each family of $\mathcal{J}$-ω^*-open sets and $x \in \bigcup_{\gamma\in\Delta} \mathcal{M}_\gamma$. Then, there is $\gamma_\circ \in \Delta$ and an open set $\mathcal{F}$ such that $x \in \mathcal{F} \cap \mathcal{M}_{\gamma_\circ}$ and $cl^*\mathcal{F}\backslash\mathcal{M}_{\gamma_\circ}$ is a countable set. Since, $cl^*\mathcal{F}\backslash(\bigcup_{\gamma\in\Delta} \mathcal{M}_\gamma) \subseteq cl^*\mathcal{F}\backslash\mathcal{M}_{\gamma_\circ}$. Thus, $\bigcup_{\gamma\in\Delta} \mathcal{M}_\gamma$ $\mathcal{J}$ -ω^*-open. □

We denote $\mathcal{T}_{\mathcal{J}\omega^*}$ to the family of all $\mathcal{J}$-ω^*-open.

Corollary 3.2. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space. Then, $\mathcal{T}_{\mathcal{J}\omega^*}$ form topology on $\mathcal{X}$. Hence, $(X, \mathcal{T}_{\mathcal{J}\omega^*}, \mathcal{J})$ is an ideal topological space.*

Proof. It follows from Remark 3.1, Theorem 3.4 and Theorem 3.5. □

The new topology of the Corollary 3.2, known as $\mathcal{J}$-ω^*-topology.

Proposition 3.1. *Every ω^*-open set in any ideal space $(\mathcal{X}, \mathcal{T}, J)$ is $\mathcal{J}$-ω^*-open.*

Proof. Let $\mathcal{M}$ be an ω^*-open subset of $\mathcal{X}$ and x belong to $\mathcal{M}$. Consequently, there is an open set $\mathcal{F}$ containing x such that $cl\mathcal{F}\backslash\mathcal{M}$ is a countable set. Since $cl^*\mathcal{F} \subseteq cl\mathcal{F}$, then $cl^*\mathcal{F}\backslash\mathcal{M} \subseteq cl\mathcal{F}\backslash\mathcal{M}$ and hence $cl^*\mathcal{F}\backslash\mathcal{M}$ is a countable set. Therefore, $\mathcal{M}$ is an $\mathcal{J}$-ω^*-open set. □

The converse of Proposition 3.1, on the other hand does not have to be correct as demonstrated by the following example:

Example 3.1. In $(R, \mathcal{T})$ with $\mathcal{T} = \{\emptyset, Q, R\}$ and $\mathcal{J} = P(R)$. Then, the set $\mathcal{M} = Q$ is not ω^*-open but it is $\mathcal{J}$-ω^*-open. Since, for each $x \in Q$ there is $Q \in \mathcal{T}$ with $cl^*Q = Q$ but $clQ = R$.

Proposition 3.2. *In any ideal space* $(\mathcal{X}, \mathcal{T}, \mathcal{J})$, $\mathcal{T}_\theta \subseteq \mathcal{T}_{\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$.

Proof. From [[12], Theorem 3.2] we have $\mathcal{T}_\theta \subseteq \mathcal{T}_{\omega^*}$ and by Proposition 3.1, $\mathcal{T}_{\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$ it remains to show that $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-open set. If $\mathcal{M}$ is empty, then $\mathcal{M} \in \mathcal{T}_\omega$. Otherwise, for any arbitrary point x in $\mathcal{M}$; there exists $\mathcal{F} \in \mathcal{T}$ containing x such that $cl^*\mathcal{F} \backslash \mathcal{M}$ is a countable set. Since $\mathcal{F} \backslash \mathcal{M} \subseteq cl^*\mathcal{F} \backslash \mathcal{M}$. Therefore, $\mathcal{F} \backslash \mathcal{M}$ is countable, implying that $\mathcal{M} \in \mathcal{T}_\omega$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$. $\square$

In general, the converse of Proposition 3.2, is not true. As illustrated in the following examples:

Example 3.2. Let $\mathcal{X} = \{a, b, c, d\}$ with $\mathcal{T} = \{\emptyset, \mathcal{X}, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. The set $\mathcal{M} = \{a, d\}$ is an $\mathcal{J}$ -ω^*-open set, but it is not θ-open.

Example 3.3. In the space R with topology $\mathcal{T} = \{\emptyset, R, Q\}$ and $\mathcal{J} = \{\emptyset\}$, the set $\mathcal{M} = Q \in \mathcal{T}_\omega$. But, $clQ = cl^*Q = R$ then $cl^*Q \backslash \mathcal{M} = Irr$ is uncountable. Hence, M is not $\mathcal{J}$ -ω^*-open.

Proposition 3.3. *If* $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ *is any ideal space such that* $\mathcal{X}$ *is a locally countable space, then* $\mathcal{T}_{\mathcal{J}\omega^*} = P(\mathcal{X})$.

Proof. Let $\mathcal{M}$ be any subset of $\mathcal{X}$. If $\mathcal{M} = \emptyset$, then $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Otherwise, for any $x \in \mathcal{M}$, the set $\mathcal{X}$ is open which contain x, and $cl^*\mathcal{X} = \mathcal{X}$ is countable, so $cl^*\mathcal{X} \backslash \mathcal{M}$ is also countable, therefore, $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} = P(\mathcal{X})$. $\square$

Corollary 3.3. *Every* $\mathcal{J}$-ω^**-open set is* α-$\mathcal{J}_\omega$*-open (resp. pre-*$\mathcal{J}_\omega$*-open, b-*$\mathcal{J}_\omega$*-open and* β-$\mathcal{J}_\omega$*-open).*

Proof. Proposition 3.2 and Theorem 2.1 provide the proof. $\square$

The following example shows that the converse of Corollary 3.3, is not true:

Example 3.4. From [[21], Example 3.1], consider R be a space with $\mathcal{T} = \{\emptyset, R, Q\}$ and $\mathcal{J} = \{\emptyset\}$. Then, $\mathcal{N} = Q \cup \{\sqrt{2}\}$ is an α-$\mathcal{J}_\omega$ *–open* set, since $int_\omega \mathcal{N} = Q$, $cl^*(int_\omega \mathcal{N}) = cl(Q) = R$. Therefore, $\mathcal{N} \subseteq int_\omega(cl^*(int_\omega \mathcal{N}))$. Thus, $\mathcal{N}$ is *pre*-$\mathcal{J}_\omega$-open (*resp.* b-$\mathcal{J}_\omega$ -open and β-$\mathcal{J}_\omega$ -open). But, $\mathcal{N} \notin \mathcal{T}_{\mathcal{J}\omega^*}$.

Corollary 3.4. *Every* $\mathcal{J}$-ω^**-open set is* α-ω*-open (resp., pre-*ω*-open, b-*ω*-open and* β-ω*–open).*

Proof. It follows from Proposition 3.2 and Lemma 2.1. □

However, as shown in the following example, the converse of Corollary 3.4 is incorrect:

Example 3.5. In the space R with $\mathcal{T} = \{\emptyset, R, Q\}$ and ideal $\mathcal{J} = \{\emptyset\}$. If the set $\mathcal{P} = Q$, then $\mathcal{P}$ is α-ω-open (*resp. pre*-ω-open, b-ω-open and β-ω-open). Since $int_\omega \mathcal{P} = Q$, $clint_\omega \mathcal{P} = R$, $int_\omega (clint_\omega \mathcal{P}) = R$. Thus, $\mathcal{P} \subseteq int_\omega (clint_\omega \mathcal{P})$. This implies that $\mathcal{P}$ is α-ω–open and from Lemma 2.1, $\mathcal{P}$ is *pre*-ω-open, b-ω-open and β-ω-open. But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$.

The examples below show that the concept of $\mathcal{J}$ -ω^*-open is independent of the classes open (*preopen*, $\mathcal{J}$-*open*, α-$\mathcal{J}$-open, *pre*-$\mathcal{J}$-open, *semi*-$\mathcal{J}$-open, b-$\mathcal{J}$-open and *strongly* β-$\mathcal{J}$-open) sets.

Example 3.6.

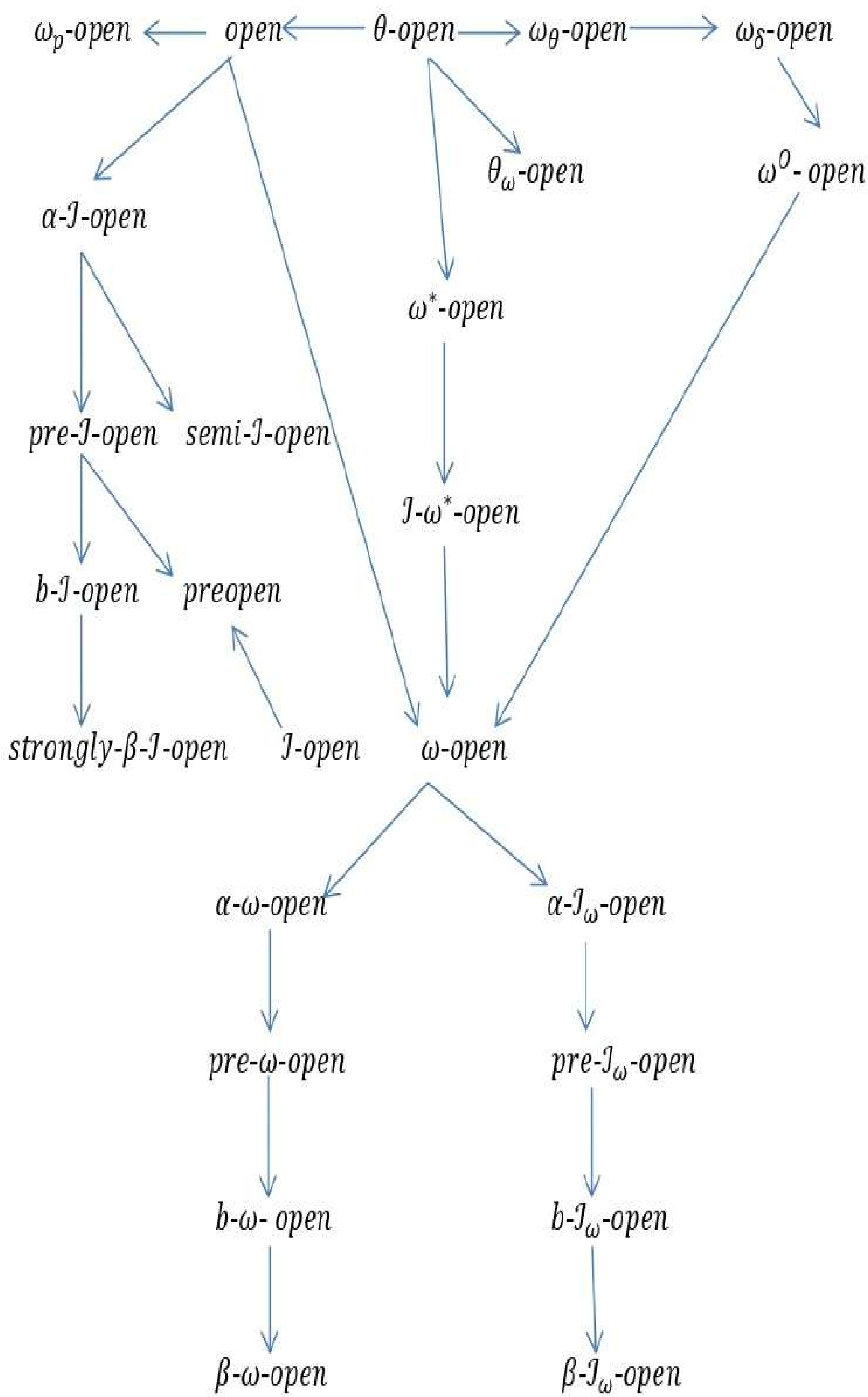
1. In the space R with $\mathcal{T} = \{\emptyset, R, Q\}$ and ideal $\mathcal{J} = \{\emptyset\}$. If the set $\mathcal{P} = Q$, then $\mathcal{P}$ is open (*peropen*, α-$\mathcal{J}$-*open*, *pre*-$\mathcal{J}$-open, *semi*-$\mathcal{J}$-open, b-$\mathcal{J}$-open and *strongly* β-$\mathcal{J}$-open). But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$. Since for each $x \in Q$, there is $Q \in \mathcal{T}$ and $cl\,(Q) = cl^*\,(Q) = R$.
2. Let $\mathcal{X} = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{a\}, \mathcal{X}\}$ and $\mathcal{J} = \{\emptyset, \{a\}\}$. Then, the set $\mathcal{M} = \{a, c\}$ is $\mathcal{J}$ -ω^*-open but not *open*, $\mathcal{J}$-open, *semi*-$\mathcal{J}$-open and α-$\mathcal{J}$-open.
3. Let $\mathcal{X} = \{a, b\}$, $\mathcal{T} = \{\emptyset, \{a\}, \mathcal{X}\}$ and $\mathcal{J} = \{\emptyset\}$. Then, the set $\mathcal{M} = \{b\}$ is $\mathcal{J}$ -ω^*-open but not *pre*-$\mathcal{J}$-open, b-$\mathcal{J}$-open and *strongly*β-$\mathcal{J}$-open.
4. In R with usual topology $\mathcal{T}_u$ and $\mathcal{J} = \mathcal{F}$ (all finite subsets of R's ideal). Then, $\mathcal{P} = Q$ is $\mathcal{J}$-open since $\mathcal{P}^* = Q^* = R$. Implies that, $\mathcal{P} \subseteq int(\mathcal{P}^*)$ but $\mathcal{P}$ is not $\mathcal{J}$-ω^*-open since $cl^*\,(Q) = R$ and $cl^*\,(Q) \backslash Q$ is not countable.

We have examples that demonstrate the independence of the notion of $\mathcal{J}$-ω^*-open set with each of the classes ω_p-open, ω_θ-open, ω_δ-open, ω^o -open and θ_ω-open is independent.

Example 3.7.

1. In the space$(R, \mathcal{T})$ with $\mathcal{T} = \{\emptyset, Q, R\}$ and $\mathcal{J} = \{\emptyset\}$. Then, the set $\mathcal{P} = Q$ is ω_p-open (*resp.* ω_θ-open, ω_δ-open, ω^o -open and θ_ω-open). But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$.
2. In the indiscrete space $(R, \mathcal{T}_{ind})$ and $\mathcal{J} = \{\emptyset\}$. Let $\mathcal{P} = R \backslash \{0\}$ is $\mathcal{J}$ -ω^*-open but it is not ω_θ-open, ω_δ-open, ω^o -open and θ_ω-open.
3. In Example 3.6.(3), assume $\mathcal{M} = \{b\}$ is $\mathcal{J}$ -ω^*-open but not ω_p-open since $\{b\} \notin \mathcal{T}$. Since $cl_\omega\,(\mathcal{M}) = \{b\}$, then $intcl_\omega\,(\mathcal{M}) = \emptyset$. This implies that, $\mathcal{M} \nsubseteq intcl_\omega\,(\mathcal{M})$.

Thus, from Proposition 3.1, Proposition 3.2, Corollary 3.3, Corollary 3.4, Example 3.6 and Example 3.7 we have the following diagram:



ω_p-open
open
θ-open
ω_θ-open
ω_δ-open
θ_ω-open
ω^O- open
α-$\mathcal{J}$-open
ω^*-open
pre-$\mathcal{J}$-open
semi-$\mathcal{J}$-open
$\mathcal{J}$-ω^*-open
b-$\mathcal{J}$-open
preopen
strongly-β-$\mathcal{J}$-open
$\mathcal{J}$-open
ω-open
α-ω-open
α-$\mathcal{J}_\omega$-open
pre-ω-open
pre-$\mathcal{J}_\omega$-open
b-ω- open
b-$\mathcal{J}_\omega$-open
β-ω-open
β-$\mathcal{J}_\omega$-open

4. Some other properties of $\mathcal{J}$-ω^*-open sets

This section investigates further aspects of $\mathcal{J}-\omega^*$-open sets and the topology $\mathcal{T}_{\mathcal{J}\omega^*}$, beginning with the following definition.

Definition 4.1. *A point x of a subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is said to be an $\mathcal{J}$ -$*$-condensation point, if $cl^*\mathcal{F} \cap \mathcal{M}$ is an uncountable set for each open set $\mathcal{F}$ containing x. The set of all $\mathcal{J}$-$*$-condensation points of a set $\mathcal{M}$ is denoted by $\mathcal{J}$-cond$^*(\mathcal{M})$.*

Theorem 4.1. *A subset $\mathcal{M}$ of an ideal space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is $\mathcal{J}$-ω^*-closed if and only if $\mathcal{J}$-cond$^*(\mathcal{M}) \subseteq \mathcal{M}$.*

Proof. Let $\mathcal{M}$ be an $\mathcal{J}$-ω^*-*closed* subset of $\mathcal{X}$ and $x \in \mathcal{J}$-*cond*$^*(\mathcal{M})$. On contrary we suppose that $x \notin \mathcal{M}$, there exists an open set $\mathcal{F}$ containing x such that $cl^*\mathcal{F} \backslash (\mathcal{X} \backslash \mathcal{M})$ is countable. This means that, $cl^*\mathcal{F} \cap \mathcal{M}$ is countable. Hence, $x \notin \mathcal{J}$-*cond*$^*(\mathcal{M})$ which is a contradiction. Then, $\mathcal{J}$-*cond*$^*(\mathcal{M}) \subseteq \mathcal{M}$.

Conversely, suppose that $\mathcal{J}$-*cond*$^*(\mathcal{M}) \subseteq \mathcal{M}$ and $x \notin \mathcal{M}$, then there is an *open* set $\mathcal{F}$ containing x such that $cl^*\mathcal{F} \cap \mathcal{M}$ is countable. This indicates that, $cl^*\mathcal{F} \backslash (\mathcal{X} \backslash \mathcal{M})$ is countable. So, $\mathcal{X} \backslash \mathcal{M}$ is $\mathcal{J}$-ω^*-open. Therefore, $\mathcal{M}$ is $\mathcal{J}$-ω^*-*closed.* □

Corollary 4.1. *Each countable subset of any ideal space is $\mathcal{J}$-ω^*-closed.*

Proof. If $\mathcal{M}$ is countable, then $\mathcal{J}$-*cond** $(\mathcal{M}) = \emptyset$. So, by Theorem 4.1, $\mathcal{M}$ is $\mathcal{J}$-ω^*-*closed.* □

Proposition 4.1. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is any ideal space, then $\mathcal{T}_{coc} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. If $\mathcal{M} \in \mathcal{T}_{coc}$, then $\mathcal{X} \backslash \mathcal{M}$ is countable subset of $\mathcal{X}$, subsequently by Corollary 4.1, $\mathcal{X} \backslash \mathcal{M}$ is $\mathcal{J}$-ω^*-*closed.* Therefore, $\mathcal{M}$ is $\mathcal{J}$ -ω^*-open subset of $\mathcal{X}$. □

Theorem 4.2. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is a $*$- hyperconnected space, then $\mathcal{T}_{\mathcal{J}\omega^*}$ is the co-countable topology on $\mathcal{X}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. If $\mathcal{M}$ is an empty set, then $\mathcal{M} \in \mathcal{T}_{coc}$. Otherwise, we choose any arbitrary point x in $\mathcal{M}$, and an open set $\mathcal{F}$ containing x such that $cl^*\mathcal{F} \backslash \mathcal{M} = \mathcal{C}$ where $\mathcal{C}$ is a countable set. Since $\mathcal{X}$ is $*$-hyperconnected, so $cl^*\mathcal{F} = R$ and $\mathcal{M} = R \backslash \mathcal{C}$. Thus, $\mathcal{M} \in \mathcal{T}_{coc}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{coc}$. By Proposition 4.1, we have $\mathcal{T}_{coc} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. Therefore, $\mathcal{T}_{\mathcal{J}\omega^*} = \mathcal{T}_{coc}$. □

The opposite of Theorem 4.2 is generally incorrect, as illustrated in the next example:

Example 4.1. Let $\mathcal{X} = \{a, b, c, d\}$, $\mathcal{T} = \{\emptyset, \mathcal{X}, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}\}$. Then, by Proposition 3.3, $\mathcal{T}_{\mathcal{J}\omega^*} = P(\mathcal{X}) = \mathcal{T}_{coc}$. Clearly $(\mathcal{X}, \mathcal{T})$ is not a hyperconnected space and from [[7], Remark 3], $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is not $*$-hyperconnected space.

The following example shows that the requirement $*$- *hyperconnected* in Theorem 4.2, cannot be replaced by *hyperconnected*:

Example 4.2. Let $\mathcal{X} = R$, $\mathcal{T} = \{\emptyset, R, Q\}$ and $\mathcal{J} = P(R)$. Then, the space $(\mathcal{X}, \mathcal{T})$ is *hyperconnected* since $Q \in \mathcal{T}$ and $clQ = R$. As a result, $Q \in \mathcal{T}_{\mathcal{J}\omega^*}$ but $\mathcal{X} \backslash Q = Irr$ which is not countable. Thus, $Q \notin \mathcal{T}_{coc}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \neq \mathcal{T}_{coc}$.

In the Example 4.2, we can see that even if the space $\mathcal{X}$ is *hyperconneceted* $\mathcal{T}_{\mathcal{J}\omega^*} \neq \mathcal{T}_{coc}$, if $\mathcal{J} = \{\emptyset\}$. Consequently, $\mathcal{T}_{\mathcal{J}\omega^*} = \mathcal{T}_{coc}$ is yields the following result:

Corollary 4.2. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ is a hyperconnected space and $\mathcal{J} = \{\emptyset\}$, then $\mathcal{T}_{\mathcal{J}\omega^*}$ is the co-countable topology on $\mathcal{X}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$ .If $\mathcal{M}$ is empty, then $\mathcal{M} \in \mathcal{T}_{coc}$. Otherwise, if $\mathcal{M} \neq \emptyset$, let $x \in \mathcal{M}$ there is an open set $\mathcal{G}$ containing x such that $cl^*\mathcal{F} \backslash \mathcal{M} = \mathcal{C}$ where $\mathcal{C}$ is a countable set. Since $\mathcal{X}$ is *hyperconnected*, so $cl\mathcal{F} = R$. Since $cl^*\mathcal{F} = cl\mathcal{F}$ then $cl^*\mathcal{F} = R$ and $\mathcal{M} = R \backslash \mathcal{C}$. Thus, $\mathcal{M} \in \tau_{coc}$. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{coc}$. However, according to Proposition 4.1, we have $\mathcal{T}_{coc} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. As a result, $\mathcal{T}_{\mathcal{J}\omega^*} = \mathcal{T}_{coc}$. $\square$

Theorem 4.3. *If $\mathcal{T}$ and $\mathcal{P}$ are two topologies on $\mathcal{X}$ and $\mathcal{J}$ is any ideal on $\mathcal{X}$ such that $\mathcal{T} \subseteq \mathcal{P}$, then $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{P}_{\mathcal{J}\omega^*}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. If $\mathcal{M} = \emptyset$, then $\mathcal{M} \in \mathcal{P}_{\mathcal{J}\omega^*}$. Otherwise, if $\mathcal{M} \neq \emptyset$. Then, for each $x \in \mathcal{M}$, there is $\mathcal{F} \in \mathcal{T}$ containing x such that $cl^*_{\mathcal{T}}\mathcal{F} \backslash \mathcal{M}$ is a countable subset of $\mathcal{X}$. Since $\mathcal{T} \subseteq \mathcal{P}$ so $\mathcal{F} \in \mathcal{P}$ then $cl^*_{\mathcal{P}}\mathcal{F} \backslash \mathcal{M} \subseteq cl^*_{\mathcal{T}}\mathcal{F} \backslash \mathcal{M}$. Hence, $cl^*_{\mathcal{P}}\mathcal{F} \backslash \mathcal{M}$ is also a countable subset of $\mathcal{X}$. Thus, $\mathcal{M} \in \mathcal{P}_{\mathcal{J}\omega^*}$. Therefore, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{P}_{\mathcal{J}\omega^*}$. $\square$

Corollary 4.3. *If $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space, then $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq (\mathcal{T}^*)_{\mathcal{J}\omega^*}$.*

Proof. Since $\mathcal{T} \subseteq \mathcal{T}^*$, so by Theorem 4.3, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq (\mathcal{T}^*)_{\mathcal{J}\omega^*}$. $\square$

However, as the examples below show, the converse of Theorem 4.3 and Corollary 4.3, are not true:

Example 4.3.
1. Let $\mathcal{X} = N$, $\mathcal{J} = \{\emptyset\}$, $\mathcal{T} = \{\emptyset, \{0\}, N\}$ and $\sigma = \{\emptyset, \{1\}, N\}$. Then, by Proposition 3.3, $\mathcal{T}_{\mathcal{I}\omega^*} = P(\mathcal{X}) = \sigma_{\mathcal{I}\omega^*}$, but neither $\mathcal{T} \subseteq \sigma$ nor $\sigma \subseteq \mathcal{T}$.
2. In the space R with topology $\mathcal{T} = \{\emptyset, R, Irr\}$ and $\mathcal{J} = P(R)$. Then, the set $\mathcal{P} = Q \in (\mathcal{T}^*)_{\mathcal{J}\omega^*}$. But, $\mathcal{P} \notin \mathcal{T}_{\mathcal{J}\omega^*}$. Since $R \in \mathcal{T}$, so $cl^*(R) = R$. Implies that, $cl^*(R) \backslash \mathcal{P} = Irr$ which is uncountable.

Proposition 4.2. *Let $(\mathcal{X}, \mathcal{T})$ be a topological space and $\mathcal{J}, \mathcal{K}$ be two ideals on $\mathcal{X}$ in which $\mathcal{J} \subseteq \mathcal{K}$ Then, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\mathcal{K}\omega^*}$.*

Proof. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. If $\mathcal{M} = \emptyset$, then there is nothing to prove. Otherwise, for each $x \in \mathcal{M}$ there exists an open set $\mathcal{F}$ containing x such that $cl^*_{\mathcal{J}}\mathcal{F}\backslash\mathcal{M}$ is countable. Since $cl^*_{\mathcal{K}}\mathcal{F} \subseteq cl^*_{\mathcal{J}}\mathcal{F}$. As a result, $cl^*_{\mathcal{K}}\mathcal{F}\backslash\mathcal{M}$ is also countable. Hence, $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\mathcal{K}\omega^*}$. $\square$

The following example demonstrates that the converse of Proposition 4.2 is incorrect:

Example 4.4. Consider $(\mathcal{X}, \mathcal{T})$ where $\mathcal{X} = N$ and $\mathcal{T}$ is the indiscrete topology on $\mathcal{X}$. Let $\mathcal{J} = \{\emptyset, \{1\}\}$ and $\mathcal{K} = \{\emptyset, \{2\}\}$. Then, every $\mathcal{K}$- ω^*-open set is $\mathcal{J}$ -ω^*-*open* when either $\mathcal{K}$ is not subfamily of $\mathcal{J}$.

Corollary 4.4. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space in which each open subset of it is $*$-dense in itself . Then, $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. From Proposition 3.1, we have $\mathcal{T}_{\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. It remains only to show that $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\omega^*}$. Let $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Then, for each $x \in \mathcal{M}$, there exists an open set $\mathcal{F}$ containing x such that $cl^*\mathcal{F}\backslash\mathcal{M}$ is a countable set. Since $\mathcal{F} \subseteq \mathcal{F}^*$, then according to Lemma 2.2, $cl^*\mathcal{F} = cl\mathcal{F}$. As a result, $cl\mathcal{F}\backslash\mathcal{M}$ is countable. Consequently, $\mathcal{M} \in \mathcal{T}_{\omega^*}$. So, we get $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$. $\square$

Proposition 4.3. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space if $\mathcal{J} = \{\emptyset\}$. Then, $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. Since $\mathcal{J} = \{\emptyset\}$, then $\mathcal{T} = \mathcal{T}^*$ and $cl^*\mathcal{G} = cl\mathcal{G}$. So, $\mathcal{T}_{\omega^*} = \mathcal{T}_{\mathcal{J}\omega^*}$. $\square$

Theorem 4.4. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be a $R\mathcal{J}$-space. Then, $\tau_\omega = \tau_{I\omega^*}$.*

Proof. From Proposition 3.2, it follows we have $\mathcal{T}_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_\omega$. So, it remains only to show that $\mathcal{T}_\omega \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$. Let $\mathcal{M} \in \mathcal{T}_\omega$. Then, for each point x belonging to $\mathcal{M}$, there exists an open set $\mathcal{F}$ containing x such that $\mathcal{F}\backslash\mathcal{M}$ is a countable set. Since $\mathcal{X}$ is $R\mathcal{J}$-space and $x \in \mathcal{F}$, there is an open set $\mathcal{L}$ such that $x \in \mathcal{L} \subseteq cl^*\mathcal{L} \subseteq \mathcal{F}$. Implying that, $cl^*\mathcal{L}\backslash\mathcal{M} \subseteq \mathcal{F}\backslash\mathcal{M}$. So, $cl^*\mathcal{L}\backslash\mathcal{M}$ is a countable set. Hence, $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. Therefore, $\mathcal{T}_\omega = \mathcal{T}_{\mathcal{J}\omega^*}$. $\square$

Theorem 4.5. *Let $(\mathcal{X}, \mathcal{T}, \mathcal{J})$ be an ideal space. Then, $(\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{J}\omega^*} \subseteq \mathcal{T}_{\mathcal{J}\omega^*}$.*

Proof. Let $x \in \mathcal{M} \in (\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{J}\omega^*}$. Then, by Theorem 3.1, there is $\mathcal{U}_x \in \mathcal{T}_{\mathcal{J}\omega^*}$ and a countable set $\mathcal{H}_x$ such that $x \in \mathcal{U}_x$, $x \notin \mathcal{H}_x$ and $cl^*_{\mathcal{T}_{\mathcal{J}\omega^*}} \mathcal{U}_x\backslash\mathcal{H}_x \subseteq \mathcal{M}$. According to Theorem 3.1, there exists $\mathcal{G}_x \in \mathcal{T}$ and a countable set $\mathcal{K}_x$ such that $cl^*_{\mathcal{T}}\mathcal{G}_x\backslash\mathcal{K}_x \subseteq \mathcal{U}_x$. Since, $\mathcal{H}_x \cup \mathcal{K}_x$ is countable. Also, $cl^*_{\mathcal{T}}\mathcal{G}_x\backslash\mathcal{H}_x \cup \mathcal{K}_x \subseteq \mathcal{U}_x\backslash\mathcal{H}_x \subseteq cl^*_{\mathcal{T}_{\mathcal{J}\omega^*}} \mathcal{U}_x\backslash\mathcal{H}_x \subseteq \mathcal{M}$. Therefore, by Theorem 3.1, $\mathcal{M} \in \mathcal{T}_{\mathcal{J}\omega^*}$. $\square$

Theorem 4.6. *Let $\mathcal{Y}$ be a subset of a space $(\mathcal{X}, \mathcal{T}, \mathcal{J})$. Then, $(\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{Y}} \subseteq (\mathcal{T}_{\mathcal{Y}})_{\mathcal{J}\omega^*}$.*

Proof. If $\mathcal{M} \in (\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{Y}}$. Then, there is an $\mathcal{J}$ -ω^*-open set $\mathcal{F}$ in $\mathcal{X}$ such that $\mathcal{M} = \mathcal{F} \cap \mathcal{Y}$. For each point x in $\mathcal{M}$, there exists an open set $\mathcal{V}$ containing x such that $cl^*\mathcal{V}\backslash\mathcal{F}$ is countable. Since $\mathcal{U} = \mathcal{V} \cap \mathcal{Y} \in \mathcal{T}_{\mathcal{Y}}$, so $x \in \mathcal{U}$ and according to Lemma 2.3, $cl^*_{\mathcal{Y}}\,\mathcal{U} \subseteq cl^*\mathcal{V}$. Thus, $cl^*_{\mathcal{Y}}\,\mathcal{U}\backslash\mathcal{M} = cl^*_{\mathcal{Y}}\,\mathcal{U}\backslash(\mathcal{F} \cap \mathcal{Y}) = cl^*_{\mathcal{Y}}\,\mathcal{U}\backslash\mathcal{F} \subseteq cl^*\mathcal{V}\backslash\mathcal{F}$. This implies that, $cl^*_{\mathcal{Y}}\,\mathcal{U}\backslash\mathcal{M}$ is countable. Therefore, $\mathcal{M} \in (\mathcal{T}_{\mathcal{Y}})_{\mathcal{J}\omega^*}$. Hence, $(\mathcal{T}_{\mathcal{J}\omega^*})_{\mathcal{Y}} \subseteq (\mathcal{T}_{\mathcal{Y}})_{\mathcal{J}\omega^*}$. □

5. Conclusion

The ideal topological space was first introduced by Kuratowski and Vaidyanathaswamy. In ideal space, a variety of open sets were introduced, including the *α-$\mathcal{J}$-open* (*resp.*, *semi-$\mathcal{J}$-open*, β-$\mathcal{J}$-open) set. In this study, we introduce $\mathcal{J}$-ω^*-open sets as a new set in ideal space that constructs a new topology on $(\mathcal{X}, \mathcal{T})$ known as $\mathcal{T}_{\mathcal{J}\omega^*}$ that is stronger than $\mathcal{T}_{\omega^*}$ (ω^*-topology) and weaker than $\mathcal{T}_\omega$ (ω-topology). Additionally, we investigate the relationships of $\mathcal{J}$-ω^*-open sets with some other classes of sets. Finally, we discussed several basic properties. In the future, researchers will be able to define topological structures including separation axioms, compactness, and connectedness for the practical application via $\mathcal{J}$-ω^*-open sets.

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A study on co-intersection graphs of rings

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Abstract. Let R be a ring and $\mathcal{I}^\star(R)$ be the set of all nontrivial left ideals of R. The Co-intersection graph of ideals of R, denoted by $\Omega(R)$, is an undirected simple graph with the vertex set $\mathcal{I}^\star(R)$, and two distinct vertices I and J are adjacent if and only if $I + J \neq R$.

This paper derives a sufficient and necessary condition for $\Omega(R)$ to be a connected graph. We characterize the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian and Hamiltonian. Furthermore, the bad (and nice) decision number of $\Omega(\mathbb{Z}_n)$ are studied in the paper.

Keywords: co-intersection graph, connectivity, decision number, bad function.
MSC 2020: 05C25, 05C40, 05C45, 05C69

1. Introduction

The idea to associate a graph to a ring first appeared in [5]. He let all elements of the ring be vertices of the graph and was interested mainly in coloring. In [4], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the nonzero zero-divisors. There are many papers on assigning a graph to a ring R, for instance, see [4, 3, 12, 11, 2, 1]. Also, the intersection graphs and co-intersection graphs of some algebraic structures such as groups, rings, and modules have been studied by several authors, see [2, 9, 7, 10]. The co-intersection graph of submodules is introduced in [9].

The paper is organized as follows. Some definitions and preliminaries are introduced in Section 2. We devote Section 3 to study for connectivity of the co-intersection graph. Also, we characterize all the values of n for which $\Omega(\mathbb{Z}_n)$

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is Eulerian and characterize some values of n for which $\Omega(\mathbb{Z}_n)$ is Hamiltonian in this section. Finally, the bad decision number, and the nice decision number of $\Omega(\mathbb{Z}_n)$ are studied in Section 4.

2. Preliminaries

This section gives some definitions of ring theory and graph theory. Also, we introduce the Co-intersection graph of a ring R and give some basic concepts about rings and maximal left ideals.

We mean from a nontrivial ideal of R is a nonzero proper left ideal of R. The set $\mathcal{I}^\star(R)$ is a set of all nontrivial left ideals of R. A nonzero ring R is called *simple* if it has no nontrivial two-sided ideal. The term *null ring* is used to refer a ring R, in which $x \times y = 0$, for all $x, y \in R$.

By $\mathrm{Max}(R)$ and $\mathrm{Min}(R)$, we denote the set of all nonzero maximal left ideals of R and all nonzero minimal left ideals of R respectively.

A graph G is an ordered pair $G = (V, E)$, consisting of a nonempty set V of vertices, and a set $E \subseteq [V]^2$ of edges, where $[V]^2$ is the set of all 2-element subsets of V. Two vertices $u, v \in V$ are *adjacent* if $uv \in E$ (for simplicity we use uv instead of subset $\{u, v\}$). The *neighbourhood* of a vertex $u \in V$ is $N(u) = \{v \in V | uv \in E\}$, and the *closed neighbourhood* of u is $N[u] = N(u) \cup \{u\}$. The degree of a vertex u in a graph G is the size of set $N(u)$, which is denoted by $deg(u)$. We denote by $\delta(G)$ the minimum degree of the vertices of G. The *complete* graph with n vertices is denoted by K_n, which is a graph with n vertices in which any two distinct vertices are adjacent. A *null graph* is a graph containing no edges. Let G be a graph, suppose that $x, y \in V(G)$, a *walk* between u and v is a sequence $u = v_0 - v_1 - \cdots - v_k = v$ of vertices of G such that for every i with $1 \leq i \leq k$, the vertices v_{i-1} and v_i are adjacent. A (u, v)*-path* between u and v is a walk between u and v without repeated vertices. Two vertices u and v of G are said to be *connected* if there is a (u, v)-path in G. A graph G is called *connected* if every pair of its vertices are connected. If vertices u and v are connected in G, the distance between u and v in G, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G. In graph G, a *tour* is a closed walk that traverses each edge of G at least once. A graph is *Eulerian* if it contains a tour which traverses each edge exactly once [6].

A *cycle* in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. A *Hamilton cycle* is a spanning cycle, and a graph which contains such a cycle is said to be *Hamiltonian*.

If $G = (V, E)$ is a finite graph, define $f(U) = \sum_{u \in U} f(u)$, for a function $f : V \to \{-1, 1\}$ and $U \subseteq V$. A function $f : V \to \{-1, 1\}$ is called a *bad function* of G, if $f(N(v)) \leq 1$ for each $v \in V$ [13]. The maximum value of $f(V)$, taken over all bad functions f, is called the *bad decision number* of G, which is denoted by $\beta_D(G)$. The function f is called a *nice function*, if $f(N[v])) \leq 1$ for each $v \in V$. The maximum value of $f(V)$, taken over all nice functions f is called the *nice decision number* of G, and denoted by $\overline{\beta_D(G)}$.

Definition 2.1. *Let R be a ring. The Co-intersection graph $\Omega(R)$ of R, is an undirected simple graph whose the vertex set $V(\Omega(R)) = \mathcal{I}^\star(R)$ is a set of all nontrivial ideals of R and two distinct vertices I, J are adjacent if and only if $I + J \neq R$.*

Remark 2.1. Let $\mathbb{Z}_n$ be the ring of integers modulo n. Suppose that m_1, m_2 are two factor of n. Then, $\langle m_1\rangle + \langle m_2\rangle = \langle (m_1, m_2)\rangle$, where (m_1, m_2) is the greatest common divisor of m_1, m_2.

Example 2.1. Suppose that $R = \mathbb{Z}_{50}$. Then, $\mathcal{I}^\star(R) = \{\langle 2\rangle, \langle 5\rangle, \langle 10\rangle, \langle 25\rangle\}$ and the co-intersection graph $\Omega(R)$ is as follow:

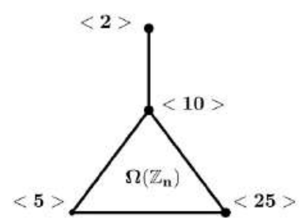

Figure 1: The Co-intersection Graph $\Omega(\mathbb{Z}_{50})$.

3. Connectivity, eulerianity and hamiltonicity

This section derives a sufficient and necessary condition for $\Omega(R)$ to be a connected graph. Also, we determine the values of n for which $\Omega(\mathbb{Z}_n)$ is a connected graph. Further, we characterize the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian and Hamiltonian. Before presenting and proving results, we give the following lemma.

Lemma 3.1. *Let R be a ring and I, J be two distinct maximal left ideals of R. Then, I and J are not adjacent.*

Proof. Since I and J are two distinct maximal left ideals of R, therefore $I+J = R$. So I and J are not adjacent. $\square$

Lemma 3.2. *Let R be a ring with co-intersection $\Omega(R)$ and J be a nontrivial left ideal of R. If $deg(J)$ is finite, then R is a left Artinian ring.*

Proof. Since $deg(J) < \infty$, so J is a left Artinian R-modules. Otherwise, there exists a descending chain $J \supset I_1 \supset \cdots \supset I_n \supset \cdots$ of left ideals of R belong to J. Thus, $J + I_i = J \neq R$ for each i and this is a contradiction. Also, R/J is a left Artinian R-modules. Otherwise, there exists a descending chain $R/J \supset I_1/J \supset \cdots \supset I_n/J \supset \cdots$ of left submodules of R/J. Thus, $J + I_i = I_i \neq R$ for each i and this is a contradiction. Hence, according to [8, Proposition 4.5], R is a left Artinian R-module and the proof is complete. $\square$

The following proposition can be obtained in a similar way in [9, Theorem 2.1] about the connectivity.

Proposition 3.1. *Let R be a ring and $\mathcal{I}^\star(R) \neq \emptyset$. Then, $\Omega(R)$ is disconnected if and only if R has at least two maximal left ideals, and every nontrivial left ideal is a maximal left ideal.*

Corollary 3.1. *The graph $\Omega(\mathbb{Z}_n)$ is disconnected if and only if $n = pq$, where p and q are distinct primes.*

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for some $k \in \mathbb{N}$. According to Remark 2.1, $\mathbb{Z}_n$ has at least two maximal ideals and every nontrivial ideal is a maximal ideal if and only if $k = 2$ and $\alpha_i = 1$. Then, by Proposition 3.1, $\Omega(\mathbb{Z}_n)$ is disconnected if and only if $\mathbb{Z}_n$ has at least two maximal ideals and every nontrivial ideal is a maximal ideal if and only if $k = 2$ and $\alpha_i = 1$.

Corollary 3.2. *Let R be a ring and $\mathcal{I}^\star(R) \neq \emptyset$. If $\Omega(R)$ is disconnected then* $\mathrm{Max}(R) = \mathrm{Min}(R)$.

Proof. By Proposition 3.1, as $\Omega(R)$ is disconnected thus $\mathcal{I}^\star(R) = \mathrm{Max}(R)$. If $I \in \mathrm{Max}(R) = \mathcal{I}^\star(R)$, there is no nontrivial left ideal $J \subsetneq I$, then $I \in \mathrm{Min}(R)$. Also, $\mathrm{Min}(R) \subseteq \mathcal{I}^\star(R) = \mathrm{Max}(R)$. □

Corollary 3.3. *Let R be a ring. If $\Omega(R)$ is disconnected then, $\Omega(R)$ is a null graph.*

Proof. By Lemma 3.1 and Proposition 3.1, the proof is complete. □

Lemma 3.3. *Let R be a ring. If $\mathrm{Max}(R) \cap \mathrm{Min}(R) \neq \emptyset$, then $\mathcal{I}^\star(R) = \mathrm{Max}(R) = \mathrm{Min}(R)$ and thus $\Omega(R)$ is a null graph.*

Proof. Suppose that $\mathfrak{m} \in \mathrm{Max}(R) \cap \mathrm{Min}(R)$, then for each $I \in \mathcal{I}^\star(R)$, $I + \mathfrak{m} = R$. So, $\Omega(R)$ is disconnected and according to Corollary 3.2, $\mathrm{Max}(R) = \mathrm{Min}(R)$. Also, by Corollary 3.3 $\Omega(R)$ is a null graph. □

Proposition 3.2. *Let R be a commutative ring. Then, the graph $\Omega(R)$ is disconnected if and only if $R = R_1 \times R_2$ where each $R_i (i = 1, 2)$ is either a field or a null ring with prime number of elements.*

Proof. For the proof of the necessity part, suppose that, the graph $\Omega(R)$ is disconnected. Then, according to Proposition 3.1 and its proof, there are two maximal ideals I and J of R such that $I + J = R$ and $I \cap J =< 0 >$, as they are minimal ideal too, from Corollary 3.2. Then, $R = I \oplus J \cong \frac{R}{J} \times \frac{R}{I}$ where $\frac{R}{J}$ and $\frac{R}{I}$ are simple commutative rings, as I and J are maximal ideal.

Conversely, let $R = R_1 \times R_2$ where R_1, R_2 are simple commutative rings. If both R_1 and R_2 are two fields, then R has only two nontrivial ideals, $I = R_1 \times \{0_{R_2}\}$ and $\{0_{R_1}\} \times R_2$ and they are maximal ideals and hence according to Lemma 3.1, $\Omega(R)$ is disconnected. If both R_1 and R_2 are two null rings with prime number of elements, let $(R_1,+) \cong (\mathbb{Z}_p,+)$ and $(R_2,+) \cong (\mathbb{Z}_q,+)$, where p , q are prime numbers. If $p \neq q$, then $(R,+) \cong (\mathbb{Z}_p \times \mathbb{Z}_q,+) \cong (\mathbb{Z}_{pq},+)$, which has only two nontrivial subgroups $(\bar{p}),(\bar{q})$. These two subsets are also only nontrivial ideals of the null ring R, and then $\Omega(R)$ is disconnected.

If R_1 is a field and R_2 is a null rings with prime number of elements, such that $(R_2,+) \cong (\mathbb{Z}_p,+)$. In this case $\{(0_{R_1}) \times R_2, R_1 \times (\bar{0})\} = \mathcal{I}^\star(R)$. As $1 \in (0_{R_1}) \times R_2 + R_1 \times (\bar{0})$, then $\Omega(R)$ is disconnected. □

In the following, we characterize all the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian; further, some values of n for which $\Omega(\mathbb{Z}_n)$ is Hamiltonian are characterized.

At the first, we give a lemma about the number of vertices of $\Omega(\mathbb{Z}_n)$, and characterize $deg(I)$ for each $I \in \mathcal{I}^\star(\mathbb{Z}_n)$ and also minimum degree $\delta(\Omega(\mathbb{Z}_n))$.

Lemma 3.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, a = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where p_i's are all distinct prime numbers, and also $0 \leq \beta_i \leq \alpha_i$. If $I =< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and suppose that $\mathfrak{B}_a = \{t_j | 1 \leq t_j \leq k, \beta_{t_j} \neq 0\}$ is the ordered set of all indices t_j, such that $\beta_{t_j} \neq 0$, then*

$$deg(I) = \sum_{j=1}^{|\mathfrak{B}_a|} \left(\alpha_{t_j} \prod_{\substack{i=1 \\ i \notin \{t_1, \cdots, t_j\}}}^{k} (\alpha_i + 1) \right) - 2. \tag{1}$$

Also, the number of vertices of $G = \Omega(\mathbb{Z}_n)$ is $|\mathcal{I}^\star(\mathbb{Z}_n)| = \prod_{i=1}^{k} (\alpha_i + 1) - 2$ and

$$\delta(G) = \alpha_{t_0} \prod_{\substack{i=1 \\ i \neq t_0}}^{k} (\alpha_i + 1) - 2$$

wherein $\alpha_{t_0} = \min\{\alpha_i | 1 \leq i \leq k\}$.

Proof. Assume that $b|n$ and $b \neq n$. Then, $J =< b > \neq I$ and I are adjacent if there exist some $t_j \in \mathfrak{B}_a$ such that $p_{t_j} | b$. But there are

$$\alpha_{t_1} \prod_{\substack{i=1 \\ i \neq t_1}}^{k} (\alpha_i + 1)$$

factors of n in the form $b = p_{t_1} b'$ (two of them are n and a), and there are

$$\alpha_{t_2} \prod_{\substack{i=1 \\ i \notin \{t_1, t_2\}}}^{k} (\alpha_i + 1)$$

factors of n in the form $b = p_{t_2} b'$ such that $p_{t_1} \nmid b'$ and so on. It is obvious that these factors of n are distinct. As $< n >, < a >$ are not adjacent to $I =< a >$, thus 2 units are deducted from the total. The proof of other statements are obvious. □

Example 3.1. Let $n = 2^{10} \times 3^5 \times 5^2 \times 7$, $a = 2^5 \times 3^2 \times 5$ and $G = \Omega(\mathbb{Z}_n)$. Thus, $I =< a >$ is a nontrivial ideal of $\mathbb{Z}_n$. Then, according to the Lemma 3.4,
$deg(< a >) = (10 \times 6 \times 3 \times 2) + (5 \times 3 \times 2) + (2 \times 2) - 2 = 392$.
Also, $\delta(G) = (1 \times 11 \times 6 \times 3) - 2 = 196$.

Proposition 3.3 ([6, Theorem 3.7])**.** *A connected graph is Eulerian if and only if all of its vertices have even degree.*

In the next proposition, we characterize all the values of n for which graphs of $\mathbb{Z}_n$ are Eulerian.

Proposition 3.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers. Then, $\Omega(\mathbb{Z}_n)$ is Eulerian if and only if $\alpha_i = 1$ for each $1 \leq i \leq k$, or each α_i is even ($1 \leq i \leq k$).*

Proof. According to Proposition 3.3, it is enough to show that all vertices of $\Omega(\mathbb{Z}_n)$ have even degree if and only if $\alpha_i = 1$ for each $1 \leq i \leq k$, or each α_i is even $(1 \leq i \leq k)$.

With the same notation in Lemma 3.4, if $\alpha_i = 1$ for each $1 \leq i \leq k$, then for each factor $a \neq n$ of n, there is some $i_0 \in \{1, 2, \cdots, k\} \setminus \mathfrak{B}_a$. Thus

$$2 = (\alpha_{i_0} + 1) \mid \prod_{\substack{i=1 \\ i \notin \{t_1, \cdots, t_j\}}}^{k} (\alpha_i + 1)$$

for each $1 \leq j \leq |\mathfrak{B}_a|$, and hence $deg(I)$ is even. Also, it is obvious that $deg(I)$ is even if α_i is even for each $1 \leq i \leq k$. Conversely, if there exist an α_i greater than 1 and also $\{s_1, \cdots, s_m | \alpha_{s_i} \text{ is odd}\}$ is the nonempty set of all s_i such that α_{s_i} is odd, then the ideal $I =< a >=< p_{s_1} \cdots p_{s_m} >$ is a nontrivial ideal of $\mathbb{Z}_n$. We show that $deg(I)$ is odd. In this case, it is obvious that the summand

$$\alpha_{s_j} \prod_{\substack{i=1 \\ i \notin \{s_1, \cdots, s_j\}}}^{k} (\alpha_i + 1)$$

in Equation 1 is even for each $1 \leq j \leq m - 1$ and is odd for $j = m$ and thus $deg(I)$ is odd. The proof is complete. □

Proposition 3.5 ([6], Theorem 18.4)**.** *Let G be a simple graph of minimum degree δ, where $\delta \geq \frac{n}{2}$ and $n \geq 3$. Then, G is Hamiltonian.*

Proposition 3.6. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers. If $k = 1, \alpha_1 \geq 4$ or $k \geq 2, \alpha_i \geq 3$ for each $1 \leq i \leq k$, then $\Omega(\mathbb{Z}_n)$ is Hamiltonian.*

Proof. If $k = 1, \alpha_1 \geq 4$, then $\Omega(\mathbb{Z}_n)$ is a complete graph with at least 3 vertices ([9, Example 2.14]) and thus is Hamiltonian.

Now, assume that $k \geq 2, \alpha_i \geq 3$ for each $1 \leq i \leq k$. Let $\alpha_{t_0} = \min\{\alpha_i | 1 \leq i \leq k\}$. Therefore, $\Omega(\mathbb{Z}_n)$ has $n \geq 3$ vertices and also

$$(2\alpha_{t_0} - (\alpha_{t_0} + 1)) \prod_{\substack{i=1 \\ i \neq t_0}}^{k} (\alpha_i + 1) \geq 2.$$

Hence,

$$\delta(\mathbb{Z}_n) = \alpha_{t_0} \prod_{\substack{i=1 \\ i \neq t_0}}^{k} (\alpha_i + 1) - 2 \geq \frac{\prod_{i=1}^{k} (\alpha_i + 1)}{2} - 1 = \frac{n}{2}.$$

Therefore, by Proposition 3.5, $\Omega(\mathbb{Z}_n)$ is Hamiltonian. □

4. The decision number of $\Omega(\mathbb{Z}_n)$

In this section, the bad decision number and the nice decision number of $G = \Omega(\mathbb{Z}_n)$ are investigated for each n. Some lemma's are presented in the following, and the results are combined to a single theorem at the end of the section.

Lemma 4.1. *Let $n = p^\alpha$, $\alpha \geq 3$, and also $G = \Omega(\mathbb{Z}_n)$. Thus,*

$$\beta_D(G) = \begin{cases} 0, & \text{for odd} \quad \alpha \geq 5 \ , \\ 2, & \text{for} \quad \alpha = 3 \ , \\ -1, & \text{for even} \quad \alpha. \end{cases} \qquad \overline{\beta_D(G)} = \begin{cases} 0, & \text{for odd} \quad \alpha \ , \\ 1, & \text{for even} \quad \alpha. \end{cases}$$

Proof. We know $G = \Omega(\mathbb{Z}_n)$ is the complete graph $K_{\alpha-1}$ for $n = p^\alpha$. Thus, at least $\lceil \frac{\alpha-1}{2} \rceil$ of the vertices must be signed by -1, for any bad function f and $\alpha > 3$. In the other side, there is a bad function f over G, such that exactly $\lceil \frac{\alpha-1}{2} \rceil$ of the vertices are signed by -1. Further, it is obvious that $\beta_D(\Omega(\mathbb{Z}_3)) = \beta_D(K_2) = 2$. Similarly, at least $\lfloor \frac{\alpha-1}{2} \rfloor$ of the vertices must be signed by -1, for any nice function f. □

Lemma 4.2. *Let $k \geq 2$, α_k be an odd number, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and also $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are lower than or equal to* 2.

Proof. If $\alpha_i = 1$, for all $1 \leq i \leq k$, then let $v_0 = p_2 \cdots p_k$. Note that, $| V(G) | = \prod_{i=1}^{k} (\alpha_i + 1) - 2$ is an even number, and $N(< v_0 >) = V(G) \setminus \{< v_0 >, < p_1 >\}$. If f is a bad function, then $f(N(< v_0 >))$ is at most equal to 0, because of $| N(< v_0 >) |$ is even. Also, $f(N[< v_0 >])$ is at most equal to 1 for a nice function f. Thus, $f(V(G))$ is at most equal to 2 for any bad or nice function f.

If there is an $\alpha_i \geq 2$, then let $v_0 = p_1 p_2 \cdots p_k$. If f is a bad function then $f(N(< v_0 >))$ is at most equal to 1. If f is a nice function then, $f(N[< v_0 >]) \leq 0$ because of $N[< v_0 >] = V(G)$ and $| V(G) |$ is even. Hence, for any bad or nice function f, $f(V(G)) \leq 2$. □

The upper bound presented in Lemma 4.2 is sharp. As $\overline{\beta_D(\Omega(\mathbb{Z}_{pq}))} = \beta_D(\Omega(\mathbb{Z}_{pq})) = 2$.

Lemma 4.3. *Let $k \geq 2$, α_k be an odd number, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are greater than or equal to 0.*

Proof. Let $m = \frac{\alpha_k - 1}{2}$. Define the function $f : V \to \{-1, 1\}$ as:

$$f(< a >) = \begin{cases} -1, & \text{if} \quad p_k^{m+1} | a \ , \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that $< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. We show that $f(N(< a >)) \leq 1$ and then the function f is a bad function. Let $A = \{i | a_i \neq 0\}$.

- If $a_k = 0$:
  There are $X = \left(\prod_{i \in A} (\alpha_i + 1) - 1\right) \prod_{i \notin A, i \neq k} (\alpha_i + 1)(m + 1) - 1$ elements in $N(< a >)$, such that have value 1 under the function f. Also, There are X elements in $N(< a >)$, such that have value -1 under the function f. Hence, $f(N(< a >)) = 0$.

- If $m \neq 0$ and $a_k \neq 0$:
  In this case, $Y = \left(\prod_{i \in A, i \neq k} (\alpha_i + 1)(m + 1) - 1\right) \prod_{i \notin A} (\alpha_i + 1)$ elements of $N(< a >)$ have value 1 under the function f, and Y elements of $N(< a >)$ have value -1 under the function f. Therefore, $f(N(< a >)) = 0$.

- If $m = 0$ and $a_k \neq 0$: In this case, there are $\prod_{i \neq k} (\alpha_i + 1) - 2$ elements of $N(< a >)$ with value -1 and $\prod_{i \in A, i \neq k} (\alpha_i + 1) - 1$ elements of $N(< a >)$ with value 1 under f. Thus, $f(N(< a >)) = \prod_{i \in A, i \neq k} (\alpha_i + 1) \left(1 - \prod_{i \notin A} (\alpha_i + 1)\right) + 1 \leq 1$. Also, $f(N[a]) \leq 0$, as $f(< a >) = -1$.

On the other side, $f(V) = 0$, as exactly the half of the vertices of G have value 1 under the f. Hence, $\beta_D(G) \geq 0$. Furthermore, it is obvious that $f(N[a]) \leq 1$ in all 3 cases, hence f is a nice function and $\overline{\beta_D(G)} \geq 0$. □

The following Lemma, present an upper bound for decision numbers in the case of all of the prime factors of n have even exponent in the prime decomposition of n.

Lemma 4.4. *Let $k \geq 2$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, α_i's are all even numbers, and also $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are lower than or equal to 1.*

Proof. Let $v = p_1 p_2 \cdots p_k$, and f be a bad function. Note that, $| V(G) | = \prod_{i=1}^{k} (\alpha_i + 1) - 2$ is an odd number. We have, $N(< v >) = V(G) \setminus \{< v >\}$ and $| N(< v >) |$ is even, thus $f(N(< v >)) \leq 0$ and $f(V(G)) \leq 1$. Further, If f is a nice function then, $f(V(G)) = f(N[< v >]) \leq 1$. Hence, $f(V(G))$ is at most equal to 1. □

In the next example we show that the upper bound presented in Lemma 4.4 is sharp.

Example 4.1. Let $n = p_1^2 p_2^2 p_3^2$, and $G = \Omega(\mathbb{Z}_n)$. Define the function f over $V(G)$ as: $f(p_i^{a_i}) = f(p_1^{a_1} p_2^{a_2} p_3^{a_3}) = 1$, where $1 \leq i \leq 3$ and $a_1 a_2 a_3 \neq 0$. Otherwise, $f(v) = -1$.

It is easy to check that the function f is a bad (and nice) function, and $f(V(G)) = +13 - 12 = 1$. Hence, $\beta_D(G), \overline{\beta_D(G)} = 1$.

Lemma 4.5. *Let $k \geq 2$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, α_i's are all even numbers, and also $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are greater than or equal to -1.*

Proof. Let $m_i = \frac{\alpha_i}{2}$ for each $1 \leq i \leq k$. Define the function $f : V \to \{-1, 1\}$ as:

$$f(< a >) = \begin{cases} 1, & \text{if} \quad p_1^{\alpha_1} \cdots p_i^{\alpha_i} | a \quad \text{and} \quad p_{i+1}^{m_{i+1}} \nmid a \text{ for some } 0 \leq i \leq k-1 \ , \\ -1, & \text{otherwise.} \end{cases}$$

Suppose that $< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. We show that $f(N(< a >)) \leq 1$ and then the function f is a bad function. Let $A = \{i | a_i \neq 0\}$, and $t = \min\{i | a_i \neq 0\}$.

According to the definition of f,

$$X = \sum_{i=t}^{k} m_i \prod_{j=i+1}^{k} (\alpha_j + 1) + \sum_{i=1}^{t-1} m_i \left(\prod_{j \in A, j > i} (\alpha_j + 1) - 1 \right) \prod_{j \notin A, j > i} (\alpha_j + 1)$$

elements of $N[< a >]$ have value -1 under the function f, and

$$\begin{aligned} Y = {} & \sum_{i=t}^{k} m_i \prod_{j=i+1}^{k} (\alpha_j + 1) - \prod_{j \notin A, j > t} (\alpha_j + 1) \\ & + \sum_{i=1}^{t-1} m_i \left(\prod_{j \in A, j > i} (\alpha_j + 1) - 1 \right) \prod_{j \notin A, j > i} (\alpha_j + 1) \end{aligned}$$

elements of $N[a]$ have value 1 under the function f. Therefore, if $f(< a >) = 1$, then X elements of $N(< a >)$ have value -1, and $Y - 1$ elements of $N(< a >)$ have value $+1$. If $f(< a >) = -1$, then $X - 1$ elements of $N(< a >)$ have value -1, and Y elements of $N(< a >)$ have value $+1$. Thus,

$$f(N(< a >)) = \begin{cases} Y - 1 - X = -\sum_{i=t}^{k} \prod_{j \notin A, j > t} (\alpha_j + 1) - 1, & \text{if} \quad f(< a >) = 1 \ , \\ Y - X + 1 = -\sum_{i=t}^{k} \prod_{j \notin A, j > t} (\alpha_j + 1) + 1, & \text{if} \quad f(< a >) = -1. \end{cases}$$

Consequently, $f(N(< a >)) \leq 0$ and $f(N[< a >]) \leq -1$. Hence, f is both bad function and nice function.

By the definition of the function f,

$$Z = \sum_{i=1}^{k} m_i \prod_{j=i+1}^{k} (\alpha_j + 1)$$

vertices of G have value -1, and $Z - 1$ elements of $N(< a >)$ have value $+1$. Hence, $\beta_D(G), \overline{\beta_D(G)} \geq -1$. □

Finally, the following theorem can immediately be concluded from the above discussions.

Theorem 4.1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and also $G = \Omega(\mathbb{Z}_n)$. We have*

$$\beta_D(G) = \begin{cases} -1, & \text{if} \quad k = 1, \alpha_1 \text{ is an even number,} \\ -1 \text{ or } 1, & \text{if} \quad \alpha_i \text{ is an even number, for all } 1 \leq i \leq k \ , \\ 0 \text{ or } 2, & \text{otherwise.} \end{cases}$$

$$\overline{\beta_D(G)} = \begin{cases} 1, & \text{if} \quad k = 1, \alpha_1 \text{ is an even number,} \\ -1 \text{ or } 1, & \text{if} \quad \alpha_i \text{ is an even number, for all } 1 \leq i \leq k \ , \\ 0 \text{ or } 2, & \text{otherwise.} \end{cases}$$

5. Conclusion

In this paper, we have obtained a sufficient and necessary condition for $\Omega(R)$ to be a connected graph. Likewise, we characterized the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian and Hamiltonian. Finally, the bad (and nice) decision number of $\Omega(\mathbb{Z}_n)$ has been presented. In our future work, we will introduce new results of connected graphs that are very useful in networks and computer sciences.

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Refinements of unitary invariant norm inequalities for matrices

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Abstract. In this paper, we first establish an arithmetic-geometric mean inequality of unitary invariant norm for matrices, which is an improvement of the result proposed by Zou and He [Linear Algebra Appl., 436(2012), 3354-3361]. Then, we use it to refine the existing inequality. Moreover, we derive two unitarily invariant norm inequalities for matrices, which refine the result of Cao and Wu.
Keywords: positive semidefinite matrix, convex function, unitarily invariant norm, arithmetic-geometric mean inequality.
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1. Introduction

Let M_n be the space of $n \times n$ complex matrices. A norm $||\cdot||$ is called unitarily invariant norm, if $||UAV|| = ||A||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The singular values $s_j(A)(j = 1, 2, \cdots, n)$ of A are the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$, arranged in a decreasing order. The Ky Fan k-norm $||\cdot||_{(k)}$ is defined as $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1, \cdots, n$ and the Schatten p-norm $||\cdot||_p$ is defined as $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{\frac{1}{p}} = (\mathrm{tr}|A|^p)^{\frac{1}{p}}, 1 \leq p < \infty$.

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In what follows, $||\cdot||$ always denotes unitarily invariant norms including Schatten p-norm $||\cdot||_p$ and Ky Fan k-norm $||\cdot||_{(k)}$.

For $A, B, X \in M_n$ and A, B are positive semidefinite, Bhatia and Davis [1] presented

$$||A^{\frac{1}{2}}XB^{\frac{1}{2}}|| \le \left|\left|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right|\right| \le \left|\left|\frac{AX+XB}{2}\right|\right|, \tag{1.1}$$

where $0 \le v \le 1$. Letting

$$\varphi(v) = ||A^vXB^{1-v}+A^{1-v}XB^v||,$$

inequality (1.1) can be rewritten as

$$\varphi\left(\frac{1}{2}\right) \le \varphi(v) \le \varphi(0).$$

For $A, B, X \in M_n$ and A, B are positive semidefinite, the function $\varphi(v)$ is a continuous convex function on [0,1], attains its minimum at $v = \frac{1}{2}$ and maximum at $v = 0$ and $v = 1$. Consequently, it is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$, moreover, $\varphi(v) = \varphi(1-v)$ for $v \in [0,1]$ (see[2]). Using the convexity of the function $\varphi(v)$, Zou and He [3] obtained a strengthening of the arithmetic-geometric mean inequality $\varphi\left(\frac{1}{2}\right) \le \varphi(0)$ as follows:

$$\varphi\left(\frac{1}{2}\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{2}\right)\right) \le \varphi(0), 0 \le v \le 1. \tag{1.2}$$

Bhatia and Kittaneh [4] derived if $A, B \in M_n$ are positive semidefinite, then

$$||AB|| \le \frac{1}{4}||(A+B)^2||. \tag{1.3}$$

Zou and He [3] gave a stronger version of inequality (1.3) as follows:

$$||AB|| + \left(\int_0^1 g(v)dv - 2||AB||\right) \le \frac{1}{4}||(A+B)^2||, \tag{1.4}$$

where $g(v) = ||A^{\frac{1}{2}+v}B^{\frac{3}{2}-v} + A^{\frac{3}{2}-v}B^{\frac{1}{2}+v}||$.

Kaur and Singh [5] proved that for $A, B, X \in M_n$, if A and B are positive definite, then for any unitarily invariant norm

$$\frac{1}{2}||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu|| \le \left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|, \tag{1.5}$$

where $\frac{1}{4} \le \nu \le \frac{3}{4}$ and $\alpha \in [\frac{1}{2}, \infty)$.

Replacing A, B by A^2, B^2 in (1.5) and taking $u = 2\nu$, we can obtain

$$\frac{1}{2}||A^uXB^{2-u} + A^{2-u}XB^u|| \le \left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right|\right|, \tag{1.6}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$ and $\alpha \in [\frac{1}{2}, \infty)$.

Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm, the function

$$\psi(u) = ||A^u XB^{2-u} + A^{2-u} XB^u||$$

is convex on $[\frac{1}{2}, \frac{3}{2}]$ and attains its minimum at $u = 1$. So, it is decreasing on $[\frac{1}{2}, 1]$ and increasing on $[1, \frac{3}{2}]$(see[2]). Using the convexity of the function $\psi(u)$, Cao and Wu [6] obtained a refinement of inequality (1.6)

$$\begin{aligned} ||A^u XB^{2-u} + A^{2-u} XB^u|| &\leq 2(4r_0 - 1)||AXB| \\ &\quad + 2(1 - 2r_0)||A^{\frac{1}{2}} XB^{\frac{3}{2}} + A^{\frac{3}{2}} XB^{\frac{1}{2}}|| \\ &\leq 2(4r_0 - 1)||AXB|| \\ &\quad + 4(1 - 2r_0)\left|\left|(1 - \alpha)AXB + \alpha\left(\frac{A^2 X + XB^2}{2}\right)\right|\right|, \end{aligned} \tag{1.7}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$ and $r_0 = \min\{\frac{u}{2}, 1 - \frac{u}{2}\}$.

For more information on this topic, the reader is referred to [7-9] and the references therein. In this paper, we first improve the inequality (1.2). As an application of our result, we refine the inequality (1.4). Finally, we establish improved versions of inequality (1.7) by using the convexity of function $\psi(u)$.

2. Main results

In this section, we show four lemmas which will turn out to be useful in the proof of our results.

Lemma 2.1 ([10])**.** *Let $A, B, X \in M_n$ such that A and B are positive semidefinite, then for every unitarily invariant norm*

$$\varphi(v) \leq 2r_0 \left(\varphi\left(\frac{1}{2}\right) - \varphi(0)\right) + \varphi(0),$$

where $0 \leq v \leq 1$ and $r_0 = \min\{v, 1 - v\}$.

Lemma 2.2 ([10])**.** *Let f be a real valued convex function on the interval $[a, b]$, then*

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}.$$

Lemma 2.3 ([4])**.** *Let $A, B \in M_n$ be positive semidefinite, then*

$$||A^{\frac{1}{2}}(A + B)B^{\frac{1}{2}}|| \leq \frac{1}{2}||(A + B)^2||.$$

Lemma 2.4 ([10])**.** *Let f be a real valued convex function on an interval $[a,b]$ which contains (x_1,x_2), then*

$$f(x) \leq \frac{f(x_2)-f(x_1)}{x_2-x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2-x_1}, x \in (x_1,x_2).$$

Theorem 2.1. *Let $A, B, X \in M_n$ such that A and B are positive semidefine, then for any unitarily invariant norm*

$$\varphi\left(\frac{1}{2}\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{2}\right)\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{4}\right)\right) \leq \varphi(0), \tag{2.1}$$

where $\varphi(v) = ||A^v X B^{1-v} + A^{1-v} X B^v||$ and $0 \leq v \leq 1$.

Proof. For $0 \leq v \leq \frac{1}{4}$, by Lemma 2.1, we have

$$\varphi(v) \leq 4\left(\varphi\left(\frac{1}{4}\right) - \varphi(0)\right)v + \varphi(0).$$

Thus,

$$\int_0^{\frac{1}{4}} \varphi(v)dv \leq 4\left(\varphi\left(\frac{1}{4}\right) - \varphi(0)\right)\int_0^{\frac{1}{4}} vdv + \int_0^{\frac{1}{4}} \varphi(0)dv.$$

By a small calculation, we have

$$\int_0^{\frac{1}{4}} \varphi(v)dv \leq \frac{1}{8}\left(\varphi\left(\frac{1}{4}\right) + \varphi(0)\right). \tag{2.2}$$

For $\frac{1}{4} \leq v \leq \frac{1}{2}$, by Lemma 2.1, we obtain

$$\varphi(v) \leq 4\left(\varphi\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{4}\right)\right)\left(v - \frac{1}{4}\right) + \varphi\left(\frac{1}{4}\right).$$

Consequently

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \varphi(v)dv \leq 4\left(\varphi\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{4}\right)\right)\int_{\frac{1}{4}}^{\frac{1}{2}}\left(v - \frac{1}{4}\right)dv + \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi\left(\frac{1}{4}\right)dv,$$

which implies

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \varphi(v)dv \leq \frac{1}{8}\left(\varphi\left(\frac{1}{4}\right) + \varphi\left(\frac{1}{2}\right)\right). \tag{2.3}$$

For $\frac{1}{2} \leq v \leq \frac{3}{4}$, by Lemma 2.1, we obtain

$$\varphi(v) \leq 4\left(\varphi\left(\frac{3}{4}\right) - \varphi\left(\frac{1}{2}\right)\right)\left(v - \frac{1}{2}\right) + \varphi\left(\frac{1}{2}\right).$$

Thus,

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v)dv \leq 4\left(\varphi\left(\frac{3}{4}\right) - \varphi\left(\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{\frac{3}{4}}\left(v - \frac{1}{2}\right)dv + \int_{\frac{1}{2}}^{\frac{3}{4}} \varphi\left(\frac{1}{2}\right)dv,$$

by a small calculation, we have

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v)dv \leq \frac{1}{8}\left(\varphi\left(\frac{3}{4}\right) + \varphi\left(\frac{1}{2}\right)\right). \tag{2.4}$$

For $\frac{3}{4} \leq v \leq 1$, by Lemma 2.1, we obtain

$$\varphi(v) \leq 4\left(\varphi(1) - \varphi\left(\frac{3}{4}\right)\right)\left(v - \frac{3}{4}\right) + \varphi\left(\frac{3}{4}\right).$$

Thus,

$$\int_{\frac{3}{4}}^{1} \varphi(v)dv \leq 4\left(\varphi(1) - \varphi\left(\frac{3}{4}\right)\right)\int_{\frac{3}{4}}^{1}\left(v - \frac{3}{4}\right)dv + \int_{\frac{3}{4}}^{1} \varphi\left(\frac{3}{4}\right)dv,$$

which implies

$$\int_{\frac{3}{4}}^{1} \varphi(v)dv \leq \frac{1}{8}\left(\varphi(1) + \varphi\left(\frac{3}{4}\right)\right). \tag{2.5}$$

It follows from (2.2)-(2.5) and $\varphi(0) = \varphi(1), \varphi\left(\frac{1}{4}\right) = \varphi\left(\frac{3}{4}\right)$ that

$$\begin{aligned}\int_0^1 \varphi(v)dv &= \int_0^{\frac{1}{4}} \varphi(v)dv + \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi(v)dv + \int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v)dv + \int_{\frac{3}{4}}^{1} \varphi(v)dv \\ &\leq \frac{1}{4}\left(\varphi(0) + \varphi\left(\frac{1}{2}\right) + 2\varphi\left(\frac{1}{4}\right)\right),\end{aligned}$$

and so

$$4\int_0^1 \varphi(v)dv \leq \varphi(0) + \varphi\left(\frac{1}{2}\right) + 2\varphi\left(\frac{1}{4}\right),$$

which is equivalent to

$$\varphi\left(\frac{1}{2}\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{2}\right)\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{4}\right)\right) \leq \varphi(0).$$

This completes the proof. □

Remark 2.1. Theorem 2.1 is sharper than inequality (1.2).

By Lemma 2.2, we have

$$\varphi\left(\frac{1}{4}\right) \leq 2\int_0^{\frac{1}{2}} \varphi(v)dv \tag{2.6}$$

and

$$\varphi\left(\frac{3}{4}\right) \leq 2\int_{\frac{1}{2}}^1 \varphi(v)dv. \tag{2.7}$$

It follows from (2.6), (2.7) and $\varphi\left(\frac{1}{4}\right) = \varphi\left(\frac{3}{4}\right)$ that

$$2\varphi\left(\frac{1}{4}\right) \leq 2\int_0^1 \varphi(v)dv,$$

that is

$$\int_0^1 \varphi(v)dv \geq \varphi\left(\frac{1}{4}\right).$$

Thus,

$$\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{4}\right) \geq 0.$$

Obviously, Theorem 2.1 is also an improvement of arithmetic-geometric mean inequality $\varphi\left(\frac{1}{2}\right) \leq \varphi(0)$.

Theorem 2.2. *Let $A, B \in M_n$ be positive semidefinite, then for any unitarily invariant norm*

$$||AB|| + \left(\int_0^1 g(v)dv - 2||AB||\right) + \left(\int_0^1 g(v)dv - ||A^{\frac{3}{4}}B^{\frac{5}{4}} + A^{\frac{5}{4}}B^{\frac{3}{4}}||\right)$$
$$\leq \frac{1}{4}||(A+B)^2||,$$

where $g(v) = ||A^{\frac{1}{2}+v}B^{\frac{3}{2}-v} + A^{\frac{3}{2}-v}B^{\frac{1}{2}+v}||$.

Proof. By (2.1), taking $X = A^{\frac{1}{2}}B^{\frac{1}{2}}$, we have

$$2||AB|| + 2\left(\int_0^1 g(v)dv - 2||AB||\right) + 2\left(\int_0^1 g(v)dv - ||A^{\frac{3}{4}}B^{\frac{5}{4}} + A^{\frac{5}{4}}B^{\frac{3}{4}}||\right)$$
$$\leq ||A^{\frac{1}{2}}(A+B)B^{\frac{1}{2}}||. \tag{2.8}$$

By Lemma 2.3, it easily follows from (2.8) that

$$2||AB|| + 2\left(\int_0^1 g(v)dv - 2||AB||\right) + 2\left(\int_0^1 g(v)dv - ||A^{\frac{3}{4}}B^{\frac{5}{4}} + A^{\frac{5}{4}}B^{\frac{3}{4}}||\right)$$
$$\leq \frac{1}{2}||(A+B)^2||.$$

This completes the proof. □

Remark 2.2. Obviously, Theorem 2.2 is a refinement of inequality (1.4).

In the following, we utilize the convexity of the function $\psi(u) = ||A^u XB^{2-u} + A^{2-u}XB^u||$ to present two matrix inequalities for unitarily invariant norms that lead to improved versions of inequality (1.7).

Theorem 2.3. *Let $A, X, B \in M_n$ such that A and B are positive semidefinite, then for any unitarily invariant norm*

$$\begin{aligned} ||A^u XB^{2-u} + A^{2-u}XB^u|| &\leq 2(4r_0 - 1)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\ &\quad + 2(3 - 8r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|, \\ &\quad u \in [\frac{1}{2}, \frac{3}{4}] \cup [\frac{5}{4}, \frac{3}{2}] \end{aligned}$$

and

$$\begin{aligned} ||A^u XB^{2-u} + A^{2-u}XB^u|| &\leq 8(1 - 2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| \\ &\quad + 2(8r_0 - 3)||AXB||, \ u \in (\frac{3}{4}, \frac{5}{4}), \end{aligned}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$ and $r_0 = \min\{\frac{u}{2}, 1 - \frac{u}{2}\}$.

Proof. For $\frac{1}{2} \leq u \leq \frac{3}{4}$, by the convexity of the function $\psi(u) = ||A^u XB^{2-u} + A^{2-u}XB^u||$ and Lemma 2.4, we obtain

$$\psi(u) \leq \frac{\psi(\frac{3}{4}) - \psi(\frac{1}{2})}{\frac{1}{4}}u - \frac{\frac{1}{2}\psi(\frac{3}{4}) - \frac{3}{4}\psi(\frac{1}{2})}{\frac{1}{4}},$$

which is equivalent to

$$\psi(u) \leq (4u - 2)\psi\left(\frac{3}{4}\right) + (3 - 4u)\psi\left(\frac{1}{2}\right). \tag{2.9}$$

Combining (1.6) with (2.9), we get

$$\begin{aligned} ||A^u XB^{2-u} + A^{2-u}XB^u|| &\leq (4u - 2)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\ &\quad + 2(3 - 4u)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|. \end{aligned}$$

Hence

$$\begin{aligned} ||A^u XB^{2-u} + A^{2-u}XB^u|| &\leq 2(4r_0 - 1)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\ &\quad + 2(3 - 8r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|. \end{aligned} \tag{2.10}$$

For $\frac{3}{4} < u \leq 1$, by the convexity of the function $\psi(u)$ and Lemma 2.4, we have

$$\psi(u) \leq \frac{\psi(1) - \psi(\frac{3}{4})}{1 - \frac{3}{4}} u - \frac{\frac{3}{4}\psi(1) - 1\psi(\frac{3}{4})}{1 - \frac{3}{4}},$$

which is equivalent to

$$\psi(u) \leq (4u - 3)\psi(1) + (4 - 4u)\psi\left(\frac{3}{4}\right). \tag{2.11}$$

Combining (1.6) with (2.11), we get

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 8(1-u)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(4u-3)||AXB||.\end{aligned}$$

Hence

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 8(1-2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(8r_0-3)||AXB||.\end{aligned} \tag{2.12}$$

For $1 < u < \frac{5}{4}$, similarly, we have

$$\psi(u) \leq (4u-4)\psi\left(\frac{5}{4}\right) + (5-4u)\psi\left(1\right),$$

that is

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 2(4u-4)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(5-4u)||AXB||.\end{aligned}$$

Consequently

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 8(1-2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(8r_0-3)||AXB||.\end{aligned} \tag{2.13}$$

For $\frac{5}{4} \leq u \leq \frac{3}{2}$, we have

$$\psi(u) \leq (4u-5)\psi\left(\frac{3}{2}\right) + (6-4u)\psi\left(\frac{5}{4}\right),$$

that is

$$\begin{aligned}||A^u XB^{2-u} + A^{2-u}XB^u|| \leq{}& 2(4u-5)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|\\ &+ (6-4u)||A^{\frac{5}{4}}XB^{\frac{3}{4}} + A^{\frac{3}{4}}XB^{\frac{5}{4}}||.\end{aligned}$$

Consequently

$$\begin{aligned} ||A^uXB^{2-u} + A^{2-u}XB^u|| &\leq 2(4r_0 - 1)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\ &\quad + 2(3 - 8r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|. \end{aligned} \tag{2.14}$$

It follows from (2.10),(2.12),(2.13),(2.14) and $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$, $r_0 = \min\{\frac{u}{2}, 1 - \frac{u}{2}\}$ that

$$\begin{aligned} ||A^uXB^{2-u} + A^{2-u}XB^u|| &\leq 2(4r_0 - 1)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\ &\quad + 2(3 - 8r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|, \\ &\quad u \in [\frac{1}{2}, \frac{3}{4}] \cup [\frac{5}{4}, \frac{3}{2}] \end{aligned}$$

and

$$\begin{aligned} ||A^uXB^{2-u} + A^{2-u}XB^u|| &\leq 8(1 - 2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| \\ &\quad + 2(8r_0 - 3)||AXB||,\ u \in (\frac{3}{4}, \frac{5}{4}). \end{aligned}$$

This completes the proof. □

Remark 2.3. Theorem 2.3 is sharper than inequality (1.7).

Note that, inequality (1.7) is equivalent to

$$\psi(u) \leq 2(1-u)\psi\left(\frac{1}{2}\right) + (2u-1)\psi(1),\ \frac{1}{2} \leq u \leq 1 \tag{2.15}$$

and

$$\psi(u) \leq (3-2u)\psi(1) + 2(u-1)\psi\left(\frac{3}{2}\right),\ 1 \leq u \leq \frac{3}{2}. \tag{2.16}$$

For $\frac{1}{2} \leq u \leq \frac{3}{4}$, compared with inequality (2.15), then

$$\begin{aligned} &2(1-u)\psi\left(\frac{1}{2}\right) + (2u-1)\psi(1) - \left((4u-2)\psi\left(\frac{3}{4}\right) + (3-4u)\psi\left(\frac{1}{2}\right)\right) \\ &= (2u-1)\left(\psi\left(\frac{1}{2}\right) - 2\psi\left(\frac{3}{4}\right) + \psi(1)\right). \end{aligned}$$

Since $\psi(u) = ||A^uXB^{2-u} + A^{2-u}XB^u||$ is convex on $[\frac{1}{2}, \frac{3}{2}]$, it follows by a slope argument that

$$\frac{\psi(1) - \psi\frac{3}{4})}{1 - \frac{3}{4}} \geq \frac{\psi(\frac{3}{4}) - \psi(\frac{1}{2})}{\frac{3}{4} - \frac{1}{2}},$$

that is

$$\psi\left(\frac{1}{2}\right)-2\psi\left(\frac{3}{4}\right)+\psi(1)\geq 0.$$

So,

$$2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1)\geq(4u-2)\psi\left(\frac{3}{4}\right)+(3-4u)\psi\left(\frac{1}{2}\right). \tag{2.17}$$

For $\frac{3}{4}<u\leq 1$,

$$\begin{aligned}&2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1)-\left((4u-3)\psi(1)+(4-4u)\psi\left(\frac{3}{4}\right)\right)\\&=(2-2u)\left(\psi\left(\frac{1}{2}\right)-2\psi\left(\frac{3}{4}\right)+\psi(1)\right)\geq 0.\end{aligned}$$

So,

$$2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1)\geq(4u-3)\psi(1)+(4-4u)\psi\left(\frac{3}{4}\right). \tag{2.18}$$

For $1<u<\frac{5}{4}$, compared with inequality (2.16), then we have

$$\begin{aligned}&(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)-\left((4u-4)\psi\left(\frac{5}{4}\right)+(5-4u)\psi(1)\right)\\&=2(u-1)\left(\psi\left(\frac{3}{2}\right)-2\psi\left(\frac{5}{4}\right)+\psi(1)\right).\end{aligned}$$

Since $\psi(u)=||A^uXB^{2-u}+A^{2-u}XB^u||$ is convex on $[\frac{1}{2},\frac{3}{2}]$, it follows by a slope argument that

$$\frac{\psi(\frac{3}{2})-\psi(\frac{5}{4})}{\frac{3}{2}-\frac{5}{4}}\geq\frac{\psi(\frac{5}{4})-\psi(1)}{\frac{5}{4}-1},$$

that is

$$\psi\left(\frac{3}{2}\right)-2\psi\left(\frac{5}{4}\right)+\psi(1)\geq 0.$$

So,

$$(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)\geq(4u-4)\psi\left(\frac{5}{4}\right)+(5-4u)\psi(1). \tag{2.19}$$

For $\frac{5}{4}\leq u\leq\frac{3}{2}$, we have

$$\begin{aligned}&(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)-\left((4u-5)\psi\left(\frac{3}{2}\right)+(6-4u)\psi\left(\frac{5}{4}\right)\right)\\&=(3-2u)\left(\psi\left(\frac{3}{2}\right)-2\psi\left(\frac{5}{4}\right)+\psi(1)\right)\geq 0.\end{aligned}$$

So,

$$(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)\geq(4u-5)\psi\left(\frac{3}{2}\right)+(6-4u)\psi\left(\frac{5}{4}\right). \tag{2.20}$$

By (2.17)-(2.20), we can conclude that Theorem 2.3 is better than inequality (1.7).

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Nondestructive evaluation of interface defects in layered media

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Abstract. In a layered thermal conductor, the inaccessible interface could be damaged by mechanical solicitation, chemical infiltration, aging. In this case, the original thermal properties of the specimen are modified. The defect occurs typically in form of *delamination*. The present paper deals with nondestructive evaluation of interface thermal conductance h from the knowledge of the surface temperature when the specimen is heated in some controlled way. The goal is achieved by expanding h in powers of the thickness of the upper layer. The mathematical analysis of the model produces exact formulas for the first coefficients of h which are tested on simulated and real data. The evaluation of interface flaws comes from reliable approximation of h.

Keywords: imperfect interface, thermal contact conductance, heat equation, inverse problem.
MSC 2020: 35F51 41A99 80A23

1. Introduction

In a layered conductor, the inaccessible interface $\tilde{\Sigma}$ could be damaged by mechanical solicitation, chemical infiltration, aging. In this case, the original thermal properties of the specimen are modified. The present paper deals with nondestructive evaluation of defects in $\tilde{\Sigma}$ from the knowledge of the surface temperature when the specimen is heated by applying a voltage or by means of a lamp system or a laser. Temperature is measured with an infrared camera in the typical framework of Active Thermography [19]. The mathematical model consists of a system of two Boundary Value Problems (BVPs) for the Laplace-transformed heat equation. The evaluation of defects affecting the interface requires the approximate solution of a non linear Inverse Heat Conduction Problem.

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1.1 Layered domains

Consider a composite body made up of two thermally conducting layers $\tilde{B}^+$ and $\tilde{B}^-$divided by a very thin and irregular interspace $\tilde{S}$ filled up with air or other poorly conductive materials (see Figure 1 (a)). As long as $\tilde{B}^-$ is heated by an external source, heat flows through $\tilde{S}$ mainly in correspondence to possible contact spots between the conducting layers. As strong as the layers are pressed together, their contact area depends on nonflatness and roughness of the contacting surfaces. Assume that the effect of $\tilde{S}$ on heat transfer between the two layers $\tilde{B}^-$ and $\tilde{B}^+$ is equivalent to the effect of a smooth thin interspace S of constant thickness d_S and virtual thermal conductivity κ_S. In this case a model with three layers is obtained ($B^+ \cup S \cup B^-$ see Figure 1 (b)) where the opposite sides of S have different temperature but there is no thermal gap between adjacent layers B^+, S and S, B^-. It is shown in [10] that heat conduction in $B^+ \cup S \cup B^-$ is correctly modeled in terms of transmission conditions on a two-dimensional interface $\tilde{\Sigma}$ that separates the conducting layers. Indeed, the thin domain S shrinks to the surface $\tilde{\Sigma}$ (a rigorous analysis of limits of the form $\lim_{d_S \to 0} \frac{k_S}{d_S}$ in a similar geometry is in [8], Sect 7) so that the specimen is, finally, $\tilde{\Omega} = \tilde{\Omega}^+ \cup \tilde{\Sigma} \cup \tilde{\Omega}^-$ (see, Figure 1 (c)).

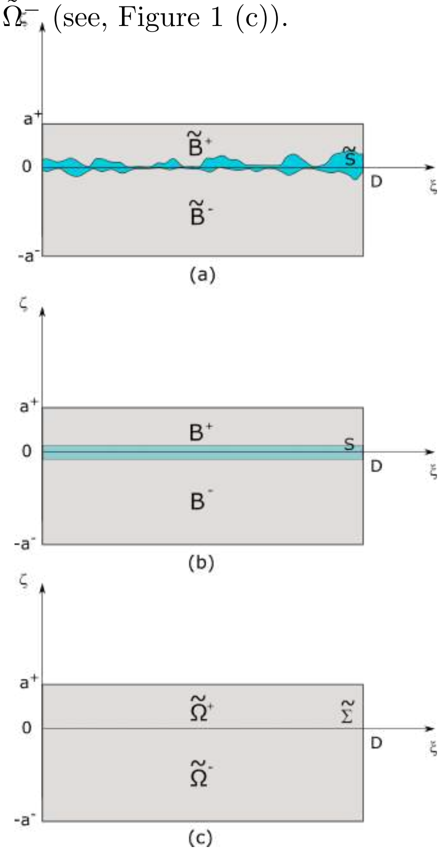

Figure 1: layered domain: from the interspace $\tilde{S}$ to the interface $\tilde{\Sigma}$

1.2 Types of interfaces and thermal parameters

Interfaces can be classified as perfect or imperfect according to their thermal properties [16]. In case of *perfect interfaces*, temperature and normal heat flux are continuous in $\tilde{\Sigma}$ while the model of *Low Conductivity Imperfect* (LCI) interface allows for a temperature jump with continuous heat flux.

The *Thermal Contact Resistance* (TCR) $\bar{r}$ (see for example [12] Ch 3) is a non negative parameter proportional to the temperature gap between the two sides of $\tilde{\Sigma}$. Its inverse $\bar{h} = \frac{1}{\bar{r}}$ is referred to as *Thermal Contact Conductance* (TCC).

In perfect interfaces the parameter $\bar{r}$ is zero (very small in practice) and $\bar{h}$ is infinite (actually large).

In LCIs, the resistance is $\bar{r} >> 0$. In the limit case of infinite $\bar{r}$ the interface is perfectly insulating and $\bar{h} = 0$.

A defect affecting $\tilde{\Sigma}$ gives rise to anomalies in the thermal behavior of the interface. We focus on the case in which the undamaged interface is perfect ($\bar{r}$ is very small) and the defect is an inclusion between the layers (see for example [23]). The occurrence of a similar defect produces locally a larger TCR $r = \bar{r} + \delta r$ ($\delta r > 0$ non constant on $\tilde{\Sigma}$). The extension of TCR and TCC to the perturbed non constant case is not rigorously founded but it is in agreement with experimental data and widely used among practitioners (see for example [1, 26, 2]). Hence, $h = \frac{1}{\bar{r}+\delta r} = \bar{h} + \delta h$ ($\delta h < 0$ non constant on $\tilde{\Sigma}$) plays the role of exchange coefficient in Robin transmission conditions (5) and (6) in section 2. In this case, there is no appreciable temperature gap between the opposite sides of $\tilde{\Sigma}$ except on the damaged area where we expect that the numerical value of $\frac{\kappa_a}{h}$ (κ_a is the thermal conductivity of the inclusion) gives a good approximation of the thickness of the defect [7]. We can reasonably simplify the problem by assuming that the defect is actually a *delamination* described by the graph of a function of two variables whose level sets are convex. More precisely, in applicative literature, "the delamination zone is often taken as a square, circular or elliptical domain so as to confirm a satisfactory compromise between the realistic representation of the geometry of the real delamination and the simple insertion of the artificial damage" [9]. We apply this concept in section 7.2. A detailed analysis of nondestructive inspection of impact-damaged composite structures is in [25]. The interface is usually filled with air. Treating the interface in terms of thermal conductance only is anyhow justified because its size is small enough to prevent the occurrence of convective motions which would require a mixed thermal/fluid model.

1.3 The direct model and the inverse problem

In this section, we describe briefly the specific model and the approach used here to solve the inverse problem. The lower layer $\tilde{\Omega}^-$ is heated by means of thermal flux coming from below, e.g. by a lamp, kept on for a time interval of $\tau_{\max}$ seconds. Heat passes through the interface $\tilde{\Sigma}$ so that the temperature of $\tilde{\Omega}^+$ changes during heating. Heat transfer through the interface is modeled by means of Robin transmission boundary conditions (see for example [21, 6]). A sequence $\tilde{\psi}$ of temperature maps is taken, in the meanwhile, on the external surface of $\tilde{\Omega}^+$. This setting is usually referred to as *transmission mode* [5] in Long Pulse Thermography. Details of this mathematical model, based on the heat equation in normalized dimensionless variables, are in section 3. In the new variables, layers are named Ω^+ and Ω^- while the interface is Σ and the whole specimen is $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$.

It is remarkable that h is independent of time (at least in the time scale of $t_{\max}$) so that it is convenient to apply Laplace's transform to equations and boundary conditions (see section 4). In this way we obtain a system of two BVPs for elliptic equations in Ω^+ and Ω^- (connected by Robin transmission conditions) whose solutions U^+ and U^- are the Laplace transform of the temperatures of the two layers. At this point, since our specimen is composed by thin layers, we introduce the formal expansion of h, U^+ and U^- in even powers of the normalized thickness γ of $\tilde{\Omega}^+$. Our goal is to write the coefficients h_k (of the expansion in γ^{2k}) in terms of the available data (incomplete thermal boundary data). We accomplish this task by means of a generalization of Thin Plate Approximation (TPA). In particular, we show that the coefficients of the expansion of the trace of U^- on Σ fulfills a family of Neumann problems for elliptic PDEs at least for $k = 0, 1$ (see (30) and (53)). In this way, transmission conditions for $k = 0, 1$ on the positive side of Σ become ordinary Robin conditions for BVPs in Ω^+ so that we are in a position to derive the explicit expressions of h_0 and h_1 in terms of $\tilde{\psi}$. A similar model has been studied in [3] where a flaw (of unknown depth) is evaluated from the knowledge of a complete set of thermal data at the boundary. A stationary two-dimensional case is studied in [1] using reciprocity functional approach. A problem of *reflection mode* [5] in Long Pulse Thermography of a single layer specimen is solved in [14].

We recall that TPA is a perturbative technique for the computational solution of some inverse problems on thin domains, borrowed from [18]. In [13], TPA is compared with pre-existing methods based on reciprocity functional approach, optimization and regularization [4].

1.4 Simulations and experiments

We apply the method described in section 6 to the nondestructive evaluation of defects affecting the interface of a coated iron slab. We test the first order approximation (i.e. $h \approx h_0 + \gamma^2 h_1$) both in 2D simulations and in case of real data in a full 3D model. The approximation of h (real data are processed) shown in figure 6 improves the reconstruction obtained in [24] where the trace of U^- on Σ is heuristically overwritten by its background temperature.

2. Geometry, notation, direct model and inverse problem

Let $\tilde{\Omega}$ be the parallelepiped $(0,D)\times(0,D)\times(-a^-,a^+)$ in the 3D space (ξ,η,ζ). Let $\tilde{\Omega}^+$ be $(0,D)\times(0,D)\times(0,a^+)$ and $\tilde{\Omega}^-$ be $(0,D)\times(0,D)\times(-a^-,0)$. Let $\tilde{\Sigma}=\{(\xi,\eta)\in(0,D)\times(0,D)\ ,\ \zeta=0\}$. Clearly $\tilde{\Omega}=\tilde{\Omega}^+\cup\tilde{\Sigma}\cup\tilde{\Omega}^-$.

To fix ideas, assume that $\frac{a^++a^-}{D} << 1$. The geometry of the problem is summarized in Figure 2. The thermal behavior of each layer $\tilde{\Omega}^\pm$ is determined by its conductivity $\tilde{\kappa}^\pm$, density $\rho^\pm$ and specific heat $c^\pm$. Heat transfer through the interface $\tilde{\Sigma}$ depends on its thermal contact conductance $\tilde{h}(\xi,\eta)$.

Let $v^\pm(\xi,\eta,\zeta,\tau)$ with $(\xi,\eta,\zeta)\in\tilde{\Omega}^\pm$ and $\tau>0$ the temperature increase (with respect to an initial and surrounding temperature V_0) in $\tilde{\Omega}^\pm$ obtained by applying, for a time interval $(0,\tau_{\max})$, a heat flux $\tilde{\phi}(\xi,\eta,\tau)$ to $\tilde{\Omega}^-$ (more precisely, $\tilde{\phi}(\xi,\eta,\tau)=0$ for $\tau>\tau_{\max}$) . Clearly, $v(\xi,\eta,\zeta,0)=0$. Assume that the vertical sides of the composite domain are insulated while the horizontal sides exchange heat with the environment. The thermal contact conductances of top ($\zeta=a^+$) and bottom side ($\zeta=-a^-$) are the positive constants $\tilde{h}^+$ and $\tilde{h}^-$ respectively.

2.1 The direct model

Given the constant parameters $a^\pm$, D, $\tilde{\kappa}^\pm$, $\rho^\pm$, $c^\pm$ and $\tilde{h}^\pm$ and given interface thermal conductance $\tilde{h}(\xi,\eta)$, the functions $v^\pm$ fulfill an Initial Boundary Value Problem (IBVP) for the heat equation in the composite domain $\tilde{\Omega}$ (we write down this IBVP later in dimensionless variables).

$$\rho^-c^-v^-_\tau=\tilde{\kappa}^-(v^-_{\xi\xi}+v^-_{\eta\eta}+v^-_{\zeta\zeta})\ ,\ (\xi,\eta,\zeta)\in\tilde{\Omega}^-,\ \tau>0, \tag{1}$$

$$-\tilde{\kappa}^-v^-_\zeta(\xi,\eta,-a^-)+\tilde{h}^-v^-(\xi,\eta,-a^-)=\tilde{\phi}(\xi,\eta,\tau), \tag{2}$$

$$\rho^+c^+v^+_\tau=\tilde{\kappa}^+(v^+_{\xi\xi}+v^+_{\eta\eta}+v^+_{\zeta\zeta})\ ,\ (\xi,\eta,\zeta)\in\tilde{\Omega}^+,\ \tau>0, \tag{3}$$

$$\tilde{\kappa}^+v^+_\zeta(\xi,\eta,a^+,\tau)+\tilde{h}^+v^+(\xi,\eta,a^+,\tau)=0 \tag{4}$$

and $v^\pm_\nu=0$ on the *vertical* sides of $\tilde{\Omega}^\pm$, with *transmission conditions*

$$\tilde{\kappa}^-v^-_\zeta(\xi,\eta,0,\tau)+\tilde{h}(\xi,\eta)(v^-(\xi,\eta,0,\tau)-v^+(\xi,\eta,0,\tau))=0, \tag{5}$$

$$\tilde{\kappa}^-v^-_\zeta(\xi,\eta,0,\tau)=\tilde{\kappa}^+v^+_\zeta(\xi,\eta,0,\tau) \tag{6}$$

Initial data are

$$v^-(\xi,\eta,\zeta,0)=0\ ,\ (\xi,\eta,\zeta)\in\tilde{\Omega}^-, \tag{7}$$

$$v^+(\xi,\eta,\zeta,0)=0\ ,\ (\xi,\eta,\zeta)\in\tilde{\Omega}^+. \tag{8}$$

2.2 The interface inverse problem

Assumed that $\tilde{h}(\xi,\eta)$ is unknown, the goal is to approximate $\tilde{h}$ by using the knowledge of $\tilde{\phi}$ and the available additional (boundary) dataset $\tilde{\psi}(\xi,\eta,\tau)=v^+(\xi,\eta,a^+,\tau)$ for $\tau\in(0,\tau_{\max})$.

3. Dimensionless variables

We introduce the standard set of dimensionless variables $z = \frac{\zeta}{a^+}$, $x = \frac{\xi}{D}$, $y = \frac{\eta}{D}$ and $t = \frac{\tau}{T}$ where $T = \frac{\rho^+ c^+ D^2}{\tilde{\kappa}^+}$. We set also $\kappa^\pm = \frac{\tilde{\kappa}^\pm}{D^2}$ and $\beta = \frac{\alpha^-}{\alpha^+}$ where the numbers $\alpha_\pm = \frac{\kappa_\pm}{\rho_\pm c_\pm}$ are the diffusivities of upper and lower slabs respectively.

Rewrite the geometry of the problem in the new variables. Here $b = \frac{a^-}{a^+}$.
Let Ω be the parallelepiped $(0,1) \times (0,1) \times (-b,1)$ in the 3D space (x,y,z).
Let Ω^+ be $(0,1) \times (0,1) \times (0,1)$ and Ω^- be $(0,1) \times (0,1) \times (-b,0)$.
Let $\Sigma = \{(x,y) \in (0,1) \times (0,1)\ ,\ z = 0\}$. Clearly $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$.
Define

$$
\begin{aligned}
u^\pm(x,y,z,t) &\equiv v^\pm(Dx, Dy, a^+ z, Tt), \\
\psi(x,y,t) &= \tilde{\psi}(Dx, Dy, Tt), \\
\gamma\phi(x,y,t) &= \tilde{\phi}(Dx, Dy, Tt), \\
\gamma h(x,y) &= \tilde{h}(Dx, Dy).
\end{aligned} \tag{9}
$$

As for the (known a priori) constant thermal conductances of top and bottom sides of Ω, we set $\tilde{h}^+ = \gamma h^+$ and $\tilde{h}^- = \gamma h^-$ respectively. The scaling factor γ (defined at the end of section 2.1) is functional to the power expansions of $u^\pm$ and h in what follows.

In dimensionless variables and taking into account (9), system (1)-(8) becomes

IBVP$^-$

$$\gamma^2 u_t^- = \beta\gamma^2(u_{xx}^- + u_{yy}^-) + \beta u_{zz}^-\ ,\ (x,y,z) \in \Omega^-,\ t > 0, \tag{10}$$

$$-D\kappa^- u_z^-(x,y,-b,t) + \gamma^2 h^- u^-(x,y,-b,t) = \gamma^2 \phi(x,y,t) \tag{11}$$

($u_\nu = 0$ on the *vertical* sides of Ω^-)

IBVP$^+$

$$\gamma^2 u_t^+ = \gamma^2(u_{xx}^+ + u_{yy}^+) + u_{zz}^+\ ,\ (x,y,z) \in \Omega^+,\ t > 0, \tag{12}$$

$$D\kappa^+ u_z^+(x,y,1) + \gamma^2 h^+ u^+(x,y,1) = 0 \tag{13}$$

($u_\nu = 0$ on the *vertical* sides of Ω^+)

with *transmission conditions*

$$D\kappa^- u_z^-(x,y,0) + \gamma^2 h(x,y)(u^-(x,y,0) - u^+(x,y,0)) = 0 \tag{14}$$

$$\kappa^- u_z^-(x,y,0) = \kappa^+ u_z^+(x,y,0). \tag{15}$$

Initial data are

$$u^-(x,y,z,0) = 0\ ,\ (x,y,z) \in \Omega^-, \tag{16}$$

$$u^+(x,y,z,0) = 0\ ,\ (x,y,z) \in \Omega^+. \tag{17}$$

Remark. If ϕ and h are continuous functions and $H^1(\Omega)$ is a product Hilbert space equipped with a suitable norm, the system (10)-(16) admits a unique solution $(u^+, u^-) \in L^2(0,T;H^1(\Omega))$, stable with respect to error on h (see [15]).

4. Laplace transform of the direct problem

First, define (for all real positive numbers s) the Laplace transform of $u^{\pm}(x,y,z,t)$ as

$$U^{s\pm}(x,y,z) = \int_0^\infty u^{\pm}(x,y,z,t)e^{-st}dt \tag{18}$$

while

$$\Phi^s(x,y) = \int_0^\infty \phi(x,y,t)e^{-st}dt.$$

We know that the bounded function $u(x,y,0,t)$ is decreasing for $t > t_{\max} + \delta t$ where $\delta t > 0$ depends on thickness and diffusivity of the specimen. The temperature data $\psi(x,y,t)$ can be extended formally for $t > t_{\max}$ to a bounded function ψ_∞ decreasing to zero without any sensitive effect in the calculation of the Laplace transform of ψ_∞ (in a suitable range of s). Hence, in what follows it is

$$\Psi^s(x,y) = \int_0^\infty \psi_\infty(x,y,t)e^{-st}dt.$$

Standard calculations change (10)-(16) into the following system of elliptic BVPs

BVP^-

$$\gamma^2 sU^{s-} = \beta\gamma^2({U^{s}}^-_{xx} + U^-_{yy}) + \beta {U^{s}}^-_{zz}\ ,\ (x,y,z) \in \Omega^-, \tag{19}$$

$$-D\kappa^- {U^{s}}^-_z(x,y,-b) + \gamma^2 h^- U^{s-}(x,y,-b) = \gamma^2\Phi^s(x,y) \tag{20}$$

and ${U^{s}}^-_\nu = 0$ on the *vertical* sides of Ω^-

BVP^+

$$\gamma^2 sU^{s+} = \gamma^2({U^{s}}^+_{xx} + {U^{s}}^+_{yy}) + {U^{s}}^+_{zz}\ ,\ (x,y,z) \in \Omega^+, \tag{21}$$

$$D\kappa^+ {U^{s}}^+_z(x,y,1) + \gamma^2 h^+ U^{s+}(x,y,1) = 0 \tag{22}$$

and ${U^{s}}^+_\nu = 0$ on the *vertical* sides of Ω^+ with *transmission conditions*

$$D\kappa^- {U^{s}}^-_z(x,y,0) + \gamma^2 h(x,y)(U^{s-}(x,y,0) - U^{s+}(x,y,0)) = 0, \tag{23}$$

$$\kappa^- {U^{s}}^-_z(x,y,0) = \kappa^+ {U^{s}}^+_z(x,y,0). \tag{24}$$

In what follows $U^{s\pm}$, Ψ^s and Φ^s are written simply $U^{\pm}$, Ψ and Φ. The dependence on the parameter s is implicit. The actual choice of values of s is discussed in section 7.1.

5. The inverse problem

After introducing dimensionless variables and applying Laplace's transform, the inverse problem defined in the end of section 2 is formulated the following way:

Interface Inverse Problem Assumed that the coefficient $h(x, y)$ in (23) is unknown, it must be recovered from the knowledge of Φ and the available additional (boundary) data $\Psi(x, y) = U^+(x, y, 1)$).

Mathematical remark It is immediate to realize that the external flux $U^\pm_\nu$ is known on the whole boundary of Ω while $U^\pm$ is given only on the top boundary of Ω^+ (incomplete Neumann to Dirichlet (NTD) map). A wide mathematical literature about uniqueness and stability of solutions of inverse problems for parabolic and elliptic PDEs is available, but we did not find any theorem fitting our Interface Inverse Problem in presence of incomplete NTD map. A rigorous solution of this aspect of the problem is out of the goal of the present research. Actually, this is a work in progress starting from the useful suggestions in [22] (a single domain instead of a layered one) and [11] (full NTD map, continuous temperature and discontinuous flux at the interface).

6. Thin plate approximation

First, we stress that, when h is given, the solutions U^+ and U^- of (19)-(23) depend on γ^2. If also h is unknown, any approximation based on the direct model (19)-(23) also depends on γ^2. Since the parameter γ is assumed small, we introduce the following formal expansions:

$$
\begin{aligned}
U^-(x, y, z) &= U_0^-(x, y, z) + \gamma^2 U_1^-(x, y, z) + ..., \\
U^+(x, y, z) &= U_0^+(x, y, z) + \gamma^2 u_1^+(x, y, z) + ..., \\
h(x, y) &= h_0(x, y) + \gamma^2 h_1(x, y) + ...
\end{aligned} \tag{25}
$$

6.1 Order zero of the expansion of h

Consider the terms of order zero:

$$U^-_{0zz} = 0,$$

for all (x,y,z), and

$$U^-_{0z}(x, y, -b) = U^-_{0z}(x, y, 0) = U^+_{0z}(x, y, 0) = U^+_{0z}(x, y, 1) = 0.$$

It means that U_0^- and U_0^+ do not depend on z.

First order terms are:

$$
\begin{aligned}
U^-_{1z}(x,y,0) &= -\frac{h_0(x,y)}{D\kappa^-}(U^-_0(x,y) - U^+_0(x,y)),\\
U^+_{1z}(x,y,0) &= \frac{\kappa^-}{\kappa^+}U^-_{1z}(x,y,0),\\
U^-_{1z}(x,y,-b) &= \frac{h^-}{D\kappa^-}U^-_0(x,y) - \frac{\Phi}{D\kappa^-},\\
U^+_{1z}(x,y,1) &= -\frac{h^+}{D\kappa^+}U^+_0(x,y),\\
U^-_{1zz} &= \frac{s}{\beta}U^-_0 - (U^-_{0xx} + U^-_{0yy}),\\
U^+_{1zz} &= sU^+_0 - (U^+_{0xx} + U^+_{0yy}).
\end{aligned} \tag{26}
$$

Since $f_z(a_2) = f_z(a_1) + \int_{a_1}^{a_2} f_{zz}(s)ds$, we have

$$-h_0(U^-_0 - U^+_0) + h^+U^+_0 + D\kappa^+sU^+_0 - D\kappa^+\Delta U^+_0 = 0 \tag{27}$$

and

$$h_0(U^-_0 - U^+_0) + h^-U^-_0 - \Phi + D\kappa^-\frac{bs}{\beta}U^-_0 - bD\kappa^-\Delta U^-_0 = 0. \tag{28}$$

The sum of the last two equation does not depend on the unknown $h_0(x,y)$. Hence, if we assume

$$U^+_0(x,y) = \Psi(x,y) \tag{29}$$

(i.e. $U^+_k(x,y,1) = 0$ for $k > 0$) we have the following elliptic PDE in $U^-_0(x,y,t)$

$$(\frac{h^-}{D\kappa^-} + \frac{bs}{\beta})U^-_0 - b\Delta U^-_0 = F_0 \tag{30}$$

where

$$F_0(x,y) = \frac{\Phi(x,y)}{D\kappa^-} - (\frac{h^+}{D\kappa^-} + \frac{\kappa^+}{\kappa^-}s)\Psi(x,y) + \frac{\kappa^+}{\kappa^-}\Delta\Psi \tag{31}$$

with Neumann boundary conditions

$$U^-_{0x}(0,y) = U^-_{0x}(1,y) = U^-_{0y}(x,0) = U^-_{0y}(x,1) = 0. \tag{32}$$

Hence, solving (27), we obtain

$$h_0(x,y) = \frac{(h^+ + D\kappa^+s)\Psi(x,y) - D\kappa^+\Delta\Psi(x,y)}{U^-_0 - \Psi}. \tag{33}$$

6.2 First order of the expansion of h

In what follows, $Lf = f_{xx} + f_{yy}$. We derive the following first order relation in Ω^+:

$$U^+_{1zz} = A^+_0(x, y), \tag{34}$$

where

$$A^+_0(x, y) = s\Psi(x, y) - L\Psi(x, y) \tag{35}$$

so, that

$$\begin{aligned} U^+_{1z}(x, z) &= A^+_0(x, y)z + B^+_0(x, y), \\ U^+_1(x, z) &= A^+_0(x, y)\frac{z^2}{2} + B^+_0(x, y)z + C^+_0(x, y), \end{aligned} \tag{36}$$

where

$$B^+_0(x, y) = -\frac{h_0(x, y)}{D\kappa^+}(U^-_0(x, y) - \Psi(x, y)). \tag{37}$$

Since we assumed $U^+_0(x, y) = U^+(x, y, 1) = \Psi(x, y)$ (see (29), it is $U^+_1(x, y, 1) \equiv 0$ so that

$$C^+_0(x, y) = -A^+_0(x, y)\frac{1}{2} - B^+_0(x, y) \tag{38}$$

and

$$U^+_1(x, y, z) = A^+_0(x, y)\frac{z^2 - 1}{2} + B^+_0(x, y)(z - 1). \tag{39}$$

Analogously, in Ω^- we have

$$U^-_{1zz} = A^-_0(x, y) \tag{40}$$

with

$$A^-_0(x, y) = \frac{s}{\beta}U^-_0(x, y) - LU^-_0(x, y). \tag{41}$$

so, that

$$\begin{aligned} U^-_{1z} &= A^-_0(x, y)z + B^-_0(x, y), \\ U^-_1(x, y, z) &= A^-_0(x, y)\frac{z^2}{2} + B^-_0(x, y)z + C^-_0(x, y), \end{aligned} \tag{42}$$

where

$$B^-_0(x, y) = -\frac{h_0(x, y)}{D\kappa^-}(U^-_0(x, y) - \Psi(x, y)). \tag{43}$$

Observe that the term $U^-_1(x, y, 0) = C^-_0(x, y)$ is still undetermined and it will be obtained by solving an equation having the same form of (30).

6.3 An equation for C_0^-

Second order terms in equations (19) and (21) are

$$U^+_{2zz} = A^+_1(x, y, z) \tag{44}$$

and

$$U^-_{2zz} = A^-_1(x, y, z), \tag{45}$$

where

$$A^+_1(x, y, z) = sU^+_1(x, y, z) - LU^+_1(x, y, z) \tag{46}$$

and

$$A^-_1(x, y, z) = \frac{s}{\beta}U^-_1(x, y, z) - LU^-_1(x, y, z). \tag{47}$$

As for boundary conditions, we have

$$\begin{aligned} U^+_{2,z}(x, y, 1) &= -\frac{h^+}{D\kappa^+}U^+_1(x, y, 1) \equiv 0 \quad from\ (29), \\ U^-_{2,z}(x, y, -b) &= \frac{h^-}{D\kappa^-}U^-_1(x, y, -b), \\ U^-_{2,z}(x, y, 0) &= -\frac{h_1(x, y)}{D\kappa^-}(U^-_0(x, y) - \Psi(x, y)) \\ &\quad - \frac{h_0(x, y)}{D\kappa^-}(U^-_1(x, y, 0) - U^+_1(x, y, 0)), \\ U^+_{2,z}(x, y, 0) &= -\frac{h_1(x, y)}{D\kappa^+}(U^-_0(x, y) - \Psi(x, y)) \\ &\quad - \frac{h_0(x, y)}{D\kappa^+}(U^-_1(x, y, 0) - U^+_1(x, y, 0)). \end{aligned} \tag{48}$$

In order to lighten the notation, in what follows we stress the dependance on the variable z only. Since

$$U^+_{2z}(1) = U^+_{2z}(0) + \int_0^1 U^+_{2zz}(z)dz$$

and

$$U^-_{2z}(0) = U^-_{2z}(-b) + \int_{-b}^0 U^-_{2zz}(z)dz,$$

we have (recall that we assumed $U^+_1(x, y, 1) = 0$)

$$-h_1(U^-_0 - \Psi) - h_0(C^-_0 + A^+_0\frac{1}{2} + B^+_0) + D\kappa^+\int_0^1 (sU^+_1(z) - LU^+_1(z))dz = 0 \tag{49}$$

and

$$
\begin{aligned}
&- h_1(U_0^- - \Psi) - h_0(C_0^- + \frac{A_0^+}{2} + B_0^+)\\
(50) \qquad &= h^- U_1^-(-b) + D\kappa^- \int_{-b}^{0} (\frac{s}{\beta} U_1^-(z) - LU_1^-(z))dz.
\end{aligned}
$$

A substitution gives

$$
\begin{aligned}
&h^- U_1^-(-b) + D\kappa^- \int_{-b}^{0} (\frac{s}{\beta} U_1^-(z) - LU_1^-(z))dz\\
(51) \qquad &+ D\kappa^+ \int_0^1 (sU_1^+(z) - LU_1^+(z))dz = 0.
\end{aligned}
$$

We plug in (51) the expressions (derived in previous section)

$$
(52) \qquad
\begin{aligned}
U_1^+(x,y,z) &= A_0^+(x,y)\frac{z^2-1}{2} + B_0^+(x,y)(z-1),\\
U_1^-(x,y,z) &= A_0^-(x,y)\frac{z^2}{2} + B_0^-(x,y)z + C_0^-(x,y),
\end{aligned}
$$

where $A_0^\pm$ and $B_0^\pm$ are known . We get the following equation in C_0^-:

$$
(53) \qquad (\frac{h^-}{D\kappa^-} + \frac{bs}{\beta})C_0^- - bLC_0^- = F_1
$$

where

$$
(54) \qquad F_1(x,y) = \frac{\kappa^+}{\kappa^-}N_3 + N_2 + \frac{h^-}{D\kappa^-}N_1
$$

where

$$
\begin{aligned}
N_1 &= -A_0^+(x,y)\frac{b^2}{2} + B_0^+(x,y)b,\\
N_2 &= -\frac{s}{\beta}(\frac{-b^3}{6}A_0^-(x,y) - \frac{b^2}{2}B_0^-(x,y)) + (\frac{b^3}{6}LA_0^-(x,y) - \frac{b^2}{2}LB_0^-(x,y)),\\
N_3 &= +\frac{s}{3}A_0^+ + \frac{s}{2}B_0^+ - \frac{LA_0^+}{3} - \frac{LB_0^+}{2}
\end{aligned}
$$

with Neumann boundary conditions

$$
(55) \qquad C_{0x}^-(0,y) = C_{0x}^-(1,y) = C_{0y}^-(x,0) = C_{0y}^-(x,1) = 0.
$$

Finally, we get

$$
(56) \qquad h_1 = \frac{-h_0(C_0^- + \frac{A_0^+}{2} + B_0^+) - D\kappa^+ s(\frac{A_0^+}{3} + \frac{B_0^+}{2}) - D\kappa^+(\frac{LA_0^+}{3} + \frac{LB_0^+}{2})}{U_0^- - \Psi}
$$

and, consequently, we have the first order thin plate approximation of h

$$
(57) \qquad h(x,y) \approx h_0(x,y) + \gamma^2 h_1(x,y).
$$

7. Simulations and inversion of real data

Figure 2 shows the geometry of the two-dimensional model used for testing the TPA solution (57), i.e. equations (33) and (56).

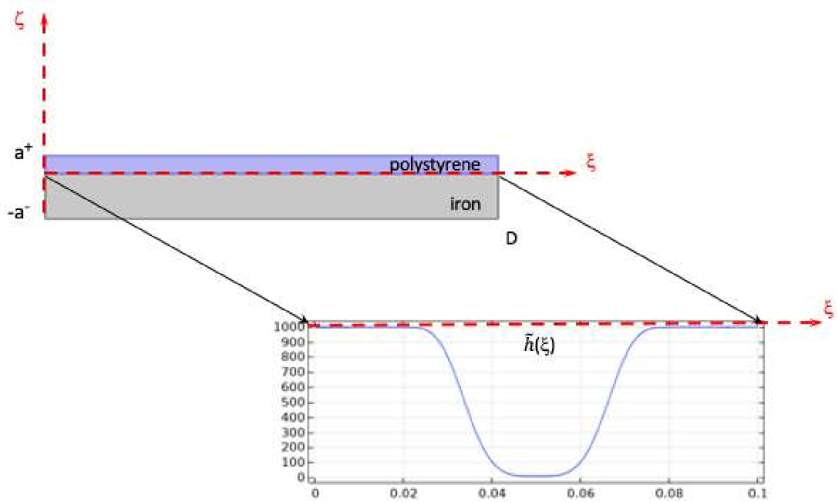

Figure 2: 2D model

A slab of (non-expanded) polystyrene, having thermal conductivity $\tilde{\kappa}^+ = 0.12\ \mathrm{Wm^{-1}K^{-1}}$, density $\rho^+ = 1050\ \mathrm{kg\ m^3}$ and specific heat $c^+ = 1100\ \mathrm{J\ kg^{-1}K^{-1}}$, is superimposed to an iron slab ($\tilde{\kappa}^- = 80\ \mathrm{Wm^{-1}K^{-1}}$, $\rho^- = 7800\ \mathrm{kg\ m^3}$, $c^- = 500\ \mathrm{J\ kg^{-1}K^{-1}}$). The imperfect contact in the central region is simulated by a thermal resistance among the slabs, i.e. by an heat exchange coefficient $h(x)$ whose value is high ($1000\ \mathrm{W\ m^2\ K^{-1}}$) where the resistance is negligible and low ($10\ \mathrm{W\ m^2\ K^{-1}}$) where the contact is bad. This kind of representation has been demonstrated to approximate reasonably well, for example, a detachment creating an air gap between the two slabs. The assumed shape of $\tilde{h}(\xi)$ is:

$$\tilde{h}(\xi) = H_a - H_b e^{-\theta(\xi - D/2)^4}$$

with $H_a = 1000$, $H_b = 990$, $\theta = 10^7$, this last to obtain an extension of the detachment region of the order of 2 cm.

The 2D model is used to simulate the production of "experimental" data on the upper surface $\zeta = a^+$, when the bottom surface $\zeta = -a^-$ is uniformly heated by a constant flux $\tilde{\phi}$. On that surface is also $h^- = 0$, because $\tilde{\phi}$ is the net flux across it. The direct problem is solved by the finite element method (FEM).

7.1 Reconstruction procedure

After the transformation into dimensionless variables, we fix a real value of the frequency parameter s and compute the Laplace transform of the data. The choice of s does not appear to be critical at all. In fact, a reasonable approach is to choose s high enough to make the product $\psi(x,t)\exp(-st)$ close to zero for any value of the coordinate x, but not too high to lose the information in the data. If, in the transformed t variable, s is such that the exponential becomes, say, 5×10^{-6} for $t = t_{\max}$, this means that: $s = \frac{6\log 10 - \log 5}{t_{\max}}$ or, in terms of the actual time τ, $s = \frac{D^2(6\log 10 - \log 5)}{a^+ \tau_{\max}}$. If $\tau_{\max} = 300$ s, as in the simulation, $s \approx 4000$. With such an s value, for instance, the product $\Psi(x,t)\exp(-st)$ far from the damage (i.e. in values of x corresponding to higher temperatures) is like in Figure 3.

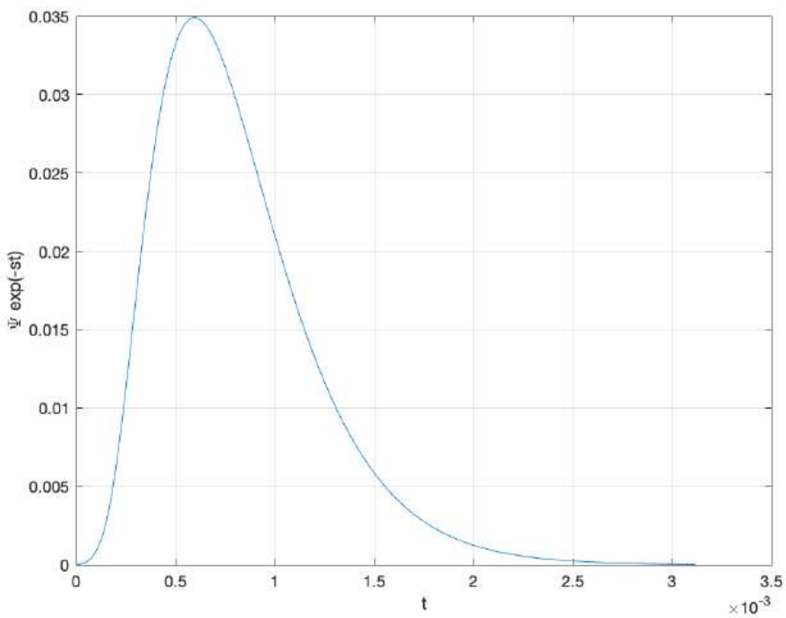

Figure 3: Test of Laplace transform

The numerical procedure involves the following steps.

1. Laplace transformation of the time-dependent data. $\tilde{\psi}(\xi,\tau)$, defined on the line $\xi \in [0,D]$ for $\tau > 0$ is transformed into $\psi(x,t)$ in the dimensionless variables introduced in section 3. Furthermore, $\Psi(x)$ is the Laplace transformation of ψ at the chosen s value. In this phase, the derivative Ψ_{xx} is also computed, by performing a smoothing on the first-order derivative Ψ_x.

2. Ψ and Ψ_{xx} are used to compute the function $F(x)$, allowing the computational solution of the differential equation for U_0^-: $\frac{bs}{\beta}U_0^- - bU_{0xx}^- = F(x)$, where h^- has been taken 0 as in the direct problem. The result, U_0^-, is stored and used to compute the zero-order heat exchange by (33).

3. The third steps computes the coefficients $A_0^{\pm}$, $B_0^{\pm}$, C_0^{+}, $A_{0xx}^{\pm}$ and $B_{0xx}^{\pm}$, necessary to obtain the function $F_1(x)$ needed to solve the unidimensional equation in C_0^-.

4. The last step numerically solves $\frac{bs}{\beta}C_0^- - bC_{0xx}^- = F_1(x)$ and allows to compute the first-order heat exchange coefficient.

At the end of the procedure, we are able to compute $h(x)$ by means of (57) and, eventually, $\tilde{h}(x) = \gamma h(x)$. The result is shown in Figure 4.

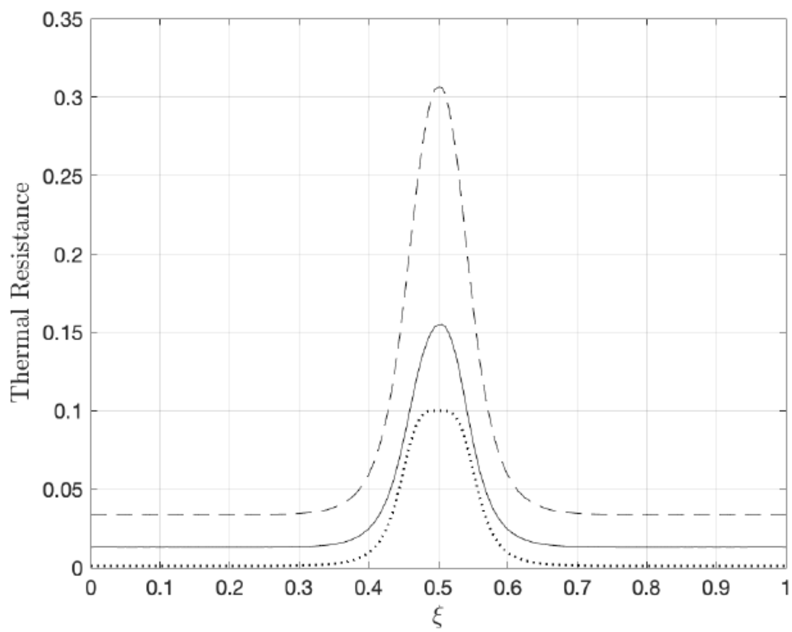

Figure 4: Reconstructed thermal resistance: zero-order $\tilde{h}_0^{-1}$ (dashed line) and first-order $(\tilde{h}_0 + \gamma^2\tilde{h}_1)^{-1}$ (solid line)

Figure 4 superimposes the true, unknown thermal resistance at the interface $\zeta = 0$ (dotted line) with those computed at zero and first order (dashed and solid line, respectively).

7.2 Experimental data

The inversion procedure outlined in the previous sections has been applied to real experimental data taken from [24] where zero order TPA had been computed starting from a rough heuristic evaluation of the temperature of the lower face of the interface. A composite solid consisting in the superposition of two parallelepipeds of square section are heated from below while a thermographic camera acquires thermal shots from above, at a rate of two photograms per second. The lower plate is made of iron ($\kappa_- = 80 \frac{W}{m\ {}^\circ K}$, $c_- = 500 \frac{J}{Kg\ {}^\circ K}$, $\rho_- = 7800 \frac{Kg}{m^3}$) while the upper one is realized in non expanded polystyrene ($\kappa_+ = 0.12 \frac{W}{m\ {}^\circ K}$, $c_+ = 1100 \frac{J}{Kg\ {}^\circ K}$, $\rho_+ = 1050 \frac{Kg}{m^3}$). The side of the squares is $D = 10\ cm$. The thicknesses of the lower and upper plates are $a_+ = 0.4\ cm$ and $a_- = 1.0\ cm$, respectively.

A square dig of side 2.0 cm and thickness $\delta = 0.2\ cm$ was made in the center of the iron plate to simulate the imperfect contact at the metal/plastic interface. Heating was provided by fixing a thermal wire electrical resistance on the iron bottom surface by means of aluminium tape. The resistance is connected to a DC power supply that provided 18 W (3.6 V $\times$ 5.0 A). Such a condition can be simply simulated by a constant flux $\tilde{\phi}$ at $z = -a^-$, i.e. mathematically by a Neumann condition there. Such a simplification does not affect the temperature behavior with the exception of very early times.

The procedure for computing $\tilde{\phi}$ is the following.

1. Record the temperature values versus time on a number of positions "far" from the damaged region (easily visible, although qualitatively, from thermal images at a suitable time). Those temperatures should, in principle, be very close to one another. Compute the average value as a function of time.

2. Obtain $\tilde{\phi}$ in the unidimensional problem to obtain a good agreement among $\tilde{u}(a^+, t)$ and the experimental values of the previous item.

The other parameters involved in the TPA formula are known, or readily available, being the measured temperature, the thermo-physical characteristics of the materials involved and the geometric quantities. The heat exchange h^+ at the surface $z = a^+$ can be guessed or obtained experimentally [20] but, anyway, it is not a critical one.

Figure 5 compares, on a section $\eta = 0$, the actual thickness of the rectangular defect (dotted line) with those obtained by the TPA procedure at order zero (dashed line) and one (solid line). The curves actually represent a smooth fitting of the quantity obtained by the inversion. It clearly appears that two terms of the expansion are sufficient to have a good quantitative estimate of the damage depth. The width of the square dig is also reasonably obtained, with soft sides (instead of sharp ones) as commonly happens in problems involving heat diffusion.

Figure 6 shows a 3D reconstruction of the defect.

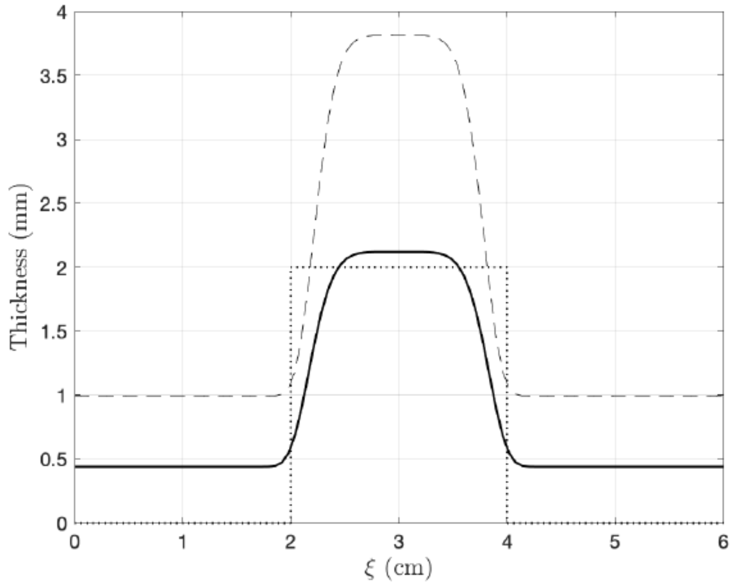

Figure 5: Reconstructed thickness: actual value (dotted line), zero-order (dashed line) and first-order (solid line)

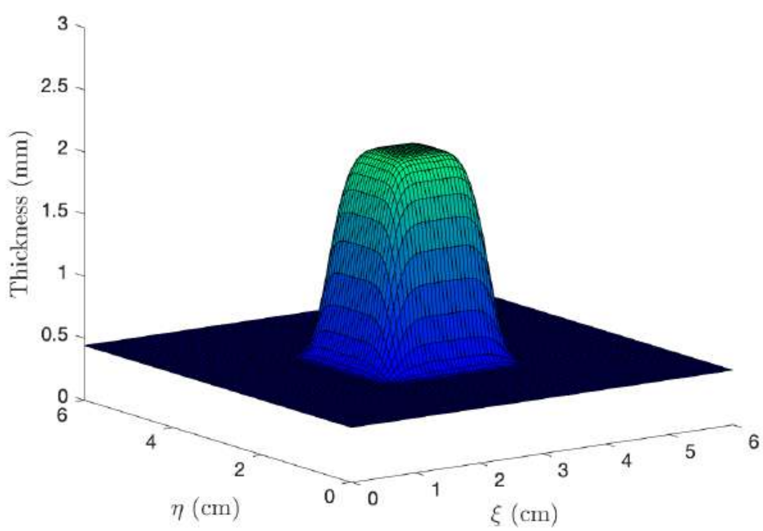

Figure 6: 3D reconstruction of the defect thickness

Remark. The dimensionless parameter $\gamma = \frac{a^+}{D}$ gives a measure of how much $\tilde{\Omega}^+$ is "geometrically thin". The slabs $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$ can be considered "thermally thin" when their Biot numbers $Bi^{\pm} = \frac{a^{\pm}(\tilde{h}^{\pm})}{\tilde{\kappa}^{\pm}}$ are much smaller than one. A value $Bi \approx 0.1$ is often assumed as a limit value for thermal thinness [12], in the context of the applicability of the well known lumped capacitance method used to approximated the temperature behavior in a solid where the spatial uniformity of the temperature is not violated at any time instant. Here, the meaning of the Biot number is somewhat different: it can be easily demonstrated by a Taylor expansion of the temperature on the domain $\tilde{\Omega}^+$ that if $Bi << 1$ the zero-order term of the TPA is sufficient to obtain an approximation for the exchange coefficient h at the interface. In the case at hand is $Bi^+ \approx 0.17$, so at least one more term of the TPA is needed.

7.3 A remark on Laplace transformation

The recourse to Laplace transformation of the data and, consequently, of the equations involved in the inversion has several practical advantages. As the investigated quantity (damage thickness, thermal resistance, or equivalent heat exchange at the interface) is inherently not time dependent, a time-domain approach should require to identify a characteristic time, or a time range, where such quantity appears to be nearly time-independent. This means that we are forced to solve the problem for all experimental times with a view to discard the most of them. In other words, we do not exploit all the information available in the data.

Laplace transformation, on the other hand, uses the whole available data, by performing a sort of weighted average with exponential weights. Indeed, once the Laplace parameter s has been chosen as discussed in section 7.1, data can be truncated at a previous time, say 200 seconds instead of 300, with a negligible effect on the final result. That is perfectly consistent with the weighted-average interpretation just introduced.

8. Conclusions

This paper deals with nondestructive evaluation of detachment-like defects affecting highly conductive inaccessible interface in the layered specimen Ω. Such defects give rise locally to a thermal resistance r whose imaging gives the required evaluation of the flaw.

The method proposed is based on the expansion

$$h = \frac{1}{r} = h_0 + \gamma^2 h_1 + O(\gamma^4),$$

where γ is the normalized thickness of the upper layer of the specimen Ω. The coefficients h_0 and h_1 are explicitly calculated by means of a perturbative method extending to layered objects a technique, known as Thin Plate Approximation, widely used to solve inverse problems in slabs.

The mathematical novelty consists in the setting of the elliptic BVPs (30) and (53) on the interface whose solutions are the coefficients of the expansion

$$U^-(x,y,0) = U_0^-(x,y) + \gamma^2 U_1^-(x,y) + O(\gamma^4). \tag{58}$$

Once we know $U^-(x,y,0)$, the problem should be reduced to the determination of the Robin coefficient in the inaccessible side of a slab. As for computation, a quite challenging step is the numerical evaluation of the derivatives of data function Ψ required to get h_0 and h_1.

Future work is mainly in the following two directions:

(i) to obtain a better theoretical foundation of the inverse problem (stability estimates, existence and uniqueness);

(ii) to fit the method to different real objects (curved geometries, failures of insulating interfaces)

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Trend modeling and multi-step taxi demand prediction

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Abstract. At present, there is a serious mismatch between the supply and demand of taxis, and reasonable demand forecasting can effectively reduce the supply-demand gap, which is an important foundation for taxi scheduling. This article proposes three modeling methods for taxi demand cycle trends, namely the Fourier series based method, the principal component analysis trend based method, and the average trend based method. Finally, based on a weighted combination of three periodic features, a multi-step prediction model for taxi demand was established. On actual data, the method proposed in this paper achieved an MAE error of 1.91, indicating that it can effectively predict taxi multi-step demand. Furthermore, after comparison, the method proposed in this paper outperforms other comparative methods in predicting taxi demand.

Keywords: trend modeling, fourier series, principal component analysis, average trend method, multi-step prediction.
MSC 2020: 68T09

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Introduction

The taxi industry provides urban transportation services. But, due to the information lag, there is an imbalance in the supply and demand of taxis in some areas. For example, there are many taxi queues waiting for passengers in areas such as airports, train station, etc., while in some places there are few taxis waiting for service. Therefore, analyze taxi passenger data is of great importance.

Conducting real-time and detailed statistical analysis of the taxi carrying data by the use of big data, cloud computing and artificial intelligence among others, to predict taxi demand will greatly help improve the efficiency of taxi operation [1, 2], which is of great significance for alleviating urban traffic pressure.

Traffic data is essentially a type of time series data, and traffic prediction problems actually belong to a type of time series prediction problem. Therefore, some researchers use basic time series prediction methods, such as exponential smoothing [3], Kalman filter algorithm [4], spectral analysis [5], differential integrated moving average auto regressive model [6], as the basis for traffic prediction.

With the continuous deepening of research on traffic prediction problems, researchers have begun the use of machine learning methods for traffic prediction. The main methods include: *support vector regression* (SVR) [7], *genetic algorithm support vector machine model* (GASVM) [8], *k-nearest neighbor* (KNN) [9]. With the continuous breakthroughs in deep learning technology in tasks such as speech recognition and image recognition, researchers have begun to attempt to use deep learning methods [10, 11, 12] to solve traffic prediction problems. The main methods include: *multi-layer perceptron* (MLP) [12], *deep belief network* (DBN) [13], *convolution neural network* (CNN) [14], *recurrent neural network* (RNN) [15], *long short term memory network* (LSTM) [16], etc.

The popular demand forecast these days for taxis is mainly the short-term forecasting, which predicts the demand for taxis at such a time as 5 minutes, 10 minutes, or 30 minutes. While this article will continue to study the short-term forecasting, to predict the demand for taxis for the whole next day in the sequence of 1 hour, 2 hours, 3 hours, $\cdots$, 24 hours, due to the strong daily periodicity of taxi demand, we will first extract the cycle characteristics of taxi demand based on Fourier series [17], *principal component analysis* (PCA) trend [18] and average trend methods [19]. Finally, based on weighted combinations, a prediction model is established. Based on the cycle characteristics of the past 5 days as input, and the data of the last day as model output, the weighted parameters of the model can be calculated. Finally, this model can be used to predict the demand data for taxis for the next day.

1. Method

1.1 General idea

In the transportation industry, through the observation of the taxi demand curve, we can find that the taxi demand curve of different working days has certain similarity and periodic regularity. This paper will propose three different taxi demand cycle trend calculation methods to predict taxi demand for the next day. These three trends are based on Fourier series, principal component analysis and average trend method. This article will propose a multi-step taxi demand prediction model based on periodic trend weighted combination.

The specific definition of the taxi demand prediction model is as follows

$$d(t) = \theta_1 \times fourier(t\%T) + \theta_2 \times pca(t\%T) + \theta_3 \times at(t\%T) + e(t), \tag{1}$$

where T is the interval numbers of one day, $fourier(t\%T)$ is periodic term of Fourier series, $pca(t\%T)$ is periodic term based on PCA trend, $at(t\%T)$ is average trend period term, $\theta_1, \theta_2, \theta_3$ are weight coefficients of three periodic terms, $d(t)$ is taxi demand value at time t, $e(t)$ is error term. The meaning of this model is to first extract daily cycle trends based on three methods, and then find the weighted sum of the demand values at the same time point of the three daily trends to predict the demand value at a certain time in the future day.

In time series prediction problems, Fourier series method, average trend method, and principal component analysis method have achieved good results in periodic trend modeling. Therefore, this article combines these three methods and integrates them based on ridge regression to predict taxi demand.

1.2 Periodic trend based on Fourier series

In this paper, we will use Fourier transform, a mathematical tool widely used in the field of signal processing, to build a periodic term with time t as the variable, which is used to describe the periodic law of the taxi demand curve [20]. The construction of this cycle item can help us describe the changes in the daily taxi demand curve more accurately and predict the number of taxis that citizens may need in a specific working day. Its specific definition is as follows

$$fourier(t) = a_0 + \sum_{m=1}^{M} a_m \cos(m \times \omega \times t) + b_m \sin(m \times \omega \times t), \tag{2}$$

where a_m and b_m are parameters to be solved. T is the number of time intervals included in a day, or the number of time intervals included in a cycle, M is the order term of the Fourier series, and we usually take this value as 10. $\omega = 2\pi/T$ is the fundamental frequency component of the Fourier series.

The a_m and b_m can be found as follows

$$a_0 = \frac{1}{T} \sum_{t=1}^{T} f(t), \tag{3}$$

$$a_m = \frac{1}{T}\sum_{t=1}^{T} f(t)\cos(m \times \omega \times t), \tag{4}$$

$$b_m = \frac{1}{T}\sum_{t=1}^{T} f(t)\sin(m \times \omega \times t). \tag{5}$$

1.3 Periodic trends based on principal component analysis

In this article, we will explore how to use principal component analysis [21] to extract periodic patterns in demand time series, with the aim of discovering patterns from data and transforming them into actionable information. In this article, we will focus on the application of principal component analysis methods and how to select appropriate periodic parameters to analyze time series data. Firstly, construct a data matrix D so that $d_{i,j}$ represents the taxi demand for the i-th time period on the j-th day. So,

$$D = \begin{bmatrix} d_{1,1} & \cdots & d_{1,N} \\ \cdots & \cdots & \cdots \\ d_{T,1} & \cdots & d_{T,N} \end{bmatrix}. \tag{6}$$

Due to the wide use of PCA algorithm, we can easily find relevant information in the literature. Here, we will not elaborate on the principles and steps of the PCA algorithm. Based on the data matrix D obtained, we can call the PCA algorithm module to reduce its dimensionality. Firstly, normalize the data

$$d_{i,j} = \frac{d_{i,j} - \mu_j}{\sigma_j}, \tag{7}$$

where μ_j is the average value of the data on day j-th, and σ_j is the standard deviation of the data on day j-th. Then calculate the covariance matrix of the data matrix as follows

$$Cov = \frac{1}{N} D^T D. \tag{8}$$

Find all the eigenvalues of the covariance matrix Cov and arrange them from maximum to minimum. Select the eigenvectors corresponding to the first K features and arrange them in rows to form a transformation matrix W. Use the transformation matrix W to reduce the dimensionality of the data. In the process of using PCA algorithm, we can obtain the fractional matrix $S \in R^{T\times N}$ and coefficient matrix $C \in R^{N\times N}$. To obtain the final trend, we can take the first K columns of the matrix S and then the first K rows of the coefficient matrix C. Then, we perform matrix multiplication on these two matrices to obtain the results of the PCA periodic law. The calculation formula is as follows

$$pca = mean(S^K \times C^K). \tag{9}$$

1.4 Periodic trend based on average method

We will propose a modeling method for taxi demand cycle patterns based on the average trend method [22]. The core idea of the average trend method is to calculate the average of demand time series from different days, and ultimately obtain the periodic trend of the time series. Firstly, we construct a data matrix so that $d_{i,j}$ represents the taxi demand for the i-th period on the j-th day. So there is a data matrix

$$D = \begin{bmatrix} d_{1,1} & \cdots & d_{1,N} \\ \cdots & \cdots & \cdots \\ d_{T,1} & \cdots & d_{T,N} \end{bmatrix}. \tag{10}$$

The formula for calculating the periodic pattern using the average trend method is as follows

$$at = [\frac{1}{N}\sum_{n=1}^{N} d_{1,n}, \frac{1}{N}\sum_{n=1}^{N} d_{2,n}, \cdots, \frac{1}{N}\sum_{n=1}^{N} d_{T,n}]. \tag{11}$$

1.5 Working days and holidays

Through long-term data analysis and statistics of the transportation industry, we find that there are obvious differences between the taxi demand curve on weekdays and weekends, which is due to the different travel needs of people on weekdays and weekends. On the other hand, we also find that there are similarities between taxi demand curve in different working days or weekends, because people travel regularly in different working days or weekends.

Therefore, in the calculation process, we divide the data into two parts: one is for weekdays, and the other is for holidays. When calculating the cycle pattern of working days, we only consider the data of working days, and when calculating the cycle pattern of weekends and holidays, we only consider the data of holidays.

1.6 Periodic model integration

We propose a periodic model integration method based on ridge regression [23]. The method of integrating three periodic terms is usually through weighted summation. Therefore, linear regression is used as the ensemble model. Due to the problem of singular matrix values, this paper adopts ridge regression to solve this problem. Therefore, the final integrated model is ridge regression. After calculating the three periodic trends, we need to determine the weights of the three periodic trends. The following data matrix is defined, and the matrix A of the three periodic trend calculation results is defined as follows

$$A = \begin{bmatrix} fourier(1) & pca(1) & at(1) & 1 \\ fourier(2) & pca(2) & at(2) & 1 \\ \cdots & \cdots & \cdots & \cdots \\ fourier(T) & pca(T) & at(T) & 1 \end{bmatrix}. \tag{12}$$

Define the vector d containing demand values as follows

$$d = [d(\tau+1), d(\tau+2), \cdots, d(\tau+T)]^T, \tag{13}$$

where τ is the starting date of the training set to predict the demand. The goal is to minimize the error between the weighted curve of three trends and the actual curve as much as possible [23]. The minimization problem is as follows

$$\min_{\theta}(||d - A\theta||^2 + \lambda||\theta||^2). \tag{14}$$

This paper employed the least squares method, obtain the coefficients and optimize the above objective function. The analytical solution for calculating the coefficients is as follows

$$\hat{\theta} = (A^T A + \lambda I)^{-1} A^T d. \tag{15}$$

1.7 Evaluation index

The evaluation index used in this paper is the *mean absolute error* (MAE). The MAE value of area is calculated as follows

$$MAE = \frac{1}{T}\sum_{t=1}^{T} |d_{i,t} - dr_{i,t}|, \tag{16}$$

where $dr_{i,t}$ is the observed demand for taxis in the i-th area on day t, and $d_{i,t}$ is the predicted demand for taxis in the i-th area on day t. In order to measure the relative value of error relative to the time series, we also defined the following indicators to measure the size of demand

$$MAX_i = max\{dr_{i,t} | t = 1, 2, \cdots, T\}, \tag{17}$$

where $dr_{i,t}$ is the observed demand for taxis in the i-th area on time t, and MAX_i is the maximum value of the demand time series.

2. Numerical experiment

2.1 Dataset

The dataset for this article is New York green taxi travel data [24], which covers the period from June 1st, 2017 to June 30th, 2017. There are 265 areas in the data. Table 1 shows the main fields of this data.

Group the above table according to the departure time (time interval of 1 hour) and boarding location ID, and aggregate the number of passengers to obtain our final demand matrix. The Figure 1 shows the transformation rule of the demand curve of 166th area from June 5th, 2017 to June 9th, 2017. It can be seen from the figure that this demand curve has a very strong daily cycle law. There is strong similarity in time series of different days.

Table 1: Field description of the New York green taxi dataset

Field name	Field Meaning
VendorID	A code indicating the LPEP provider that provided the record.
lpep_pickup_datetime	The date and time when the meter was engaged
lpep_dropoff_datetime	The date and time when the meter was disengaged.
Passenger_count	The number of passengers in the vehicle. This is a driver-entered value
Trip_distance	The elapsed trip distance in miles reported by the taximeter
PULocationID	TLC Taxi Zone in which the taximeter was engaged
DOLocationID	TLC Taxi Zone in which the taximeter was disengaged

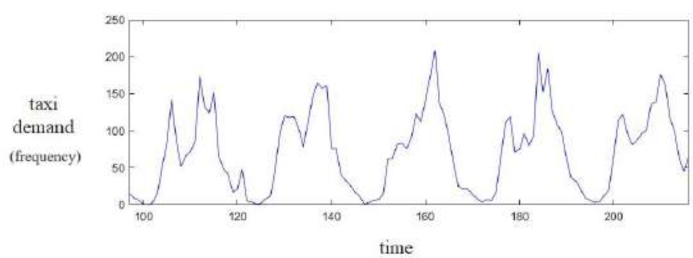

Figure 1: Taxi demand for 166th area from June 5th, 2017 to June 9th, 2017

2.2 Fourier period term

The nonlinear least square method is used to solve the parameters of Fourier series. The input variable is time, and the output variable is demand value. Taking the 166th area as an example, the fitting effect is shown in the figure 2, where the red day curve is the result of Fourier period calculation, and the blue data points are the actual value of taxi demand.

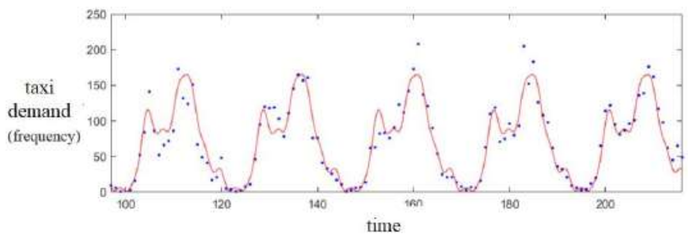

Figure 2: Fitting diagram of Fourier series period component. Taxi demand data is from 166th area, from June 5th, 2017 to June 9th, 2017. The red curve is the fitting result of Fourier series, and the blue data point is the actual taxi demand value

The residual plot of the original time series after removing the fourier period trend is shown as Figure 3, from which it can be seen that the Fourier series method can effectively extract the periodic trend of taxi demand.

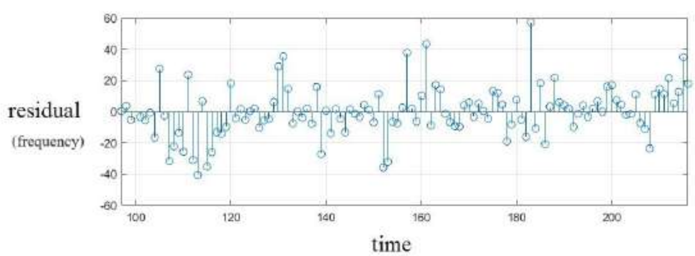

Figure 3: Residual curve after removing the Fourier period term. The taxi demand data comes from the 166th area, from June 5th, 2017 to June 9th, 2017.

2.3 PCA cycle term

The PCA method is used to extract the cycle trend of taxi demand on workdays, and the calculation results are shown in Figure 4. The green curve is the calculation results of cycle items, and the blue data points are the actual data of taxi demand:

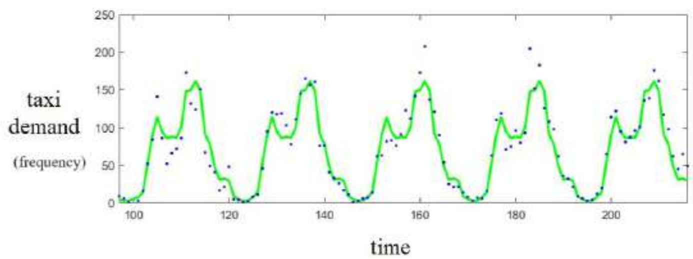

Figure 4: PCA cycle term of taxi demand curve. Taxi demand data comes from 166th area, from June 5th, 2017 to June 9th, 2017. The green curve is the fitting result of Fourier series, and the blue data points are the actual taxi demand values.

The residual plot of the original time series after excluding the PCA cycle trend is in Figure 5. From the graph, it can be seen that the principal component analysis method can effectively extract the periodic trend of taxi demand.

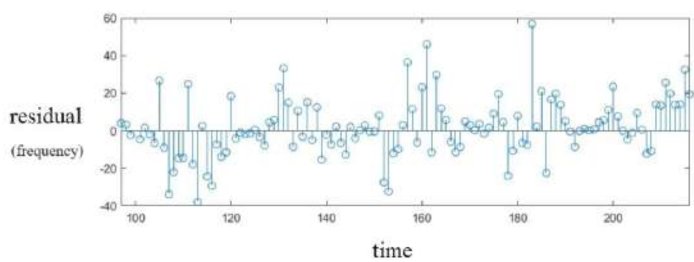

Figure 5: Taxi demand curve residual items after removing PCA cycle items. Taxi demand data comes from 166th area, from June 5th, 2017 to June 9th, 2017

2.4 Average trend method

The average trend method is used to extract the cycle trend of taxi demand on workdays. The calculation results are shown in the Figure 6. The green curve is the calculation results of cycle items, and the blue data points are the actual data of taxi demand.

The residual plot of the original time series after removing the average trend period term is as in Figure 7. From the graph, it can be seen that the average trend method can effectively extract the periodic trend of taxi demand.

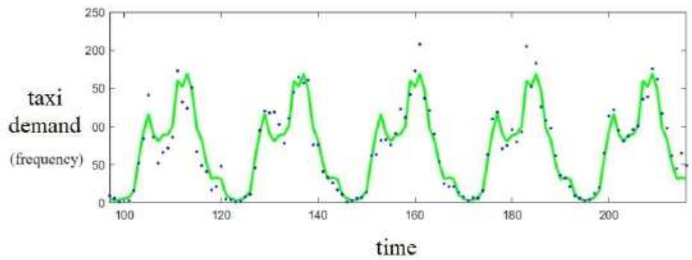

Figure 6: Simple average cycle term of the taxi demand curve. Taxi demand data comes from 166th area, from June 5th, 2017 to June 9th, 2017. The green curve is the fitting result of Fourier series, and the blue data points are the actual taxi demand values.

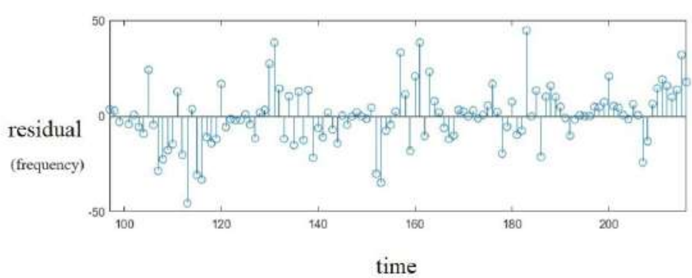

Figure 7: Residual sequence after removing the simple average period component. The taxi demand data comes from 166th area, from June 5th, 2017 to June 9th, 2017.

2.5 Prediction accuracy

We used three methods to calculate the periodic trend term, and then used the least squares method to obtain the weights of each periodic term. Finally, after calculation, we obtained the predicted and observed values on the test set, as shown in Figure 8.

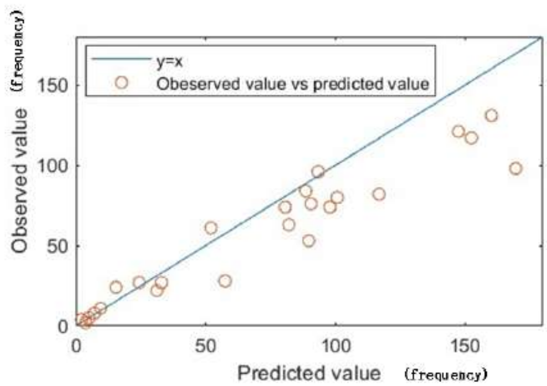

Figure 8: Predicted and true values on the test set. Taxi demand data comes from 166th area.

This paper uses data from June 5th, 2017 to June 9th, 2017 as the training set for trend extraction, uses data from June 12th, 2017 as the weight solution dataset, and uses data from June 13th, 2017 as the test set, periodic trend extraction is performed, and the weight obtained by the least squares method is used to calculate the predicted value of taxi demand. Figure 9 shows the prediction results of our method on the test set. The red curve represents the predicted results, and the blue dots represent the observed values.

In order to provide a more detailed analysis of the accuracy of the prediction algorithm in this article, we conducted experiments in most areas. This paper uses data from June 5th, 2017 to June 9th, 2017 as the trend extraction dataset, and uses data from June 12th, 2017 as the weight solution dataset. This paper uses data from June 13th, 2017 as the test set. Table 2 shows the prediction accuracy of some areas. It can be seen that the method proposed in this article can predict the time series of taxi demand for the next day. It can also be seen that the method proposed in this article achieved an average MAE prediction accuracy of 1.91, with an average of 14.93 for the maximum observed value.

To verify the predictive effect of our method on non-working days, this paper uses data from June 3th, 2017, and June 4th, 2017 as trend extraction datasets, and uses data from June 10th, 2017 as weight solving datasets. This paper uses data from June 11th, 2017 (non-working days) as the test set. Table 3 shows the prediction accuracy of some areas. It can be seen that the method proposed in this article can predict the time series of weekend taxi demand. It can also

Table 2: Prediction accuracy of the first 80 areas for the next day. The data from June 5th, 2017 to June 9th, 2017 was used as the trend extraction dataset, the data from June 12th, 2017 was used as the weight solution dataset, and the data from June 13th, 2017 (working days) was used as the test dataset.

Area id	MAE (frequency)	Max value (frequency)	Area id	MAE (frequency)	Max value (frequency)
1	0.03	0	41	11.52	169
2	0.00	0	42	9.24	101
3	0.67	5	43	5.88	65
4	0.00	0	44	0.00	0
5	0.00	0	45	0.00	0
6	0.06	0	46	0.07	0
7	17.06	154	47	1.39	7
8	0.29	2	48	0.00	0
9	0.12	1	49	5.43	39
10	0.48	2	50	0.00	0
11	0.10	1	51	0.73	4
12	0.00	0	52	5.21	33
13	0.00	0	53	0.62	3
14	1.13	6	54	0.94	7
15	0.08	0	55	3.37	8
16	0.48	2	56	1.80	8
17	5.02	24	57	0.57	3
18	1.56	6	58	0.08	0
19	0.16	1	59	0.02	0
20	1.41	4	60	0.47	2
21	0.25	1	61	5.40	32
22	0.41	5	62	1.47	11
23	0.13	0	63	0.30	2
24	2.63	23	64	0.06	0
25	10.97	82	65	8.36	74
26	0.95	6	66	9.57	93
27	0.00	0	67	0.57	5
28	0.88	4	68	0.00	0
29	0.60	4	69	3.02	13
30	0.00	0	70	1.74	7
31	0.80	4	71	0.88	5
32	0.81	2	72	0.74	5
33	11.11	97	73	0.60	6
34	1.14	5	74	10.84	206
35	0.94	4	75	16.70	213
36	1.73	17	76	1.24	7
37	2.15	18	77	1.01	8
38	0.00	0	78	1.61	5
39	0.75	2	79	0.00	0
40	5.64	25	80	5.60	33

Table 3: Prediction accuracy of non-working days in the first 80 areas. The data from June 3rd, 2017 to June 4th, 2017 was used as the trend extraction dataset, the data from June 10th, 2017 was used as the weight solution dataset, and the data from June 11th, 2017 (non-working days) was used as the test dataset.

Area id	MAE (frequency)	Max value (frequency)	Area id	MAE (frequency)	Max value (frequency)
1	0.09	1	41	30.50	176
2	0.00	0	42	12.10	118
3	0.67	5	43	3.71	30
4	0.00	0	44	0.00	0
5	0.00	0	45	0.07	1
6	0.07	0	46	0.09	0
7	25.70	249	47	1.54	7
8	0.58	7	48	0.00	0
9	0.29	1	49	8.28	63
10	0.13	1	50	0.00	0
11	0.15	1	51	0.84	5
12	0.00	0	52	6.15	45
13	0.00	0	53	0.43	2
14	1.37	7	54	0.79	6
15	0.17	1	55	2.17	12
16	0.42	1	56	1.53	12
17	6.62	57	57	0.70	4
18	2.11	8	58	0.00	0
19	0.19	1	59	0.09	0
20	1.30	7	60	0.58	3
21	0.85	8	61	4.79	50
22	0.51	3	62	2.53	19
23	0.46	6	63	0.42	1
24	3.39	19	64	0.08	1
25	15.57	98	65	8.41	64
26	1.33	6	66	11.17	83
27	0.00	0	67	0.14	1
28	1.54	13	68	0.00	0
29	0.78	6	69	3.41	12
30	0.00	0	70	2.67	11
31	1.83	17	71	0.72	3
32	0.83	3	72	1.04	3
33	11.99	115	73	0.39	2
34	0.67	5	74	20.27	221
35	1.22	11	75	20.65	119
36	4.37	70	76	1.46	8
37	5.49	36	77	0.42	2
38	0.14	1	78	1.34	5
39	1.18	6	79	0.00	0
40	7.42	38	80	6.50	97

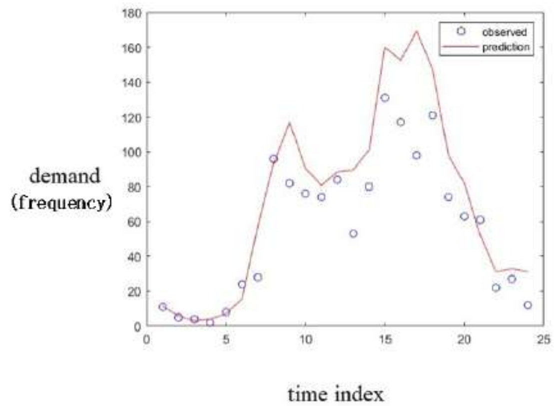

Figure 9: Predicted and actual values for 166th area on the test set as of June 13th, 2017.

be seen that the method proposed in this article achieved an average MAE prediction accuracy of 2.52, with an average of 19.53 for the maximum observed value. For the current taxi demand prediction, the MAE value is around 10% of the maximum value, which is considered a relatively good result [25, 26, 27]. Therefore, it can be seen that the method proposed in this article can effectively predict multi-step demand on both working and non-working days.

2.6 Comparison with other models

In the field of taxi demand prediction, many scholars have conducted research using CNN and RNN [25, 26, 27]. The long-term taxi demand prediction model in this article can also be predicted using CNN and RNN. Due to the differences between the issues in these literature and those in this article, there are certain differences between the comparative model and these methods used in this article. This article compares the proposed method with multi-step prediction models based on CNN [28] and RNN [29]. Figure 10 is the structural diagram of CNN and RNN models.

The CNN network consists of a convolutional layer, a fully connected layer, and a reshape layer. The RNN network consists of a recurrent layer, a fully connected layer, and a reshape layer. The input and output are shown in Figure 11. The input of the CNN network consists of the requirements of all areas in the past 72 steps (each representing one hour), and the output is the requirements of each area in the next 24 steps. The input and output of RNN also take the same form.

At the same time, we will also compare the method proposed in this article with the multi-step prediction model in Facebook's Prophet method [30, 31]. The Prophet model is a universal method for modeling periodic time series. In this article, the prophet model consists of three components. The first item is the growth trend item, the second item is the daily cycle item, and the third item is the weekly cycle item. The definition form of its model composition is as follows

$$y(t) = g(t) + day(t) + week(t) + \epsilon_t, \tag{18}$$

where $y(t)$ is the value of the time series, $g(t)$ is the trend term, $day(t)$ is the daily periodic term, $week(t)$ is the weekly periodic term, and the last term is the error term. These three items are combined through addition. These three parameters are fitted based on historical data and Bayesian methods. After obtaining the parameters, the model can make long-term predictions. Figure 12 shows the fitting results of the prophet model on 166th area. From the graph, it can be seen that the model can extract daily trends and distinguish between working and non-working days. In Figure 12, the minimum value is 60 and the maximum value is 68. That is to say, the demand for taxis remains basically unchanged within a month, and its slope is a random number close to zero. The first term of the Prophet model is a linear term, the second term is a weekly cycle term, and the third term is a daily cycle term. The first item is its mean, the third item has a negative value, and the third item reflects the fluctuation of the demand curve near the mean.

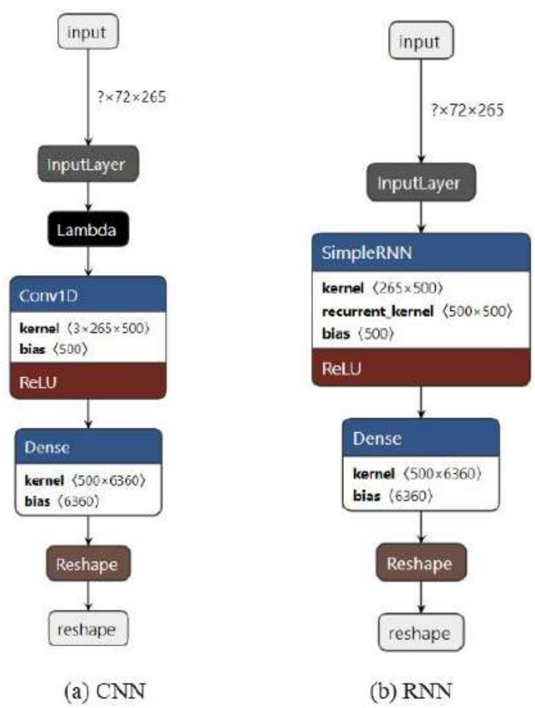

Figure 10: Structure diagram of CNN and RNN models

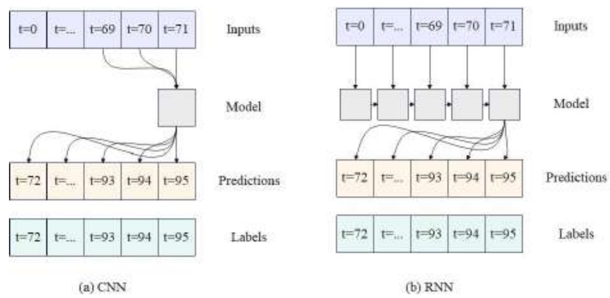

Figure 11: Input and output structure diagram of CNN and RNN models

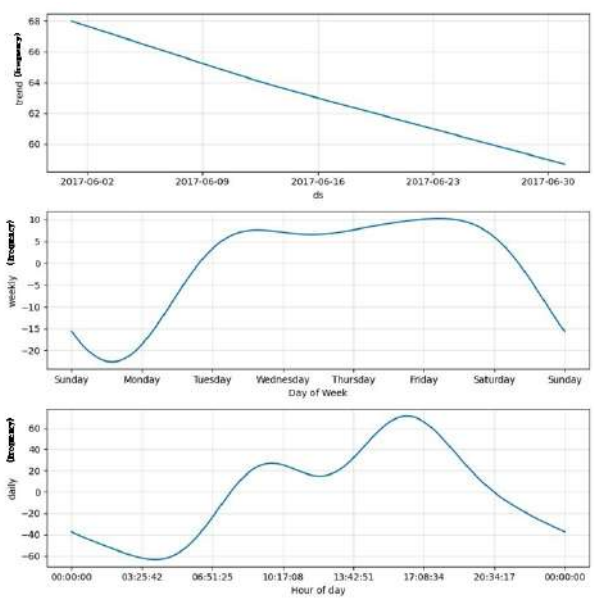

Figure 12: The fitting results of the prophet model on 166th area

This article uses data from June 1st to 29th, 2017 as the training set, and data from June 30th, 2017 as the testing set. We conducted tests on all areas and obtained the average MAE values of different methods across all areas. For convolutional neural networks, the convolutional layer adopts 1-dimensional convolution. The convolutional kernel width is set to 3, and the output size of the convolutional layer is set to 500. The activation function is set to Relu, and the output size of the fully connected layer is set to 24×265. The final output size is set to (24, 265). Set the batch size to 32 and the number of iterations to 300. For recurrent neural networks, the size of the hidden layer is set to 500, and the activation function is Relu. The output size of the fully connected layer is set to 24×265. The final output size of the model is set to (24, 265). Set the batch size to 32 and the number of iterations to 300. For the Prophet model, we set the cycle to daily and weekly, with a time interval of 1 hour, to predict data for the next 24 hours. The growth trend method adopts a linear trend, and the periodic model adopts a trigonometric function. The comparison results are shown in Table 4. The average MAE value of this method in all areas is 2.22, which is relatively smaller than other methods. It can be seen that the method proposed in this article is relatively superior in predicting taxi demand.

Table 4: Average MAE of different methods across all 265 areas.

Method	MAE
The method of this article	2.22
CNN	2.54
RNN	2.31
Prophet	2.56

3. Conclusion

The three taxi demand trend modeling methods proposed in this article can effectively extract the periodic trends of taxi demand time series. These three methods are: Fourier series based method, principal component analysis based method, and average trend based method. This article integrates three trend features for multi-step prediction of taxi demand. The average absolute error of this method reached a prediction accuracy of 1.91 on weekdays and 2.52 on weekends. The method proposed in this article can effectively predict the demand for taxis in the future. Besides, the method proposed in this article outperforms several currently popular methods in predicting taxi demand.

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Strong edge-coloring of planar graphs with girth at least seven

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Abstract. A *strong edge-coloring* of a graph G is that two edges e_1 and e_2 that are adjacent to each other or adjacent to the same edge must be colored with distinct colors. In this paper we prove that every planar graph G with girth $g \geq 7$ and maximum degree $\Delta \geq 5$ has a strong edge-coloring using at most $3\Delta - 1$ colors. In addition, we prove that every planar graph G without adjacent $7-cycles$, with girth $g \geq 7$ and the maximum degree $\Delta \geq 4$ has a strong edge-coloring using at most $3\Delta - 1$ colors.
Keywords: strong edge-coloring, planar graph, discharging method.
MSC 2020: 05C15

1. Introduction

All graphs considered in this paper are finite, loopless and undirected. Let G be a simple undirected graph. A vertex of degree k, at least k or at most k is denoted by a $k-vertex$, a $k^+-vertex$ or a $k^--vertex$ respectively. A neighbor of v of degree k, at most k or at least k is denoted by a $k-neighbor$, a $k^--neighbor$ or a $k^+-neighbor$, respectively.

A *strong edge-coloring* of a graph G is that two edges e_1 and e_2 that are adjacent to each other or adjacent to the same edge must be colored with distinct colors. The *strong chromatic index* of G is denoted by $\chi'_s(G)$, which is the minimum number of colors for a strong edge-coloring of G.

We denote the minimum and maximum degree of vertices in G by $\delta(G)$ and $\Delta(G)$ (δ and Δ for short), respectively. The degree of vertex v in G is denoted by $d_G(v)$. The *girth* of a graph G, denoted by $g(G)$ (g for short),

$*$. Corresponding author

is the length of its shortest cycle. Define that the *maximum average degree* of a graph G is the largest average degree of its subgraphs and is denoted by $mad(G) = \max_{H\subseteq G}\{\frac{2|E(H)|}{|V(H)|}\}$. The *distance* of two edges e_1 and e_2 refers to the length of the shortest path from u to v, where u is an arbitrary endvertice of e_1 and v is an arbitrary endvertice of e_2. So, equivalently, a strong edge-coloring is an assignment of colors to all edges such that every two edges with distance at most 1 receive distinct colors.

Using greedy algorithm, we may easily see that $\chi'_s(G) \le 2\Delta^2 - 2\Delta + 1$ for every graph G. In 1989, Erdős and Nešetřil [3] conjectured the following upper bounds.

Conjecture 1.1 ([2],[3])**.** For every graph G with maximum degree Δ,

$$\chi'_s(G) \le \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even,} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{if } \Delta \text{ is odd.} \end{cases}$$

These bounds would be tight, as Erdős and Nešetřil [3] gave examples of graphs that get these bounds. For the case when $\Delta = 2$, this conjecture is clearly true. For the case of $\Delta = 3$, Andersen [1] and Horák *et al.*[5] proved this conjecture to be correct, independently. Moreover, for subcubic graph G, that is, maximum degree is at most 3, Faudree *et al.*[4] proposed some conjectures. Steger and Yu [10]showed that $\chi'_s(G) \le 9$ for every subcubic bipartite graph G. For subcubic planar graph G with girth at least 6, Hudák et al.[8] proved the same result above. This conjecture is still open for $\Delta \ge 4$. For $\Delta = 4$, the best bound is 21, which was recently established by Huang, Santana, and Yu [9]. The bound of 21 is still one larger than the conjectured bound of 20. For every planar graph G with $\Delta = 4$, Wang *et al.*[11] proved that $\chi'_s(G) \le 19$ and Jian-Bo Lv *et al.*[13] proved that if $mad(G) < \frac{61}{18}$ (*resp.* $\frac{7}{2}, \frac{18}{5}, \frac{15}{4}, \frac{51}{13}$), then $\chi'_s(G) \le 16$ (*resp.* $17, 18, 19, 20$).

Recently, a great deal of research has been done on planar graphs with different values of girth. In 2014, Hudák *et al.*[8] proved that $\chi'_s(G) \le 3\Delta$ for every planar graph G with $g \ge 7$. For every planar graph G with $g \ge 10\Delta - 4$, Wang *et al.*[12] further reduced this bound and proved that $\chi'_s(G) \le 2\Delta - 1$. By maximum degree restriction, Choi *et al.*[6] showed two results, namely, $\chi'_s(G) \le 3\Delta$ for every planar graph G with $g \ge 6$ and $\Delta \ge 7$, and $\chi'_s(G) \le 3\Delta - 3$ for every graph G with $g \ge 8$ and $\Delta \ge 9$. Guo *et al.*[7] also came to two conclusions that $\chi'_s(G) \le 3\Delta - 2$ for every planar graph G with $g \ge 8$ and $\Delta \ge 4$, and $\chi'_s(G) \le 3\Delta - 3$ for every planar graph G with $g \ge 10$ and $\Delta \ge 5$.

In this paper, we take into account the girth and the maximum degree of planar graphs and prove the following results.

Theorem 1.2. *If G is a planar graph with $g \ge 7$ and $\Delta \ge 5$, then $\chi'_s(G) \le 3\Delta - 1$.*

Theorem 1.3. *If G is a planar graph without adjacent $7-cycles$, with $g \ge 7$ and $\Delta \ge 4$, then $\chi'_s(G) \le 3\Delta - 1$.*

Note that $mad(G) < \frac{2g}{g-2}$ for every planar graph. Thus, when $g \geq 7$, we have that $mad(G) < \frac{14}{5}$. Therefore, there is the following corollary.

Corollary 1.4. *If G is a planar graph with $mad(G) < \frac{14}{5}$ and $\Delta \geq 5$, then $\chi'_s(G) \leq 3\Delta - 1$.*

By adding the condition that $\Delta \geq 5$, our results improve the bound of Hudák et al.[8] and are reduce by one color.

Let G^* be obtained by removing all vertices of degree one in graph G. The paper is organized as follows. In Section 2, we assume that G is a minimal counterexample with the fewest edges to Theorem 1.2. We first prove some structural properties of the minimal counterexample G and its subgraph G^*. Next, we use the discharging method to show that G^* cannot exist. In Section 3, with a weaker maximum degree restriction and a stronger cycle constraint, we still obtain the same bound.

2. Proof of Theorem 1.2

In this section, G is a counterexample to Theorem 1.2 with the $|V(G)|$ minimized, subject to that, assume that $|E(G)|$ is as small as possible. It is obvious that G and G^* are connected. A *strong partial edge-coloring* of a graph G is a proper edge-coloring of a proper subgraph G' such that every two edges of G' with distance at most 1 in G receive different colors. Suppose that G has a strong partial edge-coloring. For every uncolored edge e of G, we use $A(e)$ to denote the set of colors that are available at the edge e. The $2-$*neighborhood* of an edge e refers to the set of edges whose distance at most 2 from e.

We first state some structural properties regarding G and G^* as follows.

Lemma 2.1. $\delta(G^*) \geq 2$. *Moreover, $d_{G^*}(v) = 2$ if and only if $d_G(v) = 2$.*

Proof. Suppose to the contrary that $\delta(G^*) \leq 1$. If $\delta(G^*) = 0$, then G is a star since G and G^* are connected. Clearly, G has a strong edge-coloring with Δ colors, a contradiction. If $\delta(G^*) = 1$, then there must be $d_{G^*}(v) = 1$. Then, there must be $d_G(v) > d_{G^*}(v) = 1$, otherwise v will not appear in G^*. Therefore, v must have at least one $1-$neighbor in G, denoted by v_1, as shown in Fig.1(1). By the minimality of G, $G - v_1$ has a strong edge-coloring ϕ with $(3\Delta - 1)$ colors. Note that there are at most $2\Delta - 2$ colored edges in the $2-$neighborhood of the edge vv_1. Therefore, $|A(vv_1)| \geq \Delta + 1 \geq 6$. Thus, we can extend ϕ to G, a contradiction. So, $\delta(G^*) \geq 2$.

If $d_G(v) = 2$, then $d_{G^*}(v) = 2$ since $d_G(v) \geq d_{G^*}(v)$. Suppose that $d_{G^*}(v) = 2$. We assume that $d_G(v) > 2$. Then, v has at least one $1-$neighbor v_1 in G, as shown in Fig.1(2). By the minimality of G, $G - v_1$ has a strong edge-coloring ϕ using $(3\Delta - 1)$ colors. Clearly, there are at most $3\Delta - 3$ colored edges in the $2-$neighborhood of the edge vv_1. Hence, $|A(vv_1)| \geq 2$, which means that we can extend ϕ to G, a contradiction. So $d_G(v) \leq 2$. Since $d_G(v) \geq d_{G^*}(v)$, $d_G(v) = 2$. □

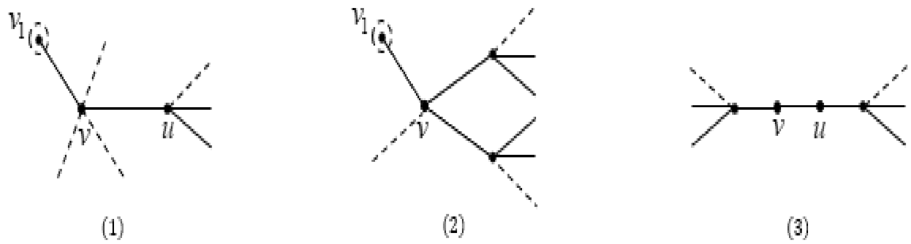

Fig.1
(The solid lines represent the edges that exist in G.
The dashed lines represent the edges that might exist in G.)

Lemma 2.2. *Let v be a $2-$vertex in G^*. Then, both of neighbors of v in G^* are 3^+-neighbor.*

Proof. Suppose otherwise that v has a $2-$neighbor, say u, in G^*. Since $d_{G^*}(v) = d_{G^*}(u) = 2$, by Lemma 2.1, $d_G(v) = d_G(u) = 2$, as shown in Fig.1(3). By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Since there are at most 2Δ colored edges in the $2-$neighborhood of the edge uv, $|A(uv)| \geq \Delta - 1 \geq 4$. Then, we can color uv with one of the available colors, a contradiction. □

Lemma 2.3. *Let v be a $3-$vertex in G^*. Then, v has at least two 3^+-neighbors in G^*.*

Proof. Suppose otherwise that v has at most a 3^+-neighbor in G^*. Let u_1, u_2 be two $2-$neighbors of v in G^*. By Lemma 2.1, $d_G(u_1) = d_G(u_2) = 2$. Assume that $d_G(v) > d_{G^*}(v)$. Then, v has at least one $1-$neighbor v_1 in G, as shown in Fig.2(1). By the minimality of G, $G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. It is easy to see that vv_1 has at most 2Δ colored edges within distance one. Thus, $|A(vv_1)| \geq \Delta - 1 \geq 4$. Then, we can color vv_1 with one of the available colors, a contradiction. Therefore, $d_G(v) = d_{G^*}(v) = 3$, as shown in Fig.2(2). Let ϕ be a strong $(3\Delta - 1)$-edge-coloring of $G - vu_1$. Note that vu_1 in G has at most $2\Delta + 2$ colored edges in its $2-$neighborhood. Hence, $|A(vu_1)| \geq \Delta - 3 \geq 2$, which implies that vu_1 has at least one available color, a contradiction. □

By Lemma 2.3, a $3-$*vertex* v in G^* is adjacent to at most one $2-$*neighbor* in G^*. We call a $3-$*vertex* v *weak* if it is adjacent to a $2-$*vertex*, otherwise we call it *strong*.

Lemma 2.4. *Let v be a weak $3-$vertex in G^*. Then, $d_G(v) = d_{G^*}(v) = 3$.*

Proof. Suppose to the contrary that $d_G(v) > d_{G^*}(v)$. Then, v has at least one $1-$neighbor in G, denoted by v_1. Let u be a $2-$neighbor of v in G^*. By Lemma 2.1, $d_G(u) = 2$, as shown in Fig.2(3). By the minimality of G, $G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Note that vv_1 in G has at most $3\Delta - 2$ colored edges in its $2-$neighborhood. So $|A(vv_1)| \geq 3\Delta - 1 - (3\Delta - 2) = 1$, which implies that vv_1 has at least one available color, a contradiction. □

Lemma 2.5. *Assume that v is a weak $3-$vertex in G^*. Then, v is not adjacent to a weak $3-$vertex.*

Proof. Suppose otherwise that v has a weak $3-$neighbor, say v_1, in G^*. By Lemma 2.4, $d_G(v) = d_{G^*}(v) = 3$ and $d_G(v_1) = d_{G^*}(v_1) = 3$. Let u be the $2-$neighbor of v in G^*, as shown in Fig.2(4). By the minimality of G, $G - uv$ has a strong $(3\Delta-1)$-edge-coloring ϕ. Then, uv in G has at most $2\Delta+3$ colored edges in its $2-$neighborhood. So, $|A(uv)| \geq 3\Delta-1-(2\Delta+3) = \Delta-4 \geq 1$. Thus, we can extend ϕ to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. □

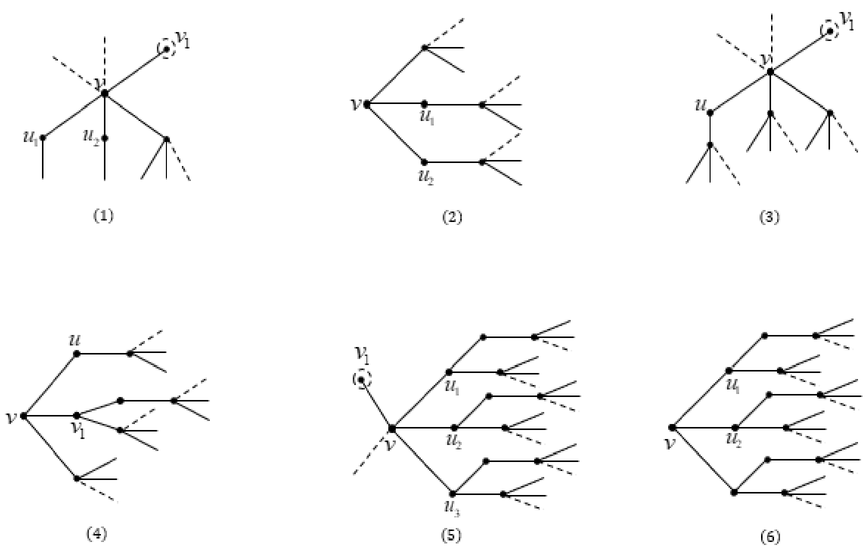

Fig.2

Lemma 2.6. *Let v be a strong $3-$vertex in G^*. Then, v has at most two weak $3-$neighbors in G^*.*

Proof. Suppose otherwise that the three neighbors of v are all weak $3-$neighbors in G^*. Let u_1, u_2, u_3 be three weak $3-$neighbors of v in G^*. By Lemma 2.4, $d_G(u_1) = d_G(u_2) = d_G(u_3) = 3$. Assume that $d_G(v) > d_{G^*}(v)$. Then, v has at least one $1-$neighbor v_1 in G, as shown in Fig.2(5). By the minimality of G,

$G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. It is easy to see that vv_1 has at most $\Delta + 5$ colored edges in its $2-$neighborhood. Thus, $|A(vv_1)| \geq 2\Delta - 6 \geq 4$. Then, we can color vv_1 with one of the available colors, a contradiction. Therefore, $d_G(v) = d_{G^*}(v) = 3$, as shown in Fig.2(6). Let ϕ be a strong $(3\Delta - 1)$-edge-coloring of $G - vu_1$. Note that vu_1 in G has at most $\Delta + 8$ colored edges in its $2-$neighborhood. Hence, $|A(vu_1)| \geq 2\Delta - 9 \geq 1$, which implies that vu_1 has at least one available color, a contradiction. $\square$

Lemma 2.7. *Every $4 - vertex$ v in G^* has at most three $2 - neighbors$.*

Proof. Suppose otherwise that the four neighbors of v are all $2-$vertices. Let u be one of neighbors of v in G^*, as shown in Fig.3(1). By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Note that uv in G has at most $2\Delta + 2$ colored edges in its $2-$neighborhood. Thus, $|A(uv)| \geq 3\Delta - 1 - (2\Delta + 2) = \Delta - 3 \geq 2$. So, ϕ can be extended to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. $\square$

If a $4-vertex$ has just three $2-neighbors$, we call it a $4_3-vertex$. Otherwise, if a $4 - vertex$ has at most two $2 - neighbors$, we call it a $4_2 - vertex$.

Lemma 2.8. *If v is a $4_3 - vertex$ in G^*, then $d_G(v) = d_{G^*}(v) = 4$.*

Proof. Suppose otherwise that that $d_G(v) > 4$. Then, v has at least one $1-$neighbor in G, denoted by v_1. Let u be a $2-$neighbor of v in G^*. By Lemma 2.1, all three $2-$neighbors of v have degree 2 in G, as shown in Fig.3(2). By the minimality of G, $G - v_1$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Note that vv_1 in G has at most $2\Delta + 1$ colored edges in its $2-$neighborhood. So $|A(vv_1)| \geq \Delta - 2 \geq 3$, which implies that vv_1 has at least one available color, a contradiction. $\square$

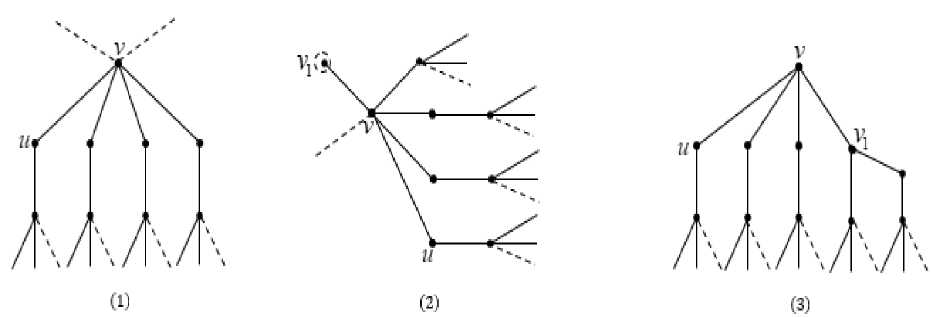

Fig.3

Lemma 2.9. *Assume that v is a $4_3-vertex$ in G^*. Then, v is not adjacent to a weak $3-vertex$.*

Proof. Suppose otherwise that v has a weak $3-$neighbor, say v_1, in G^*. By Lemma 2.7, $d_G(v) = d_{G^*}(v) = 4$. Let u be one of $2-$neighbors of v in G^*, as shown in Fig.3(3). By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Then, uv in G has at most $\Delta+7$ colored edges in its $2-$neighborhood. So, $|A(uv)| \geq 3\Delta - 1 - (\Delta + 7) = 2\Delta - 8 \geq 2$. Thus, we can extend ϕ to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. □

The total charge remains unchanged when we transfer the charge between vertices and faces. Now we will use discharging method and Euler's formula to get a contradiction and complete the proof of Theorem 1.2. We assign the initial charge $\rho(v) = \frac{1}{2}d_{G^*}(v) - 3$ for each vertex $v \in V(G^*)$ and $\rho(f) = d_{G^*}(f) - 3$ for each face $f \in F(G^*)$.

By Euler's formula, we have the following equality.

$$\sum_{v\in V(G^*)} \rho(v) + \sum_{f\in F(G^*)} \rho(f) = \sum_{v\in V(G^*)} (\frac{1}{2}d(v) - 3) + \sum_{f\in F(G^*)} (d(f) - 3) = -6$$

We will design appropriate discharging rules and redistribute charges among vertices and faces so that the final charges of every vertex and every face are non-negative. The discharging rules are shown as follows.

(R1) Every vertex receives $\frac{4}{7}$ from the incident face.

(R2) Every weak $3-vertex$ sends $\frac{3}{7}$ to the adjacent $2-vertex$.

(R3) Every $4^+-vertex$ sends $\frac{3}{7}$ to the adjacent $2-vertex$.

(R4) Every $4^+-vertex$ sends $\frac{3}{28}$ to the adjacent weak $3-vertex$.

(R5) Every strong $3-vertex$ sends $\frac{3}{28}$ to the adjacent weak $3-vertex$.

Let $\rho'(x)$ denote the finial charge of each element x in $V(G^*) \cup F(G^*)$ after the discharging process. We first consider the final charge of each face. By (R1), $\rho'(f) = d_{G^*}(f) - 3 - \frac{4}{7} \times d_{G^*}(f) = \frac{3}{7} \times d_{G^*}(f) - 3 \geq 0$. So, the final charge of each face is at least 0.

Next, we consider the final charge of each vertex v. Let $d_{G^*}(v) = k$. By (R1), it can get $\frac{4}{7} \times d_{G^*}(v) = \frac{4}{7} \times k$ from faces incident to v.

Assume that $d_{G^*}(v) = 2$. By Lemma 2.2, (R2) and (R3), we have that $\rho'(v) = \frac{1}{2} \times 2 - 3 + \frac{4}{7} \times 2 + \frac{3}{7} \times 2 = 0$.

Assume that $d_{G^*}(v) = 3$. If v is a weak $3-vertex$, by Lemma 2.5, (R4) and (R5), we have that $\rho'(v) = \frac{1}{2} \times 3 - 3 + \frac{4}{7} \times 3 + \frac{3}{28} \times 2 - \frac{3}{7} = 0$. If v is a strong $3-vertex$, then $\rho'(v) \geq \frac{1}{2} \times 3 - 3 + \frac{4}{7} \times 3 - \frac{3}{28} \times 2 = 0$ by Lemma 2.6 and (R5).

Assume that $d_{G^*}(v) = 4$. If v is a $4_2 - vertex$, then by (R3) and (R4), we have that $\rho'(v) \geq \frac{1}{2} \times 4 - 3 + \frac{4}{7} \times 4 - \frac{3}{7} \times 2 - \frac{3}{28} \times 2 = \frac{3}{14} > 0$. If v is a $4_3 - vertex$, then $\rho'(v) = \frac{1}{2} \times 4 - 3 + \frac{4}{7} \times 4 - \frac{3}{7} \times 3 = 0$ by Lemma 2.8 and (R3).

Assume that $d_{G^*}(v) = k \geq 5$. By (R3) and (R4), we have $\rho'(v) \geq \frac{1}{2} \times k - 3 + \frac{4}{7} \times k - \frac{3}{7} \times k = \frac{9}{14} \times k - 3 \geq \frac{9}{14} \times 5 - 3 = \frac{3}{14} > 0$.

Hence, the final charge of each vertex is at least 0.

By Euler's formula, we can obtain the following contradiction:

$$0 \leq \sum_{v \in V(G^*)} (\frac{1}{2} d(v) - 3) + \sum_{f \in F(G^*)} (d(f) - 3) = -6$$

Therefore, such a minimal counterexample to Theorem 1.2 does not exist.

3. Proof of Theorem 1.3

In this section, we still assume that G is a counterexample to Theorem 1.3 with the $|V(G)|$ minimized, subject to that, assume that $|E(G)|$ is as small as possible. We use the same method as Theorem 1.2 to prove Theorem 1.3. It is obvious that G satisfies the following structures.

Lemma 3.1. (1) $\delta(G^*) \geq 2$. *Moreover,* $d_{G^*}(v) = 2$ *if and only if* $d_G(v) = 2$.

(2) *Let* v *be a* $2 - vertex$ *in* G^*. *Then, both of neighbors of* v *in* G^* *are* $3^+ - neighbor$.

(3) *Let* v *be a* $3 - vertex$ *in* G^*. *Then,* v *has at least two* $3^+ - neighbors$ *in* G^*.

(4) *Let* v *be a weak* $3 - vertex$ *in* G^*. *Then,* $d_G(v) = d_{G^*}(v) = 3$.

(5) *Every* $4 - vertex$ v *in* G^* *has at most three* $2 - neighbors$.

Lemma 3.2. *Assume that* v *is a weak* $3 - vertex$ *in* G^*. *Then,* v *has at least one* $4^+ - neighbor$ *or one strong* $3 - neighbor$ *in* G^*.

Proof. Suppose otherwise that the other two neighbors of v in G^* are weak $3-$vertices, denoted by v_1, v_2. By Lemma 3.1(4), $d_G(v) = d_G(v_1) = d_G(v_2) = 3$. Let u be the $2-$neighbor of v in G^*. By the minimality of G, $G - uv$ has a strong $(3\Delta - 1)$-edge-coloring ϕ. Then, uv in G has at most $\Delta + 6$ colored edges in its $2-$neighborhood. So, $|A(uv)| \geq 3\Delta - 1 - (\Delta + 6) = 2\Delta - 7 \geq 1$. Thus, we can extend ϕ to a strong $(3\Delta - 1)$-edge-coloring of G, a contradiction. □

The total charge remains unchanged when we transfer the charge between vertices and faces. Now we assign the initial charge $\rho(v) = \frac{1}{2} d_{G^*}(v) - 3$ for each vertex $v \in V(G^*)$ and $\rho(f) = d_{G^*}(f) - 3$ for each face $f \in F(G^*)$. The discharging rules are shown as follows.

(R1) Every vertex receives $\frac{4}{7}$ from the incident $7 - face$.

(R2) Every vertex receives $\frac{5}{8}$ from the incident $8^+ - face$.

(R3) Every $4^+ - vertex$ sends $\frac{3}{7}$ to the adjacent $2 - vertex$.

(R4) Every $4^+ - vertex$ sends $\frac{3}{28}$ to the adjacent weak $3 - vertex$.

(R5) Every strong $3 - vertex$ sends $\frac{3}{28}$ to the adjacent weak $3 - vertex$.

(R6) Every weak $3 - vertex$ sends $\frac{3}{7}$ to the adjacent $2 - vertex$.

Let $\rho'(x)$ denote the finial charge of each element x in $V(G^*) \cup F(G^*)$ after the discharging process. We first consider the final charge of each face.

If $d_{G^*}(f) = 7$, then by (R1), we have that $\rho'(f) \geq d_{G^*}(f) - 3 - \frac{4}{7} \times 7 = 0$. If $d_{G^*}(f) \geq 8$, then we have that $\rho'(f) \geq d_{G^*}(f) - 3 - \frac{5}{8} \times d_{G^*}(f) = \frac{3}{8} \times d_{G^*}(f) - 3 \geq 0$ by (R2). Obviously, the final charge of each face is at least 0.

Next, we consider the final charge of each vertex. Let $d_{G^*}(v) = k$. Since there is no adjacent $7 - faces$ in G, by (R1) and (R2), it can at least get $\frac{4}{7} \times \lfloor \frac{k}{2} \rfloor + \frac{5}{8} \times \lceil \frac{k}{2} \rceil$ from 7^+-faces incident to v.

Assume that $d_{G^*}(v) = 2$. By Lemma 3.1, (R3) and (R6), $\rho'(v) \geq \frac{1}{2} \times 2 - 3 + \frac{5}{8} + \frac{4}{7} + \frac{3}{7} \times 2 = \frac{3}{56} > 0$.

Assume $d_{G^*}(v) = 3$. If v is a weak $3 - vertex$, then by Lemma 3.2, (R4) and (R6), we have that $\rho'(v) \geq \frac{1}{2} \times 3 - 3 + \frac{5}{8} \times 2 + \frac{4}{7} - \frac{3}{7} + \frac{3}{28} = 0$. If v is a strong $3 - vertex$, then $\rho'(v) \geq \frac{1}{2} \times 3 - 3 + \frac{5}{8} \times 2 + \frac{4}{7} - \frac{3}{28} \times 3 = 0$ by (R5).

Assume that $d_{G^*}(v) = 4$. By Lemma 3.1, (R3) and (R4), $\rho'(v) \geq \frac{1}{2} \times 4 - 3 + \frac{5}{8} \times 2 + \frac{4}{7} \times 2 - \frac{3}{7} \times 3 - \frac{3}{28} = 0$.

Assume that $d_{G^*}(v) = k \geq 5$. By (R3) and (R4), $\rho'(v) \geq \frac{1}{2} \times k - 3 + \frac{5}{8} \times \lceil \frac{k}{2} \rceil + \frac{4}{7} \times \lfloor \frac{k}{2} \rfloor - \frac{3}{7} \times k \geq \frac{75}{112} \times k - 3 \geq \frac{39}{112} > 0$.

Hence, the final charge of each vertex is at least 0.

By Euler's formula, we can obtain the following contradiction:

$$0 \leq \sum_{v \in V(G^*)} (\frac{1}{2} d(v) - 3) + \sum_{f \in F(G^*)} (d(f) - 3) = -6$$

Therefore, such a minimal counterexample to Theorem 1.3 does not exist.

Acknowledgements

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Predator and prey dynamics with Beddington-DeAngelis functional response with in kinesis model

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Abstract. In mathematical ecology, the study of interactions that are reactive-diffusive in nature between different species and their relevant systems has been researched extensively. However, there is still room for contribution on this rich topic. Therefore, we study a spatial-temporal prey-predator model which includes kinesis terms representing plankton dynamics under info-chemical mediated trophic interactions. The Beddington-DeAngelis functional response is coupled with a simplified two species approach within the model to describe the grazing pressure of zooplankton (M) on phytoplankton (P).This pressure is controlled through an external info-chemical (C). The mutual interference by predators within the ecosystem is implemented through the Beddington-DeAngelis functional response, a distinctive feature of this response type. This feature is utilized in this study to indicate the effect of changes in prey density in relation to predator density. In our model, a stability analysis is performed between the two aforementioned species to provide a system dynamics comparison. The critical conditions for kinesis are derived on the basis that increases in the reproduction coefficient decrease the diffusion. This means that species prefer to stay in good conditions to facilitate the reproduction process, but are likely to escape in bad conditions. The kinesis terms within our Phytoplankton-Zooplankton model impact factors such as survival and traveling wave behavior. Numerical experiments are performed in this work to examine the traveling waves and the monotonic dependence of the reproduction coefficient in the species population. Moreover, the possible benefits of purposeful kinesis are demonstrated.

Keywords: bifurcation analysis, stability analysis, predator-prey dynamics, plankton model, reaction-diffusion with kinesis model, travelling waves.

MSC 2020: 35A24

1. Introduction

Phytoplankton are the primary source of carbon dioxide transfer to the ocean and capture carbon dioxide through the process of photosynthesis. Carbon capture, and modelling processes involved, has recently become a topic of increasing interest, given its potential role in countering global warming, although little novel mathematical research into this aspect has been published in recent years. We wanted to improve the modelling of the interactions between a particular group of phytoplankton and its main predator. Phytoplankton need light for photosynthesis. This limits their viable depth to less than 200m [27]. The vertical distribution of phytoplankton is highly heterogeneous, but empirical research has shown that profiles of certain chemicals (info-chemicals), for instance dimethyl sulfide (DMS), closely resemble chlorophyll maxima (i.e. clusters of phytoplankton) as seen in [28]. Predators of plankton (e.g. Copepods) are known to travel vertically to follow prey distribution. This suggests that Copepods may use vertical gradients of info-chemicals to locate prey and remain within their profitable foraging zones. Lewis [52] developed a non-spatial model involving Copepods. Further investigation showed that small increases in the ability of Copepods to sense info-chemicals could increase their longevity in the system, and hence increased sensitivity to info-chemicals can be an evolutionarily advantageous a strategy for these predators. The phenomenon of vertically migrating zooplankton has been studied by many (e.g., [29], [30], [31], [65]), including a spatial heterogeneity which lead to the development of reaction-advection-diffusion models. The Beddington DeAngelis functional response is an essential tool in the field of plankton modelling. Although it is similar to responses such as the Holling type II functional response, it includes a term that accounts for mutual interference by predators. This allows for the prediction of predators per capita feeding rates on the prey, as well as providing better descriptions of predator-prey abundances and their relation to predator feeding activity within their respective predator-prey systems. In plankton models, the Beddington DeAngelis functional response can be used to perform a detailed mathematical analysis of the intra-species competition among predators [21]. Many ecologists have proposed the prey dependent predator-prey model, based on the assumption that the predators rate of prey capture is independent of prey density. However, some biologists disagree with that in many instances, particularly when predators must search for food and thus must share or start competing for food, the predator prey models deliver results should really be predator dependent. The Beddington DeAngelis type functional response outperformed the others in several circumstances. The functional response of a predator is the rate at which it consumes prey as a function of food density. Understanding the underlying dynamic relationships between prey and predator in the Beddington DeAngelis model is crucial for the description of ecosystem dynamics. [21, 26] implemented the effect of this functional response to describe mutual interference by predators within their predatorprey ecosystem model. Later this approach was used to highlight the effect of changes in prey density on the predator density attached per unit time in Sarwardi [25]. [58] and [48] introduced the classical PDE model which defines population dispersal, and is used to model kinesis, as can be observed in [42]. For those kinesis models, the diffusion is dependent on only localized information rather than including non-localized information. The local information, which is to be considered in cases such as taxis movement. A connection between the reproduction rate and diffusion coefficient has been established in which the reproduction coefficient can be presented as Darwinian Fitness; increase in migration should increase Darwinian Fitness [47], [54], and [41]. In this work, we aim to explore a predator-prey diffusion model of plankton with kinesis using partial differential equations (PDEs). [64] analytically explained the random population dispersal mechanics for living organisms by introducing the diffusion law, enabling an understanding of the spatial distribution of population density in linear and two-dimensional forms. Over the years, many scientists have studied diffusion to model biological, chemical, and physical processes. In particular, Alan Turing determined the causes of d-patterns in a variety of non-equilibrium situations when dealing with reaction-diffusion [4]. The classical predator-prey model was defined by Lotka and Volterra in 1920. In n our study, we investigate five key aspects related to the kinesis-diffusion terms, which provide a parameterisation of small-scale distribution. These terms account for horizontal movement in two dimensions, primarily influenced by the circular distribution and flows observed in plankton. Here's an overview of the sections covered in our study:

- General Description and Mathematical Model: The first section provides an introduction and outlines the mathematical model used in our research.
- Equilibrium Location and Analysis: The second section focuses on the location and analysis of equilibrium points within the model.
- Time-Series Behavior: Section three delves into the time-series behavior of the system, examining its dynamic evolution over time.
- Bifurcation Behavior at Different Carrying Capacity Levels: In the fourth section, we explore how the system's behavior changes at various carrying capacity levels, particularly focusing on bifurcation phenomena.
- Hydra Effect of Both Predator and Prey: Section five discusses the "hydra effect" observed in both predator and prey populations and its implications for the ecosystem.
- Analysis of Kinesis in the Reaction-Diffusion System: The sixth section provides an in-depth analysis of kinesis within the reaction-diffusion system and its impact on the overall equilibrium.
- Discussion of Findings and Conclusion: The final section offers a comprehensive discussion of our research findings. We conclude by discussing how

a small growth rate can lead to reduced phytoplankton density and potentially destabilize the model. Additionally, we explore the role of rapid responses to increases in fast-growing prey, which can contribute to the emergence of limit cycles in the dynamical system.

2. Mathematical model

The core goal of our work is to analyze the qualitative behavior of two microorganism species (phytoplankton, grazing zooplankton) interacting on two trophic levels exposed to a predator (meso-zooplankton copepod) and to examine the interaction between this trophic. The analysis will focus on a comparison between the latest obtained results and the current available results in an attempt to understand the main difference among two different functional response types in the predator-prey model and to illustrate how grazing induced by Dimethyl sulphide has a stabilizing effect on the modelled system.

2.1 General model and description

The model used is described using PDEs which include a horizontal diffusion term as shown below:

$$\frac{\partial P}{\partial t} = F_{\Delta}(P, M) := D_P \frac{\partial^2 P}{\partial x^2} + rP\left(1 - \frac{P}{K}\right) - \frac{aPM}{EM + P + b}, \tag{1}$$

$$\frac{\partial M}{\partial t} = G_{\Delta}(P, M) := D_M \frac{\partial^2 M}{\partial x^2} + \frac{\gamma a PM}{EM + P + b} - mM - \nu \frac{aPM^2}{EM + P + b}, \tag{2}$$

In the given model (Eq. 2), P and M represent the densities of phytoplankton and zooplankton within a closed homogeneous system. Similar to the approach taken by [52], the model assumes logistic growth for phytoplankton, characterized by an intrinsic growth rate denoted as r and a carrying capacity represented by K. This carrying capacity reflects the limits imposed by nutrient availability and self-shading effects on phytoplankton growth. Zooplankton in this model feed on phytoplankton based on the Beddington DeAngelis functional response, a mathematical framework used to describe predator-prey interactions. This functional response is employed to provide more detailed insights into predator-prey dynamics and how they influence predator feeding behavior. The Beddington DeAngelis functional response has been used in various ecological studies to elucidate the impact of changes in prey density on predator density over time. For instance, [26] applied this concept to illustrate mutual interference among predators in an ecosystem, while [25] examined how alterations in prey density affect the per capita feeding rates of predators. Haque [21] demonstrated that the Beddington DeAngelis functional response is suitable for conducting a comprehensive mathematical analysis of intra-specific competition among predators. This response model reflects the saturation of grazing rates at higher phytoplankton densities, with phytoplankton biomass converted into zooplankton biomass with an efficiency factor denoted as γ. Additionally, the parameter E accounts for predator interference within the system. The parameter m represents zooplankton mortality, primarily caused by copepods, but it also considers mortality due to processes such as sinking and additional predation by other zooplankton or higher trophic levels. The parameter ν has a slightly different interpretation compared to its usage in [1] and [52]. It reflects an increase in copepod predation on zooplankton in response to the immediate release of info-chemicals when phytoplankton are grazed. Thus, ν can represent both heightened copepod sensitivity and response to chemical cues and improved copepod search efficiency at higher chemical concentrations. Importantly, copepods and info-chemicals are not explicitly modeled as variables but are incorporated into the system through the interaction term involving ν. The parameters a and b respectively represent the clearance rate of zooplankton at low food densities and its half-saturation density. Typical parameter values are summarized in Table(2.1). Notably, this model differs from the one presented by [1] and [52] primarily in the choice of functional response type. Additionally, Laplacian terms, represented by D_P and D_M, are included in the model, reflecting the diffusion of phytoplankton and zooplankton, respectively, with strengths defined as D_P and D_M. These terms account for the spatial movement of these populations, and this aspect of the model is consistent with the previous work in [1].

Table 1: Model Parameter Values

Parameter	Value	Unit	Source
r	$0-5$	day^{-1}	[8]
K	$0-1000$	μg C I^{-1}	[9, 10]
a	0.3	μg CI^{-1} day^{-1}	[11, 12]
b	0.05	μg C I^{-1}	[11, 12]
γ	0.3	day^{-1}	[13]
m	0.3	day^{-1}	
ν	$0.01-0.2$	day^{-1}	
E	0.2	day^{-1}	

2.2 Location of equilibria

The equilibrium points $P(t) = P_e$ and $M(t) = M_e$ ofEq.(1-2), corresponding to $dP/dt = dM/dt = 0$, can be shown to include the trivial state $(P_{e,0}, M_{e,0}) =$

$(0,0)$, the zooplankton-free equilibrium $(P_{e,mf}, M_{e,mf}) = (K, 0)$, and the coexistence state that satisfies the polynomial.

$$r\nu P_e^3+(-E\gamma r-K\nu r+\nu br)P_e^2+(E\gamma Kr-K\nu 180br-\gamma Ka+K*m)P_e-mkb \tag{3}$$

and

$$M_e = \gamma r\frac{\nu r P_e^3 + (-E\gamma r - K\nu r + \nu br)P_e^2 + (E\gamma 187Kr - K\nu br - \gamma Ka + Km)}{ak} \tag{4}$$

In general, Eq.(3) will have three roots. Following [6], they are given by

$$P_{e,j+1} = \frac{Kb-1}{3b} + \frac{2}{3b}\sqrt{(KEGb-1)^2 - \frac{3EGK}{r\nu}(\gamma a - mb - \nu r)}\cos\left(\frac{\theta + 2j\pi}{3}\right), \; j = 0, 1, 2, \tag{5}$$

where $\theta = \cos^{-1}\left(\frac{y_N}{h}\right)$,

$$y_N = -r\nu\frac{(Kb-1)^3}{9b^2} + \frac{r\nu(Kb-1)^2}{27b^2} + \frac{K(\gamma a - mEGK - \nu r)(Kb-1)}{3b} - mK,$$

and

$$h = \frac{2\nu rb}{27}\left(\frac{(-EGr - K\nu r + \nu br)^2 - 3\nu r(EGKr - K\nu br - GKs + Km)}{\nu^2 r^2}\right)^{\frac{3}{2}}.$$

The phenomenon of two roots merging into one occurs when the condition $y_N^2 = h^2$ is satisfied, leading to the emergence of a complex-valued root. This cubic nature of the solution, as described byEq.(5), is clearly depicted in Fig. 2 for the case where $K = 120$, $r = 1.5$, and other parameters maintain values as specified in Table. 2.1. The presence and number of coexistence equilibria in the system are notably influenced by the parameter values of K and ν. For instance, when $K = 70$, as illustrated in Fig. (2) (c), there exists only a single real root of Eq. (5) for all values of ν. However, when $K \approx 70.78$, a significant event known as a saddle-saddle bifurcation occurs. This corresponds to the moment when the two saddle node bifurcation points converge or collide.

3. Analysis of equilibrium points in the non-spatial model

In this section, we delve into the dynamics of plankton populations and engage in a discussion centered on our comprehension of the non-spatial interactions, often referred to as local interactions, within the marine ecosystem. Our focus revolves around examining the complex interactions among multiple trophic levels that take place in aquatic environments. To accomplish this, we employ a reaction-diffusion model as a tool for our investigation.

$$\begin{pmatrix} P(t) \\ M(t) \end{pmatrix} = \begin{pmatrix} P_e \\ M_e \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} e^{\lambda t}. \tag{6}$$

FromEq.(6), a uniform solution is said to be linearly stable when $\lambda \leq 0$ and unstable otherwise. Substituting (6) into (2) and linearising about $\epsilon_1 = \epsilon_2 = 0$, we obtain the linear eigenvalue problem

$$\lambda \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} a_{11} &= r(1 - \frac{2rP_e}{K}) - \frac{abM_e + EaM_e^2}{(b + P_e + EM)}, \\ a_{12} &= -aP_e \frac{2EM_e + P_e + b}{(EM_e + P_e + b)}, \\ a_{21} &= 2\frac{(EM_e + b)}{(EM_e + P_e + b)^2}, \\ a_{22} &= -m - \frac{(P_e + b)\gamma a P_e}{(EM_e + P_e + b)} - \frac{EM_e^2 \nu a P_e - 2M_e \nu a P_e^2 - 2M_e b \nu a P_e}{(EM_e + P_e + b)}. \end{aligned} \tag{8}$$

The eigenvalues can then be readily obtained

$$\lambda_\pm = \frac{1}{2}\left[a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right]. \tag{9}$$

The model in Eq.(2) posses three different equilibria; Table. (2) provides a description of the stability of each equilibria. The mathematical model presented in Eq.(2) in the absence of diffusion is firstly considered by us, i.e. $D_P = D_M = 0$, which is similar to the first approach by [1]. The summary of the equilibrium stability is given in Table. (2): Note that Eq.(3) is a cubic polynomial, and all its

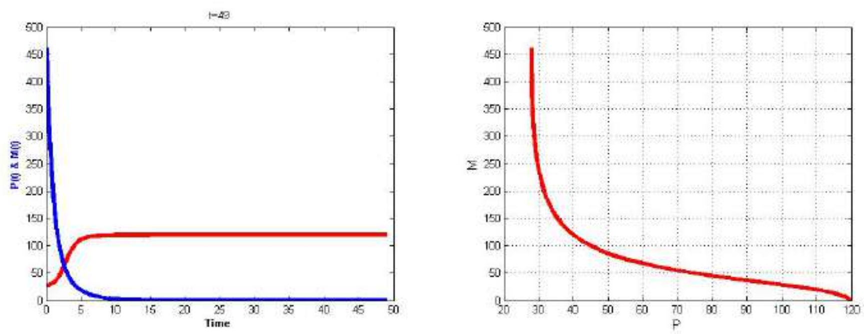
Table 2: Biologically Relevant Possible Equilibria of the System given by Eq. (2)

Equilibrium	Definition	Value in parametrized system	Description	Hyperbolic Eigenvalues
E_0	(P_e, M_e)	$(0,0)$	Trivial (extinct)	stable node point
E_1	(P_e, M_e)	$(K,0)$	The carrying capacity of phytoplankton	stable node
E_3	(P_e, M_e)	Eq.3 and 4	equilibrium Coexistence point	different stability behaviours

roots can be found by using **Cardan's method** [6]. Consequently, the obtained roots are utilized to determine the roots of the second species in Eq.(4). The stability of the coexistence point determines the behavior of the system given in Eq.(2).

4. Numerical exploration of the model

his section of the study will delve into the impact of varying parameters such as K, ν, and r on the stability of the system. In the subsequent subsection, we will introduce and define these parameters more explicitly, elucidating the specific ranges and values that could result in distinct system behaviors. This will be detailed further in the bifurcation and stability analysis section. Moreover, we will present the system's phase portrait and the equilibrium values associated with each intersection of the nullclines. The plane is inherently divided by several nullclines into distinct regions, each of which provides information about how the system behaves at different points within the plane [32]. These regions and their descriptions collectively offer a comprehensive understanding of how the system changes across various points in the plane.



(a) (b)

Figure 1: The time series behavior and phase portrait of the system are representative of the parameter setting where $\nu = 0.145$.

4.1 Bifurcation analysis of the phytoplankton-zooplankton model across various carrying capacity (K) and info-chemical (ν) levels

It is evident that the carrying capacity plays a crucial role in determining the maximum population density for plankton in each model [1]. In this work, an interesting finding relates the carrying capacity to the info-chemical parameter DMS, effectively introducing two control parameters instead of one. Fig. (2) illustrates four cases for different values of the carrying capacity (K), while keeping all other parameters fixed at the values provided in Table. (2.1). In Fig. (2(a)), (2(b)), and (2(c)), the system exhibits hysteresis behavior. Specifically, when $K = 1000$, there is an overlapping Hopf bifurcation at $\nu = 0.036$. Initially, as the info-chemical interaction parameter ν decreases from 1000 to 70, a supercritical Hopf bifurcation (Hp) occurs. During this phase, the system transitions from a stable limit cycle around the unstable coexistence state to a single stable coexistence state. Subsequently, a saddle-node bifurcation results in a region with bi-stability, where two stable coexistence states coexist alongside one unstable (saddle) coexistence state. The local stability of the stable equilibria shifts from a focus to a node, and the eigenvalues change from complex to real values. At this point, the system acquires the monotonicity property, meaning that the solution approaches a stable equilibrium in a monotonous manner, referred to as over-damped oscillations [14]. Finally, a second saddle-node bifurcation takes place, leaving only the larger stable coexistence state in the system. This outcome aligns with the findings of [1] and has been interpreted by [52] as the threshold at which persistent phytoplankton bloom formation becomes possible. In this context, persistent bloom formation implies that P_e (the phytoplankton equilibrium point) remains stable and approaches K. In Fig. (2(d)), the system exhibits less hysteresis behavior, with only one stable focus root type across all ν values. This variation is attributed to the influence of DMS on the predation of grazers. The bifurcation analysis for the behavior of zooplankton was con-

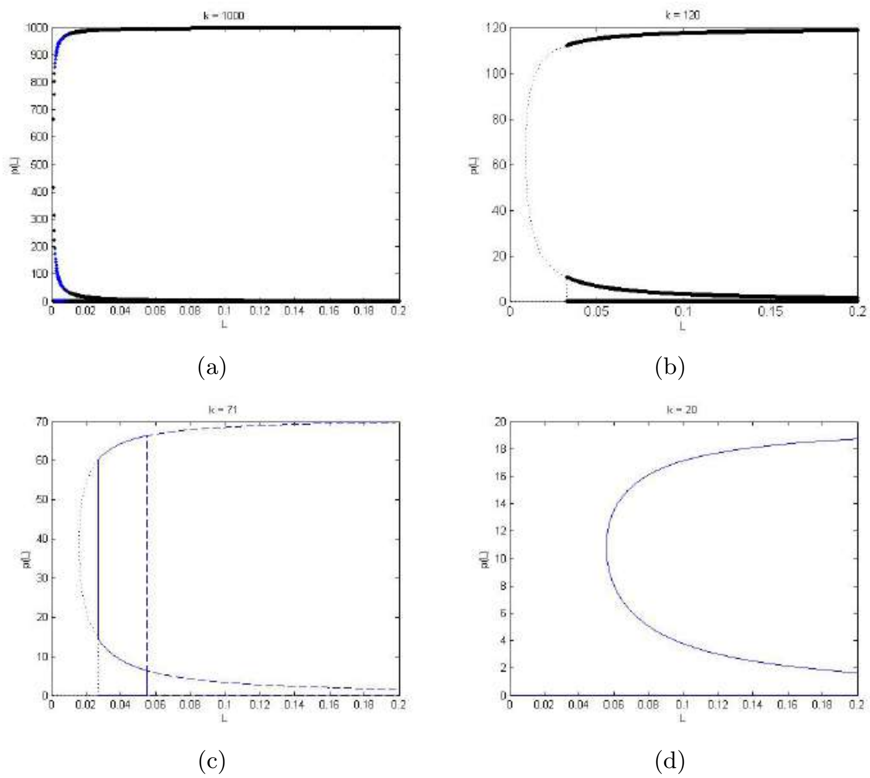


Figure 2: Bifurcation diagram correspond to different values of K in prey (phytoplankton) analysed system

ducted while keeping all other parameters fixed at the values specified in Table 21.1. In Fig. (3(a)), there is an overlap in the bifurcation behavior, specifically, a Hopf bifurcation occurs within the same range, with a value of approximately

0.036. Additionally, the two limit points correspond to a saddle-node bifurcation. In the case of Fig. (3(b)), both Hopf and saddle-node bifurcations can be observed. Fig. (3(c)) demonstrates that the system undergoes a Hopf bifurcation at $\nu = 0.01934$, after which the system's roots indicate a stable sink/node behavior. Lastly, in Fig. (3(d)), which examines the influence of DMS on grazer predation, the equilibrium type remains a 'stable focus' for all values of ν. The

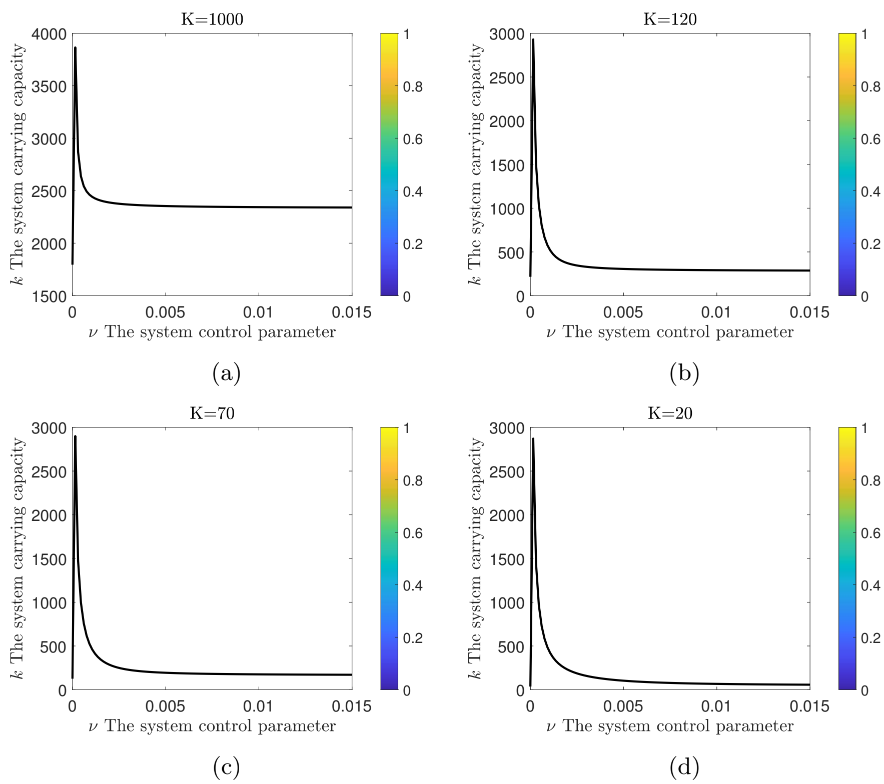


Figure 3: Bifurcation Diagram corresponding to Different Values of K in Predator (zooplankton).

model described inEq. (2) demonstrates the presence of a limit cycle for various values of K, as depicted in Figures 2 and 3. These findings align with the expected essential characteristics of the system.

4.2 Phytoplankton and zooplankton heat-maps

A phytoplankton bloom is characterized by a significant increase in the concentration of phytoplankton in a specific area. This phenomenon typically occurs when environmental conditions are favorable for enhanced reproduction, such as a continuous nutrient supply and suitable survival conditions. The formation of a phytoplankton bloom can occur within a specific range of parameter combinations involving K and ν. When copepod predation on zooplankton intensifies, it reduces the grazing pressure on phytoplankton, creating conditions conducive to bloom formation. The solution to Eq. (3) provides the roots for the saddle-node bifurcation and identifies the bifurcation position. It's essential to note that the region between the area with one real root and the area with three real and distinct roots is defined by satisfying Cardan's third condition, namely $y_N^2 = h^2$, effectively separating these regions as outlined in Eq. (3). Phytoplankton blooms can have a lasting impact on ecosystems [22, 23], and such occurrences have been referred to as the "hydra effect." The outcomes displayed in Fig. (4) depict the maximum population density of phytoplankton concerning variations in the carrying capacity. Generally, a phytoplankton bloom is characterized by a rapid proliferation of phytoplankton populations. These blooms tend to occur when there's an abundance of sunlight and nutrients available, creating favorable conditions for plant growth and reproduction. In such scenarios, the plants proliferate to the point where they become widespread, altering the water's color in which they reside [24]. Fig. (4) investigates two independent parameters: the carrying capacity and the infochemical concentration, determined using the polynomial in Eq. (3). This analysis suggests the potential occurrence of a phytoplankton bloom. A small, dark region on the left side of the saddle-node curves depicted in Fig. (2) (a) corresponds to a low phytoplankton population. Fig. (4) (a) readily illustrates the low values of P_e (the phytoplankton equilibrium point) for various combinations of K and ν, while the area to the right of the curve indicates higher phytoplankton populations, signifying the potential for a phytoplankton bloom.

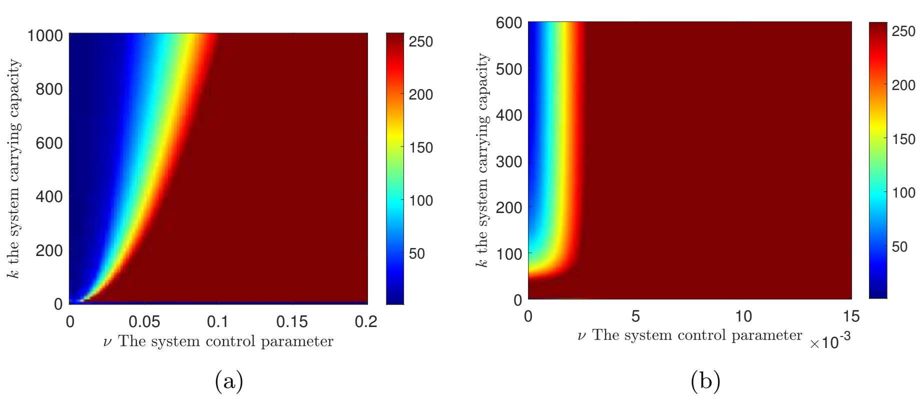


Figure 4: The heatmap in panel (a) pertains to the population of phytoplankton when subjected to the grazing pressure exerted by zooplankton. Panel (b) illustrates the population of zooplankton.

5. Analysis of predator-prey diffusion model with kinesis

In this section, we analyse the predator-prey diffusion model with kinesis that was first defined by [42] as follows:

$$\boxed{\partial_t u_i = D_{0i}\nabla \cdot \left(e^{-\alpha_i r_i(u_1,\ldots,u_k,s)}\nabla u_i\right) + r_i(u_1,\ldots,u_k,s)u_i,} \tag{10}$$

where:

u_i is the ith species-population density,

s is the abiotic characteristics of the living conditions,

r_i is the reproduction coefficient,

$D_{0i} > 0$ is the equilibrium diffusion coefficient which is defined when the reproduction coefficient is 0,

$\alpha_i > 0$ defines the relation between the diffusion coefficient on the reproduction coefficient.

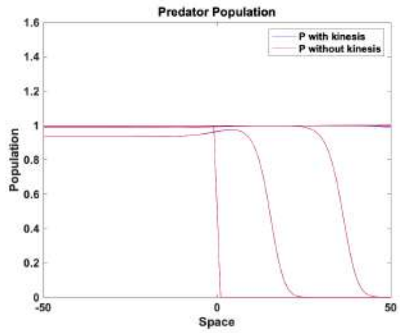
We can define $D_i = D_{0i}e^{-\alpha r_i}$ as the diffusion depending on reproduction coefficient. It has been shown in [42] that, the diffusion depends on well-being and it can be measured by the reproduction coefficient. In this section, we will present the new predator-prey plankton model with kinesis and compare the results with basic Kinesis model. The PDE model for population with constant diffusion coefficient without kinesis has been presented by (Kolmogorov, Petrovsky and Piskunov, 1937) (KPP) [50] as follows:

$$\partial_t u(t,x) = D\nabla^2 u(t,x) + (1-u(t,x))u(t,x). \tag{11}$$

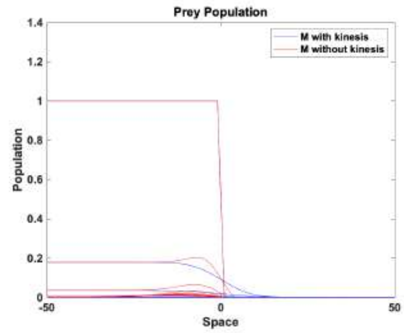
We will consider the predator-prey model presented Eq.(2) to define plankton-kinesis model as in Eq. (**??**).

$$\begin{aligned}
&\frac{\partial P}{\partial t} = F_\Delta(P,M) := D_P\nabla\cdot\left(e^{-\alpha\left(r\left(1-\frac{P}{K}\right)-\frac{aM}{EM+P+b}\right)}\nabla P\right)\\
&+ rP\left(1-\frac{P}{K}\right) - \frac{aPM}{EM+P+b},\\
&\frac{\partial M}{\partial t} = G_\Delta(P,M) := D_M\nabla\cdot\left(e^{-\alpha\left(\frac{\gamma aP}{EM+P+b}-m-\nu\frac{aPM}{EM+P+b}\right)}\nabla M\right)\\
&+ \frac{\gamma aPM}{EM+P+b} - mM - \nu\frac{aPM^2}{EM+P+b},
\end{aligned} \tag{12}$$

Figure 5 indicates that kinesis movement has no effect on the predator model. On the contrary, kinesis does affect prey population. Kinesis movement prevents extinction of prey population for a time. The MATLAB [59] function was used in this study to solve the one-dimensional system of PDE. The space interval



(a) a



(b) b

Figure 5: Predator-prey mobility under the effect of kinesis model.

was selected to be $[-50, 50]$ with zero-flux boundary conditions and with the initial conditions given below:

$$P(x,0) = P_e + \sigma \cos(wP), M(x,0) = M_e + \sigma \sin(wM) \tag{13}$$

The values of the constants are: $D = 1$, $\alpha = 1$.

In Fig.6 gives an account of the population size differences between the population with and without kinesis over time. Due to predatory causes, the prey population faces extinction within a small amount of time. The prey population without kinesis survives better than the one with kinesisin time 10, after in a while ($time20$), the population tends to survive better with kinesis movement over time. There is no time difference in the time profile of predator population with and without kinesis.Thus, there will be no difference in population size if we were to compare P with kinesis and without kinesis. Alternatively, kinesis decreases the size of prey population P with kinesis. This suggests that, initially, kinesis is not beneficial for the prey population in space. However, it starts to become beneficial and the population survives when the prey population without kinesis is dying (see Fig.6). In Fig.7, the travelling wave behaviour can be seen in the predator population in space. Initially, the predator population decreases in time and then it starts to increase and stabilizes over time. The kinesis movement affects prey population in a negative manner; it leades to population death in both conditions exponentially, and to an accelerated death in kinesis condition.

At the time 20, it can be clearly seen in space that both predator and prey population exhibit travelling wave behaviour (Fig.8). Figure 9 illustrates how the predator population dies over time in the spatial distribution, yet predators survive for a long time. M with kinesis decreases a faster, but survives better over time.

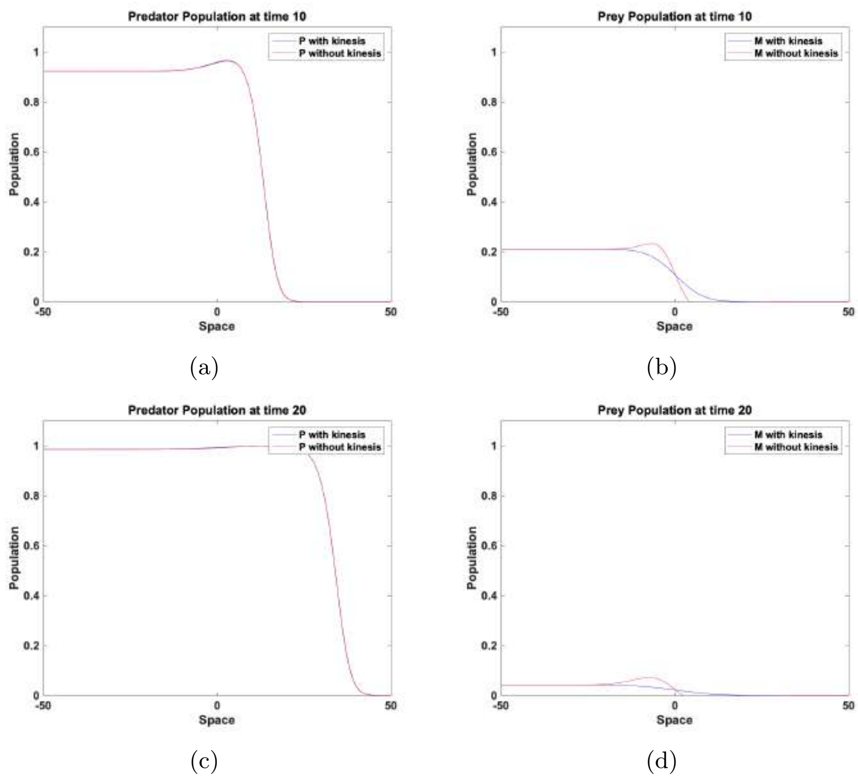


Figure 6: Prey mobility without and with kinesis.

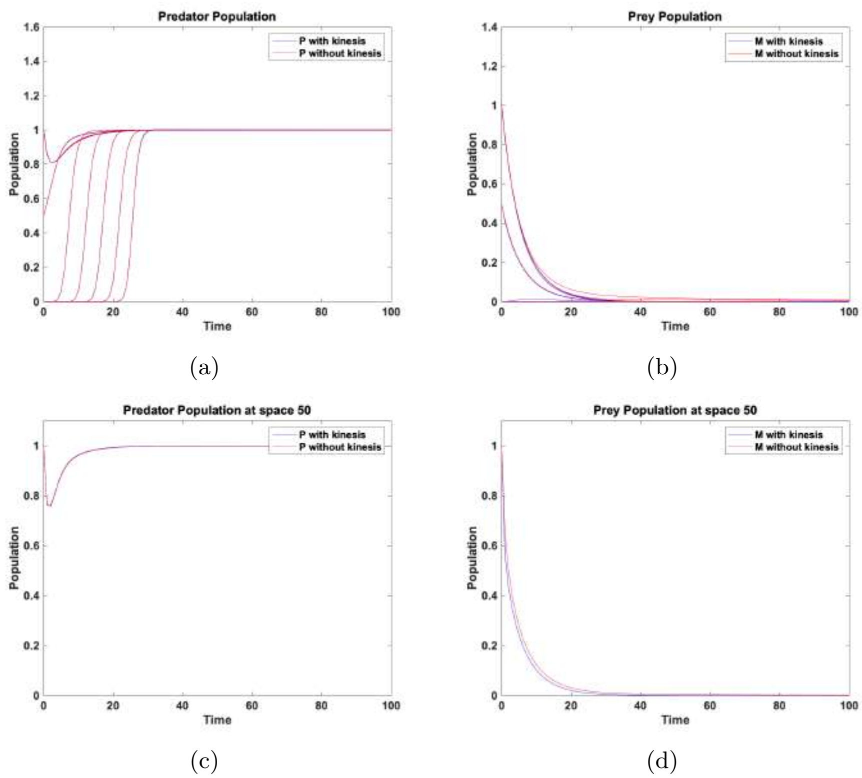


Figure 7: Predator mobility without and with kinesis.

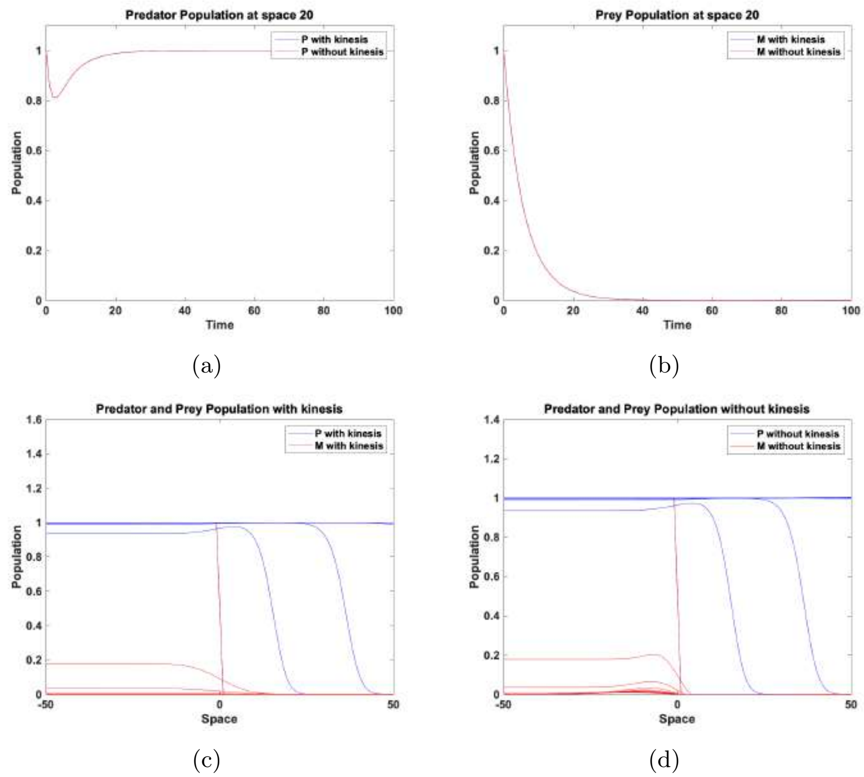


Figure 8: Predator and prey population without and with kinesis movement

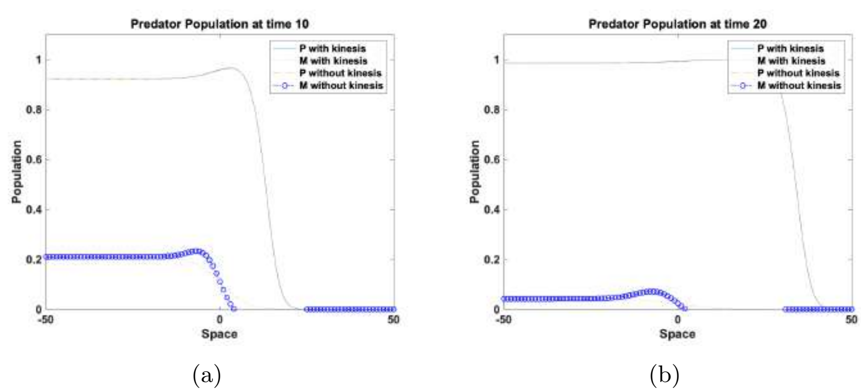


Figure 9: Predator prey population without and with kinesis movement

6. Conclusion

We gained insight into the behavior of the system described by equations Eqs. (2) by conducting a mathematical analysis involving phase plane investigations, stability assessments, and bifurcation examinations. Since we assume a uniform environment, we opted to employ the Beddington-DeAngelis functional response. This choice was motivated by the fact that it exhibits a broader spectrum of dynamic behaviors, as documented in previous studies [21]. This aligns with existing literature, which generally favors the utilization of the Beddington-DeAngelis functional response, particularly when examining interactions between two species, such as microzooplankton grazers like Oxyrrhis marina [17], [18]. In our numerical approach, we investigated the impact of the control parameter ν on the system's qualitative behavior. This investigation was made possible through the use of the phase plane tool, as demonstrated in Figs. 1(a) and 1(b). To illustrate, when setting ν to zero, we effectively transform the system into the Rosenzweig-MacArthur model [2]. In this scenario, the system becomes unstable, and we observe periodic cycles in the microzooplankton and phytoplankton population densities. This phenomenon is akin to the predator-prey interactions explored by [19] and [20], where the system exhibits a stable equilibrium, but the solution trajectories undergo substantial oscillations before returning to that equilibrium. As the control parameter increases, it leads to various stability scenarios, as depicted in the bifurcation diagram (Fig. 2) and its specific instances illustrated in Figs. 2(a), 2(b), and 2(c). This variation elucidates how infochemical signaling serves as a mechanism for enhancing copepod predation on microzooplankton. This article provides a comprehensive analysis of the system's behavior, including an examination of the location, number, and type of roots, determined using Cardan's method. Notably, this analysis helps identify crucial system parameters, particularly when $K = 70.34$, marking the point of a cusp bifurcation, where two equilibrium points merge and vanish in a saddle-node bifurcation [16]. The investigation extends to the (ν, K) plane, uncovering the phenomenon of a microzooplankton "hydra effect" on copepod predation. Additionally, the model allows for predictions regarding the occurrence and locations of phytoplankton blooms, as depicted in Fig. 3. An examination of Fig. 2 reveals that the system exhibits five distinct stability states, all of which are elaborated upon in Section 4. The text also discusses the implications of altering the growth rate and phytoplankton carrying capacity on phytoplankton behaviors, as illustrated in Figs. 4(a) and 4(b). It emphasizes how lower growth rates can shift the model's stability towards its current configuration. Furthermore, a relationship between both K and ν is identified, suggesting that both species can thrive in environments abundant with nutrients, as shown in Fig. 2. This outcome exemplifies the "hydra effect" within the predator-prey model, as mentioned in [15]. However, the primary objective of this article is to establish coherence between the model presented in Eq. 2 and the one examined in [1], both analytically and numerically.

The concept of population dispersion within a partial differential equation (PDE) model was initially introduced by [58] and further developed by [48]. The diffusion model, incorporating kinesis, has been the subject of prior research by [42]. In this context, kinesis movement influences the reproduction rate, and an interesting relationship emerges between the diffusion coefficient and the reproduction coefficient. Specifically, when reproduction rates increase over time, the diffusion coefficient decreases in contrast. This phenomenon aligns with the overarching principle that populations inherently strive for prolonged existence. Conversely, when population reproduction declines, indicating a population decline, individuals seek to disperse through kinesis movement. This behavior is driven by the imperative to escape unfavorable conditions and pursue more favorable ones. This notion of population dynamics can be likened to the concept of Darwinian fitness, as proposed by [47], [54], and [41]. According to this perspective, migration is a strategy employed to enhance Darwinian fitness, ultimately ensuring the population's survival. Consequently, populations tend to remain within beneficial areas while actively avoiding perilous conditions. To encapsulate our model's discoveries, we can outline them as follows:

- The eigenvalue problem of the predator-prey model, utilizing the Beddington DeAngelis functional response and incorporating the second condition of Cardan's method, played a crucial role in establishing a comprehensive stability analysis. This analysis is depicted in Figs. 2 and 3. The construction of these diagrams enabled us to conduct stability assessments for each value of K, corresponding to various ν values. Through this analysis, we arrived at the same conclusion as [1] concerning the case when $K = 120$. However, our study encompassed multiple scenarios for bifurcation within the system, contingent on different values of K. In all instances, it was evident that the presence of infochemicals had a stabilizing effect on what would otherwise be an unstable food web.

- By examining the behavior of the predator, denoted as M, and systematically varying the value of K as a secondary control parameter while keeping infochemicals as the primary parameter, we demonstrated the influence of DMS (dimethyl sulfide) on the predation of grazers. This analysis revealed that the populations of both species, namely phytoplankton and microzooplankton, can experience substantial simultaneous increases. This phenomenon is visually represented in Figs. 3(a), 3(b), 3(c), and 3(d).

- By investigating the growth rate of phytoplankton, we uncovered the following insights: A low growth rate results in diminished phytoplankton density, ultimately destabilizing the model, as depicted in Fig. 2(d). Conversely, a high potential growth rate enables heterotrophic protists to persist even during phases of elevated predation. However, this system is highly responsive to increases in fast-growing prey, and this heightened responsiveness may explain the existence of a limit cycle within the dynamical system.

- We examined how the inclusion of a kinesis model within a predator-prey model utilizing the Beddington DeAngelis functional response influenced the system. Fig. (5) illustrates that the predator model remained largely unaffected by kinesis movement. In contrast, kinesis had a noticeable impact on the prey population. Specifically, kinesis movement played a role in preventing the extinction of the prey population over time.

- The prey population faces a rapid risk of extinction due to predation. Initially, without the kinesis model, the prey population exhibits better survival up to time 10. However, an opposite trend emerges over time, such as at time 20, where the prey population with the kinesis model displays improved survival compared to the scenario without it.

- The size of predator populations, whether with or without kinesis, remains essentially unchanged. In contrast, the prey population with kinesis experiences an initial decrease in population size. This indicates that initially, having kinesis is not advantageous for the prey population in that particular space. However, it becomes beneficial, and the population manages to survive when the prey population without kinesis starts to decline, as illustrated in Fig. 6.

- In the model we have constructed and analyzed in this study, the predator population exhibits spatial behavior characterized by traveling waves.

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On the localization of a type B semigroup

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Abstract. This paper mainly investigates the localization of a type B semigroup. Firstly, the unique localization of a type B semigroup on its idempotent semilattice is given, and some properties of the localization of a type B semigroup are studied. It is proved that the localization of a type B semigroup on its idempotent semilattice is the maximum cancellative monoid homomorphic image. Finally, the relationships between localizations and the minimum cancellative congruence of a type B semigroup are discussed.
Keywords: type B semigroup, idempotent semilattice, cancellative monoid homomorphic image, localization.
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1. Introduction

In recent years, abundant semigroups have attracted more and more attention from semigroup scholars (see, [4-5, 7-8, 16]). As an important subclass of abundant semigroups, type B semigroups (see, [12-15, 17-19]) are called generalized inverse semigroups together with ample semigroups (see, [2-3, 6]) because of their similar properties to inverse semigroups (see, [1, 11, 23]). The localization (see, [9, 20-22]) is a good method to construct a new algebraic structure, and it plays an important role in commutative algebra. Localizations of inverse semigroups and ample semigroups have been studied by many authors (see, [9,

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21-22]). As an application of the localization, this paper will give some new characterizations of localizations of a type B semigroup.

2. Preliminaries

Firstly, some definitions, notations and known results used in this paper are provided.

In 1951, the concept of Green's relations were introduced by Green in [10]. Let $a, b \in S$, we have

$$a\mathcal{L}b \Longleftrightarrow S^1a = S^1b; \quad a\mathcal{R}b \Longleftrightarrow aS^1 = bS^1.$$

In the 1970s, Fountain extended Green's relations to Green's * relations. Let S be a semigroup. Recall, from [5] that two elements a and b in S are $\mathcal{L}^*$-[$\mathcal{R}^*$-] related if and only if they are $\mathcal{L}$-[$\mathcal{R}$-]related in some oversemigroup of S. The equivalent definitions of $\mathcal{L}^*$-relation and $\mathcal{R}^*$-relation are given as follows:

Lemma 2.1 ([5]). *Let S be a semigroup and $a, b \in S$. Then, the following statements hold:*

(1) *$a\mathcal{L}^*b$ if and only if, for all $x, y \in S^1$, $ax = ay \Leftrightarrow bx = by$;*

(2) *$a\mathcal{R}^*b$ if and only if, for all $x, y \in S^1$, $xa = ya \Leftrightarrow xb = yb$.*

Corollary 2.2 ([5]). *Let S be a semigroup and $a, e = e^2 \in S$. Then, the following statements are equivalent:*

(1) *$a\mathcal{L}^*e$ [$a\mathcal{R}^*$ e];*

(2) *$ae = a$ [$a = ea$] and for all $x, y \in S^1$, $ax = ay$ [$xa = ya$]implies $ex = ey$ [$xe = ye$].*

Obviously, let S be a semigroup. The relation $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence on S. Usually, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ on S. But, if a and b are regular elements of a semigroup S, then we obtain that $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$, and that $a\mathcal{R}^*b$ if and only if $a\mathcal{R}b$. That is, $\mathcal{L}^* \cap (RegS \times RegS) = \mathcal{L}$, $\mathcal{R}^* \cap (RegS \times RegS) = \mathcal{R}$, where $RegS$ denotes the set of all regular elements of S. For convenience, $\mathcal{L}^*{}_a$ and $\mathcal{R}^*{}_a$ denote the $\mathcal{L}^*-$class and $\mathcal{R}^*-$class containing a, respectively; $E(S)$ denotes the set of idempotents of S; a^+ and a^* denote the idempotent of the $\mathcal{L}^*-$class and $\mathcal{R}^*-$class containing a, respectively.

As in [4], a semigroup S is said to be *right (left) abundant* if each $\mathcal{L}^*-$ $(\mathcal{R}^*)-$class of S contains an idempotent. A semigroup S is *abundant* if it is both right and left abundant. A right (left) abundant semigroup S is *right (left) adequate* if $E(S)$ is a semilattice ([5]). A semigroup S is said to be *adequate* if it is both left and right adequate.

Definition 2.1 ([4]). *Let S be a right adequate semigroup. Then, S is said to be right type B, if it satisfies the following conditions:*

(B1) *for all* $e, f \in E(S^1), a \in S$, $(efa)^* = (ea)^*(fa)^*$;

(B2) *for all* $a \in S, e \in E(S)$, *if* $e \leq a^*$, *then there is* $f \in E(S^1)$ *such that* $e = (fa)^*$, *where* $\leq$ *is a natural partial order on* $E(S)$.

Definition 2.2 ([4]). *Let* S *be a left adequate semigroup. Then,* S *is left type* B, *if it satisfies the following conditions:*

$(B1')$ *for all* $e, f \in E(S^1), a \in S$, $(aef)^+ = (ae)^+(af)^+$;

$(B2')$ *for all* $a \in S, e \in E(S)$, *if* $e \leq a^+$, *then there is* $f \in E(S^1)$ *such that* $e = (af)^+$, *where* $\leq$ *is a natural partial order on* $E(S)$.

A semigroup is said to be type B *if it is both left and right type* B.

Lemma 2.3 ([12]). *Let* S *be a type* B *semigroup. The relation* σ *is defined as follows:*

$$(a, b) \in \sigma \Longleftrightarrow (\exists e \in E(S))\ eae = ebe.$$

Then, σ *is the least cancellative congruence on* S.

Definition 2.3 ([21]). *Let* T *be a monoid,* S *be a semigroup and* H *be a subsemigroup of* S. *Then,* T *is said to be a localization of* S *on* H, *if it satisfies the following conditions:*

(1) *There is a surjective homomorphism* $\phi : S \to T$ *such that* $\phi(a)$ *is inverse on* T, *for all* $a \in H$.

(2) *If there are a monoid* S' *and a homomorphism* $\alpha : S \to S'$ *such that* $\alpha(a)$ *is inverse on* S', *for all* $a \in H$, *then there is a unique homomorphism* $\psi : T \to S'$ *such that* $\psi\phi = \alpha$.

Lemma 2.4 ([9]). *Let* S *be a semigroup and* H *be a subsemigroup of* S. *If there exists a localization of* S *on* H, *then the localization is unique in the sense of isomorphism. For convenience, we denote the unique localization by* $S[H^{-1}]$.

3. The localization of a type B semigroup on its idempotent semilattice

In this section, we shall characterize the localization of a type B semigroup on its idempotents. For convenience, we denote the idempotent set $E(S)$ of a semigroup S by E.

Proposition 3.1. *Let* S *be a type* B *semigroup and* E *be its idempotent semilattice. Define a relation on set* $S \times E$ *as follows:*

$$(\forall (x, e) \in S \times E)(x, e) \sim (y, f) \Longleftrightarrow (\exists h \in E) hfxfh = heyeh,$$

then the following statements hold:

(1) *The relation $\sim$ is an equivalence relation on S.*

(2) *For all $x \in S$, $e, f \in E$, $(x, e) \sim (x, f)$.*

(3) *For all $(x, e) \in S \times E$, we denote the equivalence class containing (x, e) by x/e. Then, for all $e_1, e_2, e_3, e_4 \in E$, $e_1/e_2 \sim e_3/e_4$. In particular, for $e \in E$, we denote $\sim -$class containing all (e_1, e_2) by e/e, where $e_1, e_2 \in E$.*

(4) *Put $T = (S \times E)/\sim = \{x/e \mid x \in S\}$. Define a multiplication "$\cdot$" on T as follows:*

$$(\forall x/e, y/e \in T)\ x/e \cdot y/e = (xy)/e.$$

Then, T is a monoid whose identity element is e/e under the multiplication "$\cdot$".

Proof. (1) Obviously, "$\sim$" is reflexive and symmetric. Now, we prove that "$\sim$" is transitive. To see it, let $(x, e), (y, f), (z, g) \in S \times E$ such that $(x, e) \sim (y, f)$, $(y, f) \sim (z, g)$. Then, there exist $e_1, e_2 \in E$ such that $e_1 fxfe_1 = e_1 eyee_1$ and $e_2 gyge_2 = e_2 fzfe_2$. Hence,

$$\begin{aligned} e_1e_2fgxge_1e_2f &= e_2ge_1fxfe_1e_2g = e_2ge_1eyee_1e_2g = e_1ee_2gyge_2e_1e \\ &= e_1ee_2fzfe_2e_1e = e_1e_2fezee_1e_2f. \end{aligned}$$

Let $h = e_1e_2f \in E$. Then, $hgxgh = hezeh$. This shows that $(x, e) \sim (z, g)$. Therefore, "$\sim$" is an equivalence relation on S.

(2) For all $x \in S$, $e, f \in E$, we have that $effxfef = efxef = eefxeef = efexeef$. Let $h = ef \in E$. Then, $hfxfh = hexeh$. Therefore, $(x, e) \sim (x, f)$.

(3) Since E is the idempotent semilattice of S, we have that $h = e_1e_2e_3e_4 \in E$, for all $e_1, e_2, e_3, e_4 \in E$. Again, since $he_4e_1e_4h = he_2e_3e_2h$, we have $(e_1, e_2) \sim (e_3, e_4)$. That is, $e_1/e_2 \sim e_3/e_4$. In particular, we choose one element $e \in E$, it is easy to see that $(e_1, e_2) \in e/e$, for all $e_1, e_2 \in E$.

(4) Firstly, we prove that the multiplication operation "$\cdot$" on T is well-defined. Let $x_1/e, x_2/e, y_1/e, y_2/e \in T$ with $x_1/e = x_2/e$, $y_1/e = y_2/e$. Then, there exist $f, g \in E$ such that $fex_1ef = fex_2ef$ and $gey_1eg = gey_2eg$. Notice that $x_1^*ef \le x_1^*$, $x_2^*ef \le x_2^*$. We have that there exist $e_1, e_2 \in E(S^1)$ such that $x_1^*ef = (e_1x_1)^*$ and $x_2^*ef = (e_2x_2)^*$ from Condition (B2). Hence,

$$\begin{aligned} e_1e_2fex_1ef &= e_1e_2fex_1x_1^*ef = e_1e_2fex_1(e_1x_1)^* \\ &= e_1e_2fee_1x_1(e_1x_1)^* = e_1e_2fee_1x_1. \end{aligned}$$

Similarly, $e_1e_2fex_2ef = e_1e_2fee_2x_2$. Again, $fex_1ef = fex_2ef$. Multiplying it on the left by e_1e_2, we obtain that $e_1e_2fex_1ef = e_1e_2fex_2ef$. Thus, $e_1e_2fee_1x_1 = e_1e_2fee_2x_2$. On the other hand, it is clear that $gey_1^+ \le y_1^+$ and $gey_2^+ \le y_2^+$. Therefore, there exist $e_3, e_4 \in E(S^1)$ such that $gey_1^+ = (y_1e_3)^+$ and $gey_2^+ = (y_2e_4)^+$ from Condition (B2'), and so

$$\begin{aligned} gey_1ege_3e_4 &= gey_1^+y_1ege_3e_4 = (y_1e_3)^+y_1ege_3e_4 \\ &= (y_1e_3)^+y_1e_3ege_3e_4 = y_1e_3ege_3e_4. \end{aligned}$$

Similarly, $gey_2ege_3e_4 = y_2e_4ege_3e_4$. Again, $gey_1eg = gey_2eg$. Multiplying it on the right by e_3e_4, we obtain that $gey_1ege_3e_4 = gey_2ege_3e_4$. Thus, $y_1e_3ege_3e_4 = y_2e_4ege_3e_4$. For some $h = e_1e_2e_3e_4fg \in E$, we have

$$\begin{aligned} hex_1y_1eh &= e_1e_2e_3e_4fgex_1y_1ee_1e_2e_3e_4fg = e_3e_4ge_1e_2fee_1x_1y_1e_3ege_3e_4e_1e_2f \\ &= e_3e_4ge_1e_2fee_2x_2y_2e_4ege_3e_4e_1e_2f = e_1e_2e_3e_4fgex_2y_2ee_1e_2e_3e_4fg \\ &= hex_2y_2eh. \end{aligned}$$

Hence, $(x_1y_1)/e = (x_2y_2)/e$. This means that the multiplication operation " $\cdot$ " on T is good.

Next, we show that T is a monoid whose identity element is e/e under the multiplication " $\cdot$ ". Let $x/e, y/e, z/e \in T$. We have

$$\begin{aligned} (x/e \cdot y/e) \cdot z/e &= (xy)/e \cdot z/e = (xyz)/e \\ &= x/e \cdot (yz)/e = x/e \cdot (y/e \cdot z/e). \end{aligned}$$

This shows that T is associative under the multiplication operation " $\cdot$ ". It is clear that T is closed. Thus, T is a semigroup with respect to the multiplication " $\cdot$ ". Obviously, we have $ee(xe)ee = eexee$, for all $e \in E, x/e \in T$. Hence, $(xe, e) \sim (x, e)$. That is, $(xe)/e = x/e \cdot e/e = x/e$. On the other hand, for all $e \in E, x/e \in T$, we have $ee(ex)ee = eexee$. Thus, $(ex, e) \sim (x, e)$. That is, $(ex)/e = e/e \cdot x/e = x/e$. Therefore, T is a monoid whose identity element is e/e under the multiplication " $\cdot$ ". □

The following theorem shows that the existence of localization of a type B semigroup on its idempotent semilattice.

Theorem 3.2. *Let S be a type B semigroup and E be its idempotent semilattice. Then, there is a localization of S on E.*

Proof. Define a mapping as follows:

$$\phi : S \longrightarrow T = (S \times E)/\sim,\ x \mapsto x/e,$$

where T is a monoid which is constructed in Proposition 3.1(4). It is clear that ϕ is a surjection from S into T. For all $x, y \in S$, we have

$$\phi(xy) = (xy)/e = x/e \cdot y/e = \phi(x) \cdot \phi(y).$$

Hence, ϕ is a surjective homomorphism from S into T. By Proposition 3.1, we have $\phi(f) = f/e = e/e$, for all $f \in E$. Thus, $\phi(f)$ is an identity element of T. This means that $\phi(f)$ is inverse on T.

Suppose that there are a monoid S' and a homomorphism $\alpha : S \to S'$ such that $\alpha(f)$ is inverse on S', for all $f \in E$. Define a mapping as follows:

$$\psi : T = (S \times E)/\sim \longrightarrow S',\ x/e \mapsto \alpha(x).$$

Let $x/e, y/e \in T$ with $x/e = y/e$. Then, there exists $h \in E$ such that $hexeh = heyeh$. Let $f = eh = he \in E$. It follows that $fxf = fyf$. Hence,

$$\alpha(f)\alpha(x)\alpha(f) = \alpha(f)\alpha(y)\alpha(f).$$

Multiplying it on the left and right by $\alpha(f)^{-1}$, we have $\alpha(x) = \alpha(y)$ since $\alpha(f)$ is inverse on S'. Thus, ψ is a well defined. Let $x/e, y/e \in T$. Then,

$$\psi(x/e \cdot y/e) = \psi((xy)/e) = \alpha(xy) = \alpha(x)\alpha(y) = \psi(x/e)\psi(y/e).$$

Hence, ψ is a homomorphism. It is easy to see that $\psi\phi(x) = \psi(x/e) = \alpha(x)$, for all $x \in S$. That is, $\psi\phi = \alpha$. Finally, we prove that ψ is unique. Suppose that there exists a homomorphism $\psi' : T \to S'$ such that $\psi'\phi = \alpha$. Then, for all $x/e \in T$, we have $\psi'(x/e) = \psi'(\phi(x)) = (\psi'\phi)(x) = \alpha(x) = \psi(x/e)$. Thus, $\psi' = \psi$. To sum up, T is a localization of S on E. This completes the proof. □

4. The cancellative monoid homomorphic image of a type B semigroup

In this section, we shall characterize the relations between localizations and the minimum cancellative congruence of a type B semigroup.

By Lemma 2.4, we have the localization T of S on E is unique. we denote the localization T by $S[E^{-1}]$.

Proposition 4.1. *Let S be a type B semigroup and E be its idempotent semilattice. Then, the localization $S[E^{-1}]$ of S on E is cancellative.*

Proof. Let $x/e, y/e, z/e \in S[E^{-1}]$ with $x/e \cdot y/e = x/e \cdot z/e$. Then, $(xy)/e = (xz)/e$. Hence, there exists $h \in E$ such that $hexyeh = hexzeh$, and so

$$\begin{aligned} hexyeh = hexzeh &\Rightarrow (hex)yeh = (hex)zeh \\ &\Rightarrow (hex)^*yeh = (hex)^*zeh \\ &\Rightarrow (hex)^*heye(hex)^*h = (hex)^*heze(hex)^*h. \end{aligned}$$

Thus, $y/e = z/e$ since $(hex)^*h \in E$. This shows that $S[E^{-1}]$ is left cancellative. Dually, $S[E^{-1}]$ is right cancellative. That is, $S[E^{-1}]$ is cancellative. □

Proposition 4.2. *Let S be a type B semigroup and E be its idempotent semilattice. Then, the localization $S[E^{-1}]$ of S on E is the maximum cancellative monoid homomorphic image of S.*

Proof. Let ϕ be a surjective homomorphism from S onto $S[E^{-1}]$ such that $\phi(f)$ is inverse on $S[E^{-1}]$, for all $f \in E$. If S' is the cancellative monoid homomorphic image of S, then there exists a homomorphism $\alpha : S \to S'$. By the definition of localization, there is a unique homomorphism $\psi : S[E^{-1}] \to S$ such that $\psi\phi = \alpha$. Thus, $S[E^{-1}]$ is the maximum cancellative monoid homomorphic image of S. □

Proposition 4.3. *Let S be a type B semigroup and E be its idempotent semilattice, H be a subsemigroup of S. If $E \subseteq H \subseteq RegS$, then there is the localization $S[H^{-1}]$ of S on H with $S[H^{-1}] = S[E^{-1}]$. In particular, $S[(RegS)^{-1}] = S[E^{-1}]$.*

Proof. Since S is a type B semigroup, H is a subsemigroup of S and $E \subseteq H \subseteq RegS$, we have that $x^*\mathcal{L}x\mathcal{R}x^+$, for all $x \in H$. Again, since $S[E^{-1}]$ is the localization of S on E, there exists a surjective homomorphism $\phi : S \to S[E^{-1}]$. Hence,

$$\phi(x)\mathcal{H}(S[E^{-1}])\phi(x^*) = \phi(x^+) = e/e.$$

This means that $\phi(x)$ is inverse on $S[E^{-1}]$. On the other hand, if there are a monoid S' and a homomorphism $\alpha : S \to S'$ such that $\alpha(x)$ is inverse on S', for all $x \in H$, then $\alpha(f)$ is inverse on S', for all $f \in E \subseteq H$. By the definition of localization, there is a unique homomorphism $\psi : S[E^{-1}] \to S'$ such that $\psi\phi = \alpha$. Therefore, $S[E^{-1}]$ is the localization of S on H. That is, $S[H^{-1}] = S[E^{-1}]$.

Note that, E is an idempotent semilattice of S. we have that $RegS$ is a subsemigroup of S. Again, $E \subseteq RegS$. Therefore, $S[(RegS)^{-1}] = S[E^{-1}]$. □

Theorem 4.4. *Let S be a type B semigroup and E be its idempotent semilattice. Then, $S[E^{-1}] = S/\sigma$, where σ is the least cancellative congruence on S.*

Proof. Define a mapping as follows:

$$\varphi : S[E^{-1}] \longrightarrow S/\sigma,\ x/e \mapsto x\sigma.$$

Now, we prove that φ is an isomorphism. Let $x/e, y/e \in S[E^{-1}]$ with $x/e = y/e$. Then, there exists $h \in E$ such that $hexeh = heyeh$. Hence, $fxf = fyf$ for some $f = eh = he \in E$, and so $(x, y) \in \sigma$. That is, $x\sigma = y\sigma$. This means that φ is well defined. Let $x\sigma, y\sigma \in S/\sigma$ with $x\sigma = y\sigma$. Then, there is $g \in E$ such that $gxg = gyg$, and $gexeg = geyeg$. Thus, $x/e = y/e$. Obviously, φ is a surjection. Hence, φ is a bijection from $S[E^{-1}]$ onto S/σ. Finally, we show that φ is a homomorphism. Obviously, for all $x/e, y/e \in S[E^{-1}]$, we have

$$\varphi(x/e \cdot y/e) = \varphi((xy)/e) = (xy)\sigma = x\sigma \cdot y\sigma = \varphi(x/e) \cdot \varphi(y/e).$$

This completes the proof.

□

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On completeness of fuzzy metric spaces

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Abstract. Recently, p-convergence in fuzzy metric spaces, in George and Veeramani's sense, has been explored by Gregori et al. [6]. In this paper, we study consistency of Cauchyness (completeness, respectively) and p-Cauchyness (p-completeness, respectively) in fuzzy metric spaces.

Keywords: fuzzy metric, Cauchy sequence, completeness.

MSC 2020: 54A40, 54D35, 54E50

1. Introduction

Many authors have defined several concepts of fuzzy metric space in different ways [3, 4, 11, 12]. In particular, to make the topology induced by a fuzzy metric to be Hausdorff, George and Veeramani [4] gave the concept of fuzzy metric space with the help of continuous t-norms. Later, Gregori and Romaguera [10] proved that the topological space induced by a fuzzy metric is metrizable. In [13], Mihet introduced the concept of p-convergence in fuzzy metric spaces. Whereafter, some authors studied some aspects relative to p-convergence, p-Cauchy sequence and p-completeness in fuzzy metric spaces in [1, 6, 7, 8]. Specifically, Gregori et al. [6] posed an open problem of characterizing consistency of Cauchyness (completeness, respectively) and p-Cauchyness (p-completeness, respectively) in fuzzy metric spaces. Here, we will study those fuzzy metric spaces, that we call k-unequal, in which the family of p-Cauchy sequences and Cauchy sequences agree, moreover, completeness and p-completeness coincide.

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2. Preliminaries

From now on, $\mathbb{N}$ shall denote the set of positive integer numbers . Our basic reference for general topology is [2].

Definition 2.1 ([4]). *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

(i) $*$ *is associative and commutative;*

(ii) $*$ *is continuous;*

(iii) $a * 1 = a$, *for all* $a \in [0,1]$*;*

(iv) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0,1]$.

Observe that $a * b = \min\{a, b\}$ and $a * b = a \cdot b$ are two common examples of continuous t-norms.

Definition 2.2 ([4]). *An ordered triple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is a nonempty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X \times X \times (0, +\infty)$ *satisfying the following conditions, for all* $x, y, z \in X$ *and* $s, t \in (0, +\infty)$*:*

(i) $M(x, y, t) > 0$*;*

(ii) $M(x, y, t) = 1$ *if and only if* $x = y$*;*

(iii) $M(x, y, t) = M(y, x, t)$*;*

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*;*

(v) *the function* $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ *is continuous.*

If $(X, M, *)$ is a fuzzy metric space, then we will call $(M, *)$, or simply M, a *fuzzy metric on* X*.*

Definition 2.3 ([4]). *Let* $(X, M, *)$ *be a fuzzy metric space and let* $r \in (0, 1), t > 0$ *and* $x \in X$. *The set*

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the open ball with center x *and radius* r *with respect to* t.

George and Veeramani [4] proved that $\{B_M(x, r, t) | x \in X, t > 0, r \in (0, 1)\}$ forms a base of a topology τ_M in X.

Proposition 2.1 ([4]). *Let* $(X, M, *)$ *be a fuzzy metric space. A sequence* $\{x_n\}$ *in* X *converges to* $x_0 \in X$ *if and only if* $\lim_n M(x_n, x_0, t) = 1$, *for all* $t > 0$.

Definition 2.4 ([4]). *Let (X,d) be a metric space. Define $a*b = a\cdot b$, for all $a,b\in[0,1]$, and let M_d be the real value mapping on $X\times X\times(0,+\infty)$ defined by*

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}.$$

Then, $(X,M_d,\cdot)$ is a fuzzy metric space and $(M_d,\cdot)$ is called the standard fuzzy metric induced by d.

Definition 2.5 ([5]). *Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called Cauchy if for each $r\in(0,1)$ and each $t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-r$, for all $n,m\geq n_0$. X is called complete if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is called complete.*

Definition 2.6 ([9]). *A fuzzy metric M on X is said to be stationary, if M does not depend on t, i.e. if, for all $x,y\in X$ and t, $M(x,y,t)$ is constant. In this case we write $M(x,y)$ instead of $M(x,y,t)$.*

Definition 2.7 ([6]). *We say that the fuzzy metric space $(X,M,*)$ is principal (or simply, M is principal) if $\{B_M(x,r,t)|r\in(0,1)\}$ is local base at $x\in X$, for each $x\in X$ and each $t>0$.*

Definition 2.8 ([6, 13]). *Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be point convergent to $x_0\in X$ if $\lim_n M(x_n,x_0,t_0)=1$ for some $t_0>0$. In such a case we say that $\{x_n\}$ is p-convergent to x_0 for $t_0>0$, or, simply, $\{x_n\}$ is p-convergent.*

Remark 2.1 ([6, 13]). Clearly, $\{x_n\}$ is convergent to $x_0\in X$ if and only if $\{x_n\}$ is p-convergent to x_0, for all $t>0$.

Definition 2.9 ([6]). *Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be p-Cauchy if for each $r\in(0,1)$, there are $n_0\in\mathbb{N}$ and $t_0>0$ such that $M(x_n,x_m,t_0)>1-r$, for all $n,m\geq n_0$, i.e. $\lim_{m,n} M(x_n,x_0,t_0)=1$ for some $t_0>0$. In such a case we say that $\{x_n\}$ is p-Cauchy for $t_0>0$, or, simply, $\{x_n\}$ is p-Cauchy.*

Remark 2.2 ([6]). It is not hard to see that $\{x_n\}$ is a Cauchy sequence if and only if $\{x_n\}$ is p-Cauchy, for all $t>0$ and, obviously, p-convergent sequences are p-Cauchy.

Definition 2.10 ([6]). *A fuzzy metric space $(X,M,*)$ is called p-complete if every p-Cauchy sequence in X is p-convergent to some point of X. In such a case M is called p-complete.*

3. Main results

We start this section with the following definition.

Definition 3.1. *A fuzzy metric space $(X, M, *)$ is said to be k-unequal if $k(1 - M(x, y, kt)) \geq 1 - M(x, y, t)$ whenever $x, y \in X, t > 0$ and $k > 1$. In such a case M is called k-unequal.*

Now, we recall several examples, which were given in [6].

Example 3.1. (a) A stationary fuzzy metric M_1 is principal.

(b) The well-known standard fuzzy metric M_2 is principal.

(c) $M_3(x, y, t) = e^{-\frac{d(x,y)}{t}}$, where d is a metric on X, is principal.

(d) $M_4(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ is a fuzzy metric on $(0, +\infty)$, which is principal.

Example 3.2. Consider the above examples. It is an easy exercise to verify that M_1, M_2 and M_4 are all k-unequal. Now, we only prove that M_3 is also k-unequal. If $x = y$, then it is clear that $k(1 - M_3(x, y, kt)) = 1 - M_3(x, y, t)$. Let $x \neq y$ and $k > 1$. Suppose that $k(1 - M_3(x, y, kt)) < 1 - M_3(x, y, t)$, namely $k(1 - e^{-\frac{d(x,y)}{kt}}) < 1 - e^{-\frac{d(x,y)}{t}}$. Then, $k \cdot \frac{e^{\frac{d(x,y)}{kt}} - 1}{e^{\frac{d(x,y)}{kt}}} < \frac{e^{\frac{d(x,y)}{t}} - 1}{e^{\frac{d(x,y)}{t}}}$, which means that $ke^{\frac{d(x,y)}{t}}(e^{\frac{d(x,y)}{kt}} - 1) < e^{\frac{d(x,y)}{kt}}(e^{\frac{d(x,y)}{t}} - 1)$. Notice that $e^{\frac{d(x,y)}{kt}} > e^0 = 1$. We deduce that $ke^{\frac{d(x,y)}{t}} - ke^{\frac{(k-1)d(x,y)}{kt}} < e^{\frac{d(x,y)}{t}} - 1$, that is $(k-1)z^k - kz^{k-1} + 1 < 0$, where $z = e^{\frac{d(x,y)}{kt}}$. Set $f(z) = (k-1)z^k - kz^{k-1} + 1$, $z \in (1, +\infty)$. Then, $f(z) < 0$, for all $z > 1$. Since $f'(z) = (k-1)kz^{k-1} - k(k-1)z^{k-2} = (k-1)kz^{k-2}(z-1) > 0$, for all $z > 1$, we conclude that f is a strictly increasing function on $(1, +\infty)$. Note that, f is a continuous function on $[1, +\infty)$. We get that $f(z) > f(1) = 0$, for all $z > 1$, which is a contradiction. So, M_3 is k-unequal.

Theorem 3.1. *Let $(X, M, *)$ be a fuzzy metric space. If M is k-unequal, then M is principal.*

Proof. Let $B_M(x, \varepsilon, s)$ be an open ball with center x and radius ε with respect to s, where $x \in X$, $\varepsilon \in (0, 1)$ and $s > 0$. Put $t > 0$. In case $0 < t < s$. Take $r = \varepsilon$. Then, $x \in B_M(x, r, t) \subseteq B_M(x, \varepsilon, s)$. In case $t \geq s$. Then, $\frac{t}{s} \geq 1$. Hence, there exists $r = \frac{\varepsilon s}{2t}$ such that $x \in B_M(x, r, t) \subseteq B_M(x, \varepsilon, s)$. In fact, let $y \in B_M(x, r, t)$. Since M is k-unequal, we have that

$$\frac{s}{t}(M(x, y, s) - 1) + 1 \geq M(x, y, \frac{t}{s} \cdot s) = M(x, y, t) > 1 - r.$$

Thus

$$M(x, y, s) > 1 - \frac{rt}{s} = 1 - \frac{\varepsilon}{2} > 1 - \varepsilon,$$

which follows that $y \in B_M(x, \varepsilon, s)$. Consequently, M is principal. □

The converse of the preceding theorem is not true, in general. We illustrate this fact with the next example.

Example 3.3. Let $X = (0, 1)$. Denote $a * b = a \cdot b$, for all $a, b \in [0, 1]$. Define the function M on $X \times X \times (0, +\infty)$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ xyt, & x \neq y, t \leq 1, \\ xy, & x \neq y, t > 1. \end{cases}$$

Then, $(X, M, *)$ is a principal fuzzy metric space (see [6]). Choose $x_0 = 0.95, y_0 = 0.96, t_0 = 0.875$ and $k_0 = 2$. Then

$$k_0(1-M(x_0, y_0, k_0t_0)) = 2(1-M(0.95, 0.96, 2\cdot 0.875)) = 2(1-0.95\cdot 0.96) = 0.176,$$

and

$$1 - M(x_0, y_0, t_0) = 1 - M(0.95, 0.96, 0.875) = 1 - 0.95 \cdot 0.96 \cdot 0.875 = 0.202.$$

So, $k_0(1 - M(x_0, y_0, k_0t_0)) < 1 - M(x_0, y_0, t_0)$, which means that M is not k-unequal.

Due to Example 3.2 and Theorem 3.1, the following chain of implications is fulfilled obviously.

$$\text{stationary} \Rightarrow k\text{-unequal} \Rightarrow \text{principal}$$

At the end of paper [6], Gregori et al. posed an open problem of characterizing those fuzzy metric spaces where the family of p-Cauchy sequences and Cauchy sequences agree, or further, when it is satisfied that completeness is equivalent to p-completeness.

Next, we will solve the above open problem by the following results.

Theorem 3.2. *Let $\{x_n\}$ be a sequence in a k-unequal fuzzy metric space $(X, M, *)$. Then, $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is p-Cauchy.*

Proof. Suppose that $\{x_n\}$ is Cauchy. Then, by Remark 2.2 we deduce that $\{x_n\}$ is p-Cauchy.

Conversely, suppose that $\{x_n\}$ is p-Cauchy for $t_0 > 0$. Let $\varepsilon \in (0, 1)$ and $t > 0$. Pick $\varepsilon_1 = \min\{\frac{t\varepsilon}{t_0}, \varepsilon\}$. Then, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t_0) > 1 - \varepsilon_1$, for all $n, m \geq n_0$. If $t > t_0$, then

$$M(x_n, x_m, t) \geq M(x_n, x_m, t_0) > 1 - \varepsilon_1 \geq 1 - \varepsilon.$$

If $0 < t \leq t_0$, then $\frac{t_0}{t} \geq 1$. Since M is k-unequal, we obtain that

$$\frac{t}{t_0}(M(x_n, x_m, t) - 1) + 1 \geq M(x_n, x_m, \frac{t_0}{t} \cdot t) = M(x_n, x_m, t_0) > 1 - \varepsilon_1.$$

It follows that

$$M(x_n, x_m, t) > 1 - \frac{t_0}{t} \cdot \varepsilon_1 \geq 1 - \frac{t_0}{t} \cdot \frac{t\varepsilon}{t_0} = 1 - \varepsilon.$$

So, $\{x_n\}$ is Cauchy. The proof is finished. □

Proposition 3.1 ([6]). *Let $(X, M, *)$ be a principal fuzzy metric space. If X is p-complete, then X is complete.*

It was shown in [6] that the converse of the above proposition is false, in general. Nevertheless, the next proposition can be obtained.

Proposition 3.2. *Let $(X, M, *)$ be a k-unequal fuzzy metric space. If X is complete, then X is p-complete.*

Proof. Let $\{x_n\}$ be a p-Cauchy sequence. According to Theorem 3.2, we have that $\{x_n\}$ is Cauchy. Hence, $\{x_n\}$ converges to some point $x_0 \in X$. Due to Remark 2.1, we obtain that $\{x_n\}$ is p-convergent to x_0. We are done. □

With Theorem 3.1, Proposition 3.1 and Proposition 3.2, we get the next corollary.

Corollary 3.1. *Let $(X, M, *)$ be a k-unequal fuzzy metric space. Then, X is complete if and only if X is p-complete.*

Since stationary fuzzy metric does not depend on t, obviously "p-Cauchy sequences and Cauchy sequences" and also "p-completeness and completeness" are equivalent concepts in stationary fuzzy metrics.

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Generalization of fuzzy Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions

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Abstract. In the present paper, we present the very 1st time the generalized notion of $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function in mixed kind, which is the generalization of 22 functions, which are presented in sequel manner.Our aim is to establish generalized Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions via Fuzzy Riemann Integrals by applying several techniques in which power mean inequality and Hölder's inequality are included. Moreover, we would obtain various results with respect to the convexity of function as special cases and also recapture several established results of different authors of different papers.
Keywords: Ostrowski inequality, convex functions, fuzzy sets, fuzzy Riemann integral, power mean inequality, Hölder's inequality.
MSC 2020: 26A51, 47A30, 33B15, 26D15, 26D20, 03E72

1. Introduction and definitions

About the features of convex functions, we code some lines from [18] "Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non-linear programming problems. The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex

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set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test".

The theory of convex functions is a crucial area of mathematics that has applications in a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. This theory is also highly relevant in other areas of science, such as economics, finance, engineering, and management sciences.

The importance of convex functions for the generalization of integral inequalities due to the variety of their nature the notion have been established. Integral inequalities are satisfied by many convex functions. Among these, the well known is Ostrowski inequality. To generalize the Ostrowski's inequality, we need to generalize the concept of convex functions, in this way we can easily see the generalizations and its particular cases. From the literature, we remind few definitions for various convex(concave) functions [2].

Definition 1.1. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as convex(concave), if

$$g(ty + (1-t)z) \leq (\geq) tg(y) + (1-t)g(z), \tag{1.1}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Here we remind definition of P-convex(concave) function see [5].

Definition 1.2. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as P-convex(concave), if function g is a non-negative, then we have

$$g(ty + (1-t)z) \leq (\geq) g(y) + g(z), \tag{1.2}$$

$\forall\ y, z \in K, t \in [0, 1]$.

The definition of quasi-convex function is extracted from [9].

Definition 1.3. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is called a quasi-convex(concave), if

$$g(ty + (1-t)z) \leq (\geq) \max\{g(y), g(z)\} \tag{1.3}$$

$\forall\ y, z \in K, t \in [0, 1]$.

We present definition of s-convex(concave) functions in the 1st kind as follows (see [16]).

Definition 1.4. Suppose $s \in (0, 1]$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as s-convex(concave) in the 1st kind, if

$$g(ty + (1-t)z) \leq (\geq) t^s g(y) + (1 - t^s) g(z), \tag{1.4}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Remark 1.5. Note that in this definition we also included $s = 0$. Further if we put $s = 0$, we get quasi-convexity (see Definition 1.3).

We also present definition of s-convex(concave) functions in the second kind from [16].

Definition 1.6. Suppose $s \in (0, 1]$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as s-convex(concave) in the 2nd kind, if

$$g(ty + (1 - t)z) \leq (\geq) t^s g(y) + (1 - t)^s g(z), \tag{1.5}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Remark 1.7. In the similar manner, we have slightly improved definition of 2nd kind convexity by including $s = 0$. Further if we put $s = 0$, we easily get P-convexity (see Definition 1.2).

The following definition of m-convex(concave) function is extracted from [10]

Definition 1.8. Suppose $m \in [0, 1]$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as m-convex (concave), if

$$g(ty + m(1 - t)z) \leq (\geq) tg(y) + m(1 - t)g(z) \tag{1.6}$$

$\forall\ y, z \in [0, \infty)$, $t \in [0, 1]$.

Remark 1.9. For $m = 1$ the above definition recaptures the concept of standard convex(concave) functions in the interval K and for $m = 0$ the concept star-shaped functions.

Following definition is extracted from [10]

Definition 1.10. Let $(m_1, m_2) \in (0, 1]^2$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as (m_1, m_2)-convex(concave), if

$$g(m_1 ty + m_2(1 - t)z) \leq (\geq) m_1 tg(y) + m_2(1 - t)g(z), \tag{1.7}$$

$\forall\ y, z \in K, t \in [0, 1]$.

In [15], Mihesan stated (α, m)-convexity as in the following:

Definition 1.11. Suppose $(\alpha, m) \in [0, 1]^2$. A function $g : [0, \infty) \to \mathbb{R}$ is known as (α, m)-convex(concave), if

$$g(ty + m(1 - t)z) \leq (\geq) t^\alpha g(y) + m(1 - t^\alpha)g(z) \tag{1.8}$$

$\forall\ y, z \in [0, \infty)$, $t \in [0, 1]$. Above function can also be written as (m, s)-convex(concave) function in the 1st kind.

Firstly, we introduce a new class of (m, s)-convex(concave) function in the 2nd kind that is given below:

Definition 1.12. Let $(m, s) \in (0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, s)-convex(concave) in the 2nd kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^s g(y) + m(1-t)^s g(z) \tag{1.9}$$

$\forall\ y, z \in K,\ t \in [0, 1]$.

A new class of (s, r)-convex(concave) functions in the mixed kind is extracted from [7].

Definition 1.13. Suppose $(s, r) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (s, r)-convex(concave) in the mixed kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{rs} g(y) + (1-t^r)^s g(z), \tag{1.10}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Definition 1.14 ([6])**.** Suppose $(\alpha, \beta) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (α, β)-convex(concave) in the 1st kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha} g(y) + (1-t^{\beta}) g(z), \tag{1.11}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Definition 1.15 ([6])**.** Suppose $(\alpha, \beta) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (α, β)-convex(concave) in the 2nd kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha} g(y) + (1-t)^{\beta} g(z), \tag{1.12}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Secondly, we introduce a new class of (m, s, r)-convex(concave) functions in mixed kind which is given below:

Definition 1.16. Let $(m, s, r) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, s, r)-convex(concave) in the mixed kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{rs} g(y) + m(1-t^r)^s g(z), \tag{1.13}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Thirdly, we introduce a new class of (m, α, β)-convex(concave) functions in the 1st kind which is given below:

Definition 1.17. Let $(m, \alpha, \beta) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, α, β)-convex(concave) in the 1st kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha} g(y) + m(1-t^{\beta}) g(z), \tag{1.14}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Fourthly, we introduce a new class of (m, α, β)-convex(concave) functions in the 2nd kind which is given below:

Definition 1.18. Let $(m, \alpha, \beta) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, α, β)-convex(concave) in the 2nd kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha} g(y) + m(1-t)^{\beta} g(z), \tag{1.15}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Following definition is extracted from [10]

Definition 1.19. Let $(\alpha, m_1, m_2) \in (0, 1]^3$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as (α, m_1, m_2)-convex(concave), if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha} g(y) + m_2(1-t^{\alpha}) g(z), \tag{1.16}$$

$\forall\ y, z \in K, t \in [0, 1]$. Above function can also be written as (m_1, m_2, s)-convex(concave) function in the 1st kind.

Fifthly, we introduce a new class of (m_1, m_2, s)-convex(concave) functions in the 2nd kind which is given below:

Definition 1.20. Let $(m_1, m_2, s) \in (0, 1]^3$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as (m_1, m_2, s)-convex(concave) in the 2nd kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{s} g(y) + m_2(1-t)^{s} g(z), \tag{1.17}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Sixthly, we introduce a new class of (m_1, m_2, s, r)-convex(concave) functions in mixed kind which is given below:

Definition 1.21. Let $(m_1, m_2, s, r) \in (0, 1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m_1, m_2, s, r)-convex(concave) in the mixed kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{rs} g(y) + m_2(1-t^{r})^{s} g(z), \tag{1.18}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Seventhly, we introduce a new class of $(m_1, m_2, \alpha, \beta)$-convex(concave) functions in the 1st kind which is given below:

Definition 1.22. Let $(m_1, m_2, \alpha, \beta) \in (0, 1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m_1, m_2, \alpha, \beta)$-convex(concave) in the 1st kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha} g(y) + m_2(1-t^{\beta}) g(z), \tag{1.19}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Eighthly, we introduce a new class of $(m_1, m_2, \alpha, \beta)$-convex(concave) functions in the 2nd kind which is given below:

Definition 1.23. Suppose $(m_1, m_2, \alpha, \beta) \in (0,1]^4$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m_1, m_2, \alpha, \beta)$-convex(concave) in the 2nd kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha} g(y) + m_2(1-t)^{\beta} g(z), \tag{1.20}$$

$\forall\ y, z \in K, t \in [0,1]$.

Upcoming definition is $(\alpha, \beta, \gamma, \mu)$-convex(concave) function which is extracted from [7].

Definition 1.24. Let $(\alpha, \beta, \gamma, \mu) \in [0,1]^4$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(\alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha\gamma} g(y) + (1-t^{\beta})^{\mu} g(z), \tag{1.21}$$

$\forall\ y, z \in K, t \in [0,1]$.

Ninthly, we introduce a new class of $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) functions in mixed kind that is given below:

Definition 1.25. Let $(m, \alpha, \beta, \gamma, \mu) \in [0,1]^5$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha\gamma} g(y) + m(1-t^{\beta})^{\mu} g(z), \tag{1.22}$$

$\forall\ y, z \in K, t \in [0,1]$.

Tenthly and Finally we introduce a new class of function which would be called class of $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex(concave) function in mixed kind and containing all above classes of functions. This definition is used sequentially in this paper.

Definition 1.26. Let $(m_1, m_2, \alpha, \beta, \gamma, \mu) \in (0,1]^6$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha\gamma} g(y) + m_2(1-t^{\beta})^{\mu} g(z), \tag{1.23}$$

$\forall\ y, z \in K, t \in [0,1]$.

Remark 1.27. In Definition 1.26, we have the following cases.

(i) If we choose $m_1 = 1, m_2 = m$ in (1.23), we get $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) function in the mixed kind.

(ii) If we choose $m_1 = m_2 = 1$ in (1.23), we get $(\alpha, \beta, \gamma, \mu)$-convex(concave) function in the mixed kind.

(iii) If we choose $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get $(m_1, m_2, \alpha, \beta)$-convex(concave) function in the 2nd kind.

(iv) If we choose $\gamma = \mu = 1$ in (1.23), we get $(m_1, m_2, \alpha, \beta)$-convex(concave) function in the 1st kind.

(v) If we choose $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get (m_1, m_2, s, r)-convex(concave) function in mixed kind.

(vi) If we choose $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get (m_1, m_2, s)-convex(concave) function in the 2nd kind.

(vii) If we choose $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get (m_1, m_2, s)-convex(concave) function in the 1st kind.

(viii) If we choose $m_1 = 1$, $m_2 = m$, $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get (m, α, β)-convex(concave) function in the 2nd kind.

(ix) If we choose $m_1 = 1$, $m_2 = m$ and $\gamma = \mu = 1$ in (1.23), we get (m, α, β)-convex(concave) function in the 1st kind.

(x) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get (m, s, r)-convex(concave) function in the mixed kind.

(xi) If we choose $m_1 = m_2 = 1$, $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get (α, β)-convex(concave) function in the 2nd kind.

(xii) If we choose $m_1 = m_2 = 1$ and $\gamma = \mu = 1$ in (1.23), we get (α, β)-convex(concave) function in the 1st kind.

(xiii) If we choose $m_1 = m_2 = 1$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get (s, r)-convex(concave) function in the mixed kind.

(xiv) If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get (m, s)-convex(concave) function in the 2nd kind.

(xv) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get (m, s)-convex(concave) function in the 1st kind.

(xvi) If we choose $\alpha = \beta = \gamma = \mu = 1$ in (1.23), we get (m_1, m_2)-convex(concave) function.

(xvii) If we choose $m_1 = 1$, $m_2 = m$ and $\alpha = \beta = \gamma = \mu = 1$ in (1.23), we get m-convex(concave) function.

(xviii) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get s-convex(concave) function in the 2nd kind.

(xix) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = s$ and $\gamma = \mu = 1$ in (1.23), we get s-convex(concave) function in the 1st kind.

(xx) If we choose $m_1 = m_2 = 1$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get s-convex(concave) function in the 1st kind.

(xxi) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = 0$, and $\gamma = \mu = 1$ in (1.23), we get quasi-convex(concave) function.

(xxii) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in (1.23), we get P-convex(concave) function.

(xxiii) If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in (1.23), gives us ordinary convex(concave) function.

In almost each field of science, inequalities act an important role. Although it is very vast discipline but our focus is mainly on Ostrowski's like inequalities.

In 1938, [17] Ostrowski proved the below interesting inequality for differentiable mappings with bounded derivatives. It is well known in the literature as Ostrowski inequality.

Proposition 1.28. *Suppose $g : K \to \mathbb{R}$ is a differentiable mapping in the interior K^o of K, where $j, k \in K^o$ with $j < k$. If $|g'(y)| \leq \mathfrak{M}\ \forall\ y \in [j,k]$ where $\mathfrak{M} > 0$ is constant. Then*

$$\left| g(y) - \frac{1}{k-j}\int_j^k g(\tau)d\tau \right| \leq \mathfrak{M}(k-j)\left[\frac{1}{4} + \frac{\left(y - \frac{j+k}{2}\right)^2}{(k-j)^2}\right]. \tag{1.24}$$

The value $\frac{1}{4}$ is the best possible constant that this can not be replaced by the smallest one.

Since fuzziness is a natural reality different than randomness and determinism, Anastassiou extends Ostrowski like inequalities into the fuzzy setting in 2003 [1]. Congxin and Ming [3] introduced the concepts of fuzzy Riemann integrals. Fuzzy Riemann integral is a closed interval whose end points are the classical Riemann integrals.

2. Preliminaries with notations

Under this heading, we remind few basic definitions and notations that would help us in the sequel manner.

Definition 2.1 ([3]). *$\rho : \mathbb{R} \to [0,1]$ is called a fuzzy number if satisfies the below properties*

1. *ρ is normal (i.e, there exists an $y_0 \in \mathbb{R}$ such that $\rho(y_0) = 1$).*
2. *ρ is a convex fuzzy set, i.e., $yt + (1-t)z) \geq \min\{\rho(y), \rho(z)\}$, $\forall\ y, z \in \mathbb{R}$, $t \in [0,1]$ (ρ is called a convex fuzzy subset).*

3. *ρ is upper semi continuous on $\mathbb{R}$, i.e., $\forall\, y_0 \in \mathbb{R}$ and $\forall\, \epsilon > 0$, $\exists$ neighborhood $V(y_0) : \rho(y) \leq \rho(y_0) + \epsilon$, $\forall\, y \in V(y_0)$.*
4. *The set $[\rho]^0 = \overline{\{y \in \mathbb{R} : \rho(y) > 0\}}$ is compact where $\overline{A}$ denotes the closure of A.*

$\mathbb{R}^F$ denotes the set of all fuzzy numbers. For $\alpha \in (0,1]$ and $\rho \in \mathbb{R}^F$, $[\rho]^\alpha = \{y \in \mathbb{R} : \rho(y) \geq \alpha\}$. Then, from (1) to (4) it follows that the α-level set $[\rho]^\alpha$ is a closed interval $\forall\, \alpha \in [0,1]$. Moreover, $[\rho]^\alpha = [\rho_-^{(\alpha)}, \rho_+^{(\alpha)}]$ $\forall\, \alpha \in [0,1]$, where $\rho_-^{(\alpha)} \leq \rho_+^{(\alpha)}$ and $\rho_-^{(\alpha)}, \rho_+^{(\alpha)} \in \mathbb{R}$, i.e., $\rho_-^{(\alpha)}$ and $\rho_+^{(\alpha)}$ are the endpoints of $[\rho]^\alpha$.

Definition 2.2 ([4])**.** *Let $\rho, \varrho \in \mathbb{R}^F$ and $a \in \mathbb{R}$. Then, the addition and scalar multiplication are defined by the equations, respectively.*

1. *$[\rho \oplus \varrho]^\alpha = [\rho]^\alpha + [\varrho]^\alpha$;*
2. *$[a \odot \rho]^\alpha = a[\rho]^\alpha$;*

$\forall\, \alpha \in [0,1]$ where $[\rho]^\alpha + [\varrho]^\alpha$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $a[\rho]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Proposition 2.1 ([11])**.** *Let $\rho, \varrho \in \mathbb{R}^F$ and $a \in \mathbb{R}$. Then, the given properties holds:*

1. *$1 \odot \rho = \rho$.*
2. *$\rho \oplus \varrho = \varrho \oplus \rho$.*
3. *$a \odot \rho = \rho \odot a$.*
4. *$[\rho]^{\alpha_1} \subseteq [\rho]^{\alpha_2}$ whenever $0 \leq \alpha_2 \leq \alpha_1 \leq 1$.*
5. *For any α_n converging increasingly to $\alpha \in (0,1]$, $\bigcap_{n=1}^{\infty} [\rho]^{\alpha_n} = [\rho]^\alpha$.*

Definition 2.3 ([3])**.** *Let $D : \mathbb{R}^F \times \mathbb{R}^F \to \mathbb{R}_+ \cup \{0\}$ be a function, defined as*

$$D(\rho, \varrho) = \sup_{\alpha \in [0,1]} \max \left\{ \left| \rho_-^{(\alpha)}, \varrho_-^{(\alpha)} \right|, \left| \rho_+^{(\alpha)}, \varrho_+^{(\alpha)} \right| \right\}$$

$\forall\, \rho, \varrho \in \mathbb{R}^F$, Then, D is metric on $\mathbb{R}^F$.

Proposition 2.2 ([3])**.** *Let $\rho, \varrho, \sigma, e \in \mathbb{R}^F$ and $a \in \mathbb{R}$, we have*

1. *$(\mathbb{R}^F, D)$ is a complete metric space.*
2. *$D(\rho \oplus \sigma, \varrho \oplus \sigma) = D(\rho, \varrho)$.*
3. *$D(a \odot \rho, a \odot \varrho) = |a| D(\rho, \varrho)$.*
4. *$D(\rho \oplus \varrho, \sigma \oplus e) = D(\rho, \sigma) + D(\varrho, e)$.*

5. $D(\rho \oplus \varrho, \widetilde{0}) \leq D(\rho, \widetilde{0}) + D(\varrho, \widetilde{0})$.

6. $D(\rho \oplus \varrho, \sigma) \leq D(\rho, \sigma) + D(\varrho, \widetilde{0})$,

where $\widetilde{0} \in \mathbb{R}^F$ *is stated as* $\widetilde{0}(y) = 0 \ \forall \ y \in \mathbb{R}$.

Definition 2.4 ([4])**.** *Let* $y, z \in \mathbb{R}^F$ *if* $\exists \ \theta \in \mathbb{R}^F$ *such that* $y = z \oplus \theta$, *then* θ *is* H*-difference of* y *and* z *denoted by* $\theta = y \ominus z$.

Definition 2.5 ([4])**.** *Let* $T := [y_0, y_0 + \gamma] \subseteq \mathbb{R}$, *with* $\gamma > 0$. *A function* $g : T \to \mathbb{R}^F$ *is* H*-differentiable at* $y \in T$ *if* $\exists \ g'(y) \in \mathbb{R}^F$ *i.e., both limits (with respect to the metric* D*)*

$$\lim_{h \to 0^+} \frac{g(y+h) \ominus g(y)}{h}, \quad \lim_{h \to 0^+} \frac{g(y) \ominus g(y-h)}{h}$$

exists and are equal to $g'(y)$. *We call* g' *the derivative or* H*-derivative of* g *at* y. *If* g *is* H*-differentiable at any* $y \in T$, *we call* g *differentiable or* H*-differentiable and it has* H*-derivative over* T *the function* g'.

Definition 2.6 ([8])**.** *Let* $g : [j, k] \to \mathbb{R}^F$ *if* $\forall \ \zeta > 0$, $\exists \ \eta > 0$, *for any partition* $P = \{[\rho, \varrho]; \delta\}$ *of* $[j, k]$ *with norm* $\Delta(P) < \eta$, *we have*

$$D\left(\sum_{P}^{*} (\varrho - \rho) \odot g(\delta, K)\right) < \zeta,$$

then we say that g *is Fuzzy-Riemann integrable to the interval* $K \in \mathbb{R}^F$, *we write it as*

$$K := (FR) \int_j^k g(y)dy.$$

For some recent results connected with Fuzzy-Riemann integrals (see [12, 13]).

The main purpose of this paper is to establish generalized fuzzy Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function in mixed kind and we obtain various results with respect to the convexity of function as special cases and also recapture several previous established results of different authors of different papers [19] and [14].

3. Generalized fuzzy Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions

Regarding to prove our main results, we require the below Lemma.

Lemma 3.1. *Let $g : K \subset \mathbb{R} \to \mathbb{R}^F$ be differentiable mapping on K^o where $m_1, m_2j, m_2k \in K$ with $m_2j < m_2k$. If $g' \in C^F[m_2j, m_2k] \cap L^F[m_2j, m_2k]$, then*

$$\begin{aligned}
&\frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du \\
&\oplus \frac{(m_1y - m_2j)^2}{k-j} \odot (FR) \int_0^1 t \odot g'(m_1ty + m_2(1-t)j)dt \\
&= m_2 \odot g(m_1y) \oplus \frac{(m_2k - m_1y)^2}{k-j} \odot (FR) \int_0^1 t \odot g'(m_1ty + m_2(1-t)k)dt, \qquad (3.1)
\end{aligned}$$

$\forall\ y \in (m_2j, m_2k)$.

Proof. We obtain the required result by using similar techniques of proof of Lemma 3.1 of [19]. □

Remark 3.1. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$, we recapture Lemma 3.1 of [19].

Remark 3.2. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.1, we recapture Lemma 3.1 of [14].

Theorem 3.1. *Let all the suppositions of Lemma 3.1 be true and assuming that $D(g'(y), \widetilde{0})$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function on $[m_2j, m_2k]$ and $D(g'(y), \widetilde{0}) \leq M$. Then*

$$\begin{aligned}
&D\left(m_2 \odot g(m_1y), \frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du\right) \\
&\leq M\left(\frac{m_1}{\alpha\gamma + 2} + \frac{m_2}{\beta} B\left(\frac{2}{\beta}, \mu + 1\right)\right) I(y), \qquad (3.2)
\end{aligned}$$

$\forall\ y \in (m_2j, m_2k)$ and $\beta > 0$, where $I(y) = \frac{(m_1y - m_2j)^2 + (m_2k - m_1y)^2}{k-j}$.

Proof. From the Lemma 3.1 and using Proposition 2.2, then we have

$$\begin{aligned}
&D\left(m_2 \odot g(m_1y), \frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du\right) \\
&\leq D\left(\frac{(m_1y - m_2j)^2}{k-j} \odot (FR) \int_0^1 tg'(m_1ty + m_2(1-t)j)dt, \widetilde{0}\right) \\
&+ D\left(\frac{(m_2k - m_1y)^2}{k-j} \odot (FR) \int_0^1 tg'(m_1ty + m_2(1-t)k)dt, \widetilde{0}\right) \\
&= \frac{(m_1y - m_2j)^2}{k-j} D\left((FR) \int_0^1 tg'(m_1ty + m_2(1-t)j)dt, \widetilde{0}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(m_2k-m_1y)^2}{k-j}D\Bigg((FR)\int_0^1 tg'(m_1ty+m_2(1-t)k)dt,\widetilde{0}\Bigg)\\
&\leq \frac{(m_1y-m_2j)^2}{k-j}\int_0^1 tD\Big(g'(m_1ty+m_2(1-t)j),\widetilde{0}\Big)dt\\
&+\frac{(m_2k-m_1y)^2}{k-j}\int_0^1 tD\Big(g'(m_1ty+m_2(1-t)k),\widetilde{0}\Big)dt. \qquad (3.3)
\end{aligned}
$$

Since $D(g'(y),\widetilde{0})$ be $(m_1,m_2,\alpha,\beta,\gamma,\mu)$-convex function & $D(g'(y),\widetilde{0})\leq M$, we have

$$
\begin{aligned}
D\left(g'(m_1ty+m_2(1-t)j),\widetilde{0}\right) &\leq m_1t^{\alpha\gamma}D\left(g'(y),\widetilde{0}\right)+m_2\left(1-t^{\beta}\right)^{\mu}D\left(g'(j),\widetilde{0}\right)\\
&\leq M\left[m_1t^{\alpha\gamma}+m_2\left(1-t^{\beta}\right)^{\mu}\right], \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
D\left(g'(m_1ty+m_2(1-t)k),\widetilde{0}\right) &\leq m_1t^{\alpha\gamma}D\left(g'(y),\widetilde{0}\right)+m_2\left(1-t^{\beta}\right)^{\mu}D\left(g'(k),\widetilde{0}\right)\\
&\leq M\left[m_1t^{\alpha\gamma}+m_2\left(1-t^{\beta}\right)^{\mu}\right]. \qquad (3.5)
\end{aligned}
$$

Now, using (3.4) and (3.5) in (3.3) we get (3.2). □

Note. Where B is Beta function and it is stated as $B(l,m)=\int_0^1 t^{l-1}(1-t)^{m-1}dt=\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$. Since $\Gamma(l)=\int_0^{\infty}e^{-u}u^{l-1}du$.

Remark 3.3. Some remarks about Theorem 3.1 are following as special cases:

(i) If we choose $m_1=1$, $m_2=m$ in Theorem 3.1, we can get inequality for $(m,\alpha,\beta,\gamma,\mu)$-convex function in the mixed kind.

(ii) If we choose $\beta=\gamma=1$ and $\mu=\beta$ in Theorem 3.1, we can get inequality for (m_1,m_2,α,β)-convex function in the 2nd kind.

(iii) If we choose $\gamma=\mu=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,α,β)- convex function in the 1st kind.

(iv) If we choose $\gamma=r$, $\alpha=\mu=s$ and $\beta=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,s,r)-convex function in mixed kind.

(v) If we choose $\alpha=\mu=s$ and $\beta=\gamma=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,s)-convex function in the 2nd kind.

(vi) If we choose $\gamma=s$ and $\alpha=\beta=\mu=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,s)-convex function in the 1st kind.

(vii) If we choose $m_1=1$, $m_2=m$, $\beta=\gamma=1$ and $\mu=\beta$ in Theorem 3.1, we can get inequality for (m,α,β)-convex function in the 2nd kind.

(viii) If we choose $m_1 = 1$, $m_2 = m$ and $\gamma = \mu = 1$ in Theorem 3.1, we can get inequality for (m, α, β)-convex function in the 1st kind.

(ix) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Theorem 3.1, we can get inequality for (m, s, r)-convex function in the mixed kind.

(x) If we choose $m_1 = m_2 = 1$, $\beta = \gamma = 1$ and $\mu = \beta$ in Theorem 3.1, we can get inequality for (α, β)-convex function in the 2nd kind.

(xi) If we choose $m_1 = m_2 = 1$ and $\gamma = \mu = 1$ in Theorem 3.1, we can get inequality for (α, β)-convex function in the 1st kind.

(xii) If we choose $m_1 = m_2 = 1$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Theorem 3.1, we can get inequality for (s, r)-convex function in the mixed kind.

(xiii) If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Theorem 3.1, we can get inequality for (m, s)-convex function in the 2nd kind.

(xiv) If we choose $\alpha = \beta = \gamma = \mu = 1$ in Theorem 3.1, we can get inequality for (m_1, m_2)-convex function.

(xv) If we choose $m_1 = 1$, $m_2 = m$ and $\alpha = \beta = \gamma = \mu = 1$ in Theorem 3.1, we can get inequality for m-convex function.

(xvi) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Theorem 3.1, we can get inequality for s-convex function in the 2nd kind.

(xvii) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = s$ and $\gamma = \mu = 1$ in Theorem 3.1, we can get inequality for s-convex function in the 1st kind.

(xviii) If we choose $m_1 = m_2 = 1$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in Theorem 3.1, we can get inequality for s-convex function in the 1st kind.

(xix) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = 0$, and $\gamma = \mu = 1$ in Theorem 3.1,we can get inequality for quasi-convex function.

(xx) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in Theorem 3.1, we can get inequality for P-convex function.

(xxi) If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Theorem 3.1, we can get inequality for ordinary convex function.

Remark 3.4. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.1, we recapture the main Theorem 3.2 of [19].

Remark 3.5. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.1, we recapture the main Theorem 3.1 of [14].

Remark 3.6. By choosing suitable values of $m_1, m_2, \alpha, \beta, \gamma, \mu$ in Theorem 3.1, we recapture all results of Corollary 3.1 of [14].

Theorem 3.2. *Let all the suppositions of Lemma* 3.1 *be true and assuming that* $[D(g'(y),\widetilde{0})]^q$ *is* $(m_1, m_2, \alpha, \beta, \gamma, \mu)$*-convex function on* $[m_2 j, m_2 k]$*,* $q \geq 1$ *&* $D(g'(y),\widetilde{0}) \leq M$*. Then,*

$$D\bigg(m_2 \odot g(m_1 y), \frac{1}{k-j} \odot (FR)\int_{m_2 j}^{m_2 k} g(u)du\bigg)$$

$$\leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{m_1}{\alpha\gamma+2} + \frac{m_2}{\beta}B\left(\frac{2}{\beta}, \mu+1\right)\right)^{\frac{1}{q}} I(y). \tag{3.6}$$

$\forall\ y \in (m_2 j, m_2 k)$ *and* $\beta > 0$*.*

Proof. From the inequality (3.3) & appling power mean inequality, we have

$$D\bigg(m_2 \odot g(m_1 y), \frac{1}{k-j} \odot (FR)\int_{m_2 j}^{m_2 k} g(u)du\bigg)$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k-j}\int_0^1 tD\Big(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\Big)dt$$

$$+ \frac{(m_2 k - m_1 y)^2}{k-j}\int_0^1 tD\Big(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\Big)dt$$

$$\leq \frac{(m_1 y - m_2 j)^2}{k-j}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\Big(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\Big)\right]^q dt\right)^{\frac{1}{q}}$$

$$+ \frac{(m_2 k - m_1 y)^2}{k-j}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\Big(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\Big)\right]^q dt\right)^{\frac{1}{q}}. \tag{3.7}$$

Since $[D(g'(y),\widetilde{0})]^q$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function & $D(g'(y),\widetilde{0}) \leq M$, we have

$$\left[D\Big(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\Big)\right]^q$$

$$\leq m_1 t^{\alpha\gamma}\left[D\left(g'(y), \widetilde{0}\right)\right]^q + m_2\left(1-t^\beta\right)^\mu\left[D\left(g'(j), \widetilde{0}\right)\right]^q$$

$$\leq M^q\left[m_1 t^{\alpha\gamma} + m_2\left(1-t^\beta\right)^\mu\right], \tag{3.8}$$

$$\left[D\Big(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\Big)\right]^q$$

$$\leq m_1 t^{\alpha\gamma}\left[D\left(g'(y), \widetilde{0}\right)\right]^q + m_2\left(1-t^\beta\right)^\mu\left[D\left(g'(k), \widetilde{0}\right)\right]^q$$

$$\leq M^q\left[m_1 t^{\alpha\gamma} + m_2\left(1-t^\beta\right)^\mu\right]. \tag{3.9}$$

Now, using (3.8) and (3.9) in (3.7) we get (3.6). □

Remark 3.7. All remarks hold for Theorem 3.2 as we have given remarks (i) to (xxi) for Theorem 3.1.

Remark 3.8. If we choose $q = 1$ in Theorem 3.2, we obtain the our main Theorem 3.1.

Remark 3.9. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.2, we recapture the Theorem 3.4 of [19].

Remark 3.10. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.2, we recapture the Theorem 3.2 of [14].

Remark 3.11. By choosing suitable values of $m_1, m_2, \alpha, \beta, \gamma, \mu$ in Theorem 3.2, we recapture all results of Corollary 3.2 and Remarks 3.1 of [14].

Theorem 3.3. *Let all the suppositions of Lemma* 3.1 *be true and assuming that* $[D(g'(y), \widetilde{0})]^q$ *is* $(m_1, m_2, \alpha, \beta, \gamma, \mu)$*-convex function on* $[m_2j, m_2k]$*,* $p, q > 1$ *&* $D(g'(y), \widetilde{0}) \leq M$*. Then,*

$$D\left(m_2 \odot g(m_1y), \frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du\right)$$

$$\leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\frac{m_1}{\alpha\gamma + 1} + \frac{m_2}{\beta} B\left(\frac{1}{\beta}, \mu + 1\right)\right)^{\frac{1}{q}} I(y), \tag{3.10}$$

$\forall\ y \in (m_2j, m_2k)$ *and* $\beta > 0$*. Where* $p^{-1} + q^{-1} = 1$*.*

Proof. From inequality (3.3) & by Hölder's inequality, we have

$$D\left(m_2 \odot g(m_1y), \frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du\right)$$

$$\leq \frac{(m_1y - m_2j)^2}{k-j} \int_0^1 tD\left(g'(m_1ty + m_2(1-t)j), \widetilde{0}\right)dt$$

$$+ \frac{(m_2k - m_1y)^2}{k-j} \int_0^1 tD\left(g'(m_1ty + m_2(1-t)k), \widetilde{0}\right)dt$$

$$\leq \frac{(m_1y-m_2j)^2}{k-j} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[D\left(g'(m_1ty+m_2(1-t)j), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}$$

$$+ \frac{(m_2k-m_1y)^2}{k-j} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[D\left(g'(m_1ty+m_2(1-t)k), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}. \tag{3.11}$$

Since $[D(g'(y), \widetilde{0})]^q$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function & $D(g'(y), \widetilde{0}) \leq M$, we have

$$\left[D\left(g'(m_1ty + m_2(1-t)j), \widetilde{0}\right)\right]^q$$

$$\leq m_1 t^{\alpha\gamma} \left[D\left(g'(y), \widetilde{0}\right)\right]^q + m_2\left(1 - t^\beta\right)^\mu \left[D\left(g'(j), \widetilde{0}\right)\right]^q$$

$$\leq M^q \left[m_1 t^{\alpha\gamma} + m_2\left(1 - t^\beta\right)^\mu\right], \tag{3.12}$$

$$
\begin{aligned}
&\left[D\bigg(g'(m_1ty+m_2(1-t)k),\widetilde{0}\bigg)\right]^q \\
&\leq m_1t^{\alpha\gamma}\left[D\left(g'(y),\widetilde{0}\right)\right]^q+m_2\left(1-t^\beta\right)^\mu\left[D\left(g'(k),\widetilde{0}\right)\right]^q \\
&\leq M^q\left[m_1t^{\alpha\gamma}+m_2\left(1-t^\beta\right)^\mu\right]. \qquad (3.13)
\end{aligned}
$$

Now, using (3.12) and (3.13) in (3.11), we get (3.10). □

Remark 3.12. All remarks hold for Theorem 3.3 as we have given remarks (i) to (xxi) for Theorem 3.1.

Remark 3.13. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.3, we recapture the Theorem 3.3 of [19].

Remark 3.14. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.3, we recapture the Theorem 3.3 of [14].

Remark 3.15. By choosing suitable values of $m_1, m_2, \alpha, \beta, \gamma, \mu$ in Theorem 3.3, we recapture all results of Corollary 3.3 and Remarks 3.2 of [14].

4. Conclusion

As we all know Ostrowski inequality is one of the most celebrated inequalities. In this paper, we presented 1st time the generalized notion of $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function in mixed kind, which contains the generalization of many functions as convex, P-convex, quasi-convex, s-convex in the 1^{st} kind, s-convex in the 2^{nd} kind, m-convex, (m_1, m_2)-convex, (m, s)-convex in the 1st kind, (m, s)-convex in the 2nd kind, (s, r)-convex in mixed kind, (α, β)-convex in the 1^{st} kind, (α, β)-convex in the 2^{nd} kind, (m, s, r)-convex in mixed kind, (m, α, β)-convex in the 1st kind, (m, α, β)-convex in the 2nd kind, (m_1, m_2, s)-convex function in the 1st kind, (m_1, m_2, s)-convex function in the 2nd kind, (m_1, m_2, s, r)-convex in mixed kind, $(m_1, m_2, \alpha, \beta)$-convex in the 1st kind, $(m_1, m_2, \alpha, \beta)$-convex in the 2nd kind, $(\alpha, \beta, \gamma, \mu)$-convex in mixed kind, $(m, \alpha, \beta, \gamma, \mu)$-convex in mixed kind. We proved the generalized Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions via Fuzzy Riemann Integrals by using Hölder's and power mean inequality. Further that we obtained several results with respect to the convexity of function as special cases and recaptured various established results of different authors of different papers [19] and [14].

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On a class of half-discrete Hilbert-type inequalities in the whole plane involving some classical special constants

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Abstract. In this paper, we first define a new half-discrete kernel function in the whole plane, which involves some exponent functions and unifies some homogeneous and non-homogeneous kernels. By employing some techniques of real analysis, a new half-discrete Hilbert-type inequality with the newly defined kernel function, as well as its equivalent forms are established. Furthermore, the constant factors of the newly obtained inequalities are proved to be optimal. At last, assigning special values to the parameters, we get some interesting Hilbert-type inequalities involving hyperbolic functions, and with the constant factors related to Euler numbers, Bernoulli numbers, and Catalan constant.
Keywords: Hilbert-type inequality, half-discrete, Bernoulli number, Euler number, Catalan constant.

1. Introduction

Suppose that $p > 1$, and $f(x), \mu(x)$ are two non-negative measurable functions defined on a measurable set E. Define

$$L_{p,\mu}(E) := \left\{ f : \|f\|_{p,\mu} := \left[\int_E f^p(x)\mu(x)\mathrm{d}x \right]^{1/p} < \infty \right\}.$$

Specially, if $\mu(x) \equiv 1$, then we have the following abbreviations: $\|f\|_p := \|f\|_{p,\mu}$ and $L_p(E) := L_{p,\mu}(E)$. Additionally, suppose that $p > 1$, $a_n, \nu_n > 0$, $n \in F \subseteq \mathbb{Z}$, $\boldsymbol{a} = \{a_n\}_{n\in F}$. Define

$$l_{p,\nu} := \left\{ \boldsymbol{a} : \|\boldsymbol{a}\|_{p,\nu} := \left(\sum_{n\in F} a_n^p \nu_n \right)^{1/p} < \infty \right\}.$$

Specially, if $\nu_n \equiv 1$, then we have $\|a\|_p := \|a\|_{p,\nu}$ and $l_p := l_{p,\nu}$.

Consider two real-valued sequences: $\boldsymbol{a} = \{a_m\}_{m\in\mathbb{N}^+} \in l_2$ and $\boldsymbol{b} = \{b_n\}_{n\in\mathbb{N}^+} \in l_2$, then

$$(1) \qquad \sum_{n\in\mathbb{N}^+} \sum_{m\in\mathbb{N}^+} \frac{a_m b_n}{m+n} < \pi \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2,$$

where the constant factor π is the best possible. Inequality (1) was proposed by D. Hilbert in his lectures on integral equations in 1908, and in 1911, Schur proved the integral analogy of inequality (1) as follows:

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\pi\|f\|_2\|g\|_2, \tag{2}$$

where $f,g\geq 0,\ f,g\in L_2(\mathbb{R}^+)$, and the constant factor π is the best possible.

Inequalities (1) and (2) are usually known as Hilbert's inequality [1]. In the past twenty years, by the introduction of some parameters and special functions such as the Beta function, some extended forms of (1) and (2) were established, such as the following[2]:

$$\sum_{n\in\mathbb{N}^+}\sum_{m\in\mathbb{N}^+}\frac{a_m b_n}{(m+n)^\lambda}<B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{3}$$

where $0<\lambda\leq\min\{p,q\}$, $\mu_m=m^{p-\lambda-1}$, $\nu_n=n^{q-\lambda-1}$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $B(x,y)$ is the Beta function [3, 4], that is,

$$B(x,y):=\int_0^\infty\frac{z^{x-1}}{(1+z)^{x+y}}\mathrm{d}z\ (x,y>0).$$

In addition, Yang [5] proved the following extended form of (2) in 2004:

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\frac{f(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y<\frac{\pi}{\lambda\sin r\pi}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4}$$

where $r,s,\lambda>0$, $r+s=1$, $\mu(x)=x^{p(1-\lambda r)-1}$, $\nu(x)=x^{q(1-\lambda s)-1}$. With regard to some other extensions of (1) and (2), we refer to [6, 7, 8, 9, 10, 11, 12, 13, 14]. Such extended inequalities as (3) and (4) are usually named as Hilbert-type inequality. Furthermore, by constructing new kernel functions, introducing parameters, and considering the reverse form, coefficient refinement and multi-dimensional extension, a great many Hilbert-type inequalities were established in the past 20 years (see, [15, 16, 17, 18, 19, 20, 21, 22, 23]).

It should be noted that, in addition to the discrete and integral forms, Hilbert-type inequality sometimes appears in half-discrete form. The first half-discrete Hilbert-type inequality was put forward by Hardy et al. (see, Theorem 351 of [1]). However, the constant factor was not proved to be the best possible. Until recently, researchers established some new half-discrete Hilbert-type inequalities with the best possible constant factors, such as [24]

$$\int_{x\in\mathbb{R}^+}f(x)\sum_{n\in\mathbb{N}^+}\frac{a_n}{(1+nx)^\lambda}\mathrm{d}x<B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\|f\|_{2,\mu}\|a\|_{2,\nu}, \tag{5}$$

where $\mu(x)=x^{\frac{\lambda}{2}-1}$, $\nu_n=n^{\frac{\lambda}{2}-1}$. Regarding some other half-discrete Hilbert-type inequalities, we refer to [25, 26, 27, 28, 29, 30].

The objective of this work is to establish a class of half-discrete Hilbert-type inequalities with the kernel functions related to some hyperbolic functions. Our motivation mainly comes from the following integral Hilbert-type inequalities [31, 32]:

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\operatorname{csch}(xy)f(x)g(y)\mathrm{d}x\mathrm{d}y<\frac{\pi^2}{4}\|f\|_{p,\mu}\|g\|_{q,\nu}\ , \tag{6}$$

$$\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}}\frac{f(x)g(y)}{|\mathrm{e}^{pxy}-\mathrm{e}^{-qxy}|}\mathrm{d}x\mathrm{d}y<\left(\frac{\pi}{pq\sin\frac{\pi}{p}}\right)^2\|f\|_{p,\hat{\mu}}\|g\|_{q,\hat{\nu}}, \tag{7}$$

where $\mu(x)=x^{-(p+1)}$, $\nu(y)=y^{-(q+1)}$, $\hat{\mu}(x)=|x|^{-(p+1)}$, $\hat{\nu}(y)=|y|^{-(q+1)}$.

In this work, we will establish the following Hilbert-type inequalities involving hyperbolic secant function and hyperbolic cosecant function:

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\operatorname{sech}\left(\sqrt[2m+1]{\frac{n}{x}}\right)a_n\mathrm{d}x<\frac{E_m}{2^{2m}}(2m+1)\pi^{2m+1}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{8}$$

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\left|\operatorname{csch}\left(\sqrt[2m+1]{xn}\right)\right|a_n\mathrm{d}x<\frac{B_m}{m}(2m+1)(2^{2m}-1)\pi^{2m}\|f\|_{p,\hat{\mu}}\|\boldsymbol{a}\|_{q,\hat{\nu}}, \tag{9}$$

where $\mu(x)=|x|^{2p-1}$, $\nu_n=|n|^{-1}$, $\hat{\mu}(x)=|x|^{\frac{p}{2m+1}-1}$, $\hat{\nu}_n=|n|^{\frac{q}{2m+1}-1}$, $E_m(m\in\mathbb{N})$ is the Euler number, and $B_m(m\in\mathbb{N}^+)$ is the Bernoulli number.

More generally, we will construct a new kernel function involving several exponent functions with multiple parameters, which unifies some homogeneous and non-homogeneous kernels, and then a half-discrete Hilbert-type inequality and its equivalent forms are established. Detailed lemmas will be presented in Section 2, and main results and some corollaries will be presented in Section 3 and Section 4, respectively.

2. Some Lemmas

Lemma 2.1. *Let $\tau,\eta\in\{1,-1\}$, and $\tau\neq-1$ when $\eta=1$. Suppose that $c>a\geq b>d>0$, and $ab=cd$ when $\tau\eta=1$. Define*

$$K(z):=\frac{|\ a^z+\tau b^z\ |}{|\ c^z+\eta d^z\ |}\ (z\neq 0). \tag{10}$$

Then, $K(z)$ decreases on $\mathbb{R}^+$, and increases on $\mathbb{R}^-$.

Proof. If $\tau=1,\eta=1$, then we have $ab=cd$, and

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}z}&=\frac{(ac)^z\log\frac{a}{c}+(bd)^z\log\frac{b}{d}+(ad)^z\log\frac{a}{d}+(bc)^z\log\frac{b}{c}}{(c^z+d^z)^2}\\&:=L(z)(c^z+d^z)^{-2}.\end{aligned}$$

Since $c > a \geq b > d > 0$, we have $bc > ad$, and $ac > bd$.
If $z \in \mathbb{R}^+$, we have

$$L(z) < (ac)^z \log \frac{a}{c} + (ac)^z \log \frac{b}{d} + (bc)^z \log \frac{a}{d} + (bc)^z \log \frac{b}{c} = 0.$$

If $z \in \mathbb{R}^-$, we have

$$L(z) > (bd)^z \log \frac{a}{c} + (bd)^z \log \frac{b}{d} + (ad)^z \log \frac{a}{d} + (ad)^z \log \frac{b}{c} = 0.$$

It implies that $\frac{\mathrm{d}K}{\mathrm{d}z} < 0$ for $z \in \mathbb{R}^+$, and $\frac{\mathrm{d}K}{\mathrm{d}z} > 0$ for $z \in \mathbb{R}^-$. Thus, $K(z)$ decreases on $\mathbb{R}^+$ and increases on $\mathbb{R}^-$ for $\tau = 1, \eta = 1$.

If $\tau = 1, \eta = -1$, $z \in \mathbb{R}^+$, then we have

$$\frac{\mathrm{d}K}{\mathrm{d}z} = -\frac{(ac)^z \log \frac{c}{a} + (bd)^z \log \frac{b}{d} + (ad)^z \log \frac{a}{d} + (bc)^z \log \frac{c}{b}}{(c^z - d^z)^2} < 0.$$

If $\tau = 1, \eta = -1$, $z \in \mathbb{R}^-$, then we have

$$\frac{\mathrm{d}K}{\mathrm{d}z} = \frac{(ad)^z \log \frac{a}{d} + (bc)^z \log \frac{c}{b} + (ac)^z \log \frac{c}{a} + (bd)^z \log \frac{b}{d}}{(c^z - d^z)^2} > 0.$$

Therefore, $K(z)$ decreases on $\mathbb{R}^+$ and increases on $\mathbb{R}^-$ for $\tau = 1, \eta = -1$.

If $\tau = -1, \eta = -1$, then $ab = cd$, and we have

$$\begin{aligned} \frac{\mathrm{d}K}{\mathrm{d}z} &= \frac{(ac)^z \log \frac{a}{c} + (bd)^z \log \frac{b}{d} - (ad)^z \log \frac{a}{d} - (bc)^z \log \frac{b}{c}}{(c^z - d^z)^2} \\ &:= g(z) \left[\left(\sqrt{\frac{c}{d}} \right)^z - \left(\sqrt{\frac{d}{c}} \right)^z \right]^{-2}, \end{aligned} \tag{11}$$

where $g(z) = g_1(z) + g_2(z) - g_3(z) - g_4(z)$, and

$$\begin{aligned} g_1(z) &= \left(\frac{a}{d}\right)^z \log \frac{a}{c} = \left(\sqrt{\frac{ac}{bd}}\right)^z \log \sqrt{\frac{ad}{bc}}, \\ g_2(z) &= \left(\frac{b}{c}\right)^z \log \frac{b}{d} = \left(\sqrt{\frac{bd}{ac}}\right)^z \log \sqrt{\frac{bc}{ad}}, \\ g_3(z) &= \left(\frac{a}{c}\right)^z \log \frac{a}{d} = \left(\sqrt{\frac{ad}{bc}}\right)^z \log \sqrt{\frac{ac}{bd}}, \\ g_4(z) &= \left(\frac{b}{d}\right)^z \log \frac{b}{c} = \left(\sqrt{\frac{bc}{ad}}\right)^z \log \sqrt{\frac{bd}{ac}}. \end{aligned}$$

It follows that

$$\frac{\mathrm{d}g_1}{\mathrm{d}z} = \left(\sqrt{\frac{ac}{bd}}\right)^z \left[\log^2\sqrt{\frac{a}{b}} - \log^2\sqrt{\frac{c}{d}}\right],$$
$$\frac{\mathrm{d}g_2}{\mathrm{d}z} = \left(\sqrt{\frac{bd}{ac}}\right)^z \left[\log^2\sqrt{\frac{a}{b}} - \log^2\sqrt{\frac{c}{d}}\right],$$
$$\frac{\mathrm{d}g_3}{\mathrm{d}z} = \left(\sqrt{\frac{ad}{bc}}\right)^z \left[\log^2\sqrt{\frac{a}{b}} - \log^2\sqrt{\frac{c}{d}}\right],$$
$$\frac{\mathrm{d}g_4}{\mathrm{d}z} = \left(\sqrt{\frac{bc}{ad}}\right)^z \left[\log^2\sqrt{\frac{a}{b}} - \log^2\sqrt{\frac{c}{d}}\right].$$

Therefore, we get

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}z} &= \left[\log^2\sqrt{\frac{a}{b}} - \log^2\sqrt{\frac{c}{d}}\right] \\ &\quad\times \left[\left(\sqrt{\frac{ac}{bd}}\right)^z + \left(\sqrt{\frac{bd}{ac}}\right)^z - \left(\sqrt{\frac{ad}{bc}}\right)^z - \left(\sqrt{\frac{bc}{ad}}\right)^z\right] \\ &= \left[\log^2\sqrt{\frac{a}{b}} - \log^2\sqrt{\frac{c}{d}}\right]\left[\left(\frac{c}{b}\right)^z + \left(\frac{b}{c}\right)^z - \left(\frac{a}{c}\right)^z - \left(\frac{c}{a}\right)^z\right].\end{aligned}$$

Let $h(t) := t^z + t^{-z}$, then it can be shown that $h(t)$ increases on $[1, \infty)$ for arbitrary $z \in \mathbb{R}^+$. Since $\frac{c}{b} \geq \frac{c}{a} > 1$, we have $h\left(\frac{c}{b}\right) \geq h\left(\frac{c}{a}\right)$, that is,

$$\left(\frac{c}{b}\right)^z + \left(\frac{b}{c}\right)^z - \left(\frac{a}{c}\right)^z - \left(\frac{c}{a}\right)^z \geq 0.$$

Additionally, in view of $\frac{c}{d} \geq \frac{a}{b} \geq 1$, we have $\log^2\sqrt{\frac{a}{b}} - \log^2\sqrt{\frac{c}{d}} \leq 0$. Thus, we obtain $\frac{\mathrm{d}g}{\mathrm{d}z} \leq 0$ on $\mathbb{R}^+$, which leads to

$$g(z) \leq g(0) = \log\frac{a}{c} + \log\frac{b}{d} - \log\frac{a}{d} - \log\frac{b}{c} = 0 \ (z \in \mathbb{R}^+).$$

By (11), we have $\frac{\mathrm{d}K}{\mathrm{d}z} \leq 0$ ($z \in \mathbb{R}^+$), and it implies that $K(z)$ decreases on $\mathbb{R}^+$. Similarly, it can be proved that $K(z)$ increases on $\mathbb{R}^-$. Thus, we proved Lemma 2.1 in the case of $\tau = -1, \eta = -1$. □

Lemma 2.2. *Let $\tau, \eta \in \{1, -1\}$, and $\tau \neq -1$ when $\eta = 1$. Suppose that $c > a \geq b > d > 0$, and $ab = cd$ when $\tau\eta = 1$. Let λ be such that $\lambda \geq 1$, and $\lambda \neq 1$ for $\tau = 1, \eta = -1$. $K(z)$ is defined via* (10)*, and*

$$\begin{aligned}(12)\qquad \kappa(a, b, c, d, \tau, \eta, \lambda) :=& \sum_{j=0}^{\infty}\left[\frac{(-\eta)^j}{(j\log\frac{c}{d} + \log\frac{c}{a})^\lambda} + \frac{\tau(-\eta)^j}{(j\log\frac{c}{d} + \log\frac{c}{b})^\lambda}\right] \\ &+ \sum_{j=0}^{\infty}\left[\frac{(-\eta)^j}{(j\log\frac{c}{d} + \log\frac{b}{d})^\lambda} + \frac{\tau(-\eta)^j}{(j\log\frac{c}{d} + \log\frac{a}{d})^\lambda}\right].\end{aligned}$$

Then

$$\int_{z\in\mathbb{R}} K(z)\,|z|^{\lambda-1}\,\mathrm{d}z = \Gamma(\lambda)\kappa\,(a,b,c,d,\tau,\eta,\lambda)\,. \tag{13}$$

Proof. Expanding $\frac{1}{c^z+\eta d^z}$ ($z \in \mathbb{R}^+$) into power series, and observing that $c > d > 0$, we obtain

$$\frac{1}{c^z+\eta d^z} = \frac{c^{-z}}{1+\eta(c^{-1}d)^z} = c^{-z}\sum_{j=0}^{\infty}(-\eta)^j\left(\frac{d}{c}\right)^{jz}.$$

By Lebesgue term-by-term integration theorem, we get

$$\begin{aligned} \int_0^{\infty} K(z)z^{\lambda-1}\mathrm{d}z &= \sum_{j=0}^{\infty}(-\eta)^j\Bigg[\int_0^{\infty}\left(\frac{d}{c}\right)^{jz}\left(\frac{a}{c}\right)^{z} z^{\lambda-1}\mathrm{d}z \\ &\qquad + \tau\int_0^{\infty}\left(\frac{d}{c}\right)^{jz}\left(\frac{b}{c}\right)^{z} z^{\lambda-1}\mathrm{d}z\Bigg] \\ &:= \sum_{j=0}^{\infty}(-\eta)^j(J_1+\tau J_2). \end{aligned} \tag{14}$$

Let $z = \frac{u}{j\log\frac{c}{d}+\log\frac{c}{a}}$ ($j \in \mathbb{N}$), then we have

$$J_1 = \frac{1}{(j\log\frac{c}{d}+\log\frac{c}{a})^{\lambda}}\int_0^{\infty} e^{-u}u^{\lambda-1}\mathrm{d}u = \frac{\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{a})^{\lambda}}. \tag{15}$$

Similarly, we can obtain

$$J_2 = \frac{1}{(j\log\frac{c}{d}+\log\frac{c}{b})^{\lambda}}\int_0^{\infty} e^{-u}u^{\lambda-1}\mathrm{d}u = \frac{\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{b})^{\lambda}}. \tag{16}$$

Plug (15) and (16) back into (14), then we obtain

$$\int_0^{\infty} K(z)z^{\lambda-1}\mathrm{d}z = \sum_{j=0}^{\infty}\left[\frac{(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{a})^{\lambda}} + \frac{\tau(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{b})^{\lambda}}\right]. \tag{17}$$

Since $c > a \geq b > d > 0$, we have $\frac{1}{d} > \frac{1}{b} \geq \frac{1}{a} > \frac{1}{c} > 0$. From the above discussion, we get

$$\begin{aligned} \int_{-\infty}^{0} K(z)\,|z|^{\lambda-1}\,\mathrm{d}z &= \int_0^{\infty} K(-z)z^{\lambda-1}\mathrm{d}z \\ &= \sum_{j=0}^{\infty}\left[\frac{(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{b}{d})^{\lambda}} + \frac{\tau(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{a}{d})^{\lambda}}\right]. \end{aligned} \tag{18}$$

Combining (17) and (18), and using (12), we get (13). Lemma 2.2 is proved. □

Lemma 2.3. *Let $\tau, \eta \in \{1, -1\}$, and $\tau \neq -1$ when $\eta = 1$. Let*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\},$$

$\beta \in \Omega$, and $\gamma \in \mathbb{R}^+ \cap \Omega$. Suppose that $c > a \geq b > d > 0$, and $ab = cd$ when $\tau\eta = 1$. Let λ be such that $\lambda \geq 1$, $\lambda\gamma \leq 1$, and $\lambda \neq 1$ for $\tau = 1, \eta = -1$. Let $K(z)$ be defined via (10), and for an arbitrary positive natural number s which is large enough, define

$$\tilde{\boldsymbol{a}} := \{\tilde{a}_n\}_{n \in \mathbb{Z}^0} := \left\{ |n|^{\lambda\gamma - 1 - \frac{2\gamma}{qs}} \right\}_{n \in \mathbb{Z}^0},$$

$$\tilde{f}(x) := \begin{cases} |x|^{\lambda\beta - 1 + \frac{2\beta}{ps}}, & x \in E \\ 0, & x \in \mathbb{R} \setminus E \end{cases},$$

where $\mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$, and $E := \{x : |x|^{\operatorname{sgn}\beta} < 1\}$. Then

$$\begin{aligned} \tilde{I} :&= \sum_{n \in \mathbb{Z}^0} \tilde{a}_n \int_{x \in \mathbb{R}} K\left(x^\beta n^\gamma\right) \tilde{f}(x) \mathrm{d}x = \int_{x \in \mathbb{R}} \tilde{f}(x) \sum_{n \in \mathbb{Z}^0} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x \\ &> \frac{s}{|\beta\gamma|} \left[\int_{[-1,1]} K(z) \, |z|^{\lambda - 1 + \frac{2}{ps}} \, \mathrm{d}z + \int_{\mathbb{R} \setminus [-1,1]} K(z) \, |z|^{\lambda - 1 - \frac{2}{qs}} \, \mathrm{d}z \right]. \end{aligned} \tag{19}$$

Proof. Let

$$E^+ := \{x : x \in E \cap \mathbb{R}^+\}, E^- := \{x : x \in E \cap \mathbb{R}^-\}.$$

Then

$$\tilde{I} = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &:= \int_{x \in E^-} \tilde{f}(x) \sum_{n \in \mathbb{Z}^+} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x, \\ I_2 &:= \int_{x \in E^-} \tilde{f}(x) \sum_{n \in \mathbb{Z}^-} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x, \\ I_3 &:= \int_{x \in E^+} \tilde{f}(x) \sum_{n \in \mathbb{Z}^+} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x, \\ I_4 &:= \int_{x \in E^+} \tilde{f}(x) \sum_{n \in \mathbb{Z}^-} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x. \end{aligned}$$

In view of $\lambda\gamma \le 1$, it follows that $\tilde{a}_n = |n|^{\lambda\gamma-1-\frac{2\gamma}{qs}}$ decreases with respect to n if $n \in \mathbb{Z}^+$. In addition, for $x \in E^-$, $n \in \mathbb{Z}^+$, we have $x^\beta n^\gamma < 0$. By Lemma 2.1, it can be proved that $K\left(x^\beta n^\gamma\right)$ decreases with respect to n if $n \in \mathbb{Z}^+$. Therefore,

$$I_1 > \int_{x\in E^-} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_1^\infty K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}}\,\mathrm{d}y\mathrm{d}x := W_1.$$

Similarly, we can obtain

$$I_2 > \int_{x\in E^-} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_{-\infty}^{-1} K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}}\,\mathrm{d}y\mathrm{d}x := W_2,$$
$$I_3 > \int_{x\in E^+} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_1^\infty K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}}\,\mathrm{d}y\mathrm{d}x := W_3,$$
$$I_4 > \int_{x\in E^+} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_{-\infty}^{-1} K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}}\,\mathrm{d}y\mathrm{d}x := W_4.$$

We first consider the case where $\beta < 0$, that is, $\beta \in \Omega \cap \mathbb{R}^-$. Letting $x^\beta y^\gamma = z$, and observing that $x^{-\frac{\beta}{\gamma}} = -|x|^{-\frac{\beta}{\gamma}}$ $(x<0)$ and $z^{\frac{1}{r}-1} = |z|^{\frac{1}{r}-1}$ $(z<0)$, we get

$$\begin{aligned}
W_1 &= \int_{-\infty}^{-1} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_1^\infty K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}}\,\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\gamma}\int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-\infty}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\mathrm{d}x \\
&= \frac{1}{\gamma}\int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-\infty}^{-1} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\mathrm{d}x \\
&+ \frac{1}{\gamma}\int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-1}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\mathrm{d}x \\
&= \frac{s}{2|\beta\gamma|}\int_{-\infty}^{-1} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z \\
&+ \frac{1}{\gamma}\int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-1}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\mathrm{d}x.
\end{aligned} \tag{20}$$

By Fubini's theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-1}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\mathrm{d}x \\
&= \int_{-1}^{0} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \int_{-\infty}^{z^{1/\beta}} |x|^{-1+\frac{2\beta}{s}}\,\mathrm{d}x\mathrm{d}z \\
&= \frac{s}{2|\beta|}\int_{-1}^{0} K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z.
\end{aligned} \tag{21}$$

Applying (21) to (20), we get

$$W_1 = \frac{s}{2|\beta\gamma|}\left[\int_{-\infty}^{-1} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z + \int_{-1}^{0} K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z\right].$$

In addition, it can be proved that $W_1 = W_4$, and

$$W_2 = W_3 = \frac{s}{2\,|\beta\gamma|}\left[\int_1^\infty K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z + \int_0^1 K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z\right].$$

Therefore, we have

$$\begin{aligned}\tilde{I} &> W_1 + W_2 + W_3 + W_4 \\ &= \frac{s}{|\beta\gamma|}\left[\int_{[-1,1]} K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\right].\end{aligned}$$

Inequality (19) is proved for $\beta < 0$. Similarly, (19) can also be proved to be true for $\beta > 0$, and we complete the proof of Lemma 2.3. □

Lemma 2.4. *Let $s_1, s_2 > 0$, $s_1 + s_2 = 1$, $\psi(z) = \cot z$, $\phi(z) = \csc z$ and $m \in \mathbb{N}$. Then*

$$\sum_{j=0}^{\infty}\left[\frac{1}{(j+s_1)^{2m+1}} - \frac{1}{(j+s_2)^{2m+1}}\right] = \frac{\pi^{2m+1}}{(2m)!}\psi^{(2m)}(s_1\pi), \tag{22}$$

$$\sum_{j=0}^{\infty}\left[\frac{1}{(j+s_1)^{2m+2}} + \frac{1}{(j+s_2)^{2m+2}}\right] = -\frac{\pi^{2m+2}}{(2m+1)!}\psi^{(2m+1)}(s_1\pi), \tag{23}$$

$$\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{(j+s_1)^{2m+1}} + \frac{(-1)^j}{(j+s_2)^{2m+1}}\right] = \frac{\pi^{2m+1}}{(2m)!}\phi^{(2m)}(s_1\pi). \tag{24}$$

Proof. We write the partial fraction expansion of $\psi(z) = \cot z$ $(0 < z < \pi)$ as follows [4]:

$$\psi(z) = \frac{1}{z} + \sum_{j=1}^{\infty}\left(\frac{1}{z+j\pi} + \frac{1}{z-j\pi}\right).$$

Taking the $(2m)$th derivative of $\psi(z)$, we get

$$\begin{aligned}\psi^{(2m)}(z) &= (2m)!\left[\sum_{j=0}^{\infty}\frac{1}{(j\pi+z)^{2m+1}} + \sum_{j=1}^{\infty}\frac{1}{(z-j\pi)^{2m+1}}\right] \\ &= (2m)!\sum_{j=0}^{\infty}\left[\frac{1}{(z+j\pi)^{2m+1}} - \frac{1}{(j\pi+\pi-z)^{2m+1}}\right].\end{aligned} \tag{25}$$

Letting $z = s_1\pi$ in (25), and observing that $s_1 + s_2 = 1$, we obtain (22). Taking the first derivative of (25) and setting $z = s_1\pi$, we arrive at (23). Additionally, owing to the following identity:

$$2\phi(2z) = \psi\left(\frac{\pi}{2} - z\right) + \psi(z)\ \left(0 < z < \frac{\pi}{2}\right),$$

we have

$$2^{2m+1}\phi^{(2m)}(2z) = \psi^{(2m)}\left(\frac{\pi}{2} - z\right) + \psi^{(2m)}(z). \tag{26}$$

Let $u = \frac{s_1\pi}{2}$ in (26), and use (22), then we have

$$\begin{aligned}\phi^{(2m)}(s_1\pi) &= \frac{(2m)!}{\pi^{2m+1}}\sum_{j=0}^{\infty}\left[\frac{1}{(2j+s_2)^{2m+1}} - \frac{1}{(2j+1+s_1)^{2m+1}}\right] \\ &+ \frac{(2m)!}{\pi^{2m+1}}\sum_{j=0}^{\infty}\left[\frac{1}{(2j+s_1)^{2m+1}} - \frac{1}{(2j+1+s_2)^{2m+1}}\right] \\ &= \frac{(2m)!}{\pi^{2m+1}}\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{(j+s_1)^{2m+1}} + \frac{(-1)^j}{(j+s_2)^{2m+1}}\right].\end{aligned} \tag{27}$$

Equality (27) implies (24) obviously. Lemma 2.4 is proved. □

Remark 2.1. By Lemma 2.4, we have the following identities related to classical special constants:

$$\psi^{(2m)}\left(\frac{\pi}{4}\right) = 2^{2m}E_m, \tag{28}$$

$$\phi^{(2m)}\left(\frac{\pi}{2}\right) = E_m, \tag{29}$$

$$\psi^{(2m+1)}\left(\frac{\pi}{4}\right) = \frac{4^{2m+1}}{m+1}\left(1 - 2^{2m+2}\right)B_{m+1}, \tag{30}$$

$$\psi^{(2m+1)}\left(\frac{\pi}{2}\right) = \frac{2^{2m+1}}{m+1}\left(1 - 2^{2m+2}\right)B_{m+1}, \tag{31}$$

where E_m is the Euler number, $E_0 = 1$, $E_1 = 1$, $E_2 = 5$, $\cdots$, and B_{m+1} is Bernoulli number, $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \cdots$. In fact, let $s_1 = \frac{1}{4},\ s_2 = \frac{3}{4}$ in (22). In view of [4]

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2m+1}} = \frac{\pi^{2m+1}E_m}{2^{2m+2}(2m)!}, \tag{32}$$

and

$$\sum_{j=0}^{\infty}\left[\frac{1}{(4j+1)^{2m+1}} - \frac{1}{(4j+3)^{2m+1}}\right] = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2m+1}},$$

we can get (28). Similarly, let $s_1 = s_2 = \frac{1}{2}$ in (24). By (32), we have (29). Additionally, let $s_1 = \frac{1}{4},\ s_2 = \frac{3}{4}$ in (23), and observe that [4]

$$\sum_{j=0}^{\infty}\frac{1}{(2j+1)^{2m+2}} = \frac{B_{m+1}}{2(2m+2)!}\left(2^{2m+2} - 1\right)\pi^{2m+2}\ (m \in \mathbb{N}),$$

then we get (30). At last, letting $s_1 = s_2 = \frac{1}{2}$ in (23), we arrive at (31).

3. Main results

Theorem 3.1. *Let $\tau, \eta \in \{1, -1\}$, and $\tau \neq -1$ when $\eta = 1$. Let*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}$$

$\beta \in \Omega$, and $\gamma \in \mathbb{R}^+ \cap \Omega$. Suppose that $c > a \geq b > d > 0$, and $ab = cd$ when $\tau\eta = 1$. Let λ be such that $\lambda \geq 1$, $\lambda\gamma \leq 1$, and $\lambda \neq 1$ for $\tau = 1, \eta = -1$. Assume that $\mu(x) = |x|^{p(1-\lambda\beta)-1}$, $\nu_n = |n|^{q(1-\lambda\gamma)-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$, $f(x), a_n \geq 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n \in \mathbb{Z}^0} \in l_{q,\nu}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $K(z)$ and $\kappa(a, b, c, d, \tau, \eta, \lambda)$ be defined via (10) and (12), respectively. Then the following inequalities hold and are equivalent:

$$\begin{aligned} I :&= \sum_{n \in \mathbb{Z}^0} a_n \int_{x \in \mathbb{R}} K\left(x^\beta n^\gamma\right) f(x) \mathrm{d}x = \int_{x \in \mathbb{R}} f(x) \sum_{n \in \mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n \mathrm{d}x \\ &< |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda) \kappa(a, b, c, d, \tau, \eta, \lambda) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \end{aligned} \tag{33}$$

$$\begin{aligned} J_1 :&= \sum_{n \in \mathbb{Z}^0} |n|^{p\lambda\gamma - 1} \left[\int_{x \in \mathbb{R}} K\left(x^\beta n^\gamma\right) f(x) \mathrm{d}x \right]^p \\ &< \left[|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda) \kappa(a, b, c, d, \tau, \eta, \lambda) \right]^p \|f\|_{p,\mu}^p, \end{aligned} \tag{34}$$

$$\begin{aligned} J_2 :&= \int_{x \in \mathbb{R}} |x|^{q\lambda\beta - 1} \left[\sum_{n \in \mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n \right]^q \mathrm{d}x \\ &< \left[|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda) \kappa(a, b, c, d, \tau, \eta, \lambda) \right]^q \|\boldsymbol{a}\|_{q,\nu}^q, \end{aligned} \tag{35}$$

where the constant $|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda) \kappa(a, b, c, d, \tau, \eta, \lambda)$ in (33), (34) and (35) is the best possible.

Proof. For $y \in [n-1, n)$, $n \in \mathbb{N}^+$, let $\widetilde{K}\left(x^\beta y^\gamma\right) := K\left(x^\beta n^\gamma\right)$, $g(y) := a_n$, $h(y) := n$. For $y \in [n, n+1)$, $n \in \mathbb{N}^-$, let $\widetilde{K}\left(x^\beta y^\gamma\right) := K\left(x^\beta n^\gamma\right)$, $g(y) := a_n$, $h(y) := |n|$. By Hölder's inequality, we have

$$\begin{aligned} &\sum_{n \in \mathbb{Z}^0} a_n \int_{x \in \mathbb{R}} K\left(x^\beta n^\gamma\right) f(x) \mathrm{d}x = \int_{x \in \mathbb{R}} f(x) \sum_{n \in \mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n \mathrm{d}x \\ &= \int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} \widetilde{K}\left(x^\beta y^\gamma\right) f(x) g(y) \mathrm{d}x \mathrm{d}y \\ &= \int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} \left[\widetilde{K}\left(x^\beta y^\gamma\right) \right]^{1/p} [h(y)]^{(\lambda\gamma - 1)/p} |x|^{(1-\lambda\beta)/q} f(x) \\ &\qquad \times \left[\widetilde{K}\left(x^\beta y^\gamma\right) \right]^{1/q} |x|^{(\lambda\beta - 1)/q} [h(y)]^{(1-\lambda\gamma)/p} g(y) \mathrm{d}x \mathrm{d}y \\ &\leq \left\{ \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} \widetilde{K}\left(x^\beta y^\gamma\right) [h(y)]^{\lambda\gamma - 1} |x|^{p(1-\lambda\beta)/q} f^p(x) \mathrm{d}y \mathrm{d}x \right\}^{1/p} \end{aligned} \tag{36}$$

$$\times\left\{\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}}\widetilde{K}\left(x^{\beta}y^{\gamma}\right)|x|^{\lambda\beta-1}\left[h(y)\right]^{q(1-\lambda\gamma)/p}g^{q}(y)\mathrm{d}x\mathrm{d}y\right\}^{1/q}$$

$$=\left[\int_{x\in\mathbb{R}}\Psi(x)\,|x|^{p(1-\lambda\beta)/q}\,f^{p}(x)\mathrm{d}x\right]^{1/p}\left[\sum_{n\in\mathbb{Z}^{0}}\Phi(n)\,|n|^{q(1-\lambda\gamma)/p}\,a_{n}^{q}\right]^{1/q},$$

where

$$\Psi(x)=\sum_{n\in\mathbb{Z}^{0}}K\left(x^{\beta}n^{\gamma}\right)|n|^{\lambda\gamma-1}, \tag{37}$$

$$\Phi(n)=\int_{x\in\mathbb{R}}K\left(x^{\beta}n^{\gamma}\right)|x|^{\lambda\beta-1}\,\mathrm{d}x. \tag{38}$$

In view of $\lambda\gamma\leq 1$, it can be easy to show that $|n|^{\lambda\gamma-1}$ decreases if $n\in\mathbb{N}^{+}$ and increases if $n\in\mathbb{N}^{-}$. Additionally, using Lemma 2.1, and observing that $\beta\in\Omega$ and $\gamma\in\mathbb{R}^{+}\cap\Omega$, it can be proved that whether $x>0$ or $x<0$, $K\left(x^{\beta}n^{\gamma}\right)$ decreases with respect to n when $n\in\mathbb{N}^{+}$, and increases with respect to n when $n\in\mathbb{N}^{-}$. Therefore, we get

$$\Psi(x)<\int_{y\in\mathbb{R}}K\left(x^{\beta}y^{\gamma}\right)|y|^{\lambda\gamma-1}\,\mathrm{d}y. \tag{39}$$

We first consider the case where $x<0$. Let $x^{\beta}y^{\gamma}=z$. Observing that $\beta\in\Omega$ and $\gamma\in\mathbb{R}^{+}\cap\Omega$, we have $x^{-\frac{\beta}{\gamma}}=-|x|^{-\frac{\beta}{\gamma}}$ $(x<0)$ and $z^{\frac{1}{r}-1}=|z|^{\frac{1}{r}-1}$. It follows therefore that

$$\int_{y\in\mathbb{R}}K\left(x^{\beta}y^{\gamma}\right)|y|^{\lambda\gamma-1}\,\mathrm{d}y=\frac{|x|^{-\lambda\beta}}{\gamma}\int_{z\in\mathbb{R}}K(z)\,|z|^{\lambda-1}\,\mathrm{d}z. \tag{40}$$

Similarly, it can also be proved that (40) holds when $x>0$. Therefore, for arbitrary x $(x\neq 0)$, combining (39) and (40), and using (13), we have

$$\Psi(x)<\frac{|x|^{-\lambda\beta}}{\gamma}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right). \tag{41}$$

Furthermore, by similar discussion, we have

$$\Phi(n)=\frac{|n|^{-\lambda\gamma}}{|\beta|}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right). \tag{42}$$

Plugging (41) and (42) back into (36), we get (33). In what follows, we will prove (34) and (35) via (33). In fact, assuming (33) holds, and setting $\boldsymbol{b}=\{b_{n}\}_{n\in\mathbb{N}^{0}}$, where

$$b_{n}:=|n|^{p\lambda\gamma-1}\left[\int_{x\in\mathbb{R}}K\left(x^{\beta}n^{\gamma}\right)f(x)\mathrm{d}x\right]^{p-1},$$

we obtain

$$
\begin{aligned}
J_1 &= \sum_{n\in\mathbb{Z}^0} |n|^{p\lambda\gamma-1} \left[\int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x\right]^p \\
&= \sum_{n\in\mathbb{Z}^0} b_n \int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x \\
&< |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|f\|_{p,\mu} \|\boldsymbol{b}\|_{q,\nu} \\
&= |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|f\|_{p,\mu} J_1^{1/q}.
\end{aligned}
\tag{43}
$$

Inequality (43) implies (34) obviously. Moreover, let

$$
g(x) := |x|^{q\lambda\beta-1} \left[\sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n\right]^{q-1}.
$$

By (33), we get

$$
\begin{aligned}
J_2 &= \int_{x\in\mathbb{R}} |x|^{q\lambda\beta-1} \left[\sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n\right]^q \mathrm{d}x \\
&= \int_{x\in\mathbb{R}} g(x) \sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n \mathrm{d}x \\
&< |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|g\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu} \\
&= |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|\boldsymbol{a}\|_{q,\nu} J_2^{1/p}.
\end{aligned}
\tag{44}
$$

Thus, we get (35) via (33). Conversely, if (34) or (35) holds, it can also be proved that (33) is valid. In fact, we first suppose that (34) holds. By Hölder's inequality, we obtain

$$
\begin{aligned}
I &= \sum_{n\in\mathbb{Z}^0} \left[|n|^{\lambda\gamma-1/p} \int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x\right] \left[a_n |n|^{-\lambda\gamma+1/p}\right] \\
&\leq J_1^{1/p} \left[\sum_{n\in\mathbb{Z}^0} a_n^q |n|^{q(1-\lambda\gamma)-1}\right]^{1/q} = J_1^{1/p} \|\boldsymbol{a}\|_{q,\nu}.
\end{aligned}
\tag{45}
$$

Applying (34) to (45), we arrive at (33). Similarly, supposing that (35) holds, we can also get (33). Therefore, Based on the above discussions, inequalities (33), (34) and (35) are equivalent.

Lastly, it will be proved that the constant $|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)$ in (33), (34) and (35) is the best possible. In fact, assume that there exists a constant C satisfying

$$
0 < C \leq |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right),
\tag{46}
$$

and

$$I=\sum_{n\in\mathbb{Z}^0}a_n\int_{x\in\mathbb{R}}K\left(x^\beta n^\gamma\right)f(x)\mathrm{d}x=\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}K\left(x^\beta n^\gamma\right)a_n\mathrm{d}x \tag{47}$$
$$<C\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.$$

Replacing a_n and $f(x)$ in (47) by $\tilde{a}_n$ and $\tilde{f}(x)$ defined in Lemma 2.3, repectively, and using (19), we have

$$\begin{aligned}
&\int_{[-1,1]}K(z)\left|z\right|^{\lambda-1+\frac{2}{ps}}\mathrm{d}z+\int_{\mathbb{R}\setminus[-1,1]}K(z)\left|z\right|^{\lambda-1-\frac{2}{qs}}\mathrm{d}z\\
&<\frac{|\beta\gamma|}{s}\tilde{I}<\frac{|\beta\gamma|\,C}{s}\|\tilde{f}\|_{p,\mu}\|\tilde{\boldsymbol{a}}\|_{q,\nu}\\
&=\frac{|\beta\gamma|\,C}{s}\left(2\int_{E^+}x^{\frac{2\beta}{s}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(2+2\sum_{n=2}^{\infty}n^{\frac{-2\gamma}{s}-1}\right)^{\frac{1}{q}}\\
&<\frac{2\,|\beta\gamma|\,C}{s}\left(\int_{E^+}x^{\frac{2\beta}{s}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(1+\int_1^{\infty}x^{-\frac{2\gamma}{s}-1}\mathrm{d}x\right)^{\frac{1}{q}}\\
&=2\,|\beta\gamma|\,C\left(\frac{1}{2\,|\beta|}\right)^{\frac{1}{p}}\left(\frac{1}{s}+\frac{1}{2\gamma}\right)^{\frac{1}{q}}.
\end{aligned}\tag{48}$$

Applying Fatou's lemma to (48), and using (13), it follows that

$$\begin{aligned}
&\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)=\int_{z\in\mathbb{R}}K(z)\left|z\right|^{\lambda-1}\mathrm{d}z\\
&=\int_{[-1,1]}\underline{\lim}_{s\to\infty}K(z)\left|z\right|^{\lambda-1+\frac{2}{ps}}\mathrm{d}z+\int_{\mathbb{R}\setminus[-1,1]}\underline{\lim}_{s\to\infty}L\left(z\right)\left|z\right|^{\lambda-1-\frac{2}{qs}}\mathrm{d}z\\
&\leqslant\underline{\lim}_{s\to\infty}\left[\int_{[-1,1]}L\left(z\right)\left|z\right|^{\lambda-1+\frac{2}{ps}}\mathrm{d}z+\int_{\mathbb{R}\setminus[-1,1]}L\left(z\right)\left|z\right|^{\lambda-1-\frac{2}{qs}}\mathrm{d}z\right]\\
&\leqslant\underline{\lim}_{s\to\infty}\left[2\,|\beta\gamma|\,C\left(\frac{1}{2\,|\beta|}\right)^{\frac{1}{p}}\left(\frac{1}{s}+\frac{1}{2\gamma}\right)^{\frac{1}{q}}\right]=C\,|\beta|^{\frac{1}{q}}\,\gamma^{\frac{1}{p}}.
\end{aligned}$$

It implies that

$$C\geq|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right). \tag{49}$$

Combining (46) and (49), we get $C=|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)$. Therefore, the constant factor in inequality (33) is the best possible. Owing to the equivalence of (33), (34) and (35), it can be proved that the constant factors in (34) and (35) are the best possible. Theorem 3.1 is proved. □

4. Corollaries

Let $\tau=\eta=-1$, and $\lambda=2m+1(m\in\mathbb{N})$ in Theorem 3.1, then we have $ab=cd$. By (22), we have

$$\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)=\sum_{j=0}^{\infty}\left[\frac{2}{(j\log\frac{c}{d}+\log\frac{b}{d})^{2m+1}}-\frac{2}{(j\log\frac{c}{d}+\log\frac{c}{b})^{2m+1}}\right]$$
$$=\frac{2}{(2m)!}\left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1}\psi^{(2m)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right).$$

Thus, we have the following corollary.

Corollary 4.1. *Let $\beta\in\Omega$, and $\gamma\in\mathbb{R}^{+}\cap\Omega$, where*

$$\Omega:=\left\{z:z=\frac{2i+1}{2l+1},i,l\in\mathbb{Z}\right\}.$$

Suppose that $c>a\geq b>d>0$, and $ab=cd$. Let m be such that $(2m+1)\gamma\leq 1$, $m\in\mathbb{N}$. Assume that $\psi(z)=\cot z$, $\mu(x)=|x|^{p[1-(2m+1)\beta]-1}$, $\nu_n=|n|^{q[1-(2m+1)\gamma]-1}$, $n\in\mathbb{Z}^0:=\mathbb{Z}\setminus\{0\}$. Let $f(x),a_n\geq 0$ with $f(x)\in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a}=\{a_n\}_{n\in\mathbb{Z}^0}\in l_{q,\nu}$. Then

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\frac{a^{x^\beta n^\gamma}-b^{x^\beta n^\gamma}}{c^{x^\beta n^\gamma}-d^{x^\beta n^\gamma}}a_n\mathrm{d}x$$
$$<2\left|\beta\right|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1}\psi^{(2m)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{50}$$

Let $a=\mathrm{e}^{\tau_1}$, $b=\mathrm{e}^{-\tau_1}$, $c=\mathrm{e}^{\tau_2}$, $d=\mathrm{e}^{-\tau_2}$ in (50), where $0<\tau_1<\tau_2$. Then

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\sinh\left(\tau_1x^\beta n^\gamma\right)\operatorname{csch}\left(\tau_2x^\beta n^\gamma\right)a_n\mathrm{d}x$$
$$<2\mid\beta\mid^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{2\tau_2}\right)^{2m+1}\psi^{(2m)}\left(\frac{(\tau_2-\tau_1)\pi}{2\tau_2}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{51}$$

Let $\tau_2=2\alpha$, $\tau_1=\alpha$ $(\alpha>0)$ in (51). By (28), we obtain

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\operatorname{sech}(\alpha x^\beta n^\gamma)a_n\mathrm{d}x<|\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\frac{E_m}{2^{2m}}\left(\frac{\pi}{\alpha}\right)^{2m+1}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{52}$$

Setting $\beta=\gamma=\frac{1}{2m+1}$ $m\in\mathbb{N}$, and $\alpha=1$ in (52), we get

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\operatorname{sech}\left(\sqrt[2m+1]{xn}\right)a_n\mathrm{d}x<\frac{E_m}{2^{2m}}(2m+1)\pi^{2m+1}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{53}$$

where $\mu(x)=|x|^{-1}$, $\nu_n=|n|^{-1}$. Setting $\beta=-\frac{1}{2m+1}$, $\gamma=\frac{1}{2m+1}$, and $\alpha=1$ in (52), we get (8).

Let $\tau_2=3\alpha$, $\tau_1=\alpha$ $(\alpha>0)$ in (51), then we obtain

$$\begin{aligned}(54)\qquad &\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{2\cosh\left(2\alpha x^{\beta}n^{\gamma}\right)+1}\mathrm{d}x\\ &\quad<2\left|\beta\right|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{6\alpha}\right)^{2m+1}\psi^{(2m)}\left(\frac{\pi}{3}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Setting $\beta=\gamma=\frac{1}{2m+1}$, and $\alpha=\frac{1}{2}$ in (54), we get

$$\begin{aligned}(55)\qquad &\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{2\cosh\left(\sqrt[2m+1]{xn}\right)+1}\mathrm{d}x\\ &\quad<(4m+2)\left(\frac{\pi}{3}\right)^{2m+1}\psi^{(2m)}\left(\frac{\pi}{3}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},\end{aligned}$$

where $\mu(x)=|x|^{-1}$, $\nu_n=|n|^{-1}$. Let $m=0$ in (55), then (55) is transformed into

$$(56)\qquad \int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\frac{a_n}{2\cosh\left(xn\right)+1}\mathrm{d}x<\frac{2\sqrt{3}\pi}{9}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.$$

Let $\tau_2=4\alpha$, $\tau_1=\alpha$ $(\alpha>0)$ in (51), then we have

$$\begin{aligned}(57)\qquad &\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\operatorname{sech}\left(\alpha x^{\beta}n^{\gamma}\right)\operatorname{sech}\left(2\alpha x^{\beta}n^{\gamma}\right)a_n\mathrm{d}x\\ &\quad<\frac{1}{2^{6m}}\left|\beta\right|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{\alpha}\right)^{2m+1}\psi^{(2m)}\left(\frac{3\pi}{8}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Setting $\beta=\gamma=\frac{1}{2m+1}$, and $\alpha=1$ in (57), we get

$$\begin{aligned}&\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\operatorname{sech}\left(\sqrt[2m+1]{xn}\right)\operatorname{sech}\left(2\sqrt[2m+1]{xn}\right)a_n\mathrm{d}x\\ &\quad<\frac{2m+1}{2^{6m}}\pi^{2m+1}\psi^{(2m)}\left(\frac{3\pi}{8}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Let $\tau=-1$, $\eta=1$, and $\lambda=2m+2$ $(m\in\mathbb{N})$ in Theorem 3.1, By (23), we have

$$\begin{aligned}&\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)\\ =&\sum_{j=0}^{\infty}\left[\frac{1}{(j\log\frac{c}{d}+\log\frac{c}{a})^{2m+2}}+\frac{1}{(j\log\frac{c}{d}+\log\frac{a}{d})^{2m+2}}\right]\\ +&\sum_{j=0}^{\infty}\left[\frac{1}{(j\log\frac{c}{d}+\log\frac{b}{d})^{2m+2}}+\frac{1}{(j\log\frac{c}{d}+\log\frac{c}{b})^{2m+2}}\right]\\ =&\frac{1}{(2m+1)!}\left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+2}\left[\psi^{(2m+1)}\left(\frac{\pi\ln\frac{c}{a}}{\ln\frac{c}{d}}\right)+\psi^{(2m+1)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right)\right].\end{aligned}$$

Thus, we have the following corollary.

Corollary 4.2. *Let $\beta \in \Omega$, and $\gamma \in \mathbb{R}^+ \cap \Omega$, where*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}.$$

Suppose that $c > a \geq b > d > 0$, and m satisfies $(2m+2)\gamma \leq 1$ $(m \in \mathbb{N})$. Assume that $\psi(z) = \cot z$, $\mu(x) = |x|^{p[1-(2m+2)\beta]-1}$, $\nu_n = |n|^{q[1-(2m+2)\gamma]-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z}\backslash\{0\}$. Let $f(x), a_n \geq 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Then

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a^{x^\beta n^\gamma} + b^{x^\beta n^\gamma}}{\left|c^{x^\beta n^\gamma} - d^{x^\beta n^\gamma}\right|} a_n \mathrm{d}x < -|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{\ln \frac{c}{d}} \right)^{2m+2} \times \left[\psi^{(2m+1)} \left(\frac{\pi \ln \frac{c}{a}}{\ln \frac{c}{d}} \right) + \psi^{(2m+1)} \left(\frac{\pi \ln \frac{b}{d}}{\ln \frac{c}{d}} \right) \right] \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{58}$$

Let $a = b = 1$ in (58), then we get $c > 1 > d > 0$. Since

$$\psi^{(2m+1)}(z) = \psi^{(2m+1)}(\pi - z), \ z \in (0, \pi), \tag{59}$$

inequality (58) is transformed into

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\left|c^{x^\beta n^\gamma} - d^{x^\beta n^\gamma}\right|} \mathrm{d}x < -|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \times \left(\frac{\pi}{\ln \frac{c}{d}} \right)^{2m+2} \psi^{(2m+1)} \left(\frac{\pi \ln c}{\ln \frac{c}{d}} \right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{60}$$

Let $c = e^p$, $d = e^{-q}$ in (60), then (60) reduces to

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\left|e^{px^\beta n^\gamma} - e^{-qx^\beta n^\gamma}\right|} \mathrm{d}x < -|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{pq} \right)^{2m+2} \psi^{(2m+1)} \left(\frac{\pi}{p} \right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{61}$$

Let $m = 0$, $\beta = \gamma = \frac{1}{3}$ in (61), then we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\left|e^{p\sqrt[3]{xn}} - e^{-q\sqrt[3]{xn}}\right|} \mathrm{d}x < \left(\frac{\sqrt{3}\pi}{pq \sin \frac{\pi}{p}} \right)^2 \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{62}$$

where $\mu(x) = |x|^{\frac{p}{3}-1}$, $\nu_n = |n|^{\frac{q}{3}-1}$.

Let $c = e^\alpha$, $d = e^{-\alpha}$ $(\alpha > 0)$ in (60). By (31), we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \left| \operatorname{csch} \left(\alpha x^\beta n^\gamma \right) \right| a_n \mathrm{d}x < |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \frac{B_{m+1}}{m+1} (2^{2m+2} - 1) \left(\frac{\pi}{\alpha} \right)^{2m+2} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{63}$$

Setting $\beta = \gamma = \frac{1}{2m+3}$ ($m \in \mathbb{N}$), $\alpha = 1$ in (63), and replacing $m+1$ with m, we get (9). Similarly, setting $\beta = -\frac{1}{2m+3}$, $\gamma = \frac{1}{2m+3}$, $\alpha = 1$ in (63), and replacing $m+1$ with m, we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \left|\operatorname{csch}\left(\sqrt[2m+1]{\frac{n}{x}}\right)\right| a_n \mathrm{d}x < \frac{B_m}{m}(2m+1)(2^{2m}-1)\pi^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},$$

where $\mu(x) = |x|^{\frac{(4m+1)p}{2m+1}-1}$, $\nu_n = |n|^{\frac{q}{2m+1}-1}$ ($m \in \mathbb{N}^+$).

Let $a = \mathrm{e}^{\tau_1}$, $b = \mathrm{e}^{-\tau_1}$, $c = \mathrm{e}^{\tau_2}$, $d = \mathrm{e}^{-\tau_2}$ in (58), where $0 < \tau_1 < \tau_2$. Then

$$\begin{aligned}(64)\quad & \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \cosh\left(\tau_1 x^\beta n^\gamma\right) \left|\operatorname{csch}\left(\tau_2 x^\beta n^\gamma\right)\right| a_n \mathrm{d}x \\ & < -2\,|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}} \left(\frac{\pi}{2\tau_2}\right)^{2m+2} \psi^{(2m+1)}\left(\frac{(\tau_2-\tau_1)\pi}{2\tau_2}\right) \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Let $\tau_2 = 2\alpha$, $\tau_1 = \alpha$ ($\alpha > 0$) in (64). By using (30), we can also get (63). Let $\tau_2 = 4\alpha$, $\tau_1 = \alpha$ ($\alpha > 0$) in (64), then we have

$$\begin{aligned}(65)\quad & \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \left|\operatorname{csch}\left(\alpha x^\beta n^\gamma\right)\right| \operatorname{sech}\left(2\alpha x^\beta n^\gamma\right) a_n \mathrm{d}x \\ & < -\frac{1}{8^{2m+1}}\,|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}} \left(\frac{\pi}{\alpha}\right)^{2m+2} \psi^{(2m+1)}\left(\frac{3\pi}{8}\right) \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Additionally, let $a = b = \mathrm{e}^{-\alpha}$, $c = \mathrm{e}^{2\alpha}$, $d = \mathrm{e}^{-2\alpha}$ ($\alpha > 0$) in (58). By (59) and (30), we get the following inequality with the same constant factor as (63), that is,

$$\begin{aligned}(66)\quad & \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \left|\operatorname{csch}\left(\alpha x^\beta n^\gamma\right) - \operatorname{sech}\left(\alpha x^\beta n^\gamma\right)\right| a_n \mathrm{d}x \\ & < |\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}} \frac{B_{m+1}}{m+1}(2^{2m+2}-1)\left(\frac{\pi}{\alpha}\right)^{2m+2} \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Furthermore, let $a = b = \mathrm{e}^{\alpha}$, $c = \mathrm{e}^{2\alpha}$, $d = \mathrm{e}^{-2\alpha}$ ($\alpha > 0$) in (58). Then we get

$$\begin{aligned}(67)\quad & \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \left|\operatorname{csch}\left(\alpha x^\beta n^\gamma\right) + \operatorname{sech}\left(\alpha x^\beta n^\gamma\right)\right| a_n \mathrm{d}x \\ & < |\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}} \frac{B_{m+1}}{m+1}(2^{2m+2}-1)\left(\frac{\pi}{\alpha}\right)^{2m+2} \|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Let $\tau = \eta = 1$, and $\lambda = 2m+1 (m \in \mathbb{N})$ in Theorem 3.1, then we have $ab = cd$. By (24), we get

$$\begin{aligned}\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) &= \sum_{j=0}^{\infty}\left[\frac{2(-1)^j}{(j\log\frac{c}{d}+\log\frac{b}{d})^{2m+1}} + \frac{2(-1)^j}{(j\log\frac{c}{d}+\log\frac{c}{b})^{2m+1}}\right] \\ &= \frac{2}{(2m)!}\left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1}\phi^{(2m)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right).\end{aligned}$$

Therefore, Theorem 3.1 is transformed into the following corollary.

Corollary 4.3. *Let $\beta \in \Omega$, and $\gamma \in \mathbb{R}^+ \cap \Omega$, where*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}.$$

Suppose that $c > a \geq b > d > 0$, and $ab = cd$. Let m be such that $(2m+1)\gamma \leq 1$, $m \in \mathbb{N}$. Assume that $\phi(z) = \csc z$, $\mu(x) = |x|^{p[1-(2m+1)\beta]-1}$, $\nu_n = |n|^{q[1-(2m+1)\gamma]-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. Let $f(x), a_n \geq 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Then

$$\begin{aligned} \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a^{x^\beta n^\gamma} + b^{x^\beta n^\gamma}}{c^{x^\beta n^\gamma} + d^{x^\beta n^\gamma}} a_n \mathrm{d}x \\ < 2\,|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}} \left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1} \phi^{(2m)}\left(\frac{\pi \ln\frac{b}{d}}{\ln\frac{c}{d}}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \end{aligned} \tag{68}$$

Let $a = \mathrm{e}^{\tau_1}$, $b = \mathrm{e}^{-\tau_1}$, $c = \mathrm{e}^{\tau_2}$, $d = \mathrm{e}^{-\tau_2}$ in (68), where $0 < \tau_1 < \tau_2$. Then

$$\begin{aligned} \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \cosh\left(\tau_1 x^\beta n^\gamma\right) \operatorname{sech}\left(\tau_2 x^\beta n^\gamma\right) a_n \mathrm{d}x \\ < 2\,|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}} \left(\frac{\pi}{2\tau_2}\right)^{2m+1} \phi^{(2m)}\left(\frac{(\tau_2-\tau_1)\pi}{2\tau_2}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \end{aligned} \tag{69}$$

Letting $\tau_2 = \alpha$ ($\alpha > 0$), $\tau_1 = 0$ in (69), and using (29), we can also get (52). Letting $\tau_2 = 2\alpha$, $\tau_1 = \alpha$ ($\alpha > 0$) in (70), we have

$$\begin{aligned} \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \operatorname{csch}\left(\alpha x^\beta n^\gamma\right) \tanh\left(2\alpha x^\beta n^\gamma\right) a_n \mathrm{d}x \\ < \frac{1}{2^{4m}}\,|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}} \left(\frac{\pi}{\alpha}\right)^{2m+1} \phi^{(2m)}\left(\frac{\pi}{4}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \end{aligned}$$

At last, let $\tau = \eta = 1$, $a = b$, and $\lambda = 2$ in Theorem 3.1. Then, we have $cd = a^2$. Let $\frac{c}{a} = \frac{a}{d} = \mathrm{e}^\alpha$ ($\alpha > 0$), then

$$\log\frac{c}{a} = \log\frac{c}{b} = \log\frac{b}{d} = \log\frac{a}{d} = \frac{1}{2}\log\frac{c}{d} = \alpha,$$

and

$$\kappa\left(a, b, c, d, \tau, \eta, \lambda\right) = \frac{4}{\alpha^2} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} = \frac{4c_0}{\alpha^2},$$

where c_0 is the Catalan constant. Thus, Theorem 3.1 is transformed into the following corollary.

Corollary 4.4. *Let $\alpha > 0$, $\beta \in \Omega$, $\gamma \in \mathbb{R}^+ \cap \Omega$ and $\gamma \leq \frac{1}{2}$, where*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}.$$

Suppose that $\mu(x) = |x|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-2\gamma)-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. Let $f(x), a_n \geq 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n \in \mathbb{Z}^0} \in l_{q,\nu}$. Then

$$\int_{x \in \mathbb{R}} f(x) \sum_{n \in \mathbb{Z}^0} \operatorname{sech}\left(\alpha x^\beta n^\gamma \right) a_n \mathrm{d}x < \frac{4c_0}{\alpha^2} |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{70}$$

Setting $\beta = \gamma = \frac{1}{2m+1}$ $(m \in \mathbb{N}^+)$, $\alpha = 1$ in (70), we have

$$\int_{x \in \mathbb{R}} f(x) \sum_{n \in \mathbb{Z}^0} \operatorname{sech}\left(\sqrt[2m+1]{xn} \right) a_n \mathrm{d}x < 4c_0(2m+1) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{71}$$

where $\mu(x) = |x|^{\frac{2m-1}{2m+1}p-1}$, $\nu_n = |n|^{\frac{2m-1}{2m+1}q-1}$ $(m \in \mathbb{N}^+)$.

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Some algebraic identities on prime near rings with generalized derivations

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Abstract. The purpose of the present paper is to investigate the commutativity of a prime near ring N with a generalized derivation F associated with a nonzero derivation d satisfying one of the conditions: For some nonnegative integers p and q:

(i) $[F(x), y] = \pm y^p(x \circ y)y^q$;

(ii) $[x, F(y)] = \pm x^p(x \circ y)x^q$;

(iii) $F(x) \circ y = \pm y^p[x, y]y^q$;

(iv) $x \circ F(y) = \pm x^p[x, y]x^q$;

(v) $F(x) \circ y = \pm y^p(x \circ y)y^q$;

(vi) $[x, F(y)] = \pm x^p[x, y]x^q$;

(vii) $[F(x), y] = \pm y^p[x, y]y^q$;

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(viii) $x \circ F(y) = \pm x^p(x \circ y)x^q$,

for all $x, y \in N$. Moreover, we give an example which shows the necessity of primness hypothesis in the theorems.

Keywords: prime near ring, derivation, generalized derivation, commutativity.

MSC 2020: 16N60, 16W25, 16Y30

1. Introduction

A right near ring N is a triplet $(N, +, \cdot)$, where $+$ and $\cdot$ are two binary operations such that (i) $(N, +)$ is a group (not necessarily abelian), (ii) $(N, \cdot)$ is a semigroup, and (iii) $(x+y)\cdot z = x\cdot z+y\cdot z$, for all $x, y, z \in N$. Analogously, if instead of (iii), N satisfies the left distributive law, then N is said to be a left near ring. A near ring N is said to be zero-symmetric if $x0 = 0$, for all $x \in N$ (right distributivity yields that $0x = 0$). Throughout the paper, N represents a zero-symmetric right near ring with multiplicative center $Z(N)$. For any $x, y \in N$, the symbols $[x, y]$ and $(x \circ y)$ denote the Lie product $xy - yx$ and Jordan product $xy + yx$ respectively. A near ring N is said to be prime if $xNy = \{0\}$, for all $x, y \in N$ implies that $x = 0$ or $y = 0$. A near ring N is said to be 2-torsion free if $(N, +)$ has no element of order 2.

The notion of derivation in near rings was introduced by Bell and Mason [8]. An additive mapping $d : N \to N$ is said to be a derivation on N if $d(xy) = xd(y) + d(x)y$, for all $x, y \in N$ or equivalently in [20], $d(xy) = d(x)y + xd(y)$, for all $x, y \in N$. Motivated by the definition of derivation in near rings, Gölbaşi [13] defined generalized derivation in near rings as follows: An additive mapping $F : N \to N$ is said to be a right (resp. left) generalized derivation associated with a derivation d on N if $F(xy) = F(x)y + xd(y)$ (resp. $F(xy) = d(x)y + xF(y)$), for all $x, y \in N$. Moreover, F is said to be a generalized derivation associated with a derivation d on N if it is both a right generalized derivation as well as a left generalized derivation on N. All derivations are generalized derivations. There has been a great deal of work by various authors with some suitable constraints on derivations and generalized derivations to prime and semiprime rings (see [5, 10, 11, 12, 15, 17]). A number of authors have obtained some comparable results on near rings, (c.f. [1, 2, 4, 6, 8, 16, 19, 20]).

Daif and Bell [10] proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation on R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in I$, then R is commutative. Further, Dhara [12] proved that if R is a semiprime ring with a generalized derivation F associated with a derivation d satisfying $F([x, y]) = \pm[x, y]$ or $F(x \circ y) = \pm(x \circ y)$, for all $x, y \in I$, a nonzero ideal of R, then R must contain a nonzero central ideal, provided $d(I) \neq \{0\}$. Moreover, he proved that in case R is a prime ring, R must be commutative, provided $d \neq 0$. Motivated by the above results, Boua and Oukhtite [9] proved that a prime near ring N with a derivation d is a commutative ring if one of the conditions holds: (i) $d([x, y]) = \pm[x, y]$, (ii) $d(x \circ y) = \pm(x \circ y)$, for all $x, y \in N$.

Recently, Shang [19] considered the more general situations (i) $F([x,y]) = \pm x^k[x,y]x^l$, (ii) $F(x \circ y) = \pm x^k(x \circ y)x^l$, for all $x, y \in N$, $k \geq 0$, $l \geq 0$ non negative integers and proved that the prime near ring N is a commutative ring if it satisfies one of the above conditions.

In this line of investigation, we prove that a prime near ring N equipped with a generalized derivation F associated with a nonzero derivation d is a commutative ring if it satisfies one of the following conditions: For some nonnegative integers p and q: (i) $[F(x), y] = \pm y^p(x \circ y)y^q$, (ii) $[x, F(y)] = \pm x^p(x \circ y)x^q$, (iii) $F(x) \circ y = \pm y^p[x,y]y^q$, (iv) $x \circ F(y) = \pm x^p[x,y]x^q$, (v) $F(x) \circ y = \pm y^p(x \circ y)y^q$, (vi) $[x, F(y)] = \pm x^p[x,y]x^q$, (vii) $[F(x), y] = \pm y^p[x,y]y^q$ and (viii) $x \circ F(y) = \pm x^p(x \circ y)x^q$, for all $x, y \in N$.

2. Preliminary results

For developing the proof of our theorems, we shall need the following lemmas. These results appear in the case of left near rings and so it is easy to observe that they also hold for right near ring as well.

Lemma 2.1 ([14], Lemma 2.2)**.** *Let N be a near ring admitting a generalized derivation F associated with a derivation d. Then:*

(i) $F(x)y + xd(y) = xd(y) + F(x)y$, *for all* $x, y \in N$,

(ii) $F(xy) = xF(y) + d(x)y$, *for all* $x, y \in N$.

Lemma 2.2. *Let N be a near ring admitting a generalized derivation F associated with a derivation d. Then*

(i) $x(F(y)z + yd(z)) = xF(y)z + xyd(z)$, *for all* $x, y, z \in N$,

(ii) $x(yd(z) + F(y)z) = xyd(z) + xF(y)z$, *for all* $x, y, z \in N$.

Proof. (i) For all $x, y, z \in N$, we have

$$F(x(yz)) = d(x)yz + xF(yz) = d(x)yz + x(F(y)z + yd(z)). \tag{1}$$

Also

$$F((xy)z) = F(xy)z + xyd(z) = d(x)yz + xF(y)z + xyd(z). \tag{2}$$

Comparing (1) and (2), we get

$$x(F(y)z + yd(z)) = xF(y)z + xyd(z), \quad \text{for all } x, y, z \in N.$$

(ii) For all $x, y, z \in N$,

$$F(x(yz)) = xF(yz) + d(x)yz = x(yd(z) + F(y)z) + d(x)yz. \tag{3}$$

On the other hand,

$$F((xy)z) = xyd(z) + F(xy)z = xyd(z) + xF(y)z + d(x)yz. \tag{4}$$

Comparing (3) and (4), we get the result.

Lemma 2.3 ([7], Theorem 2.1)**.** *If a prime near ring N admits a nonzero derivation with $d(N) \subseteq Z(N)$, then N is a commutative ring.*

3. Main results

Theorem 3.1. *Let N be a prime near ring. If there exist non negative integers $p \geq 0$, $q \geq 0$ and F is a generalized derivation on N associated with a nonzero derivation d satisfying one of the following:*

(i) $[F(x), y] = \pm y^p(x \circ y)y^q$, *for all* $x, y \in N$,

(ii) $[x, F(y)] = \pm x^p(x \circ y)x^q$, *for all* $x, y \in N$,

then N is a commutative ring.

Proof of Theorem 3.1. (i) Suppose that

$$[F(x), y] = y^p(x \circ y)y^q, \text{ for all } x, y \in N. \tag{5}$$

Replacing x by xy in (5) and using $(xy \circ y) = (x \circ y)y$, we have

$$\begin{aligned} &[F(xy), y] = y^p(xy \circ y)y^q = y^p(x \circ y)y^{q+1} = [F(x), y]y, \\ &F(xy)y - yF(xy) = F(x)y^2 - yF(x)y, \text{ for all } x, y \in N. \end{aligned} \tag{6}$$

Applying the definition of F and Lemma 2.1, we get

$$(F(x)y + xd(y))y - y(xd(y) + F(x)y) = F(x)y^2 - yF(x)y, \text{ for all } x, y \in N.$$

Invoking Lemma 2.2, we obtain

$$F(x)y^2 + xd(y)y - yxd(y) - yF(x)y = F(x)y^2 - yF(x)y, \text{ for all } x, y \in N,$$

which reduces to

$$xd(y)y = yxd(y), \text{ for all } x, y \in N. \tag{7}$$

Substituting zx in place of x for $z \in N$ in (7) and using (7), we find that

$$zxd(y)y = yzxd(y) = zyxd(y), \text{ for all } x, y, z \in N.$$

This implies that $[y, z]xd(y) = 0$, for all $x, y, z \in N$, i.e., $[y, z]Nd(y) = \{0\}$, for all $y, z \in N$. Since N is prime, we get

$$[y, z] = 0, \text{ for all } y, z \in N \text{ or } d(y) = 0, \text{ for all } y \in N. \tag{8}$$

But $d \neq 0$, we have

$$[y, z] = 0, \text{ for all } y, z \in N, \tag{9}$$

replacing y by $yd(x)$, for any $x, y \in N$ in (9), we get

$$[yd(x), z] = 0, \text{ for all } x, y, z \in N,$$

which reduces to

$$[y, z]d(x) + y[d(x), z] = 0, \text{ for all } x, y, z \in N. \tag{10}$$

Using (8) in (10) and N is zero-symmetric, we get

$$y[d(x), z] = 0, \text{ for all } x, y, z \in N. \tag{11}$$

Again replacing y by $d(r)y$, for any $r \in N$ in (11), we have

$$d(r)y[d(x), z] = 0, \text{ for all } r, x, y, z \in N.$$

This implies that

$$d(r)N[d(x), z] = \{0\}, \text{ for all } r, x, z \in N.$$

Since N is prime, we get

$$d(r) = 0, \text{ for all } r \in N \text{ or } [d(x), z] = 0 \text{ for all } x, z \in N.$$

But $d \neq 0$, we have

$$[d(x), z] = 0, \text{ for all } x, z \in N.$$

This implies that

$$d(x) \in Z(N), \text{ for all } x \in N.$$

$$d(N) \subseteq Z(N).$$

Therefore, by Lemma 2.3, N is a commutative ring.

Now, we taking

$$[F(x), y] = -y^p(x \circ y)y^q, \text{ for all } x, y \in N. \tag{12}$$

Replacing x by xy in (12) and using $(xy \circ y) = (x \circ y)y$, we have

$$[F(xy), y] = -y^p(xy \circ y)y^q = -y^p(x \circ y)y^{q+1} = [F(x), y]y,$$

$$F(xy)y - yF(xy) = F(x)y^2 - yF(x)y, \text{ for all } x, y \in N. \tag{13}$$

Equation (13) is the same as equation (6). Now arguing in the similar manner, we can obtain the result.

(ii) By hypothesis,

$$[x, F(y)] = x^p(x \circ y)x^q, \text{ for all } x, y \in N. \tag{14}$$

Substituting yx in place of y in (14), we get

$$[x, F(yx)] = x^p(x \circ yx)x^q = x^p(x \circ y)x^{q+1} = [x, F(y)]x,$$
$$xF(yx) - F(yx)x = xF(y)x - F(y)x^2, \text{ for all } x, y \in N,$$
$$x(F(y)x + yd(x)) - (yd(x) + F(y)x)x = xF(y)x - F(y)x^2, \text{ for all } x, y \in N.$$

Using Lemma 2.2, we obtain

$$xF(y)x + xyd(x) - yd(x)x - F(y)x^2 = xF(y)x - F(y)x^2, \text{ for all } x, y \in N.$$

This reduces to,

$$xyd(x) = yd(x)x, \text{ for all } x, y \in N. \tag{15}$$

Replacing y by zy, where $z \in N$ in (15) and using it again, we arrive at

$$xzyd(x) = zyd(x)x = zxyd(x), \text{ for all } x, y, z \in N,$$

which implies that $[x, z]yd(x) = 0$, for all $x, y, z \in N$, i.e., $[x, z]Nd(x) = \{0\}$. The primness of N gives that $[x, z] = 0$ or $d(x) = 0$, for all $x \in N$. Since d is a nonzero derivation on N, then we have,

$$[x, z] = 0, \text{ for all } x, z \in N. \tag{16}$$

Similar proof follows from equation (9).

Now, we taking

$$[x, F(y)] = -x^p(x \circ y)x^q, \text{ for all } x, y \in N. \tag{17}$$

Substituting yx in place of y in (17), we get

$$[x, F(yx)] = -x^p(x \circ yx)x^q = -x^p(x \circ y)x^{q+1} = [x, F(y)]x.$$

Now arguing in the similar manner as above, we can obtain the result.

Theorem 3.2. *Let N be a prime near ring. If there exist non negative integers $p \geq 0$, $q \geq 0$ and F is a generalized derivation on N associated with a nonzero derivation d satisfying one of the following:*

(i) $F(x) \circ y = \pm y^p[x, y]y^q$, *for all* $x, y \in N$,

(ii) $x \circ F(y) = \pm x^p[x, y]x^q$, *for all* $x, y \in N$,

then N is a commutative ring.

Proof of Theorem 3.2. (i) Assume that

$$F(x) \circ y = y^p[x, y]y^q, \text{ for all } x, y \in N. \tag{18}$$

Replacing x by xy in (18) and using $[xy, y] = [x, y]y$, we get

$$F(xy) \circ y = y^p[xy, y]y^q = y^p[x, y]y^{q+1} = (F(x) \circ y)y,$$

which implies that

$$(F(x)y + xd(y))y + y(xd(y) + F(x)y) = (F(x)y + yF(x))y, \text{ for all } x, y \in N.$$

Applying Lemma 2.2, we obtain

$$F(x)y^2 + xd(y)y + yxd(y) + yF(x)y = F(x)y^2 + yF(x)y, \text{ for all } x, y \in N,$$

which reduces to,

$$yxd(y) = -xd(y)y, \text{ for all } x, y \in N. \tag{19}$$

Substituting zx for x in (19), where $z \in N$, we have

$$yzxd(y) = -zxd(y)y = (-z)(xd(y)y) = (-z)(-yxd(y)) = (-z)((-y)xd(y)).$$

Replacing y by $-y$ in the above expression, we find that

$$-yzxd(-y) = (-y)zxd(-y) = (-z)yxd(-y) = -zyxd(-y), \text{ for all } x, y, z \in N.$$

The last expression yields that $[y, z]xd(-y) = 0$, for all $x, y, z \in N$. This implies that

$$[y, z]Nd(-y) = \{0\}, \text{ for all } y, z \in N.$$

By primness of N, we get $[y, z] = 0$, for all $y, z \in N$ or $d(-y) = 0$, for all $y \in N$. Taking $d(-y) = 0$, for all $y \in N$, this imply that $d(y) = 0$, for all $y \in N$. But $d \neq 0$, so we have,$[y, z] = 0$, for all $y, z \in N$. Hence, by the same argument as in the proof of Theorem 3.1, we conclude that N is a commutative ring.

Arguing in the similar manner as above, we can obtain the results for $F(x) \circ y = -y^p[x, y]y^q$, for all $x, y \in N$.

(ii) By hypothesis, we have

$$x \circ F(y) = x^p[x, y]x^q, \text{ for all } x, y \in N. \tag{20}$$

Substituting yx for y in (20), we have

$$x \circ F(yx) = x^p[x, yx]x^q = x^p[x, y]x^{q+1} = (x \circ F(y))x,$$
$$x(F(y)x + yd(x)) + (F(y)x + yd(x))x = xF(y)x + F(y)x^2.$$

Applying Lemma 2.1 and Lemma 2.2, the last expression yields that

$$xF(y)x + xyd(x) + (yd(x) + F(y)x)x = xF(y)x + F(y)x^2.$$

This implies that

$$xyd(x) = -yd(x)x, \text{ for all } x, y \in N. \tag{21}$$

Replacing y by zy for $z \in N$ in (21), we obtain

$$xzyd(x) = -zyd(x)x = (-z)(yd(x)x) = (-z)(-xyd(x)) = (-z)((-x)yd(x)).$$

Substituting $-x$ in place of x, we arrive at $[x,z]yd(-x) = 0$, for all $x, y, z \in N$. This implies that $[x,z]Nd(-x) = \{0\}$, for all $x, z \in N$. By primness of N, we get $[x,z] = 0$, for all $x, z \in N$ or $d(-x) = 0$, for all $x \in N$. Taking $d(-x) = 0$, for all $x \in N$, this imply that $d(x) = 0$, for all $x \in N$. But $d \neq 0$, we have $[x,z] = 0$, for all $x, z \in N$. Hence, by the same argument as in the proof of Theorem 3.1, we conclude that N is a commutative ring.

Arguing in the similar manner as above, we can obtain the result for $x \circ F(y) = -x^p[x,y]x^q$, for all $x, y \in N$.

Theorem 3.3. *Let N be a prime near ring. If there exist non negative integers $p \geq 0$, $q \geq 0$ and F is a generalized derivation on N associated with a nonzero derivation d satisfying one of the following:*

(i) $F(x) \circ y = \pm y^p(x \circ y)y^q$, *for all* $x, y \in N$,

(ii) $[x, F(y)] = \pm x^p[x,y]x^q$, *for all* $x, y \in N$,

then N is a commutative ring.

Proof of Theorem 3.3. (i) Assume that

$$F(x) \circ y = y^p(x \circ y)y^q, \text{ for all } x, y \in N. \tag{22}$$

Replacing x by xy in (22), we get

$$\begin{aligned}
&F(xy) \circ y = y^p(xy \circ y)y^q = y^p(x \circ y)y^{q+1} = (F(x) \circ y)y,\\
&F(xy)y + yF(xy) = F(x)y^2 + yF(x)y,\\
&(F(x)y + xd(y))y + y(xd(y) + F(x)y) = F(x)y^2 + yF(x)y.
\end{aligned}$$

Applying Lemma 2.2, the above expression reduces to

$$yxd(y) = -xd(y)y, \text{ for all } x, y \in N. \tag{23}$$

Equation (23) is same as Equation (19), arguing in the similar manner as in Theorem 3.2, we can get the result.

Arguing in the similar manner as above, we can obtain the result for $F(x) \circ y = -y^p(x \circ y)y^q$, for all $x, y \in N$.

(ii) Suppose that

$$[x, F(y)] = x^p[x,y]x^q, \text{ for all } x, y \in N. \tag{24}$$

Substituting yx for y in (24), we obtain

$$\begin{aligned}
&[x, F(yx)] = x^p[x, yx]x^q = x^p[x,y]x^{q+1} = [x, F(y)]x,\\
&xF(yx) - F(yx)x = xF(y)x - F(y)x^2, \text{ for all } x, y \in N.
\end{aligned}$$

Applying the definition of F and Lemma 2.1(i), the above expression yields that

$$x(F(y)x + yd(x)) - (yd(x) + F(y)x)x = xF(y)x - F(y)x^2, \text{ for all } x, y \in N.$$

Using Lemma 2.2, we get

$$xF(y)x + xyd(x) - yd(x)x - F(y)x^2 = xF(y)x - F(y)x^2, \text{ for all } x, y \in N.$$

The above expression reduces to

$$xyd(x) = yd(x)x, \text{ for all } x, y \in N. \tag{25}$$

Since Equation (25) is same as Equation (15), arguing in the similar manner as in Theorem 3.1, we can get the result.

Arguing in the similar manner as above, we can obtain the result for $[x, F(y)] = -x^p[x, y]x^q$, for all $x, y \in N$.

Theorem 3.4. *Let N be a prime near ring. If there exist non negative integers $p \geq 0$, $q \geq 0$ and F is a generalized derivation on N associated with a nonzero derivation d satisfying one of the following:*

(i) $[F(x), y] = \pm y^p[x, y]y^q$*, for all* $x, y \in N$*,*

(ii) $x \circ F(y) = \pm x^p(x \circ y)x^q$*, for all* $x, y \in N$*,*

then N is a commutative ring.

Proof of Theorem 3.4. (i) Assume that

$$[F(x), y] = y^p[x, y]y^q, \text{ for all } x, y \in N. \tag{26}$$

Replacing x by xy in (26) and using $[xy, y] = [x, y]y$, we find that

$$[F(xy), y] = y^p[xy, y]y^q = y^p[x, y]y^{q+1} = [F(x), y]y.$$

This implies that

$$(F(x)y + xd(y))y - y(xd(y) + F(x)y) = F(x)y^2 - yF(x)y, \text{ for all } x, y \in N.$$

Using Lemma 2.2, we get

$$F(x)y^2 + xd(y)y - yxd(y) - yF(x)y = F(x)y^2 - yF(x)y, \text{ for all } x, y \in N,$$

which reduces to

$$xd(y)y = yxd(y), \text{ for all } x, y \in N. \tag{27}$$

Since Equation (27) is same as Equation (15), arguing in the similar manner as in Theorem 3.1, we get the result.

Arguing in the similar manner as above, we can obtain the result for $[F(x), y] = -y^p[x, y]y^q$, for all $x, y \in N$.

(ii) By hypothesis

$$x \circ F(y) = x^p(x \circ y)x^q, \text{ for all } x, y \in N. \tag{28}$$

Replacing y by yx in (28) and using $(x \circ yx) = (x \circ y)x$, we get

$$x \circ F(yx) = x^p(x \circ yx)x^q = x^p(x \circ y)x^{q+1} = (x \circ F(y))x,$$
$$x(F(y)x + yd(x)) + (yd(x) + F(y)x)x = xF(y)x + F(y)x^2.$$

Applying Lemma 2.2(i), the above expression gives

$$xF(y)x + xyd(x) + yd(x)x + F(y)x^2 = xF(y)x + F(y)x^2,$$

which reduces to,

$$xyd(x) = -yd(x)x, \text{ for all } x, y \in N. \tag{29}$$

Since Equation (29) is same as Equation (21), arguing in the similar manner as in Theorem 3.2, we can get the result.

Arguing in the similar manner as above, we can obtain the result for $x \circ F(y) = -x^p(x \circ y)x^q$, for all $x, y \in N$.

The following example demonstrates that the primness hypothesis in the Theorems 3.1, 3.2, 3.3 and 3.4 is not superfluous.

Example 3.1. Let S be a zero-symmetric right near ring. Let us consider

$$N = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid 0, a, b \in S \right\}.$$

It is easy to verify that N is a non prime zero-symmetric right near ring with respect to matrix addition and matrix multiplication. Define mappings $F, d : N \to N$ by

$$F \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } d \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, F is a nonzero generalized derivation associated with a nonzero derivation d on N satisfying for some nonnegative integers p and q:

(i) $[F(x), y] = \pm y^p(x \circ y)y^q$;

(ii) $[x, F(y)] = \pm x^p(x \circ y)x^q$;

(iii) $F(x) \circ y = \pm y^p[x, y]y^q$;

(iv) $x \circ F(y) = \pm x^p[x, y]x^q$;

(v) $F(x) \circ y = \pm y^p(x \circ y)y^q$;

(vi) $[x, F(y)] = \pm x^p[x, y]x^q$;

(vii) $[F(x), y] = \pm y^p[x, y]y^q$;

(viii) $x \circ F(y) = \pm x^p(x \circ y)x^q$, for all $x, y \in N$.

However, N is not commutative.

4. Concluding remarks

In this paper, the class of near rings involving generalized derivations satisfying some differential identities has been studied. We proved commutativity of prime near rings with differential identities on generalized derivations. This work can be further studied by considering multiplicative generalized derivations on prime near rings and semiprime near rings along with examples that illustrates the necessity of the assumptions used which is left for future work.

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Binary soft simply* alpha open sets and continuous function

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Abstract. A topological rough approximation space is defined over two different universes using binary soft relations. A new class of binary soft set and its corresponding soft topology is obtained. Continuity functions for the newly defined set are introduced. The characteristics of continuity functions between two binary soft topological spaces and that between binary soft topological space and topological rough approximation space are examined. The proposed definitions and properties are demonstrated with suitable examples.

Keywords: soft set, binary soft set, binary soft nowhere dense, continuity mappings, approximation space.
MSC 2020: 54A05, 54C05, 03B52, 03E72, 03E75

1. Introduction

Data involving uncertainties are present in various disciplines such as economics, engineering, social science, and medical science. Uncertainty in events complicates decision making in many aspects. To handle problems with uncertainty, the concept of fuzzy sets was first defined by Zadeh [34]. Though fuzzy set theory helped in solving problems with uncertainty, assigning membership values to a large number of data was challenging. To overcome such difficulties, the concepts of rough set and soft set were developed. Pawlak [26] first defined rough sets in 1982. These sets were related to upper and lower approximations and generally are crisp sets. Pawlak's rough sets are based on equivalence relations, but finding an equivalence relation among the elements of a set was difficult. Though different relations were used to define rough set theory, they had compli-

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cations in modelling problems with uncertainty. Hence, Moldtsov [22] initiated the theory of soft sets. It was further investigated by Maji et al. and other researchers [7, 8, 19, 20, 23]. Soft set theory has application in various fields like decision making, game theory, operations research, etc. Continuity functions of soft near open sets in soft topological spaces [2, 3], continuity functions of rough sets [28], fuzzy continuous functions [13, 17], and many other hybrid topological spaces are studied in literature [5, 24]. In addition, the relationship between soft sets and fuzzy sets was studied by Alcantud [4], and the relationship between soft, rough, and fuzzy sets was investigated by Feng et al. [11]-[12]. A review on soft set based parameter reduction and decision making was done by Sani Danjuma et al. [9].

Extension of theories of uncertainty over n number of different non-empty finite sets helps the decision maker to make many decisions at a time. This will help in the future advancements in different areas of research. When it comes to two universes, the rough set model was first studied in 1996 [33]. Following them, numerous studies utilising uncertainty theories were conducted over two universes [18], [27], [29], [30], [31], [32], [35], [36], [37].

In recent years, attempts have been made to generalise the soft sets over a single universe to two or more universes. The binary soft set was first defined and studied by Ackogz et al. [1]. Following them, Hussain [14] studied the topological properties of binary soft set. Further, binary soft mappings, separation axioms, connectedness, and other hybrid concepts like binary bipolar soft sets, fuzzy binary soft sets, etc. are studied by researchers in [6, 13, 15, 16]. Simply* alpha open sets are useful in the field of decision making as it contributes to attribute reduction. It was studied over a rough set by El Safty et al. [10]. Although it is examined in a rough set, it is appropriate to study the simply* alpha open set over the soft set since the soft set contains a parametrization tool. Simply* alpha open set is extended to soft set theory over two different universes using soft binary relations in the author's previous work [25]. In that work, BR-soft topological rough approximation space was obtained, the definition of BR-soft simply* alpha open sets, other related BR-soft sets are defined, and their basic properties are studied.

Decision-making becomes relatively simple if we can identify continuous mapping from one set of parameters to another set and continuous mapping among the universal set. In this paper, the notions of BR-soft simply* alpha continuous mapping, contra continuous, and irresolute are defined between soft topological rough approximation space and newly obtained soft topology. This can be used to deal with uncertainty and vagueness in many areas, like data analysis, machine learning, etc.

This paper is divided into three main sections. In Section 2, the basic definitions used in the paper is discussed. Section 3 deals with the BR-soft simply* alpha open sets and related topological notions. In Section 4, BR-soft simply* alpha mapping, continuous functions, and contra continuous functions are introduced along with theorems and examples followed by concluding remarks.

2. Preliminary

Definition 2.1. *Let S be the universe set, E be the parameter set, and k be the subset of the parameter set E. Then, a soft set is a mapping from a subset of a parameter set to the power set of the universe set.*

Definition 2.2. *Let (m_1, k) and (m_2, j) be two soft sets over a common universe S. Then, (m_1, k) is said to be a soft subset of (m_2, j), if k is a subset of j for all e belongs to k, $m_i(e)$ are identical approximations.*

Definition 2.3. *A soft set (m, k) over S is said to be an absolute soft set if for every e belongs to k, $m(e) = S$.*

Definition 2.4. *A relation between the sets S and T is a subset of the cartesian product $S \times T$, where $S \times T = \{(s,t) : s \in S, t \in T\}$.*

Definition 2.5. *Let S and T be two different nonempty finite sets. k be the subset of a parameter set E. A pair (m, k) or m_k is called a soft binary relation over S and T if (m, k) is a soft set (binary soft set or BR-soft set) over $S \times T$.*

(Throughout this paper, BR stands for binary.)

Example 2.1. Let S denote the set of three patients $\{N, Z, C\}$, and T denote the set of three diseases $\{Typhoid(Ty), Dengue(D), Pneumonia(P)\}$. Let E be the set of parameter that define the symptoms of diseases, where $E = \{e_1(fever), e_2(breathingproblem), e_3(jointpain), e_4(headache)\}, K = \{e_1, e_2\} \subseteq E$. Let $S \times T = \{(N, Ty), (N, D), (N, P), (Z, Ty), (Z, D), (Z, P), (C, Ty), (C, D), (C, P)\}$ be the universal set. Then, soft set $(m, K) = \{(e_1, \{(N, Ty), (N, P), (Z, Ty), (Z, P), (C, Ty), (C, P)\}), (e_2, \{(N, D), (N, P), (Z, D), (Z, P), (C, D), (C, P)\})\}$ denotes patients and their symptoms along with the possibility of diseases.

Definition 2.6. *A binary relation $R_{(m(s,t))}$ on S and T induced by m_k is defined by $(s,t)R_{m(s,t)}(s_1, t_1) \iff \{(s,t)m_k(s_1, t_1)\}$ for each $(s,t), (s_1, t_1) \in S \times T$.*

Definition 2.7. *The successor neighbourhood of each (s,t) in $S \times T$ is given by $R_{m(s,t)}(s,t) = \{(s_1, t_1) \in S \times T; (s,t)R_{m(s,t)}(s_1, t_1)\}$.*

Definition 2.8. *Let m_k be a soft binary relation over $S \times T$. $G \times J \subseteq S \times T$ and $(S, T, R_{m(s,t)})$ be a rough approximation space with respect to the parameter set. The approximation operators are defined as follows:*

$$\begin{aligned}\underline{S_{apr}}(G \times J) =& \{(s,t) \in S \times T; R_{m(s,t)}(s,t) \subseteq (G \times J)\},\\ \overline{S_{apr}}(G \times J) =& \{(s,t) \in S \times T; R_{m(s,t)}(s,t) \cap (G \times J) \neq \emptyset\},\end{aligned}$$

where $\underline{S_{apr}}(G \times J)$ is the lower rough soft approximation and $\overline{S_{apr}}(G \times J)$ is the upper rough soft approximation over two different universal sets. If $\underline{S_{apr}}(G \times J) = \overline{S_{apr}}(G \times J)$, then $G \times J$ is a definable soft set. If $\underline{S_{apr}}(G \times J) \neq \overline{S_{apr}}(G \times J)$, then $(G \times J)$ is a rough soft set.

Definition 2.9. *Let $(S, T, R_{m(s,t)})$ be a rough approximation space, and τ_{BR} be a soft topology obtained from soft binary relation over S, T. Thus, $(S, T, R_{m(s,t)}, \tau_{BR})$ is said to be BR-topological rough approximation space, where the elements of τ_{BR} are BR-soft open, and its complements are closed.*

Definition 2.10. *Let $(S, T, R_{m(s,t)}, \tau_{BR})$ be a BR-topological rough approximation space. For each $m_{ki} \subseteq m_k$, the BR-topological approximation operators are defined as follows:*

$$\underline{\tau}_{BR}(m_{ki}) = \cup\{m_{kj} \in \tau_{BR}; m_{kj} \subseteq m_{ki}\},$$
$$\overline{\tau}_{BR}(m_{ki}) = \cap\{m_{kj} \in \tau^c_{BR}; m_{ki} \subseteq m_{kj}\}.$$

In other words, $\underline{\tau}_{BR}, \overline{\tau}_{BR}$ is considered the interior and closure of the BR-topological approximation space, respectively.

3. BR-soft simply* alpha open set

Definition 3.1. *In a BR-topological rough approximation space, a BR-soft subset is called BR-soft nowhere dense, if $\underline{\tau}_{BR}(\overline{\tau}_{BR}(m_{ki})) = \emptyset$.*

Definition 3.2. *In a BR-topological rough approximation space, a BR-soft subset is said to be BR-soft alpha open if $m_{ki} \subseteq \underline{\tau}_{BR}(\overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki})))$ and is BR-soft alpha closed if $\underline{\tau}_{BR}(\overline{\tau}_{BR}(\underline{\tau}_{BR}(m_{ki}))) \subseteq m_{ki}$.*

Definition 3.3. *In a BR-topological rough approximation space, a BR-soft subset is called a BR-soft simply* alpha open set if $(m_{ki}) \in \{\emptyset, m_k, (m_{kj}) \cup (m_{kl}) : (m_{kj})$ is $BR - soft$ alpha open, $(m_{kl}) is BR - soft$ nowhere dense$\}$. The collection of BR-soft simply* alpha open set is denoted by $BR_S S^* \alpha O(m_{ki})$, the complement is BR-soft simply* alpha closed.*

Proposition 3.1. *Let $(S, T, R_{m(s,t)}, \tau_{BR})$ be a BR-topological rough approximation space.*

i) The arbitrary union of the BR-Soft simply alpha open set is BR-Soft simply* alpha open.*

ii) Finite intersection of BR-Soft simply alpha open set is BR-Soft simply* alpha open.*

Definition 3.4. *Let $(S, T, R_{m(s,t)}, \tau_{BR})$ be a BR-topological rough approximation space. For each $m_{ki} \subseteq m_k$, where m_{ki}, m_k are BR-soft simply* alpha open sets. Then the BR-topological approximation operators are defined as follows:*

$$\overline{BR}_S(m_{ki}) = \cap\{m_{kj} \in \tau^c_{BR}; m_{ki} \subseteq m_{kj}\},$$
$$\underline{BR}_S(m_{ki}) = \cup\{m_{kj} \in \tau_{BR}; m_{kj} \subseteq m_{ki}\},$$

where $\overline{BR}_S(m_{ki}), \underline{BR}_S(m_{ki})$ are the closure and interior of BR soft simply alpha open sets in BR-topological rough approximation space respectively.*

Theorem 3.1. *A collection of BR-Soft simply* alpha open sets forms a BR-soft topology* τ^*_{BR}.

Definition 3.5. *Let* τ^*_{BR} *be a BR-soft topology obtained from the collection of BR-soft simply* alpha open sets. For each BR-soft simply* alpha open sets* $m_{ki} \subseteq m_k$*, the BR-topological approximation operators are defined as follows:*

$$\overline{BR}_S(m_{ki}) = \cap \{m_{kj} \in (\tau^*_{BR})^c; m_{ki} \subseteq m_{kj}\},$$
$$\underline{BR}_S(m_{ki}) = \cup \{m_{kj} \in \tau^*_{BR}; m_{kj} \subseteq m_{ki}\},$$

where $\overline{BR}_S(m_{ki}), \underline{BR}_S(m_{ki})$ *are the closure and interior of BR soft simply* alpha open sets in* τ^*_{BR} *respectively.*

4. Continuous mapping of BR-soft simply* alpha open set

Definition 4.1. *Let* (S, T, τ^*_{BR}, E) *be a BR-soft topological space obtained from the collection of BR-soft simply* alpha open sets and* $(U, V, R_{m(s,t)}, \tau_{BR})$ *be a soft topological approximation space. Then,* $f : (S, T, \tau^*_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ *is said to be* $BR_S S^*\alpha$*-continuous, if* $f^{-1}(m_k)$ *is BR-soft simply* alpha open set for every* $m_k \in (U, V, R_{m(s,t)}, \tau_{BR})$.

Example 4.1. Let $U = \{b, c, d\}, V = \{e, f\}$. Then $U \times V = \{(b, e), (b, f), (c, e), (c, f), (d, e), (d, f)\}$. Let the parameter sets be $E = \{e_1, e_2\}, K = \{k_1, k_2\}$ respectively, and $m_e = \{(e_1, \{(c, e), (d, e)\}), (e_2, \{(b, e), (c, f)\})\}$ be the BR-soft set. Then, the topological rough approximation space obtained is

$$\begin{aligned}\tau_{BR} =&\{\emptyset, \{(e_1, \{(d, e)\})\}, \{(e_1, \{(c, e)\})\}, \{(e_2, \{(c, f)\})\}, \{(e_2, \{(b, e)\})\},\\ &\{(e_1, \{(c, e), (d, e)\})\}, \{(e_1, \{(d, e)\}), (e_2, \{(c, f)\})\}, \{(e_1, \{(d, e)\}),\\ &(e_2, \{(b, e)\})\}, \{(e_1, \{(c, e)\}), (e_2, \{(c, f)\})\}, \{(e_1, \{(c, e)\}), (e_2, \{(b, e)\})\},\\ &\{(e_2, \{(b, e), (c, f)\})\}, \{(e_1, \{(c, e), (d, e)\}), (e_2, \{(c, f)\})\}, \{(e_1, \{(c, e),\\ &(d, e)\}), (e_2, \{(b, e)\})\}, \{(e_1, \{(d, e)\}), (e_2, \{(b, e), (c, f)\})\}, \{(e_1, \{(c, e)\}),\\ &(e_2, \{(b, e), (c, f)\})\}, \{(e_1, \{(c, e), (d, e)\}), (e_2, \{(b, e), (c, f)\})\}\}.\end{aligned}$$

Let $S = \{2, 3, 5\}, T = \{4, 6\}$, and $S \times T = \{(2, 4), (2, 6), (3, 4), (3, 6), (5, 4), (5, 6)\}$. Let $m_k = \{(k_1, \{(3, 4), (5, 4)\}), (k_2, \{(2, 4), (3, 6)\})\}$ be the BR-soft set. Then, the topological rough approximation space obtained is

$$\begin{aligned}\tau_{BR} =&\{\emptyset, \{(e_1, \{(5, 4)\})\}, \{(e_1, \{(3, 4)\})\}, \{(e_2, \{(3, 6)\})\}, \{(e_2, \{(2, 4)\})\},\\ &\{(e_1, \{(3, 4), (5, 4)\})\}, \{(e_1, \{(5, 4)\}), (e_2, \{(3, 6)\})\}, \{(e_1, \{(5, 4)\}),\\ &(e_2, \{(2, 4)\})\}, \{(e_1, \{(3, 4)\}), (e_2, \{(3, 6)\})\}, \{(e_1, \{(3, 4)\}), (e_2, \{(2, 4)\})\},\\ &\{(e_2, \{(2, 4), (3, 6)\})\}, \{(e_1, \{(3, 4), (5, 4)\}), (e_2, \{(3, 6)\})\}, \{(e_1, \{(3, 4),\\ &(5, 4)\}), (e_2, \{(2, 4)\})\}, \{(e_1, \{(5, 4)\}), (e_2, \{(2, 4), (3, 6)\})\}, \{(e_1, \{(3, 4)\}),\\ &(e_2, \{(2, 4), (3, 6)\})\}, \{(e_1, \{(3, 4), (5, 4)\}), (e_2, \{(2, 4), (3, 6)\})\}.\end{aligned}$$

Then, the collection of BR-soft simply*alpha open sets forming soft topology $\tau_{BR} = \tau^*_{BR}$, where $\emptyset$ is the BR-soft nowhere dense set. Here, $w : S \times T \to U \times V$ and $p : E \to K$ are defined as $w(2,4) = (c,f); w(2,6) = (b,e); w(3,4) = (d,f); w(3,6) = (d,e); w(5,4) = (b,f); w(5,6) = (c,e); p(e_1) = k_1; p(e_2) = k_2$.

Let $m_k = \{(e_2, \{(b,e),(c,f)\})\}$ be a BR-soft open set in $U \times V$ and $f : (S,T,\tau^*_{BR},E) \to (U,V,R_{m(s,t)},\tau_{BR})$ is a BR-soft mapping. Then, $f^{-1}(m_k) = \{(e_2, \{(2,4),(3,6)\})\}$ is BR-soft simply*alpha open set in $S \times T$. Therefore, f is a BR-soft simply*alpha- continuous function.

Example 4.2. Consider Example 4.1, where τ'_{BR}, τ''_{BR} be two BR soft topological spaces obtained when $\{(e_1,\{(5,6)\}),(e_2,\{(2,6)\})\}, \{(e_1,\{(5,6)\})\}$ are taken as BR-soft nowhere dense sets, respectively. Let $m_k = \{(e_1,\{(d,e)\})\}$ be a BR-soft open set in τ''_{BR}. Then, $f^{-1}(m_k) = \{(e_1,\{(5,6)\})\}$ is not a BR-soft simply* alpha open set in τ'_{BR}. Therefore, f is not a BR-soft simply*alpha- continuous function.

Theorem 4.1. *For the class of BR-soft simply*alpha continuous functions, the following are equivalent:*

i) f *is BR-soft simply*alpha continuous function.*

ii) $f^{-1}(m_k)$ *is BR-soft simply*alpha closed for every BR-soft closed set* m_k*.*

Proof of Theorem 4.1. i) $\implies$ ii). Let m_k be a BR-soft closed set over $U \times V$. Then, $(m_k)^c \in SO(U \times V)$. Hence, $f^{-1}((m_k)^c) \in BR_S S^*\alpha O(S \times T)$. That is, $(f^{-1}(m_k))^c \in BR_S S^*\alpha O(S \times T)$ which implies $f^{-1}(m_k) \in BR_S S^*\alpha C(S \times T)$.

ii) $\implies$ i). Let $m_k \in SO(U \times V)$. Then, $(m_k)^c \in SC(U \times V)$. So, $f^{-1}((m_k)^c) \in BR_S S^*\alpha C(S \times T)$. That is, $(f^{-1}(m_k))^c \in BR_S S^*\alpha C(S \times T)$ implies that $f^{-1}(m_k) \in BR_S S^*\alpha O(S \times T)$. Thus, f^{-1} is $BR_S S^*\alpha$-continuous.

Theorem 4.2. *For a BR-soft simply*alpha continuous function* $f : (S,T,\tau^*_{BR},E) \to (U,V,R_{m(s,t)},\tau_{BR})$*, where* m_k *is any BR-soft subset.*

i) $\underline{BR}(f^{-1}(\underline{\tau}_{BR}(m_k))) \subseteq f^{-1}(m_k)$

ii) $\underline{BR}(f^{-1}(\underline{\tau}_{BR}(m_k))) \subseteq f^{-1}(\underline{\tau}_{BR}(m_k))$

iii) $f^{-1}(\underline{\tau}_{BR}(m_k)) \subseteq \overline{BR}(f^{-1}(\underline{\tau}_{BR}(m_k)))$

Proof of Theorem 4.2. *i)* Since f is BR-soft simply* alpha continuous, and $\underline{\tau}_{BR}(m_k)$ is BR-soft open in $U \times V$, $f^{-1}(\underline{\tau}_{BR}(m_k))$ is BR-soft simply* alpha open in $S \times T$. We know that $\underline{\tau}_{BR}(m_k) \subseteq m_k$, which implies that $f^{-1}(\underline{\tau}_{BR}(m_k)) \subseteq f^{-1}(m_k)$. Since BR-soft simply* alpha interior of m_k is the largest open subset of m_k. Thus, $\underline{BR}(f^{-1}(\underline{\tau}_{BR}(m_k))) \subseteq f^{-1}(m_k)$. Hence the proof.

ii) The proof is obvious from (i).

iii) Let m_k be any BR-soft subset, and $\underline{\tau}_{BR}(m_k)$ is the largest BR-soft open subset of m_k. Since f is BR-soft simply* alpha continuous, $f^{-1}(\underline{\tau}_{BR}(m_k))$ is BR-soft simply* alpha open. Thus, we have

$$f^{-1}(\underline{\tau}_{BR}(m_k)) \subseteq \overline{BR}(f^{-1}(\underline{\tau}_{BR}(m_k))).$$

Definition 4.2. *Let (S, T, τ^*_{BR}, E) be a BR-soft topological space obtained from the collection of BR-soft simply* alpha open sets and $(U, V, R_{m(s,t)}, \tau_{BR})$ be a soft topological rough approximation space. Then, $f : (S, T, \tau^*_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ is said to be BR soft semi-continuous, if $f^{-1}(m_k)$ is BR-soft semi open set for every $m_k \in (U, V, R_m(s, t), \tau_{BR})$.*

Definition 4.3. *Let (S, T, τ^*_{BR}, E) be a BR-soft topological space obtained from the collection of BR-soft simply* alpha open sets and $(U, V, R_{m(s,t)}, \tau_{BR})$ be a soft topological rough approximation space. Then, $f : (S, T, \tau^*_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ is said to be BR soft beta-continuous, if $f^{-1}(m_k)$ is BR-soft beta open set for every $m_k \in (U, V, R_{m(s,t)}, \tau_{BR})$.*

Theorem 4.3. *Every BR-soft semi continuous is BR-soft simply*alpha-continuous.*

Proof of Theorem 4.3. Under a BR-soft semi continuous function, the inverse image of every BR-soft open set is BR-soft semi open. Since every BR-soft semi open is BR-soft simply*alpha open, inverse image of BR-soft open set is BR-soft simply*alpha open.

Theorem 4.4. *Every BR-soft simply*alpha-continuous function is BR-soft beta continuous.*

Proof of Theorem 4.4. Under a BR-soft beta continuous function, the inverse image of every BR-soft open set is BR-soft beta open. Since every BR-soft beta open is BR-soft simply*alpha open, inverse image of BR-soft open set is BR-soft simply*alpha open.

Definition 4.4. *Let (S, T, τ'_{BR}, E), and (U, V, τ''_{BR}, E) be two BR-soft topological spaces obtained from the collection of BR-soft simply*alpha open sets. Then, $f : (S, T, \tau'_{BR}, E) \to (U, V, \tau''_{BR}, E)$ is BR-soft simply* alpha-irresolute, if $f^{-1}(m_k)$ is BR-soft simply* alpha open for every BR-soft simply* alpha open set $m_k \in \tau''_{BR}$.*

Example 4.3. Consider Example 4.1, where τ'_{BR} be a BR-soft topological space obtained from the collection of BR-soft simply*alpha open sets where $\{e_1, \{(3, 4), (5, 4)\}, (e_2, \{(2, 4), (3, 6)\})\}$ is BR-soft open and $\{(e_1, \{(5, 6)\}), (e_2, \{(2, 6)\})\}$ is a BR-soft nowhere dense set. Let τ''_{BR} be a BR-soft topological space obtained from the collection of BR-soft simply*alpha open sets where $\{e_1, \{(c, e), (d, e), (d, f)\}, (e_2, \{(b, e), (b, f), (c, f)\})\}$ is BR-soft open and empty set is BR-soft nowhere dense set.

Let $m_k = \{e_1, \{(d,e),(d,f)\},(e_2,\{(b,e),(b,f)\})\}$ be BR-soft simply* alpha open set in τ''_{BR}. Then, $f^{-1}(m_k) = \{(e_1,\{(5,4),(5,6)\}),(e_2,\{(2,6),(3,6)\})\}$ is also BR-soft simply* alpha open set in τ'_{BR}.

Example 4.4. Consider Example 4.1, where τ'_{BR} is a BR-soft topological space obtained from the collection of BR-soft simply*alpha open sets where $\{(e_1,\{(5,6)\}),(e_2,\{(2,6)\})\}$ is a BR-soft nowhere dense set.

Similarly, τ''_{BR} is a BR-soft topological space obtained from the collection of BR-soft simply*alpha open sets where $\{(e_1,\{(d,f)\})\}$ is a BR-soft nowhere dense.

Let $m_k = \{e_1,\{(b,f),(c,e)\},(e_2,\{(c,f),(d,e)\})\}$ and $f^{-1}(m_k) = \{(e_1,\{(5,4),(5,6)\}),(e_2,\{(2,4),(3,6)\})\}$. Here, m_k is BR-soft simply*alpha open set in τ''_{BR} but $f^{-1}(m_k)$ is not a BR-soft simply*alpha open set in τ'_{BR}. Thus, f is not a BR-soft simply* alpha-irresolute.

Theorem 4.5. *Every BR-soft simply* alpha-irresolute is BR-soft simply* alpha-continuous.*

Proof of Theorem 4.5. Let $f : (S,T,\tau'_{BR},E) \to (U,V,\tau''_{BR},E)$ is BR-soft simply* alpha-irresolute. Let m_k be a BR-soft open set in (U,V,τ''_{BR},E). Then m_k is BR-soft simply*alpha open set in (U,V,τ''_{BR},E). Since f is BR-soft simply* alpha-irresolute mapping, $f^{-1}(m_k)$ is a BR-soft simply* alpha open set in (S,T,τ'_{BR},E). Hence f is BR-soft simply* alpha-continuous mapping.

Definition 4.5. *A mapping $f : (S,T,\tau'_{BR},E) \to (U,V,\tau''_{BR},E)$ is said to be a BR-soft simply*alpha open map if the image of every BR-soft open set in (S,T,τ'_{BR},E) is BR-soft simply* alpha open in (U,V,τ''_{BR},E).*

Definition 4.6. *A mapping $f : (S,T,\tau'_{BR},E) \to (U,V,\tau''_{BR},E)$ is said to be a BR-soft simply*alpha closed map if the image of every BR-soft closed set in (S,T,τ'_{BR},E) is a BR-soft simply* alpha closed set in (U,V,τ''_{BR},E).*

Definition 4.7. *A map $f : (S,T,\tau'_{BR},E) \to (U,V,R_{m(s,t)},\tau_{BR})$ is called contra BR-soft simply* alpha continuous if $f^{-1}(m_k)$ is BR-soft simply* alpha closed in (S,T,τ'_{BR},E) for every BR-soft open set m_k of $(U,V,R_{m(s,t)},\tau_{BR})$.*

Example 4.5. Considering Example 4.1, $f : (S,T,\tau'_{BR},E) \to (U,V,R_{m(s,t)},\tau_{BR})$ be a map where τ_{BR} is the topological rough approximation space over $U \times V$. Then, BR-soft topology over $S \times T$ is obtained by taking $\{(e_1,\{(5,6)\}),(e_2,\{(2,6)\})\}$ as nowhere dense set. Let $m_k = \{(e_2,\{(c,f)\})\}$ is BR-soft open in τ_{BR} and $f^{-1}(m_k) = \{(e_2,\{(2,4)\})\}$ is BR-soft simply* alpha closed. Thus, f is contra BR-soft simply* alpha continuous.

Theorem 4.6. *Let arbitrary union of BR-soft simply* alpha open set is BR-soft simply* alpha open. Then, the following statements are equivalent for a map $f : (S,T,\tau'_{BR},E) \to (U,V,R_{m(s,t)},\tau_{BR})$*

i) f is BR-soft simply alpha contra continuous.*

ii) *For every BR-soft closed set* m_k *of* $(U, V, R_{m(s,t)}, \tau_{BR})$, $f^{-1}(m_k)$ *is BR-soft simply* alpha open in* (S, T, τ'_{BR}, E).

Proof of Theorem 4.6. i) $\implies$ ii). Let m_k be BR-soft closed set of $(U, V, R_{m(s,t)}, \tau_{BR})$ over $U \times V$. Then, $U \times V - m_k$ is BR-soft open in $(U, V, R_{m(s,t)}, \tau_{BR})$. By (i), $f^{-1}(U \times V - m_k) = S \times T - f^{-1}(m_k)$ BR-soft simply* alpha closed in $S \times T$. Thus $f^{-1}(m_k)$ is BR-soft simply* alpha open in (S, T, τ'_{BR}, E).

ii) $\implies$ i). Let m_{ki} be BR-soft open in $(U, V, R_{m(s,t)}, \tau_{BR})$. Then, $U \times V - m_{ki}$ is BR-soft closed in $(U, V, R_{m(s,t)}, \tau_{BR})$. By (ii), $f^{-1}(U \times V - m_{ki})=$ $f^{-1}(U \times V - m_{ki}) = S \times T - f^{-1}(m_{ki})$ is BR-soft simply* alpha open in $S \times T$. Thus, $f^{-1}(m_{ki})$ is BR-soft simply* alpha closed in (S, T, τ'_{BR}, E).

Theorem 4.7. *Let* $f : (S, T, \tau'_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ *be contra BR-soft simply* alpha continuous. Then,* $\underline{BR}(f^{-1}(\overline{\tau}_{BR}(m_k))) \subset f^{-1}(m_k)$ *for every* m_k *in* $(U, V, R_{m(s,t)}, \tau_{BR})$.

Proof of Theorem 4.7. Let f be a contra BR-soft simply* alpha continuous function. Let $\overline{\tau}_{BR}(m_k)$ is BR-soft closed in $(U, V, R_{m(s,t)}, \tau_{BR})$. Then $f^{-1}(\overline{\tau}_{BR}(m_k))$ is BR-soft simply* alpha open in (S, T, τ'_{BR}, E). Also, we know that $\underline{BR}(m_k) \subset m_k$, such that $\underline{BR}(f^{-1}(\overline{\tau}_{BR}(m_k))) \subset f^{-1}(m_k)$.

Theorem 4.8. *If* $f : (S, T, \tau'_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ *is contra BR-soft simply* alpha continuous, then the following statements hold:*

i) *f is contra BR-soft simply* alpha continuous*

ii) *For every* $(s, t) \in S \times T$ *and every BR-soft closed set* m_k *of* $(U, V, R_{m(s,t)}, \tau_{BR})$ *containing* $f(s, t)$*, there exists a BR-soft simply* alpha open set* m_{ki} *such that* $(s, t) \in m_{ki}$ *and* $f(m_{ki}) \subseteq m_k$*, if arbitrary union of BR-soft simply* alpha open sets is BR-soft simply* alpha open.*

iii) *The inverse image of each BR-soft open set in* $(U, V, R_{m(s,t)}, \tau_{BR})$ *is BR-soft simply* alpha closed in* (S, T, τ'_{BR}, E).

Proof of Theorem 4.8. i) $\implies$ ii). Let f be a contra BR-soft simply* alpha continuous. Let $(s, t) \in S \times T$ and m_k be a BR-soft closed set in $U \times V$ containing $f(s, t)$. So, $(s, t) \in f^{-1}(m_k)$, which is BR-soft simply* alpha open in $S \times T$. Let $f^{-1}(m_k) = m_{ki}$. Hence, $(s, t) \in m_{ki}$. Thus, $f(m_{ki}) = ff^{-1}(m_k) \subset m_k$.

ii) $\implies$ i). Let m_k be BR-soft closed in $U \times V$. Let $(s, t) \in f^{-1}(m_k)$. Thus, $f(s, t) \in m_k$. Hence, there exists a BR-soft simply* alpha open set m_{kj} containing (s, t) such that $f(m_{kj}) \subset m_k$. That is, $(s, t) \in m_{kj} \subset f^{-1}(m_k)$. Therefore, $f^{-1}(m_k)$ is a BR-soft simply*alpha open set in (S, T, τ'_{BR}, E).

iii) $\implies$ i. The proof is obvious.

Theorem 4.9. *Let* $f : (S, T, \tau'_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ *be a contra BR-soft simply* alpha continuous function and* $g : (U, V, R_{m(s,t)}, \tau_{BR}) \to (P, Q, \tau''_{BR}, E)$ *be a BR-soft continuous function. Then* $g \circ f : (S, T, \tau'_{BR}, E) \to (P, Q, \tau''_{BR}, E)$ *is contra BR-soft simply* alpha continuous.*

Proof of Theorem 4.9. Let m_k be any BR-soft open set over $P \times Q$. Since g is BR-soft continuous, $g^{-1}(m_k)$ is BR-soft open cover over $U \times V$. Since f is contra BR-soft simply* alpha continuous, $f^{-1}(g^{-1}(m_k)) = (g \circ f)^{-1}(m_k)$ is BR-soft simply* alpha closed over $S \times T$. Thus $g \circ f$ is a contra BR-soft simply* alpha continuous function.

Theorem 4.10. *Let $f : (S, T, \tau'_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ be a BR-soft simply* alpha irresolute and $g : (U, V, R_{m(s,t)}, \tau_{BR}) \to (P, Q, \tau''_{BR}, E)$ be a contra BR-soft simply* alpha continuous function. Then $(g \circ f) : (S, T, \tau'_{BR}, E) \to (P, Q, \tau''_{BR}, E)$ is contra BR-soft simply* alpha continuous function.*

Proof of Theorem 4.10. Let m_k be any BR-soft open set over $P \times Q$. Since g is contra BR-soft simply* alpha continuous, $g^{-1}(m_k)$ is BR-soft simply* alpha closed over $U \times V$. Since f is BR-soft simply* alpha irresolute, $f^{-1}(g^{-1}(m_k)) = (g \circ f)^{-1}(m_k)$ is BR-soft simply* alpha closed over $S \times T$. Thus $g \circ f$ is a contra BR-soft simply* alpha continuous function.

Definition 4.8. *A map $f : (S, T, \tau'_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ is called perfectly BR-soft simply* alpha continuous if $f^{-1}(m_k)$ is BR-soft simply* alpha clopen in (S, T, τ'_{BR}, E) for every BR-soft open in $(U, V, R_{m(s,t)}, \tau_{BR})$.*

Theorem 4.11. *Every perfectly BR-soft simply* alpha continuous is contra BR-soft simply* alpha continuous.*

Proof of Theorem 4.11. Let $f : (S, T, \tau'_{BR}, E) \to (U, V, R_{m(s,t)}, \tau_{BR})$ be perfectly BR-soft simply* alpha continuous. Let m_k be BR-soft closed in $(U, V, R_{m(s,t)}, \tau_{BR})$. Then, $f^{-1}(m_k)$ is BR-soft simply* alpha clopen, and hence $f^{-1}(m_k)$ is BR-soft simply* alpha open. Thus, f is contra BR-soft simply* alpha continuous.

The converse of the above theorem need not be true, as can be seen from the following example:

Example 4.6. From Example 4.5, it is shown that $f^{-1}(m_k)$ is BR-soft simply* alpha closed but not BR-soft simply* alpha open.

5. Conclusion

A new class of binary soft sets, namely BR-soft simply* alpha open sets, was studied over two different universes. This is followed by the study of the continuous functions of the defined new class of set. Definitions of BR-soft simply* alpha continuous function, BR-soft simply* alpha contra continuous, BR-soft simply* alpha perfectly continuous, BR-soft simply* alpha open map, BR-soft simply* alpha closed map, and BR-soft simply* alpha irresolute are introduced and studied. The properties and results of the definitions are illustrated with examples. The definitions of such a new class of sets and the study of their continuous functions can lead to simplification in the decision making process in various fields of research and may help in further developments. In addition to this study, other topological properties of the defined set are being studied.

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Structural invariants of the product maximal graph

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Abstract. In this paper, some structural properties of the product maximal graph like matching, vertex covering, edge covering and cordial labeling are studied. Furthermore, the number of triangles of $\Gamma_{pm}(R)$ are calculated. The isomorphism between the product maximal graph of cartesian product of two commutative rings and cartesian product of two product maximal graphs of commutative rings and its relations is interpreted with an example.

Keywords: product maximal graph, covering, matching, labeling, cartesian product.
MSC 2020: 05C25, 05C78, 05C15, 05C40

1. Introduction

A graph G is an ordered pair (V, E), where $V = V(G)$ is a set of elements called vertices, $E = E(G)$ is a set of elements called edges and each edge is an unordered pair of vertices (its ends or end vertices or end-points). Graph theory has become a very popular and promptly increasing area of discrete mathematics for its numerous theoretical development and manifold applications to the practical problems. Graphs constructed from algebraic structures have been studied extensively by many authors and have become a major field of research.

Groups as graphs contain the most merging combination which is used repeatedly in the algebraic graph theory. The graphs from groups include power graph, commuting graph, non-commuting graph etc., Another important kind of graph construction is the construction of graphs from rings. The study of graphs from rings contributes to the interplay between the ring invariants and the graph structure. Graphs from rings are introduced by Beck. I [2] and it is

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named as the zero-divisor graphs of a finite commutative ring. The complete summary of graphs from rings and the results are found in [4]. Some graphs [11, 12, 13] may represent the molecular structure of certain chemical compound and it is mainly associated with the different molecular biology.

Moreover, D. Kalamani and G. Ramya [10] defined a new graph from ring called product maximal graph. It is a graph of a finite commutative ring with unity whose vertices are all the elements of ring R and two distinct vertices are adjacent if and only if the product of two vertices are in maximal ideals of R. They also extended the graph properties such as domination number [14] and some graph theoretic properties [9] of the product maximal graph.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labelled graphs serve useful mathematical models for a broad range of applications. Graph labeling is useful in network problems because each network node has a different transmission capacity for sending or receiving messages in wired or wireless link. Most of the details related to graph labeling and different methods of labeling like cordial labeling, graceful labeling, hormonius labeling are in [8]. G. Ramya and D. Kalamani [15] establish that the commuting graph of the subset of the dihedral group admits sum cordial, signed product cordial and divisor cordial labeling. Further notation and terminologies are followed from Frank Harary [7] and Douglas B. West [6] for graphs and from Dummit and Foote [5] for algebra concepts.

2. Preliminaries

In this section, the essential definitions of the matching, covering, labeling, cordial labeling and cartesian product are specified. Also some basic properties of the covering and product maximal graph are given.

Definition 2.1. *Let R be a finite commutative ring with unity and $M_1, M_2, \ldots, M_r$ be the maximal ideals of R. The* ***product maximal graph*** *of a commutative ring R is the graph whose vertices are the elements of R and two distinct vertices u and v are adjacent if and only if the product $uv \in M_i, i = 1, 2, \ldots, r$ and it is denoted by $\Gamma_{pm}(R)$.*

Definition 2.2. *A subset $\mathscr{M}$ of the edge set E is called a* ***matching or edge independent set*** *in G if no two edges of $\mathscr{M}$ are adjacent in a graph G. The two ends of an edge in $\mathscr{M}$ are said to be matched under $\mathscr{M}$.*

Definition 2.3. *A matching $\mathscr{M}$ is a* ***maximum matching*** *if a graph G has no matching $\mathscr{M}'$ with $|\mathscr{M}'| > |\mathscr{M}|$. The number of edges in a maximum matching of G is called the* ***matching number*** *(edge independent number) of a graph G. It is denoted by $\alpha'(G)$.*

Definition 2.4. *A set S of vertices which covers all the edges of a graph G is called* ***vertex cover****, in the sense that every edge of G is incident with some vertex in S. A vertex cover with minimum cardinality is the minimum vertex* *cover, the cardinality of minimum vertex cover is called the* ***vertex covering number*** *and it is denoted by $\beta(G)$ for the graph G.*

Definition 2.5. *A set S of edges which covers all the vertices of a graph G is called* ***edge cover*** *of G. A minimum edge cover is one with minimum cardinality. The cardinality of a minimum edge cover of a graph G is called the* ***edge covering number*** *and it is denoted by $\beta'(G)$. The graph without isolated vertices have an edge cover.*

Definition 2.6. *Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called* ***binary vertex labeling*** *of G and $f(v)$ is called the label of the vertex v of G under f.*

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having label 0 and 1 respectively under f and $e_f(0)$, $e_f(1)$ be the number of edges having label 0 and 1 respectively under f^*.

Definition 2.7. *A vertex labeling $f : V(G) \to \{0, 1\}$ and the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(uv) = |f(u) - f(v)|$. Such labeling is called* ***cordial labeling*** *if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.*

Definition 2.8. *A binary vertex labeling of a graph G with induced edge labeling $f^* : E(G) \to \{0, 1\}$ is defined by $f^*(uv) = (f(u) + f(v))(mod 2)$ is named as* ***sum cordial labeling*** *if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is sum cordial if it admits sum cordial labeling.*

Definition 2.9. *A vertex labeling, $f : V(G) \to \{-1, 1\}$ of a graph G with induced edge labeling $f^* : E(G) \to \{-1, 1\}$ defined by $f^*(uv) = f(u)f(v)$ is called a* ***signed product cordial labeling*** *if $|v_f(-1) - v_f(1)| \leq 1$ and $|e_f(-1) - e_f(1)| \leq 1$. A graph G is signed product cordial if it admits signed product cordial labeling.*

Definition 2.10. *A* ***divisor cordial labeling*** *of a graph G with the vertex set V is a bijection $f : V \to \{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and 0 otherwise. A graph G is divisor cordial if it admits divisor cordial labeling.*

Definition 2.11. *An* ***isomorphism*** *from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$.*

Definition 2.12. *The* ***cartesian product*** *of G and H, written as $G \times H$ is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') iff*

- *$u = u'$ and $vv' \in E(H)$ or*

- $v = v'$ *and* $uu' \in E(G)$.

Theorem 2.1 ([6]). *In a graph G, the subset S is an independent set iff $\overline{S}$ is a vertex cover and hence $\alpha(G) + \beta(G) = n$.*

Theorem 2.2 ([6]). *If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n$, where n is the number of vertices in G.*

Lemma 2.1 ([10]). *Let R be a finite commutative ring with unity and $M_1, M_2, \dots, M_r$ be the maximal ideals of R. Let $\Gamma_{pm}(R)$ be a product maximal graph of R. Then, the degree of vertex v of the graph $\Gamma_{pm}(R)$ is given by*

$$deg(v) = \begin{cases} n-1, & v \in M_i, i = 1, 2, \dots, r, \\ m, & otherwise, \end{cases}$$

where m and n are the cardinalities of $M = \bigcup_{i=1}^{r} M_i$ and R respectively.

Lemma 2.2 ([10]). *Let $\Gamma_{pm}(R)$ be the product maximal graph of a finite commutative ring R. Then, the independent number is $\alpha(\Gamma_{pm}(R)) = n - m$, where m and n are the cardinalities of $M = \bigcup_{i=1}^{r} M_i$ and R respectively.*

3. Main results

In this section, the number of triangles of the product maximal graph and some graph theoretic properties like matching, covering, cordial labeling are discussed. Also the product maximal graph of a finite commutative ring which is isomorphic to the product maximal graph of cartesian product of two commutative rings is found. Moreover, the relation between product maximal graph of cartesian product of two rings and cartesian product of two product maximal graphs are established.

Theorem 3.1. *Let R be a commutative ring of order n, where n is not a prime then the number of triangles for the product maximal graph $\Gamma_{pm}(R)$ is $mC_2(n - m) + mC_3$, where m is the number of elements in $\bigcup_{i=1}^{r} M_i$.*

Proof. We know that the elements which are in maximal ideals form the complete subgraph.

Note that, the number of triangles for the complete graph is mC_3,

$$\therefore \triangle(K_m) = mC_3.$$

Now, the elements which are in non-maximal ideals are adjacent to the elements in maximal ideals. Therefore, $(n-m)$ vertices of non-maximal elements form the triangles with mC_2 vertices of maximal elements. Hence, $\triangle(\Gamma_{pm}(R)) = mC_2(n - m) + mC_3$. $\square$

Corollary 3.1. *If p is a prime, then the product maximal graph of a finite commutative ring of order p has no triangles.*

3.1 Matching number of the product maximal graph

Graph matching has applications in flow networks, scheduling and planning, modeling bonds in chemistry, the stable marriage problem, neural networks in artifical intelligance. A matching is a subset of the edge set such that no two edges have a common vertex. Any matching with the largest size in G is called a maximum matching. A maximal matching in a graph is a matching that cannot be enlarged by adding an edge. Every maximum matching is a maximal matching but the converse need not hold. The matching number of $\Gamma_{pm}(R)$ is shown in Theorem 3.3.

Theorem 3.2. *Let R be the finite commuatative ring with unity and $M_1, M_2, \ldots,$ M_r be the maximal ideals of R then the matching number of the product maximal graph is*

$$\alpha'(\Gamma_{pm}(R)) = \begin{cases} \frac{n}{2}, & n \text{ is even}, \\ m, & n \text{ is odd}, \end{cases}$$

where n is the number of vertices in $\Gamma_{pm}(R)$ and m is the number of elements in $M = \bigcup_{i=1}^{r} M_i$.

Proof. Let $E(\Gamma_{pm}(R))$ be the edge set of the product maximal graph.

The matching set $\mathscr{M}$ is the subset of the edge set $E(\Gamma_{pm}(R))$ and the end points of the edges of $\mathscr{M}$ are obtained in the following ways:

(i) both ends are in maximal ideals.

(ii) one end is in maximal ideal and other end is in non-maximal ideal.

Denote the subset of $\mathscr{M}$ defined by (i) as $\mathscr{M}_1$ and the subset defined by (ii) as $\mathscr{M}_2$. Clearly the matching set $\mathscr{M}$ is the disjoint union of two subsets $\mathscr{M}_1$ and $\mathscr{M}_2$ of the edge set $E(\Gamma_{pm}(R))$. The edges in $\mathscr{M}_1$ and $\mathscr{M}_2$ are all independent.

Case 1. n is even. In this case, $|M| \geq |\overline{M}|$ and m is even.

If $|M| = |\overline{M}|$, then the number of edges in $\mathscr{M}_1$ various from 0 to $\frac{m}{2}$ and the number of edges in $\mathscr{M}_2$ various from 0 to $\frac{n}{2}$. Combining $\mathscr{M}_1$ and $\mathscr{M}_2$, the matching set $\mathscr{M}$ has k edges, where $\frac{m}{2} \leq k \leq \frac{n}{2}$. From this, it is clear that the maximum matching set $\mathscr{M}$ has $\frac{n}{2}$ edges and the matching number is $\frac{n}{2}$

$$\alpha'(\Gamma_{pm}(R)) = \frac{n}{2}.$$

If $|M| > |\overline{M}|$, then the number of edges in $\mathscr{M}_1$ various from $m - \frac{n}{2}$ to $\frac{m}{2}$ and the number of edges in $\mathscr{M}_2$ various from 0 to $n - m$. Combining $\mathscr{M}_1$ and $\mathscr{M}_2$, the matching set $\mathscr{M}$ has k edges, where $\frac{m}{2} \leq k \leq \frac{n}{2}$. From this, it is clear that the maximum matching set $\mathscr{M}$ has $\frac{n}{2}$ edges and the matching number is $\frac{n}{2}$

$$\alpha'(\Gamma_{pm}(R)) = \frac{n}{2}.$$

Case 1. n is odd. In this case, $|M| < |\overline{M}|$ and m is odd.

The number of edges in $\mathscr{M}_1$ various from 0 to $\frac{m-1}{2}$ and the number of edges in $\mathscr{M}_2$ various from 1 to m. Combining $\mathscr{M}_1$ and $\mathscr{M}_2$, the matching set $\mathscr{M}$ has k edges, where $\frac{m+1}{2} \leq k \leq m$. From this, it is clear that the maximum matching set $\mathscr{M}$ has m edges and the matching number is m.

Hence, the maximum matching number of the product maximal graph is $\alpha'(\Gamma_{pm}(R)) = m$. $\square$

Example 3.1. The maximum matching of the product maximal graph $\Gamma_{pm}(\mathbb{Z}/8\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/9\mathbb{Z})$ are shown below:

- Let $R = \mathbb{Z}/8\mathbb{Z}$. The maximal ideal of $\mathbb{Z}/8\mathbb{Z}$ is $M_1 =< 2 >$. Here, n is even and $|M| = |\overline{M}|$ then the matching set $\mathscr{M}$ has 4 edges. $\mathscr{M} = \{(0,1),(2,7),(3,6),(4,5)\}$ is one of the maximum matching set and hence $\alpha'(\Gamma_{pm}(\mathbb{Z}/8\mathbb{Z})) = 4$ and the matching is shown in Figure 1 with colored lines.

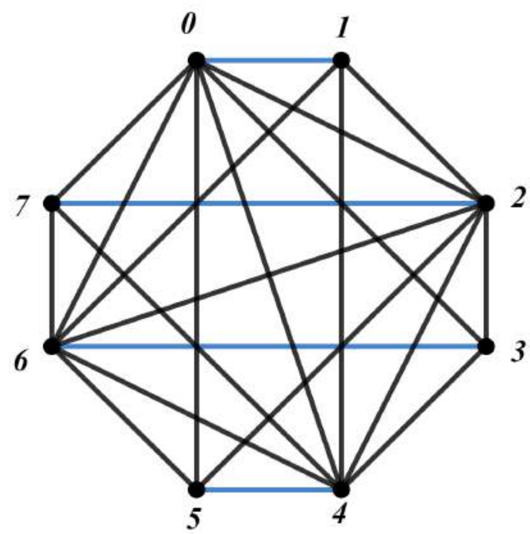

Figure 1: The Graph $\Gamma_{pm}(\mathbb{Z}/8\mathbb{Z})$ with matching.

- Let $R = \mathbb{Z}/9\mathbb{Z}$. The maximal ideal of $\mathbb{Z}/9\mathbb{Z}$ is $M_1 =< 3 >$. Here, n is odd and $|M| < |\overline{M}|$ then the matching set $\mathscr{M}$ may have either 2 or 3 edges. One of the maximum matching set of the graph is $\mathscr{M} = \{(0,1),(3,5),(6,8)\}$ and hence $\alpha'(\Gamma_{pm}(\mathbb{Z}/9\mathbb{Z}) = 3$ and the matching is shown in Figure 2 with colored lines.

3.2 Vertex covering and edge covering of the product maximal graph

Graph covering is one of the classical topics in graph theory. The vertex covering problem, matching number problem are said to be classical optimization problem

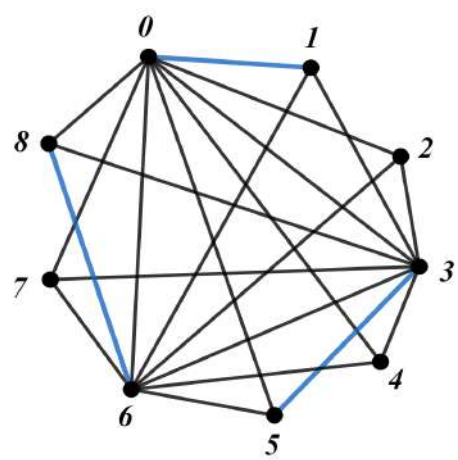

Figure 2: The Graph $\Gamma_{pm}(\mathbb{Z}/9\mathbb{Z})$ with matching.

in computer science. A covering graph is a subgraph that has either all the vertices or all the edges belonging to another graph. Edge covering refers to a subgraph that has all of the vertices. Vertex covering number and edge covering number of the product maximal graph are shown in the following theorems.

Theorem 3.3. *Let $\Gamma_{pm}(R)$ be the product maximal graph of the finite commutative ring R with unity. The vertex covering number of the product maximal graph is $\beta(\Gamma_{pm}(R)) = m$.*

Proof. Let n be the number of vertices in the product maximal graph and m be the number of elements in $M = \bigcup_{i=1}^{r} M_i$.

The vertex cover is the subset of the vertex set of the product maximal graph which covers all the edges of the graph $\Gamma_{pm}(R)$. By Theorem 2.1, the complement of the independent set is a vertex covering set of the graph $\Gamma_{pm}(R)$. By Lemma 2.3, the independent set for the product maximal graph is $I = \{v \in \Gamma_{pm}(R) | v \notin M\}$.

The independence number of $\Gamma_{pm}(R)$ is $n - m$. i.e., $|I| = n - m$. Now, the complement of the independent set of the product maximal graph is $\overline{I} = \{v \in \Gamma_{pm}(R) | v \in M\}$ and its cardinality is m.
Hence, the vertex covering number of the product maximal graph is m.

$\beta(\Gamma_{pm}(R)) = m$. □

Theorem 3.4. *Let R be the finite commutative ring with unity and $M_1, M_2, \ldots, M_r$ be the maximal ideals of R. The edge covering number of the product maximal graph is*

$$\beta'(\Gamma_{pm}(R)) = \begin{cases} \frac{n}{2}, & \textit{if } n \textit{ is} \quad \textit{even}, \\ n - m, & \textit{if } n \textit{ is} \quad \textit{odd}. \end{cases}$$

Proof. An edge cover is the subset of the edge set which covers all the vertices in the graph $\Gamma_{pm}(R)$. By Theorem 2.2, the edge covering number of $\Gamma_{pm}(R)$ is the complement of the matching number in the product maximal graph.

$\therefore$ The edge covering number of $\Gamma_{pm}(R)$ is

$$\beta'(\Gamma_{pm}(R)) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ n-m, & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

Example 3.2. Let $R = \mathbb{Z}/12\mathbb{Z}$ whose maximal ideals are $M_1 =< 2 >$ and $M_2 =< 3 >$.

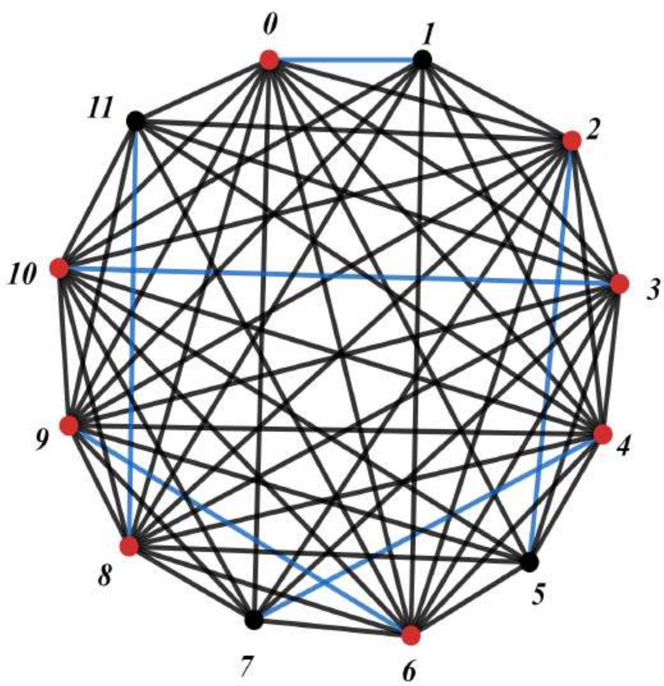

Figure 3: The Graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ with vertex covering and edge covering.

- The subset $\{0, 2, 3, 4, 6, 8, 9, 10\}$ of the vertex set $V(\Gamma_{pm}(R))$ covers all the edges of the graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ and it is the minimum vertex cover. Hence, the vertex covering number $\beta(\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})) = 8$ and the graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ is shown in Figure 3 with colored vertices.
- The subset $\{(0,1), (2,5), (4,7), (8,1), (10,3), (6,9)\}$ of the edge set $E(\Gamma_{pm}(R))$ covers all the vertices of the graph $\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})$ and it is the minimum edge cover. Hence, the edge covering number $\beta'(\Gamma_{pm}(\mathbb{Z}/12\mathbb{Z})) = 6$ and is shown in Figure 3 with colored lines.

3.3 Cordial labeling of the product maximal graph

Cahit [3] has introduced a weeker version of both graceful and harmonious labeling. The following theorem shows that the graph $\Gamma_{pm}(R)$ satisfies sum cordial, signed product cordial and divisor cordial labeling if $n = p^2$, where p is a prime.

Theorem 3.5. *The product maximal graph $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ admits sum cordial labeling if $p = 2$ or $p \equiv 3(mod\, 4)$, where p is a prime number.*

Proof. Let $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ be the product maximal graph, where $\mathbb{Z}/p^2\mathbb{Z}$ is the finite commutative ring with unity and $< p >$ is the maximal ideal of $\mathbb{Z}/p^2\mathbb{Z}$. Assume that, for $i < j$ then, $v_i < v_j$.

Let the vertex bijective mapping $f : V(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ be defined as

$$f(v_i) = \begin{cases} 0, & \text{if } 1 \ \le \ i \ \le \ \lceil \frac{p^2}{2} \rceil, \\ 1, & \text{if } \lceil \frac{p^2}{2} \rceil + 1 \ \le i \ \le p^2. \end{cases}$$

Case 1. $p = 2$. Clearly $v_f(0) = v_f(1) = p$. The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ is given by $f^*(v_iv_j) = |f(v_i) + f(v_j)|(mod 2)$, where $1 \le i, j \le p^2$ and $i \ne j$.

We have $e_f(0) = p$ and $e_f(1) = p + 1$. Thus, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

$\therefore$ The product maximal graph satisfies the sum cordial labeling if $p = 2$.

Case 2: $p \equiv 3(mod\, 4)$. Clearly, $v_f(0) = \lceil \frac{p^2}{2} \rceil$ and $v_f(1) = \lfloor \frac{p^2}{2} \rfloor$.

The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ is given by $f^*(v_iv_j) = |f(v_i) + f(v_j)|(mod 2)$, where $1 \le i, j \le p^2$ and $i \ne j$. We have $e_f(0) = \lfloor \frac{p(p-1)(1+2p)}{4} \rfloor$ and $e_f(1) = \lceil \frac{p(p-1)(1+2p))}{4} \rceil$. Thus, $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

$\therefore$ The product maximal graph satisfies the sum cordial labeling if $p \equiv 3(mod\, 4)$. □

Theorem 3.6. *The product maximal graph $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ admits signed product cordial labeling if $p = 2$ or $p \equiv 3(mod\, 4)$, where p is a prime number.*

Proof. Assume that $i < j$ then $v_i < v_j$.

Let the vertex bijective mapping $f : V(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, -1\}$ be define as

$$f(v_i) = \begin{cases} 1, & \text{if } 1 \ \le \ i \ \le \ \lceil \frac{p^2}{2} \rceil, \\ -1, & \text{if } \lceil \frac{p^2}{2} \rceil + 1 \ \le i \ \le p^2. \end{cases}$$

Case 1. $p = 2$. Clearly $v_f(1) = v_f(-1) = p$.

The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, -1\}$ is given by $f^*(v_iv_j) = f(v_i)f(v_j)$, where $1 \le i, j \le p^2$ and $i \ne j$.

We have $e_f(1) = p$ and $e_f(-1) = p + 1$. Thus, $|v_f(1) - v_f(-1)| \le 1$ and $|e_f(1) - e_f(-1)| \le 1$.

$\therefore$ The product maximal graph satisfies the signed product cordial labeling if $p = 2$.

Case 2. $p \equiv 3(mod\ 4)$. Clearly, $v_f(1) = \lceil \frac{p^2}{2} \rceil$ and $v_f - (1) = \lfloor \frac{p^2}{2} \rfloor$.

The edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, -1\}$ is given by $f^*(v_iv_j) = f(v_i)f(v_j)$, where $1 \leq i, j \leq p^2$ and $i \neq j$.

We have $e_f(1) = \lfloor \frac{p(p-1)(1+2p)}{4} \rfloor$ and $e_f(-1) = \lceil \frac{p(p-1)(1+2p)}{4} \rceil$. Thus, $|v_f(1) - v_f(-1)| \leq 1$ and $|e_f(1) - e_f(-1)| \leq 1$.

$\therefore$ The product maximal graph satisfies the signed product cordial labeling if $p \equiv 3(mod\ 4)$. $\square$

Theorem 3.7. *The product maximal graph $\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})$ admits divisor cordial labeling if $p = 2$ or $p \equiv 3(mod 4)$, where p is a prime number.*

Proof. Let the vertex bijective mapping $f : V(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{1, 2, \ldots, p^2\}$ be define as $f(v_i) = i$, where $1 \leq i \leq p^2$.

Then, the edge labeling $f^* : E(\Gamma_{pm}(\mathbb{Z}/p^2\mathbb{Z})) \longrightarrow \{0, 1\}$ is given by $f^*(v_iv_j) = f(v_i)|f(v_j)$ or $f(v_i)|f(v_j)$, where $1 \leq i, j \leq p^2$ and $i \neq j$.

We have $e_f(0) = \lceil \frac{p(p-1)(1+2p)}{4} \rceil$ and $e_f(1) = \lfloor \frac{p(p-1)(1+2p)}{4} \rfloor$.

$\therefore |e_f(1) - e_f(-1)| \leq 1$.

The product maximal graph satisfies the divisor cordial labeling if $p = 2$ and $p \equiv 3(mod\ 4)$. $\square$

4. Cartesian product

In this section, the product maximal graph of the cartisean product $R \times S$ of two finite commutative rings R and S and the cartesian product of the product maximal graphs $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$ are studied and its relation is also discussed in the subsequent theorems.

4.1 Product maximal graph of the cartesian product of two rings

Let R and S be two finite commutative rings with unity whose orders are n_1 and n_2 respectively. Then, the cartesian product $R \times S$ of two rings is also a finite commutative ring with unity whose order is n_1n_2. If I and J are the ideals of R and S respectively then every ideal of $R \times S$ is of the form $I \times J$ [1].

Let $M_i, i = 1, 2, \ldots, r$ and $N_j, j = 1, 2, \ldots, s$ be the maximal ideals of R and S respectively then the maximal ideals of $R \times S$ are of the form $R \times N_j$ and $M_i \times S$. Note that the number of maximal ideals of $R \times S$ is equal to the sum of the number of maximal ideals of R and S.

The next theorem explains the isomorphism between the product maximal graph of cartesian product of $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$ and the product maximal graph $\mathbb{Z}/n_1n_2\mathbb{Z}$. It is proved that $\mathbb{Z}/n_1n_2\mathbb{Z} \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ whenever $(n_1, n_2) = 1$ in [5].

Theorem 4.1. *Let $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$ be the product maximal graphs of $\mathbb{Z}/n_1n_2\mathbb{Z}$ and $\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z}$ respectively then $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z}) \cong \Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$ whenever $(n_1, n_2) = 1$.*

Proof. Let $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$ be the product maximal graph of $\mathbb{Z}/n_1n_2\mathbb{Z}$ and $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$ be the product maximal graph of cartesian product of $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$, where $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$ are the integers modulo n_1 and n_2 respectively.

Let the vertex mapping $f : V(\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})) \longrightarrow V(\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z}))$ be defined as $[x]_{n_1n_2} \longrightarrow ([x]_{n_1}, [x]_{n_2})$, where $[x]_{n_1}$ is the residue class of $x \ mod \ n_1$. Obviously the function f is bijective.

Let $M_1, M_2, \ldots, M_r$ and $N_1, N_2, \ldots, N_s$ be the maximal ideals of $\mathbb{Z}/n_1\mathbb{Z}$ and $\mathbb{Z}/n_2\mathbb{Z}$ respectively and $M_1, M_2, \ldots, M_r, N_1, N_2, \ldots, N_s$ be the maximal ideals of $\mathbb{Z}/n_1n_2\mathbb{Z}$.

Let x and y be any two vertices in $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$. If x and y are adjacent in $\Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z})$ then either x or y is an element of the maximal ideal of $\mathbb{Z}/n_1n_2\mathbb{Z}$. Without loss of generality, we assume that x is an element of the maximal ideal of $\mathbb{Z}/n_1n_2\mathbb{Z}$.

Since f is bijective, $f(x)$ is an element of the maximal ideal of $\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z}$. This implies that $f(x)$ is adjacent to all other elements in $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z}\times\mathbb{Z}/n_2\mathbb{Z})$. It means that $f(x)$ and $f(y)$ are adjacent in $\Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z})$.

$$\therefore \Gamma_{pm}(\mathbb{Z}/n_1n_2\mathbb{Z}) \cong \Gamma_{pm}(\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}).$$

□

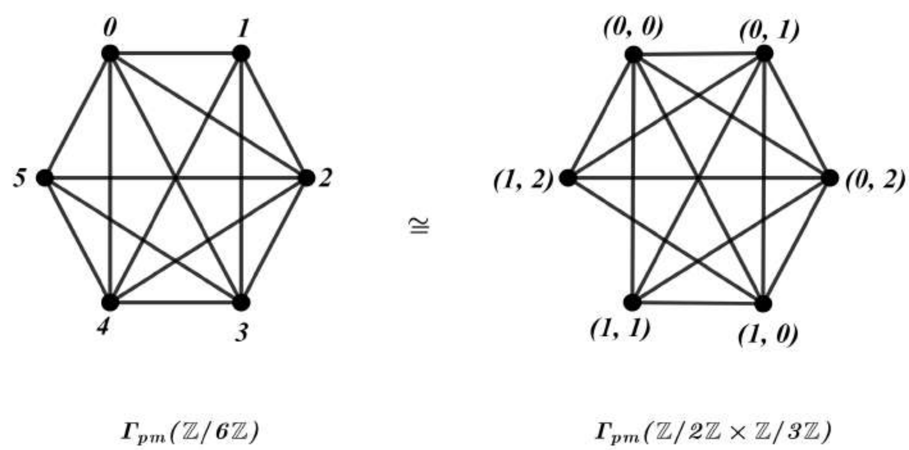

Figure 4: $\Gamma_{pm}(\mathbb{Z}/6\mathbb{Z}) \cong \Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$.

Example 4.1. Consider $\Gamma_{pm}(\mathbb{Z}/6\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$ be the product maximal graphs. The isomorphism between $\Gamma_{pm}(\mathbb{Z}/6\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z})$ is shown in Figure 4.

4.2 Cartesian product of two product maximal graphs $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$

In this section, the graph theoretic property like cartesian product of two graphs is explained and it is applied for product maximal graph of the commutative ring.

Theorem 4.2. *Let $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$ be the two product maximal graphs of commutative rings R and S respectively. Then, the degree of the vertex (x, y) of the cartesian product $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ is $deg(x, y) = deg(x) + deg(y)$, where $x \in \Gamma_{pm}(R)$ and $y \in \Gamma_{pm}(S)$.*

Proof. Let $M_1, M_2, \ldots, M_r$ and $N_1, N_2, \ldots, N_s$ be the maximal ideals of R and S respectively. By Lemma 2.3, the degree of the vertex x of the product maximal graph $\Gamma_{pm}(R)$ is

$$deg(x) = \begin{cases} n_1 - 1, & x \in M_i, i = 1, 2, \ldots r, \\ m, & \text{otherwise}, \end{cases} \tag{1}$$

where $m = |\bigcup_{i=1}^{r} M_i|$. Similarly, the degree of the vertex y of the product maximal graph $\Gamma_{pm}(S)$ is

$$deg(y) = \begin{cases} n_2 - 1, & y \in N_j, i = 1, 2, \ldots, s \\ n, & \text{otherwise}, \end{cases} \tag{2}$$

where $n = |\bigcup_{j=1}^{s} N_j|$.

The degree of the vertex (x, y) of the cartesian product of two graphs $\Gamma_{pm}(R)$ and $\Gamma_{pm}(S)$ is

$$deg(x, y) = \begin{cases} n_1 + n_2 - 1, & x \in M_i, y \in N_j \\ n_1 - 1 + n, & x \in M_i, y \notin N_j \\ m + n_2 - 1, & x \notin M_i, y \in N_j \\ m + n, & x \notin M_i, y \notin N_j. \end{cases} \tag{3}$$

Comparing (1), (2) and (3), we conclude that

$$deg(x, y) = deg(x) + deg(y).$$

□

The following example explains the degree of the cartesian product of two graphs $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

Example 4.2. Consider the cartesian product $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ of two product maximal graphs $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$. The graphs of $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$, $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ are shown in Figure 5. Table 1 shows the degrees of some of the vertices (x, y) of $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$, $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

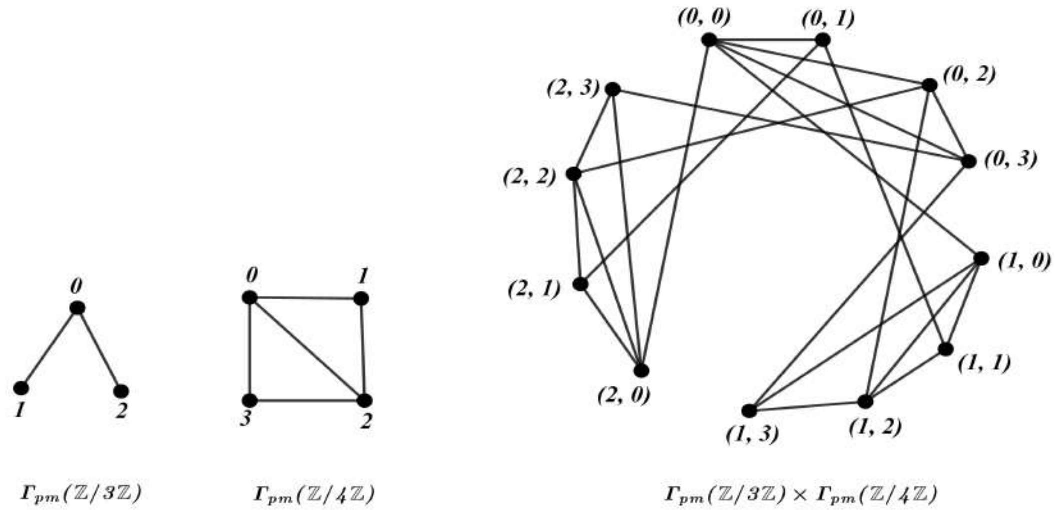

Figure 5: Cartesian product of two product maximal graph $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

Table 1: Degrees of $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$, $\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

$\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z})$	$\Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$	$\Gamma_{pm}(\mathbb{Z}/3\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$
$deg(0) = 2$	$deg(0) = 3$	$deg(0,0) = 5$
$deg(1) = 1$	$deg(1) = 2$	$deg(1,1) = 3$
$deg(2) = 1$	$deg(2) = 3$	$deg(2,2) = 4$

Theorem 4.3. *Let $\Gamma_{pm}(R \times S)$ be the product maximal graph of cartesian product R and S and $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ be the cartesian product of two product maximal graphs then $\Gamma_{pm}(R \times S)$ is not isomorphic to $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$.*

Proof. Let $M_i, i = 1, 2, \ldots, r$ and $N_j, j = 1, 2, \ldots, s$ be the maximal ideals of R and S respectively and $T_1, T_2, \ldots, T_t$ be the maximal ideals of $R \times S$, where T_k is either in $R \times N_j$ or $M_i \times S$.

Since the number of vertices of $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ is $n_1 n_2$ then the degree of the vertex (x, y) in $\Gamma_{pm}(R \times S)$ is

$$deg(x,y) = \begin{cases} n_1 n_2 - 1, & (x,y) \in T_k, k = 1, 2, \ldots, t, \\ l, & \text{otherwise}, \end{cases} \tag{4}$$

where l be the number of elements in $l = |\bigcup_{k=1}^{t} T_k|$. By Theorem 4.2, the degree of the vertex (x, y) in the cartesian product of two product maximal graphs $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ is

$$deg(x, y) = deg(x) + deg(y) = n_1 + n_2. \tag{5}$$

But from (4) and (5) the degrees are not equal i.e., the degree of any vertex in $\Gamma_{pm}(R \times S)$ is not the same as the degree of that vertex in $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$. Hence, the graphs $\Gamma_{pm}(R \times S)$ and $\Gamma_{pm}(R) \times \Gamma_{pm}(S)$ are not isomorphic. $\square$

Example 4.3. Consider the product maximal graph $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$. Then, Figure 6 shows that there is no isomorphism between $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$. The degree of every vertex in two graphs are given in the Table 2 .

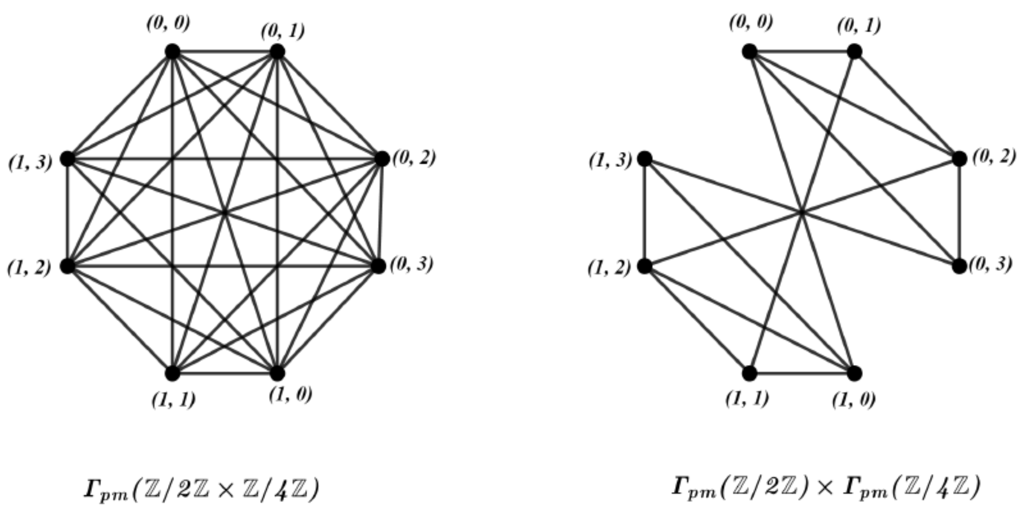

Figure 6: $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ is not isomorphic to $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$.

Table 2: Degrees of $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$ and $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$

(x, y)	**degree of** $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$	**degree of** $\Gamma_{pm}(\mathbb{Z}/2\mathbb{Z}) \times \Gamma_{pm}(\mathbb{Z}/4\mathbb{Z})$
$deg(0, 0)$	7	4
$deg(0, 1)$	7	3
$deg(0, 2)$	7	4
$deg(0, 3)$	7	3
$deg(1, 0)$	7	4
$deg(1, 1)$	6	3
$deg(1, 2)$	7	4
$deg(1, 3)$	6	3

Conclusion

In this paper, the product maximal graph and its graph theoretic properties like matching, covering and some cordial labeling are studied. Also, the relation between the product maximal graph of cartesian product of finite commutative ring and cartesian product of two product maximal graphs are discussed with theorems and suitable examples.

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Degree sum exponent distance energy of non-commuting graph for dihedral groups

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Abstract. The non-commuting graph is defined on a finite group G, denoted by Γ_G, with $G\backslash Z(G)$ is the vertex set of Γ_G and $v_p \neq v_q \in G\backslash Z(G)$ are adjacent whenever they do not commute in G. In this paper, we focus on Γ_G for dihedral groups of order $2n$, D_{2n}, where $n \geq 3$. We show the spectrum, spectral radius and energy of the graph corresponding to the degree sum exponent distance matrix and analyze the hyperenergetic property. Moreover, we then present the correlation between the obtained energy and the adjacency energy.

Keywords: non-commuting graph, the energy of a graph, dihedral group, degree sum exponent distance matrix.

1. Introduction

Let G be a group and $Z(G)$ be a center of G. The non-commuting graph of G, denoted by Γ_G, has vertex set $G\backslash Z(G)$ and two distinct vertices v_p, v_q in Γ_G are connected by an edge whenever $v_pv_q \neq v_qv_p$ ([1]).

The non-commuting graphs have been studied by many authors for various kinds of groups. Abdollahi et al. [1] discussed Γ_G for a non-abelian group G and stated that it is always connected with diameter 2. Consequently, the distance between two vertices in Γ_G is well defined, and it is the length of the shortest path between v_p and v_q. Moreover, this discussion continues by examining the isomorphic properties of two non-commuting graphs related to the isomorphic properties of the corresponding groups. Darafsheh [6] proved the conjecture

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that two non-commuting graphs which are isomorphic imply that the groups are also isomorphic as well. Likewise, Abdollahi and Shahverdi [2] stated that if Γ_G is isomorphic to Γ_G of the alternating group A_n, then $G \cong A_n$. Besides, they presented this conjecture as verified for Γ_G with the simple groups of Lie type.

Afterward, Tolue et al. [28] extended the study of Γ_G and introduced the new concept of $g-$non-commuting graph of finite groups that involve the commutator between two members of the group. If two groups are isoclinic and the numbers of their center are the same, then their associated $g-$non-commuting graphs are isomorphic. Moreover, Khasraw, et al. [15] presented the mean distance of Γ_G for the dihedral groups.

Moreover, Γ_G on n vertices can be interpreted with the adjacency matrix of Γ_G. It is $A(\Gamma_G) = [a_{pq}]$ of size $n \times n$ whose entries $a_{pq} = 1$ for adjacent v_p and v_q; otherwise, $a_{pq} = 0$. For the identity matrix of order n, I_n, the characteristic polynomial of Γ_G is defined as $P_{A(\Gamma_G)}(\lambda) = det\,(\lambda I_n - A(\Gamma_G))$, and its roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the eigenvalues of Γ_G. The spectrum of Γ_G is $Spec(\Gamma_G) = \left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m}\right\}$, with $k_1, k_2, \ldots, k_m$ are the respective multiplicities of $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Energy of Γ_G is calculated by adding all the absolute values of $\lambda_1, \lambda_2, \ldots, \lambda_n$. Gutman [10] pioneered this definition in 1978. The graph energy on n vertices with a value more than $E_A(K_n)$ can be stated as hyperenergetic, or it can be said that $E(\Gamma_G) > 2(n-1)$ [16]. In addition, the adjacency energy bounds of the graph can be found at [7] and graphs with self-loops can be seen at [11]. Additionally, Sun et al. have demonstrated that the clique path has the maximum distance of eigenvalues and energy in their work [27]. It has been shown that the adjacency energy is not equal to an odd integer [4] and is never equal to its square root [18].

In 2008, Indulal et al. [12] introduced the graph matrix whose entries depend on the distance between two vertices. They showed the distance energy of graphs. For the degree product distance energy, the readers can refer [13]. Moreover, the discussion of the degree sum exponent distance of graphs can be found in [14].

In this work, the set of vertex for Γ_G is the non-abelian dihedral group of order $2n$, D_{2n} where $n \geq 3$ which denoted by $D_{2n} = \left\langle a, b : a^n = b^2 = e, bab = a^{-1}\right\rangle$ [3]. The center of D_{2n} and the centralizer of v, where $v \in D_{2n}$ are denoted by $Z(D_{2n})$ and $C_{D_{2n}}(v)$, respectively. Therefore, we have

$$Z(D_{2n}) = \begin{cases} \{e\}, & \text{if } n \text{ is odd} \\ \left\{e, a^{\frac{n}{2}}\right\}, & \text{if } n \text{ is even,} \end{cases} \quad C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}, \text{ and}$$

$$C_{D_{2n}}(a^i b) = \begin{cases} \{e, a^i b\}, & \text{if } n \text{ is odd} \\ \left\{e, a^{\frac{n}{2}}, a^i b, a^{\frac{n}{2}+i} b\right\}, & \text{if } n \text{ is even.} \end{cases}$$

Energy studies of the commuting and non-commuting graphs involving D_{2n} as the set of vertex have been carried out by several authors. Romdhini and Nawawi [21, 22] and Romdhini et al. [23] formulated the energy of Γ_G by considering the eigenvalues of the degree sum, degree subtraction, and neighbors degree sum matrices, meanwhile, [17] presented the adjacency energy. The degree exponent sum, maximum and minimum degree energies were shown in [24, 25].

In studies of correlations between molecules containing heteroatoms and their total electron energy, Gowtham and Swamy [9] reports a correlation coefficient of 0.952 between Sombor energy values and total electron energy. The authors of Redzepovic and Gutman [20] also developed a numerical approach to compare a graph's Sombor energy with its adjacency energy, and it remains an open problem for mathematical verification. Based on these two papers, the authors take the initiative to apply it to Γ_G. Then, this paper is dedicated to formulating the energy based on the degree sum exponent distance matrix $DSED$ for Γ_G on D_{2n} and comparing the results obtained and the adjacency energy.

2. Preliminaries

In this part, we begin with the definition of $DSED$-matrix. Suppose that d_{pq} is the distance between vertex v_p and v_q in Γ_G and d_{v_p} is the degree of vertex v_p.

Definition 2.1 ([14]). *The degree sum exponent distance matrix of Γ_G is an $n \times n$ matrix $DSED(\Gamma_G) = [dsed_{pq}]$ whose (p, q)-th entry is*

$$dsed_{pq} = \begin{cases} \left(d_{v_p} + d_{v_q}\right)^{d_{pq}}, & \text{if } v_p \neq v_q \\ 0, & \text{if } v_p = v_q. \end{cases}$$

The $DSED-$energy of Γ_G is given by

$$E_{DSED}(\Gamma_G) = \sum_{i=1}^{n} |\lambda_i|,$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ represent the eigenvalues (not necessarily distinct) of $DSED(\Gamma_G)$.

The degree sum exponent distance spectral radius of Γ_G is

$$\rho_{DSED}(\Gamma_G) = max\{|\lambda| : \lambda \in Spec(\Gamma_G)\}. \tag{1}$$

From the fact that Γ_G has $2n-1$ and $2n-2$ vertices for odd and even n, respectively, then Γ_G can be classified as hyperenergetic whenever the $DSED-$energy fulfil the following terms:

$$E_{DSED}(\Gamma_G) > \begin{cases} 4(n-1), & \text{for odd } n \\ 4(n-1) - 2, & \text{for even } n, \end{cases}. \tag{2}$$

We now supply some previous results in support of the theorems derived in Section 3. Obtaining the graph energy requires formulating the characteristic polynomial of Γ_G. Here is an essential result that assists in formulating the characteristic polynomial of Γ_G.

Theorem 2.1 ([8]). *If* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is a square matrix with four block matrices and* $|A| \neq 0$, *then*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A| \left| D - CA^{-1}B \right|.$$

Lemma 2.1 ([5]). *If* K_n *is the complete graph on n vertices, then its adjacency matrix is* $(J-I)_n$, *and the spectrum is* $\{(n-1)^{(1)}, (-1)^{(n-1)}\}$.

This article concerned on D_{2n} of order $2n$, D_{2n}, where $n \geq 3$. Let $G_1 = \{a^i : 1 \leq i \leq n\} \backslash Z(D_{2n})$ and $G_2 = \{a^i b : 1 \leq i \leq n\}$. Now, the degree of every vertex of Γ_G for $G = G_1 \cup G_2$ is determined as follows:

Theorem 2.2 ([15]). *Let* Γ_G *be the non-commuting graph on* G, *where* $G = G_1 \cup G_2$. *Then*

1. $d_{a^i} = n$, *and*
2. $d_{a^i b} = \begin{cases} 2(n-1), & \text{if } n \text{ is odd} \\ 2(n-2), & \text{if } n \text{ is even.} \end{cases}$.

Thus, we can see the isomorphism between Γ_G and some common graph types in the theorem as given below:

Theorem 2.3 ([15]). *Let* Γ_G *be a non-commuting graph for* G.

1. *If* $G = G_1$, *then* $\Gamma_G \cong \bar{K}_s$, *for* $s = |G_1|$.
2. *If* $G = G_2$, *then* $\Gamma_G \cong \begin{cases} K_n, & \text{if } n \text{ is odd} \\ K_n - \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$,

where $\frac{n}{2}K_2$ *denotes* $\frac{n}{2}$ *copies of* K_2.

In order to compare the $DSED$ and adjacency energies of Γ_G for D_{2n}, here we write the adjacency energy from Mahmoud et al. [17] as given below:

Theorem 2.4 ([17]). *The adjacency energy of* Γ_G, *where* $G = G_1 \cup G_2$, $E_A(\Gamma_G)$ *is*

1. *for odd* n, $E_A(\Gamma_G) = (n-1) + \sqrt{5n^2 - 6n + 1}$, *and*
2. *for even* n, $E_A(\Gamma_G) = \begin{cases} 8, & \text{if } n = 4 \\ (n-2) + \sqrt{5n^2 - 12n + 4}, & \text{if } n > 4 \end{cases}$.

To define the elements of $DSED-$matrix, we need to determine the distance for every pair of vertices in Γ_G, for $G = G_1 \cup G_2$. The discussion is in Theorem 2.5 below:

Theorem 2.5 ([26]). *For two distinct vertices v_p, v_q in Γ_G, where $G = G_1 \cup G_2$, the distance between v_p and v_q is*

1. *for the odd n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 1, & \text{otherwise,} \end{cases}$ *and*

2. *for the even n,* $d_{pq} = \begin{cases} 2, & \text{if } (v_p, v_q \in G_1) \text{ or } \left(v_p \in G_2, v_q \in \left\{a^{\frac{n}{2}+i}b\right\},\right. \\ & \left.\text{or vice versa}\right) \\ 1, & \text{otherwise.} \end{cases}$.

3. Characteristic polynomial of some matrices

Several properties need to be performed in order to provide $DSED-$energy of Γ_G, for $G = G_1 \cup G_2$ in Section 4. In this section, we derive three theorems of the solution of the determinant of a particular matrix.

Lemma 3.1 ([19]). *If $a, b, c,$ and d are real numbers, and J_n is an $n \times n$ matrix whose all entries are equal to one, then the determinant of*

$$\begin{vmatrix} (\lambda + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\lambda + b)I_{n_2} - bJ_{n_2} \end{vmatrix}$$

can be simplified as

$$(\lambda + a)^{n_1 - 1}(\lambda + b)^{n_2 - 1}\left((\lambda - (n_1 - 1)\, a)\,(\lambda - (n_2 - 1)\, b) - n_1 n_2 cd\right),$$

where $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

Theorem 3.1. *For real numbers a, b, the characteristic polynomial of an $n \times n$ matrix*

$$M = \begin{bmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{bmatrix}$$

can be simplified as

$$P_M(\lambda) = (\lambda - a - (n-1)b)(\lambda - a + b)^{n-1}.$$

Proof. Let a, b are real numbers and M is a square matrix of order n as

$$M = \left[(a - b)I_n + bJ_n\right].$$

Then, we get the characteristic polynomial of M as

$$P_M(\lambda) = |\lambda I_n - M| = |(\lambda - a + b)I_n - bJ_n|\,. \tag{3}$$

The first step, we apply $R_i' = R_i - R_1$, for $2 \leq i \leq n$. Consequently, Equation 3 is as the following:

$$P_M(\lambda) = \begin{vmatrix} \lambda - a & -bJ_{1\times(n-1)} \\ -(\lambda - a + b)J_{(n-1)\times 1} & (\lambda - a + b)I_{(n-1)} \end{vmatrix}. \tag{4}$$

The next step is replacing C_1 by $C_1' = C_1 + C_2 + C_3 + \ldots + C_n$, then Equation 4 can be written as

$$P_M(\lambda) = \begin{vmatrix} \lambda - a - (n-1)b & -bJ_{1\times(n-1)} \\ 0_{(n-1)\times 1} & (\lambda - a + b)I_{(n-1)} \end{vmatrix}. \tag{5}$$

It is obvious from Equation 5, $P_M(\lambda)$ is an upper triangle matrix. Thus, it can be simplified as given below:

$$P_M(\lambda) = (\lambda - a - (n-1)b)(\lambda - a + b)^{n-1},$$

and we complete the proof. $\square$

Theorem 3.2. *For real numbers a, b, the characteristic polynomial of an $n \times n$ matrix*

$$M = \begin{bmatrix} U & V \\ V & U \end{bmatrix},$$

where $U = [b(J-I)_{\frac{n}{2}}]$ and $V = [b(J-I)_{\frac{n}{2}} + aI_{\frac{n}{2}}]$, can be simplified as

$$P_M(\lambda) = (\lambda - a + 2b)^{\frac{n}{2}-1}\left(\lambda - a - (n-2)\,b\right)(\lambda + a)^{\frac{n}{2}}.$$

Proof. For real numbers s, t, suppose that M is an $n \times n$ matrix

$$M = \begin{bmatrix} U & V \\ V & U \end{bmatrix} = \begin{bmatrix} 0 & \ldots & b & a & \ldots & b \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b & \ldots & 0 & b & \ldots & a \\ a & \ldots & b & 0 & \ldots & b \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b & \ldots & a & b & \ldots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} b(J-I)_{\frac{n}{2}} & b(J-I)_{\frac{n}{2}} + aI_{\frac{n}{2}} \\ b(J-I)_{\frac{n}{2}} + aI_{\frac{n}{2}} & b(J-I)_{\frac{n}{2}} \end{bmatrix}.$$

Then, equation $P_M(\lambda) = |\lambda I_n - M|$ can be written as follows:

$$P_M(\lambda) = \begin{vmatrix} (\lambda + b)I_{\frac{n}{2}} - bJ_{\frac{n}{2}} & -aI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} \\ -bI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} & (\lambda + b)I_{\frac{n}{2}} - bJ_{\frac{n}{2}} \end{vmatrix}. \tag{6}$$

To solve the determinant in Equation 6, it is necessary to perform row and column operations. The first step is replacing $R_{\frac{n}{2}+i}$ by $R'_{\frac{n}{2}+i} = R_{\frac{n}{2}+i} - R_i$, where $1 \leq i \leq \frac{n}{2}$. Consequently, Equation 6 is as the following:

$$P_M(\lambda) = \begin{vmatrix} (\lambda+b)I_{\frac{n}{2}} - bJ_{\frac{n}{2}} & -aI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} \\ -(\lambda+a)I_{\frac{n}{2}} & (\lambda+a)I_{\frac{n}{2}} \end{vmatrix}. \tag{7}$$

Next, the second step is replacing C_i by $C'_i = C_i + C_{\frac{n}{2}+i}$, where $1 \leq i \leq \frac{n}{2}$. Hence, Equation 7 can be written as follows:

$$P_M(\lambda) = \begin{vmatrix} (\lambda-a+2b)I_{\frac{n}{2}} - 2bJ_{\frac{n}{2}} & -aI_{\frac{n}{2}} - b(J-I)_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda+a)I_{\frac{n}{2}} \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}. \tag{8}$$

Bearing in mind Theorem 2.1 and since $C = 0$, it implies Equation 8 can be simplified to

$$P_M(\lambda) = |A|\,|D|. \tag{9}$$

We first consider $|A|$ using Theorem 3.1 as follows:

$$|A| = (\lambda - a + 2b)^{\frac{n}{2}-1}(\lambda - a - (n-2)\,b). \tag{10}$$

Meanwhile, as a result of D as a diagonal matrix, as a consequence, we derive:

$$|D| = (\lambda + a)^{\frac{n}{2}}. \tag{11}$$

Therefore, by substituting Equations 10 and 11 to Equation 9, we obtain

$$P_M(\lambda) = (\lambda - a + 2b)^{\frac{n}{2}-1}(\lambda - a - (n-2)\,b)\,(\lambda + a)^{\frac{n}{2}}. \qquad \square$$

Theorem 3.3. *For real numbers a, b, c, d, the characteristic polynomial of a $(2n-2)\times(2n-2)$ matrix:*

$$M = \begin{bmatrix} a(J-I)_{n-2} & cJ_{(n-2)\times\frac{n}{2}} & cJ_{(n-2)\times\frac{n}{2}} \\ cJ_{\frac{n}{2}\times(n-2)} & d(J-I)_{\frac{n}{2}} & d(J-I)_{\frac{n}{2}} + bI_{\frac{n}{2}} \\ cJ_{\frac{n}{2}\times(n-2)} & d(J-I)_{\frac{n}{2}} + bI_{\frac{n}{2}} & d(J-I)_{\frac{n}{2}} \end{bmatrix},$$

can be simplified as

$$P_M(\lambda) = (\lambda + a)^{n-3}(\lambda - b + 2d)^{\frac{n}{2}-1}(\lambda + b)^{\frac{n}{2}}$$
$$\left(\lambda^2 - (b + (n-2)d + a(n-3))\,\lambda + a(n-3)\,(b + (n-2)d) - n(n-2)c^2\right).$$

4. Degree sum exponent distance energy of non-commuting graph for dihedral groups

This section will present the results of non-commuting graph energy for D_{2n}, using the corresponding $DSED$-matrix. Since for $n = 1$ and $n = 2$, D_{2n} is abelian, then strictly it is for $n \geq 3$. The following is an example of Γ_G for D_{2n}, where $n = 4$.

Example 4.1. Let $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ and $Z(D_8) = \{e, a^2\}$, where $C_{D_8}(a^i) = \{e, a, a^2, a^3\}$, $C_{D_8}(b) = \{e, a^2, b, a^2b\} = C_{D_8}(a^2b)$, $C_{D_8}(ab) = \{e, a^2, ab, a^3b\} = C_{D_8}(a^3b)$. For $G = D_8\backslash Z(D_8)$, according to each element's centralizer in G, as a consequence, Γ_G is presented in Figure 1.

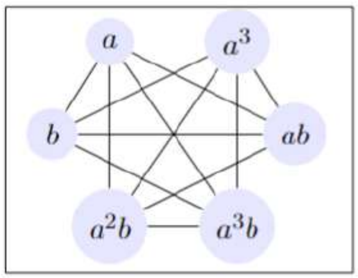

Figure 1: Non-commuting graph for D_8

The vertex degree of a and a^3 is four. Similarly, for $1 \leq i \leq 4$, and the degree of b, ab, a^2b, and a^3b is also four. The distance between a and b, between a^2b and a^3b, and between a^3 and ab are found to be equal, i.e. equal to one, otherwise it is two.

In the next theorem, we derive $DSED-$energy of Γ_G in terms of $G = G_1$ and $G = G_2$.

Theorem 4.1. *Let Γ_G be the non-commuting graph on G.*

1. *If $G = G_1$, then $E_{DSED}(\Gamma_G)$ is undefined, and*

2. *If $G = G_2$, then* $E_{DSED}(\Gamma_G) = \begin{cases} 4(n-1)^2, & \text{if } n \text{ is odd} \\ 4n(n-2)^2, & \text{if } n \text{ is even.} \end{cases}$.

Proof. 1. For $G = G_1$ case, by Theorem 2.3, $\Gamma_G \cong \bar{K}_m$, where $m = |G_1|$. Then, Γ_G consists of m isolated vertices which implies the distance of every pair vertices of G_1 is undefined.

2. For the second case when $G = G_2$, we first proceed for odd n. Again, by Theorem 2.3, $\Gamma_G \cong K_n$. Then, for every v_p of Γ_G, $d_{v_p} = (n-1)$ and every pair of vertices are at distance 1. Now, the $DSED-$matrix of Γ_G is $DSED(\Gamma_G) =$

$dsed_{pq}$, with $(p,q)-$entry if $v_p \neq v_q$ is $((n-1)+(n-1))^1 = 2(n-1)$, and zero if $v_p = v_q$. Hence,

$$\begin{aligned} DSED(\Gamma_G) &= \begin{bmatrix} 0 & 2(n-1) & 2(n-1) & \dots & 2(n-1) \\ 2(n-1) & 0 & 2(n-1) & \dots & 2(n-1) \\ 2(n-1) & 2(n-1) & 0 & \dots & 2(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 0 \end{bmatrix} \\ &= 2(n-1)A(K_n). \end{aligned}$$

In other words, $DSED(\Gamma_G)$ is the product of $2(n-1)$ and $A(K_n)$. Therefore, from Lemma 2.1, the $DSED-$energy of Γ_G is $2(n-1).2(n-1) = 4(n-1)^2$.

Meanwhile for the even n, by Theorem 2.3, $\Gamma_G \cong K_n - \frac{n}{2}K_2$, then every vertex has degree $(n-2)$ and the distance between every pair a^ib and $a^{\frac{n}{2}+i}$ for all $1 \leq i \leq n$ is 2, and 1, otherwise. Thus, $DSED(\Gamma_G) = dsed_{pq}$ and for $v_p \neq v_q$,

$$dsed_{ij} = \begin{cases} 4(n-2)^2, & \text{if } v_p = a^ib,\ v_q = a^{\frac{n}{2}+i}b,\ 1 \leq i \leq n \\ 2(n-2), & \text{if } v_p = a^ib,\ v_q \neq a^{\frac{n}{2}+i}b,\ 1 \leq i \leq n\ . \\ 0, & otherwise. \end{cases}$$

Now, we can construct $DSED(\Gamma_G)$ as follows:

$$DSED(\Gamma_G) = \begin{bmatrix} 0 & \dots & 2(n-2) & 4(n-2)^2 & \dots & 2(n-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-2) & \dots & 0 & 2(n-2) & \dots & 4(n-2)^2 \\ 4(n-2)^2 & \dots & 2(n-2) & 0 & \dots & 2(n-2) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-2) & \dots & 4(n-2)^2 & 2(n-2) & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(n-2)(J-I)_{\frac{n}{2}} & 2(n-2)(J-I)_{\frac{n}{2}} + 4(n-2)^2 I_{\frac{n}{2}} \\ 2(n-2)(J-I)_{\frac{n}{2}} + 4(n-2)^2 I_{\frac{n}{2}} & 2(n-2)(J-I)_{\frac{n}{2}} \end{bmatrix}.$$

In this case, we have four block matrices of $DSED(\Gamma_G)$:

$$DSED(\Gamma_G) = \begin{bmatrix} U & V \\ V & U \end{bmatrix}, \tag{12}$$

where U and V are $\frac{n}{2} \times \frac{n}{2}$ matrices. Matrix U consists of zero diagonal entries, otherwise, the entries are $2(n-2)$, while the diagonal entries of V are $4(n-2)^2$ and the non-diagonal entries are $2(n-2)$. By Theorem 3.2 with $a = 4(n-2)^2$ and $b = 2(n-2)$, Equation 12 is

(13)

$$P_{DSED(\Gamma_G)}(\lambda) = \left(\lambda + 4(n-2)^2\right)^{\frac{n}{2}} \left(\lambda - 4(n-2)(n-3)\right)^{\frac{n}{2}-1} \left(\lambda - 6(n-2)^2\right).$$

Therefore, using the roots of Equation 13, the $DSED-$energy of Γ_G is

$$\begin{aligned} E_{DSED}(\Gamma_G) &= \left(\frac{n}{2}\right)\left|-4(n-2)^2\right| + \left(\frac{n}{2}-1\right)\left|4(n-2)(n-3)\right| + \left|6(n-2)^2\right| \\ &= 4n(n-2)^2. \end{aligned}$$

□

Our next proposition will provide us with the characteristic polynomial of Γ_G for $G = G_1 \cup G_2$.

Theorem 4.2. *Let Γ_G be the non-commuting graph on G on $G = G_1 \cup G_2$, where $n \geq 3$. Then, the characteristic polynomial of Γ_G is*

1. *for n is odd:*

$$\begin{aligned} P_{DSED(\Gamma_G)}(\lambda) =& (\lambda + 4n^2)^{n-2}(\lambda + 4(n-1))^{n-1} \\ & ((\lambda - 4n^2(n-2))(\lambda - 4(n-1)^2) - (n-1)n(3n-2)^2), \end{aligned}$$

2. *for n is even:*

$$\begin{aligned} &P_{DSED(\Gamma_G)}(\lambda)=(\lambda+4n^2)^{n-3}(\lambda-8(n-2)(2n-5))^{\frac{n}{2}-1}(\lambda+16(n-2)^2)^{\frac{n}{2}} \\ &(\lambda^2 - (20(n-2)^2+4n^2(n-3))\lambda+80n^2(n-3)(n-2)^2-n(n-2)(3n-4)^2). \end{aligned}$$

Proof. 1. Let n is odd, from Theorem 2.2, we have $d_{a^i} = n$ and $d_{a^ib} = 2(n-1)$, for $1 \leq i \leq n$. Following Theorem 2.5, we then obtain the distance of every pair of vertices. Since $Z(D_{2n}) = \{e\}$, then there are $2n-1$ vertices for Γ_G, where $G = G_1 \cup G_2$. The vertex set consists of $n-1$ vertices of a^i, for $i = 1, 2, \ldots, n-1$, and n vertices of a^ib, $i = 1, 2, \ldots, n$. Then, from Definition 2.1, $DSED(\Gamma_G)$ is an $(2n-1) \times (2n-1)$ matrix as the following:

$$DSED(\Gamma_G) = \begin{bmatrix} 0 & \ldots & 4n^2 & 3n-2 & \ldots & 3n-2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 4n^2 & \ldots & 0 & 3n-2 & \ldots & 3n-2 \\ 3n-2 & \ldots & 3n-2 & 0 & \ldots & 4(n-1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 3n-2 & \ldots & 3n-2 & 4(n-1) & \ldots & 0 \end{bmatrix}.$$

It can be partitioned into four block matrices:

$$DSED(\Gamma_G) = \begin{bmatrix} 4n^2(J-I)_{n-1} & (3n-2)J_{(n-1)\times n} \\ (3n-2)J_{(n-1)\times n} & 4(n-1)(J-I)_n \end{bmatrix}. \tag{14}$$

Now, the characteristic polynomial of Equation 14 is

$$\begin{aligned} P_{DSED(\Gamma_G)}(\lambda) &= |\lambda I_{2n-1} - DSED(\Gamma_G)| \\ &= \begin{vmatrix} (\lambda + 4n^2)I_{n-1} - 4n^2 J_{n-1} & -(3n-2)J_{(n-1)\times n} \\ -(3n-2)J_{n\times(n-1)} & (\lambda + 4(n-1))I_n - 4(n-1)J_n) \end{vmatrix}. \end{aligned}$$

According to Lemma 3.1, with $a = 4n^2$, $b = 4(n-1)$, $c = d = 3n-2$, and $n_1 = n-1$, $n_2 = n$, then we obtain the formula of $P_{DSED(\Gamma_G)}(\lambda)$, and we obtain the desired outcome.

2. Let us prove the even n case. Based on Theorem 2.2, we know that $d_{(a^i)} = n$ and $d_{(a^ib)} = 2(n-2)$, for all $1 \leq i \leq n$. Since $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, then there are $2n-2$ vertices in Γ_G. The vertex set contains $n-2$ vertices of a^i, for $1 \leq i < \frac{n}{2}, \frac{n}{2} < i < n$, and n vertices of a^ib, for $1 \leq i \leq n$. Following the result of Theorem 2.5 and by Definition 2.1, then matrix $DSED(\Gamma_G)$ of size $(2n-2) \times (2n-2)$ is as given below:

$$\begin{bmatrix}
0 & \dots & 4n^2 & 3n-4 & \dots & 3n-4 & 3n-4 & \dots & 3n-4 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
4n^2 & \dots & 0 & 3n-4 & \dots & 3n-4 & 3n-4 & \dots & 3n-4 \\
3n-4 & \dots & 3n-4 & 0 & \dots & 4(n-2) & 16(n-2)^2 & \dots & 4(n-2) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
3n-4 & \dots & 3n-4 & 4(n-2) & \dots & 0 & 4(n-2) & \dots & 16(n-2)^2 \\
3n-4 & \dots & 3n-4 & 16(n-2)^2 & \dots & 4(n-2) & 0 & \dots & 4(n-2) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
3n-4 & \dots & 3n-4 & 4(n-2) & \dots & 16(n-2)^2 & 4(n-2) & \dots & 0
\end{bmatrix}.$$

Now, we provide nine block matrices of $DSED(\Gamma_G)$ as follows:

$$\begin{bmatrix}
4n^2(J-I)_{n-2} & (3n-4)J_{(n-2)\times\frac{n}{2}} & (3n-4)J_{(n-2)\times\frac{n}{2}} \\
(3n-4)J_{\frac{n}{2}\times(n-2)} & 4(n-2)(J-I)_{\frac{n}{2}} & 4(n-2)(J-I)_{\frac{n}{2}} + 16(n-2)^2 I_{\frac{n}{2}} \\
(3n-4)J_{\frac{n}{2}\times(n-2)} & 4(n-2)(J-I)_{\frac{n}{2}} + 16(n-2)^2 I_{\frac{n}{2}} & 4(n-2)(J-I)_{\frac{n}{2}}
\end{bmatrix}.$$

By Theorem 3.3 with $r = 4n^2$, $s = 16(n-2)^2$, $t = 3n-4$, $u = 4(n-2)$, we then obtain the required result. □

As a result of Theorem 4.2, we proceed to the two following theorems.

Theorem 4.3. *Let Γ_G be a non-commuting graph on G, where $G = G_1 \cup G_2$, then $DSED-$spectral radius for Γ_G is*

1. *for n is odd:*

$$\begin{aligned}\rho_{DSED}(\Gamma_G) =& 2n^2(n-2) + 2(n-1)^2 + \\ & \sqrt{\left(2n^2(n-2) - 2(n-1)^2\right)^2 + n(n-1)(3n-2)^2},\end{aligned}$$

2. *for n is even:*

$$\begin{aligned}\rho_{DSED}(\Gamma_G) =& 10(n-2)^2 + 2n^2(n-3) + \\ & \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}.\end{aligned}$$

Proof. 1. Consider the first case for odd n, $DSED(\Gamma_G)$ has four eigenvalues, where it follows the result of Theorem 4.2 (1). They are $\lambda_1 = -4n^2$ of multiplicity $(n-2)$ and $\lambda_2 = -4(n-1)$ of multiplicity $(n-1)$. The quadratic formula gives the other two eigenvalues, which are

$$\lambda_3, \lambda_4 = 2n^2\,(n-2) + 2\,(n-1)^2 \pm \sqrt{\left(2n^2\,(n-2) - 2\,(n-1)^2\right)^2 + (n-1)\,n\,(3n-2)^2}.$$

They are positive real numbers. Hence, the spectrum of Γ_G as the following:

$$\begin{aligned} Spec(\Gamma_G) = \Bigg\{ & \left(2n^2\,(n-2) + 2\,(n-1)^2 + \sqrt{\left(2n^2\,(n-2) - 2\,(n-1)^2\right)^2 + (n-1)\,n\,(3n-2)^2}\right)^1, \\ & \left(2n^2\,(n-2) + 2\,(n-1)^2 - \sqrt{\left(2n^2\,(n-2) - 2\,(n-1)^2\right)^2 + (n-1)\,n\,(3n-2)^2}\right)^1, \\ & (-4(n-1))^{n-1}, \left(-4n^2\right)^{n-2} \Bigg\}. \end{aligned}$$

By determining the maximum absolute eigenvalues, consequently, we derive the spectral radius of Γ_G as the desired result.

2. We may consider the even n case, it follows from Theorem 4.2 (2), $DSED(\Gamma_G)$ has five eigenvalues. Hence, we get $\lambda_1 = -4n^n$ of multiplicity $(n-3)$, the second is $\lambda_2 = 8(n-2)(2n-5)$ of multiplicity $\frac{n}{2}-1$, and the third is $\lambda_3 = -16(n-2)^2$ of multiplicity $\frac{n}{2}$. From the quadratic formula we have $\lambda_4, \lambda_5 = 10\,(n-2)^2 + 2n^2\,(n-3) \pm \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}$.

Hence, the spectrum of Γ_G as the following:

$$\begin{aligned} Spec(\Gamma_G) = \Bigg\{ & \left(10\,(n-2)^2 + 2n^2\,(n-3) + \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}\right)^1, \\ & \left(10\,(n-2)^2 + 2n^2\,(n-3) - \sqrt{(10(n-2)^2 - 2n^2(n-3))^2 + n(n-2)(3n-4)^2}\right)^1, \\ & (8(n-2)(2n-5))^{\frac{n}{2}-1}, \left(-4n^2\right)^{n-3}, \left(-16(n-2)^2\right)^{\frac{n}{2}} \Bigg\}. \end{aligned}$$

Now, for $i = 1, 2, 3, 4$, the maximum of $|\lambda_i|$ is $DSED-$spectral radius of Γ_G. $\square$

Theorem 4.4. *Let Γ_G be a non-commuting graph on G, where $G = G_1 \cup G_2$, then $DSED-$energy for Γ_G is*

1. *for n is odd: $E_{DSED}(\Gamma_G) = 8n^2\,(n-2) + 8\,(n-1)^2$*
2. *for n is even: $E_{DSED}(\Gamma_G) = 8n^2\,(n-3) + 8\,(n-2)^2 + 8n(n-2)^2$.*

Proof. 1. The proving part of Theorem 4.3 (1) was given the spectrum of Γ_G for odd n, then the $DSED-$energy of Γ_G can be calculated as follows:

$$\begin{aligned} E_{DSED}(\Gamma_G) = & (n-2)\left|-4n^2\right| + (n-1)\left|-4(n-1)\right| + \\ & \left|2n^2\,(n-2) + 2\,(n-1)^2 \pm \sqrt{\left(2n^2\,(n-2) - 2\,(n-1)^2\right)^2 + (n-1)\,n\,(3n-2)^2}\right| \\ = & 8n^2\,(n-2) + 8\,(n-1)^2 \end{aligned}$$

2. Let n is even, by Theorem 4.3 (2), the $DSED-$energy of Γ_G is derived as follows:

$$\begin{aligned} E_{DSED}(\Gamma_G) =&(n-3)\left|-4n^2\right| + \left(\frac{n}{2}-1\right)\left|-8(n-2)\right| + \left(\frac{n}{2}\right)\left|-16(n-2)^2\right| + \\ &\left|2n^2(n-3)+2(n-2)^2 \pm \sqrt{\left(2n^2(n-2)-2(n-1)^2\right)^2 + (n-1)n(3n-2)^2}\right| \\ =&8n^2(n-3)+8(n-2)^2+8n(n-2)^2. \end{aligned}$$

□

Example 4.2. Following Example 4.1, we can construct 6×6 degree sum exponent distance matrix of Γ_G as follows:

$$DSED(\Gamma_G) = \begin{bmatrix} 0 & 64 & 8 & 8 & 8 & 8 \\ 64 & 0 & 8 & 8 & 8 & 8 \\ 8 & 8 & 0 & 8 & 64 & 8 \\ 8 & 8 & 8 & 0 & 8 & 64 \\ 8 & 8 & 64 & 8 & 0 & 8 \\ 8 & 8 & 8 & 64 & 8 & 0 \end{bmatrix}$$

Here $P_{DSED(\Gamma_G)}(\lambda)$ is derived as follows:

$$P_{DSED(\Gamma_G)}(\lambda) = (\lambda-48)^2(\lambda+64)^3(\lambda-96).$$

As a result of using Maple, we have determined that

$$Spec(\Gamma_G) = \left\{(96)^1, (48)^2, (-64)^3\right\}.$$

Therefore, the $DSED-$energy of Γ_G is as follows:

$$E_{DSED}(\Gamma_G) = (1)|96| + (2)|48| + (3)|-64| = 384.$$

5. Discussion

As in the previous result of Theorem 4.4 for $G = G_1 \cup G_2$, in the following, we get the classification of the $DSED-$Energy of Γ_G for D_{2n}.

Corollary 5.1. *Graph Γ_G associated with the degree sum exponent distance matrix is hyperenergetic.*

Moreover, based on the facts obtained in the previous section, the energies in Theorem 4.4 yield the following fact:

Corollary 5.2. *$DSED-$energy of Γ_G is always an even integer.*

The fact in Corollary 5.2 corresponds with the well-known statement from [4] and [18]. Furthermore, as a comparison of the energies from Theorems 2.4 and 4.4, as a consequence, we derive the following conclusion:

Corollary 5.3. $E_{DSED}(\Gamma_G) > E_A(\Gamma_G)$.

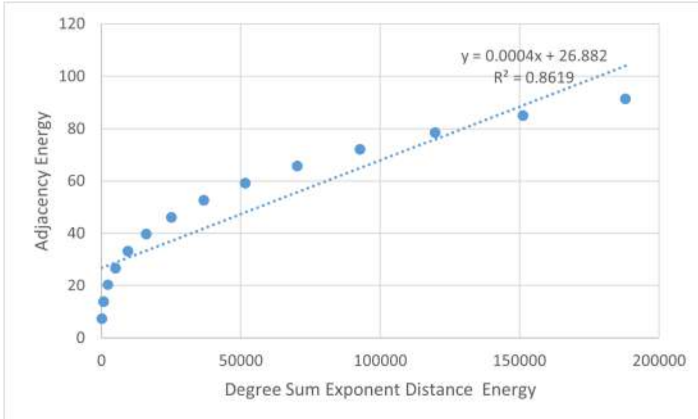

Figure 2: Correlation of $E_{DSED}(\Gamma_G)$ with $E_A(\Gamma_G)$ for odd n

In our graph, the $DSED-$energy of Γ_G for D_{2n}, where $n \geq 3$ is always greater than the adjacency energy. In addition, it can be seen from Figures 2 and 3 that $E_{DSED}(\Gamma_G)$ has a significant correlation with $E_A(\Gamma_G)$, with a correlation coefficient of 0.8619 for odd n, 0.865 for even n. Those results state that $E_{DSED}(\Gamma_G)$ and $E_A(\Gamma_G)$ have a strong correlation between them and comply with the result from [9]. However, it is slightly different from the claim from [20].

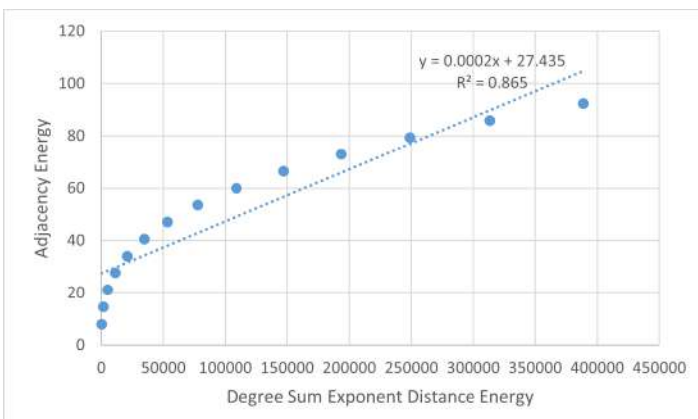

Figure 3: Correlation of $E_{DSED}(\Gamma_G)$ with $E_A(\Gamma_G)$ for even n

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On divisor labeling of co-prime order graphs of finite groups

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Abstract. The co-prime order graph of a finite group G is an undirected graph whose vertex set is G and two distinct vertices $u, v \in G$ are adjacent if $gcd(o(u), o(v)) = 1$ or a prime number. Labeling a graph is the process of assigning integers to its vertices and/or edges subject to certain conditions. In other words, vertex (edge) labeling is a function of the set of vertices (edges) to a set of labels (generally integers). A graph Γ is a divisor graph if all its vertices can be labeled with positive integers such that two distinct vertices x and y are adjacent if and only if $x|y$ or $y|x$. This paper focuses on some conditions under which the co-prime order graphs of finite groups, especially abelian groups and permutation groups, are divisor graphs.

Keywords: divisor graph, co-prime order graph, labeling.

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1. Introduction

For a graph Γ, we denote its vertex set and edge set by $V(\Gamma)$ and $E(\Gamma)$ respectively. In a directed graph, we use (u, v) for a directed edge from u to v, the in-degree of a vertex v is the number of edges coming to the vertex v and the out-degree of a vertex v is the number of edges going out from the vertex v. Further, in a digraph D, a vertex with zero in-degree (out-degree) is called a transmitter (receiver), whereas a vertex v with positive in-degree and positive out-degree is called a transitive vertex if $(u, w) \in E(D)$ whenever (u, v) and (v, w) belong to $E(D)$, see [1]. If G is a graph whose vertex set is V and S is a non-empty subset of V, then the subgraph of G having vertex set S and edge set as the set of those edges of G that have both ends in S is called the subgraph of G induced by S. For more details of graph theory, the reader may refer to Bondy and Murty [2].

Singh and Santosh [3] conceptualized divisor graphs for non-empty sets of integers. Assume that S is a finite non-empty set of integers. The divisor graph $G(S)$ of S is a graph with vertex set S such that two distinct vertices x and y are adjacent if either $x|y$ or $y|x$. Further, the divisor digraph $D(S)$ of S has vertex set S and (x, y) is an arc of $D(S)$ if $x|y$. A graph Γ is called a divisor graph if Γ is isomorphic to $G(S)$ for some finite non-empty set S of integers. Chartrand et al. [4] studied the divisor graphs in terms of non-empty sets of positive integers. The term divisor graph used in the paper is in the same sense as in [4]. Thus, if Γ is a divisor graph, then there exists a function $f\colon V(\Gamma) \to \mathbb{N}$ such that Γ is isomorphic to $G(f(V(\Gamma)))$. Such a function f is called a divisor labeling of the graph Γ. Divisor graphs associated with algebraic structures have also caught the attention of researchers. Osba and Alkam [5] worked on the necessary and sufficient conditions for the zero-divisor graphs of a class of rings to be divisor graph. Recently, Takshak et al. [6] showed that the power graph of a finite group is always a divisor graph but the converse is not true.

In 2021, Banerjee [7] introduced the co-prime order graph of a group G as the graph whose vertex set is G and two distinct vertices x, y are adjacent if $gcd(o(x), o(y))$ is either 1 or a prime number. Since then many researchers [8, 9, 10, 11] have studied co-prime order graphs and have shown their utility in characterizing finite groups.

In this paper, we shall find out some conditions under which the co-prime order graphs of finite groups (especially abelian groups and permutation groups) are/are not divisor graphs. All graphs considered in this paper are finite and simple.

2. Preliminaries

In this section, we state some relevant notations and basic results used in the paper. If G is a group and g is an arbitrary element of G, then their orders are denoted by $o(G)$ and $o(g)$ respectively. S_n denotes the permutation group of degree n. $\Theta(G)$ shall denote the co-prime order graph of the group G. Further, $\Gamma_1 \vee \Gamma_2$ represents the join of graphs Γ_1 and Γ_2. The complete graph on n vertices is denoted by K_n and $K_{n_1,n_2,\ldots,n_k}$ denotes the complete k-partite graph.

Now we state some well-known results on divisor graphs.

Theorem 2.1 ([4]). *Let Γ be a graph. Then Γ is a divisor graph if and only if there exists an orientation D of Γ such that every vertex of D is a transmitter, a receiver or a transitive vertex.*

Theorem 2.2 ([4]). *Every induced subgraph of a divisor graph is a divisor graph.*

Theorem 2.3 ([4]). *If Γ_1 and Γ_2 are two divisor graphs, then $\Gamma_1 \vee \Gamma_2$ is a divisor graph.*

Theorem 2.4. *Let Γ_1 and Γ_2 be two divisor graphs whose vertex sets are disjoint, then $\Gamma_1 \cup \Gamma_2$ is also a divisor graph.*

Theorem 2.5 ([4, 5]). *A graph that contains the following (Figure 1) induced subgraph is not a divisor graph.*

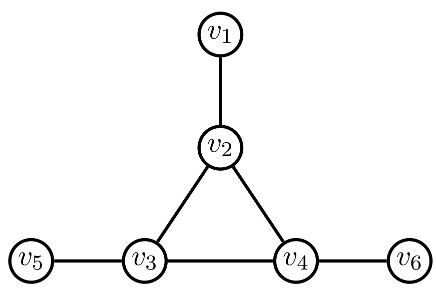

Figure 1

3. Main results

We begin this section with the following observation:

Let Γ be a graph having $\{a_1, a_2, \ldots, a_{n_1}, b_1, b_2, \ldots, b_{n_2}, c_1, c_2, \ldots, c_{n_3}, d_1, d_2, \ldots, d_{n_4}, e_1, e_2, \ldots, e_{n_5}\}$ as the vertex set s.t. its orientation is represented by Figure 2.

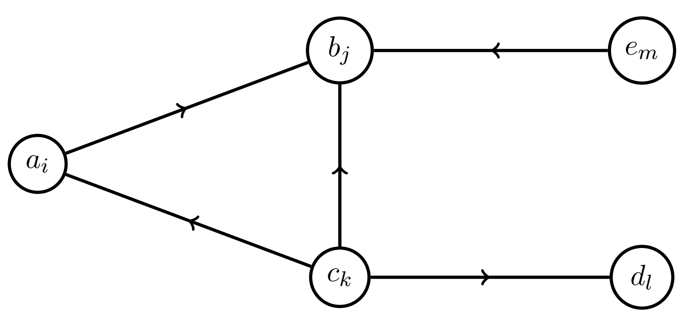

Figure 2

It is obvious that each a_i is a transitive vertex. Further, c_k's and e_m's are transmitters and b_j's and d_l's are receivers. Thus, each of the vertices of Γ is either a receiver, a transmitter or a transitive vertex. Hence, Γ is a divisor graph by Theorem 2.1.

Theorem 3.1. *Let S be a subset of a finite group such that the order of its every element divides ${p_1}^m {p_2}^n$, where p_1 and p_2 are distinct primes and $m, n \in \mathbb{N}$, then $\Theta(S)$ is a divisor graph.*

Proof. Firstly, consider the case wherein there exist $x_i, y_j, z_k, \alpha_l, \beta_r, \gamma_s, \delta_t \in S$ such that

- $o(x_i) = 1$ or p_1 or p_2, where $1 \leq i \leq n_1$;
- $o(y_j) = {p_1}^2$ or ${p_1}^3 \ldots$ or ${p_1}^m$, where $1 \leq j \leq n_2$;
- $o(z_k) = {p_2}^2$ or ${p_2}^3 \ldots$ or ${p_2}^n$, where $1 \leq k \leq n_3$;
- $o(\alpha_l) = p_1 p_2$, where $1 \leq l \leq n_4$;
- $o(\beta_r) = {p_1}^2 p_2$ or ${p_1}^3 p_2 \ldots$ or ${p_1}^m p_2$, where $1 \leq r \leq n_5$;
- $o(\gamma_s) = p_1 {p_2}^2$ or $p_1 {p_2}^3 \ldots$ or $p_1 {p_2}^n$, where $1 \leq s \leq n_6$;
- $o(\delta_t) = {p_1}^2 {p_2}^2$ or ${p_1}^2 {p_2}^3 \ldots$ or ${p_1}^m {p_2}^n$, where $1 \leq t \leq n_7$.

Now, let us partition the vertex set of graph $\Theta(S)$ into three mutually disjoint sets A, B and C, where

$A = \{x_1, x_2, \ldots, x_{n_1}\}$,
$B = \{y_1, y_2, \ldots, y_{n_2}, z_1, z_2, \ldots, z_{n_3}, \alpha_1, \alpha_2, \ldots, \alpha_{n_4}, \beta_1, \beta_2, \ldots, \beta_{n_5}, \gamma_1, \gamma_2, \ldots, \gamma_{n_6}\}$,
$C = \{\delta_1, \delta_2, \ldots, \delta_{n_7}\}$.

Let Γ_1, Γ_2 (Figure 3) and Γ_3 denote the subgraphs of $\Theta(G)$ induced by A, B and C respectively. So, we have

$$\Theta(S) = \Gamma_1 \vee (\Gamma_2 \cup \Gamma_3).$$

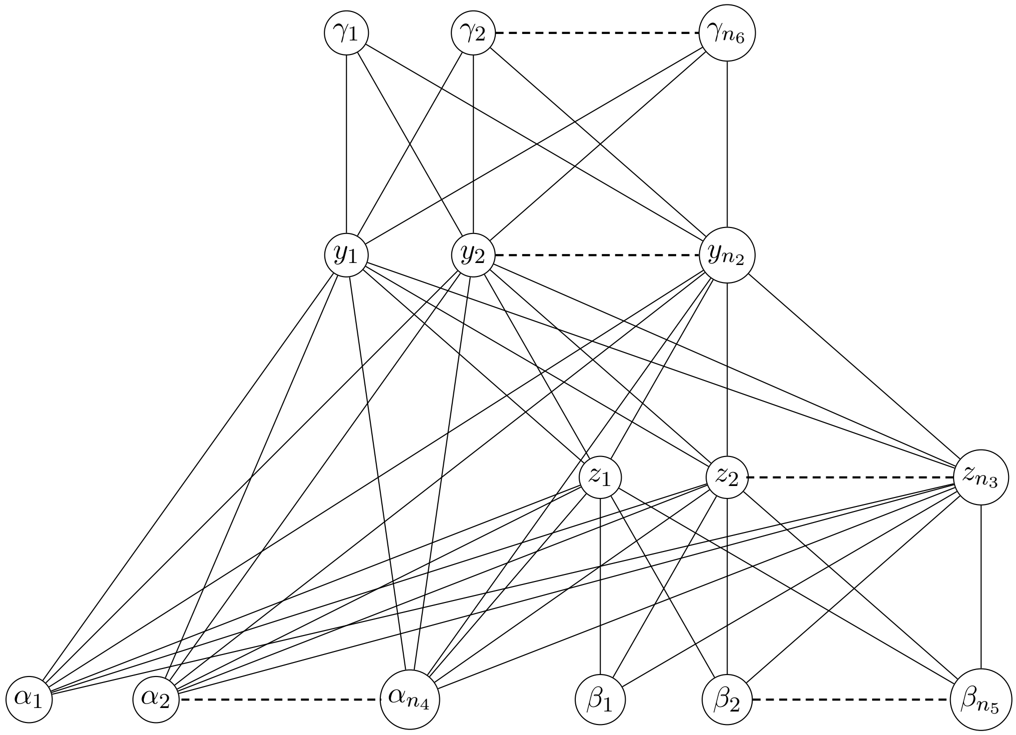

Figure 3: Subgraph Γ_2

Consider the following orientation of Γ_2 :
For $j \in \{1, 2, \ldots, n_2\}$, $k \in \{1, 2, \ldots, n_3\}$, $l \in \{1, 2, \ldots, n_4\}$, $r \in \{1, 2, \ldots, n_5\}$ and $s \in \{1, 2, \ldots, n_6\}$, we take $(y_j, z_k), (y_j, \alpha_l), (y_j, \gamma_s)$, (α_l, z_k) and (β_r, z_k) as edges of Γ_2.

As this orientation of Γ_2 is similar to that of Γ (Figure 2), it is a divisor graph. Further, as $\Gamma_1 \cong K_{n_1}$ and $\Gamma_3 \cong n_7 K_1$, so Γ_1 and Γ_3 are also divisor graphs. Hence, $\Theta(S)$ is a divisor graph in this case.

In each of the remaining cases, the co-prime order graph of G is nothing but an induced subgraph of $\Theta(G)$ considered in the above case, hence a divisor graph by Theorem 2.2. $\square$

Corollary 3.1. *If order of every element of a finite group G divides ${p_1}^m {p_2}^n$, where p_1 and p_2 are distinct prime numbers and $m, n \in \mathbb{N}$, then $\Theta(G)$ is a divisor graph.*

Corollary 3.2. *If G is a group of order ${p_1}^m {p_2}^n$, where p_1 and p_2 are distinct prime numbers and $m, n \in \mathbb{N}$, then $\Theta(G)$ is a divisor graph.*

The following result can be proved by proceeding as in Theorem 3.1:

Theorem 3.2. *Let G be a finite group s.t. $o(G) = p^m$, where p is a prime number and $m \in \mathbb{N}$, then $\Theta(G)$ is a divisor graph.*

Theorem 3.3. *Assume that the order of every element of a finite group G divides $p_1p_2p_3$, where p_1, p_2 and p_3 are distinct prime numbers, then $\Theta(G)$ is a divisor graph.*

Proof. As in Theorem 3.1, it is sufficient to prove the result in the following case:

Let there exist $x_i, y_j, z_k, \alpha_l, \beta_m \in G$ s.t.

- $o(x_i) = 1$ or p_1 or p_2 or p_3, where $1 \leq i \leq n_1$;
- $o(y_j) = p_1p_2$, where $1 \leq j \leq n_2$;
- $o(z_k) = p_1p_3$, where $1 \leq k \leq n_3$;
- $o(\alpha_l) = p_2p_3$, where $1 \leq l \leq n_4$;
- $o(\beta_m) = p_1p_2p_3$, where $1 \leq m \leq n_5$.

Now, we partition the vertex set of graph $\Theta(G)$ into three mutually disjoint subsets $\{x_1, x_2, \ldots, x_{n_1}\}$, $\{y_1, y_2, \ldots, y_{n_2}, z_1, z_2, \ldots, z_{n_3}, \alpha_1, \alpha_2, \ldots, \alpha_{n_4}\}$ and $\{\beta_1, \beta_2, \ldots, \beta_{n_5}\}$. Let Γ_4, Γ_5 and Γ_6 respectively denote the subgraphs of $\Theta(G)$ induced by these sets. It follows that $\Theta(G) = \Gamma_4 \vee (\Gamma_5 \cup \Gamma_6)$.

Further, Γ_4, Γ_5 and Γ_6 are divisor graphs as $\Gamma_4 \cong K_{n_1}, \Gamma_5 \cong K_{n_2,n_3,n_4}$ and $\Gamma_6 \cong n_5K_1$. Hence, $\Theta(G)$ is also a divisor graph. □

Corollary 3.3. *Let G be a group of order ${p_1}^{m_1}{p_2}^{m_2}{p_3}^{m_3}$ such that it has no element of order ${p_1}^2$ or ${p_2}^2$ or ${p_3}^2$, then $\Theta(G)$ is a divisor graph.*

Theorem 3.4. *Assume that a finite group G contains at least one element of order $p_1p_2, p_1p_3, p_1p_4, p_1p_2p_3, p_1p_2p_4$ and $p_1p_3p_4$ each, where p_1, p_2 and p_3 are distinct prime numbers. Then $\Theta(G)$ is not a divisor graph.*

Proof. Let x_1, x_2, x_3, x_4, x_5 and x_6 be elements of G whose orders are $p_1p_2, p_1p_3, p_1p_4, p_1p_2p_3, p_1p_2p_4$ and $p_1p_3p_4$ respectively. Then, the subgraph of $\Theta(G)$ induced by the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ is isomorphic to the graph represented by Figure 1. So, by Theorem 2.2, $\Theta(G)$ is not a divisor graph. □

Corollary 3.4. *Let G be an abelian group such that $o(G) = n$ and ${p_1}^{m_1}{p_2}^{m_2} \ldots {p_k}^{m_k}$ be prime power decomposition of n. Then, for $k \geq 4$, $\Theta(G)$ is not a divisor graph.*

Theorem 3.5. *If a finite group G contains at least one element of order $p_1p_2, p_1p_3, {p_1}^2, {p_1}^2p_2, {p_1}^2p_3$ and $p_1p_2p_3$ each, where p_1, p_2 and p_3 are distinct prime numbers, then $\Theta(G)$ is not a divisor graph.*

Proof. Let $x_1, x_2, x_3, x_4, x_5, x_6 \in G$ such that their orders are $p_1p_2, p_1p_3, {p_1}^2, {p_1}^2p_2, {p_1}^2p_3$ and $p_1p_2p_3$ respectively. Considering the subgraph of $\Theta(G)$ induced by the set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and proceeding as in Theorem 3.4, it follows that $\Theta(G)$ is not a divisor graph. □

Corollary 3.5. *If a group G contains at least one element of order ${p_1}^2 p_2 p_3$, where p_1, p_2 and p_3 are distinct prime numbers, then $\Theta(G)$ is not a divisor graph.*

Corollary 3.6. *Let G be an abelian group of order n and ${p_1}^{m_1} {p_2}^{m_2} {p_3}^{m_3}$ be prime power decomposition of n. If G contains at least one element of order ${p_1}^2$ or ${p_2}^2$ or ${p_3}^2$, then $\Theta(G)$ is not a divisor graph.*

Corollary 3.7. *If $n \geq 10$, then $\Theta(S_n)$ is not a divisor graph.*

Proof. Consider $x_1 = (1,2)(3,4,5), x_2 = (1,2)(3,4,5,6,7), x_3 = (1,2,3,4), x_4 = (1,2,3,4)(5,6,7), x_5 = (1,2,3,4)(5,6,7,8,9)$ and $x_6 = (1,2)(3,4,5)(6,7,8,9,10)$. Then, x_1, x_2, x_3, x_4, x_5 and x_6 are elements of S_n with orders $6, 10, 4, 12, 20$ and 30 respectively and using the above theorem, it can be concluded that $\Theta(S_n)$ is not a divisor graph for $n \geq 10$. □

The following result is an implication of the results discussed above:

Theorem 3.6. *Let G be an abelian group of order n and ${p_1}^{m_1} {p_2}^{m_2} \dots {p_k}^{m_k}$ be the prime decomposition of n, then $\Theta(G)$ is divisor graph if and only if $k \leq 3$, with the condition that if $k = 3$, then G contains no element of order ${p_1}^2$ or ${p_2}^2$ or ${p_3}^3$.*

Theorem 3.7. *If a finite group G contains no element whose order is other than $1, p_1, p_2, p_3, p_4, p_1p_2, p_1p_3, p_1p_4, p_2p_3, {p_1}^2, {p_2}^2, {p_1}^3, {p_1}^2 p_2, {p_1}^2 p_3$, where p_1, p_2, p_3 and p_4 are distinct prime numbers, then $\Theta(G)$ is a divisor graph.*

Proof. Let there exist $x_i, y_j, z_k, w_s, \alpha_l, \beta_r, \gamma_m, \delta_n, u_q \in G$ s.t.

- $o(x_i) = 1$ or p_1 or p_2 or p_3 or p_4, where $1 \leq i \leq n_1$;
- $o(y_j) = p_1p_4$, where $1 \leq j \leq n_2$;
- $o(z_k) = p_2p_3$, where $1 \leq k \leq n_3$;
- $o(w_s) = {p_2}^2$, where $1 \leq s \leq n_4$;
- $o(\alpha_l) = p_1p_3$, where $1 \leq l \leq n_5$;
- $o(\beta_r) = p_1p_2$, where $1 \leq r \leq n_6$;
- $o(\gamma_m) = {p_1}^2 p_2$, where $1 \leq m \leq n_7$;
- $o(\delta_n) = {p_1}^2 p_3$, where $1 \leq n \leq n_8$;
- $o(u_q) = {p_1}^2$ or ${p_1}^3$, where $1 \leq q \leq n_9$.

We write $V(\Theta(G)) = D \cup E \cup F$, where D, E and F are three mutually disjoint sets given by:

$D = \{x_1, x_2, \dots, x_{n_1}\}$,
$E = \{y_1, y_2, \dots, y_{n_2}, z_1, z_2, \dots, z_{n_3}, w_1, w_2, \dots, w_{n_4}\}$,
$F = \{\alpha_1, \alpha_2, \dots, \alpha_{n_5}, \beta_1, \beta_2, \dots, \beta_{n_6}, \gamma_1, \gamma_2, \dots, \gamma_{n_7}, \delta_1, \delta_2, \dots, \delta_{n_8}, u_1, u_2, \dots, u_{n_9}\}$.

Consider Γ_7, Γ_8 and Γ_9 (Figure 4), the subgraphs of $\Theta(G)$ induced by D, E and F respectively. Clearly, Γ_7 and Γ_8 are divisor graphs as $\Gamma_7 \cong K_{n_1}$ and $\Gamma_8 \cong K_{n_2,n_3,n_4}$. Also, we have $\Theta(G) = (\Gamma_7 \vee \Gamma_8) \vee \Gamma_9$.

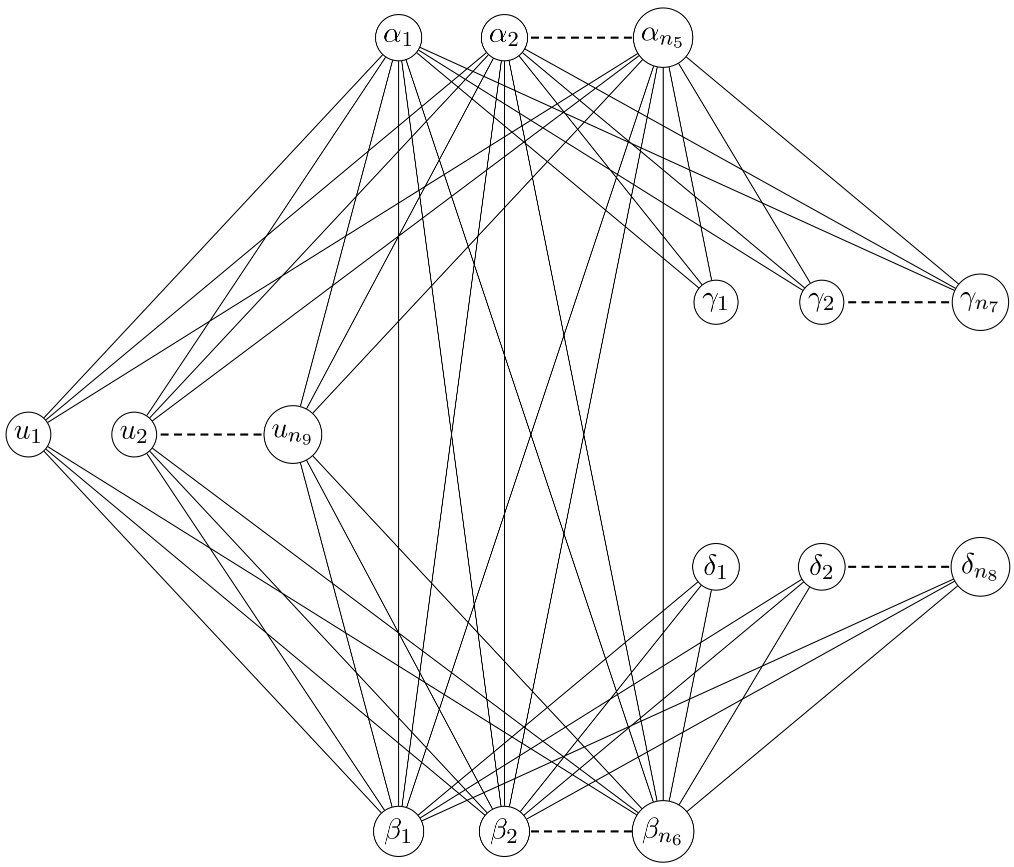

Figure 4: Subgraph Γ_9

Consider an orientation of the subgraph Γ_9 as stated below:
For every $l\in\{1, 2, \ldots, n_5\}$, $r\in\{1, 2, \ldots, n_6\}$, $m \in \{1, 2, \ldots, n_7\}$, $n \in \{1, 2, \ldots, n_8\}$ and $q \in \{1, 2, \ldots, n_9\}$, we take $(\alpha_l, u_q), (\alpha_l, \beta_r), (\alpha_l, \gamma_m)$, (u_q, β_r) and (δ_n, β_r) as edges of Γ_9. Then, proceeding as in Theorem 3.1, it can be shown that the subgraph Γ_9, and hence $\Theta(G)$, is a divisor graph. $\square$

Corollary 3.8. *For $n \leq 9$, then $\Theta(S_n)$ is a divisor graph.*

Proof. It is easy to check that for $n \leq 9$, the order of each element of S_n belongs to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 20\}$. In the above theorem, if we take $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $p_4 = 7$ then, $\Theta(S_n)$ becomes an induced subgraph of $\Theta(G)$. Thus, S_n is a divisor graph for $n \leq 9$. $\square$

It follows from the Corollary 3.7 and Corollary 3.8 that:

Theorem 3.8. *$\Theta(S_n)$ is a divisor graph if and only if $n \leq 9$.*

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Common fixed point for compatible self-maps in an orbitally complete b-metric space

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Abstract. A common fixed point theorem is obtained for three self-maps on a b-metric space, satisfying a rational type condition, through the notions of orbital completeness, orbital continuity and the compatibility.
Keywords: b-metric space, Orbital completeness, Orbital continuity, Unique common fixed point.
MSC 2020: 47H10, 54H25, 55M20

1. Introduction

In the last few decades, fixed point theorems were developed in a metric space, normed linear space, topological space etc., while the conditions on the underlying mappings are usually metrical or compact type conditions. Further, new algebraic structures were also formulated to improve the results. For instance, the following notion of b-metric space is a generalization of a metric space, due to Bakhtin [2].

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Definition 1.1. *Let $s \geq 1$, X be a nonempty set and $\rho_s : X \times X \to [0, +\infty)$ be such that*

$(b1)$ $\rho_s(x, y) = 0$ *if and only if* $x = y$

$(b2)$ $\rho_s(x, y) = \rho_s(y, x)$, *for all* $x, y \in X$

$(b3)$ $\rho_s(x, y) \leq s[\rho_s(x, z) + \rho_s(y, z)]$, *for all* $x, y, z \in X$.

Then, ρ_s is called a b-metric on X, and the pair (X, ρ_s) denotes a b-metric space.

If $s = 1$, the condition (b_3) reduces to the the triangle inequality of a metric. Thus metric space is a particular case of a b-metric space, when $s = 1$. However, a b-metric space is not necessarily a metric space. For instance, consider the pair (X, ρ_s), where $X = \mathbb{R}$ and $\rho_s(x, y) = |x - y|^2$, for all $x, y \in \mathbb{R}$. Then, the conditions (b_1) and (b_2) are obvious. Further, $\rho_s(x, y) = |x - y|^2 = |x - z + z - y|^2 \leq 2(|x - z|^2 + |z - y|^2) = 2[\rho_s(x, z) + \rho_s(y, z)]$, for all $x, y \in X$. Thus $(\mathbb{R}, \rho_s)$ is a b-metric space with $b = 2$. Since $\rho_s(1, 3) + \rho_s(1, 0) = 5$ and $\rho_s(0, 3) = 9$, the triangle inequality fails to hold good, showing that ρ_s is not a metric. Thus the class of b-metric spaces contains that of metric spaces.

Definition 1.2. *A b-ball in a b-metric space (X, ρ_s) is defined by*

$$B_{\rho_s}(x, r) = \{y \in X : \rho_s(x, y) < r\}.$$

The family of all b-balls forms a basis for topology, which is called the b-metric topology $\tau(\rho_s)$ on X.

Definition 1.3. *Let (X, ρ_s) be a b-metric space with parameter s. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in X is said to be*

(a) *b-convergent, with limit p, if it converges to p in the b-metric topology* $\tau(\rho_s)$

(b) *b-Cauchy, if* $\lim_{n,m\to\infty} \rho_s(x_n, x_m) = 0$

(c) *b-complete, if every b-Cauchy sequence in X is b-convergent in it.*

Remark 1.1. A b-metric is not jointly continuous in its coordinate variables x and y, even though a metric d is known to be continuous (see, Example 2.13, [8]).

Definition 1.4. *Let (X, ρ_s) be a b-metric space with parameter s. Given $x_0 \in X$, and self-maps A, S and T on X, if there exist points x_0, x_1, x_2, $\ldots$, x_n, $\ldots$ such that*

$$y_{2n-1} = Sx_{2n-2} = Ax_{2n-1},\ y_{2n} = Tx_{2n-1} = Ax_{2n} \text{ for } n = 1, 2, \ldots, \tag{1.1}$$

then, the sequence $\langle Ax_n \rangle_{n=1}^{\infty}$ is called an (S,T)-orbit with respect to A at x_0 or simply an (S,T,A)-orbit at x_0, and is denoted by $\mathscr{O}_{S,T,A}(x_0)$.

The pair (S,T) is said to be asymptotically regular with respect to A at x_0 , if $\lim_{n\to\infty} \rho_s(Ax_n, Ax_{n+1}) \to 0$, and (S,T) is asymptotically regular with respect to A, if it is asymptotically regular with respect to A at each $x_0 \in X$. The b-metric space X is said to be (S,T,A)-orbitally b-complete at x_0, if every b-Cauchy sequence in $\mathscr{O}_{S,T,A}(x_0)$ converges in X. The space X is said to be (S,T,A)-orbitally b-complete, if it is (S,T,A)-orbitally b-complete at each x_0.

Definition 1.5. *Let (X,ρ_s) be a b-metric space with parameter s. A self-map $T : X \to X$ is said to be continuous at $p \in X$, if $\lim_{n\to\infty} \rho_s(Tp_n, Tp) = 0$ whenever $\langle p_n\rangle_{n=1}^{\infty} \subset X$ $\lim_{n\to\infty} \rho_s(p_n, p) = 0$. And, T is continuous on X, if it is continuous at every $x_0 \in X$.*

Definition 1.6. *The self-map A is (S,T) orbitally continuous at x_0 or simply orbitally continuous at x_0, if it is continuous on some (S,T,A)-orbit at x_0.*

Self-maps A and S on a metric space (X,d) are commuting, if $Asx = SAx$, for all $x \in X$. As a weaker form of it, Sessa [7] introduced weakly commuting maps A and S on X with the choice $d(ASx, SAx) \le d(Ax, Sx)$, for all $x \in X$. Gerald Jungck [4] introduced compatible maps as a generalization for weakly commuting maps as follows:

Definition 1.7. *Self-maps f and r on a metric space (X,d) are said to be compatible, if*

$$\lim_{n\to\infty} d(ASp_n, SAp_n) = 0, \tag{1.2}$$

whenever there exists a sequence $\langle p_n\rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} Ap_n = \lim_{n\to\infty} Sp_n = z, \text{ for some } z \in X. \tag{1.3}$$

In [1], the following notion was introduced:

Definition 1.8. *Let (X,d) be a metric space. Self-maps T and A on X are (T,A)-weak compatible, if*

$$\lim_{n\to\infty} ATp_n = Tz, \text{ and } \lim_{n\to\infty} TAp_n = \lim_{n\to\infty} T^2p_n = Tz, \tag{1.4}$$

whenever there exists a sequence $\langle p_n\rangle_{n=1}^{\infty} \subset X$ with the choice (1.3).

Note that, compatible maps T and A are (T,A)-weak compatible. However, the converse is not true. For example, let $X = (-\infty, +\infty)$ with usual metric $d(x,y) = |x-y|$, for all $x, y \in X$.

As the compatibility of a pair of self-maps on a b-metric space is just similar to that in metric space, we skip its discussion. In this paper, we establish a common fixed point theorem for three self-maps on a b-metric space, which satisfy a rational inequality, through the notions of orbital completeness, orbital continuity and the compatibility.

2. Main results

We use the following results from [5]:

Lemma 2.1. *Let (X, ρ_s) be a b-metric space with parameter s. Suppose that $\langle x_n \rangle_{n=1}^{\infty}$ is b-convergent with limit x, and $\langle y_n \rangle_{n=1}^{\infty}$ is b-convergent with limit y in X. Then*

$$\frac{1}{s^2}\rho_s(x, y) \le \liminf_{n\to\infty} \rho_s(x_n, y_n) \le \limsup_{n\to\infty} \rho_s(x_n, y_n) \le s^2 \rho_s(x, y). \tag{2.1}$$

In particular, if $x = y$, then $\lim_{n\to\infty} \rho_s(x_n, y_n) = 0$. Further, for each $z \in X$, we have

$$\frac{1}{s}\rho_s(x, z) \le \liminf_{n\to\infty} \rho_s(x_n, z) \le \limsup_{n\to\infty} \rho_s(x_n, z) \le s\rho_s(x, z). \tag{2.2}$$

The following is the main result of this paper:

Theorem 2.1. *Let A, S and T be self-maps on a b-metric space (X, ρ_s) with $s \ge 1$, satisfying the inclusions:*

$$S(X) \subset A(X) \text{ and } T(X) \subset A(X) \tag{2.3}$$

and the rational inequality

$$\begin{aligned} \rho_s(Sx, Ty) \le a\,\rho_s(Ax, Ay) &+ \beta \cdot \frac{\rho_s(Ay, Ty)[1 + \rho_s(Ax, Sx)]}{1 + \rho_s(Ax, Ay)} \\ &+ \gamma \cdot \frac{\rho_s(Ay, Ty) + \rho_s(Ay, Sx)}{1 + \rho_s(Ay, Ty)\rho_s(Ay, Sx)}, \text{ for all } x, y \in X, \end{aligned} \tag{2.4}$$

where a, β and γ are non-negative numbers, not all being zero, such that

$$s^4 a + (s^4 + 1)\beta + (s^5 + s^4 + s)\gamma < 1. \tag{2.5}$$

Then, (S, T) is asymptotically regular with respect to A at each $x_0 \in X$. Suppose that

- (a) *the space X is (S, T, A)-orbitally b-complete,*
- (b) *A is orbitally continuous.*

If one of the pairs (A, S) and (A, T) is compatible, then S, T and A have a unique common fixed point.

Proof. Given $x_0 \in X$, in view of (2.3), we see that $Sx_0 = Ax_1$ for some $x_1 \in X$ and $Tx_1 = Ax_2$ for some $x_2 \in X$ and so on. Thus inductively we choose points $x_1, x_2, \dots, x_n, \dots$ in X with the choice (1.1).

Writing $x = x_{2n-2}$, $y = x_{2n-1}$ in (2.4) and using (1.1),

$$
\begin{aligned}
\rho_s(y_{2n-1}, y_{2n}) &= \rho_s(Sx_{2n-2}, Tx_{2n-1}) \\
&\le \alpha\, \rho_s(Ax_{2n-2}, Ax_{2n-1}) \\
&\quad + \beta \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1})[1 + \rho_s(Ax_{2n-2}, Sx_{2n-2})]}{1 + \rho_s(Ax_{2n-2}, Ax_{2n-1})} \\
&\quad + \gamma \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1}) + \rho_s(Ax_{2n-1}, Sx_{2n-2})}{1 + \rho_s(Ax_{2n-1}, Tx_{2n-1})\rho_s(Ax_{2n-1}, Sx_{2n-2})} \\
&= \alpha\, \rho_s(y_{2n-2}, y_{2n-1}) \\
&\quad + \beta \cdot \frac{\rho_s(y_{2n-1}, y_{2n})[1 + \rho_s(y_{2n-2}, y_{2n-1})]}{1 + \rho_s(y_{2n-2}, y_{2n-1})} \\
&\quad + \gamma \cdot \frac{\rho_s(y_{2n-1}, y_{2n}) + \rho_s(y_{2n-1}, y_{2n-1})}{1 + \rho_s(y_{2n-1}, y_{2n})\rho_s(y_{2n-1}, y_{2n-1})} \\
&\le \frac{\alpha}{1 - \beta - \gamma} \cdot \rho_s(y_{2n-2}, y_{2n-1}) < q.\rho_s(y_{2n-2}, y_{2n-1}),
\end{aligned}
\tag{2.6}
$$

where

$$
q = \frac{\alpha + \beta + (s+1)\gamma}{1 - \beta - s\gamma}. \tag{2.7}
$$

Similarly, in view of (1.1), the inequality (2.4) with $x = x_{2n-2}$ and $y = x_{2n-3}$, gives

$$
\begin{aligned}
\rho_s(y_{2n-2}, y_{2n-1}) &= \rho_s(y_{2n-1}, y_{2n-2}) \\
&= \rho_s(Sx_{2n-2}, Tx_{2n-3}) \\
&\le \alpha\, \rho_s(Ax_{2n-2}, Ax_{2n-3}) \\
&\quad + \beta \cdot \frac{\rho_s(Ax_{2n-3}, Tx_{2n-3})[1 + \rho_s(Ax_{2n-2}, Sx_{2n-2})]}{1 + \rho_s(Ax_{2n-2}, Ax_{2n-3})} \\
&\quad + \gamma \cdot \frac{\rho_s(Ax_{2n-3}, Tx_{2n-3}) + \rho_s(Ax_{2n-3}, Sx_{2n-2})}{1 + \rho_s(Ax_{2n-3}, Tx_{2n-3})\rho_s(Ax_{2n-3}, Sx_{2n-2})} \\
&= \alpha\, \rho_s(y_{2n-3}, y_{2n-2}) + \beta \cdot \frac{\rho_s(y_{2n-3}, y_{2n-2})[1 + \rho_s(y_{2n-2}, y_{2n-1})]}{1 + \rho_s(y_{2n-2}, y_{2n-3})} \\
&\quad + \gamma \cdot \frac{\rho_s(y_{2n-3}, y_{2n-2}) + \rho_s(y_{2n-3}, y_{2n-1})}{1 + \rho_s(y_{2n-3}, y_{2n-2})\rho_s(y_{2n-3}, y_{2n-1})} \\
&\le \alpha\, \rho_s(y_{2n-3}, y_{2n-2}) + \beta\, \rho_s(y_{2n-3}, y_{2n-2}) + \beta\, \rho_s(y_{2n-2}, y_{2n-1}) \\
&\quad + \gamma\rho_s(y_{2n-3}, y_{2n-2}) + s\gamma[\rho_s(y_{2n-3}, y_{2n-2}) + \rho_s(y_{2n-2}, y_{2n-1})]
\end{aligned}
$$

so, that

$$
\rho_s(y_{2n-2}, y_{2n-1}) \le q.\rho_s(y_{2n-3}, y_{2n-2}). \tag{2.8}
$$

Thus from (2.6) and (2.8), it follows that

$$
\rho_s(y_{n-1}, y_n) \le q\rho_s(y_{n-2}, y_{n-1}), \text{ for all } n.
$$

By induction,

$$\rho_s(y_n, y_{n+1}) \le q\rho_s(y_{n-1}, y_n) \le q^2\rho_s(y_{n-2}, y_{n-1}) \le \cdots \le q^{n-1}\rho_s(y_1, y_2),\ n \ge 1. \tag{2.9}$$

Since $q < 1/s^4 < 1$, (2.9) implies that $\rho_s(y_n, y_{n+1}) \to 0$ as $n \to \infty$. Thus, (S, T) is asymptotically regular with respect to A at x_0.

Now, for all $m > n$, employing the condition (b_3) repeatedly and using (2.9),

$$\begin{aligned}\rho_s(y_n, y_m) &\le s[\rho_s(y_n, y_{n+1}) + \rho_s(y_{n+1}, y_m)]\\ &\le s\rho_s(y_n, y_{n+1}) + s^2[\rho_s(y_{n+1}, y_{n+2}) + \rho_s(y_{n+2}, y_m)]\\ &\le s\rho_s(y_n, y_{n+1}) + s^2\rho_s(y_{n+1}, y_{n+2}) + s^3[\rho_s(y_{n+2}, y_{n+3}) + \rho_s(y_{n+3}, y_m)]\\ &\dots\\ &\le s\rho_s(y_n, y_{n+1}) + s^2\rho_s(y_{n+1}, y_{n+2}) + \cdots + s^{m-n}\rho_s(y_{m-1}, y_m)\\ &\le \left[sq^{n-1} + s^2q^n + \cdots + s^{m-n}q^{m-2}\right]\rho_s(y_1, y_2)\\ &= sq^{n-1}\left[1 + sq + \cdots + (sq)^{m-n-1}\right]\rho_s(y_1, y_2)\\ &\le \frac{sq^{n-1}}{1 - sq} \cdot \rho_s(y_1, y_2).\end{aligned}$$

Proceeding the limit as $n \to \infty$ in this, we see that $\rho_s(y_n, y_m) \to 0$. Thus $\langle y_n \rangle_{n=1}^{\infty}$ is a b-Cauchy sequence.

Since X is (S, T, A)-orbitally b-complete at x_0, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$. That is,

$$\lim_{n\to\infty} Ax_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = z. \tag{2.10}$$

In view of the condition (b) of the theorem, from (2.10) we get that

$$\lim_{n\to\infty} A^2x_{2n+1} = \lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} A^2x_{2n+2} = \lim_{n\to\infty} ATx_{2n+1} = Az. \tag{2.11}$$

First, we suppose that (A, S) is compatible. Then, from (2.11), it follows that

$$\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} ASx_{2n} = Az. \tag{2.12}$$

Now, from (2.4) with $x = Ax_{2n}$ and $y = x_{2n-1}$,

$$\begin{aligned}\rho_s(SAx_{2n}, Tx_{2n-1}) \le{}& \alpha\,\rho_s(A^2x_{2n}, Ax_{2n-1})\\ &+ \beta \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1})[1 + \rho_s(A^2x_{2n}, SAx_{2n})]}{1 + \rho_s(A^2x_{2n}, Ax_{2n-1})}\\ &+ \gamma \cdot \frac{\rho_s(Ax_{2n-1}, Tx_{2n-1}) + \rho_s(Ax_{2n-1}, SAx_{2n})}{1 + \rho_s(Ax_{2n-1}, Tx_{2n-1})\rho_s(Ax_{2n-1}, SAx_{2n})},\end{aligned}$$

which, in view of (2.1), (2.10), (2.11) and (2.12), gives

$$\begin{aligned}
\frac{1}{s^2}\rho_s(Az,z) &\le \liminf_{n\to\infty}\rho_s(SAx_{2n},Tx_{2n-1}) \le \limsup_{n\to\infty}\rho_s(SAx_{2n},Tx_{2n-1})\\
&\le \limsup_{n\to\infty}\Bigg[\alpha\,\rho_s(A^2x_{2n},Ax_{2n-1})\\
&\qquad + \beta\cdot\frac{\rho_s(Ax_{2n-1},Tx_{2n-1})[1+\rho_s(A^2x_{2n},SAx_{2n})]}{1+\rho_s(A^2x_{2n},Ax_{2n-1})}\\
&\qquad\qquad + \gamma\cdot\frac{\rho_s(Ax_{2n-1},Tx_{2n-1})+\rho_s(Ax_{2n-1},SAx_{2n})}{1+\rho_s(Ax_{2n-1},Tx_{2n-1})\rho_s(Ax_{2n-1},SAx_{2n})}\Bigg]\\
&\le s^2\Bigg[\alpha\,\rho_s(Az,z)+\beta\cdot\frac{\rho_s(z,z)[1+\rho_s(Az,Az)]}{1+\rho_s(Az,z)}\\
&\qquad + \gamma\cdot\frac{\rho_s(z,z)+\rho_s(z,Az)}{1+\rho_s(z,z)\rho_s(z,Az)}\Bigg]\\
&= s^2(\alpha+\gamma)\rho_s(z,Az)
\end{aligned}$$

so that $\rho_s(Az,z)\le s^4(\alpha+\gamma)\rho_s(z,Az)$ and hence $Az=z$.

On one hand, writing $x=Ax_{2n}$ and $y=z$ in (2.4),

$$\begin{aligned}
\rho_s(SAx_{2n},Tz) &\le \alpha\,\rho_s(A^2x_{2n},Az)+\beta\cdot\frac{\rho_s(Az,Tz)[1+\rho_s(A^2x_{2n},SAx_{2n})]}{1+\rho_s(A^2x_{2n},Az)}\\
&\qquad + \gamma\cdot\frac{\rho_s(Az,Tz)+\rho_s(Az,SAx_{2n})}{1+\rho_s(Az,Tz)\rho_s(Az,SAx_{2n})}
\end{aligned}$$

Using (2.2), (2.10), (2.11) and (2.12), this gives

$$\begin{aligned}
\frac{1}{s}\rho_s(Az,Tz) &\le \liminf_{n\to\infty}\rho_s(SAx_{2n},Tz)\\
&\le \limsup_{n\to\infty}\rho_s(SAx_{2n},Tz)\\
&\le s\Bigg[\alpha\rho_s(Az,Tz)+\beta\cdot\frac{\rho_s(Az,Tz)[1+\rho_s(Az,Az)]}{1+\rho_s(Az,Az)}\\
&\qquad + \gamma\cdot\frac{\rho_s(Az,Tz)+\rho_s(Az,Az)}{1+\rho_s(Az,Tz)\rho_s(Az,Az)}\Bigg]
\end{aligned}$$

so that $\rho_s(Az,Tz)\le s^2(\alpha+\beta+\gamma)\rho_s(Az,Tz)$ or $\rho_s(Az,Tz)=0$ and hence $Az=Tz$. Thus

$$Az=Tz=z. \tag{2.13}$$

On the other hand, writing $x=z$ and $y=z$ in (2.4), and using (2.13),

$$\begin{aligned}
\rho_s(Sz,z)=\rho_s(Sz,Tz) &\le \alpha\,\rho_s(Az,Az)+\beta\cdot\frac{\rho_s(Az,Tz)[1+\rho_s(Az,Sz)]}{1+\rho_s(Az,Az)}\\
&\qquad + \gamma\cdot\frac{\rho_s(Az,Tz)+\rho_s(Az,Sz)}{1+\rho_s(Az,Tz)\rho_s(Az,Sz)}\\
&= \rho_s(Az,Sz)
\end{aligned}$$

so that $\rho_s(Sz, z) = 0$ or $Sz = z$. In other words, z is a common fixed point of A, S and T. Similarly, a common fixed point of A, S and T is obtained, if (A, T) is compatible.

It is not hard to establish the uniqueness of the common fixed point. □

Corollary 2.1. *Let T be a self-map on a b-metric space (X, ρ_s) with $s \geq 1$, satisfying the inequality*

$$\rho_s(Tx, Ty) \leq \alpha\, \rho_s(x, y) + \beta \cdot \frac{\rho_s(y, Ty)[1 + \rho_s(x, Tx)]}{1 + \rho_s(x, y)} + \gamma \cdot \frac{\rho_s(y, Ty) + \rho_s(y, Tx)}{1 + \rho_s(y, Ty)\rho_s(y, Tx)}, \text{ for all } x, y \in X, \tag{2.14}$$

where α, β and γ are non-negative numbers, not all being zero, such that

$$s^4\alpha + (s^4 + 1)\beta + (s^5 + s^4 + s)\gamma < 1. \tag{2.15}$$

If the space X is T-orbitally b-complete, then T has a unique fixed point.

Proof. We write $S = T$ and $A = I_X$ in Theorem 2.1, where I_X is the identity self-map on X. Note that I_X commutes with every map and hence (I, T) is compatible. Since every continuous function is T-orbitally continuous, by Theorem 2.1, T has a unique fixed point. □

The following result was proved in [6]:

Theorem 2.2. *Let T be a self-map on a complete b-metric space (X, ρ_s) with $s \geq 1$, satisfying the inequality* (2.14)*, where α, β and γ are non-negative numbers, not all being zero, such that*

$$s\alpha + \beta + \gamma < 1. \tag{2.16}$$

Then, T has a unique fixed point.

Remark 2.1. It may be noted that a complete b-metric space is T-orbitally b-complete at each of its points, and $s\alpha+\beta+\gamma < s^4\alpha+(s^4+1)\beta+(s^5+s^4+s)\gamma < 1$, a unique fixed point of T follows from Corollary 2.1. Therefore, Corollary 2.1 is a generalization of Theorem 2.2.

Since every complete metric space is orbitally complete, the following result of Dass and Gupta [3] follows from Corollary 2.1 with $s = 1$ and $\gamma = 0$:

Corollary 2.2. *Let T be a self-map on a complete metric space (X, d) satisfying the inequality*

$$d(Tx, Ty) \leq \alpha\, d(x, y) + \beta \cdot \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \text{ for all } x, y \in X, \tag{2.17}$$

where α and β are non-negative numbers, not both being zero, such that

$$\alpha + 2\beta < 1. \tag{2.18}$$

Then, T has a unique fixed point.

3. Conclusions

In the introductory section of this paper, a brief account of b-metric space and its relation with metric space is presented along with its topological properties. The highlights of Theorem 2.1 for three compatible self-maps on a b-metric space satisfying a rational type condition are the notions of asymptotic regularity, orbital completeness and orbital continuity. Also, the main result of this paper is an elegant extension of theorems of Sarwar and Rahman [6], and Dass and Gupta [3].

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Simultaneous approximation of translation operators

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Abstract. Let $(m_n)_{n\in\mathbb{N}}$ be an unbounded sequence of complex numbers and $(a_v)_{v\in\mathbb{N}}$ be a sequence of numbers in the unit circle

$$C(0,1)=\{z\in\mathbb{C}\mid |z|=1\},$$

where $\mathbb{N}$ is the set of natural numbers.

We shall prove that there is an entire function f so that, for every entire function g there is a subsequence (λ_n), $n\in\mathbb{N}$ of $(m_n)_{n\in\mathbb{N}}$ such that, for every compact subset $L\subseteq\mathbb{C}$ and for every $v\in\mathbb{N}$,

$$\sup_{z\in L}|f(z+\lambda_n a_v)-g(z)|\to 0 \quad\text{as}\quad n\to\infty.$$

In relation with other results about hypercyclic operators, the new element in this paper is that we achieve the approximation with the same sequence (λ_n), for all numbers a_v $(v=1,2,\ldots)$.
Keywords: hypercyclic operator, common hypercyclic vectors, translation operator, simultaneous approximation.
MSC 2020: 47A16, 30H05, 30E10, 41A28

1. Introduction

We denote $\mathcal{H}(\mathbb{C})$ the set of entire functions endowed with the topology $\mathcal{T}_u$ of uniform convergence on compacta.

Let $a\in\mathbb{C}$. We denote $t_a:\mathbb{C}\to\mathbb{C}$ the translation function, which is given by the formula $t_a(z)=z+a$, for every $z\in\mathbb{C}$.

We consider the translation operator $T_a:\mathcal{H}(\mathbb{C})\to\mathcal{H}(\mathbb{C})$, that is, the operator defined by the formula $T_a(f)=f\circ t_a$, for every $f\in\mathcal{H}(\mathbb{C})$. The operator T_a is a linear and continuous operator.

We write $T_a^1=T_a$ and

$$T_a^{n+1}=T_a\circ T_a^n,\quad\text{for}\quad n=1,2,\ldots.$$

Birkhoff proved [4] that there is $f\in\mathcal{H}(\mathbb{C})$ so that

$$\overline{\{T_a^n(f),\ \ n\in\mathbb{N}\}}=\mathcal{H}(\mathbb{C}),\quad\text{where}\quad a\in\mathbb{C}\setminus\{0\}.$$

His proof was constructive.

Let $(a_n)_{n\in\mathbb{N}}$ be an unbounded sequence of complex numbers. Luh [12] proved that there is $f\in\mathcal{H}(\mathbb{C})$ so that

$$\overline{\{T_{a_n}(f), n\in\mathbb{N}\}}=\mathcal{H}(\mathbb{C}).$$

Gethner and Shapiro [8] and Grosse-Erdmann [9] have also proved the above results by using the Baire's Category Theorem. In particular, let $\mathcal{U}((T_{a_n}))$ be the set of entire functions that are universal (or hypercyclic) for the sequence (T_{a_n}), that is,

$$\mathcal{U}((T_{a_n}))=\{f\in\mathcal{H}(\mathbb{C}) \mid \overline{\{T_{a_n}(f)\mid n\in\mathbb{N}\}}=\mathcal{H}(\mathbb{C})\}.$$

Then, the set $\mathcal{U}((T_{a_n}))$ is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. Let $(b_m)_{m\in\mathbb{N}}$ be a sequence of non-zero complex numbers. Based on the previous result, the set $\bigcap_{m\in\mathbb{N}}\mathcal{U}((T_{b_m a_n}))$ is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$.

Costakis and Sambarino [6] established a notable strengthening of Birkhoff's result. More specifically, they proved that the set

$$\bigcap_{a\in\mathbb{C}-\{0\}}\{f\in\mathcal{H}(\mathbb{C}) \mid \overline{\{T_a^n(f), n\in\mathbb{N}\}}=\mathcal{H}(\mathbb{C})\}$$

contains a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. Note that each set in the last intersection is $\mathcal{U}(T_a):=\mathcal{U}((T_a^n))=\mathcal{U}((T_{an}))$.

The important element here is the uncountable range of a.

Furthermore, Costakis [5] proved a more general result, that is, the set $\bigcap_{b\in C(0,1)}\mathcal{U}(T_{ba_n})$ contains a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$, where a_n is an unbounded and specific sequence of complex numbers.

Let us apply this result, in certain cases.

Let $(\theta_v)_{v\in\mathbb{N}}$ be a sequence of distinct numbers in $[0,1)$ and $(m_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers so that $m_n\to\infty$. We shall consider the numbers $w_n(\theta_v)=m_n e^{2\pi i\theta_v}$, $n,v\in\mathbb{N}$. That is, for every $v\in\mathbb{N}$ we shall consider the sequence $(w_n(\theta_v))_{n\in\mathbb{N}}$. Of course, we have $w_n(\theta_v)\to\infty$ as $n\to+\infty$, for every $v\in\mathbb{N}$.

We now set:

$$E_v=\{f\in\mathcal{H}(\mathbb{C})|\overline{\{f(\cdot+w_n(\theta_v)):n\in\mathbb{N}\}}=\mathcal{H}(\mathbb{C})\}, \quad \text{for every} \quad v\in\mathbb{N}.$$

Based on Grosse-Erdmann's result we conclude that, for every $v\in\mathbb{N}$ the set E_v is G_δ and dense in $\mathcal{H}(\mathbb{C})$. Hence, the set $E:=\bigcap_{v=1}^{+\infty}E_v$ is a G_δ dense subset of $\mathcal{H}(\mathbb{C})$, so it is non-empty by Baire's Category Theorem, given that the space $\mathcal{H}(\mathbb{C})$ is a complete metric space. Let us see in more detail what this result means.

Let $f\in E$. Then, for every $v\in\mathbb{N}$ and $g\in\mathcal{H}(\mathbb{C})$ there is a subsequence $(\lambda_n^v)=(\lambda_n(v,g))$ of $(w_n(\theta_v))$, that depends on g and v so that, for every compact set $K\subseteq\mathbb{C}$ one has

$$\sup_{z\in K}|f(z+\lambda_n^v)-g(z)|\to 0 \quad \text{as} \quad n\to\infty.$$

So, this convergence depends on the specific sequence $\lambda_n^v = \lambda_n(v, g)$, $n \in \mathbb{N}$, and the sequence λ_n^v depends on the specific number $\theta_v \in [0, 1)$. In the present paper we shall examine whether we can have this convergence *without* the dependence on the specific number $\theta_v \in [0, 1)$.

With this aim, we shall introduce the set of entire functions that achieve simultaneous approximation on all numbers θ_v, $v \in \mathbb{N}$, where $\theta_v \in [0, 1)$, for every $v \in \mathbb{N}$ with the same sequence of indices. More specifically, we shall consider the set SA (standing for Simultaneous Approximation) defined as

$$SA = \{f \in \mathcal{H}(\mathbb{C}) : \quad \text{for every} \quad g \in \mathcal{H}(\mathbb{C})$$

there is a subsequence $(\lambda_n)_{n\in\mathbb{N}}$ of $(m_n)_{n\in\mathbb{N}}$ such that

$$\sup_{z\in K} |f(z + \lambda_n e^{2\pi i\theta_v}) - g(z)| \to 0 \quad \text{as} \quad n \to \infty,$$

for every compact set $K \subset \mathbb{C}$ and every $v \in \mathbb{N}\}$.

Of course $SA \subseteq E$.

We prove that the set SA is a G_δ-dense subset of $\mathcal{H}(\mathbb{C})$, so it is non-empty. In order to prove that SA is a G_δ, dense subset of $\mathcal{H}(\mathbb{C})$ we shall introduce one other set $V \subseteq \mathcal{H}(\mathbb{C})$ and we prove that V is a G_δ, dense subset of $\mathcal{H}(\mathbb{C})$ and $SA = V$. Other articles dealing with translation operators or sequences of translation operators on $\mathcal{H}(\mathbb{C})$ are [3] and [10].

Also, there are some papers concerning common hypercyclic vectors for translation operators; see the papers [1], [5], [6], [7], [14], [15], [16], as well as Chapter 11 in the book [11]. The notion of simultaneous hypercyclicity/universality was formally introduced (for finitely many operators) in [2].

Whenever we refer to a topology in the $\mathcal{H}(\mathbb{C})$ space, we always mean the topology of uniform convergence on compacta.

In the following Section 2 we prove some helpful propositions in order to prove our main result Theorem 2.6.

2. The main result

First of all, we shall prove a proposition which is the key in order to prove our main result.

We fix $g \in \mathcal{H}(\mathbb{C})$.

We also fix some natural numbers $n_0 \geq 2$, v_0, N_0, and some real numbers $\theta_1, \theta_2, \ldots, \theta_{n_0}$ where $\theta_i \in [0, 1)$ for each $i = 1, \ldots, n_0$ and $\theta_i \neq \theta_j$, for every $i, j \in A_{n_0} = \{1, \ldots, n_0\}$, $i \neq j$. For every natural number m we use the set

$$V_g(m, v_0, N_0, n_0) = \Big\{ f \in \mathcal{H}(\mathbb{C}) \Big/ \sup_{|z|\leq v_0} \Big| f(z + m e^{2\pi i\theta_j}) - g(z) \Big| < \frac{1}{N_0}, \quad \text{for every } j = 1, \ldots, n_0 \Big\}.$$

For every $m \in \mathbb{N}$, $j \in A_{n_0}$ we use the set

$$\widetilde{V}_g(m, v_0, N_0, j) = \Big\{ f \in \mathcal{H}(\mathbb{C}) \Big| \sup_{|z| \le v_0} \Big| f(z + me^{2\pi i \theta_j}) - g(z) \Big| < \frac{1}{N_0} \Big\}.$$

Of course, we have

$$V_g(m, v_0, N_0, n_0) = \bigcap_{j=1}^{n_0} \widetilde{V}_g(m, v_0, N_0, j), \tag{1}$$

based on the above definitions.

It is easy to see that the sets $\widetilde{V}_g(m, v_0, N_0, j)$ are open in $\mathcal{H}(\mathbb{C})$, for every $m \in \mathbb{N}$, $j = 1, \ldots, n_0$, so the set $V_g(m, v_0, N_0, n_0)$ is open in $\mathcal{H}(\mathbb{C})$, for every $m \in \mathbb{N}$, according to the above relation (1). Therefore, the set $\bigcup_{m=1}^{+\infty} V_g(m, v_0, N_0, n_0)$ is open in $\mathcal{H}(\mathbb{C})$.

For a function $h : \mathbb{C} \to \mathbb{C}$ and $A \subseteq \mathbb{C}$, we shall denote $\|h\|_A := \sup\{|h(z)| : z \in A\}$.

Proposition 2.1. *Under the above notations, we have that the set* $\bigcup_{m=1}^{+\infty} V_g(m, v_0, N_0, n_0)$ *is dense in* $\mathcal{H}(\mathbb{C})$.

Proof. We fix a function $h \in \mathcal{H}(\mathbb{C})$, a compact set $K \subseteq \mathbb{C}$ and an $\varepsilon > 0$. It suffices to show that there are $f \in \mathcal{H}(\mathbb{C})$ and $m_0 \in \mathbb{N}$, so that

$$f \in V_g(m_0, v_0, N_0, n_0) \ \text{ and } \ \|f - h\|_K < \varepsilon. \tag{1}$$

We set $D_v = \{z \in \mathbb{C} \mid |z| \le v\}$, for every $v \in \mathbb{N}$. We also choose $v_1 \in \mathbb{N}$ so that

$$D_{v_0} \cup K \subseteq D_{v_1}. \tag{2}$$

Let us assume that $m \in \mathbb{N}$ satisfies

$$D_{v_1} \cap (D_{v_1} + me^{2\pi i \theta_j}) \neq \emptyset,$$

for some $j \in A_{n_0}$ (if it exists). We remind that $A_{n_0} = \{1, 2, \ldots, n_0\}$.
This means that there also exist $z_j, w_j \in D_{v_1}$, so that

$$w_j = z_j + me^{2\pi i \theta_j}, \ \text{ for some } \ j \in A_{n_0}. \tag{3}$$

According to (3), we shall have:

$$|w_j - z_j| = m\,, \ \text{ and this gives } \ m \le 2v_1.$$

Therefore, for every $m \in \mathbb{N}$ and $m > 2v_1$, we have

$$D_{v_1} \cap (D_{v_1} + me^{2\pi i \theta_j}) = \emptyset, \ \text{ for every } \ j \in A_{n_0}. \tag{4}$$

Let $j_1, j_2 \in A_{n_0}$, so that $j_1 \neq j_2$.

Let $m \in \mathbb{N}$ so that

$$(D_{v_1} + me^{2\pi i\theta_{j_1}}) \cap (D_{v_1} + me^{2\pi i\theta_{j_2}}) \neq \emptyset \quad \text{(if it exists)}.$$

This means that there are $z_1, w_1 \in D_{v_1}$ so that

$$z_1 + me^{2\pi i\theta_{j_1}} = w_1 + me^{2\pi i\theta_{j_2}}. \tag{5}$$

By (5) we have:

$$|z_1 - w_1| = m\big|e^{2\pi i(\theta_{j_2} - \theta_{j_1})} - 1\big|. \tag{6}$$

From (6) we deduce that:

$$m \leq \frac{2v_1}{\big|e^{2\pi i(\theta_{j_2} - \theta_{j_1})} - 1\big|}. \tag{7}$$

So, for every $m \in \mathbb{N}$ satisfying

$$m > \frac{2v_1}{\big|e^{2\pi i(\theta_{j_2} - \theta_{j_1})} - 1\big|}$$

we have:

$$(D_{v_1} + me^{2\pi i\theta_{j_1}}) \cap (D_{v_1} + me^{2\pi i\theta_{j_2}}) = \emptyset. \tag{8}$$

We set

$$M_0 = \min\left\{\big|e^{2\pi i(\theta_{j_2} - \theta_{j_1})} - 1\big| : j_1, j_2 \in A_{n_0} : j_1 \neq j_2\right\}.$$

We fix now some natural number m_0 so that $m_0 > \max\{2v_1, \frac{2v_1}{M_0}\}$. Then, by (4) and (8) we derive

$$D_{v_1} \cap (D_{v_1} + m_0 e^{2\pi i\theta_j}) = \emptyset, \quad \text{for every} \quad j \in A_{n_0}$$

and

$$(D_{v_1} + m_0 e^{2\pi i\theta_{j_1}}) \cap (D_{v_1} + m_0 e^{2\pi i\theta_{j_2}}) = \emptyset, \quad \text{for every} \quad j_1, j_2 \in A_{n_0}, \quad j_1 \neq j_2. \tag{9}$$

Now, we set

$$L := D_{v_1} \cup \left(\bigcup_{j=1}^{n_0} (D_{v_1} + m_0 e^{2\pi i\theta_j})\right).$$

Because of (9) we have that the set L is a union of $n_0 + 1$ disjoint closed discs with the same radius v_1.

This means that the set L is a compact set with connected complement. We shall consider the function $F : L \to \mathbb{C}$, defined as follows:

$$F(z) = \begin{cases} h(z), & \text{if } z \in D_{v_1} \\ g(z - m_0 e^{2\pi i\theta_j}), & \text{if } z \in D_{v_1} + m_0 e^{2\pi i\theta_j}, \text{ for some } j \in A_{n_0}. \end{cases}$$

Of course, F is continuous on L and holomorphic on $\overset{\circ}{L}$, the interior of L. So, according to Mergelyan's Approximation Theorem (see, e.g., [[13], Chapter 20]) there is a complex polynomial f, so that

$$\|F - f\|_L < \min\left\{\varepsilon, \frac{1}{N_0}\right\}. \tag{10}$$

Based on the definition of F and (10), we have:

$$\|f - h\|_K < \varepsilon, \tag{11}$$

because of relation (2), and the definition of L.

Let us suppose $w \in D_{v_1}$. Then, for $j \in A_{n_0}$, $w + m_0 e^{2\pi i\theta_j} \in D_{v_1} + m_0 e^{2\pi i\theta_j}$.

We also set $z = w + m_0 e^{2\pi i\theta_j}$. Then, $F(z) = g(z - m_0 e^{2\pi i\theta_j}) = g(w)$. By (10) we have, for every $w \in D_{v_1}$ and $j \in A_{n_0}$ that

$$\left|f(w + m_0 e^{2\pi i\theta_j}) - g(w)\right| < \frac{1}{N_0}.$$

This yields that $f \in V_g(m_0, v_0, N_0, n_0)$, because of relation (2) and the fact that f is a polynomial (so entire). By this fact and (11) the proof of this proposition is complete now because relation (1) is satisfied. $\square$

Now, we shall fix an unbounded sequence $(m_s)_{s\in\mathbb{N}}$ of complex numbers. With the notation of the previous Proposition 2.1 we shall consider the set:

$$V_g(m_s, v_0, N_0, n_0) = \left\{ f \in \mathcal{H}(\mathbb{C}) \,\middle|\, \sup_{|z|\le v_0} \left| f(z + m_s e^{2\pi i\theta_j}) - g(z)\right| < \frac{1}{N_0}, \right.$$
$$\left. \text{for every } \; j \in A_{n_0} \right\},$$

for every $s \in \mathbb{N}$.

The sets $V_g(m_s, v_0, N_0, n_0)$ are open for every $s \in \mathbb{N}$, so the set $\bigcup_{s=1}^{+\infty} V_g(m_s, v_0, N_0, n_0)$ is open in $\mathcal{H}(\mathbb{C})$.

As in Proposition 2.1, we now state the following proposition:

Proposition 2.2. *The set $\bigcup_{s=1}^{+\infty} V_g(m_s, v_0, N_0, n_0)$ is dense in $\mathcal{H}(\mathbb{C})$.*

Proof. The proof is similar to that of Proposition 2.1 and for this reason the proof is omitted. $\square$

Indeed, the only property of $\{1, 2, \dots\}$ used in the proof of the last proposition is its non-boundedness.

Recall that the space $\mathcal{H}(\mathbb{C})$ is separable, and so we can fix a dense sequence $(p_k)_{k\in\mathbb{N}}$ of $\mathcal{H}(\mathbb{C})$ (for example $(p_k)_{k\in\mathbb{N}}$ be an enumeration of all complex polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$). For every $v, N, k, n, s \in \mathbb{N}$, $n \ge 2$ we shall consider the set:

$$V_{p_k}(m_s, v, N, n) = \left\{ f \in \mathcal{H}(\mathbb{C}) \middle| \sup_{|z| \leq v} \left| f(z + m_s e^{2\pi i \theta_j}) - p_k(z) \right| < \frac{1}{N}, \right.$$
$$\left. \text{for every } \; j \in A_n \right\}.$$

The sets $V_{p_k}(m_s, v, N, n)$ are open in $\mathcal{H}(\mathbb{C})$, for every $v, N, k, n, s \in \mathbb{N}$, $n \geq 2$, so that the set $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ is open for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$. According to Proposition 2.2, we have that the sets $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ are dense in $\mathcal{H}(\mathbb{C})$, for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$.

We shall also consider the set:

$$V = \bigcap_{v=1}^{+\infty} \bigcap_{N=1}^{+\infty} \bigcap_{k=1}^{+\infty} \bigcap_{n=2}^{+\infty} \left(\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n) \right).$$

Under the above notation, we shall establish the following assertion

Proposition 2.3. *The set V is a G_δ-dense subset of $\mathcal{H}(\mathbb{C})$, so V is non-empty.*

Proof. The set V is a G_δ subset of $\mathcal{H}(\mathbb{C})$ due to its definition, because the sets $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ are open for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$. Based on Proposition 2.2, the sets $\bigcup_{s=1}^{+\infty} V_{p_k}(m_s, v, N, n)$ are dense for every $v, N, k, n \in \mathbb{N}$, $n \geq 2$. Hence, the conclusion follows from Baire's Category Theorem because the space $\mathcal{H}(\mathbb{C})$ is a complete metric space. □

We now connect the previous set V with the set of entire functions that succeed simultaneous approximation with respect to a countable set of real numbers.

We shall state here the respective data. Let $(\theta_n)_{n \in \mathbb{N}}$ be a sequence of real numbers, so that $\theta_n \in [0, 1)$ and $\theta_{j_1} \neq \theta_{j_2}$, for every $j_1, j_2 \in \mathbb{N}$, $j_1 \neq j_2$, $n \in \mathbb{N}$. Let $(m_s)_{s \in \mathbb{N}}$ be a fixed sequence of complex numbers which is unbounded.

Let $\Theta := \{\theta_n : n \in \mathbb{N}\}$. Of course, the set Θ and the set:

$$m = \{m_s : s \in \mathbb{N}\}$$

consisting of all the terms of the sequence $(m_s)_{s \in \mathbb{N}}$ are also infinite.

We shall consider the set:

$$SA = \{ f \in \mathcal{H}(\mathbb{C}) \, |, \text{ for every } g \in \mathcal{H}(\mathbb{C}),$$

there is a sequence $(\lambda_n)_{n \in \mathbb{N}}$ so that $\lambda_n \in m$, for every $n \in \mathbb{N}$, so that for every $a \in \Theta$ and for every compact set $K \subseteq \mathbb{C}$ it holds that

$$\sup_{z \in K} \left| f(z + \lambda_n e^{2\pi i a}) - g(z) \right| \to 0 \;\text{ as }\; n \to \infty \}.$$

The method to prove that $SA \neq \emptyset$ is the following:
We shall prove that $SA = V$ and given that $V \neq \emptyset$ we shall also have $SA \neq \emptyset$.

In order to prove that $SA = V$ we show that $SA \subseteq V$ and $V \subseteq SA$. This is the subject of the following two propositions.

Proposition 2.4. *It holds $SA \subseteq V$.*

Proof. If $SA = \emptyset$, then the result is obvious. We suppose that $SA \neq \emptyset$. Let $f \in SA$. We fix $v_0, N_0, n_0, k_0 \in \mathbb{N}$, $n_0 \geq 2$.

Because $f \in SA$ for $g = p_{k_0}$ there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$, so that $\lambda_n \in m$, for every $n \in \mathbb{N}$ and, for every $a \in \Theta$ and every compact set $K \subseteq \mathbb{C}$, we have

$$\sup_{z\in K} |f(z+\lambda_n e^{2\pi i a}) - p_{k_0}(z)| \to 0 \ \text{ as } n\to\infty.$$

So, for $K = D_{v_0}$ we have that

$$\sup_{|z|\leq v_0} |f(z+\lambda_n e^{2\pi i\theta_j}) - p_{k_0}(z)| \to 0 \ \text{ as } \ n\to\infty,$$

for every $j \in A_{n_0}$.

This entails that for every $j \in A_{n_0}$ there is some $n_j \in \mathbb{N}$, so that

$$\sup_{|z|\leq v_0} \left|f(z+\lambda_n e^{2\pi i\theta}) - p_{k_0}(z)\right| < \frac{1}{N_0}, \ \text{ for every } \ n \in \mathbb{N}, \ \ n \geq n_j.$$

Let $\widetilde{n} = \max\{n_j | j \in A_{n_0}\}$. With this selection we obtain

$$\sup_{|z|\leq v_0} \left|f(z+\lambda_n e^{2\pi i\theta_j}) - p_{n_0}(z)\right| < \frac{1}{N_0},$$

for every $j \in A_{n_0}$, for every $n \in \mathbb{N}$, $n \geq \widetilde{n}$.

This implies that $f \in V_{p_{k_0}}(\lambda_{\widetilde{n}}, v_0, N_0, n_0)$, or equivalently, $f \in \bigcup_{s=1}^{+\infty} V_{p_{k_0}}(m_s, v_0, N_0.n_0)$ because $\lambda_{\widetilde{n}} \in m$, that implies $f \in V$ and the result is proven. □

Proposition 2.5. *It holds that $V \subseteq SA$.*

Proof. We know that $V \neq \emptyset$. Let $f \in V$. We shall prove that $f \in SA$.

We fix $g \in \mathcal{H}(\mathbb{C})$. We shall show that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$, so that $\lambda_n \in m$, for every $n \in \mathbb{N}$, and so that for every $a \in \Theta$ and every compact set $K \subseteq \mathbb{C}$

$$\sup_{z\in K} \left|f(z+\lambda_n e^{2\pi i a}) - g(z)\right| \to 0 \ \text{ as } \ n\to\infty.$$

Based on the above mentioned properties, we shall now construct the respective sequence $(\lambda_n)_{n\in\mathbb{N}}$. We shall fix some $n_0 \in \mathbb{N}$, $n_0 \geq 2$.

Given that the sequence $(p_k)_{k\in\mathbb{N}}$ of complex polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathcal{H}(\mathbb{C})$, there is some $k_0 \in \mathbb{N}$ so that

$$\|g - p_{k_0}\|_{D_{n_0}} < \frac{1}{2n_0}. \tag{1}$$

Since $f \in V$ we have $f \in \bigcup_{s=1}^{+\infty} V_{p_{k_0}}(m_s, n_0, 2n_0, n_0)$. This means that there is some $s_{n_0} \in \mathbb{N}$ so that $f \in V_{p_{k_0}}(m_{s_{n_0}}, n_0, 2n_0, n_0)$, or equivalently,

$$\sup_{|z|\leq n_0} \left|f(z+m_{s_{n_0}} e^{2\pi i\theta_j}) - p_{k_0}(z)\right| < \frac{1}{2n_0}, \ \text{ for every } \ j \in A_{n_0}. \tag{2}$$

By (1), (2) and the triangle inequality we have:

$$(3) \qquad \sup_{|z|\leq n_0} \left| f(z+m_{s_{n_0}} e^{2\pi i \theta_j}) - g(z) \right| < \frac{1}{n_0}, \quad \text{for every} \quad j \in A_{n_0}.$$

According to the previous procedure, for every $n \in \mathbb{N}$, $n \geq 2$, we can choose some $s_n \in \mathbb{N}$ such that

$$(4) \qquad \sup_{|z|\leq n} \left| f(z+m_{s_n} e^{2\pi i \theta_j}) - g(z) \right| < \frac{1}{n}, \quad \text{for every} \quad j \in A_n.$$

We shall now prove that for the sequence $(s_n)_{n\in\mathbb{N}}$ one has the following: $\sup_{z\in K} \left| f(z+m_{s_n} e^{2\pi i a}) - g(z) \right| \to 0$ as $n\to\infty$, for every compact set $K \subseteq \mathbb{C}$ and for every $a \in \Theta$.

With this aim, fix some $\varepsilon_0 > 0$.

There are $v_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$, so that $K \subseteq D_v$, for every $v \in \mathbb{N}$, $v \geq v_0$ and $a_0 = \theta_{n_0}$. Let us choose $N_0 \in \mathbb{N}$ such that $\frac{1}{N_0} < \varepsilon_0$.

Let $M_0 = \max\{v_0, n_0, N_0, 2\}$. For every $n \in \mathbb{N}$, $n \geq M$, we have $n \geq v_0$, so $K \subseteq D_n$. Of course, $a_0 \in \{\theta_1, \theta_2, \ldots, \theta_n\}$, for every $n \in \mathbb{N}$, $n \geq M_0$, because $a_0 = \theta_{n_0}$ and $n_0 \leq M_0 \leq n$. With this, we also get

$$\frac{1}{n} \leq \frac{1}{M_0} \leq \frac{1}{N_0} < \varepsilon_0, \quad \text{for every} \quad n \in \mathbb{N}, \quad n \geq M_0.$$

Then, for every $n \in \mathbb{N}$, $n \geq M_0$, it follows from (4) that

$$\sup_{z\in K} \left| f(z+m_{s_n} e^{2\pi i a_0}) - g(z) \right| \leq \sup_{|z|\leq n} \left| f(z+m_{s_n} e^{2\pi i a_0}) - g(z) \right| < \frac{1}{n} < \varepsilon_0.$$

This yields that

$$\sup_{z\in K} \left| f(z+m_{s_n} e^{2\pi i a_0}) - g(z) \right| \to 0 \quad \text{as} \quad n\to\infty.$$

So, for every $a \in \Theta$ and every compact set $K \subseteq \mathbb{C}$ we have:

$$\sup_{z\in K} \left| f(z+m_{s_n} e^{2\pi i a}) - g(z) \right| \to 0 \quad \text{as} \quad n\to\infty.$$

Since this is the case for arbitrary $g \in \mathcal{H}(\mathbb{C})$ we conclude that $f \in SA$ and the proof of this proposition is complete. □

Based on the above results, we are ready now to state and prove the main result of this paper, that is Theorem 2.6.

Theorem 2.6. *The set SA is a G_δ dense subset of $\mathcal{H}(\mathbb{C})$. In particular, the set SA is non-empty.*

Proof. Based on Proposition 2.4 and 2.5 we have that $SA = V$. We have also proved in Proposition 2.3 that the set V is a G_δ and dense subset of $\mathcal{H}(\mathbb{C})$. So, the result follows. □

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L-fuzzy ideal theory on bounded semihoops

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Abstract. This article mainly focuses on the L-fuzzy ideal theory on bounded semihoops. Firstly, we propose two classes of L-fuzzy ideals on bounded semihoop and prove that each L-fuzzy strong ideal is an L-fuzzy ideal but an L-fuzzy ideal may not be an L-fuzzy strong ideal. We also get some properties and equivalent descriptions of L-fuzzy strong ideal. Secondly, we introduce the notion of L-fuzzy prime ideal and the second type of L-fuzzy prime ideal on bounded semihoops. Moreover, we discuss the relationship between these two types of L-fuzzy prime ideals. Finally, we present the concept of L-fuzzy maximal ideal on bounded semihoops and obtain some properties.

Keywords: bounded semihoop, L-fuzzy (strong) ideal, L-fuzzy prime ideal, L-fuzzy maximal ideal.

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1. Introduction

Classical logic can no longer fully adapt to people's reasoning and thinking activities in the development of today's era, and then non-classical logic came into being. Non-classical logic has become a useful tool for computers to deal with uncertain and fuzzy information. Various logical algebras have been introduced as the semantical systems of non-classical logic systems, for instance, MV-algebras [5], BL-algebras [8], MTL-algebras [16] and residuated lattices [?]. Semihoops [7] are generalization of hoops [1], which were originally proposed by Bosbach. Semihoops, as the basic residuated structures, contain all logical algebras that satisfy the residuated law. Recent years, many scholars have conducted research on semihoops and obtain some important conclusions. For example, Borzooei and Kologani [16] studied the relationships between various filters on semihoops in 2015. In 2019, Niu [13] proposed the tense operators on bounded semihoops and Zhang [17] introduced the derivations and differential filters on semihoops. In 2020, Niu and Xin [14] established the ideal theory on bounded semihoops. Since semihoops are the fundamental residuated structures, the study of semihoops is important for fuzzy logic and some corresponding algebras.

In 1965, Zadeh [18] proposed the concept of fuzzy subset of a nonempty set X as a function $f\colon X \to I$, where $I = [0, 1]$ is the unit interval of real numbers. This marked the formation of fuzzy mathematics as a new discipline. The concept of fuzzy groups was introduced by Rosenfied [15] in 1971, fuzzy algebras have developed rapidly, especially fuzzy ideals on logical algebras. For example, in 2005, Liu and Li [11] proposed the definition of fuzzy filters on BL-algebras. In 2017, Liu [12] studied the ideal and fuzzy ideal in residuated lattices and obtained some important conclusions. In 2019, Borzooei [2] introduced the concept of fuzzy filters in pseudo hoops. However, we find that the current study of fuzzy ideals is limited to chain structures, but ignores that not all elements are comparable in some structures. For instance, there exists incomparable elements in lattice structures. Therefore, we try to associate semihoops with lattice structures and establish the L-fuzzy ideal theory.

This article is structured as follows: In Section 2, we summarize some fundamental definitions and conclusions on bounded semihoops, which will be used in the sequel chapters. In Section 3, we will propose two types of L-fuzzy ideals and discuss their relationship. We also study properties and equivalent characterizations of L-fuzzy strong ideal. In the remaining sections, we will introduce several special classes of L-fuzzy ideals on bounded semihoops, including L-fuzzy prime ideal, the second type of L-fuzzy prime ideal and L-fuzzy maximal ideal.

2. Preliminaries

In this section, we recall some definitions and conclusions, which will be used in the following sections.

Definition 2.1 ([7]). *An algebra* $(S, \odot, \to, \wedge, 1)$ *of type* $(2, 2, 2, 0)$ *is called a semihoop if it satisfies:*

$(S1)$ $(S, \wedge, 1)$ *is a* $\wedge$*-semilattice and it has a upper bound 1;*

$(S2)$ $(S, \odot, 1)$ *is a commutative monoid;*

$(S3)$ $(\alpha \odot \beta) \to \theta = \alpha \to (\beta \to \theta)$*, for any* $\alpha, \beta, \theta \in S$.

In a semihoop $(S, \odot, \to, \wedge, 1)$, we define $\alpha \leq \beta$ if and only if $\alpha \to \beta = 1$, for any $\alpha, \beta \in S$. It is easy to check that $\leq$ is a partial order relation on S and we get $\alpha \leq 1$, for all $\alpha \in S$.

Proposition 2.1 ([7]). *Let* S *be a semihoop. Then, the following properties hold:*

(1) $\alpha \odot \beta \leq \theta$ *if and only if* $\alpha \leq \beta \to \theta$*, for every* $\alpha, \beta, \theta \in S$*;*

(2) $\alpha \odot \beta \leq \alpha, \beta$*, for any* $\alpha, \beta \in S$*;*

(3) $1 \to \alpha = \alpha, \alpha \to 1 = 1$*, for all* $\alpha \in S$*;*

(4) $\alpha^n \leq \alpha$*, for every* $\alpha \in S$*,* $n \in \mathbb{N}^+$*;*

(5) $\alpha \odot (\alpha \to \beta) \leq \beta$*, for any* $\alpha, \beta \in S$*;*

(6) $\alpha \leq \beta$ *implies* $\alpha \odot \theta \leq \beta \odot \theta$*,* $\beta \to \theta \leq \alpha \to \theta$ *and* $\theta \to \alpha \leq \theta \to \beta$*, for every* $\alpha, \beta, \theta \in S$*;*

(7) $\alpha \leq (\alpha \to \beta) \to \beta$*, for any* $\alpha, \beta \in S$*;*

(8) $\alpha \to (\beta \to \theta) = \beta \to (\alpha \to \theta)$*, for every* $\alpha, \beta, \theta \in S$.

A semihoop $(S, \odot, \to, \wedge, 1)$ is called a bounded semihoop if there exists an element $0 \in S$ such that $0 \leq \alpha$, for all $\alpha \in S$. We denote a bounded semihoop $(S, \odot, \to, \wedge, 0, 1)$ by S.

Example 2.1 ([3]). Let $S = \{0, m, n, 1\}$ be a chain with $0 < m < n < 1$. We define $\odot$ and $\to$ on S as follows:

$\odot$	0	m	n	1
0	0	0	0	0
m	0	0	m	m
n	0	m	n	n
1	0	m	n	1

$\to$	0	m	n	1
0	1	1	1	1
m	m	1	1	1
n	0	m	1	1
1	0	m	n	1

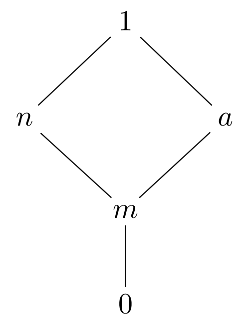
Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop.

Example 2.2 ([3]). Let $S = \{0, m, n, a, 1\}$. Define $\odot$ and $\to$ as follows:

$\boxtimes$	0	m	n	a	1
0	0	0	0	0	0
m	0	m	m	m	m
n	0	m	n	m	n
a	0	m	m	a	a
1	0	m	n	a	1

$\rightarrow$	0	m	n	a	1
0	1	1	1	1	1
m	0	1	1	1	1
n	0	a	1	a	1
a	0	n	n	1	1
1	0	m	n	a	1

It's Hasse diagram is as follows:



Then, $(S, \odot, \rightarrow, \wedge, 0, 1)$ is a bounded semihoop.

In a bounded semihoop S, we define $\star$: $\alpha^\star = \alpha \rightarrow 0$, for any $\alpha \in S$. A bounded semihoop is said to have the Double Negation Property or (DNP) for short if it satisfies $\alpha^{\star\star} = \alpha$, for all $\alpha \in S$.

Proposition 2.2 ([3]). *Let S be a bounded semihoop. Then, we have the following statements hold: for any $\alpha, \beta \in S$,*

(1) $1^\star = 0$, $0^\star = 1$;

(2) $\alpha \leq \alpha^{\star\star}$;

(3) $\alpha^{\star\star\star} = \alpha^\star$;

(4) $\alpha \odot \alpha^\star = 0$;

(5) $\beta^\star \leq \beta \rightarrow \alpha$;

(6) $\alpha \leq \beta$ *implies* $\beta^\star \leq \alpha^\star$;

(7) *if S has (DNP), then* $\alpha \rightarrow \beta = \beta^\star \rightarrow \alpha^\star$;

(8) $\alpha \rightarrow \beta \leq \beta^\star \rightarrow \alpha^\star$;

(9) *if S has (DNP), then* $\alpha^\star \rightarrow \beta = \beta^\star \rightarrow \alpha$.

Definition 2.2 ([14]). *Assume that S is a bounded semihoop. The binary operation $\oplus$ is defined by $\alpha \oplus \beta = \alpha^\star \rightarrow \beta$, for any $\alpha, \beta \in S$.*

Proposition 2.3 ([14]). *Let S be a bounded semihoop. Then, the following properties hold:*

(1) $\alpha \leq \beta$ *implies* $\alpha \oplus \theta \leq \beta \oplus \theta$, *for every* $\alpha, \beta, \theta \in S$;

(2) $\alpha \leq \alpha \oplus \beta$, *for any* $\alpha, \beta \in S$;

(3) $\alpha \oplus \alpha^\star = 1$, *for all* $\alpha \in S$;

(4) $0 \oplus \alpha = \alpha$, $\alpha \oplus 0 = \alpha^{\star\star}$, *for all* $\alpha \in S$;

(5) $\alpha \oplus \beta = 1$ *if and only if* $\alpha^\star \leq \beta$, *for any* $\alpha, \beta \in S$;

(6) $\alpha^\star \odot \beta^\star = (\alpha \oplus \beta)^\star$ *if* S *has (DNP), for any* $\alpha, \beta \in S$;

(7) $\alpha^\star \oplus \beta^\star = (\alpha \odot \beta)^\star$ *if* S *has (DNP), for any* $\alpha, \beta \in S$.

Proposition 2.4 ([3]). *Let* S *be a bounded semihoop and for any* $\alpha, \beta \in S$, *we define:* $\alpha \vee \beta = [(\alpha \to \beta) \to \beta] \wedge [(\beta \to \alpha) \to \alpha]$. *Then, the following conditions are equivalent:*

(1) $\vee$ *is an associative operation on* S;

(2) $\alpha \leq \beta$ *implies* $\alpha \vee \theta \leq \beta \vee \theta$, *for all* $\alpha, \beta, \theta \in A$;

(3) $\alpha \vee (\beta \wedge \theta) \leq (\alpha \vee \beta) \wedge (\alpha \vee \theta)$, *for all* $\alpha, \beta, \theta \in A$;

(4) $\vee$ *is the join operation on* A.

Definition 2.3 ([3]). *A bounded semihoop is a bounded* $\vee$*-semihoop if it satisfies one of the equivalent conditions of Proposition* 2.9.

Definition 2.4 ([14]). *Let* S *be a bounded semihoop. A nonempty subset* D*of* S *is called an ideal if it satisfies:*

$(D1)$ *for any* $\alpha, \beta \in S$, $\alpha \leq \beta$ *and* $\beta \in D$*imply* $\alpha \in D$;

$(D2)$ *for any* $\alpha, \beta \in D$, $\alpha \oplus \beta \in D$.

Definition 2.5 ([3]). *Let* S *be a bounded semihoop. A nonempty subset* F*of* S *is called a filter if it satisfies:*

$(F1)$ *for any* $\alpha, \beta \in S$, $\alpha \leq \beta$ *and* $\alpha \in F$*imply* $\beta \in F$;

$(F2)$ *for any* $\alpha, \beta \in F$, $\alpha \odot \beta \in F$.

We denote the set of all ideals of S by $D(S)$.

Definition 2.6 ([14]). *Let* S *be a bounded semihoop. A proper ideal* D*of* S *is called a prime ideal if* $P \cap Q \subseteq D$ *implies* $P \subseteq D$ *or* $Q \subseteq D$, *for any* $P, Q \in D(S)$.

Proposition 2.5 ([14]). *Let* S *be a bounded* $\vee$*-semihoop with DNP. Then, every maximal ideal of* S *is prime ideal.*

3. L-fuzzy ideals

Definition 3.1. *Let S be a semihoop and $\rho{:}S \to [0,1]$ be a fuzzy subset on A. Then, ρ is called a fuzzy ideal of S, if for all $\alpha, \beta \in S$satifies:*

$(FI1)$ $\alpha \leq \beta$ *implies* $\rho(\alpha) \geq \rho(\beta)$;

$(FI2)$ $\rho(\alpha \oplus \beta) \geq min\{\rho(\alpha), \rho(\beta)\}$.

Let $(S, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop and $(L, \sqcap, \sqcup, 0, 1)$ be a complete lattice. The map ρ: $S \to L$ is called an L-fuzzy subset of S. Let ρ and χ be two L-fuzzy subsets, then $\rho \wedge \chi$ and $\rho \vee \chi$ are L-fuzzy subsets, where $(\rho \wedge \chi)(\alpha) = \rho(\alpha) \sqcap \chi(\alpha)$ and $(\rho \vee \chi)(\alpha) = \rho(\alpha) \sqcup \chi(\alpha)$, for all $\alpha \in S$.

We can induce the partial order relation on $(L, \sqcap, \sqcup, 0, 1)$ with $\leq$. Define four types of level sets by $\rho_t^1 = \{\alpha \in S | \rho(\alpha) \geq t\}$, $\rho_t^2 = \{\alpha \in S | \rho(\alpha) \ngeq t\}$, $\rho_t^3 = \{\alpha \in S | \rho(\alpha) > t\}$, $\rho_t^4 = \{\alpha \in S | \rho(\alpha) \ngtr t\}$, for any $t \in L$.

Definition 3.2. *Let S be a bounded semihoop. The binary operation $\boxplus$ is defined by $\alpha \boxplus \beta = \alpha^\star \to \beta^{\star\star}$, for any $\alpha, \beta \in S$.*

If S is a bounded semihoop with DNP, then we have $\alpha \oplus \beta = \alpha \boxplus \beta$, for any $\alpha, \beta \in S$. The following example will illustrate that the two binary operations $\oplus$ and $\boxplus$ are different.

Example 3.1. Let $S = \{0, e, b, c, d, 1\}$ with $0 < e < 1$, $0 < b < c < d < 1$, where e and b are incomparable. Define $\boxtimes$ and $\to$as follows,

$\odot$	0	e	b	c	d	1
0	0	0	0	0	0	0
e	0	0	0	0	e	e
b	0	0	0	0	b	b
c	0	0	0	0	c	c
d	0	e	b	c	d	d
1	0	e	b	c	d	1

$\to$	0	e	b	c	d	1
0	1	1	1	1	1	1
e	c	1	c	1	1	1
b	c	c	1	1	1	1
c	c	c	c	1	1	1
d	0	e	b	c	1	1
1	0	e	b	c	d	1

Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. In the bounded semihoop S, $\alpha \oplus \beta \neq \alpha \boxplus \beta$since $e \oplus b = e^\star \to b = (e \to 0) \to b = c \to b = c$, $e \boxplus b = e^\star \to b^{\star\star} = (e \to 0) \to ((b \to 0) \to 0) = c \to (c \to 0) = c \to c = 1$.

Since there are incomparable elements in lattice L, we will define two types of L-fuzzy ideals on bounded semihoops. The infimum and supremum of two elements $x, y \in L$ be denoted by $x \sqcap y$ and $x \sqcup y$.

Definition 3.3. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy strong ideal if it satisfies: for any $\alpha, \beta \in S$,*

$(LFD1)$ $\alpha \leq \beta$ *implies* $\rho(\alpha) \geq \rho(\beta)$;

$(LFD2)$ $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.

Definition 3.4. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy ideal if it satisfies: for any $\alpha, \beta \in S$,*

$(LFD1^{'})$ *$\alpha \leq \beta$ implies $\rho(\alpha) \geq \rho(\beta)$ or $\rho(\alpha)$ and $\rho(\beta)$ are incomparable;*

$(LFD2^{'})$ *$\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.*

Obviously, each L-fuzzy strong ideal of S is an L-fuzzy ideal.

In the following we will explain the difference between fuzzy ideals and L-fuzzy ideals through definitions:

(1) Since all elements on [0,1] are comparable, $\alpha \leq \beta$ implies $\rho(\alpha) \geq \rho(\beta)$ in Definition 3.1($FD1$). However, not all elements in the lattice are comparable, so based on this feature we propose L-fuzzy strong ideals and L-fuzzy ideals.

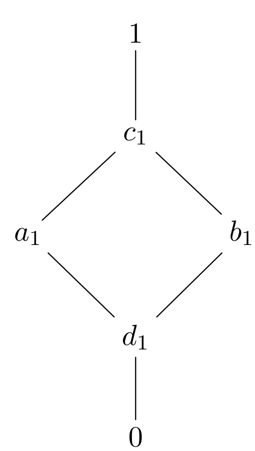
(2) Since all elements on [0,1] are comparable, $\rho(\alpha \oplus \beta) \geq min\{\rho(\alpha), \rho(\beta)\}$ in Definition 3.1($FD2$). However, not all elements in the lattice are comparable, but a lower bound exists for any two elements in the lattice. Thus, $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$ is satisfied in Definitions 3.3($LFD2$) and 3.4($LFD2^{'}$), where $\rho(\alpha) \sqcap \rho(\beta)$ is the infimum of $\rho(\alpha)$ and $\rho(\beta)$.

Example 3.2 ([10])**.** Let $S = \{0, m, n, p, 1\}$ with $0 < m < p < 1$, $0 < n < p < 1$, where m and n are incomparable. Define $\odot$ and $\to$ as follows,

$\odot$	0	m	n	p	1
0	0	0	0	0	0
m	0	m	0	m	m
n	0	0	n	n	n
p	0	m	n	p	p
1	0	m	n	p	1

$\to$	0	m	n	p	1
0	1	1	1	1	1
m	n	1	n	1	1
n	m	m	1	1	1
p	0	m	n	1	1
1	0	m	n	p	1

Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. Let $L = \{0, c_1, a_1, b_1, d_1, 1\}$ be a complete lattice. It's Hasse diagram is as follows:



We define an L-fuzzy subset ρ of $S$by

$$\rho(\alpha) = \begin{cases} c_1, & \text{if } \alpha = 0 \\ b_1, & \text{if } \alpha = m \\ d_1, & \text{if } \alpha = n, p, 1 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy strong ideal of S.

Remark 3.1. Let S be a bounded semihoop. Then, an L-fuzzy ideal of Smay not be an L-fuzzy strong ideal.

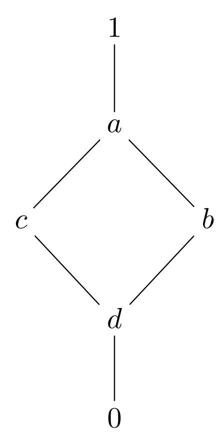
The following example will illustrate Remark 3.1.

Example 3.3 ([9])**.** Let $S = \{0, n, e, p, q, r, m, 1\}$ with $0 < n < e < 1$, $0 < p < q < m < 1$, $0 < r < m < 1$, where e and p are incomparable, q and r are incomparable. Define $\odot$ and $\to$ as follows,

$\odot$	0	n	e	p	q	r	m	1
0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	n
e	0	0	e	0	e	0	0	e
p	0	0	0	0	0	p	p	p
q	0	0	e	0	e	p	q	q
r	0	0	0	p	p	r	r	r
m	0	0	e	p	q	r	m	m
1	0	n	e	p	q	r	m	1

$\to$	0	n	e	p	q	r	m	1
0	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1
e	r	r	1	r	1	r	1	1
p	q	q	q	1	1	1	1	1
q	p	p	q	r	1	r	1	1
r	e	e	e	q	q	1	1	1
m	n	n	e	p	q	r	1	1
1	0	n	e	p	q	r	m	1

We can see that $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. Let $L = \{0, c, d, b, a, 1\}$ be a complete lattice. It's Hasse diagram is as follows:



We define an L-fuzzy subset ρ of S by

$$\rho(\alpha) = \begin{cases} a, & \text{if } \alpha = 0, n \\ b, & \text{if } \alpha = p \\ c, & \text{if } \alpha = e \\ d, & \text{if } \alpha = q, r, m, 1 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy ideal of Sbut it is not an L-fuzzy strong ideal since $e < q$but $\rho(e) = c$ and $\rho(q) = d$ are incomparable.

Definition 3.5. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy strong filter if it satisfies: for each $\alpha, \beta \in S$,*

$(LFF1)$ *$\alpha \leq \beta$ implies $\rho(\alpha) \leq \rho(\beta)$;*

$(LFF2)$ *$\rho(\alpha \odot \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.*

Definition 3.6. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is called an L-fuzzy filter if it satisfies: for each $\alpha, \beta \in S$,*

$(LFF1')$ *$\alpha \leq \beta$ implies $\rho(\alpha) \leq \rho(\beta)$ or $\rho(\alpha)$ and $\rho(\beta)$ are incomparable;*

$(LFF2')$ *$\rho(\alpha \odot \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$.*

Example 3.4. In Example 3.2, we define an L-fuzzy subset ρ of S by

$$\rho(\alpha) = \begin{cases} c_1, & \text{if } \alpha = p, 1 \\ b_1, & \text{if } \alpha = n \\ d_1, & \text{if } \alpha = 0, m \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy strong filter of S.

Example 3.5. In Example 3.3, we define an L-fuzzy subset ρ of S by

$$\rho(\alpha) = \begin{cases} a, & \text{if } \alpha = 1, m \\ b, & \text{if } \alpha = r \\ c, & \text{if } \alpha = q, e \\ d, & \text{if } \alpha = p, n, 0 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy filter of Sbut it is not an L-fuzzy strong filter since $p < q$but $\rho(q) = c$ and $\rho(p) = d$ are incomparable.

Proposition 3.1. *Let S be a bounded semihoop with DNP.*

(1) *If ρ is an L-fuzzy strong ideal of S, then $\rho^\star$ is an L-fuzzy strong filter of S;*

(2) *If ρ is an L-fuzzy strong filter of S, then $\rho^\star$ is an L-fuzzy strong ideal of S;*

(3) *If ρ is an L-fuzzy ideal of S, then $\rho^\star$ is an L-fuzzy filter of S;*

(4) *If ρ is an L-fuzzy filter of S, then $\rho^\star$ is an L-fuzzy ideal of S;*

where $\rho^\star(\alpha) = \rho(\alpha^\star)$, for any $\alpha \in S$.

Proof. (1) Assume that ρ is an L-fuzzy strong ideal of S. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$, then $\beta^\star \leq \alpha^\star$. By Definition 3.3($LFD1$), we get $\rho(\alpha^\star) \leq \rho(\beta^\star)$, so $\rho^\star(\alpha) \leq \rho^\star(\beta)$. Since S is a bounded semihoop with DNP, by Proposition 2.3(7), we have $\alpha^\star \boxplus \beta^\star = \alpha^\star \oplus \beta^\star = (\alpha \odot \beta)^\star$. By Definition 3.3($LFD2$), $\rho((\alpha \odot \beta)^\star) = \rho(\alpha^\star \boxplus \beta^\star) \geq \rho(\alpha^\star) \sqcap \rho(\beta^\star)$, then $\rho^\star(\alpha \odot \beta) \geq \rho^\star(\alpha) \sqcap \rho^\star(\beta)$. Therefore, $\rho^\star$ is an L-fuzzy strong filter.

(2) Let ρ be an L-fuzzy strong filter of S. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$, then $\beta^\star \leq \alpha^\star$. By Definition 3.5($LFF1$), we get $\rho(\beta^\star) \leq \rho(\alpha^\star)$, so $\rho^\star(\beta) \leq \rho^\star(\alpha)$. Since S is a bounded semihoop with DNP, by Proposition 2.3(6), we have $\alpha^\star \odot \beta^\star = (\alpha \oplus \beta)^\star = (\alpha \boxplus \beta)^\star$. By Definition 3.5($LFF2$), $\rho((\alpha \boxplus \beta)^\star) = \rho(\alpha^\star \odot \beta^\star) \geq \rho(\alpha^\star) \sqcap \rho(\beta^\star)$, then $\rho^\star(\alpha \boxplus \beta) \geq \rho^\star(\alpha) \sqcap \rho^\star(\beta)$. Therefore, $\rho^\star$ is an L-fuzzy strong ideal.

(3)The proof of the conclusion is similar to (1).

(4)The proof of the conclusion is similar to (2). □

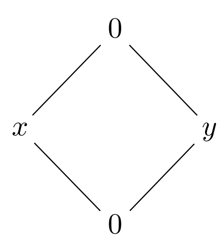
Proposition 3.2. *Assume that S is a bounded semihoop and ρ, χ are two L-fuzzy strong ideals of S. Then, $\rho \wedge \chi$ is an L-fuzzy strong ideal.*

Proof. The proof of this proposition is obvious. □

Remark 3.2. Assume that S is a bounded semihoop and ρ, χ are two L-fuzzy strong ideals of S. Then, $\rho \vee \chi$may not be an L-fuzzy strong ideal.

The following example will illustrate Remark 3.2.

Example 3.6. Let S be a bounded semihoop in Example 3.2 and $L = \{0, x, y, 1\}$ be a complete lattice. The Hasse diagram of L is as follows:



We define two L-fuzzy subsets by

$$\rho_1(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ x, & \text{if } \alpha = n \\ 0, & \text{if } \alpha = m, p, 1 \end{cases}$$

and

$$\rho_2(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ y, & \text{if } \alpha = m \\ 0, & \text{if } \alpha = n, p, 1 \end{cases}$$

for any $\alpha \in S$. Then, ρ_1 and ρ_2 are two L-fuzzy strong ideals of S. Moreover, we get an L-fuzzy subset $\rho_1 \vee \rho_2$ by

$$(\rho_1 \vee \rho_2)(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ y, & \text{if } \alpha = m \\ x, & \text{if } \alpha = n \\ 0, & \text{if } \alpha = p, 1 \end{cases}$$

for any $\alpha \in S$. Then, $\rho_1 \vee \rho_2$ is not an L-fuzzy strong ideal since $(\rho_1 \vee \rho_2)(n) = x$ and $(\rho_1 \vee \rho_2)(m) = y$ are incomparable.

Corollary 3.1. *Let S be a bounded semihoop and ρ, χ be two L-fuzzy strong ideals of S. If $\rho \subseteq \chi$, then $\rho \vee \chi$ is an L-fuzzy strong ideal.*

Proof. The proof is clearly. □

Proposition 3.3. *Given a bounded semihoop S.*

(1) *An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the level set $\rho_t^1 = \{\alpha \in S | \rho(\alpha) \geq t\} (\neq \emptyset)$ is an ideal, for any $t \in L$.*

(2) *If Lsatisfies $y = \vee\{x \in L | x < y\}$, for any $y \in L$, then an L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the level set $\rho_t^3 = \{\alpha \in S | \rho(\alpha) > t\} (\neq \emptyset)$ is an ideal, for any $t \in L$.*

Proof. (1)Assume that ρ is an L-fuzzy strong ideal. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$ and $\beta \in \rho_t^1$, then $\rho(\beta) \geq t$. By Definition 3.3($LFD1$), $\rho(\alpha) \geq \rho(\beta) \geq t$, so $\rho(\alpha) \geq t$, then $\alpha \in \rho_t^1$. Let $\alpha, \beta \in \rho_t^1$, that is $\rho(\alpha) \geq t$ and $\rho(\beta) \geq t$, so $\rho(\alpha) \sqcap \rho(\beta) \geq t$. By Definition 3.3($LFD2$), $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta) \geq t$, so $\rho(\alpha \boxplus \beta) \geq t$, then $\alpha \boxplus \beta \in \rho_t^1$. Hence, $\rho_t^1 (\neq \emptyset)$ is an ideal of S.

Conversely, let $\rho_t^1 (\neq \emptyset)$ be an ideal. Taking $t = \rho(\alpha) \sqcap \rho(\beta)$, we get $\alpha \in \rho_t$ and $\beta \in \rho_t$, for any $\alpha, \beta \in S$. Since ρ_t^1 is an ideal of S, thus $\alpha \boxplus \beta \in \rho_t^1$, so $\rho(\alpha \boxplus \beta) \geq t = \rho(\alpha) \sqcap \rho(\beta)$. Let $\alpha, \beta \in S$such that $\alpha \leq \beta$. Taking $t = \rho(\beta)$, then $\beta \in \rho_{\rho(\beta)}^1$. Since $\rho_{\rho(\beta)}^1$ is an ideal of S, thus $\alpha \in \rho_{\rho(\beta)}^1$, that is $\rho(\alpha) \geq \rho(\beta)$. Therefore, ρ is an L-fuzzy strong ideal of S.

(2)The proof is similar to (1). □

By Proposition 3.3, we easily obtain that an L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the complement of $\rho_t^2 (\neq \emptyset)$ is an ideal. Similarly, an L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if the complement of $\rho_t^4 (\neq \emptyset)$ is an ideal.

Proposition 3.4. *Let S be a bounded semihoop with DNP. An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if for any $\alpha, \beta \in S$, the following conditions hold:*

(1) $\rho(0) \geq \rho(\alpha)$;

(2) $\rho(\alpha) \sqcap \rho(\alpha^{\star} \odot \beta) \leq \rho(\beta)$.

Proof. For all $\alpha \in S$, we have $0 \leq \alpha$. Since ρ is an L-fuzzy strong ideal of S, by Definition 3.3($LFD1$), we get $\rho(0) \geq \rho(\alpha)$. Since $\beta \to (\alpha \boxplus (\alpha^{\star} \odot \beta)) = \beta \to (\alpha^{\star} \to (\alpha^{\star} \odot \beta)^{\star\star}) = (\beta \odot \alpha^{\star}) \to (\beta \odot \alpha^{\star})^{\star\star} = (\beta \odot \alpha^{\star}) \to (\beta \odot \alpha^{\star}) = 1$, for any $\alpha, \beta \in S$, thus $\alpha \boxplus (\alpha^{\star} \odot \beta) \geq \beta$, then $\rho(\alpha \boxplus (\alpha^{\star} \odot \beta)) \leq \rho(\beta)$. By Definition 3.3($LFD2$), $\rho(\alpha) \sqcap \rho(\alpha^{\star} \odot \beta) \leq \rho(\alpha \boxplus (\alpha^{\star} \odot \beta)) \leq \rho(\beta)$. Hence, $\rho(\alpha) \sqcap \rho(\alpha^{\star} \odot \beta) \leq \rho(\beta)$.

Conversely, for any $\alpha, \beta \in S$such that $\alpha \leq \beta$, so $\beta^{\star} \leq \alpha^{\star}$, then $\alpha \odot \beta^{\star} \geq \alpha \odot \alpha^{\star} = 0$, so $\rho(\beta^{\star} \odot \alpha) \leq \rho(0)$. By (1), $\rho(0) \geq \rho(\beta^{\star} \odot \alpha)$, then $\rho(0) = \rho(\beta^{\star} \odot \alpha)$. By (2), $\rho(\alpha) \leq \rho(\beta) \sqcap \rho(\beta^{\star} \odot \alpha) = \rho(\beta) \sqcap \rho(0) = \rho(\beta)$, so $\rho(\alpha) \geq \rho(\beta)$. Since $\alpha^{\star} \odot (\alpha \boxplus \beta) = \alpha^{\star} \odot (\alpha^{\star} \to \beta^{\star\star}) = \alpha^{\star} \odot (\alpha^{\star} \to \beta) \leq \beta$, thus $\rho(\alpha^{\star} \odot (\alpha \boxplus \beta)) \geq \rho(\beta)$, then $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap \rho(\alpha^{\star} \odot (\alpha \boxplus \beta)) \geq \rho(\alpha) \sqcap \rho(\beta)$. Therefore, ρ is an L-fuzzy strong ideal of S. □

Proposition 3.5. *Let S be a bounded semihoop with DNP. An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if for any $\alpha, \beta \in S$, the following conditions hold:*

(1) $\rho(0) \geq \rho(\alpha)$;

(2) $\rho(\alpha) \sqcap \rho(\alpha^{*} \to \beta^{\star})^{\star} \leq \rho(\beta)$.

Proof. By Proposition 2.3(6), $\alpha^{\star} \odot \beta = \alpha^{\star} \odot \beta^{\star\star} = (\alpha \boxplus \beta^{\star})^{\star} = (\alpha^{*} \to \beta^{\star\star\star})^{\star} = (\alpha^{\star} \to \beta^{\star})^{\star}$, for every $\alpha, \beta \in S$. Thus, by Proposition 3.4, the conclusion holds. □

Proposition 3.6. *Let S be a bounded semihoop. An L-fuzzy subset ρ of S is an L-fuzzy strong ideal if and only if for any $\alpha, \beta \in S$, the following conditions hold:*

(1) $\rho(\alpha \wedge \beta) \geq \rho(\alpha)$;

(2) $\rho(\alpha \boxplus \beta) \geq \rho(\alpha) \sqcap f(\beta)$.

Proof. The proof is clearly. □

Lemma 3.1. *Assume that S is a bounded semihoop and ρ is an L-fuzzy strong ideal of S. Then, $\rho(\alpha^{\star\star}) = \rho(\alpha)$, for each $\alpha \in S$.*

Proof. By Proposition 2.2(2), $\alpha \leq \alpha^{**}$. Since ρ is an L-fuzzy strong ideal, by Definition 3.3($LFD1$), we obtain $\rho(\alpha) \geq \rho(\alpha^{\star\star})$. By Definition 3.3($LFD2$), $\rho(\alpha^{\star\star}) = \rho(\alpha \boxplus 0) \geq \rho(\alpha) \sqcap \rho(0) = \rho(\alpha)$, so $\rho(\alpha^{\star\star}) \geq \rho(\alpha)$. Therefore, $\rho(\alpha^{\star\star}) = \rho(\alpha)$, for all $\alpha \in S$. □

Given a nonempty subset $D$of S and $x, y \in L$ such that $x > y$. Define an L-fuzzy set $\rho^{D}_{x,y}$ by

$$\rho^{D}_{x,y}(\alpha) = \begin{cases} x, & \text{if } \alpha \in D \\ y, & \textit{others} \end{cases}$$

for any $\alpha \in S$.

Proposition 3.7. *Let S be a bounded semihoop and D be a nonempty subset of S. Then, $\rho^D_{x,y}$ is an L-fuzzy strong ideal if and only if D is an ideal.*

Proof. Assume that $\rho^D_{x,y}$ is an L-fuzzy strong ideal of S. Let $\alpha, \beta \in D$, then $\rho^D_{x,y}(\alpha) = \rho^D_{x,y}(\beta) = x$, so $\rho^D_{x,y}(\alpha \boxplus \beta) \geq \rho^D_{x,y}(\alpha) \sqcap \rho^D_{x,y}(\beta) = x$, then $\alpha \boxplus \beta \in D$. Let $\alpha \leq \beta$ and $\beta \in D$, for any $\alpha, \beta \in S$, then $\rho^D_{x,y}(\alpha) \geq \rho^D_{x,y}(\beta)$ and $\rho^D_{x,y}(\beta) = x$, so $\rho^D_{x,y}(\alpha) \geq x$, then $\alpha \in D$. Therefore, D is an ideal of S.

Conversely, let D be an ideal of S.

Firstly, suppose $\alpha, \beta \in S$, then we discuss the following two situations.

Case (1). If $\alpha, \beta \in D$, then $\alpha \boxplus \beta \in D$ and $\rho^D_{x,y}(\alpha) = \rho^D_{x,y}(\beta) = x$, so $\rho^D_{x,y}(\alpha \boxplus \beta) = x = \rho^D_{x,y}(\alpha) \sqcap \rho^D_{x,y}(\beta)$.

Case (2). If $\alpha \notin D$ or $\beta \notin D$, then $\rho^D_{x,y}(\alpha) = y$ or $\rho^D_{x,y}(\beta) = y$, so $\rho^D_{x,y}(\alpha \boxplus \beta) = y = \rho^D_{x,y}(\alpha) \sqcap \rho^D_{x,y}(\beta)$.

Hence, $\rho^D_{x,y}(\alpha \boxplus \beta) \geq \rho^D_{x,y}(\alpha) \sqcap \rho^D_{x,y}(\beta)$, for any $\alpha, \beta \in S$.

Secondly, let $\alpha, \beta \in S$ and $\alpha \leq \beta$, then we also discuss the following two situations.

Case (1). If $\beta \in D$, so $\alpha \in D$ and $\rho^D_{x,y}(\beta) = x = \rho^D_{x,y}(\alpha)$.

Case (2). If $\beta \notin D$, then $\rho^D_{x,y}(\beta) = y$, so $\alpha \notin I$, then $\rho^D_{x,y}(\alpha) \geq \rho^D_{x,y}(\beta) = y$.

Hence, $\rho^D_{x,y}(\alpha) \geq \rho^D_{x,y}(\beta)$, for any $\alpha, \beta \in S$satisfying $\alpha \leq \beta$. Therefore, $\rho^D_{x,y}$ is an L-fuzzy strong ideal. $\square$

Let S and T be two bounded semihoops. The map h: $S \to T$ is said to be a homomorphism if $h(\alpha \to \beta) = h(\alpha) \to h(\beta)$, $h(\alpha \odot \beta) = h(\alpha) \odot h(\beta)$, $h(\alpha \wedge \beta) = h(\alpha) \wedge h(\beta)$, $h(0) = 0_L$, for any $\alpha, \beta \in S$. We also get $h(1) = 1_L$ and $h(\alpha^\star) = (h(\alpha))^\star$, for all $\alpha \in S$.

Let L_1 and L_2 be two complete lattices. The map h: $L_1 \to L_2$ is said to be a lattice-homomorphism if $h(\alpha \sqcap \beta) = h(\alpha) \sqcap h(\beta)$, $h(\alpha \sqcup \beta) = h(\alpha) \sqcup h(\beta)$, $h(0) = 0_{L_2}$, $h(1) = 1_{L_2}$, for any $\alpha, \beta \in S$.

Proposition 3.8. *Let S and T be two bounded semihoops, ρ be an L-fuzzy strong ideal of T and h: $S \to T$ be a homomorphism. Then, ρh is also an L-fuzzy strong ideal of S.*

Proof. Since h is a homomorphism, thus $(\rho h)(0) = \rho(h(0)) = \rho(0) \geq \rho(h(\alpha)) = (\rho h)(\alpha)$, for all $\alpha \in S$. Since $(\rho h)(\alpha) \sqcap (\rho h)((\alpha^\star \to \beta^\star)^\star) = \rho(h(\alpha)) \sqcap \rho(h((\alpha^\star \to \beta^\star)^\star)) = \rho(h(\alpha)) \sqcap \rho((h(\alpha^\star \to \beta^\star))^\star) = \rho(h(\alpha)) \sqcap \rho((h(\alpha^\star) \to h(\beta^\star))^\star) = \rho(h(\alpha)) \sqcap \rho(((h(\alpha))^\star \to (h(\beta))^\star)^\star) \leq \rho(h(\beta)) = (\rho h)(\beta)$, for any $\alpha, \beta \in S$. Hence, by Proposition 3.3, ρh is an L-fuzzy strong ideal. $\square$

Proposition 3.9. *Let L_1 and L_2 be two complete lattices, ρ be an L_1-fuzzy strong ideal of S and h: $L_1 \to L_2$ be a lattice-homomorphism. Then, $h\rho$ is also an L_2-fuzzy strong ideal of S.*

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[dr, "\rho h"'] & T \arrow[d, "\rho"] \\ & L \end{tikzcd}$$

Proof. Since h is a lattice-homomorphism and ρ be an L_1-fuzzy strong ideal of S, thus $(h\rho)(\alpha \wedge \beta) = h(\rho(\alpha \wedge \beta)) \geq h(\rho(\alpha)) = (h\rho)(\alpha)$, for all $\alpha, \beta \in S$. Moreover, $(h\rho)(\alpha \boxplus \beta) = h(\rho(\alpha \boxplus \beta) \geq h(\rho(\alpha) \sqcap \rho(\beta)) = h(\rho(\alpha)) \sqcap h(\rho(\beta)) = (h\rho)(\alpha) \sqcap (h\rho)(\beta)$, for all $\alpha, \beta \in S$. Therefore, by Proposition 3.6, $h\rho$ is also an L_2-fuzzy strong ideal of S. $\square$

$$\begin{tikzcd} S \arrow[r, "\rho"] \arrow[dr, "h\rho"'] & L_1 \arrow[d, "h"] \\ & L_2 \end{tikzcd}$$

Proposition 3.10. *Let S be a bounded semihoop with DNP, L be a complete lattice, ρ be an L-fuzzy strong ideal of S and H be an up-set sublattice of L. Then, $\rho^{-1}(H)$ is an ideal of S.*

Proof. We will prove the proposition in the following parts:

(i) Since H is a sublattice of L, thus there exists $\alpha \in S$such that $\rho(\alpha) = x \in H$, so $\alpha \in \rho^{-1}(H)$, then $\rho^{-1}(H)$ is a non-empty set of S.

(ii) Let $\alpha, \beta \in S$ with $\alpha \leq \beta$ and $\beta \in \rho^{-1}(H)$, then $\rho(\beta) \in H$ and $\rho(\alpha) \geq \rho(\beta)$. Since H is an up-set, thus $\rho(\alpha) \in H$, so $\alpha \in \rho^{-1}(H)$.

(iii) For any $\alpha, \beta \in \rho^{-1}(H)$, then $\rho(\alpha), \rho(\beta) \in H$. Since H is an up-set sublattice of L, thus $\rho(\alpha) \sqcap \rho(\beta) \in H$. From Definition 3.3($LFD2$) and S has with DNP, $\rho(\alpha \boxplus \beta) = \rho(\alpha \oplus \beta) \geq \rho(\alpha) \sqcap \rho(\beta)$, then $\rho(\alpha \oplus \beta) \in H$, so $\alpha \oplus \beta \in \rho^{-1}(H)$. Therefore, $\rho^{-1}(H)$ is an ideal of S. $\square$

Definition 3.7. *Let S be a bounded semihoop and ρ be an L-fuzzy strong ideal of A. The smallest L-fuzzy strong ideal containing ρ is called the L-fuzzy strong ideal generalized by ρ, written $[\rho]$.*

Proposition 3.11. *Let S be a bounded semihoop and ρ be an L-fuzzy subset of S. Then, $[\rho](\alpha) = \sqcup\{\rho(\alpha_1) \sqcap \rho(\alpha_2) \sqcap \cdots \sqcap \rho(\alpha_n) | \alpha \leq \alpha_1 \boxplus \alpha_2 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \alpha_2, \cdots, \alpha_n \in S\}$.*

Proof. Let $f(\alpha) = \sqcup\{\rho(\alpha_1) \sqcap \rho(\alpha_2) \sqcap \cdots \sqcap \rho(\alpha_n) | \alpha \leq \alpha_1 \boxplus \alpha_2 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \alpha_2, \cdots, \alpha_n \in S\}$.

First, we prove that $f(\alpha)$ is an L-fuzzy strong ideal of S. Obviously, $f(0) \geq f(\alpha)$, for all $\alpha \in S$. Let $\alpha, \beta \in S$, if there are $a_1, \cdots, a_n, b_1, \cdots, b_m \in S$such that $\alpha \leq a_1 \boxplus \cdots \boxplus a_n$ and $\alpha^* \odot \beta \leq b_1 \boxplus \cdots \boxplus b_m$, then $\beta \leq \alpha \boxplus (\alpha^* \odot \beta) =$

$(a_1 \boxplus \cdots \boxplus a_n) \boxplus (b_1 \boxplus \cdots \boxplus b_m)$, so $f(\beta) \geq \rho(a_1) \sqcap \cdots \sqcap \rho(a_n) \sqcap \rho(b_1) \sqcap \cdots \sqcap \rho(b_m)$. Since $f(\alpha) \sqcap f(\alpha^* \odot \beta) = (\sqcup\{\rho(a_1) \sqcap \cdots \sqcap \rho(a_n) | \alpha \leq a_1 \boxplus \cdots \boxplus a_n, a_1, \cdots, a_n \in S\}) \sqcap (\sqcup\{\rho(b_1) \sqcap \cdots \sqcap \rho(b_m) | \alpha^* \odot \beta \leq b_1 \boxplus \cdots \boxplus b_m, b_1, \cdots, b_m \in S\}) = \sqcup\{\rho(a_1) \sqcap \cdots \sqcap \rho(a_n) \sqcap \rho(b_1) \sqcap \cdots \sqcap \rho(b_m) | \alpha \leq a_1 \boxplus \cdots \boxplus a_n, \alpha^* \odot \beta \leq b_1 \boxplus \cdots \boxplus b_m, a_1, \cdots, a_n, b_1, \cdots, b_m \in S\}$, thus $f(\alpha) \sqcap f(\alpha^* \odot \beta) \leq f(\beta)$. Therefore by Proposition 3.4, we have f is an L-fuzzy strong ideal of S.

Next, since $\alpha \leq \alpha \boxplus \alpha$, we have $f(\alpha) \geq \rho(\alpha) \sqcap \rho(\alpha) = \rho(\alpha)$. Thus, $f$contains ρ.

Finally, suppose ω is also an L-fuzzy strong ideal of Ssuch ωcontains ρ. Then, for any $\alpha \in S$, $f(\alpha) = \sqcup\{\rho(\alpha_1) \sqcap \cdots \sqcap \rho(\alpha_n) | \alpha \leq \alpha_1 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \cdots, \alpha_n \in S\} \leq \sqcup\{\omega(\alpha_1) \sqcap \cdots \sqcap \omega(\alpha_n)\} | \alpha \leq \alpha_1 \boxplus \cdots \boxplus \alpha_n, \alpha_1, \cdots, \alpha_n \in S\} \leq \omega(\alpha)$. Therefore, f is an L-fuzzy strong ideal generated by ρ, that is $[\rho] = f$. □

4. L-fuzzy prime ideals

In this part, we will introduce the concept of L-fuzzy prime ideals on bounded semihoops and study some of their properties.

Definition 4.1. *Let S be a bounded semihoop. An L-fuzzy strong ideal ρ of S is called an L-fuzzy prime ideal if $\rho(\alpha \wedge \beta) \leq \rho(\alpha) \sqcup \rho(\beta)$, for any $\alpha, \beta \in S$.*

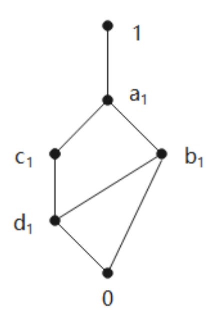
Example 4.1 ([14]). Let $S = \{0, r, m, n, 1\}$ be a chain with $0 < r < m < n < 1$. Define $\odot$ and $\to$ on S in the following:

$\boxtimes$	0	r	m	n	1
0	0	0	0	0	0
r	0	0	0	0	r
m	0	0	0	r	m
n	0	0	r	r	n
1	0	r	m	n	1

$\to$	0	r	m	n	1
0	1	1	1	1	1
r	r	1	1	1	1
m	m	n	1	1	1
n	r	n	n	1	1
1	0	r	m	n	1

Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop.

Let $L = \{0, a_1, b_1, c_1, d_1, 1\}$ be a lattice. It's Hasse diagram is as follows:



Define an L-fuzzy subset ρ of S by

$$\rho(\alpha)=\begin{cases}1, & \text{if } \alpha=0\\ a_1, & \text{if } \alpha=r\\ c_1, & \text{if } \alpha=m\\ d_1, & \text{if } \alpha=n\\ 0, & \text{if } \alpha=1\end{cases}$$

for $\alpha\in S$. We can see that ρ is an L-fuzzy prime ideal of S.

Proposition 4.1. *Suppose that S is a bounded semihoop. An L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if $\rho(\alpha\wedge\beta)=\rho(\alpha)$ or $\rho(\alpha\wedge\beta)=\rho(\beta)$, for any $\alpha,\beta\in S$.*

Proof. Let ρ be a L-fuzzy prime ideal of S, then $\rho(\alpha)\sqcup\rho(\beta)\geq\rho(\alpha\wedge\beta)$, for every $\alpha,\beta\in S$, so $\rho(\alpha)\geq\rho(\alpha\wedge\beta)$ or $\rho(\beta)\geq\rho(\alpha\wedge\beta)$. Since $\alpha\wedge\beta\leq\alpha$ and $\alpha\wedge\beta\leq\beta$, thus $\rho(\alpha\wedge\beta)\geq\rho(\alpha)$, $\rho(\alpha\wedge\beta)\geq\rho(\beta)$. Therefore, $\rho(\alpha\wedge\beta)=\rho(\alpha)$ or $\rho(\alpha\wedge\beta)=\rho(\beta)$.

Conversely, the proof is obviously. □

Proposition 4.2. *Suppose that S is a bounded semihoop.*

(1) *An L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the level set $\rho_t^1=\{\alpha\in S|\rho(\alpha)\geq t\}(\neq\emptyset)$ is a prime ideal, for any $t\in L$.*

(2) *If Lsatisfies $y=\vee\{x\in L|x<y\}$, for any $y\in L$, then an L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the level set $\rho_t^3=\{\alpha\in S|\rho(\alpha)>t\}(\neq\emptyset)$ is a prime ideal, for any $t\in L$.*

Proof. (1) Let ρ be an L-fuzzy prime ideal of S. By Proposition 3.3(1), we get that ρ_t^1 is an ideal of S. For any $\alpha,\beta\in S$satisfying $\alpha\wedge\beta\in\rho_t^1$, then $\rho(\alpha\wedge\beta)\geq t$, so $t\leq\rho(\alpha\wedge\beta)\leq\rho(\alpha)\sqcup\rho(\beta)$, so $\rho(\alpha)\sqcup\rho(\beta)\geq t$. Since $\rho(\alpha)\sqcup\rho(\beta)=\rho(\alpha)$ or $\rho(\alpha)\sqcup\rho(\beta)=\rho(\beta)$, thus $\rho(\alpha)\geq t$ or $\rho(\beta)\geq t$, that is $\alpha\in\rho_t^1$ or $\beta\in\rho_t^1$. Therefore, by the definition of prime ideal, we get that ρ_t^1 is a prime ideal.

Conversely, let ρ_t^1 be a prime ideal. By Proposition 3.3(1), we get ρ is an L-fuzzy strong ideal. Taking $t=\rho(\alpha\wedge\beta)$, so $\alpha\wedge\beta\in\rho^1_{\rho(\alpha\wedge\beta)}$, for $\alpha,\beta\in S$. So $\alpha\in\rho^1_{\rho(\alpha\wedge\beta)}$ and $\beta\in\rho^1_{\rho(\alpha\wedge\beta)}$, then $\rho(\alpha)\geq\rho(\alpha\wedge\beta)$ and $\rho(\beta)\geq\rho(\alpha\wedge\beta)$, so $\rho(\alpha)\sqcup\rho(\beta)\geq\rho(\alpha\wedge\beta)$. Hence, ρ is an L-fuzzy prime ideal.

(2) The proof is similar to part (1). □

By Proposition 4.2, we obtain that an L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the complement of $\rho_t^2(\neq\emptyset)$ is a prime ideal. Similarly, an L-fuzzy strong ideal ρ of S is an L-fuzzy prime ideal if and only if the complement of $\rho_t^4(\neq\emptyset)$ is a prime ideal.

Proposition 4.3. *Assume that S is a bounded semihoop and ρ is an L-fuzzy strong ideal of S. Then, the following conditions are equivalent:*

(1) *ρ is an L-fuzzy prime ideal of S;*

(2) *$\rho(\alpha \wedge \beta) = \rho(0)$ implies $\rho(\alpha) = \rho(0)$ or $\rho(\beta) = \rho(0)$, for any $\alpha, \beta \in S$.*

Proof. (1) $\Rightarrow$ (2) Let ρ be an L-fuzzy prime ideal of S, then $\rho(\alpha \wedge \beta) \leq \rho(\alpha) \sqcup \rho(\beta)$, for any $\alpha, \beta \in S$. Suppose $\alpha, \beta \in S$such that $\rho(\alpha \wedge \beta) = \rho(0)$, so $\rho(0) \geq \rho(\alpha) \sqcup \rho(\beta)$. Since ρ is an L-fuzzy strong ideal, thus $\rho(0) \leq \rho(\alpha) \sqcup \rho(\beta)$, then $\rho(\alpha) \sqcup \rho(\beta) = \rho(0)$. Hence, $\rho(\alpha) = \rho(0)$ or $\rho(\beta) = \rho(0)$.

(2) $\Rightarrow$ (1) For any $\alpha, \beta \in S$satisfying $\alpha \wedge \beta \in \rho_t^1$, then $\rho(\alpha \wedge \beta) \geq t$, taking $t = \rho(0)$. Since ρ is an L-fuzzy strong ideal, thus $\rho(\alpha \wedge \beta) \leq t = \rho(0)$, then $\rho(\alpha \wedge \beta) = \rho(0)$, so $\rho(\alpha) = \rho(0) \geq t$ or $\rho(\beta) = \rho(0) \geq t$, that is $\alpha \in \rho_t^1$ or $\beta \in \rho_t^1$, then ρ_t^1 is a prime ideal. By Proposition 4.2, we obtain that ρ is an L-fuzzy prime ideal. $\square$

Proposition 4.4. *Suppose that S is a bounded semihoop and D is an ideal of S and ρ is an L-fuzzy strong ideal of S. Then, $\rho_{x,y}^D$ is an L-fuzzy prime ideal if and only if D is a prime ideal.*

Proof. Given an L-fuzzy prime ideal $\rho_{x,y}^D$. By Proposition 4.1, $\rho_{x,y}^D(\alpha \wedge \beta) = \rho_{x,y}^D(\alpha)$ or $\rho_{x,y}^D(\alpha \wedge \beta) = \rho_{x,y}^D(\beta)$. Let for any $\alpha \wedge \beta \in D$, that is $\rho_{x,y}^D(\alpha \wedge \beta) = x$, then $\rho_{x,y}^D(\alpha) = x$ or $\rho_{x,y}^D(\beta) = x$, so $\alpha \in D$ or $\beta \in D$. Therefore, D is a prime ideal.

Conversely, let D be a prime ideal of S. For any $\alpha, \beta \in S$, if $\alpha \wedge \beta \in D$, then $\alpha \in D$ and $\beta \in D$, in other words, $\rho_{x,y}^D(\alpha) = x$ or $\rho_{x,y}^D(\beta) = x$, so $\rho_{x,y}^D(\alpha \wedge \beta) = x = \rho_{x,y}^D(\alpha) \sqcup \rho_{x,y}^D(\beta)$. If $\alpha \wedge \beta \notin D$, then $\alpha \notin D$ and $\beta \notin D$, that is $\rho_{x,y}^D(\alpha) = y$ and $\rho_{x,y}^D(\beta) = y$, so $\rho_{x,y}^D(\alpha \wedge \beta) = y = \rho_{x,y}^D(\alpha) \sqcup \rho_{x,y}^D(\beta)$ and so $\alpha \wedge \beta \notin D$. Therefore, $\rho_{x,y}^D$ is an L-fuzzy prime ideal. $\square$

Proposition 4.5. *Suppose that S is a bounded semihoop and ρ is an L-fuzzy subset of S. Define a map $\rho^\square : S \to L$by $\rho^\square(\alpha) = \rho(\alpha) \sqcup w$, for any $\alpha \in S$, $w \in L$satisfying $w < \rho(0)$. Then, ρ is an L-fuzzy prime ideal if and only if $\rho^\square$ is an L-fuzzy prime ideal.*

Proof. Let ρ be an L-fuzzy prime ideal of S, then $\rho(0) \geq \rho(\alpha)$, for every $\alpha \in S$, so $\rho^\square(\alpha) = \rho(\alpha) \sqcup w \leq \rho(0) \sqcup w = \rho^\square(0)$. Since $\rho^\square(\alpha) \sqcap \rho^\square(\alpha^\star \odot \beta) = (\rho(\alpha) \sqcup w) \sqcap (\rho(\alpha^\star \odot \beta) \sqcup w) = (\rho(\alpha) \sqcap \rho(\alpha^\star \odot \beta)) \sqcup w \leq \rho(\beta) \sqcup w = \rho^\square(\beta)$, for any $\alpha, \beta \in S$. So by Proposition 3.4, $\rho^\square$ is an L-fuzzy strong ideal. Since ρ is an L-fuzzy prime ideal, thus $\rho(\alpha) \sqcup \rho(\beta) \geq \rho(\alpha \wedge \beta)$, then $\rho^\square(\alpha \wedge \beta) = \rho(\alpha \wedge \beta) \sqcup w \leq (\rho(\alpha) \sqcup \rho(\beta)) \sqcup w = (\rho(\alpha) \sqcup w) \sqcup (\rho(\beta) \sqcup w) = \rho^\square(\alpha) \sqcup \rho^\square(\beta)$. Therefore, $\rho^\square$ is an L-fuzzy prime ideal of S.

Conversely, given an L-fuzzy prime ideal $\rho^\square$, so $\rho^\square(0) \geq \rho^\square(\alpha)$, so $\rho(0) \sqcup w \geq \rho(\alpha) \sqcup w$, then $\rho(0) \geq \rho(\alpha)$. Since $\rho^\square(\alpha) \sqcap \rho^\square(\alpha^\star \odot \beta) \leq \rho^\square(\beta)$, thus $(\rho(\alpha) \sqcup w) \sqcap (\rho(\alpha^\star \odot \beta) \sqcup w) \leq (\rho(\beta) \sqcup w)$, then $\rho(\alpha) \sqcap \rho(\alpha^\star \odot \beta) \leq \rho(\beta)$, so ρ is an L-fuzzy strong ideal. Since $\rho^\square(\alpha \wedge \beta) \leq \rho^\square(\alpha) \sqcup \rho^\square(\beta)$, thus $\rho(\alpha \wedge \beta) \sqcup w \leq (\rho(\alpha) \sqcup w) \sqcup (\rho(\beta) \sqcup w) = (\rho(\alpha) \sqcup \rho(\beta)) \sqcup w$, so $\rho(\alpha \wedge \beta) \leq \rho(\alpha) \sqcup \rho(\beta)$. Therefore, ρ is an L-fuzzy prime ideal of S. $\square$

5. The second type of L-fuzzy prime ideals

Definition 5.1. *Let S be a bounded semihoop. An L-fuzzy strong ideal ρ is called the second type of L-fuzzy prime if ρ is non-constant and $\rho((\alpha \to \beta)^\star) = \rho(0)$ or $\rho((\beta \to \alpha)^\star) = \rho(0)$, for any $\alpha, \beta \in S$.*

Example 5.1. Let S be a bounded semihoop in Example 3.5 and L be a lattice in Example 3.2. Define two L-fuzzy subsets ρ and $\chi$by

$$\rho(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0, m \\ x, & \text{if } \alpha = n, p, 1 \end{cases}$$

and

$$\chi(\alpha) = \begin{cases} x, & \text{if } \alpha = 0, n \\ 0, & \text{if } \alpha = m, p, 1 \end{cases}$$

for any $\alpha \in S$. Through verification, we can see that ρ and χ are the second type of L-fuzzy prime ideals.

Lemma 5.1. *Given a bounded semihoop S. Then, $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star \geq \alpha$, $(\alpha \wedge \beta) \boxplus (\beta \to \alpha)^\star \geq \beta$, for any $\alpha, \beta \in S$.*

Proof. Since $(\alpha \odot \beta)^\star = (\alpha \odot \beta) \to 0 = \alpha \to (\beta \to 0) = \alpha \to \beta^\star$, for every $\alpha, \beta \in S$, thus $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star = (\alpha \wedge \beta)^\star \to (\alpha \to \beta)^{\star\star\star} = (\alpha \wedge \beta)^\star \to (\alpha \to \beta)^\star = ((\alpha \wedge \beta)^\star \odot (\alpha \to \beta))^\star = ((\alpha \wedge \beta)^\star \odot (\alpha \to \beta)) \to 0 = ((\alpha \to \beta) \odot (\alpha \wedge \beta)^\star) \to 0 = (\alpha \to \beta) \to ((\alpha \wedge \beta)^\star \to 0) = (\alpha \to \beta) \to (\alpha \wedge \beta)^{\star\star}$. Since $(\alpha \wedge \beta)^{\star\star} \geq \alpha \wedge \beta$, thus $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star = (\alpha \to \beta) \to (\alpha \wedge \beta)^{\star\star} \geq (\alpha \to \beta) \to (\alpha \wedge \beta) = ((\alpha \to \beta) \to \alpha) \wedge ((\alpha \to \beta) \to \beta) \geq \alpha \wedge \alpha = \alpha$, then $(\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star \geq \alpha$. Similarly, $(\alpha \wedge \beta) \boxplus (\beta \to \alpha)^\star \geq \beta$. $\square$

Proposition 5.1. *Let S be a bounded semihoop. Then, the second type of L-fuzzy prime ideal ρ of S is an L-fuzzy prime ideal.*

Proof. Suppose that ρ is the second type of L-fuzzy prime ideal of S, then $\rho((\alpha \to \beta)^\star) = \rho(0)$ or $\rho((\beta \to \alpha)^\star) = \rho(0)$, for any $\alpha, \beta \in S$. Since ρ is an L-fuzzy strong ideal and by Lemma 5.1, thus $\rho((\alpha \wedge \beta)) \sqcap \rho((\alpha \to \beta)^\star) \leq \rho((\alpha \wedge \beta) \boxplus (\alpha \to \beta)^\star) \leq \rho(\alpha)$, so $\rho((\alpha \wedge \beta)) \sqcap \rho((\alpha \to \beta)^\star) = \rho((\alpha \wedge \beta)) \sqcap \rho(0) = \rho((\alpha \wedge \beta)) \leq \rho(\alpha)$, then $\rho((\alpha \wedge \beta)) \leq \rho(\alpha)$. The same to be, $\rho((\alpha \wedge \beta)) \leq \rho(\beta)$. So $\rho((\alpha \wedge \beta)) \leq \rho(\alpha) \sqcup \rho(\beta)$. Therefore, the conclusion holds. $\square$

Definition 5.2. *A bounded semihoop S is called a bounded prelinearity semihoop if it satisfies $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$, for any $\alpha, \beta \in S$.*

Proposition 5.2. *Let S be a bounded prelinearity semihoop. Then, an L-fuzzy prime ideal ρ of S is the second type of L-fuzzy prime ideal.*

Proof. Since S be a bounded prelinearity semihoop, thus $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$, then $(\alpha \to \beta) = 1$ or $(\beta \to \alpha) = 1$, then $(\alpha \to \beta)^\star = 1^\star = 0$ and $(\beta \to \alpha)^\star = 1^\star = 0$, so $(\alpha \to \beta)^\star \wedge (\beta \to \alpha)^\star = 0$. Since ρ is an L-fuzzy prime ideal of S, thus $\rho(0) = \rho(0 \wedge 0) = \rho((\alpha \to \beta)^\star \wedge (\beta \to \alpha)^\star) \leq \rho((\alpha \to \beta)^\star) \sqcup \rho((\beta \to \alpha)^\star)$, so $\rho(0) \leq \rho((\alpha \to \beta)^\star)$ or $\rho(0) \leq \rho((\beta \to \alpha)^\star)$, by Proposition 3.4(1), we get $\rho(0) \geq \rho((\alpha \to \beta)^\star)$ and $\rho(0) \geq \rho((\beta \to \alpha)^\star)$, then $\rho(0) = \rho((\alpha \to \beta)^\star)$ or $\rho(0) = \rho((\beta \to \alpha)^\star)$. Therefore, the conclusion holds. □

If there exist $\alpha, \beta \in S$such that $(\alpha \to \beta) \vee (\beta \to \alpha) \neq 1$, then an L-fuzzy prime ideal may not be the second type of L-fuzzy prime ideal, by the following example will illustrate.

Example 5.2 ([10])**.** Let $S = \{0, m, n, r, p, q, 1\}$ with $0 < m < n < q < 1$, $0 < r < p < q < 1$ and L be a complete lattice in Example 3.6. Define $\odot$ and $\to$as follows,

$\odot$	0	m	n	r	p	q	1
0	0	0	0	0	0	0	0
m	0	m	m	0	0	m	m
n	0	m	m	0	0	m	n
r	0	0	0	r	r	r	r
p	0	0	0	r	r	r	p
q	0	m	m	r	r	q	q
1	0	m	n	r	p	q	1

$\to$	0	m	n	r	p	q	1
0	1	1	1	1	1	1	1
m	p	1	1	p	p	1	1
n	p	q	1	p	p	1	1
r	n	n	n	1	1	1	1
p	n	n	n	q	1	1	1
q	0	n	n	p	p	1	1
1	0	m	n	r	p	q	1

We can see that $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop but S is not satisfy $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$, for any $\alpha, \beta \in S$since $(r \to n) \vee (n \to r) = n \vee p \neq 1$. We define an L-fuzzy subset by

$$\rho(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ x, & \text{if } \alpha = m, n \\ y, & \text{if } \alpha = r, p \\ 0, & \text{if } \alpha = q, 1 \end{cases}$$

for all $\alpha \in S$. Then, ρ is an L-fuzzy prime ideal but it is not the second type of L-fuzzy prime since $\rho((m \to r)^\star) = \rho(p^\star) = \rho(n) = x \neq 1 = \rho(0)$, $\rho((r \to m)^\star) = \rho(n^\star) = \rho(p) = y \neq 1 = \rho(0)$.

In a bounded semihoop S, we denote that $FD(S)$ is the L-fuzzy strong ideal set of S. A partial order relation $\preceq$ is defined by $\rho \preceq \chi$if $\rho(\alpha) \leq \chi(\alpha)$, for all $\alpha \in S$, $\rho, \chi \in FD(S)$.

Proposition 5.3. *Assume that S is a bounded semihoop and ρ, χ are L-fuzzy strong ideals of S and satisfying $\rho \preceq \chi$ and $\rho(0) = \chi(0)$. If ρ is the second type of L-fuzzy prime ideal of S, then χ is also the second type of L-fuzzy prime ideal.*

Proof. Let ρ be the second type of L-fuzzy prime ideal, then $\rho((\alpha \to \beta)^\star) = \rho(0)$ or $\rho((\beta \to \alpha)^\star) = \rho(0)$, for any $\alpha, \beta \in S$. So $\chi(0) = \rho(0) \le \rho((\alpha \to \beta)^\star) \le \chi((\alpha \to \beta)^\star)$ or $\chi(0) = \rho(0) \le \rho((\beta \to \alpha)^\star) \le \chi((\beta \to \alpha)^\star)$, then $\chi(0) \le \chi((\alpha \to \beta)^\star)$ or $\chi(0) \le \chi((\beta \to \alpha)^\star)$. Since χ is an L-fuzzy strong ideal, thus $\chi(0) \ge \chi((\alpha \to \beta)^\star)$ or $\chi(0) \ge \chi((\beta \to \alpha)^\star)$, then $\chi(0) = \chi((\alpha \to \beta)^\star)$ or $\chi(0) = \chi((\beta \to \alpha)^\star)$. Therefore, χ is the second type of L-fuzzy prime ideal. $\square$

Proposition 5.4. *Assume that S is a bounded semihoop and ρ is the second type of L-fuzzy prime ideal of S. If $w < \rho(0)$, for any $w \in L$, then $\rho^\square$ is the second type of L-fuzzy prime ideal.*

Proof. By Proposition 4.5, we get that $\rho^\square$ is an L-fuzzy strong ideal. Since $\rho(\alpha) \le \rho(\alpha) \sqcup w = \rho^\square(\alpha)$, for all $\alpha \in S$, thus $\rho \preceq \rho^\square$. Since $\rho(0) = \rho(0) \sqcup w = \rho^\square(0)$. Therefore, by Proposition 5.3, $\rho^\square$ is the second type of L-fuzzy prime ideal. $\square$

6. L-fuzzy maximal ideals

Definition 6.1. *Let S be a bounded semihoop. A proper L-fuzzy strong ideal ρ of S is called an L-fuzzy maximal ideal if ρ_t^1 is non-trivial implies ρ_t^1 is a maximal ideal, for any $t \in L$.*

Example 6.1 ([10])**.** Let $S = \{0, m, n, a, p, q, 1\}$ with $0 < m < n < 1$, $0 < a < p < q < 1$, where n and a are incomparable. Define $\odot$ and $\to$as bellow:

$\odot$	0	m	n	a	p	q	1
0	0	0	0	0	0	0	0
m	0	m	m	m	m	m	m
n	0	m	m	m	m	m	n
a	0	m	m	a	a	a	a
p	0	m	m	a	a	a	p
q	0	m	m	a	a	q	q
1	0	m	n	a	p	q	1

$\to$	0	m	n	a	p	q	1
0	1	1	1	1	1	1	1
m	0	1	1	1	1	1	1
n	0	q	1	q	1	1	1
a	0	n	n	1	1	1	1
p	0	n	n	q	1	1	1
q	0	n	n	p	p	1	1
1	0	m	n	a	p	q	1

Then, $(S, \odot, \to, \wedge, 0, 1)$ is a bounded semihoop. Let $L = \{0, x, y, 1\}$ be a complete lattice with $0 < x < y < 1$. We define an L-fuzzy subset ρ by

$$\rho(\alpha) = \begin{cases} x, & \text{if } \alpha = 0, \\ 0, & \text{if } \alpha \neq 0, \end{cases}$$

for any $\alpha \in S$. Then, ρ is an L-fuzzy maximal ideal of S.

Proposition 6.1. *Assume that S is a bounded semihoop and ρ is an L-fuzzy maximal ideal of S. If $\rho(\alpha) < \rho(\beta)$ and $\rho^1_{\rho(\beta)} \neq \rho^1_{\rho(\alpha)}$, then $\rho^1_{\rho(\beta)} = \{0\}$ or $\rho^1_{\rho(\alpha)} = S$, for any $\alpha, \beta \in S$.*

Proof. Since $\rho(\alpha) < \rho(\beta)$, thus $\rho^1_{\rho(\beta)} \subset \rho^1_{\rho(\alpha)}$, for any $\alpha, \beta \in S$. If $\rho^1_{\rho(\beta)} \neq \{0\}$, since ρ is an L-fuzzy maximal ideal, then $\rho^1_{\rho(\beta)}$ is a maximal ideal, so $\rho^1_{\rho(\alpha)} = S$. Therefore, $\rho^1_{\rho(\beta)} = \{0\}$ or $\rho^1_{\rho(\alpha)} = S$, for any $\alpha, \beta \in S$. □

Proposition 6.2. *Let S be a bounded semihoop, L be a complete lattice and $\rho : S \to L$ is a non-constant L-fuzzy strong ideal of S. Then, the following statements are equivalent:*

(1) *ρ is an L-fuzzy maximal ideal of S;*

(2) *$\rho^1_{\rho(0)}$ is a maximal ideal of S;*

(3)
$$\rho(\alpha) = \begin{cases} \rho(0), & \text{if } \alpha \in \rho^1_{\rho(0)} \\ \rho(\alpha_1), & \text{if } \alpha \notin \rho^1_{\rho(0)} \end{cases}$$
for some $\alpha_1 \in S$ with $\rho(\alpha_1) < \rho(0)$.

Proof. (1) ⇒ (2) Let ρ is an L-fuzzy maximal ideal of S. By Proposition 3.4(1), we have $\rho(0) \geq \rho(\alpha)$, for any $\alpha \in S$. Since ρ is not constant, thus there exists $\alpha_1 \neq \rho(0)$, so $\rho(\alpha_1) < \rho(0)$, then $\alpha_1 \notin \rho^1_{\rho(0)}$. Since $0 \in \rho^1_{\rho(0)}$, thus $\rho^1_{\rho(0)} \neq \emptyset$ and $\rho^1_{\rho(0)} \neq S$, so $\rho^1_{\rho(0)}$ is a maximal ideal of S.

(2) ⇒ (3) Let $\rho^1_{\rho(0)}$ be a maximal ideal of S. Since ρ is an L-fuzzy strong ideal of S, for any $\alpha \in \rho^1_{\rho(0)}$, we have $\rho(\alpha) \geq \rho(0)$ and $\rho(\alpha) \leq \rho(0)$by Proposition 3.4(1), then $\rho(\alpha) = \rho(0)$. Since ρ is not constant, thus there is $\alpha_1 \in S$such that $\rho(\alpha_1) \neq \rho(0)$, so $\rho(\alpha_1) < \rho(0)$. Suppose that there exists $\alpha_2 \in S$such that $\rho(\alpha_2) \neq \rho(0)$ and $\rho(\alpha_2) \neq \rho(\alpha_1)$. We will discuss the following cases:

(i) If $\rho(\alpha_1) < \rho(\alpha_2) < \rho(0)$ or $\rho(\alpha_2) < \rho(\alpha_1) < \rho(0)$, then $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_2)} \subset \rho^1_{\rho(\alpha_1)}$ or $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_1)} \subset \rho^1_{\rho(\alpha_2)}$. From Proposition 3.3(1), $\rho^1_{\rho(\alpha_1)}$ and $\rho^1_{\rho(\alpha_2)}$ are ideals, which contradicts $\rho^1_{\rho(0)}$ be a maximal ideal of S.

(ii) If $\rho(\alpha_1)$ and $\rho(\alpha_2) < \rho(0)$ are incomparable, then $\alpha_1 \notin \rho^1_{\rho(\alpha_2)}$ and $\alpha_2 \notin \rho^1_{\rho(\alpha_1)}$, so $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_1)} \subset S$ and $\rho^1_{\rho(0)} \subset \rho^1_{\rho(\alpha_2)} \subset S$, which contradicts $\rho^1_{\rho(0)}$ be a maximal ideal of S.

Then, $\rho(\alpha_2) = \rho(0)$ or $\rho(\alpha_2) = \rho(\alpha_1)$. Therefore, the conclusion holds.

(3) ⇒ (1) Suppose
$$\rho(\alpha) = \begin{cases} \rho(0), & \text{if } \alpha \in \rho^1_{\rho(0)} \\ \rho(\alpha_1), & \text{if } \alpha \notin \rho^1_{\rho(0)} \end{cases}$$
for some $\alpha_1 \in S$ with $\rho(\alpha_1) < \rho(0)$. Then, $\rho^1_t \in \{\rho^1_{\rho(0)}, S, \emptyset\}$, for any $t \in L$, so ρ is an L-fuzzy maximal ideal of S. □

Corollary 6.1. *Let S be a bounded semihoop and $\rho : S \to [0,1]$ be a fuzzy maximal ideal of S. Then, ρ has exactly two values.*

Proposition 6.3. *Let S be a bounded $\vee$-semihoop with DNP and ρ be an L-fuzzy strong ideal on S. If ρ is an L-fuzzy maximal ideal of S, then ρ is an L-fuzzy prime ideal of S.*

Proof. Let ρ be an L-fuzzy maximal ideal of S. Then, for any $t \in L$such that ρ_t^1 is non-trivial implies ρ_t^1 is a maximal ideal of S. From Proposition 2.5, so ρ_t^1 is a prime ideal of S. Therefore, from Proposition 4.4(1), we have that ρ is an L-fuzzy prime ideal of S. □

Proposition 6.4. *Suppose that S is a bounded semihoop and ρ is a proper L-fuzzy strong ideal of S. If s and t are incomparable, for any $s, t \in L$, then the following conditions hold:*

(1) *ρ_s^1 and ρ_t^1 are proper ideals;*

(2) *if ρ is an L-fuzzy maximal ideal, then ρ_s^1 and ρ_t^1 are maximal ideals.*

Proof. (1) Let $\alpha, \beta \in S$such that $\rho(\alpha) = s$ and $\rho(\beta) = t$. From Proposition 3.3(1), we have ρ_s^1 and ρ_t^1 are two ideals. Since $\rho(\alpha) = s$ and $\rho(\beta) = t$ are incomparable, thus $\rho(\alpha) \neq \rho(0)$ and $\rho(\beta) \neq \rho(0)$, so $\{0\} \subset \rho_s^1$ and $\{0\} \subset \rho_t^1$. Moreover, $\alpha \notin \rho_t^1$ and $\beta \notin \rho_s^1$, then $\rho_s^1 \subset S$ and $\rho_t^1 \subset S$. Therefore, ρ_s^1 and ρ_t^1 are proper ideals.

(2) Let ρ be an L-fuzzy maximal ideal of S. From (1), ρ_s^1 and ρ_t^1 are proper ideals, so ρ_s^1 and ρ_t^1 are non-trivial. Thus, ρ_s^1 and ρ_t^1 are maximal ideals by Definition 6.1. □

7. Conclusion

In this paper, we associate bounded semihoops with lattice structures and establish L-fuzzy ideals theory on bounded semihoop. In particular, we obtain several important conclusions. (1)Let S be a bounded semihoop and L be a complete lattice. Then, each L-fuzzy strong ideal is an L-fuzzy ideal but an L-fuzzy ideal may not be an L-fuzzy strong ideal. (2)Let S be a bounded semihoop, L be a complete lattice and $\rho : S \to L$ be an L-fuzzy set of S. (i)If ρ is an L-fuzzy strong ideal(filter) of S, then $\rho^\star$ is an L-fuzzy strong filter(ideal). (ii)If ρ is an L-fuzzy ideal(filter) of S, then $\rho^\star$ is an L-fuzzy filter(ideal), where $\rho^\star(\alpha) = \rho(\alpha^\star)$, for any $\alpha \in S$. (3)We establish equivalence descriptions between L-fuzzy strong ideals and ideals using four types of level sets. (4)Let S be a bounded semihoop, ρ be an L-fuzzy strong ideal of S and H be an up-set sublattice of L. Then, $\rho^{-1}(H)$ is an ideal of S. (5)Let S be a bounded semihoop. Then, each the second type of L-fuzzy prime ideal is an L-fuzzy prime ideal but an L-fuzzy prime ideal may not be the second type of L-fuzzy prime ideal unless S is a bounded prelinearity semihoop. (6)Let S be a bounded semihoop and $\rho : S \to [0, 1]$ be an L-fuzzy maximal ideal of S. Then, ρ has exactly two values. (7) Let S be a bounded $\vee$-semihoop with DNP and ρ be an L-fuzzy strong ideal on S. If ρ is an L-fuzzy maximal ideal of S, then ρ is an L-fuzzy prime ideal of S.

Since semihoops are the fundamental residuated structures, these properties and conclusions in this article can be applied to other residuated structures.

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Population dynamics of a modified predator-prey model with economic harvesting

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Abstract. The dynamical behaviors of a predator-prey model with commercial harvesting are studied in the present work. The model is developed from the Leslie predator-prey model with harvesting on predator, which is established by differential-algebra equations. The harvesting is considered from an economic perspective, and the impacts of the harvesting profit on the dynamics of our model are investigated. Firstly, basing on the parameterisation approach of differential-algebra system, the local stability of positive equilibrium point is studied. Further, by treating the harvesting profit as a bifurcation parameter, the Hopf bifurcation occurring at the equilibrium point is analyzed, and we find a qualitative change in the dynamics. Besides, the stability of centre is also considered. Some computer simulations using Matlab software are presented to support the analytical results. Lastly, we relate the results on mathematics and dynamics with the biology, and interpret these results in terms of ecosystem stability and destruction.

Keywords: predator-prey, differential-algebra, local stability, Hopf bifurcation, centre, harvesting profit.

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1. Introduction

Predator-prey interactions are the fundamental blocks of any complex biological and ecological systems, as well as generalized competitive and cooperative systems [1]. As a result, the dynamic relationship between the populations of predators and preys is an important research theme in the areas of applied mathematics and theoretical ecology (see, [2, 3]). Actually, as with the dynamic theory of differential equations has been widely used in these research areas, in the past few years the dynamics of predator-prey system (usually formulated by differential equations) also has become an interesting subject in itself, since many complicated dynamical behaviors have been discovered in this subject, for instances, instability, stability switches, limit cycle, oscillations, various kinds of bifurcations, chaos, and so on [2-5]. Especially, in this work, by combining the dynamic theories of differential-algebra system and differential equations, we aim to present a complete dynamical analysis for a modified Leslie's predator-prey model with commercial harvesting, which takes the form of differential-algebra equations. The establishing process of our model is introduced as follows.

The fundamental model that we consider is the following predator-prey model introduced by Leslie [6], which is a system of nonlinear ordinary differential equations:

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1 - ay(t)\right), \\ \dot{y}(t) = y(t)\left(r_2 - b\dfrac{y(t)}{x(t)}\right), \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ represent the densities of preys and predators at time $t \geq 0$, respectively; besides, the parameters r_1, a, r_2, and b are positive constants, which stand for the intrinsic growth rate of prey species, the catch rate at which the predator population kills its preys, the intrinsic growth rate of predator species, and the conversion rate of consumed preys into the newborns of predator species, respectively. For more details on the biological significance of model (1.1), refer to the literature [6, 7].

In reality, biological populations are often harvested to satisfy people's demands for material life [8, 9]. For predator-prey system, in order to avoid the extinction of prey population, harvesting of predator population is commonly practiced, which is effective in controlling the population size of predators. So we consider human harvesting effort $E(t)$ on the predator species in model (1.1), and then we have

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1 - ay(t)\right), \\ \dot{y}(t) = y(t)\left(r_2 - b\dfrac{y(t)}{x(t)} - E(t)\right). \end{cases} \tag{1.2}$$

Subsequently, the number of predators harvested by people is $E(t)y(t)$, which is regarded as the market supply here. We assume that the market is quite capable of absorbing all the catches. Referring to Refs. [10, 11], the selling price and market supply move in opposite directions, and harvesting cost also moves inversely to the population density of harvested population. In light of these rules, we let the unit selling price $\tilde{p}$ and the unit harvesting cost $\tilde{c}$ respectively be $p/[l + E(t)y(t)]$ and $c/y(t)$, where p, l and c are positive parameters, p/l is the maximum unit selling price, and c is the harvesting cost for unit population density of predators. And then, we can show that $\tilde{p} \to p/l$ as $E(t)y(t) \to 0$, and $\tilde{p} \to 0$ as $E(t)y(t) \to +\infty$, which indicate that the selling price will decrease when the supply $E(t)y(t)$ increases. Moreover, $\tilde{c} \to +\infty$ as $y(t) \to 0$, and $\tilde{c} \to 0$ as $y(t) \to +\infty$, which imply that the harvesting cost will increase when the population density of predators becomes small. It is easy to imagine that, when the predators are rare, people must make more effort to capture them. In this way, the total revenue from harvesting is $[p/(l+E(t)y(t))] \cdot E(t)y(t)$ and the total harvesting cost is $[c/y(t)] \cdot E(t)y(t)$. Consequently, the net economic revenue is $[p/(l + E(t)y(t))] \cdot E(t)y(t) - [c/y(t)] \cdot E(t)y(t)$. On the basis of model (1.2), so we can establish the following modified predator-prey model with economic harvesting, which is a differential-algebra system:

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1 - ay(t)\right), \\ \dot{y}(t) = y(t)\left(r_2 - b\dfrac{y(t)}{x(t)} - E(t)\right), \\ \quad 0 = E(t)y(t)\left(\dfrac{p}{l + E(t)y(t)} - \dfrac{c}{y(t)}\right) - v, \end{cases} \tag{1.3}$$

where v denotes people's harvesting profit. In addition, when time $t = 0$, the initial values of system (1.3) should be positive. That is,

$$x(0) > 0, \;\; y(0) > 0, \;\; E(0) > 0. \tag{1.4}$$

In recent years, dynamical behaviors of harvested predator-prey models are reported in Refs. [46-50]. The literature [45] has investigated the nontrivial equilibrium solution and transcritical bifurcation of a three dimensional intraguild predator-prey model with Michaelis-Menten type of harvesting in predator. Besides, the stability of equilibria, limit cycle, saddle-node bifurcation and Bogdanov-Takens bifurcation in several predator-prey systems with nonlinear prey harvesting are discussed in Refs. [46, 47]. Das et al. [48] have studied the endangeredness, resilience and extinction of a predator-prey system under prey harvesting and predator harvesting, respectively. Kashyap et al. [49] have explored the coexistence, ecologically feasible steady states and local codimension one bifurcations of a predator-prey system with predator harvesting. Moreover, local and global stability at the interior equilibrium points of a harvested three species predator-prey model (prey, predator, and super predator) have been considered in Ref. [50]. Clearly, these harvested predator-prey models [46-50] are modelled by systems of differential equations. In contrast, our harvested predator-prey model (1.3) is established by differential-algebra equations. Compared with the familiar harvested predator-prey models expressed by differential equations, the superiority of our modified model (1.3) is that it not only involves population interactions in the harvested predator-prey system but also investigates the harvesting from an economic viewpoint. Some relevant modified models are presented in the publications [21, 24, 26, 29, 32]. By employing Rouche's theorem [22] as well as the centre manifold reduction methods [23, 25], Refs. [21, 24] have analyzed the existence of time-delay-induced Hopf bifurcation phenomena and the stability of bifurcating periodic orbits in delayed modified predator-prey models. Moreover, the authors [26, 29, 32] have discussed the local stability of equilibrium points and bifurcations (flip bifurcation and N-S bifurcation) in several discrete modified predator-prey models by applying the center manifold theory and the bifurcation theory of discrete systems in Refs. [27, 28, 30, 31]. Different from the literature [21, 24, 26, 29, 32], we will investigate the impact of the harvesting profit v on the dynamics (including the local stability of equilibrium point, Hopf bifurcation and stability of centre) in the modified predator-prey model (1.3), and then afterwards we propose an appropriate scope for the profit to guarantee the maintenance of long-term sustainable development of our biological system. Besides, it is notable that the relevant differential-algebra predator-prey models [21, 24, 26, 29, 32] are all established under the assumptions that the price $\tilde{p}$ and cost $\tilde{c}$ are constants, which results in that the harvesting variable $E(t)$ can be explicitly solved out from the algebra equation, and then the differential-algebra models can be easily reduced to the systems of differential equations. Apparently, our differential-algebra model (1.3) has overcome the shortages.

Furthermore, it is worth noting that there are many essential distinctions between differential-algebra system and the system of differential equations, see the literature [34-38] for more details. In the sense of index, the system of ordinary differential equations is a special case of differential-algebraic system, since the index of the former is zero, while the index of the latter is nonzero. Obviously, it is a leap from a zero index system to a nonzero index one. In fact, the dynamics of differential-algebra system is much more difficult to investigate than the corresponding system of differential equations (see, [35-37]). Hence, in a certain meaning, our work supplements and enhances the research in the previous publications [12-21, 24, 26, 29, 32, 45-50] on the dynamic analysis for predator-prey models.

We organize the rest of this paper as follows. In the next section, we deduce the Jacobian matrix of model (1.3) and investigate the corresponding characteristic equation, which give the local stability results for the equilibrium point. In Section 3, we study the Hopf bifurcation of our model in detail basing on the previous section. To complement Sections 2 and 3, the stability of the centre is further explored in Section 4. Moreover, some numerical simulations are presented in Section 5 to make the derived findings more complete. Finally, in Section 6 we discuss the theoretical results and summarize the research work of this article.

2. Stability analysis for equilibrium point

In this section, combining the parameterisation approach [39, 40] with Routh-Hurwitz stability criteria [2, 3], we study local stability of the equilibrium point of model (1.3). At first, we prove the positiveness of the solutions of model (1.3).

Lemma 2.1. *The trajectories of model* (1.3) *with initial values* (1.4) *and* $v > 0$ *stay in* $\mathrm{R}^3_+ = \{(x(t), y(t), E(t)) \mid x(t) > 0,\, y(t) > 0,\, E(t) > 0\}$*, for* $\forall\, t > 0$.

Proof. In view of model (1.3), we have

$$\frac{\mathrm{d}x(t)}{x(t)} = (r_1 - ay(t))\ \mathrm{d}t.$$

Due to the initial value $x(0) > 0$, by integrating above equation in the interval $[0, t]$, we obtain

$$x(t) = x(0)\ \exp\left\{\int_0^t (r_1 - ay(s))\ \mathrm{d}s\right\} > 0,\ \text{ for } \forall\, t > 0.$$

Similarly, we can get

$$y(t) = y(0)\ \exp\left\{\int_0^t \left(r_2 - b\frac{y(s)}{x(s)} - E(s)\right)\ \mathrm{d}s\right\} > 0,\ \text{ for } \forall\, t > 0.$$

Furthermore, $E(t)$ is also positive for $\forall\, t > 0$, since the harvesting profit $v > 0$ here. □

Lemma 2.1 suggests that only the positive equilibrium point of model (1.3) is required to be considered. If $X_0 := (x_0, y_0, E_0)^T$ is an equilibrium point of model (1.3), then we have

$$\begin{cases} r_1 - ay_0 = 0, \\ r_2 - b\dfrac{y_0}{x_0} - E_0 = 0, \\ \dfrac{pE_0y_0}{l + E_0y_0} - cE_0 - v = 0. \end{cases}$$

By means of solving this set of linear equations, model (1.3) has an equilibrium point:

$$X_0(v) = (x_0, y_0, E_0)^T = \left(\frac{by_0}{r_2 - E_0}, \frac{r_1}{a}, E_0\right)^T,$$

where $E_0 = \left\{(py_0 - vy_0 - cl) \pm \sqrt{(cl + vy_0 - py_0)^2 - 4clvy_0}\,\right\}/2cy_0$.

To make such an equilibrium point X_0 is positive, in this paper we need to suppose that

$$r_2 > E_0,\ py_0 > cl + vy_0,\ (cl + vy_0 - py_0)^2 \geq 4clvy_0. \tag{2.1}$$

On the basis of the theory of differential-algebra system [35-37], near the point of X_0, model (1.3) can be locally equivalent to

$$\begin{cases} \dot{x}(t) = x(t)\left(r_1 - ay(t)\right), \\ \dot{y}(t) = y(t)\left(r_2 - b\dfrac{y(t)}{x(t)} - E(t)\right), \\ \dot{E}(t) = f_3(x(t), y(t), E(t)), \\ \quad 0 = E(t)y(t)\left(\dfrac{p}{l + E(t)y(t)} - \dfrac{c}{y(t)}\right) - v, \end{cases} \tag{2.2}$$

where the function f_3 satisfies $f_3(X_0) = 0$. The explicit expression of f_3 is not required to be defined, refer to Eq. (A.5) in Appendix.

For the purpose of discussions, we denote

$$f(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \end{pmatrix} = \begin{pmatrix} x(t)\left(r_1 - ay(t)\right) \\ y(t)\left(r_2 - b\dfrac{y(t)}{x(t)} - E(t)\right) \\ f_3(x(t), y(t), E(t)) \end{pmatrix},$$

$$g(X) = E(t)y(t)\left(\frac{p}{l + E(t)y(t)} - \frac{c}{y(t)}\right) - v, \; X = (x(t), y(t), E(t))^T. \tag{2.3}$$

So, system (2.2) can be written as

$$\begin{cases} \dot{X} = f(X), \\ 0 = g(X). \end{cases} \tag{2.4}$$

In the following, we consider the parameterisation ψ [39, 40] for system (2.4):

$$X = \psi(Y) = X_0 + U_0 Y + V_0 h(Y) \quad \text{and} \tag{2.5}$$

$$g(\psi(Y)) = 0, \tag{2.6}$$

where $Y = (y_1, y_2)^T \in \mathrm{R}^2$, $U_0 = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}$, I_2 denotes an identity matrix of dimension 2×2, $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $h : \mathrm{R}^2 \to \mathrm{R}$ is a smooth mapping. Consequently, by the parameterised system (A.5) in Appendix, the Taylor expansions of the parameterised system of system (2.2) at X_0 takes the form of

$$\dot{Y} = U_0^T D_X f(X_0) D_Y \psi(0) Y + o(|Y|), \tag{2.7}$$

where D denotes the differential operator, and $D_X f(X)$ represents the Jacobian matrix of function $f(X)$ regarding X. With respect to the derivation process of the formula (2.7), refer to Appendix. □

Summarizing the above analysis, we have the following results.

Theorem 2.1. *For model* (1.3),

(*i*) *if*

$$\left(\frac{by_0}{x_0} - \frac{plE_0 y_0}{ply_0 - c(l + E_0 y_0)^2}\right)^2 \geq \frac{4aby_0^2}{x_0},$$

then, when $by_0/x_0 > plE_0y_0/[ply_0 - c(l + E_0y_0)^2]$, *the equilibrium point* X_0 *is a stable node*; *when* $by_0/x_0 < plE_0y_0/[ply_0 - c(l + E_0y_0)^2]$, *the equilibrium point* X_0 *is an unstable node*;

(*ii*) *if*

$$\left(\frac{by_0}{x_0} - \frac{plE_0 y_0}{ply_0 - c(l + E_0 y_0)^2}\right)^2 < \frac{4aby_0^2}{x_0},$$

then, when $by_0/x_0 > plE_0y_0/[ply_0 - c(l + E_0y_0)^2]$, *the equilibrium point* X_0 *is a sink*; *when* $by_0/x_0 < plE_0y_0/[ply_0 - c(l + E_0y_0)^2]$, *the equilibrium point* X_0 *is a source.*

Proof. We can derive the following Jacobian matrix P of system (2.7) in view of Eqs. (2.7), (A.2) and (A.3) (in Appendix) that

$$
\begin{aligned}
\mathrm{P} &= \begin{pmatrix} D_{y_1} f_1(\psi(Y)) & D_{y_2} f_1(\psi(Y)) \\ D_{y_1} f_2(\psi(Y)) & D_{y_2} f_2(\psi(Y)) \end{pmatrix}\Bigg|_{Y=0} \\
&= U_0^T D_X f(X_0) D_Y \psi(0) = U_0^T D_X f(X_0) \begin{pmatrix} D_X g(X_0) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_2 \end{pmatrix} \\
&= \begin{pmatrix} D_x f_1(X_0) & D_y f_1(X_0) - \dfrac{plE_0 \cdot D_E f_1(X_0)}{ply_0 - c(l+E_0 y_0)^2} \\ D_x f_2(X_0) & D_y f_2(X_0) - \dfrac{plE_0 \cdot D_E f_2(X_0)}{ply_0 - c(l+E_0 y_0)^2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -ax_0 \\ \dfrac{by_0^2}{x_0^2} & -\dfrac{by_0}{x_0} + \dfrac{plE_0 y_0}{ply_0 - c(l+E_0 y_0)^2} \end{pmatrix}, \qquad (2.8)
\end{aligned}
$$

where $D_X f_1(X_0) = \big(0, -ax_0, 0\big)$, $D_X f_2(X_0) = \big(by_0^2/x_0^2, -by_0/x_0, -y_0\big)$, $D_X g(X_0) = \big(0, plE_0/(l+E_0y_0)^2, [ply_0 - c(l+E_0y_0)^2]/(l+E_0y_0)^2\big)$. Hence, from Eq. (2.8), the characteristic equation of matrix P is

$$
\lambda^2 + \left(\frac{by_0}{x_0} - \frac{plE_0 y_0}{ply_0 - c(l+E_0y_0)^2}\right)\lambda + \frac{aby_0^2}{x_0} = 0. \qquad (2.9)
$$

For case (i), if $by_0/x_0 > plE_0y_0/[ply_0 - c(l+E_0y_0)^2]$, then Eq. (2.9) has two negative real roots. Hence, X_0 is a stable node. Conversely, X_0 is an unstable node iff Eq. (2.9) has two positive real roots. For case (ii), if $by_0/x_0 > plE_0y_0/[ply_0 - c(l+E_0y_0)^2]$, then Eq. (2.9) has two complex roots which have negative real parts, and therefore X_0 is a sink. On the contrary, X_0 is a source iff the two complex roots of Eq. (2.9) have positive real parts. And then, in view of Eq. (2.9), we are easy to derive Theorem 2.1 on the grounds of Routh-Hurwitz stability criteria [2, 3]. $\square$

Remark 2.1. *By analyzing the eigenvalues of characteristic equation* (2.9), *Hopf bifurcation can take place in model* (1.3) *under certain conditions, which will be discussed in the following section.*

3. Hopf bifurcation analysis

In this section, by choosing the economic profit v as a variable bifurcation parameter, we investigate the Hopf bifurcation in model (1.3) on the grounds of the Hopf bifurcation theorem developed by Guckenheimer and Holmes [33].

When $\Delta = \{by_0/x_0 - plE_0y_0/[ply_0 - c(l+E_0y_0)^2]\}^2 - 4aby_0^2/x_0 < 0$, it is clear that Eq. (2.9) has the following complex roots:

$$
\lambda_{1,2}(v) := \alpha(v) \pm i\omega(v),
$$

where $\alpha(v) = -(1/2)\{by_0/x_0 - plE_0y_0/[ply_0 - c(l+E_0y_0)^2]\}$, $\omega(v) = \{aby_0^2/x_0 - (1/4)[by_0/x_0 - plE_0y_0/(ply_0 - c(l + E_0y_0)^2)]^2\}^{1/2}$. Besides, in view of Eq. (2.9), the bifurcation value v_0 of variable v firstly needs to meet the equation

$$\frac{by_0}{x_0(v)} = \frac{plE_0(v)y_0}{ply_0 - c(l + E_0(v)y_0)^2}. \tag{3.1}$$

Further, in order to guarantee the existence of Hopf bifurcation in model (1.3), we assume that the following transversality conditions in the literatrue [33] are satisfied throughout this section:

$$\begin{aligned} &\alpha(v_0) = 0,\ \alpha'(v_0) = \Bigg(- \frac{b^2y_0^2}{x_0^2(r_2 - E_0(v_0))^2} - \frac{ply_0}{ply_0 - c(l + E_0(v_0)y_0)^2} \\ &- \frac{2pcly_0^2E_0(v_0)(l+E_0(v_0)y_0)}{[ply_0 - c(l+E_0(v_0)y_0)^2]^2}\Bigg) \cdot E_0'(v_0) \neq 0,\ \ \omega(v_0) := \omega_0 = \sqrt{\frac{ab}{x_0}}y_0 \neq 0, \end{aligned} \tag{3.2}$$

where $E_0'(v_0) = -\frac{1}{2c} \pm \frac{v_0y_0 - py_0 - cl}{2c\sqrt{(cl+v_0y_0-py_0)^2 - 4clv_0y_0}}$. So Hopf bifurcation takes place if the quantity v attains the critical value v_0.

To derive the detailed information about the Hopf bifurcation, in the light of the Hopf bifurcation theorem in Ref. [33], we need to make system (2.7) equivalent to the following normal form:

$$\begin{cases} \dot{y}_1 = -\omega_0 y_2 + \frac{1}{2}a_{11}^1 y_1^2 + a_{12}^1 y_1 y_2 + \frac{1}{2}a_{22}^1 y_2^2 + \frac{1}{6}a_{111}^1 y_1^3 + \frac{1}{2}a_{112}^1 y_1^2 y_2 \\ \qquad + \frac{1}{2}a_{122}^1 y_1 y_2^2 + \frac{1}{6}a_{222}^1 y_2^3 + o(|Y|^4)\,, \\ \dot{y}_2 = \omega_0 y_1 + \frac{1}{2}a_{11}^2 y_1^2 + a_{12}^2 y_1 y_2 + \frac{1}{2}a_{22}^2 y_2^2 + \frac{1}{6}a_{111}^2 y_1^3 + \frac{1}{2}a_{112}^2 y_1^2 y_2 \\ \qquad + \frac{1}{2}a_{122}^2 y_1 y_2^2 + \frac{1}{6}a_{222}^2 y_2^3 + o(|Y|^4)\,. \end{cases} \tag{3.3}$$

Subsequently, we should first of all calculate the following third order Taylor series developments of system (2.7):

$$\begin{cases} \dot{y}_1 = f_{1y_1}(X_0)y_1 + f_{1y_2}(X_0)y_2 + \frac{1}{2}f_{1y_1y_1}(X_0)y_1^2 + f_{1y_1y_2}(X_0)y_1y_2 \\ \qquad + \frac{1}{2}f_{1y_2y_2}(X_0)y_2^2 + \frac{1}{6}f_{1y_1y_1y_1}(X_0)y_1^3 + \frac{1}{2}f_{1y_1y_1y_2}(X_0)y_1^2y_2 \\ \qquad + \frac{1}{2}f_{1y_1y_2y_2}(X_0)y_1y_2^2 + \frac{1}{6}f_{1y_2y_2y_2}(X_0)y_2^3 + o(|Y|^4), \\ \dot{y}_2 = f_{2y_1}(X_0)y_1 + f_{2y_2}(X_0)y_2 + \frac{1}{2}f_{2y_1y_1}(X_0)y_1^2 + f_{2y_1y_2}(X_0)y_1y_2 \\ \qquad + \frac{1}{2}f_{2y_2y_2}(X_0)y_2^2 + \frac{1}{6}f_{2y_1y_1y_1}(X_0)y_1^3 + \frac{1}{2}f_{2y_1y_1y_2}(X_0)y_1^2y_2 \\ \qquad + \frac{1}{2}f_{2y_1y_2y_2}(X_0)y_1y_2^2 + \frac{1}{6}f_{2y_2y_2y_2}(X_0)y_2^3 + o(|Y|^4). \end{cases} \tag{3.4}$$

The coefficients of (3.4) are calculated as follows. From Eq. (2.3), we have

$$D_X f_1(X) = (r_1 - ay,\ -ax,\ 0)\,,\ D_X f_2(X) = \left(\frac{by^2}{x^2},\ r_2 - \frac{2by}{x} - E,\ -y\right),$$

$$D_X g(X) = \left(0,\ \frac{plE}{(l+Ey)^2},\ \frac{ply - c(l+Ey)^2}{(l+Ey)^2}\right). \tag{3.5}$$

In view of Eqs. (A.2) and (A.3) in Appendix, we can derive

$$D_Y\psi(Y) = \begin{pmatrix} D_X g(X) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{plE}{(l+Ey)^2} & \dfrac{ply - c(l+Ey)^2}{(l+Ey)^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\dfrac{plE}{ply - c(l+Ey)^2} \end{pmatrix} := \big(D_{y_1}\psi(Y),\ D_{y_2}\psi(Y)\big). \tag{3.6}$$

By Eqs. (2.7), (3.5) and (3.6), we get

$$\begin{aligned} f_{1y_1}(X) &= D_X f_1(X) D_{y_1}\psi(Y) = r_1 - ay, \\ f_{1y_2}(X) &= D_X f_1(X) D_{y_2}\psi(Y) = -ax, \\ f_{2y_1}(X) &= D_X f_2(X) D_{y_1}\psi(Y) = \frac{by^2}{x^2}, \\ f_{2y_2}(X) &= D_X f_2(X) D_{y_2}\psi(Y) = r_2 - \frac{2by}{x} - E + \frac{plEy}{ply - c(l+Ey)^2}. \end{aligned} \tag{3.7}$$

Substituting $X = X_0$ into Eq. (3.7), we obtain

$$\begin{aligned} & f_{1y_1}(X_0) = 0,\ \ f_{1y_2}(X_0) = -ax_0,\ \ f_{2y_1}(X_0) = \frac{by_0^2}{x_0^2}, \\ & f_{2y_2}(X_0) = -\frac{by_0}{x_0} + \frac{plE_0y_0}{ply_0 - c(l+E_0y_0)^2} = 0. \end{aligned} \tag{3.8}$$

By Eq. (3.7), we have

$$D_X f_{1y_1}(X) = (0, -a, 0), D_X f_{1y_2}(X) = (-a, 0, 0), D_X f_{2y_1}(X) = \left(-\frac{2by^2}{x^3}, \frac{2by}{x^2}, 0\right),$$

$$\begin{aligned} D_X f_{2y_2}(X) = \Bigg(& \frac{2by}{x^2},\ -\frac{2b}{x} + \frac{plE}{ply - c(l+Ey)^2} - \frac{p^2l^2Ey}{[ply - c(l+Ey)^2]^2} \\ & + \frac{2pclE^2y(l+Ey)}{[ply - c(l+Ey)^2]^2}, -1 + \frac{ply}{ply - c(l+Ey)^2} + \frac{2pclEy^2(l+Ey)}{[ply - c(l+Ey)^2]^2}\Bigg). \end{aligned} \tag{3.9}$$

In view of Eqs. (2.7), (3.6) and (3.9), we obtain

$$
\begin{aligned}
&f_{1y_1y_1}(X)=D_X f_{1y_1}(X)D_{y_1}\psi(Y)=0, f_{1y_1y_2}(X)=D_X f_{1y_1}(X)D_{y_2}\psi(Y)=-a,\\
&f_{1y_2y_2}(X)=D_X f_{1y_2}(X)D_{y_2}\psi(Y)=0, f_{2y_1y_1}(X)=D_X f_{2y_1}(X)D_{y_1}\psi(Y)=-\frac{2by^2}{x^3},\\
&f_{2y_1y_2}(X)=D_X f_{2y_1}(X)D_{y_2}\psi(Y)=\frac{2by}{x^2},\\
&f_{2y_2y_2}(X)=D_X f_{2y_2}(X)D_{y_2}\psi(Y)=-\frac{2b}{x}+\frac{2plE}{ply-c(l+Ey)^2}\\
&\qquad -\frac{2p^2l^2Ey}{[ply-c(l+Ey)^2]^2}+\frac{2pclE^2y(l+Ey)}{[ply-c(l+Ey)^2]^2}-\frac{2p^2cl^2E^2y^2(l+Ey)}{[ply-c(l+Ey)^2]^3}.
\end{aligned}
\tag{3.10}
$$

Substituting $X = X_0$ into Eq. (3.10), which yields

$$
\begin{aligned}
&f_{1y_1y_1}(X_0) = 0,\ \ f_{1y_1y_2}(X_0) = -a,\ \ f_{1y_2y_2}(X_0) = 0,\\
&f_{2y_1y_1}(X_0) = -\frac{2by_0^2}{x_0^3},\ \ f_{2y_1y_2}(X_0) = \frac{2by_0}{x_0^2},\\
&f_{2y_2y_2}(X_0) = -\frac{2p^2l^2E_0y_0}{[ply_0-c(l+E_0y_0)^2]^2}+\frac{2pclE_0^2y_0(l+E_0y_0)}{[ply_0-c(l+E_0y_0)^2]^2}\\
&\qquad -\frac{2p^2cl^2E_0^2y_0^2(l+E_0y_0)}{[ply_0-c(l+E_0y_0)^2]^3}.
\end{aligned}
\tag{3.11}
$$

Besides, in view of Eqs. (3.6) and (3.10), we have

$$
\begin{aligned}
&D_X f_{1y_1y_1}(X_0) = D_X f_{1y_1y_2}(X_0) = D_X f_{1y_2y_2}(X_0) = (0,0,0),\\
&D_X f_{2y_1y_1}(X_0)=\left(\frac{6by_0^2}{x_0^4},-\frac{4by_0}{x_0^3},0\right),\ D_X f_{2y_1y_2}(X_0)=\left(-\frac{4by_0}{x_0^3},\frac{2b}{x_0^2},0\right),\\
&D_X f_{2y_2y_2}(X_0)=\left(\frac{2b}{x_0^2},\frac{2plE_0(3clE_0+4E_0^2y_0-2pl)}{[ply_0-c(l+E_0y_0)^2]^2}\right.\\
&\quad+\frac{2p^2l^2E_0y_0(2pl-9cE_0^2y_0)}{[ply_0-c(l+E_0y_0)^2]^3}+\frac{8pc^2lE_0^3y_0(l+E_0y_0)^2-16p^2cl^3E_0^2y_0}{[ply_0-c(l+E_0y_0)^2]^3}\\
&\quad+\frac{6p^2cl^2E_0^2y_0^2(l+E_0y_0)[pl-2cE_0(l+E_0y_0)]}{[ply_0-c(l+E_0y_0)^2]^4},\\
&\frac{2pl}{ply_0-c(l+E_0y_0)^2}+\frac{2ply_0[4clE_0+5cE_0^2y_0-pl]}{[ply_0-c(l+E_0y_0)^2]^2}+\frac{8pc^2lE_0^2y_0^2(l+E_0y_0)^2}{[ply_0-c(l+E_0y_0)^2]^3}\\
&\quad\left.-\frac{2p^2l^2E_0y_0^2(6cl+7cE_0y_0)}{[ply_0-c(l+E_0y_0)^2]^3}-\frac{12p^2c^2l^2E_0^2y_0^3(l+E_0y_0)^2}{[ply_0-c(l+E_0y_0)^2]^4}\right),
\end{aligned}
$$

$$
D_Y\psi(0)=\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & -\dfrac{plE_0}{ply_0-c(l+E_0y_0)^2}\end{pmatrix}:=\big(D_{y_1}\psi(0),\ D_{y_2}\psi(0)\big).
\tag{3.12}
$$

Furthermore, Eqs. (2.7) and (3.12) can give that

$$
\begin{aligned}
&f_{1y_1y_1y_1}(X_0) = D_X f_{1y_1y_1}(X_0) D_{y_1}\psi(0) = 0,\\
&f_{1y_1y_1y_2}(X_0) = D_X f_{1y_1y_1}(X_0) D_{y_2}\psi(0) = 0,\\
&f_{1y_1y_2y_2}(X_0) = D_X f_{1y_1y_2}(X_0) D_{y_2}\psi(0) = 0,\\
&f_{1y_2y_2y_2}(X_0) = D_X f_{1y_2y_2}(X_0) D_{y_2}\psi(0) = 0,\\
&f_{2y_1y_1y_1}(X_0) = D_X f_{2y_1y_1}(X_0) D_{y_1}\psi(0) = \frac{6by_0^2}{x_0^4},\\
&f_{2y_1y_1y_2}(X_0) = D_X f_{2y_1y_1}(X_0) D_{y_2}\psi(0) = -\frac{4by_0}{x_0^3},\\
&f_{2y_1y_2y_2}(X_0) = D_X f_{2y_1y_2}(X_0) D_{y_2}\psi(0) = \frac{2b}{x_0^2},\\
&f_{2y_2y_2y_2}(X_0) = D_X f_{2y_2y_2}(X_0) D_{y_2}\psi(0) = \frac{2plE_0(3clE_0 + 4E_0^2y_0 - 3pl)}{[ply_0 - c(l+E_0y_0)^2]^2}\\
&\qquad + \frac{2p^2l^2E_0y_0(3pl - 14cE_0^2y_0 - 12clE_0) + 8pc^2lE_0^3y_0(l+E_0y_0)^2}{[ply_0 - c(l+E_0y_0)^2]^3}\\
&\qquad + \frac{2p^2cl^2E_0^2y_0^2(l+E_0y_0)[3pl - 10cE_0(l+E_0y_0)]}{[ply_0 - c(l+E_0y_0)^2]^4}\\
&\qquad + \frac{2p^3l^3E_0^2y_0^2(6cl + 7cE_0y_0)}{[ply_0 - c(l+E_0y_0)^2]^4} + \frac{12p^3c^2l^3E_0^3y_0^3(l+E_0y_0)^2}{[ply_0 - c(l+E_0y_0)^2]^5}.
\end{aligned}
\tag{3.13}
$$

Substituting Eqs. (3.8), (3.11) and (3.13) into Taylor series developments (3.4), we derive

$$
\begin{cases}
\dot{y}_1 = -ax_0y_2 - ay_1y_2,\\
\dot{y}_2 = \dfrac{by_0^2}{x_0^2}y_1 - \dfrac{by_0^2}{x_0^3}y_1^2 + \dfrac{2by_0}{x_0^2}y_1y_2 + \dfrac{1}{2}f_{2y_2y_2}(X_0)y_2^2 + \dfrac{by_0^2}{x_0^4}y_1^3\\
\qquad - \dfrac{2by_0}{x_0^3}y_1^2y_2 + \dfrac{b}{x_0^2}y_1y_2^2 + \dfrac{1}{6}f_{2y_2y_2y_2}(X_0)y_2^3 + o(|Y|^4).
\end{cases}
\tag{3.14}
$$

In view of the required form (3.3), we need to make a matrix transformation — viz. $Y = \mathrm{T}Z$ for system (3.14), where $Z = (z_1, z_2)^T$, $\mathrm{T}_{2\times 2}$ is an invertible matrix and satisfies

$$
\mathrm{T}^{-1}\begin{pmatrix} 0 & -ax_0 \\ \dfrac{by_0^2}{x_0^2} & 0 \end{pmatrix}\mathrm{T} = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}.
$$

By computing, we can get $\mathrm{T} = \begin{pmatrix} x_0^{\frac{3}{2}} & 0 \\ 0 & \sqrt{\dfrac{b}{a}}y_0 \end{pmatrix}$. For convenience, Z is denoted as Y. Accordingly, we obtain the normal form of system (3.14):

$$
(3.15)\quad \begin{cases} \dot{y}_1 = -\omega_0 y_2 - \sqrt{ab} y_0 y_1 y_2, \\ \dot{y}_2 = \omega_0 y_1 - \sqrt{ab} y_0 y_1^2 + \dfrac{2by_0}{\sqrt{x_0}} y_1 y_2 + \dfrac{y_0}{2}\sqrt{\dfrac{b}{a}} f_{2y_2y_2}(X_0) y_2^2 + \sqrt{abx_0} y_0 y_1^3 \\ \qquad -2by_0 y_1^2 y_2 + \dfrac{b^{\frac{3}{2}} y_0}{\sqrt{ax_0}} y_1 y_2^2 + \dfrac{by_0^2}{6a} f_{2y_2y_2y_2}(X_0) y_2^3 + o(|Y|^4). \end{cases}
$$

Summarizing the above analysis, we have the following Hopf bifurcation theorem.

Theorem 3.1. *For model* (1.3), *there exist a small neighborhood* Ω *of equilibrium point* $X_0(v)$ *as well as a small positive constant* γ.

Case I. If

$$y_0 f_{2y_2y_2y_2}(X_0) > 2 f_{2y_2y_2}(X_0),$$

then

(i) *when* $v_0 < v < v_0 + \gamma$, $X_0(v)$ *is unstable, which excludes the points in* Ω;

(ii) *when* $v_0 - \gamma < v < v_0$, *there exists a periodic orbit in* $\Omega \setminus \{X_0(v)\}$, *besides* $X_0(v)$ *is locally asymptotically stable, which attracts the points in* Ω;

Case II. If

$$y_0 f_{2y_2y_2y_2}(X_0) < 2 f_{2y_2y_2}(X_0),$$

then

(i) *when* $v_0 - \gamma < v < v_0$, $X_0(v)$ *is locally asymptotically stable, which attracts the points in* Ω;

(ii) *when* $v_0 < v < v_0 + \gamma$, *there exists a periodic orbit in* $\Omega \setminus \{X_0(v)\}$, *besides* $X_0(v)$ *is unstable, which excludes the points in* Ω.

Proof. In terms of the Hopf bifurcation theorem in the literature [33], we need to calculate the important quantity $16\varrho_0$ (see below), in view of the normal forms (3.3) and (3.15), we have

$$
\begin{aligned}
16\varrho_0 :=& \{a_{11}^1(a_{12}^1 - a_{11}^2) + a_{22}^2(a_{22}^1 - a_{12}^2) + (a_{12}^1 a_{22}^1 - a_{11}^2 a_{12}^2)\}/\omega_0 \\
&+ (a_{111}^1 + a_{122}^1 + a_{112}^2 + a_{222}^2) \\
=& \left\{ \sqrt{\frac{b}{a}} y_0 f_{2y_2y_2}(X_0) \left(0 - \frac{2by_0}{\sqrt{x_0}}\right) + 2\sqrt{ab} y_0 \cdot \frac{2by_0}{\sqrt{x_0}} \right\} \Big/ \left\{ \sqrt{\frac{ab}{x_0}} y_0 \right\} \\
&- 4by_0 + \frac{by_0^2}{a} f_{2y_2y_2y_2}(X_0) \\
=& -\frac{2by_0}{a} f_{2y_2y_2}(X_0) + \frac{by_0^2}{a} f_{2y_2y_2y_2}(X_0).
\end{aligned}
$$

Next, the two cases $16\varrho_0 > 0$ and $16\varrho_0 < 0$ need further discussion. Because the rest of the process is quite similar to Ref. [33], and therefore it is eliminated in this paper. $\square$

4. Stability analysis for centre

In view of Eq. (2.9), when $by_0/x_0 = plE_0y_0/[ply_0 - c(l + E_0y_0)^2]$ (i.e., $v = v_0$), the eigenvalues of Eq. (2.9) are a pair of imaginary roots: $\pm i\sqrt{ab/x_0}y_0$. That is to say, the equilibrium point X_0 is a centre. Nevertheless, for $v = v_0$, Theorems 2.1 and 3.1 don't include the corresponding stability result. In this section, we study the stability of the centre.

Theorem 4.1. *When $v = v_0$,*

(*i*) *if*

$$y_0 f_{2y_2y_2y_2}(X_0) > 2f_{2y_2y_2}(X_0),$$

then the centre X_0 of model (1.3) *is unstable*;

(*ii*) *if*

$$y_0 f_{2y_2y_2y_2}(X_0) < 2f_{2y_2y_2}(X_0),$$

then the centre X_0 of model (1.3) *is stable.*

Proof. First of all, we need to make system (3.15) equivalent to the following form according to the formal series approach [33, 41, 42]:

$$\begin{cases} \dot{y}_1 = -y_2 + M_2(y_1, y_2) + M_3(y_1, y_2) + o(|Y|^4), \\ \dot{y}_2 = y_1 + N_2(y_1, y_2) + N_3(y_1, y_2) + o(|Y|^4), \end{cases} \tag{4.1}$$

where $M_i(y_1, y_2)$ and $N_i(y_1, y_2)$ denote the i^{th} degree homogeneous polynomials of y_1 and y_2.

On writing $\bar{t} = \omega_0 t$ in system (3.15), and in this section $\dot{Y}$ denotes the derivative of vector function Y regarding $\bar{t}$, then (3.15) is transformed into

$$\begin{cases} \dot{y}_1 = -y_2 - \dfrac{\sqrt{ab}y_0}{\omega_0} y_1 y_2, \\ \dot{y}_2 = y_1 - \dfrac{\sqrt{ab}y_0}{\omega_0} y_1^2 + \dfrac{2by_0}{\omega_0\sqrt{x_0}} y_1 y_2 + \dfrac{y_0}{2\omega_0}\sqrt{\dfrac{b}{a}} f_{2y_2y_2}(X_0) y_2^2 + \dfrac{\sqrt{abx_0}y_0}{\omega_0} y_1^3 \\ \qquad - \dfrac{2by_0}{\omega_0} y_1^2 y_2 + \dfrac{b^{\frac{3}{2}} y_0}{\omega_0\sqrt{ax_0}} y_1 y_2^2 + \dfrac{by_0^2}{6a\omega_0} f_{2y_2y_2y_2}(X_0) y_2^3 + o(|Y|^4). \end{cases} \tag{4.2}$$

Next, we consider the following formal series for the above system (4.2):

$$V(y_1, y_2) = y_1^2 + y_2^2 + \sum_{n=3}^{\infty} V_n(y_1, y_2),$$

where $V_n(y_1, y_2)$ denotes the n^{th} degree homogeneous polynomials of y_1 and y_2. We then have

$$\begin{aligned}\left.\frac{\mathrm{d}V(y_1,y_2)}{\mathrm{d}\bar{t}}\right|_{(4.2)} &= \frac{\partial V(y_1,y_2)}{\partial y_1}\cdot\dot{y}_1+\frac{\partial V(y_1,y_2)}{\partial y_2}\cdot\dot{y}_2\\ &= \left(2y_1+\sum_{n=3}^{\infty}\frac{\partial V_j(y_1,y_2)}{\partial y_1}\right)\left(-y_2-\frac{\sqrt{ab}y_0}{\omega_0}y_1y_2\right)+\left(2y_2+\sum_{n=3}^{\infty}\frac{\partial V_j(y_1,y_2)}{\partial y_2}\right)\\ &\times\left(y_1-\frac{\sqrt{ab}y_0}{\omega_0}y_1^2+\frac{2by_0}{\omega_0\sqrt{x_0}}y_1y_2+\frac{y_0}{2\omega_0}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)y_2^2\right.\\ &\left.+\frac{\sqrt{abx_0}y_0}{\omega_0}y_1^3-\frac{2by_0}{\omega_0}y_1^2y_2+\frac{b^{\frac{3}{2}}y_0}{\omega_0\sqrt{ax_0}}y_1y_2^2+\frac{by_0^2}{6a\omega_0}f_{2y_2y_2y_2}(X_0)y_2^3+\cdots\right).\end{aligned}\tag{4.3}$$

Setting the 3^{th} degree homogeneous polynomial in Eq. (4.3) to 0, we obtain

$$\begin{aligned}&y_1\frac{\partial V_3(y_1,y_2)}{\partial y_2}-y_2\frac{\partial V_3(y_1,y_2)}{\partial y_1}\\ &=\frac{4\sqrt{ab}y_0}{\omega_0}y_1^2y_2-\frac{4by_0}{\omega_0\sqrt{x_0}}y_1y_2^2-\frac{y_0}{\omega_0}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)y_2^3.\end{aligned}\tag{4.4}$$

Let $y_1 = r\cos\theta$, $y_2 = r\sin\theta$, then by chain rule we can get

$$y_1\frac{\partial V_n(y_1,y_2)}{\partial y_2}-y_2\frac{\partial V_n(y_1,y_2)}{\partial y_1}=\frac{\partial V_n(y_1,y_2)}{\partial\theta}=r^n\cdot\frac{\mathrm{d}V_n(\cos\theta,\sin\theta)}{\mathrm{d}\theta}.\tag{4.5}$$

In view of Eqs. (4.4) and (4.5), we have

$$\begin{aligned}\frac{\mathrm{d}V_3(\cos\theta,\sin\theta)}{\mathrm{d}\theta}=&\frac{4\sqrt{ab}y_0}{\omega_0}\cos^2\theta\sin\theta-\frac{4by_0}{\omega_0\sqrt{x_0}}\cos\theta\sin^2\theta\\ &-\frac{y_0}{\omega_0}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)\sin^3\theta\\ :=&-H_3(\cos\theta,\sin\theta)=\frac{\sigma_0}{2}+\sum_{\delta=1}^{\infty}(a_\delta\cos\delta\theta+b_\delta\sin\delta\theta),\end{aligned}\tag{4.6}$$

where $(\sigma_0/2)+\sum_{\delta=1}^{\infty}(a_\delta\cos\delta\theta+b_\delta\sin\delta\theta)$ is the Fourier series of H_3. Such a $V_3(\cos\theta,\sin\theta)$ exists if and only if $\sigma_0 = 0$, viz., $\int_0^{2\pi}H_3(\cos\theta,\sin\theta)\,\mathrm{d}\theta = 0$. Indeed,

$$\int_0^{2\pi}\left\{\frac{4\sqrt{ab}y_0}{\omega_0}\cos^2\theta\sin\theta-\frac{4by_0}{\omega_0\sqrt{x_0}}\cos\theta\sin^2\theta-\frac{y_0}{\omega_0}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)\sin^3\theta\right\}\mathrm{d}\theta=0.$$

Hence, $V_3(y_1, y_2)$ exists, and by Eq. (4.6) we derive

$$\begin{aligned}V_3(y_1,y_2)=&\left(\frac{2y_0}{3\omega_0}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)-\frac{4\sqrt{ab}y_0}{3\omega_0}\right)y_1^3\\ &+\frac{y_0}{\omega_0}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)y_1y_2^2-\frac{4by_0}{3\omega_0\sqrt{x_0}}y_2^3.\end{aligned}\tag{4.7}$$

Again, setting 4^{th} degree homogeneous polynomial in Eq. (4.3) to 0, which yields

$$
\begin{aligned}
&y_1\frac{\partial V_4(y_1,y_2)}{\partial y_2}-y_2\frac{\partial V_4(y_1,y_2)}{\partial y_1}=-\frac{2\sqrt{abx_0}y_0}{\omega_0}y_1^3y_2+\frac{4by_0}{\omega_0}y_1^2y_2^2\\
&-\frac{2b^{\frac{3}{2}}y_0}{\omega_0\sqrt{ax_0}}y_1y_2^3-\frac{by_0^2}{3a\omega_0}f_{2y_2y_2y_2}(X_0)y_2^4+\frac{\sqrt{ab}y_0}{\omega_0}y_1y_2\cdot\frac{\partial V_3(y_1,y_2)}{\partial y_1}\\
&+\left(\frac{\sqrt{ab}y_0}{\omega_0}y_1^2-\frac{2by_0}{\omega_0\sqrt{x_0}}y_1y_2-\frac{y_0}{2\omega_0}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)y_2^2\right)\cdot\frac{\partial V_3(y_1,y_2)}{\partial y_2}.
\end{aligned}
\tag{4.8}
$$

Furthermore, setting $y_1=r\cos\theta$, $y_2=r\sin\theta$ in Eq. (4.8), which leads to

$$
\begin{aligned}
&\frac{\mathrm{d}V_4(\cos\theta,\sin\theta)}{\mathrm{d}\theta}=\left(-\frac{2\sqrt{abx_0}y_0}{\omega_0}+\frac{2by_0^2}{\omega_0^2}f_{2y_2y_2}(X_0)-\frac{4aby_0}{\omega_0^2}+\frac{2by_0^2}{\omega_0^2}\right)\cos^3\theta\sin\theta\\
&+\left(\frac{4by_0}{\omega_0}-\frac{4b\sqrt{ab}y_0^2}{\omega_0^2\sqrt{x_0}}-\frac{4by_0^2}{\omega_0^2\sqrt{x_0}}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)\right)\cos^2\theta\sin^2\theta\\
&+\left(\frac{by_0^2}{\omega_0^2}f_{2y_2y_2}(X_0)-\frac{2b^{\frac{3}{2}}y_0}{\omega_0\sqrt{ax_0}}+\frac{8b^2y_0^2}{\omega_0^2x_0}-\frac{by_0^2}{a\omega_0^2}\left(f_{2y_2y_2}(X_0)\right)^2\right)\cos\theta\sin^3\theta\\
&+\left(\frac{2by_0^2}{\omega_0^2\sqrt{x_0}}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)-\frac{by_0^2}{3a\omega_0}f_{2y_2y_2y_2}(X_0)\right)\sin^4\theta\\
&:=-H_4(\cos\theta,\sin\theta).
\end{aligned}
$$

Similarly, such a $V_4(\cos\theta,\sin\theta)$ exists if and only if $\int_0^{2\pi}H_4(\cos\theta,\sin\theta)\mathrm{d}\theta=0$. But,

$$
\begin{aligned}
&\int_0^{2\pi}H_4(\cos\theta,\sin\theta)\mathrm{d}\theta\\
&=-\frac{by_0\pi}{\omega_0}+\frac{b\sqrt{ab}y_0^2\pi}{\omega_0^2\sqrt{x_0}}-\frac{by_0^2\pi}{2\omega_0^2\sqrt{x_0}}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)+\frac{by_0^2\pi}{4a\omega_0}f_{2y_2y_2y_2}(X_0)\neq 0.
\end{aligned}
$$

There upon we should amend $V_4(\cos\theta,\sin\theta)$ such that

$$
\frac{\mathrm{d}V_4(\cos\theta,\sin\theta)}{\mathrm{d}\theta}=-H_4(\cos\theta,\sin\theta)+\aleph_4:=-\widetilde{H}_4(\cos\theta,\sin\theta),
$$

where $\aleph_4=\frac{1}{2\pi}\int_0^{2\pi}H_4(\cos\theta,\sin\theta)\mathrm{d}\theta=-\frac{by_0}{2\omega_0}+\frac{b\sqrt{ab}y_0^2}{2\omega_0^2\sqrt{x_0}}-\frac{by_0^2}{4\omega_0^2\sqrt{x_0}}\sqrt{\frac{b}{a}}f_{2y_2y_2}(X_0)$ $+\frac{by_0^2}{8a\omega_0}f_{2y_2y_2y_2}(X_0)$. Substituting $\omega_0=\sqrt{ab/x_0}y_0$ into $\aleph_4$, which yields $\aleph_4=\frac{1}{2a}(\frac{y_0}{2}f_{2y_2y_2y_2}(X_0)-f_{2y_2y_2}(X_0))\neq 0$. Clearly, $\int_0^{2\pi}\widetilde{H}_4(\cos\theta,\sin\theta)\mathrm{d}\theta=0$, therefore the amended $V_4(\cos\theta,\sin\theta)$ exists.

We now construct the Lyapunov function $V(y_1,y_2)=y_1^2+y_2^2+V_3(y_1,y_2)+V_4(y_1,y_2)$ for system (4.2), and further we have

$$
\left.\frac{\mathrm{d}V(y_1,y_2)}{\mathrm{d}\bar{t}}\right|_{(4.2)}=\aleph_4(y_1^2+y_2^2)^2+o\left((y_1^2+y_2^2)^2\right).
$$

If $\aleph_4 > 0$ (viz., $y_0 f_{2y_2y_2y_2}(X_0) > 2f_{2y_2y_2}(X_0)$), then the equilibrium point $(0,0)^T$ of system (4.2) is unstable, consequently the centre X_0 is unstable. On the contrary, if $\aleph_4 < 0$ (viz., $y_0 f_{2y_2y_2y_2}(X_0) < 2f_{2y_2y_2}(X_0)$), then the equilibrium point $(0,0)^T$ of system (4.2) is stable, hence the centre X_0 is stable. □

Remark 4.1. Due to $\int_0^{2\pi} H_{2\mu-1}(\cos\theta, \sin\theta)\mathrm{d}\theta = 0$, $\mu = 2, 3, \cdots$, so if $\int_0^{2\pi} H_4(\cos\theta, \sin\theta)\mathrm{d}\theta = 0$, then we should find the minimum positive integer ϑ such that $\int_0^{2\pi} H_{2\vartheta}(\cos\theta, \sin\theta)\mathrm{d}\theta \neq 0$, and then afterwards, amending the corresponding function $V_{2\vartheta}(\cos\theta, \sin\theta)$ similar to $V_4(\cos\theta, \sin\theta)$.

5. Numerical simulations

In this section, we perform several Matlab simulations to complement the analytical results above.

As an example, we consider the harvested predator-prey model (1.3) with the coefficients $r_1 = 2,\ a = 1,\ r_2 = \frac{3}{4},\ b = 1,\ p = 1,\ l = 1,\ c = \frac{4}{9}$. Then by the analysis in section 2, we can find that model (1.3) has a positive equilibrium point $X_0 = (4, 2, 0.25)$ and the bifurcation value $v_0 = 2/9$. We can check that model (1.3) satisfies the requirement (2.1), the transversality conditions for Hopf bifurcation in (3.2), as well as the condition of case (i) in Theorems 3.1 and 4.1.

In accordance with Theorems 3.1 and 4.1 (on choosing $\gamma = 0.002$), we present four groups of numerical simulations as follows:

(i) The equilibrium point X_0 is locally asymptotically stable when $v = 0.2205 < v_0$, which is verified as shown in Fig. 1. In this case, the prey species, predator species and economic harvesting are in a stable state, so the ecological balance can be maintained.

(ii) A Hopf-bifurcating periodic orbit bifurcates from the equilibrium point X_0 when $v = 0.222222 < v_0$, which is verified as shown in Fig. 2. The emergence of the periodic orbit would generate small-amplitude population oscillations in our ecosystem.

(iii) The centre X_0 is unstable when v equals to $v_0 = 2/9$, which is verified as shown in Fig. 3. Unstable center means that the aforementioned population oscillations are growing as time t goes on, i.e., the prey species, predator species and economic harvesting can't coexist in an oscillatory mode.

(iv) The equilibrium point X_0 is unstable when $v = 0.223 > v_0$, which is verified as shown in Fig. 4. At this moment, the biological populations and harvesting effort are unstable, which can result in ecological unbalance.

From Figs. 1-4, it is clear that our harvested predator-prey model can exhibit a Hopf bifurcation as the increase of the harvesting profit v, which can cause potentially dramatic variations in the dynamical behaviors of the population model. Hence, the Hopf bifurcation is biologically important.

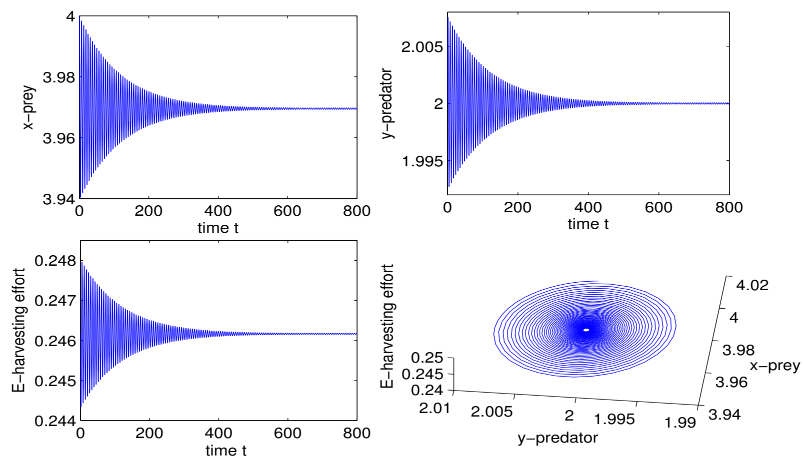

Figure 1: For the parameters of model (1.3) with the values $r_1 = 2$, $a = 1$, $r_2 = \frac{3}{4}$, $b = 1$, $p = 1$, $l = 1$, $c = \frac{4}{9}$, $x(0) = 3.9999$, $y(0) = 1.9999$, $E(0) = 0.2499$, numerical simulations show that the equilibrium point $X_0 = (4, 2, 0.25)$ of model (1.3) is locally asymptotically stable when $v = 0.2205 < v_0 = 2/9$.

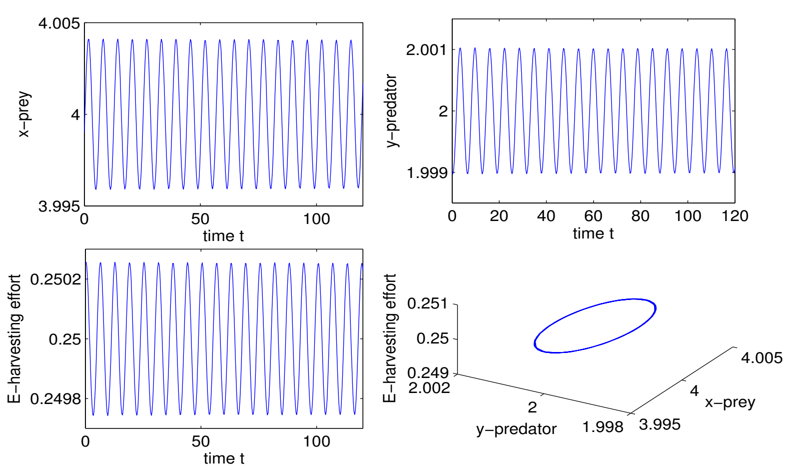

Figure 2: For the parameters of model (1.3) with the values $r_1 = 2$, $a = 1$, $r_2 = \frac{3}{4}$, $b = 1$, $p = 1$, $l = 1$, $c = \frac{4}{9}$, $x(0) = 3.999$, $y(0) = 1.999$, $E(0) = 0.249$, numerical simulations show that a periodic orbit bifurcates from the equilibrium point $X_0 = (4, 2, 0.25)$ of model (1.3) when $v = 0.222222 < v_0 = 2/9$.

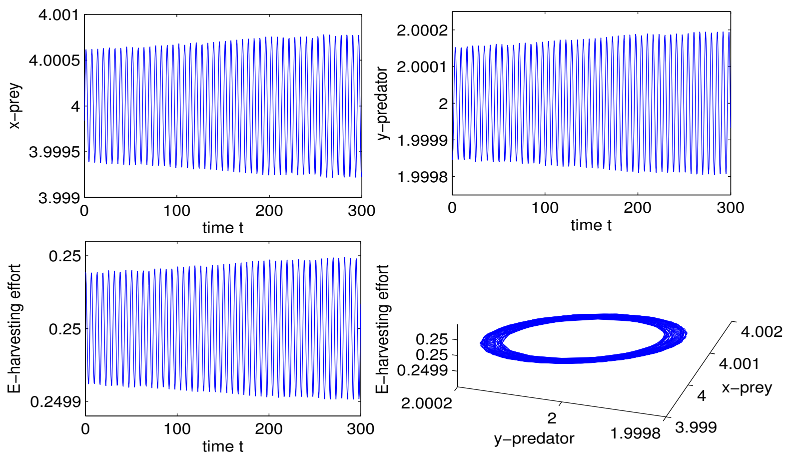

Figure 3: For the parameters of model (1.3) with the values $r_1 = 2$, $a = 1$, $r_2 = \frac{3}{4}$, $b = 1$, $p = 1$, $l = 1$, $c = \frac{4}{9}$, $x(0) = 3.99984$, $y(0) = 1.99985$, $E(0) = 0.24986$, numerical simulations show that the equilibrium point $X_0 = (4, 2, 0.25)$ of model (1.3) is an unstable centre when v equals to the bifurcation value $v_0 = 2/9$.

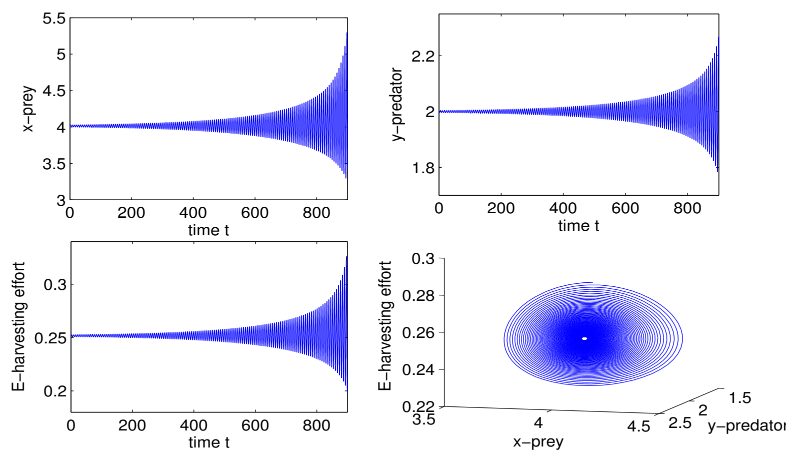

Figure 4: For the parameters of model (1.3) with the values $r_1 = 2$, $a = 1$, $r_2 = \frac{3}{4}$, $b = 1$, $p = 1$, $l = 1$, $c = \frac{4}{9}$, $x(0) = 3.9999$, $y(0) = 1.9999$, $E(0) = 0.2499$, numerical simulations show that the equilibrium point $X_0 = (4, 2, 0.25)$ of model (1.3) is unstable when $v = 0.223 > v_0 = 2/9$.

6. Concluding remarks

The present paper has studied the dynamics of a predator-prey model with external harvesting for predators. The original predator-prey model (1.1) proposed by Leslie is described by two differential equations, which has been reasonably modified as the differential-algebra predator-prey system (1.3) on the basis of the consideration of expressing the harvesting profit. The asymptotic stability of the modified predator-prey model (1.3) is investigated here, which reveals that the population model can be asymptotically stable under certain condition. In such a circumstance, the prey population, predator population and human harvesting are able to coexist in harmony. For the benefit of maintaining the ecological balance, the rational range of the harvesting profit v of human beings should be the interval $(0, v_0)$. It means that people can't exploit the biological resource too heavy. Otherwise, the ecological balance would be in danger of being damaged, and then people will completely loss their productivity eventually.

Besides, it is interesting to note that the parameterisation used in section 2 can reduce our model (1.3) described by differential-algebra equations to the system (3.14) of differential equations, which has a significant effect in this study. Refs. [34-37] suggest that Differential-Algebraic Equations have widespread applications in constrained dynamical systems, so we expect that the parameterisation can be employed to analyze the dynamics of more complex constrained systems in biology and engineering.

Finally, Refs. [43, 44] show that the impact of delays on the dynamics of a system is an interesting problem. Thus, further studies on the stability and bifurcations of differential-algebra population model (1.3) with delays can be considered.

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Appendix

Here we deduce the formula (2.7). Substituting $X = \psi(Y)$ into system (2.4), we have

$$D_Y \psi(Y) \dot{Y} = f(\psi(Y)), \tag{A.1}$$

Next, differentiating Eq. (2.5) regarding Y and then left multiplying U_0^T to the differentiated equation, which lead to

$$U_0^T D_Y \psi(Y) = I_2. \tag{A.2}$$

Differentiating Eq. (2.6) regarding Y, which yields

$$D_X g(X) D_Y \psi(Y) = 0. \tag{A.3}$$

By Eqs. (A.1)-(A.3), we get

$$\begin{pmatrix} D_X g(X) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_2 \end{pmatrix} \dot{Y}(t) = f(\psi(Y)). \tag{A.4}$$

Further, Eqs. (A.1), (A.3) and (A.4) suggest that system (2.4) can be locally equivalent to

$$\dot{Y} = U_0^T f(\psi(Y)), \tag{A.5}$$

which shows that X_0 corresponds to $Y = 0$ of system (A.5).

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On one-sided MPCEP-inverse for matrices of an arbitrary index

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Abstract. One-sided MPCEP-inverse for matrices was introduced in this paper. The MPCEP-inverse can be described by using the core part A_1 in Core-EP decomposition of A and the Moore-Penrose inverse of A. The MPCEP-inverse of A coincides with the $(A^{\dagger}A^k, (A^k)^*)$-inverse of A. In addition, the CE matrix was introduced, a necessary and sufficient condition such that a matrix A to be a CE matrix is the MPCEP-inverse of A commutes with A.

Keywords: MPCEP-inverse, Core-EP decomposition, CE matrix.

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1. Introduction

Let $\mathbb{C}$ be the complex filed. The set $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ matrices over $\mathbb{C}$. Let $A \in \mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. Notations $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$, $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$ and $\mathbb{C}_n^{CM} = \{A \in \mathbb{C}^{n\times n} | \operatorname{rank}(A) = \operatorname{rank}(A^2)\}$ will be used in the sequel. The smallest positive integer k such that $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$ is called the index of $A \in \mathbb{C}^{n\times n}$ and denoted by $\operatorname{ind}(A)$.

Let $A \in \mathbb{C}^{m\times n}$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies

$$AXA = A,\ XAX = X,\ (AX)^* = AX \text{ and } (XA)^* = XA,$$

then X is called the Moore-Penrose inverse of A [11, 15] and denoted by $X = A^\dagger$.We call X is an inner inverse of A, if we have $AXA = A$. The set $A\{1\}$ denotes the set of all inner inverse of A. We call X is a $\{1, 4\}$ inverse of A, if we have $AXA = A$ and $(XA)^* = XA$. The set $A\{1, 4\}$ denotes the set of all $\{1, 4\}$ inverse of A. The Moore-Penrose can be used to represent orthogonal projectors $P_A \triangleq AA^\dagger$ and $Q_A \triangleq A^\dagger A$ onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. Let $A, X \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, algebraic definition of the Drazin inverse as follows: if

$$X = XAX,\ XA^{k+1} = A^k \text{ and } AX = XA,$$

then X is called a Drazin inverse of A. It is unique and denoted by A^D [4]. Note that, for a square complex matrix, the algebraic definition of the Drazin inverse is equivalent to the functional definition of the Drazin inverse. If $\operatorname{ind}(A) = 1$, the Drazin inverse is called the group inverse of A and denoted by $A^\#$. The core inverse and the dual core inverse for a complex matrix were introduced by Baksalary and Trenkler [2]. Let $A \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a core inverse of A, if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where P_A is the orthogonal projector onto $\mathcal{R}(A)$. And if such a matrix exists, then it is unique (and denoted by $A^{\#\!\!\!\bigcirc}$). Baksalary and Trenkler gave several characterizations of the core inverse by using the decomposition of Hartwig and Spindelböck [7]. Let $A \in \mathbb{C}^{n\times n}$, the DMP inverse of A was introduced by using the Drazin and the Moore-Penrose inverses of A in [14], and the formula of the DMP inverse of A is $A^{D,\dagger} = A^D AA^\dagger$ [14, Theorem 2.2]. The CMP inverse of $A \in \mathbb{C}^{n\times n}$ was introduced by Mehdipour and Salemi in [13], who using the core part in core-nilpotent decomposition of A and the Moore-Penrose inverse of A, the CMP inverse of A was denoted by $A^{c,\dagger}$. Manjunatha Prasad and Mohana [12] introduced the core-EP inverse of matrix [12, Definition 3.1]. Let $A \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that $XAX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, then X is called the core-EP inverse of A. If such inverse exists, then it is unique and denoted by $A^{\dagger\!\!\!\bigcirc}$. The concept of the MPCEP-inverse of a Hilbert space operators was initially introduced by Chen, Mosić and Xu [3] and this concept was expanded on quaternion matrices by Kyrchei, Mosić and Stanimirović [8, 9]. Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If there exists a matrix $X \in \mathbb{C}^{n\times n}$ such that

$$XAX = X,\ AX = AA^{\dagger\!\!\!\bigcirc} \text{ and } XA = A^\dagger AA^{\dagger\!\!\!\bigcirc}A$$

then X is called the MPCEP-inverse of A and denoted by $A^{\dagger,\textcircled{\dagger}}$.

In [18, Theorem 2.1], Wang introduced a new matrix decomposition, namely the Core-EP decomposition of $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}\,(A) = k$. Given a matrix $A \in \mathbb{C}^{n\times n}$, then A can be written as the sum of matrices $A_1 \in \mathbb{C}^{n\times n}$ and $A_2 \in \mathbb{C}^{n\times n}$, that is $A = A_1 + A_2$, where $A_1 \in \mathbb{C}_n^{CM}$, $A_2^k = 0$ and $A_1^* A_2 = A_2 A_1 = 0$. In [18, Theorem 2.3 and Theorem 2.4], Wang proved this matrix decomposition is unique and there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*, \tag{1}$$

where $T \in \mathbb{C}^{r\times r}$ is nonsingular and $N \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent with $\mathrm{rank}\,(A^k) = r$.

Let $A, B, C \in \mathbb{C}^{n\times n}$. We say that $Y \in \mathbb{C}^{n\times n}$ is a (B,C)-inverse of A if we have

$$YAB = B,\ CAY = C,\ \mathcal{N}(C) \subseteq \mathcal{N}(Y) \text{ and } \mathcal{R}(Y) \subseteq \mathcal{R}(B).$$

If such Y exists, then it is unique (see [1, Definition 4.1] and [16, Definition 1.2]). Note that, the (B,C)-inverse was introduced in the setting of semigroups [5].

In [6, Definition 1.2] and [10, Definition 2.1], the authors introduced the one-sided (b,c)-inverse in rings. In [1, Definition 2.7], the authors introduced the one-sided (B,C)-inverse for complex matrices. Let $A, B, C \in \mathbb{C}^{n\times n}$. We call that $X \in \mathbb{C}^{n\times n}$ is a left (B,C)-inverse of A if we have $\mathcal{N}(C) \subseteq \mathcal{N}(X)$ and $XAB = B$. We call that $Y \in \mathbb{C}^{n\times n}$ is a right (B,C)-inverse of A if we have $\mathcal{R}(Y) \subseteq \mathcal{R}(B)$ and $CAY = C$.

In fact, there is an important generalized inverse was introduced in [17] by Rao and Mitra. Let $A \in \mathbb{C}^{n\times n}$. In [16], Rakić showed that Rao and Mitra's constrained inverse of A coincides with the (B,C)-inverse of A, where $B, C \in \mathbb{C}^{n\times n}$.

In 1972, Rao and Mitra introduced two different types of constraints in order to extend the concept of Bott-Duffin inverse and define a new constrained inverse $Y \in \mathbb{C}^{n\times n}$ of a matrix $A \in \mathbb{C}^{n\times n}$ in [17]. Let $B, C \in \mathbb{C}^{n\times n}$.

Constraints of type 1 :

$\mathfrak{c}:$ Y maps vectors of $\mathbb{C}^m$ into $\mathcal{R}(B)$;

$\mathfrak{r}:$ Y^* maps vectors of $\mathbb{C}^n$ into $\mathcal{R}(C^*)$;

Constraints of type 2 :

$\mathfrak{C}:$ YA is an identity on $\mathcal{R}(B)$;

$\mathfrak{R}:$ $(AY)^*$ is an identity on $\mathcal{R}(C^*)$.

Note that, Rao and Mitra denoted their inverse by $A_{\mathfrak{c}\mathfrak{r}\mathfrak{C}\mathfrak{R}}$. In fact, they defined this inverse in a broader context, where A is an $m \times n$ matrix mapping vectors of $\mathbb{C}^n$ to $\mathbb{C}^m$, where $\mathbb{C}^n$ denotes an n dimensional vector space with an inner product.

Let $A, B, C \in \mathbb{C}^{n\times n}$. A matrix $Y \in \mathbb{C}^{n\times n}$ is a $\mathfrak{cr}^{\mathfrak{CR}}$ *constrained inverse* of A if it satisfies constraints $\mathfrak{c}, \mathfrak{r}, \mathfrak{C}$ and $\mathfrak{R}$. Here the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A will be denoted by $A^{\|(B,C)}$. In the sequel, one can see that the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A coincides with the (B, C)-inverse of A, thus, we use the symbol of the (B, C)-inverse to denoted the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A.

In order to rewrite the constraints $\mathfrak{c}, \mathfrak{r}, \mathfrak{C}$ and $\mathfrak{R}$ in purely multiplicative language, we need the following fact: the condition $\mathcal{R}(Y) \subseteq \mathcal{R}(B)$ if and only if $Y = BK$, for some $K \in \mathbb{C}^{n\times n}$; the condition $\mathcal{R}(Y^*) \subseteq \mathcal{R}(C^*)$ if and only if $\mathcal{N}(C) \subseteq \mathcal{R}(Y)$ if and only if $Y = LC$, for some $L \in \mathbb{C}^{n\times n}$; the constraint C is clearly equivalent to $YAB = B$ and the constraint R is equivalent to $CAY = C$. Therefore, these constraints can be rewritten as follows:

Constraints of type 1 :

$\mathfrak{c}:\ \mathcal{R}(Y) \subseteq \mathcal{R}(B)$;

$\mathfrak{r}:\ \mathcal{R}(Y^*) \subseteq \mathcal{R}(C^*)$;

Constraints of type 2 :

$\mathfrak{C}:\ YAB = B$;

$\mathfrak{R}:\ CAY = C$.

Let $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(A) = r$. Let T, S be two subspaces of $\mathbb{C}^n$ with $\dim(T) = s \leqslant r$ and $\dim(S) = n - r$. Recall that the out inverse $A^{(2)}_{T,S}$ with prescribed the column space T and null space S is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying $AT \oplus S = \mathbb{C}^n$. It is well-known fact that the following ten kinds of generalized inverse are all special cases of the out inverse $A^{(2)}_{T,S}$ with prescribed the column space T and null space S: the Moore-Penrose inverse $A^\dagger$ [11, 15], the Drazin inverse A^D [4], the group inverse $A^\#$ [4], the core inverse $A^{\oplus}$ [2], the DMP-inverse $A^{D,\dagger}$ [14] and the core-EP inverse $A^{\oplus}$ [12]. Thus, all the results related the the out inverse $A^{(2)}_{T,S}$ with prescribed the column space T and null space S are applicable to these generalized inverses.

2. Existence criteria and expressions of one sided MPCEP-inverse

In [18, Theorem 2.3], Wang proved that A_1 can be described by using the Moore-Penrose inverse of A^k. The explicit expressions of A_1 can be found in the follows lemma.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $A = A_1 + A_2$ is the Core-EP decomposition of A, then $A_1 = A^k(A^k)^\dagger A$ and $A_2 = A - A^k(A^k)^\dagger A$.*

Motivated by the ideal of one-sided (B, C)-inverse of A, one-sided MPCEP-inverse was introduced.

Definition 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. We call that $X \in \mathbb{C}^{n\times n}$ is a left MPCEP-inverse of A if we have*

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X) \text{ and } XA^k = A^\dagger A^k. \tag{2}$$

We call that $Y \in \mathbb{C}^{n\times n}$ is a right MPCEP-inverse of A if we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k) \text{ and } (AY)^* A^k = A^k. \tag{3}$$

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^k (A^k)^\dagger$ is a left MPCEP-inverse of A.*

Proof. Let X be a left MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X) \text{ and } XA^k = A^\dagger A^k. \tag{4}$$

Then

$$\begin{aligned} X &= U(A^k)^* \text{ for some } U \in \mathbb{C}^{n\times n} \\ &= U(A^k)^*[(A^k)^*]^\dagger (A^k)^* = X[(A^k)^*]^\dagger (A^k)^* \\ &= X[A^k(A^k)^\dagger]^* = XA^k(A^k)^\dagger = A^\dagger A^k (A^k)^\dagger \end{aligned} \tag{5}$$

by (4). Thus, $A^\dagger A^k (A^k)^\dagger$ is a left MPCEP-inverse of A by (5). □

In the following theorem, a general expression of the left MPCEP-inverse of A was given.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, a general solution of the left MPCEP-inverse of A is*

$$A^\dagger A^k (A^k)^\dagger + V\left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^*\right)^-\right](A^k)^*,$$

for any $V \in \mathbb{C}^{n\times n}$, any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$.

Proof. Let X be a left MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X) \text{ and } XA^k = A^\dagger A^k. \tag{6}$$

Then

$$X = U(A^k)^* \text{ for some } U \in \mathbb{C}^{n\times n}. \tag{7}$$

Hence

$$A^\dagger A^k = XA^k = U(A^k)^* A^k \tag{8}$$

by (6) and (7). That is $A^\dagger A^k = U(A^k)^* A^k$.

Since $\operatorname{rank}\left((A^k)^* A^k\right) = \operatorname{rank}(A^k)$, so one can check that $\left((A^k)^* A^k\right)^- = (A^k)^- \left((A^k)^*\right)^-$, for any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$ as follows:

The condition $\operatorname{rank}\left((A^k)^* A^k\right) = \operatorname{rank}(A^k)$ implies $\mathcal{N}\left((A^k)^* A^k\right) = \mathcal{N}(A^k)$. We have the equality $(A^k)^* A^k [I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k] = 0$ in view of the equality $(A^k)^*A^k\left((A^k)^*A^k\right)^-(A^k)^*A^k=(A^k)^*A^k$, so $I_n-\left((A^k)^*A^k\right)^-(A^k)^*A^k\in\mathcal{N}((A^k)^*A^k)\subseteq\mathcal{N}(A^k)$, thus $A^k[I_n-\left((A^k)^*A^k\right)^-(A^k)^*A^k]=0$, that is

$$A^k=A^k\left((A^k)^*A^k\right)^-(A^k)^*A^k,$$

gives $\left((A^k)^*A^k\right)^-(A^k)^*$ is an inner inverse of A^k.

Since $\left((A^k)^*A^k\right)^-(A^k)^*\in A^k\{1\}$, so let $(A^k)^-=\left((A^k)^*A^k\right)^-(A^k)^*$, then

$$\begin{aligned}(A^k)^*A^k(A^k)^-((A^k)^*)^-(A^k)^*A^k&=(A^k)^*A^k[((A^k)^*A^k)^-(A^k)^*]((A^k)^*)^-(A^k)^*A^k\\&=(A^k)^*A^k((A^k)^*A^k)^-((A^k)^*((A^k)^*)^-(A^k)^*)A^k\\&=(A^k)^*A^k((A^k)^*A^k)^-(A^k)^*A^k\\&=(A^k)^*A^k.\end{aligned}$$

That is, for any $\left((A^k)^*\right)^-\in(A^k)^*\{1\}$ and some $(A^k)^-\in A^k\{1\}$, the equality $\left((A^k)^*A^k\right)^-=(A^k)^-\left((A^k)^*\right)^-$ holds.

Since

$$\begin{aligned}&\left\{A^\dagger\left((A^k)^\dagger\right)^*+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right]\right\}(A^k)^*A^k\\&=A^\dagger\left((A^k)^\dagger\right)^*(A^k)^*A^k+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right](A^k)^*A^k\\&=A^\dagger\left((A^k)^\dagger\right)^*(A^k)^*A^k=A^\dagger\left(A^k(A^k)^\dagger\right)^*A^k\\&=A^\dagger A^k,\end{aligned}$$

hence a general solution of $A^\dagger A^k=U(A^k)^*A^k$ is

$$A^\dagger\left((A^k)^\dagger\right)^*+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right]$$

can be written as

$$A^\dagger\left((A^k)^\dagger\right)^*+V\left[I_n-(A^k)^*A^k(A^k)^-\left((A^k)^*\right)^-\right],$$

for any $V\in\mathbb{C}^{n\times n}$, any $((A^k)^*)^-\in(A^k)^*\{1\}$ and some $(A^k)^-\in A^k\{1\}$. Let $\widetilde{X}=A^\dagger A^k(A^k)^\dagger+V\left[I_n-(A^k)^*A^k(A^k)^-((A^k)^*)^-\right](A^k)^*$. One can check $\widetilde{X}$ is a left MPCEP-inverse of A in what follows.

$$\begin{aligned}\widetilde{X}A^k&=A^\dagger A^k(A^k)^\dagger A^k+V\left[I_n-(A^k)^*A^k(A^k)^-\left((A^k)^*\right)^-\right](A^k)^*A^k\\&=A^\dagger A^k(A^k)^\dagger A^k+V\left[I_n-(A^k)^*A^k\left((A^k)^*A^k\right)^-\right](A^k)^*A^k\\&=A^\dagger A^k+V\left[I_n(A^k)^*A^k-(A^k)^*A^k\left((A^k)^*A^k\right)^-(A^k)^*A^k\right]\\&=A^\dagger A^k.\end{aligned}\tag{9}$$

Since

$$\begin{aligned}
\widetilde{X} &= A^\dagger A^k (A^k)^\dagger + V\left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^*\right)^-\right](A^k)^* \\
&= A^\dagger \left[A^k (A^k)^\dagger\right]^* + V\left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^*\right)^-\right](A^k)^* \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* + V\left[I_n - (A^k)^* A^k (A^k)^- \left((A^k)^*\right)^-\right](A^k)^* \\
&= Q(A^k)^*,
\end{aligned} \tag{10}$$

where $Q = A^\dagger[(A^k)^\dagger]^* + V[I_n - (A^k)^* A^k (A^k)^- ((A^k)^*)^-]$. Hence, (10) gives

$$\widetilde{X} = Q(A^k)^*. \tag{11}$$

The equality in (11) is equivalent to $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(\widetilde{X})$. Thus, $\widetilde{X}$ is a left MPCEP-inverse of A by $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(\widetilde{X})$ and $\widetilde{X} A^k = A^\dagger A^k$ in (9). □

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^k (A^k)^\dagger$ is a right MPCEP-inverse of A.*

Proof. Let Y be a right MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k) \text{ and } (AY)^* A^k = A^k. \tag{12}$$

Then

$$\begin{aligned}
Y &= A^\dagger A^k V \text{ for some } V \in \mathbb{C}^{n\times n} \\
&= A^\dagger A^k (A^k)^\dagger A^k V = A^\dagger \left[A^k (A^k)^\dagger\right]^* A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* A^k V \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* A^* A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* (AA^\dagger A)^* A^k V \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* A^* (AA^\dagger)^* A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^{k-1})^* A^* AA^\dagger A^k V \\
&= A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* AA^\dagger A^k V = A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* AY \\
&= A^\dagger \left[(A^k)^\dagger\right]^* \left[(AY)^* A^k\right]^* = A^\dagger \left[(A^k)^\dagger\right]^* (A^k)^* \\
&= A^\dagger A^k (A^k)^\dagger
\end{aligned} \tag{13}$$

by (12). Thus, $A^\dagger A^k (A^k)^\dagger$ is a right MPCEP-inverse of A by (13). □

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, a general solution of the right MPCEP-inverse of A is*

$$A^\dagger A^k (A^k)^\dagger + A^\dagger A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T,$$

for any $T \in \mathbb{C}^{n\times n}$, any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$.

Proof. Let Y be a right MPCEP-inverse of A. Then, by Definition 2.1, we have

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k) \text{ and } (AY)^* A^k = A^k. \tag{14}$$

Then

$$Y = A^\dagger A^k S \text{ for some } S \in \mathbb{C}^{n\times n}. \tag{15}$$

Hence

$$(A^k)^* = (A^k)^* AY = (A^k)^* AA^\dagger A^k S = (A^k)^* AY = (A^k)^* A^k S \tag{16}$$

by (14) and (15). That is $(A^k)^* = (A^k)^* A^k S$.

Since $\operatorname{rank}\left((A^k)^* A^k\right) = \operatorname{rank}(A^k)$, so $\left((A^k)^* A^k\right)^- = (A^k)^- \left((A^k)^*\right)^-$, for any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$ by the proof Theorem 2.2.

Since

$$\begin{aligned}
&(A^k)^* A^k \left\{(A^k)^\dagger + \left[I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k\right] T\right\} \\
&= (A^k)^* A^k (A^k)^\dagger + (A^k)^* A^k \left[I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k\right] T \\
&= (A^k)^* A^k (A^k)^\dagger = (A^k)^* [A^k (A^k)^\dagger]^* \\
&= (A^k)^*,
\end{aligned}$$

hence a general solution of $(A^k)^* = (A^k)^* A^k S$ is

$$(A^k)^\dagger + \left[I_n - \left((A^k)^* A^k\right)^- (A^k)^* A^k\right] T$$

can be written as

$$(A^k)^\dagger + \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T,$$

for any $T \in \mathbb{C}^{n\times n}$, any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$. Let $\widetilde{Y} = A^\dagger A^k (A^k)^\dagger + A^\dagger A^k [I_n - (A^k)^-((A^k)^*)^-(A^k)^* A^k] T$. One can check $\widetilde{Y}$ is a right MPCEP-inverse of A in what follows.

$$\begin{aligned}
\widetilde{Y} &= A^\dagger A^k (A^k)^\dagger + A^\dagger A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T \\
&= A^\dagger A^k \left\{(A^k)^\dagger + \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\} \\
&= A^\dagger A^k P,
\end{aligned} \tag{17}$$

where $P = (A^k)^\dagger + [I_n - (A^k)^-((A^k)^*)^-(A^k)^* A^k] T$. Hence, (17) gives

$$\widetilde{Y} = A^\dagger A^k P. \tag{18}$$

The following equality will be used in the sequel.

$$
\begin{aligned}
A^k &= A^k(A^k)^\dagger A^k = \left[A^k(A^k)^\dagger\right]^* A^k = \left[(A^k)^\dagger\right]^* (A^k)^* A^k \\
&= \left[(A^k)^\dagger\right]^* (A^k)^* A^k \left[(A^k)^* A^k\right]^- (A^k)^* A^k \\
&= A^k \left[(A^k)^* A^k\right]^- (A^k)^* A^k \\
&= A^k (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k
\end{aligned} \tag{19}
$$

by $\left((A^k)^*A^k\right)^- = (A^k)^-\left((A^k)^*\right)^-$, for any $((A^k)^*)^- \in (A^k)^*\{1\}$ and some $(A^k)^- \in A^k\{1\}$.

Since

$$
\begin{aligned}
(A\widetilde{Y})^*A^k &= \left\{AA^\dagger A^k(A^k)^\dagger + AA^\dagger A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\}^* A^k \\
&= \left\{A^k(A^k)^\dagger + A^k \left[I_n - (A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\}^* A^k \\
&= \left\{A^k(A^k)^\dagger + \left[A^k - A^k(A^k)^- \left((A^k)^*\right)^- (A^k)^* A^k\right] T\right\}^* A^k \\
&= \left[A^k(A^k)^\dagger\right]^* A^k \\
&= A^k
\end{aligned} \tag{20}
$$

by (19). The equality in (18) is equivalent to $\mathcal{R}(\widetilde{Y}) \subseteq \mathcal{R}(A^\dagger A^k)$. Thus, $\widetilde{Y}$ is a right MPCEP-inverse of A by $\mathcal{R}(\widetilde{Y}) \subseteq \mathcal{R}(A^\dagger A^k)$ and $(A\widetilde{Y})^*A^k = A^k$ in (20). □

In the following theorem, we will use the core part A_1 of the Core-EP decomposition to describe the left MPCEP-inverse of A.

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $X \in \mathbb{C}^{n\times n}$ is a left MPCEP-inverse of A if and only if $\mathcal{N}(A_1A^\dagger) \subseteq \mathcal{N}(X)$ and $XAA^\dagger A_1 = A^\dagger A_1$ hold.*

Proof. Firstly, we will prove $\mathcal{N}(A_1A^\dagger) = \mathcal{R}(A^k)^\perp$. Let $u \in \mathcal{N}\big((A^k)^*AA^\dagger\big)$, then

$$
\begin{aligned}
A_1A^\dagger u &= A^k(A^k)^\dagger AA^\dagger u = \left[A^k(A^k)^\dagger\right]^* AA^\dagger u \\
&= \left[(A^k)^\dagger\right]^* (A^k)^* AA^\dagger u = 0
\end{aligned} \tag{21}
$$

by Lemma 2.1. Let $v \in \mathcal{N}(A_1A^\dagger)$, then

$$
\begin{aligned}
(A^k)^*AA^\dagger v &= (A^k)^* \left[(A^k)^*\right]^\dagger (A^k)^* AA^\dagger v = (A^k)^* \left[A^k(A^k)^\dagger\right]^* AA^\dagger v \\
&= (A^k)^* A^k (A^k)^\dagger AA^\dagger v = (A^k)^* A_1 A^\dagger v = 0
\end{aligned} \tag{22}
$$

by Lemma 2.1. So, by (21) and (22) we have

$$\mathcal{N}(A_1A^\dagger) = \mathcal{N}\Big((A^k)^*AA^\dagger\Big) \tag{23}$$

Note that

$$\mathcal{R}(A^k)^\perp = \mathcal{N}\Big((A^k)^*\Big) = \mathcal{N}\Big((AA^\dagger A^k)^*\Big) = \mathcal{N}\Big((A^k)^*AA^\dagger\Big). \tag{24}$$

The equality $\mathcal{N}(A_1A^\dagger) = \mathcal{N}((A^k)^*AA^\dagger)$ in (23) gives $\mathcal{N}(A_1A^\dagger) = \mathcal{R}(A^k)^\perp$ by (24). Hence, $\mathcal{N}(A_1A^\dagger) \subseteq \mathcal{N}(X)$ if and only if $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X)$ by $\mathcal{N}(A_1A^\dagger) = \mathcal{R}(A^k)^\perp$.

Next, we will prove $XAA^\dagger A_1 = A^\dagger A_1$ if and only if $XA^k = A^\dagger A^k$. The condition $XAA^\dagger A_1 = A^\dagger A_1$ can be written as

$$XAA^\dagger A^k(A^k)^\dagger A = A^\dagger A^k(A^k)^\dagger A \tag{25}$$

by Lemma 2.1, (25) can be written as

$$XA^k(A^k)^\dagger A = A^\dagger A^k(A^k)^\dagger A \tag{26}$$

by Lemma $AA^\dagger A = A$. Post-multiplying by A^{k-1} on (26) gives

$$XA^k(A^k)^\dagger AA^{k-1} = A^\dagger A^k(A^k)^\dagger AA^{k-1},$$

that is $XA^k = A^\dagger A^k$. □

In the following theorem, we will use the core part A_1 of the Core-EP decomposition to describe the right MPCEP-inverse of A.

Theorem 2.6. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $Y \in \mathbb{C}^{n\times n}$ is a right MPCEP-inverse of A if and only if $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A_1)$ and $A_1A^\dagger AY = A_1A^\dagger$ hold.*

Proof. Firstly, we will proof $\mathcal{R}(A^\dagger A^k) = \mathcal{R}(A^\dagger A_1)$. Since, we have

$$A^\dagger A_1 = A^\dagger A^k(A^k)^\dagger A \tag{27}$$

and

$$A^\dagger A^k = A^\dagger A^k(A^k)^\dagger A^k = A^\dagger A^k(A^k)^\dagger AA^{k-1} = A^\dagger A_1A^{k-1} \tag{28}$$

by Lemma 2.1. The conditions in (27) and (28) imply $\mathcal{R}(A^\dagger A^k) = \mathcal{R}(A^\dagger A_1)$.

Since

$$\begin{aligned} & A_1A^\dagger AY = A_1A^\dagger \\ & \Leftrightarrow A^k(A^k)^\dagger AA^\dagger AY = A^k(A^k)^\dagger AA^\dagger \\ & \Leftrightarrow A^k(A^k)^\dagger AY = A^k(A^k)^\dagger AA^\dagger \\ & \Leftrightarrow (A^k)^\dagger AY = (A^k)^\dagger AA^\dagger \\ & \Leftrightarrow (A^k)^*AY = (A^k)^*AA^\dagger \\ & \Leftrightarrow (A^k)^*AY = (A^k)^*(AA^\dagger)^* \\ & \Leftrightarrow (A^k)^*AY = (AA^\dagger A^k)^* \\ & \Leftrightarrow (AY)^*A^k = A^k \end{aligned} \tag{29}$$

by Lemma 2.1. □

Theorem 2.7. *Let $A \in \mathbb{C}^{n\times n}$. If A is both left and right MPCEP-invertible, then the left MPCEP-inverse of A and the right MPCEP-inverse of A are unique. Moreover, the left MPCEP-inverse of A coincides with the right MPCEP-inverse of A.*

Proof. Let X be a left MPCEP-inverse of A and Y be a right MPCEP-inverse of A. Then

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X) \text{ and } XA^k = A^\dagger A^k. \tag{30}$$

and

$$\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k) \text{ and } (AY)^* A^k = A^k \tag{31}$$

hold. Thus, $X = U(A^k)^*$ and $Y = A^\dagger A^k V$, for some $U, V \in \mathbb{C}^{n\times n}$ by (30) and (31). Therefore,

$$\begin{aligned} X &= U(A^k)^* = U(A^k)^* AY = XAY, \\ Y &= A^\dagger A^k V = XA^k V = XAA^\dagger A^k V = XAY \end{aligned} \tag{32}$$

by (30) and (31). Hence, $X = Y$ by (32). If Z is a another right MPCEP-inverse of A, one can prove $X = Z$ in a similar way. Then, $Y = Z$ by $X = Y$ and $X = Z$, which says the right MPCEP-inverse of A is unique. One also can prove the left MPCEP-inverse of A is unique by a similar proof of the uniqueness of the right MPCEP-inverse of A. By the above proof, we can get that the left MPCEP-inverse of A coincides with the right MPCEP-inverse of A. □

The concept of the MPCEP-inverse of A will be introduced by using left MPCEP-inverse of A and right MPCEP-inverse of A. The concept of the MPCEP-inverse of a Hilbert space operators was introduced by Chen, Mosić and Xu in [3].

Definition 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. We call that $X \in \mathbb{C}^{n\times n}$ is the MPCEP-inverse of A if A is both left MPCEP-invertible and right MPCEP-invertible. That is,*

$$\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X), \mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger A^k),\ XA^k = A^\dagger A^k \textit{ and } (AX)^* A^k = A^k. \tag{33}$$

And X is denoted by the symbol $A^{\dagger,\oplus}$, that is $A^{\dagger,\oplus} = X$.

By Theorem 2.7 and Definition 2.2, we have the uniqueness of the MPCEP-inverse of A in what follows:

We have $A^{\dagger,\oplus} = A^\dagger A A^{\oplus} = A^\dagger A A^D A^k (A^k)^\dagger = A^\dagger A^D A^{k+1} (A^k)^\dagger = A^\dagger A^k (A^k)^\dagger$ by $A^{\oplus} = A^D A^k (A^k)^\dagger$. So, the MPCEP-inverse defined in Definition 2.2 coincides with ones introduced in [3] that was expanded to matrices in [8, 9].

Theorem 2.8. *Let $A \in \mathbb{C}^{n\times n}$. Then, the MPCEP-inverse of A is unique.*

The formula of the MPCEP-inverse of a complex matrix was given in the following theorem.

Theorem 2.9. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^k (A^k)^\dagger$ is the MPCEP-inverse of A.*

Proof. By Definition 2.2, a MPCEP-invertible matrix, is both left MPCEP-invertible and right MPCEP-invertible. Then, By Theorem 2.1, we have $A^\dagger A^k (A^k)^\dagger$ is a left MPCEP-inverse of A. And by Theorem 2.3, we have $A^\dagger A^k (A^k)^\dagger$ is a right MPCEP-inverse of A. The proof is finished by Theorem 2.7. □

3. Existence criteria and expressions of the MPCEP-inverse

The CMP inverse of $A \in \mathbb{C}^{n\times n}$ was introduced by Mehdipour and Salemi in [13], who using the core part in core-nilpotent decomposition of A and the Moore-Penrose inverse of A. Motivated by the above method, we have a natural question as follows: Using the core part A_1 in Core-EP decomposition of A and the Moore-Penrose inverse of A to introduce a matrix $X = A^\dagger A_1 A^\dagger$.

Question What is X ?

In the following theorem, we answer this question, we proved that $X = A^\dagger A_1 A^\dagger$ is a formula of the MPCEP-inverse.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $A = A_1 + A_2$ is the Core-EP decomposition of A. Then, the formula of the MPCEP-inverse is $X = A^\dagger A_1 A^\dagger$.*

Proof. Let $X = A^\dagger A_1 A^\dagger$. Then, by Lemma 2.1, we have

$$\begin{aligned} X &= A^\dagger A_1 A^\dagger = A^\dagger A^k (A^k)^\dagger A A^\dagger \\ &= A^\dagger [A^k (A^k)^\dagger]^* (AA^\dagger)^* = A^\dagger [AA^\dagger A^k (A^k)^\dagger]^* \\ &= A^\dagger [A^k (A^k)^\dagger]^* = A^\dagger [(A^k)^\dagger]^* (A^k)^*. \end{aligned} \tag{34}$$

The condition $\mathcal{R}(A^k)^\perp \subseteq \mathcal{N}(X)$ holds by (34). Since

$$X = A^\dagger A_1 A^\dagger = A^\dagger A^k (A^k)^\dagger A A^\dagger \tag{35}$$

so, the condition $\mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger A^k)$ holds by (35). Since

$$XA^k = A^\dagger A^k (A^k)^\dagger AA^\dagger A^k = A^\dagger A^k (A^k)^\dagger A^k = A^\dagger A^k \tag{36}$$

so, the condition $XA^k = A^\dagger A^k$ holds by (36). Since

$$(AX)^* A^k = [AA^\dagger A^k (A^k)^\dagger AA^\dagger]^* A^k = AA^\dagger A^k (A^k)^\dagger AA^\dagger A^k = A^k \tag{37}$$

so, the condition $(AX)^* A^k = A^k$ holds by (37). Thus, the proof is finished by Definition 2.2. □

The following exmaple shows that the core part in core-nilpotent decomposition of A is different from the core part in Core-EP decomposition of A. Moreover, this example also shows that the MPCEP-inverse is different from the CMP inverse.

Example 3.1. Let $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then, the core part in core-nilpotent decomposition of A is $AA^DA = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the core part in Core-EP decomposition of A is $AA^{\textcircled{\dagger}}A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Thus, $A^{c,\dagger} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by $A^{c,\dagger} = A^\dagger AA^DAA^\dagger$ and $A^{\dagger,\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ by $A^{\dagger,\textcircled{\dagger}} = A^\dagger AA^{\textcircled{\dagger}}AA^\dagger$.

The following example shows that the MPCEP-inverse can equal to the CMP inverse.

Example 3.2. Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & 4 \end{bmatrix} \in \mathbb{C}^{4\times 4}$. It is easy to check that the index of A is 2. By [18, Corollary 3.3], we have

$$A^{\textcircled{\dagger}}=A^2(A^3)^{\textcircled{\#}}=A^2(A^2)^{\textcircled{\#}}=A^2(A^2)^{\#}A^2(A^2)^{\dagger}=A^2(A^2)^{\dagger}=\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A^D,$$

which gives the core part in core-nilpotent decomposition of A equals to the core part in Core-EP decomposition of A. Moreover, the MPCEP-inverse of A equals to the CMP inverse of A.

In [18, Theorem 3.4], Wang proved that A_1 can be described by using the Core-EP inverse of A. The explicit expressions of A_1 can be found in the follows lemma.

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $A = A_1 + A_2$ is the Core-EP decomposition of A, then $A_1 = AA^{\textcircled{\dagger}}A$ and $A_2 = A - AA^{\textcircled{\dagger}}A$.*

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$. Then, the MPCEP-inverse of A is an outer inverse of A.*

Proof. Let $A = A_1 + A_2$ is the Core-EP decomposition of A and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, $X = A^\dagger A_1 A^\dagger$ by Theorem 3.1, thus

$$\begin{aligned} XAX &= A^\dagger A_1 A^\dagger A A^\dagger A_1 A^\dagger = A^\dagger A_1 A^\dagger A_1 A^\dagger \\ &= A^\dagger A^k (A^k)^\dagger A A^\dagger A^k (A^k)^\dagger A A^\dagger \\ &= A^\dagger A^k (A^k)^\dagger A^k (A^k)^\dagger A A^\dagger \\ &= A^\dagger A^k (A^k)^\dagger A A^\dagger \\ &= A^\dagger A_1 A^\dagger \\ &= X \end{aligned} \tag{38}$$

by Lemma 2.1. □

Let $A \in \mathbb{C}^{n\times n}$ and $i, m \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called an $\langle i, m\rangle$-core inverse of A, if it satisfies

$$X = A^D A X \quad \text{and} \quad A^m X = A^i (A^i)^\dagger. \tag{39}$$

The $\langle i, m\rangle$-core inverse of A is unique and denoted by $A^{\oplus}_{i,m}$.

Proposition 3.1 ([19, Proposition 1])**.** *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $i \geqslant k$, then $A^m A^{\oplus}_{i,m}$ is the orthogonal projector onto $\mathcal{R}(A^i)$ along $\mathcal{R}(A^i)^{\perp}$.*

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and $i, m \in \mathbb{N}$. If $i \geqslant k$, then $AA^{\dagger,\textcircled{\dagger}}$ is the orthogonal projector onto $\mathcal{R}(A^i)$ along $\mathcal{R}(A^i)^{\perp}$. Moreover, we have*

$$AA^{\dagger,\textcircled{\dagger}} = A_1 A^\dagger = AA^{\textcircled{\dagger}} = A^m A^{\oplus}_{i,m} = A^k (A^k)^\dagger = A^i (A^i)^\dagger, \tag{40}$$

where A_1 is the core part A_1 in Core-EP decomposition of A and $A^{\textcircled{\dagger}}$ is the Core-EP inverse of A.

Proof. By Theorem 2.9, we have $A^{\dagger,\textcircled{\dagger}} = A^\dagger A^k (A^k)^\dagger$. Then

$$AA^{\dagger,\textcircled{\dagger}} = AA^\dagger A^k (A^k)^\dagger = A^k (A^k)^\dagger. \tag{41}$$

The equality $AA^{\textcircled{\dagger}} = A^k (A^k)^\dagger$ can be got [18, Corollary 3.3]. The equality $A^m A^{\oplus}_{i,m} = A^k (A^k)^\dagger = A^i (A^i)^\dagger$ is hold by Lemma 3.1. By Lemma 2.1, we have $A_1 = A^k (A^k)^\dagger A$, then

$$\begin{aligned} A_1 A^\dagger &= A^k (A^k)^\dagger A A^\dagger = [A^k (A^k)^\dagger]^* (AA^\dagger)^* \\ &= [AA^\dagger A^k (A^k)^\dagger]^* = [A^k (A^k)^\dagger]^* \\ &= A^k (A^k)^\dagger. \end{aligned}$$

Thus, the proof is finished by (41). □

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, X can be written as the $\mathfrak{cr}^{\mathfrak{CR}}$ constrained inverse of A, where*

Constraints of type 1 :

$\mathfrak{c}:\ \mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger A_1)$;

$\mathfrak{r}:\ \mathcal{R}(X^*) \subseteq \mathcal{R}((A_1A^\dagger)^*)$;

Constraints of type 2 :

$\mathfrak{C}:\ XAA^\dagger A_1 = A^\dagger A_1$;

$\mathfrak{R}:\ A_1A^\dagger AX = A_1A^\dagger$.

Where A_1 is the core part of the Core-EP decomposition of A.

Proof. The proof of Constraints of type 1:

Let $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, $X = A^\dagger A_1 A^\dagger$ by Theorem 3.1, which gives the condition $\mathfrak{c} : \mathcal{R}(X) \subseteq \mathcal{R}(A^\dagger A_1)$. Let $u \in \mathcal{N}(A_1A^\dagger)$, then $Xu = A^\dagger A_1 A^\dagger u = 0$, which implies $\mathcal{N}(A_1A^\dagger) \subseteq \mathcal{N}(X)$. The condition $\mathfrak{r} : \mathcal{R}(X^*) \subseteq \mathcal{R}((A_1A^\dagger)^*)$ is satisfied by $\mathcal{R}(X^*) \subseteq \mathcal{R}((A_1A^\dagger)^*)$ if and only if $\mathcal{N}(A_1A^\dagger) \subseteq \mathcal{N}(X)$.

The proof of Constraints of type 2:

By Lemma 2.1, we have $A_1 = A^k(A^k)^\dagger A$. Then

$$
\begin{aligned}
XAA^\dagger A_1 &= XAA^\dagger A_1 = A^\dagger A_1 A^\dagger AA^\dagger A_1\\
&= A^\dagger A_1 A^\dagger A_1 = A^\dagger A^k(A^k)^\dagger AA^\dagger A^k(A^k)^\dagger A\\
&= A^\dagger A^k(A^k)^\dagger A^k(A^k)^\dagger A\\
&= A^\dagger A^k(A^k)^\dagger A\\
&= A^\dagger A_1,\\
A_1A^\dagger AX &= A_1A^\dagger AA^\dagger A_1A^\dagger\\
&= A_1A^\dagger A_1A^\dagger = A^k(A^k)^\dagger AA^\dagger A^k(A^k)^\dagger AA^\dagger\\
&= A^k(A^k)^\dagger A^k(A^k)^\dagger AA^\dagger\\
&= A^k(A^k)^\dagger AA^\dagger\\
&= A_1A^\dagger.
\end{aligned} \tag{42}
$$

The condition $\mathfrak{C}$ and $\mathfrak{R}$ are satisfied by (42). □

If we let $B = A^\dagger A_1$ and $C = A_1A^\dagger$, then by the proof of Theorem 3.4, we have that the MPCEP-inverse of A coincides with the $(A^\dagger A_1, A_1A^\dagger)$-inverse of A. That is, we have the following theorem.

Theorem 3.5. *Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. Then, X is the $(A^\dagger A_1, A_1A^\dagger)$-inverse of A, where A_1 is the core part of the Core-EP decomposition of A.*

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The MPCEP-inverse of A coincides with the $(A^\dagger A^k, (A^k)^*)$-inverse of A.*

Proof. One can prove this theorem by using Theorem 2.5, Theorem 2.6 and Theorem 2.7. □

The MPCEP-inverse of A can be got by using the "S" part of the Core-EP inverse and the "T" part of the CMP inverse by Theorem 3.6.

4. The CE matrix based on the Core-EP decomposition

We introduced CE matrix by mimicking the concept of EP matrix. The notation $[A, B] = AB - BA$ will be used in the sequel.

Definition 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $A = A_1 + A_2$ be the Core-EP decomposition of A as in (1). If $A^\dagger A_1 = A_1 A^\dagger$, then we call A is a CE matrix.*

Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ be the MPCEP-inverse of A. If A is a CE matrix, then X is the $(A^\dagger A_1, A_1 A^\dagger)$-inverse by Theorem 3.5.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$. Then, A is a CE matrix if and only if $[A^{\dagger,\textcircled{\dagger}}, A] = 0$.*

Proof. By Theorem 3.3, we have $AA^{\dagger,\textcircled{\dagger}} = A_1 A^\dagger$. By Theorem 2.9, we have $A^{\dagger,\textcircled{\dagger}} = A^\dagger A^k (A^k)^\dagger$. Then, $A^{\dagger,\textcircled{\dagger}} A = A^\dagger A^k (A^k)^\dagger A = A^\dagger [A^k (A^k)^\dagger A] = A^\dagger A_1$. Thus

$$A^{\dagger,\textcircled{\dagger}} A - AA^{\dagger,\textcircled{\dagger}} = A^\dagger A_1 - A_1 A^\dagger = 0$$

by the definition of the CE matrix. □

Proposition 4.1. *Let $A \in \mathbb{C}^{n\times n}$ is a CE matrix with $\operatorname{ind}(A) = k$. Then, $A^\dagger A^{k+1} = A^k$.*

Proof. By the definition of the CE matrix, we have $A^\dagger A_1 = A_1 A^\dagger$, which is equivalent to

$$A^\dagger A^k (A^k)^\dagger A = A^k (A^k)^\dagger A A^\dagger \tag{43}$$

by Lemma 2.1. Post-multiplying by A^k on (43) gives

$$\begin{aligned} & A^\dagger A^k (A^k)^\dagger A A^k = A^k (A^k)^\dagger A A^\dagger A^k \\ \Leftrightarrow\ & A^\dagger A^k (A^k)^\dagger A^k A = A^k (A^k)^\dagger A^k \\ \Leftrightarrow\ & A^\dagger A^{k+1} = A^k. \end{aligned} \tag{44}$$

Thus, $A^\dagger A^k (A^k)^\dagger A A^k = A^k (A^k)^\dagger A A^\dagger A^k$ if and only if $A^\dagger A^{k+1} = A^k$. The proof is finished by $A^\dagger A_1 = A_1 A^\dagger$ implies $A^\dagger A^k (A^k)^\dagger A A^k = A^k (A^k)^\dagger A A^\dagger A^k$. □

Proposition 4.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $A^\dagger A^{k+1} = A^k$, then $A^\dagger A^2 \in A^{\dagger,\textcircled{\dagger}}\{1, 4\}$.*

Proof. By the hypothesis of the proposition, we have $A^\dagger A^{k+1} = A^k$. From Theorem 3.3, we have $AA^{\dagger,\textcircled{\dagger}} = A_1 A^\dagger = A^k(A^k)^\dagger$. In view of Lemma 2.1, we have $A_1 = A^k(A^k)^\dagger A$. Then

$$
\begin{aligned}
AA^{\dagger,\textcircled{\dagger}} &= A_1A^\dagger = A^k(A^k)^\dagger AA^\dagger = A^\dagger A^{k+1}(A^k)^\dagger AA^\dagger \\
&= A^\dagger AA^k(A^k)^\dagger AA^\dagger = A^\dagger A[A^k(A^k)^\dagger]^*(AA^\dagger)^* \\
&= A^\dagger A[AA^\dagger A^k(A^k)^\dagger]^* = A^\dagger A[A^k(A^k)^\dagger]^* \\
&= A^\dagger AA^k(A^k)^\dagger = A^\dagger AAA^{\dagger,\textcircled{\dagger}} \\
&= A^\dagger A^2A^{\dagger,\textcircled{\dagger}}.
\end{aligned} \tag{45}
$$

The equality (45) gives $AA^{\dagger,\textcircled{\dagger}} = A^\dagger A^2A^{\dagger,\textcircled{\dagger}}$. By Theorem 3.2, we have the MPCEP-inverse of A is an outer inverse of A. Pre-multiplying by $A^{\dagger,\textcircled{\dagger}}$ on $AA^{\dagger,\textcircled{\dagger}} = A^\dagger A^2A^{\dagger,\textcircled{\dagger}}$ gives $A^{\dagger,\textcircled{\dagger}} = A^{\dagger,\textcircled{\dagger}}AA^{\dagger,\textcircled{\dagger}} = A^{\dagger,\textcircled{\dagger}}A^\dagger A^2A^{\dagger,\textcircled{\dagger}}$, that is $A^\dagger A^2$ is an inner inverse of $A^{\dagger,\textcircled{\dagger}}$. Since $A^\dagger A^2A^{\dagger,\textcircled{\dagger}} = A^\dagger A^2A^\dagger A^k(A^k)^\dagger = A^\dagger AA^k(A^k)^\dagger = A^\dagger A^{k+1}(A^k)^\dagger = A^k(A^k)^\dagger$, then $A^\dagger A^2 \in A^{\dagger,\textcircled{\dagger}}\{4\}$ by $A^k(A^k)^\dagger = [A^k(A^k)^\dagger]^*$. $\square$

5. Conclusions

One-sided MPCEP-inverse for matrices was introduced in this paper. The MPCEP-inverse can be described by using the core part A_1 in Core-EP decomposition of A and the Moore-Penrose inverse of A. The MPCEP-inverse of A coincides with the $(A^\dagger A^k, (A^k)^*)$-inverse of A, that is, the MPCEP-inverse of A is $A^{(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}((A^k)^*)}$. In addition, the CE matrix was introduced, a necessary and sufficient condition such that a matrix A to be a CE matrix is the MPCEP-inverse of A commutes with A, that is $[A^{\dagger,\textcircled{\dagger}}, A] = 0$, where $A^{\dagger,\textcircled{\dagger}}$ is the MPCEP-inverse of A. The future perspectives for research are proposed:

Part 1. The reverse order law of the MPCEP-inverse.

Part 2. The rank properties of the MPCEP-inverse.

Part 3. The weighted MPCEP-inverse of matrices.

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Inequalities for the generalized inverse trigonometric and hyperbolic functions with one parameter

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Abstract. In this paper, we show some inequalities for the generalized inverse trigonometric and hyperbolic functions with one parameter of $(2, q)$. Especially, we also present several Shafer-Fink, Wilker and Huygens type inequalities of these functions. These results are consistent with previously known results.
Keywords: generalized inverse trigonometric function; Lerch Phi function; Shafer-Fink type inequalities; Wilker and Huygens type inequalities.

1. Introduction

For $p, q \in (1, +\infty)$ and $x \in [0, 1]$, the function $\sin_{p,q}(x)$ is defined by the inverse function of

$$\sin_{p,q}^{-1}(x) = \int_0^x (1 - t^q)^{-1/p} dt.$$

The function $\sin_{p,q}^{-1}(x)$ is increasing in $[0, 1]$ onto $[0, \pi_{p,q}/2]$ where

$$\frac{\pi_{p,q}}{2} = \sin_{p,q}^{-1}(1) = \int_0^1 (1 - t^q)^{-1/p} dt = \frac{1}{q} B(1 - \frac{1}{p}, \frac{1}{q}).$$

The function $\sin_{p,q}(x)$ is defined on $[0, \pi_{p,q}/2]$ and can be extended to $(-\infty, +\infty)$. Similarly, we can define $\cos_{p,q}(x), \tan_{p,q}(x)$ and their inverses (see [11]). In the same way, we can define the generalized hyperbolic functions as follows:

$$\sinh_{p,q}^{-1}(x) = \int_0^x (1 + t^q)^{-1/p} dt, x \in \mathbb{R}.$$

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Recently, the arc lemniscate sine function and the hyperbolic arc lemniscate sine function defined by

$$\operatorname{arcsl}(x) = \int_0^x (1-t^4)^{-1/2} dt, |x| < 1 \tag{1}$$

and

$$\operatorname{arcslh}(x) = \int_0^x (1+t^4)^{-1/2} dt, x \in \mathbb{R} \tag{2}$$

are deeply studied. In fact, (1) and (2) are $\sin_{2,4}^{-1}$ and $\sinh_{2,4}^{-1}$ respectively.

Neuman used the arc lemniscate sine function and the hyperbolic arc lemniscate sine function, respectively, to define the arc lemniscate tangent function and the hyperbolic arc lemniscate tangent function, as follows(see [14], [15]):

$$\begin{aligned} \operatorname{arctl}(x) &= \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), x \in \mathbb{R}, \\ \operatorname{arctlh}(x) &= \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), |x| < 1. \end{aligned} \tag{3}$$

In [3], [4], Chen established several lemniscate function inequalities of the Wilker and Huygens type. Recently, some Shafer-Fink type inequalities for the lemniscate functions were established. In [5], inequalities of the Wilker and Huygens type involving inverse trigonometric functions were given by Chen et. al.. For more results, the reader may see references: [6], [10], [13], [16], [17]. In [18], Xu et. al. got some new bounds for the arc lemniscate functions. In particular, from the point view of bivariate means, Zhao [20, 21, 22] et. al. dealt with the arc lemniscate functions and got optimal bounds for these bivariate means.

For several functions connected to the generalized inverse lemniscate and the generalized hyperbolic inverse lemniscate functions, Yin and Lin [19] investigated monotonicity and some inequalities. By utilizing the Lerch Phi function, they provided a bound estimation of the generalized inverse lemniscate functions. Later, some inequalities of the Shafer-Fink, Wilker, and Huygens types were obtained.

The lemniscate inverse functions and the generalized inverse lemniscate functions are the generalized $(2, 4)$-trigonometric and $(2, 6)$-trigonometric functions respectively, thus are the special cases of the generalized $(2, q)$-trigonometric functions. Motivated by the work of references [1, 4, 17, 19], we mainly study the generalized $(2, q)$-trigonometric and hyperbolic functions:

$$\begin{aligned} \sin_{2,q}^{-1}(x) &= \int_0^x (1-t^q)^{-1/2} dt, |x| < 1, \\ \sinh_{2,q}^{-1}(x) &= \int_0^x (1+t^q)^{-1/2} dt, x \in \mathbb{R}. \end{aligned}$$

Previously, mathematicians focused on the study of generalized trigonometric and hyperbolic functions, the reader may refer to the literature [7, 8, 9, 12]. However, the generalized $(2,q)$-trigonometric and hyperbolic functions have rarely been studied. Here, we mainly showed several the Shafer-Fink, Wilker and Huygens type inequalities for the generalized $(2,q)$-trigonometric and hyperbolic functions.

2. Bounds of $\sin_{2,q}^{-1}(x)$

Lemma 2.1 ([19, Theorem 1.1]). *Let $-\infty < a < b < +\infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b) with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x \in (a, b)$. If f'/g' is increasing (decreasing) on (a, b), then so is f/g.*

Theorem 2.1. *For all $x \in (0, 1)$ and $q \geq 4$, we have*

$$\alpha x\Phi(x^q, 3/2, 1/q) < \sin_{2,q}^{-1}(x) < \beta x\Phi(x^q, 3/2, 1/q) \tag{4}$$

with the best possible constants $\alpha = q^{-\frac{3}{2}}$ and $\beta = \frac{B(1/2,1/q)}{q\zeta(3/2,1/q)}$ where

$$\Phi(z, s, \alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s}, \alpha \neq 0, -1 \ldots, |z| < 1,$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

$$\zeta(s, \alpha) = \Phi(1, s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$$

are Lerch Phi function, classical beta function and Hurwitz zeta function respectively. If $1 < q \leq 3$, the inequalities (4) are inverse.

Proof of Theorem 2.1. Let

$$F(x) = \frac{\sin_{2,q}^{-1}(x)}{x\Phi(x^q, 3/2, 1/q)}. \tag{5}$$

Applying the Lemma 2.1 with $f(x) = \sin_{2,q}^{-1}(x)$ and $g(x) = x\Phi(x^q, 3/2, 1/q)$ and simple computation, we get

$$f(0^+) = g(0^+) = 0, f'(x) = \frac{1}{\sqrt{1-x^q}}, g'(x) = q^{\frac{3}{2}} \sum_{n=0}^{\infty} \frac{x^{qn}}{\sqrt{qn+1}}.$$

So, we obtain

$$\frac{f'(x)}{g'(x)} = \frac{1}{q^{\frac{3}{2}} h(x^q)},$$

where

$$h(t)=\sqrt{1-t}\sum_{n=0}^{\infty}\frac{t^n}{\sqrt{qn+1}},\, t\in(0,1).$$

By differentiation, we get

$$2\sqrt{1-t}h'(t)=\sum_{n=0}^{\infty}\Big(\frac{2n+2}{\sqrt{qn+q+1}}-\frac{2n+1}{\sqrt{qn+1}}\Big)t^n.$$

Let $a_n=\frac{2n+2}{\sqrt{qn+q+1}}-\frac{2n+1}{\sqrt{qn+1}}$, then

$$a_n=\frac{(4-q)n+3-q}{(2n+2)(qn+1)\sqrt{qn+q+1}+(2n+1)(qn+q+1)\sqrt{qn+1}}.$$

If $q\geq 4$, we have $a_n<0$, thus $h'(t)<0$, it follows that $h(t)$ is strictly decreasing on $(0,1)$. This implies that $\frac{f'(x)}{g'(x)}$ is strictly increasing on $(0,1)$, by Lemma 2.1, we conclude that $F(x)$ is strictly increasing on $(0,1)$. Thus $F(0)<F(x)<F(1)$ for $x\in(0,1)$. By simple computation, we get

$$\begin{aligned}F(0^+)&=\lim_{x\to 0^+}\frac{f'(x)}{g'(x)}=q^{-\frac{3}{2}},\\ F(1^-)&=\frac{\sin_{2,q}^{-1}(1)}{\Phi(1,3/2,1/q)}=\frac{B(1/2,1/q)}{q\zeta(3/2,1/q)}.\end{aligned}\tag{6}$$

If $1<q\leq 3$, we easily complete the proof.

Remark 2.1. When $3<q<4$, the situation is more complex. Taking $q=3.1$ as an example, then by (5), we have

$$F(x)=\frac{\sin_{2,\frac{31}{10}}^{-1}(x)}{x\Phi(x^{\frac{31}{10}},\frac{3}{2},\frac{10}{31})}.$$

By (6), we get

$$F(1^-)=\frac{B(\frac{1}{2},\frac{10}{31})}{\frac{31}{10}\zeta(\frac{3}{2},\frac{10}{31})}=0.183373\ldots.$$

However, $F(0.9)=0.183419\ldots>F(1^-)$. Therefore, it is necessary to find the maximum value of $F(x)$ in $(0,1)$. This is a challenging problem and open.

3. Shafer-Fink type inequalities

Lemma 3.1. *For $q>1$, we have*

(*i*) *The function $f_1(x)=\frac{\sin_{2,q}^{-1}(x)}{x}$ is strictly increasing on $(0,1)$ with range $(1,\frac{\pi_{2,q}}{2})$, where $\frac{\pi_{2,q}}{2}=\sin_{2,q}^{-1}(1)=\frac{1}{q}B(\frac{1}{2},\frac{1}{q})$;*

(ii) The function $f_2(x) = \frac{\sinh_{2,q}^{-1}(x)}{x}$ *is strictly decreasing on* $(0, +\infty)$ *with range* $(0, 1)$.

Proof of Lemma 3.1. Lemma 2.1 allows us to simply finish the proof.

Lemma 3.2. *For* $q \geq 4$, *we have*

(i) The function $g_1(x) = \frac{x - \sqrt[q]{1-x^q}\sin_{2,q}^{-1}(x)}{\sin_{2,q}^{-1}(x) - x}$ *is strictly increasing on* $(0, 1)$ *with range* $(\frac{q+2}{q}, \frac{2}{\pi_{2,q}-2})$;

(ii) The function $g_2(x) = \frac{\sqrt[q]{1+x^q}\sinh_{2,q}^{-1}(x) - x}{x - \sinh_{2,q}^{-1}(x)}$ *is strictly decreasing on* $(0, +\infty)$ *with range* $(\frac{\pi_{2q/(q+2),q}}{2} - 1, \frac{q+2}{q})$ *where* $\frac{\pi_{2q/(q+2),q}}{2} = \frac{1}{q}B(\frac{q-2}{2q}, \frac{1}{q})$.

Proof of Lemma 3.2. (i) Let $g_1(x) = \frac{g_{11}(x)}{g_{12}(x)}$ where $g_{11}(x) = x - \sqrt[q]{1-x^q}\sin_{2,q}^{-1}(x)$ and $g_{12}(x) = \sin_{2,q}^{-1}(x) - x$. Then $g_{11}(0^+) = g_{12}(0^+) = 0$. By differentiation, we obtain

$$\frac{g_{11}'(x)}{g_{12}'(x)} = \frac{1 + x^{q-1}(1-x^q)^{\frac{1-q}{q}}\sin_{2,q}^{-1}(x) - (1-x^q)^{\frac{2-q}{2q}}}{(1-x^q)^{-\frac{1}{2}} - 1}$$

with $g_{11}'(0^+) = g_{12}(0^+) = 0$. Computing once more, we obtain

$$\frac{g_{11}''(x)}{g_{12}''(x)} = \frac{2(q-1)}{q}\frac{\sin_{2,q}^{-1}(x)}{x}(1-x^q)^{\frac{2-q}{2q}} + \frac{4-q}{q}(1-x^q)^{\frac{1}{q}}.$$

As $q \geq 4$, by lemma 3.1, $\frac{g_{11}''(x)}{g_{12}''(x)}$ is strictly increasing, as a result, $g_1(x)$ strictly increases by Lemma 2.1, it follows that $g_1(0^+) < g_1(x) < g_1(1^-)$. Simple computation yields $g_1(0^+) = \frac{q+2}{q}$ and $g_1(1^-) = \frac{2}{\pi_{2,q}-2}$.

(ii) Let $g_2(x) = \frac{g_{21}(x)}{g_{22}(x)}$ where $g_{21}(x) = \sqrt[q]{1+x^q}\sinh_{2,q}^{-1}(x) - x$ and $g_{22}(x) = x - \sinh_{2,q}^{-1}(x)$. Then $g_{21}(0^+) = g_{22}(0^+) = 0$. By differentiation, we obtain

$$\frac{g_{21}'(x)}{g_{22}'(x)} = \frac{x^{q-1}(1+x^q)^{\frac{1-q}{q}}\sinh_{2,q}^{-1}(x) + (1+x^q)^{\frac{2-q}{2q}} - 1}{1 - (1-x^q)^{-\frac{1}{2}}}$$

with $g_{11}'(0^+) = g_{12}'(0^+) = 0$. Differentiating again, we get

$$\frac{g_{21}''(x)}{g_{22}''(x)} = \frac{2(q-1)}{q}\frac{\sinh_{2,q}^{-1}(x)}{x}(1+x^q)^{\frac{2-q}{2q}} + \frac{4-q}{q}(1+x^q)^{\frac{1}{q}}.$$

As $q \geq 4$, $\frac{g_{21}''(x)}{g_{22}''(x)}$ is strictly decreasing by lemma 3.1. Hence, $g_2(x)$ is strictly decreasing by Lemma 2.1, thus, it follows $g_2(0^+) > g_2(x) > g_2(+\infty)$. The limiting values read as follows

$$g_2(0^+) = \frac{q+2}{q},$$

$$\begin{aligned} g_2(+\infty) =& \sinh_{2.q}^{-1}(+\infty) - 1 = \int_0^{+\infty} (1+t^q)^{-1/2} dt - 1 \\ =& \int_0^1 (1-s^q)^{\frac{-q-2}{2q}} ds - 1 = \frac{\pi_{2q/(q+2),q}}{2} - 1, \end{aligned}$$

where we apply the substitution $1 + t^q = \frac{1}{1-s^q}$. This completes the proof.

Theorem 3.1. *For $q \geq 4$, the following inequalities exist:*

(i) $$\frac{\pi_{2,q}}{2 + (\pi_{2,q} - 2)\sqrt[q]{1-x^q}} < \frac{\sin_{2,q}^{-1}(x)}{x} < \frac{2q+2}{q+2+q\sqrt[q]{1-x^q}}, 0 < |x| < 1;$$

(ii) $$\frac{\pi_{2q/(q+2),q}}{(\pi_{2q/(q+2),q} - 2) + 2\sqrt[q]{1+x^q}} < \frac{\sinh_{2,q}^{-1}(x)}{x} < \frac{2q+2}{q+2+q\sqrt[q]{1+x^q}}, |x| > 0.$$

Proof of Theorem 3.1. *We finished the proof by utilizing Lemma 3.2.*

4. Wilker and Huygens type inequalities

The fact that the Pochhammers symbol $(a)_n$ is defined by

$$(a)_0 = 1, (a)_n = a(a+1)\dots(a+n-1), n = 1, 2, \dots,$$

and the ordinary binomial expansion can be written with the following notation,

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \tag{7}$$

As an analogy to arc lemniscate functions which are defined in (3), $\tan_{2,q}^{-1}(x)$ and $\tanh_{2,q}^{-1}(x)$ have been defined as follows:

$$\begin{aligned} \tan_{2,q}^{-1}(x) &= \sin_{2,q}^{-1}\left(\frac{x}{\sqrt[q]{1+x^q}}\right) = \int_0^{\frac{x}{\sqrt[q]{1+x^q}}} (1-t^q)^{-1/2} dt, x \in \mathbb{R}, \\ \tanh_{2,q}^{-1}(x) &= \sinh_{2,q}^{-1}\left(\frac{x}{\sqrt[q]{1-x^q}}\right) = \int_0^{\frac{x}{\sqrt[q]{1-x^q}}} (1+t^q)^{-1/2} dt, |x| < 1. \end{aligned}$$

By using (7), we get the following power series expansions:

Lemma 4.1. *For $q > 1$, we have*

$$\sin_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{(qn+1)n!} x^{qn+1}, |x| < 1, \tag{8}$$

$$\sinh_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})_n}{(qn+1)n!} x^{qn+1}, x \in \mathbb{R}, \tag{9}$$

$$\tan_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2} + \frac{1}{q})_n}{(qn+1)n!} x^{qn+1}, x \in \mathbb{R}, \tag{10}$$

$$\tanh_{2,q}^{-1}(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} + \frac{1}{q})_n}{(qn+1)n!} x^{qn+1}, x \in \mathbb{R}, |x| < 1. \tag{11}$$

Proof of Lemma 4.1. We only prove (10), other proofs are completely similar. By simple computation, we get

$$\begin{aligned}\frac{d}{dx}\left(\tan_{2,q}^{-1}(x)\right) &= \frac{d}{dx}\int_0^{\frac{x}{\sqrt[q]{1+x^q}}} \frac{1}{\sqrt{1-t^q}}dt = (1+x^q)^{-\frac{1}{2}-\frac{1}{q}} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{2}+\frac{1}{q})_n}{n!}x^{qn}.\end{aligned}$$

Hence,

$$\tan_{2,q}^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(\frac{1}{2}+\frac{1}{q})_n}{(qn+1)n!}x^{qn+1}.$$

Lemma 4.2. *Let $q \geq 1$ be an integer. Then for all $0 < x < 1$,*

$$\text{(12)} \qquad (i) \quad \sum_{k=0}^{2p-1}(-1)^k a_k x^{qk+1} < \sinh_{2,q}^{-1}(x) < \sum_{k=0}^{2p}(-1)^k a_k x^{qk+1}$$

where

$$a_k = \frac{(\frac{1}{2})_k}{(qk+1)k!}, k = 0, 1 \dots.$$

$$\text{(13)} \qquad (ii) \quad \sum_{k=0}^{2p-1}(-1)^k b_k x^{qk+1} < \tan_{2,q}^{-1}(x) < \sum_{k=0}^{2p}(-1)^k b_k x^{qk+1}$$

where

$$b_k = \frac{(\frac{1}{2}+\frac{1}{q})_k}{(qk+1)k!}, k = 0, 1 \dots.$$

Proof of Lemma 4.2. We only prove (i). Simple computation results in

$$\begin{aligned}\frac{a_k}{a_{k+1}} &= \frac{(\frac{1}{2})_k}{(qk+1)k!}\frac{(qk+q+1)(k+1)!}{(\frac{1}{2})_{k+1}} \\ &= \frac{(qk+q+1)(2k+2)}{(qk+1)(2k+1)} > 1.\end{aligned}$$

That is to say, $a_k > a_{k+1}$. We have

$$a_k x^{qk+1} - a_{k+1}x^{q(k+1)+1} = x^{qk+1}(a_k - a_{k+1}x^q) > 0$$

because of $a_{k+1}x^q < a_{k+1} < a_k$. According to (9), we get

$$\begin{aligned}\sinh_{2,q}^{-1}(x) &= \sum_{n=0}^{\infty}(-1)^n a_n x^{qn+1} \\ \text{(14)} \qquad &= (a_0 x - a_1 x^{q+1}) + (a_2 x^{2q+1} - a_3 x^{3q+1}) + \dots \\ \text{(15)} \qquad &= a_0 x - (a_1 x^{q+1} - a_2 x^{2q+1}) - (a_3 x^{3q+1} - a_4 x^{4q+1}) + \dots.\end{aligned}$$

By using (14) and (15), we complete the proof of (i).

Theorem 4.1. *For $q \geq 2$ and $0 < x < 1$, we have*

$$\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 + \frac{\tan_{2,q}^{-1}(x)}{x} > 2. \tag{16}$$

Proof of Theorem 4.1. *For $0 < x < 1$, by using (8) and (13), we get*

$$\begin{aligned}\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 &= \left(1 + \frac{1}{2(q+1)}x^q + \frac{3}{8(2q+1)}x^{2q} + \dots\right)^2 \\ &= 1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} + \dots \\ &> 1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} \qquad (17)\end{aligned}$$

and

$$1 - \frac{q+2}{2q(q+1)}x^q < \frac{\tan_{2,q}^{-1}(x)}{x} < 1. \tag{18}$$

So, we find

$$\begin{aligned}&\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 + \frac{\tan_{2,q}^{-1}(x)}{x} - 2 \\ &> 1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} + 1 - \frac{q+2}{2q(q+1)}x^q - 2 \\ &> \frac{q-2}{2q(q+1)}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q} > 0\end{aligned}$$

since $q \geq 2$.

Theorem 4.2. *For $q \geq 3$ and $0 < x < 1$, we have*

$$\left(\frac{x}{\sin_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\tan_{2,q}^{-1}(x)} < 2. \tag{19}$$

Proof of Theorem 4.2. For $0 < x < 1$, by using (17) and (18), we get

$$\begin{aligned}&\left(\frac{x}{\sin_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\tan_{2,q}^{-1}(x)} - 2 \\ &< \frac{1}{1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q}} + \frac{1}{1 - \frac{q+2}{2q(q+1)}x^q} - 2 \\ &= \frac{\begin{aligned}x^q[(q+2)(3q^2+8q+4)x^{2q} - (q+1)(3q^3-16q-8)x^q \\ - 2(q-2)(2q+1)(q+1)^2]\end{aligned}}{4q(2q+1)(q+1)^3(1 + \frac{1}{q+1}x^q + \frac{3q^2+8q+4}{4(2q+1)(q+1)^2}x^{2q})(1 - \frac{q+2}{2q(q+1)}x^q)}.\end{aligned}$$

Let $f(t) = at^2 + bt + c$ where

$$\begin{aligned}
a &= (q+2)(3q^2+8q+4),\\
b &= -(q+1)(3q^3-16q-8),\\
c &= -2(q-2)(2q+1)(q+1)^2,\\
t &= x^q \in (0,1).
\end{aligned}$$

As $q \geq 3$, so $a > 0, b < 0, c < 0$ and $f(1^-) = -7q^4 - 2q^3 + 42q^2 + 58q + 20 < 0$, Using the property of quadratic function, we get $f(t) < 0$, for all $t \in (0,1)$. Hence,

$$\left(\frac{x}{\sin_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\tan_{2,q}^{-1}(x)} - 2 < 0.$$

The proof is complete.

Corollary 4.1. *For $q \geq 3$ and $0 < x < 1$, we have*

$$\frac{2\sin_{2,q}^{-1}(x)}{x} + \frac{\tan_{2,q}^{-1}(x)}{x} > 3. \tag{20}$$

Proof of Corollary 4.1. Another option for inequality (19) is

$$\frac{2}{\frac{1}{\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\tan_{2,q}^{-1}(x)}{x}}} > 1.$$

The arithmetic-geometric-harmonic mean inequality provides the following result:

$$\begin{aligned}
&\frac{2\sin_{2,q}^{-1}(x)}{x} + \frac{\tan_{2,q}^{-1}(x)}{x} \geq 3\sqrt[3]{\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2 \frac{\tan_{2,q}^{-1}(x)}{x}}\\
&\geq 3\frac{2}{\frac{1}{\left(\frac{\sin_{2,q}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\tan_{2,q}^{-1}(x)}{x}}} > 3.
\end{aligned}$$

In [2], Chen and Cheung proved the following inequalities:

$$\left(\frac{x}{\arcsin x}\right)^2 + \frac{x}{\arctan x} < 2, 0 < |x| < 1,$$

$$\frac{2\arcsin x}{x} + \frac{\arctan x}{x} > 3, 0 < |x| < 1.$$

So, we conject that the condition $q \geq 3$ in Theorem 4.2 and Corollary 4.1 can be changed to $q \geq 2$.

Theorem 4.3. *For $q > 1$ and $0 < x < 1$, we have*

$$\left(\frac{x}{\tanh_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\sinh_{2,q}^{-1}(x)} < 2. \tag{21}$$

Proof of Theorem 4.3. For $0 < x < 1$, by using (11) and (12), we have

$$\begin{aligned}\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2 &= \left(1 + \frac{q+2}{2q(q+1)}x^q + \dots\right)^2 \\ &= 1 + \frac{q+2}{q(q+1)}x^q + \dots \\ &> 1 + \frac{q+2}{q(q+1)}x^q\end{aligned}$$

and

$$\frac{\sinh_{2,q}^{-1}(x)}{x} > 1 - \frac{1}{2(q+1)}x^q.$$

So, we get

$$\begin{aligned}&\left(\frac{x}{\tanh_{2,q}^{-1}(x)}\right)^2 + \frac{x}{\sinh_{2,q}^{-1}(x)} - 2 \\ &< \frac{1}{1 + \frac{q+2}{q(q+1)}x^q} + \frac{1}{1 - \frac{1}{2(q+1)}x^q} - 2 \\ &= \frac{x^q\left((2q+4)x^q - (q+1)(q+4)\right)}{\left(2q+2-x^q\right)\left(q^2+q+(q+2)x^q\right)} < 0\end{aligned}$$

since

$$(2q+4)x^q - (q+1)(q+4) < (2q+4) - (q+1)(q+4) < 0.$$

This completes the proof.

Corollary 4.2. *For $q > 1$ and $0 < x < 1$, we have*

$$\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2 + \frac{\sinh_{2,q}^{-1}(x)}{x} > 2 \tag{22}$$

and

$$\frac{2\tanh_{2,q}^{-1}(x)}{x} + \frac{\sinh_{2,q}^{-1}(x)}{x} > 3. \tag{23}$$

Proof of Corollary 4.2. Inequality (21) can be rewritten as

$$\frac{2}{\frac{1}{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\sinh_{2,q}^{-1}(x)}{x}}} > 1.$$

The result of applying the arithmetic-geometric-harmonic mean inequality is

$$\frac{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2+\frac{\sinh_{2,q}^{-1}(x)}{x}}{2}\geq\sqrt{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2\frac{\sinh_{2,q}^{-1}(x)}{x}}$$
$$\geq\frac{2}{\frac{1}{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2}+\frac{1}{\frac{\sinh_{2,q}^{-1}(x)}{x}}}>1$$

and

$$\frac{2\tanh_{2,q}^{-1}(x)}{x}+\frac{\sinh_{2,q}^{-1}(x)}{x}\geq 3\sqrt[3]{\left(\frac{\tanh_{2,q}^{-1}(x)}{x}\right)^2\frac{\sinh_{2,q}^{-1}(x)}{x}}>3.$$

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Extensions of singular value inequalities for sector matrices

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Abstract. In this paper, we present singular value inequalities for matrices. As a consequence, we prove singular value inequalities for sector matrices. Moreover, we give singular value inequalities involving operator concave function, which are generalizations of some existing results.

Keywords: singular value, sector matrix, operator concave function.
MSC 2020: 15A45, 15A60, 15A18

1. Introduction

Throughout this paper, let M_n represent the set of all $n \times n$ complex matrices. I_n denotes the identity matrix. For two Hermitian matrices $A, B \in M_n$, we use $A \geq B$ to mean that $A - B$ is positive semidefinite. If the eigenvalues of matrix $A \in M_n$ are all real, the jth largest eigenvalue of A is denoted by $\lambda_j(A), j = 1, 2, \cdots, n$. The singular values $s_j(A) (j = 1, 2, \cdots, n)$ of A are the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ arrange in a decreasing order. For $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$, the matrices $\mathrm{Re}(A) = \frac{A+A^*}{2}$ and $\mathrm{Im}(A) = \frac{A-A^*}{2i}$ are called the real part and imaginary part of A, respectively. A real valued continuous function f on an interval J is called matrix concave of order n if $f(\alpha A + (1-\alpha)B) \geq \alpha f(A) + (1-\alpha)f(B)$ for any two Hermitian matrices $A, B \in M_n$ with spectrum in J and all $\alpha \in [0, 1]$. If f is operator concave function for all n, then it is called operator concave. It is well known that a continuous non-negative function f on $[0, \infty)$ is operator monotone if and only if f is operator concave.

The numerical range of $A \in M_n$ is described by

$$W(A) = \{x^*Ax | x \in C^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, we define a sector on the complex plane

$$S_\alpha = \{z \in C : \mathrm{Re}(z) > 0, |\mathrm{Im}(z)| \leq \tan\alpha \mathrm{Re}(z)\}.$$

Clearly, for some $\alpha \in [0, \frac{\pi}{2})$, if $W(A), W(B) \subset S_\alpha$, then $W(A + B) \subset S_\alpha$. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, then A is nonsingular. A matrix $A \in M_n$ is said to be sector matrix if its numerical range is contained in S_α, for some $\alpha \in [0, \frac{\pi}{2})$.

Garg and Aujla [1] proved that if $A, B \in M_n$ and $1 \le r \le 2$. Then

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \prod_{j=1}^{k} s_j(I_n + |A|^r) \prod_{j=1}^{k} s_j(I_n + |B|^r) \tag{1.1}$$

and

$$\prod_{j=1}^{k} s_j(I_n + f(|A+B|)) \le \prod_{j=1}^{k} s_j(I_n + f(|A|)) \prod_{j=1}^{k} s_j(I_n + f(|B|)), \tag{1.2}$$

where $f : [0, \infty) \to [0, \infty)$ is an operator concave function and $1 \le k \le n$.

Xue and Hu [2] showed that if $A, B \in M_n$ such that $W(A), W(B) \subset S_\alpha$. Then

$$\prod_{j=1}^{k} s_j(A+B) \le \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)) \tag{1.3}$$

and

$$\prod_{j=1}^{k} s_j(I_n + A + B) \le \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(A)) \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2}\mathrm{Re}(B)), \tag{1.4}$$

where $1 \le k \le n$.

Recently, Lin and Fu [3], Yang [4] and Nasiri and Furuichi [5] independently gave some singular value inequalities for sector matrices related to Garg and Aujla's results.

In this paper, we give some new singular value inequalities for sector matrices, which are generalizations of existing results.

2. Main results

We begin this section with the following lemmas which will turn out to be useful in the proof of our results.

Lemma 2.1 ([6]). *Let $A, B \in M_n$. There exist unitary matrices $U, V \in M_n$ such that*

$$|A+B| \le U^*|A|U + V^*|B|V.$$

Lemma 2.2 ([7]). *Let $A, B \in M_n$ be positive semidefinite matrices. Then $A\sharp B$ is the largest Hermitian matrix X such that*

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix}$$

is positive semidefinite.

Lemma 2.3 ([8]). *Let $A \in M_n$ be Hermitian matrix. Then*

$$\prod_{j=1}^{k} s_j(A) = \max |\det(U^*AU)|,$$

where maximum is taken over $n \times k$ matrices U for $U^*U = I_k$, $1 \leq k \leq n$.

Lemma 2.4 ([1]). *Let $A \in M_n$ be Hermitian matrix and B be positive definite matrix with $A < B, -A < B$. Then*

$$|\det A| < \det B.$$

Lemma 2.5 ([1]). *Let $A, B \in M_n$ be positive semidefinite matrices. Then*

$$\prod_{j=1}^{k} \lambda_j(A \sharp B) \leq (\prod_{j=1}^{k} \lambda_j(A)) \sharp (\prod_{j=1}^{k} \lambda_j(B)), 1 \leq k \leq n.$$

Lemma 2.6 ([6]). *Let $A, B \in M_n$. Then*

$$\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A) s_j(B), 1 \leq k \leq n.$$

Lemma 2.7 ([1]). *The inequality*

$$(1 + x^t)^r \leq (1 + x^r)^t$$

holds, for all $x > 0$ and $0 \leq r \leq t$.

Lemma 2.8 ([9]). *Let $A \in M_n$ be such that $W(A) \subset S_\alpha$ and $A = U|A|$ be the polar decomposition of A. Then*

$$|A| \leq \frac{\sec(\alpha)}{2} (\mathrm{Re}(A) + U^*(\mathrm{Re}(A))U).$$

Lemma 2.9 ([10]). *If $f : [0, +\infty) \to [0, +\infty)$ is operator monotone. Then*

$$f(\alpha t) \leq \alpha f(t),$$

for $\alpha \geq 1$.

Theorem 2.1. *Let $A, B \in M_n$ and $\mu > 0$. Then*

$$\prod_{j=1}^{k} s_j(|A + B|^r) \leq \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2})^{\frac{r}{2}},$$

where $1 \leq r \leq 2$ and $1 \leq k \leq n$.

Proof. For $A, B \in M_n$, by Lemma 2.1, there exist unitary matrices $V_1, V_2 \in M_n$ such that

$$|A+B| \leq V_1^*|A|V_1 + V_2 * |B|V_2. \tag{2.1}$$

By $\mu > 0$, we have

$$\begin{aligned} &\begin{bmatrix} \mu I_n & V_1^*\,|A|\,V_1 \\ V_2^*\,|B|\,V_2 & \mu I_n \end{bmatrix} \begin{bmatrix} \mu I_n & V_2^*\,|B|\,V_2 \\ V_1^*\,|A|\,V_1 & \mu I_n \end{bmatrix} \\ &= \begin{bmatrix} \mu^2 I_n + V_1^*\,|A|^2\,V_1 & \mu(V_1^*\,|A|\,V_1 + V_2^*\,|B|\,V_2) \\ \mu(V_1^*\,|A|\,V_1 \;+ V_2^*\,|B|\,V_2) & \mu^2 I_n + V_2^*\,|B|^2\,V_2 \end{bmatrix} \geq 0. \end{aligned} \tag{2.2}$$

Using (2.1), (2.2) and Lemma 2.2, we have

$$\begin{aligned} \pm\mu\,|A+B| &\leq \mu(V_1^*\,|A|\,V_1 + V_2^*\,|B|\,V_2) \\ &\leq (\mu^2 I_n + V_1^*\,|A|^2\,V_1)\sharp(\mu^2 I_n + V_2^*\,|B|^2\,V_2). \end{aligned} \tag{2.3}$$

By Lemma 2.3, there exists an $n \times k$ matrix U with $U^*U = I_k$ and

$$\prod_{j=1}^{k} s_j(|A+B|) = |\det(U^*(A+B)U)|\,, 1 \leq k \leq n. \tag{2.4}$$

By (2.3) and Lemma 2.4, we have

$$|\det(U^*\mu|A+B|U)| \leq \det[U^*((\mu^2 I_n + V_1^*\,|A|^2\,V_1)\sharp(\mu^2 I_n + V_2^*\,|B|^2\,V_2))U]. \tag{2.5}$$

Now, from (2.4), (2.5), Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned} \prod_{j=1}^{k} s_j(\mu\,|A+B|) &\leq \det[U^*((\mu^2 I_n + V_1^*\,|A|^2\,V_1)\sharp(\mu^2 I_n + V_2^*\,|B|^2\,V_2))U] \\ &\leq \max|\det[V^*((\mu^2 I_n + V_1^*\,|A|^2\,V_1)\sharp(\mu^2 I_n + V_2^*\,|B|^2\,V_2))V]| \\ &= \prod_{j=1}^{k} s_j((\mu^2 I_n + V_1^*\,|A|^2\,V_1)\sharp(\mu^2 I_n + V_2^*\,|B|^2\,V_2)) \\ &\quad (\text{by Lemma 2.3}) \\ &\leq \sqrt{\prod_{j=1}^{k} s_j(\mu^2 I_n + V_1^*\,|A|^2\,V_1) \prod_{j=1}^{k} s_j(\mu^2 I_n + V_2^*\,|B|^2\,V_2)} \\ &\leq \sqrt{\prod_{j=1}^{k} s_j(\mu^2 I_n + |A|^2) \prod_{j=1}^{k} s_j(\mu^2 I_n + |B|^2)}, 1 \leq k \leq n. \end{aligned}$$

That is

$$\begin{aligned}\prod_{j=1}^{k} s_j(|A+B|^2) &\le \frac{1}{\mu^{2k}} \prod_{j=1}^{k} s_j(\mu^2 I_n + |A|^2) \prod_{j=1}^{k} s_j(\mu^2 I_n + |B|^2)\\ &= \mu^{2k} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2}), 1 \le k \le n.\end{aligned}$$

Then

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2})^{\frac{r}{2}}.$$

This completes the proof. □

Substituting A and B with $\frac{A}{\mu}$ and $\frac{B}{\mu}$ in the inequality (1.1), respectively, we have the following inequality

$$\prod_{j=1}^{k} s_j(|A+B|^r) \le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^r}{\mu^r}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|^r}{\mu^r}). \tag{2.6}$$

Applying Lemma 2.7, we can obtain the following corollary, which is sharper than inequality (2.6).

Corollary 2.1. *Let $A, B \in M_n$ and $\mu > 0$. Then*

$$\begin{aligned}\prod_{j=1}^{k} s_j(|A+B|^r) &\le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2})^{\frac{r}{2}}\\ &\le \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^r}{\mu^r}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|^r}{\mu^r}),\end{aligned}$$

where $1 \le r \le 2$ and $1 \le k \le n$.

Proof. For $1 \le j \le n$ and $1 \le r \le 2$, by Lemma 2.7, we have

$$(1 + (\frac{s_j(A)}{\mu})^2)^{\frac{r}{2}} \le 1 + (\frac{s_j(A)}{\mu})^r.$$

That is

$$s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \le s_j(I_n + (\frac{|A|}{\mu})^r). \tag{2.7}$$

Now, from Theorem 2.1 and inequality (2.7), we get

$$\begin{aligned}\prod_{j=1}^{k} s_j(|A+B|^r) &\leq \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^2}{\mu^2})^{\frac{r}{2}} \prod_{j=1}^{k} s_j(I_n + \frac{|B|^2}{\mu^2})^{\frac{r}{2}} \\ &\leq \mu^{rk} \prod_{j=1}^{k} s_j(I_n + \frac{|A|^r}{\mu^r}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|^r}{\mu^r}).\end{aligned}$$

This completes the proof. □

Remark 2.1. Let $\mu = 1$ in Corollary 2.1. Obviously, Corollary 2.1 is a generalization of the inequality (1.1).

Using Corollary 2.1, we have the following Theorem which is a generalization of the inequality (1.3).

Theorem 2.2. *Let $A, B \in M_n$ such that $W(A), W(B) \subset S_\alpha$. Then*

$$\prod_{j=1}^{k} s_j(A+B) \leq \mu^k \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A)) s_j^2(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B)),$$

where $1 \leq k \leq n$ and $\mu > 0$.

Proof. Let U, V be unitary matrices, we have the following chain of inequalities:

$$\begin{aligned}\prod_{j=1}^{k} s_j(A+B) &= \prod_{j=1}^{k} s_j(|A+B|) \\ &\leq \mu^k \prod_{j=1}^{k} s_j(I_n + \frac{|A|}{\mu}) \prod_{j=1}^{k} s_j(I_n + \frac{|B|}{\mu}) \quad (\text{by Corollary 2.1}) \\ &\leq \mu^k \prod_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2\mu}(\mathrm{Re}(A) + U^*\mathrm{Re}(A)U)) \\ &\quad s_j(I_n + \frac{\sec(\alpha)}{2\mu}(\mathrm{Re}(B) + V^*\mathrm{Re}(B)V)) \quad (\text{by Lemma 2.8}) \\ &\leq \mu^k \prod_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A)) s_j(I_n + \frac{\sec(\alpha)}{2\mu}U^*\mathrm{Re}(A)U) \\ &\quad \prod_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B)) s_j(I_n + \frac{\sec(\alpha)}{2\mu}V^*\mathrm{Re}(B)V)\end{aligned}$$

(by (1.2))

$$= \mu^k \prod_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A)) s_j(U^*(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A))U)$$

$$\prod_{j=1}^{k} s_j(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B)) s_j(V^*(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B))V)$$

$$\leq \mu^k \prod_{j=1}^{k} s_j^2(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(A)) s_j^2(I_n + \frac{\sec(\alpha)}{2\mu}\mathrm{Re}(B))$$

(by Lemma 2.6).

This completes the proof. □

Theorem 2.3. *Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$ and $a, b > 0$. Then*

$$\prod_{j=1}^{k} s_j(I_n + f(|aA + bB|))$$

$$\leq \prod_{j=1}^{k} s_j^2(I_n + \sec(\alpha) f(\frac{a\mathrm{Re}(A)}{2})) \prod_{j=1}^{k} s_j^2(I_n + \sec(\alpha) f(\frac{b\mathrm{Re}(B)}{2})), \tag{2.8}$$

where $f : [0, +\infty) \to [0, +\infty)$ is operator concave function, $1 \leq k \leq n$.

Proof. Let U, V be unitary matrices, we have the following chain of inequalities:

$$\prod_{j=1}^{k} s_j(I_n + f(|aA + bB|))$$

$$\leq \prod_{j=1}^{k} s_j(I_n + f(a|A|)) \prod_{j=1}^{k} s_j(I_n + f(b|B|)) \quad \text{(by (1.2))}$$

$$\leq \prod_{j=1}^{k} s_j(I_n + f(\frac{a\sec(\alpha)}{2}(\mathrm{Re}(A) + U_1^*\mathrm{Re}(A)U_1)))$$

$$\prod_{j=1}^{k} s_j(I_n + f(\frac{a\sec(\alpha)}{2}(\mathrm{Re}(B) + U_2^*\mathrm{Re}(B)U_2))) \quad \text{(by Lemma 2.8)}$$

$$\leq \prod_{j=1}^{k} s_j(I_n + f(\frac{a\sec(\alpha)}{2}\mathrm{Re}(A)) s_j(I_n + f(\frac{a\sec(\alpha)}{2}U_1^*\mathrm{Re}(A)U_1))$$

$$
\begin{aligned}
&\prod_{j=1}^{k} s_j(I_n+f(\frac{b\sec(\alpha)}{2}\mathrm{Re}(B))s_j(I_n+f(\frac{a\sec(\alpha)}{2}U_2^*\mathrm{Re}(B)U_2)) \quad (\text{by}\,(1.2))\\
&\le \prod_{j=1}^{k} s_j^2(I_n+f(\frac{a\sec(\alpha)}{2}\mathrm{Re}(A))\prod_{j=1}^{k} s_j^2(I_n+f(\frac{b\sec(\alpha)}{2}\mathrm{Re}(B))\\
&\quad (\text{by}\,\text{Lemma}\,2.6)\\
&\le \prod_{j=1}^{k} s_j^2(I_n+\sec(\alpha)f(\frac{a\mathrm{Re}(A)}{2}))\prod_{j=1}^{k} s_j^2(I_n+\sec(\alpha)f(\frac{b\mathrm{Re}(B)}{2}))\\
&\quad (\text{by}\,\text{Lemma}\,2.9).
\end{aligned}
$$

This completes the proof. □

Remark 2.2. Let $f(t) = t$ and $a = b = 1$ in Theorem 2.3, we obtain the inequality (1.4).

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A remark on relative Hilali conjectures

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Abstract. S. Chouingou, M. A. Hilali, M. R. Hilali and A. Zaim have recently proved, in certain cases, a relative Hilali conjecture. This is an inequality about the dimensions of the kernel of homomorphisms of rational homotopy groups and rational homology groups, hence shall be called a Kernel-relative Hilali conjecture. In this paper we add another relative Hilali conjecture with respect to the cokernel of such homomorphisms, which shall be called a Cokernel-relative Hilali conjecture. We consider some examples for these conjectures and discuss conditions under which these conjectures hold and also conditions under which they are equivalent to each other. As byproducts of these computations, we show that $\dim(\pi_*(X)\otimes\mathbb{Q})$ and the formal dimension n_X of X have the same parity and that the Hilali conjecture holds when $\dim(\pi_*(X)\otimes\mathbb{Q}) \leq 4$.

Keywords: rational homotopy theory, rationally elliptic space, Hilali conjecture, relative Hilali conjecture.
MSC 2020: 55P62, 55N99, 55Q05

1. Introduction

The homotopy and homology ranks of a topological space X are respectively defined by $\dim(\pi_*(X)\otimes\mathbb{Q})$ and $\dim H_*(X;\mathbb{Q})$, where $\pi_*(X)\otimes\mathbb{Q} := \sum_{i\geqq 1}\pi_i(X)\otimes\mathbb{Q}$ and $H_*(X;\mathbb{Q}) := \sum_{i\geqq 0} H_i(X;\mathbb{Q})$. Since $\mathbb{Q}$ is a field, it follows from the Universal Coefficient Theorem for the homology group, involving the torsion-module $\mathrm{Tor}(A,B)$, that we have $H_i(X;\mathbb{Q}) \cong H_i(X)\otimes\mathbb{Q}$ where $H_i(X) := H_i(X;\mathbb{Z})$. So, we use $H_*(X)\otimes\mathbb{Q}$ instead of $H_*(X;\mathbb{Q})$.

A *rationally elliptic space* is a simply connected topological space X such that

$$\dim(\pi_*(X)\otimes\mathbb{Q}) < \infty \text{ and } \dim(H_*(X)\otimes\mathbb{Q}) < \infty.$$

In [9] M. R. Hilali conjectured that if X is a rationally elliptic space, then the following inequality holds:

$$\dim(\pi_*(X)\otimes\mathbb{Q}) \leqq \dim(H_*(X)\otimes\mathbb{Q})\,. \tag{1.1}$$

Namely, since X is simply connected, (1.1) means that

$$\dim\left(\bigoplus_{i\geqq 2}\pi_i(X)\otimes\mathbb{Q}\right) \leqq 1 + \dim\left(\bigoplus_{i\geqq 2} H_i(X)\otimes\mathbb{Q}\right).$$

Remark 1.1. Usually the Hilali conjecture is the following inequality, using the rational cohomology group:

$$\dim(\pi_*(X) \otimes \mathbb{Q}) \leqq \dim H^*(X; \mathbb{Q}). \tag{1.2}$$

However, since $\dim H_*(X;\mathbb{Q}) < \infty$ and $\dim H^*(X;\mathbb{Q}) < \infty$, namely $H_*(X;\mathbb{Q})$ and $H^*(X;\mathbb{Q})$ are finitely generated[1], we have $H^*(X;\mathbb{Q}) \cong Hom\left((H_*(X;\mathbb{Q}),\mathbb{Q}\right)$ (more precisely, $H^i(X;\mathbb{Q}) \cong Hom\left((H_i(X;\mathbb{Q}),\mathbb{Q}\right)$ for each i), so $H^*(X;\mathbb{Q}) \cong H_*(X;\mathbb{Q})$, hence $\dim H^*(X;\mathbb{Q}) = \dim H_*(X;\mathbb{Q})$. Therefore, (1.1) and (1.2) are the same. Since we use (rational) Hurewicz Theorem later, it is better to use homology groups instead of cohomology groups.

Remark 1.2. In [19] we showed the Hilali conjecture "modulo product", which is that for any rationally elliptic space X such that its fundamental group is an Abelian group, then there exists some integer n_0 such that for any $n \geq n_0$ the following strict inequality holds:

$$\dim\left(\pi_*(X^n) \otimes \mathbb{Q}\right) < \dim\left(H_*(X^n) \otimes \mathbb{Q}\right),$$

where X^n is the Cartesian product $X^n = \underbrace{X \times \cdots \times X}_{n}$. As to some work on such an integer n_0 and related topics, see [11, 12, 20].

In our previous paper [17] (also see [18]) we made the following conjecture, called *a relative Hilali conjecture*:

Conjecture 1.1. *For a continuous map $f : X \to Y$ of rationally elliptic spaces X and Y, the following inequality holds:*

$$\sum_{i \geqq 2} \dim\left(\operatorname{Ker}(\pi_i(f) \otimes \mathbb{Q})\right) \leqq 1 + \sum_{i \geqq 2} \dim\left(\operatorname{Ker}(H_i(f) \otimes \mathbb{Q})\right).$$

As remarked below (a remark right after Conjecture 2.1 below in §2), in the above conjecture it suffices to assume only that the source space X is rationally elliptic.

In [2, 21] S. Chouingou, M. A. Hilali, M. R. Hilali and A. Zaim have proved this relative conjecture positively in some cases. This relative conjecture is a conjecture using the kernel of the homomorphisms $\pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ and $H_*(f) \otimes \mathbb{Q} : H_*(X) \otimes \mathbb{Q} \to H_*(Y) \otimes \mathbb{Q}$. So, this shall be called *a Kernel-relative Hilali conjecture*, abusing words. In this note we add another relative conjecture, called *a Cokernel-relative Hilali conjecture*, using the cokernel of these two homomorphisms. We consider some examples for these two conjectures and we discuss conditions under which these two conjectures hold

1. If $H_*(X;\mathbb{Q})$ is not finitely generated, then we do not have $H^*(X;\mathbb{Q}) \cong H_*(X;\mathbb{Q})$. Indeed, if $H_*(X;\mathbb{Q}) = \oplus_{n\in\mathbb{N}}\mathbb{Q}$, which is note finitely generated, then $H^*(X;\mathbb{Q}) \cong Hom(H_*(X;\mathbb{Q}),\mathbb{Q}) = \prod_{n\in\mathbb{N}}\mathbb{Q}$. Thus, $H^*(X;\mathbb{Q}) \ncong H_*(X;\mathbb{Q})$.

and also conditions under which they are equivalent to each other. For example, if the above inequality (1.1) becomes equality for both X and Y, then for *any* continuous map $f : X \to Y$ the Kernel-relative Hilali conjecture holds if and only if the Cokernel-relative Hilali conjecture holds.

As byproducts of these computations and using the well-known stringent restrictions on homotopy groups, we show that if $\dim(\pi_*(X) \otimes \mathbb{Q})$ is odd (resp., even), the formal dimension n_X is odd (resp., even), and also we show that if $\dim(\pi_*(X) \otimes \mathbb{Q}) = 1, 2, 3, 4$, the Hilali conjecture holds.

In this paper we discuss without appealing to minimal models, although minimal models play important roles in rational homotopy theory.

2. Relative Hilali conjectures

In this section, for the sake of later presentation, we recall some basic ingredients of homotopical and homological aspects of a continuous map, for example, homotopical and homological Poincaré polynomial of a map.

Let $f : X \to Y$ be a continuous map of simply connected spaces X and Y of finite type. For the homomorphisms $H_i(f) \otimes \mathbb{Q} : H_i(X) \otimes \mathbb{Q} \to H_i(Y) \otimes \mathbb{Q}$ and $\pi_i(f) \otimes \mathbb{Q} : \pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$, we have the following exact sequences of finite dimensional $\mathbb{Q}$-vector spaces[2]:

$$\begin{aligned} &0 \to \operatorname{Ker}(H_i(f) \otimes \mathbb{Q}) \to H_i(X) \otimes \mathbb{Q} \\ &\qquad \to H_i(Y) \otimes \mathbb{Q} \to \operatorname{Coker}(H_i(f) \otimes \mathbb{Q}) \to 0, \quad \forall i \geqq 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &0 \to \operatorname{Ker}(\pi_i(f) \otimes \mathbb{Q}) \to \pi_i(X) \otimes \mathbb{Q} \\ &\qquad \to \pi_i(Y) \otimes \mathbb{Q} \to \operatorname{Coker}(\pi_i(f) \otimes \mathbb{Q}) \to 0, \quad \forall i \geqq 2. \end{aligned} \tag{2.2}$$

Since X and Y are simply connected, they are path-connected as well (by the definition of simply connectedness), thus we have

$$\mathbb{Q} \cong H_0(X) \otimes \mathbb{Q} \xrightarrow[\cong]{f_*} H_0(Y) \otimes \mathbb{Q} \cong \mathbb{Q},$$

so, $\operatorname{Ker}(H_0(f) \otimes \mathbb{Q}) = \operatorname{Coker}(H_0(f) \otimes \mathbb{Q}) = 0$. It follows from (2.1) and (2.2) that we get the following equalities: for $\forall i \geqq 2$

$$\begin{aligned} \dim(\operatorname{Ker}(H_i(f) \otimes \mathbb{Q})) - \dim(H_i(X) \otimes \mathbb{Q}) + \dim(H_i(Y) \otimes \mathbb{Q}) \\ - \dim(\operatorname{Coker}(H_i(f) \otimes \mathbb{Q})) = 0, \end{aligned}$$

$$\begin{aligned} \dim(\operatorname{Ker}(\pi_i(f) \otimes \mathbb{Q})) - \dim(\pi_i(X) \otimes \mathbb{Q}) + \dim(\pi_i(Y) \otimes \mathbb{Q}) \\ - \dim(\operatorname{Coker}(\pi_i(f) \otimes \mathbb{Q})) = 0. \end{aligned}$$

2. Recall that $\operatorname{Coker}(T) := B/\operatorname{Im}(T)$ for a linear map $T : A \to B$ of vector spaces.

For later use, we use the following notation.

$$\dim\left(\mathrm{Ker}(\pi_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\mathrm{Ker}(\pi_i(f)\otimes\mathbb{Q})\right),$$
$$\dim\left(\mathrm{Ker}(H_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\mathrm{Ker}(H_i(f)\otimes\mathbb{Q})\right),$$
$$\dim\left(\mathrm{Coker}(\pi_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\mathrm{Coker}(\pi_i(f)\otimes\mathbb{Q})\right),$$
$$\dim\left(\mathrm{Coker}(H_*(f)\otimes\mathbb{Q})\right) := \sum_{i\geqq 2}\dim\left(\mathrm{Coker}(H_i(f)\otimes\mathbb{Q})\right).$$

Definition 2.1. *Let $f : X \to Y$ be a continuous map of simply connected spaces X and Y.*

1. *If $\dim\left(\mathrm{Ker}(H_*(f)\otimes\mathbb{Q})\right) < \infty$ and $\dim\left(\mathrm{Ker}(\pi_*(f)\otimes\mathbb{Q})\right) < \infty$, then f is called rationally elliptic with respect to Kernel or rationally Kernel-elliptic.*
2. *If $\dim\left(\mathrm{Coker}(H_*(f)\otimes\mathbb{Q})\right) < \infty$ and $\dim\left(\mathrm{Coker}(\pi_*(f)\otimes\mathbb{Q})\right) < \infty$, then f is called rationally elliptic with respect to Cokernel or rationally Cokernel-elliptic.*
3. *If the map f is rationally elliptic with respect to both kernel and cokernel, then f is called rationally elliptic.*

Remark 2.1. Let $f : X \to Y$ be a continuous map of simply connected spaces X and Y.

1. If X is rationally elliptic, then f is rationally Kernel-elliptic.
2. If Y is rationally elliptic, then f is rationally Cokernel-elliptic.
3. If X and Y are both rationally elliptic, then f is rationally elliptic.

In our previous paper [17] (cf. [18]) we made the following conjecture, called *a relative Hilali conjecture*

Conjecture 2.1. *For a continuous map $f : X \to Y$ of simply connected rationally elliptic spaces X and Y, the following inequality holds:*

$$\sum_{i\geqq 2}\dim\left(\mathrm{Ker}(\pi_i(f)\otimes\mathbb{Q})\right) \leqq 1 + \sum_{i\geqq 2}\dim\left(\mathrm{Ker}(H_i(f)\otimes\mathbb{Q})\right).$$

It follows from the above Remark 2.1 that it suffices to require only the rational ellipticity of the source space X for the above Conjecture 2.1, which is a conjecture *as to the Kernel*. Due to Remark 2.1 (3), clearly Conjecture 2.1 can be modified as follows, *adding an inequality with respect to Cokernel*:

Conjecture 2.2. *For a rationally elliptic continuous map $f : X \to Y$ of simply connected spaces X and Y, the following inequalities hold:*

$$\sum_{i \geqq 2} \dim \left(\operatorname{Ker}(\pi_i(f) \otimes \mathbb{Q})\right) \leqq 1 + \sum_{i \geqq 2} \dim \left(\operatorname{Ker}(H_i(f) \otimes \mathbb{Q})\right), \tag{2.3}$$

$$\sum_{i \geqq 2} \dim \left(\operatorname{Coker}(\pi_i(f) \otimes \mathbb{Q})\right) \leqq 1 + \sum_{i \geqq 2} \dim \left(\operatorname{Coker}(H_i(f) \otimes \mathbb{Q})\right). \tag{2.4}$$

Here, we note that when the target space Y is contractible, the above conjecture (2.3) becomes the original Hilali conjecture. Similarly, when the source space X is contractible, the above conjecture (2.4) also becomes the original Hilali conjecture.

In order to make it clear, we call (2.3) and (2.4), respectively, *a Kernel-relative Hilali conjecture* and *a Cokernel-relative Hilali conjecture*, abusing words.

Remark 2.2. We note that if $f : A \to B$ is a linear map of two vector spaces, then we have

$$\begin{aligned}
\dim \operatorname{Coker} f &= \dim(B/\operatorname{im}(f)) \\
&= \dim B - \dim(\operatorname{im}(f)) \\
&= \dim B - \dim(A/\ker(f)) \\
&= \dim B - \dim A + \dim(\ker(f)).
\end{aligned}$$

Hence, the above (2.4) is also expressed as follows:

$$\begin{aligned}
&\sum_{i \geqq 2} \dim \left(\ker(\pi_i(f) \otimes \mathbb{Q})\right) + \sum_{i \geqq 2} \dim(\pi_i(Y) \otimes \mathbb{Q}) - \sum_{i \geqq 2} \dim(\pi_i(X) \otimes \mathbb{Q}) \\
&\leqq 1 + \sum_{i \geqq 2} \dim \left(\ker(H_i(f) \otimes \mathbb{Q})\right) + \sum_{i \geqq 2} \dim(H_i(Y) \otimes \mathbb{Q}) - \sum_{i \geqq 2} \dim(H_i(X) \otimes \mathbb{Q})).
\end{aligned}$$

It may be interesting to see whether these two conjectures are related to each other or not, namely whether (2.3) implies (2.4) and vice versa.

In [2, 21] S. Chouingou, M. A. Hilali, M. R. Hilali and A. Zaim have proved the above Kernel-relative Hilali conjecture (2.3) in some cases. Thus, it would be interesting to see whether the above Cokernel-relative Hilali conjecture also holds in these cases considered by Chouingou–Hilali–Hilali–Zaim.

3. Some examples

For discussion below, we use the following symbols for the sake of simplicity:

$$\varpi(X) := \sum_{i \geqq 2} \dim(\pi_i(X) \otimes \mathbb{Q}), \quad \eta(X) := \sum_{i \geqq 2} \dim(H_i(X) \otimes \mathbb{Q})),$$

$$\ker \varpi(f) := \sum_{i \geqq 2} \dim \left(\ker(\pi_i(f) \otimes \mathbb{Q})\right), \quad \ker \eta(f) := \sum_{i \geqq 2} \dim \left(\ker(H_i(f) \otimes \mathbb{Q})\right).$$

Thus, the Hilali conjecture is claiming

$$\varpi(X) \leq 1 + \eta(X),$$

i.e., either $\varpi(X) = 1 + \eta(X)$ or $\varpi(X) < 1 + \eta(X)$. The latter means that $\varpi(X) \leq \eta(X)$. Clearly, for any rationally elliptic space X, either $\varpi(X) \leq \eta(X)$ or $\varpi(X) > \eta(X)$. Therefore, the Hilali conjecture claims that if $\varpi(X) > \eta(X)$, then $\varpi(X)$ *exceeds* $\eta(X)$ *only by* 1; $\varpi(X) = 1 + \eta(X)$.

No counterexample to the Hilali conjecture has been found yet. If there exists a counterexample to the Hilali conjecture, then that would be a rationally elliptic space Z such that

$$\varpi(Z) = j + \eta(Z) \text{ for some integer } j \geq 2.$$

Simple typical examples for $\varpi(X) > \eta(X)$ are all the even dimensional spheres $S^{2k}(k \geq 1)$, by the following well-known results (due to Serre Finiteness Theorem [14, 15]):

$$\pi_i(S^{2k}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2k, 4k-1, \\ 0 & i \neq 2k, 4k-1, \end{cases} \qquad \pi_i(S^{2k+1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2k+1, \\ 0 & i \neq 2k+1. \end{cases}$$

$\varpi(S^{2k}) = 2$ and $\eta(S^{2k}) = 1$, thus $\varpi(S^{2k}) = 1 + \eta(S^{2k}) = 2$. $\varpi(S^{2k+1}) = \eta(S^{2k+1}) = 1$

For later computation, we recall the rational homotopy and homology groups of some familiar rationally elliptic spaces:

1. $\pi_k(\mathbb{RP}^n) = \pi_k(S^n)$ for $k > 1$. Hence, we have

$$\pi_k(\mathbb{RP}^n) \otimes \mathbb{Q} = \pi_k(S^n) \otimes \mathbb{Q}.$$

$$H_k(\mathbb{RP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{for } k = 0, n, \\ 0, & \text{for } k \neq 0, n. \end{cases}$$

So, $\varpi(\mathbb{RP}^n) = 1$ if n is odd and $\varpi(\mathbb{RP}^n) = 2$ if n is even. $\eta(\mathbb{RP}^n) = 1$. Thus, we have

$$\varpi(\mathbb{RP}^n) = \eta(\mathbb{RP}^n) = 1 \text{ for } n \text{ odd and } \varpi(\mathbb{RP}^n) = 1 + \eta(\mathbb{RP}^n) = 2 \text{ for } n \text{ even.}$$

2.

$$\pi_k(\mathbb{CP}^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & \text{for } k = 2, 2n+1, \\ 0, & \text{for } k \neq 2, 2n+1, \end{cases}$$

which follows from the long exact sequence of a fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{CP}^n$:

$$\cdots \to \pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{CP}^n) \to \pi_{k-1}(S^1) \to \pi_{k-1}(S^{2n+1}) \to \cdots,$$

$$H_k(\mathbb{CP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{for } k = 0 \text{ and } 2 \leq k \leq 2n \text{ for even } k, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\varpi(\mathbb{CP}^n) = 2$ and $\eta(\mathbb{CP}^n) = n$. Therefore, we have

$$\varpi(\mathbb{CP}^1) = 1 + \eta(\mathbb{CP}^1) = 2 \text{ and } \varpi(\mathbb{CP}^n) \leq \eta(\mathbb{CP}^n) \text{ for } n \geq 2.$$

We shall use the following terminology:

Definition 3.1. *Le X be a rationally elliptic space.*

1. *If $\varpi(X) \leq \eta(X)$, it is called a space of type $\varpi \leq \eta$, (e.g., S^{2k+1}, $\mathbb{RP}^{2k+1}$, $\mathbb{CP}^n$ for $n \geq 2$).*
2. *If $\varpi(X) = 1 + \eta(X)$, it is called a Hilali space*[3]*, (e.g., S^{2k}, $\mathbb{RP}^{2k}$. Here, $S^2 = \mathbb{CP}^1$.)*
3. *If $\varpi(X) = j + \eta(X)$ with an integer $j \geq 2$, it is called a space of type $\varpi = j + \eta(j \geq 2)$ or a non-Hilali space with $\varpi = j + \eta(j \geq 2)$.*

Or we can simplify these names as follows:

Definition 3.2. *A rationally elliptic space X such that $\varpi(X) = \gamma(X) + \eta(X)$, where $\gamma(X)$ is an integer called a homotopy–homology gap, is called a space of type $\varpi = \gamma + \eta$.*

1. *If $\gamma < 1$, it is a space of type $\varpi \leq \eta$, which shall be also called a standard space,*
2. *If $\gamma = 1$, it is a Hilali space,*
3. *If $\gamma > 1$, it is a non-Hilali space of type $\varpi = \gamma + \eta(\gamma \geq 2)$.*

Remark 3.1. The Hilali conjecture [9] claims that $\gamma \leq 1$ for any rationally elliptic space X.

Remark 3.2. Our previous result [19] about the Hilali conjecture "modulo product" (see Remark 1.2 above) means that for any rationally elliptic space X, in particular, whether it is a Hilali space or a non-Hilali space, there exists a certain integer N_0 such that for all integers $n \geq N_0$ the Cartesian product X^n of n copies of X becomes a standard space, i.e., even if $\varpi(X) > \eta(X)$, $\varpi(X^n) \leq \eta(X^n)$ for all integers $n \geq N_0$.

Example 3.1. Let Y be a rationally elliptic space. *The Kernel-relative Hilali conjecture holds for any continuous map $f : S^{2k+1} \to Y$.* Since $\pi_n(S^{2k+1}) \otimes \mathbb{Q} = 0$ for $n \neq 2k + 1$,

$$\ker\Big(f_* \otimes \mathbb{Q} : \pi_n(S^{2k+1}) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}\Big) = 0,$$

3. We call it so, since Hilali made such a conjecture.

for $n \neq 2k+1$. Hence, we consider only $f_* \otimes \mathbb{Q} : \pi_{2k+1}(S^{2k+1}) \otimes \mathbb{Q} \to \pi_{2k+1}(Y) \otimes \mathbb{Q}$, which is either injective or the zero homomorphism by the dimension reason since $\pi_{2k+1}(S^{2k+1}) \otimes \mathbb{Q} = \mathbb{Q}$ and $f_* \otimes \mathbb{Q}$ is a linear map of vector spaces over $\mathbb{Q}$.

$\ker \varpi(f) = 0$ if $f_* \otimes \mathbb{Q}$ is injective and $\ker \varpi(f) = 1$ if $f_* \otimes \mathbb{Q}$ is the zero homomorphism. Thus, $\ker \varpi(f) = 0$ or 1. As to the rational homology, it is the same, i.e., $\ker \eta(f) = 0$ or 1, hence $1 + \ker \eta(f) = 1$ or 2. Therefor we have

$$\ker \varpi(f) \leq 1 + \ker \eta(f). \tag{3.1}$$

Thus, the Kernel-relative Hilali conjecture holds for any continuous map $f : S^{2k+1} \to Y$.

Example 3.2. For a continuous map $f : S^{2k+1} \to Y$, let us consider whether the Cokernel-relative Hilali conjecture holds or not, i.e., we consider whether the following holds or not:

$$\ker \varpi(f) + \varpi(Y) - \varpi(S^{2k+1}) \leq 1 + \ker \eta(f) + \eta(Y) - \eta(S^{2k+1}). \tag{3.2}$$

Since $\varpi(S^{2k+1}) = \eta(S^{2k+1}) = 1$, the above (3.2) becomes

$$\ker \varpi(f) + \varpi(Y) \leq 1 + \ker \eta(f) + \eta(Y). \tag{3.3}$$

If Y is a space of type $\varpi \leq \eta$, then the inequality (3.3) holds due to the above (3.1). Therefore, if Y is a space of type $\varpi \leq \eta$, then the Cokernel-relative Hilali conjectures hold for any continuous map $f : S^{2k+1} \to Y$. This result still holds even if the source space S^{2k+1} is replaced by $S^{2k+1} \times S^{2m}$, because a key point in the above argument is the equality $\varpi(S^{2k+1}) = \eta(S^{2k+1})$, which is equal to 1 in this case, and we do have the equality $\varpi(S^{2k+1} \times S^{2m}) = \eta(S^{2k+1} \times S^{2m})$, which is equal to 3 in this case.

Remark 3.3. For a continuous map $f : S^{2k+1} \to Y$, we consider the cases when Y is not a space of type $\varpi \leq \eta$.

1. Let Y be a Hilali space, i.e., $\varpi(Y) = \eta(Y) + 1$. Then (3.3) becomes

$$\ker \varpi(f) \leq \ker \eta(f). \tag{3.4}$$

   Since we have that $\ker \varpi(f) = 0$ or 1 and $\ker \eta(f) = 0$ or 1, we need to check only the case when $\ker \varpi(f) = 1$, namely whether $\ker \varpi(f) = 1$ automatically implies $\ker \eta(f) = 1$ or not. $\ker \varpi(f) = 1$ implies that $f_* \otimes \mathbb{Q} : \pi_{2k+1}(S^{2k+1}) \otimes \mathbb{Q} \to \pi_{2k+1}(Y) \otimes \mathbb{Q}$ is the zero homomorphism. If we could claim that $f : S^{2k+1} \to Y$ is homotopic to a constant map, then $f_* : H_{2k+1}(S^{2k+1}; \mathbb{Q}) \to H_{2k+1}(Y; \mathbb{Q})$ is the zero homomorphism, thus $\ker \eta(f) = 1$, therefore we would get the above (3.4). However, $f_* \otimes \mathbb{Q} = 0$ for the homotopy groups *does not necessarily imply* that f is homotopic to a constant map[4]. So, we can say that if Y is a Hilali space and $f : S^{2k+1} \to Y$ is homotopic to a constant map, then the Cokernel-relative Hilali conjecture also holds.

2. Suppose that the Hilali conjecture does not hold, i.e., there is a non-Hilali space Y, i.e., there is a space Y such that $\varpi(Y) = \eta(Y) + j$ with $j \geq 2$. Then, (3.3) becomes

$$\ker \varpi(f) + j - 1 \leq \ker \eta(f), \tag{3.5}$$

which may not hold. If $j \geq 3$, then clearly (3.5) does not hold.

Now, by the above arguments, simply by the dimension reason, we can show the following corollary:

Corollary 3.1. *1. Let $\varpi(X) = 1$. Then, the Kernel-relative Hilali conjecture always holds for any continuous map $f : X \to Y$.*

2. Let $\varpi(X) = \eta(X) = 1$ and Y be of type $\varpi \leq \eta$. Then, the Kernel- and Cokernel-relative Hilali conjectures both hold for any continuous map $f : X \to Y$. (Note: In fact, $\varpi(X) = 1$ implies $\eta(X) = 1$ as we will see in §4 below.)

Remark 3.4. 1. A typical example for a space X such that $\varpi(X) = 1$ is the Eilenberg–Maclane space $K(\mathbb{Z}, n)$. A more general one for such a space is

$$K(\mathbb{Z} \oplus F_{01} \oplus \cdots \oplus F_{0m}, n_0) \times K(F_1, n_1) \times \cdots K(F_k, n_k), \tag{3.6}$$

where F_{0i} $(i = 1, \cdots, m)$ and F_i $(i = 1, \cdots, k)$ are finite abelian groups.

2. As to the case of $\eta(X) = 1$, as an example for such a space we can consider Moore space $M(G, n)$ (e.g., see [8, Example 2.40, p.143]), which is a homological analogue of Eilenberg–Maclane space, i.e., a CW complex X such that $H_n(X) \cong G$ and $\widetilde{H}_i(X) \cong 0$ for $i \neq n$. Here, we note that $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$ and $\widetilde{H}_i(X) \cong H_i(X)$ for $i \geq 1$. So, by the Künneth Theorem, a more general example of X such that $\eta(X) = 1$ is a "Moore space" version of the above (3.6). i.e.,

$$M(\mathbb{Z} \oplus F_{01} \oplus \cdots \oplus F_{0m}, n_0) \times M(F_1, n_1) \times \cdots M(F_k, n_k),$$

where $n_i \geq 2$ $(i = 0, 1, \cdots, k)$.

4. According to MathOverFlow "Maps which induce the same homomorphism on homotopy and homology groups are homotopic" (answered by Allen Hatcher), the composition of a degree one map $f : T^3 \to S^3$ with the Hopf map $g : S^3 \to S^2$ is trivial on homotopy groups, but $g \circ f$ is not homotopic to a constant map.

3. In the case of $\varpi(X) = \eta(X) = 1$, which, for example, S^{2n+1} satisfies as observed above, it does not seem to be so easy to come up with a general example of such a space X. For example, although $\varpi(K(\mathbb{Z}, n)) = 1$ for any n, if $n = 2k$ is even, then $\eta(K(\mathbb{Z}, 2k)) = \infty$ since $H^*(K(\mathbb{Z}, 2k); \mathbb{Q}) = \mathbb{Q}[\alpha]$ where $\alpha \in H^{2k}(K(\mathbb{Z}, 2k); \mathbb{Q}) \cong \mathbb{Q}$ is a generator, thus $1 = \varpi(K(\mathbb{Z}, 2k)) \neq \eta(K(\mathbb{Z}, 2k)) = \infty$. Note that, the Eilenberg–Maclane space $K(\mathbb{Z}, 2n+1)$ is rationally homotopy equivalent to the sphere S^{2n+1}. Since the Eilenberg–Maclane space $K(F, n)$ for a finite abelian group is rationally homotopy equivalent to a point for any integer n, for the following space

$$X := K(\mathbb{Z}; 2n+1) \times K(F_1, n_1) \times \cdots K(F_s, n_s) \tag{3.7}$$

with finite abelian groups $F_i (i = 1, \cdots, s)$, we have $\varpi(X) = \eta(X) = 1$.

Let $X = S^{2n_1+1} \times \cdots \times S^{2n_k+1}$, where $n_i \neq n_j$ if $i \neq j$. Then, clearly we have $\varpi(X) = k$, but we have $\eta(X) = 2^k - 1$ since $\dim(H^*(S^{2n_i+1}; \mathbb{Q})) = 2$. Hence, we have that $k = \varpi(X) < \eta(X) = 2^k - 1$ for $k \geq 2$. As to the case of $\varpi(X) = \eta(X) = 2$, an example of such a space is $\mathbb{CP}^2$ as observed above. So, we pose the following problem-conjecture:

Problem 3.1. *For each $n \geq 3$, give an example of a space X satisfying the equality $\varpi(X) = \eta(X) = n$. (See also §4.2 below). Or, we conjecture that there does not exist such a space X.*

Example 3.3. Let us consider the case when $Y = S^{2m}$ in the above Remark 3.3 (1). By the above discussion, it suffices to consider the homomorphism $f_* : H_{2k+1}(S^{2k+1}; \mathbb{Q}) \to H_{2k+1}(S^{2m}; \mathbb{Q}) = 0$, which is clearly the zero homomorphism, thus $\eta(f) = 1$. Therefore, (3.4) holds, thus the Cokernel-relative Hilali conjecture holds. Namely, *for any continuous map $f : S^{2k+1} \to S^{2m}$, the Kernel- and Cokernel-relative Hilali conjectures both hold.*

Example 3.4. Let X be homotopy equivalent to (3.7) and Y be homotopy equivalent to the following space

$$\prod_{i=1}^{k} K(\mathbb{Z}, 2m_i + 1) \times K(F_1', n_1') \times \cdots K(F_j', n_j'),$$

where $F_i' (i = 1, \cdots, j)$ is a finite abelian group. Note that, $\varpi(Y) \leq \eta(Y)$. Then, by Corollary 3.1, *for any continuous map $f : X \to Y$, the Kernel-and Cokernel-relative Hilali conjectures both hold.*

Example 3.5. Let us consider a continuous map $f : S^{2k} \to Y$ where $k \geq 1$ and Y is a simply connected rationally elliptic space. Since $\pi_n(S^{2k}) \otimes \mathbb{Q} = 0$ for $n \neq 2k, 4k+1$,

$$\ker\Big(f_* \otimes \mathbb{Q} : \pi_n(S^{2k+1}) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}\Big) = 0,$$

for $n \neq 2k, 4k+1$. Hence, we consider the following two cases:

$$(f_* \otimes \mathbb{Q})_{2k} : \pi_{2k}(S^{2k}) \otimes \mathbb{Q} \to \pi_{2k}(Y) \otimes \mathbb{Q},$$
$$(f_* \otimes \mathbb{Q})_{4k+1} : \pi_{4k+1}(S^{2k}) \otimes \mathbb{Q} \to \pi_{4k+1}(Y) \otimes \mathbb{Q},$$

each of which is either injective or the zero homomorphism by the dimension reason, in the same way as in Example 3.1. Hence, we have

$$\ker \pi(f) = \begin{cases} 0, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ are both injective,} \\ 1, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ is injecive and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ is the zero map,} \\ 1, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ is the zero map and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ is injective,} \\ 2, & \text{if } (f_* \otimes \mathbb{Q})_{2k} \text{ and } (f_* \otimes \mathbb{Q})_{4k+1} \text{ are both the zero map.} \end{cases}$$

Thus, $\ker \varpi(f) = 0, 1$ or 2. As to the homology, we consider

$$f_* : H_{2k}(S^{2k}; \mathbb{Q}) = \mathbb{Q} \to H_{2k}(Y; \mathbb{Q}),$$

which is either injective or the zero map. Hence, we have $\ker \eta(f) = 0$ or 1, hence $1 + \ker \eta(f) = 1$ or 2. Therefore, unless $(f_* \otimes \mathbb{Q})_{2k}$ and $(f_* \otimes \mathbb{Q})_{4k+1}$ are both the zero map, we have

$$\ker \varpi(f) \leq 1 + \ker \eta(f). \tag{3.8}$$

Thus, the Kernel-relative Hilali conjecture holds. If $(f_* \otimes \mathbb{Q})_{2k}$ and $(f_* \otimes \mathbb{Q})_{4k+1}$ are both the zero map *and* $f_* : H_{2k}(S^{2k}; \mathbb{Q}) = \mathbb{Q} \to H_{2k}(Y; \mathbb{Q})$ *is also the zero map*, e.g., if $f : S^{2k} \to Y$ is homotopic to a constant map, then we also have (3.8), thus the Kernel-relative Hilali conjecture holds. If $(f_* \otimes \mathbb{Q})_{2k}$ and $(f_* \otimes \mathbb{Q})_{4k+1}$ are both the zero map *and* $f_* : H_{2k}(S^{2k}; \mathbb{Q}) = \mathbb{Q} \to H_{2k}(Y; \mathbb{Q})$ *is injective*, then $\ker \varpi(f) = 2$ and $1 + \ker \eta(f) = 1$, thus the Kernel-relative Hilali conjecture *does not* hold.

For the Cokernel-relative Hilali conjecture, we consider whether the following holds or not.

$$\ker \varpi(f) + \varpi(Y) - \varpi(S^{2k}) \leq 1 + \ker \eta(f) + \eta(Y) - \eta(S^{2k}). \tag{3.9}$$

Since $\varpi(S^{2k}) = 2$ and $\eta(S^{2k}) = 1$, the above (3.9) becomes

$$\ker \varpi(f) + \varpi(Y) - 2 \leq \ker \eta(f) + \eta(Y), \tag{3.10}$$

in other words, we consider whether the following inequality holds or not

$$\ker \varpi(f) + \varpi(Y) \leq 2 + \ker \eta(f) + \eta(Y). \tag{3.11}$$

Here, we note that from the above discussion, for any space Y the following inequality always holds:

$$\ker \varpi(f) \leq 2 + \ker \eta(f). \tag{3.12}$$

1. If Y is a space of type $\varpi \leq \eta$, then the inequality (3.11) holds due to the above (3.12). Therefore, if Y is a space of type $\varpi \leq \eta$, then the Cokernel-relative Hilali conjecture *holds for any* continuous map $f : S^{2k} \to Y$.

2. If Y is not a space of type $\varpi \leq \eta$, say it is a Hilali space, i.e., $\varpi(Y) = \eta(Y) + 1$, then (3.11) becomes (3.8). In other words, in this case the Kernel-relative Hilali conjecture holds *if and only if* the Cokernel-relative Hilali conjecture holds.

3. Suppose that the Hilali conjecture does not hold, i.e., there is a non-Hilali space Y, i.e., there is a space Y such that $\varpi(Y) = \eta(Y) + j$ with $j \geq 2$. Then, (3.10) becomes

$$\ker \varpi(f) + j - 2 \leq \ker \eta(f), \tag{3.13}$$

which may not hold. If $j \geq 4$, then clearly (3.13) does not hold.

Proposition 3.1. *Let X and Y be simply connected rationally elliptic spaces of type $\varpi = \gamma_X + \eta$ and $\varpi = \gamma_Y + \eta$, respectively. Then, we have:*

1. *If $\gamma_Y \leq \gamma_X$, the Kernel-relative Hilali conjecture implies the Cokernel-relative Hilali conjecture.*

2. *If $\gamma_Y \geq \gamma_X$, the Cokernel-relative Hilali conjecture implies the Kernel-relative Hilali conjecture.*

3. *If both X and Y are Hilali spaces, the Kernel-relative Hilali conjecture holds if and only if the Cokernel-relative Hilali conjecture holds.*

Proof. The proof is simple, but we write it down.

1. Suppose that the Kernel-relative Hilali conjecture holds, i.e., $\ker \varpi(f) \leq 1 + \ker \eta(f)$. We have $\varpi(X) = \eta(X) + \gamma_X$ and $\varpi(Y) = \eta(Y) + \gamma_Y$. Since $\gamma_Y \leq \gamma_X$, we have

$$\ker \varpi(f) + \gamma_Y \leq 1 + \ker \eta(f) + \gamma_X. \tag{3.14}$$

Hence

$$\ker \varpi(f) + \varpi(Y) - \eta(Y) \leq 1 + \ker \eta(f) + \varpi(X) - \eta(X),$$

which implies

$$\ker \varpi(f) + \varpi(Y) - \varpi(X) \leq 1 + \ker \eta(f) + \eta(Y) - \eta(X), \tag{3.15}$$

which is nothing but the Cokernel-relative Hilali conjecture.

2. The Cokernel-relative Hilali conjecture, i.e., (3.15) implies (3.14). Hence, we have
$$\ker \varpi(f) + \gamma_Y - \gamma_X \leq 1 + \ker \eta(f),$$
which implies $\ker \varpi(f) \leq 1 + \ker \eta(f)$ because $\gamma_Y - \gamma_X \geq 0$. Hence, the Kernel-relative Hilali conjecture holds.

3. It is due to the above results, since $\gamma_X = \gamma_Y = 1$. $\square$

For any continuous map $f : X \to Y$, it is clear that we have $\ker \varpi(f) \leq \varpi(X)$ and $\ker \eta(f) \leq \eta(X)$, similarly we have $\operatorname{Coker} \varpi(f) \leq \varpi(Y)$ and $\operatorname{Coker} \eta(f) \leq \eta(Y)$. Let us set the gaps between these integers as follows:

$$\begin{aligned} &\ker \varpi(f) + \varpi_f = \varpi(X), \quad \ker \eta(f) + \eta_f = \eta(X), \\ &\operatorname{Coker} \varpi(f) + \operatorname{Co} \varpi_f = \varpi(Y), \quad \operatorname{Coker} \eta(f) + \operatorname{Co} \eta_f = \eta(Y). \end{aligned}$$

Proposition 3.2. *Let $f : X \to Y$ be a continuous map of simply connected rationally elliptic spaces.*

1. *If the Hilali conjecture holds for the source space X and $\eta_f \leq \varpi_f$, then the Kernel-relative Hilali conjecture holds.*

2. *If the Hilali conjecture holds for the target space Y and $\operatorname{Co} \eta_f \leq \operatorname{Co} \varpi_f$, then the Cokernel-relative Hilali conjecture holds.*

Proof. Since the second statement is proved in the same way as in the first one, we prove the first one.

$$\begin{aligned} \ker \varpi(f) &= \varpi(X) - \varpi_f \\ &\leq 1 + \eta(X) - \varpi_f \\ &\qquad\qquad \text{(since the Hilali conjecture holds for } X\text{: } \varpi(X) \leq 1 + \eta(X)) \\ &\leq 1 + \eta(X) - \eta_f \quad (\text{since } -\varpi_f \leq -\eta_f) \\ &= 1 + \eta(f). \end{aligned}$$
$\square$

Now, we observe that it follows from the fundamental homomorphism theorem on vector spaces that we have

$$\varpi_f = \operatorname{Co} \varpi_f, \qquad \eta_f = \operatorname{Co} \eta_f.$$

Corollary 3.2. *Suppose that the Hilali conjecture holds for any simply connected elliptic spaces. Let $f : X \to Y$ be a continuous map of simply connected elliptic spaces X and Y such that $\eta_f \leq \varpi_f$. Then, the Kernel- and Cokernel-relative Hilali conjectures both hold.*

4. Stringent restrictions on homotopy groups of rationally elliptic spaces

In this section we discuss some results which follow from some *stringent restrictions* on homotopy groups of rationally elliptic spaces (see [3], [4], [5]).

First we recall Halperin's theorems. For that, we set

$$\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}:=\bigoplus_{k\geq 1}\pi_{2k}(X)\otimes\mathbb{Q},\quad \pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}:=\bigoplus_{k\geq 0}\pi_{2k+1}(X)\otimes\mathbb{Q},$$
$$\varpi_{even}(X):=\dim\left(\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}\right),\quad \varpi_{odd}(X):=\dim\left(\pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}\right).$$

$$\chi^{\pi}(X):=\varpi_{even}(X)-\varpi_{odd}(X),$$

which is called the homotopical Euler–Poincaré characteristic of X and is a homotopical version of the (usual homological) Euler–Poincaré characteristic

$$\chi(X)=\chi_{\mathrm{even}}(X)-\chi_{\mathrm{odd}}(X),$$

where

$$\chi_{\mathrm{even}}(X):=\dim\left(H_{\mathrm{even}}(X;\mathbb{Q})\right),\quad \chi_{\mathrm{odd}}(X):=\dim\left(H_{\mathrm{odd}}(X;\mathbb{Q})\right),$$
$$H_{\mathrm{even}}(X;\mathbb{Q}):=\bigoplus_{k\geq 0}H_{2k}(X;\mathbb{Q}),\quad H_{\mathrm{odd}}(X;\mathbb{Q}):=\bigoplus_{k\geq 0}H_{2k+1}(X;\mathbb{Q}).$$

S. Halperin proved.

Theorem 4.1 ([7, Theorem 1])**.** *$\chi^{\pi}(X)\leq 0$ and $\chi(X)\geq 0$. Moreover, the following are equivalent:*

1. *$\chi^{\pi}(X)=0$.*
2. *$\chi(X)>0$.*
3. *$H_{odd}(X)\otimes\mathbb{Q}=0$.*

Remark 4.1. 1. In other words, $\chi^{\pi}(X)<0\Longleftrightarrow\chi(X)=0$.

2. The equivalence of the above (1), (2) and (3) was posed as a question in D. Sullivan's famous paper [16].

Let $y_1,\cdots,y_q$ be a basis of $\pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}$ and $x_1,\cdots,x_r$ be a basis of $\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}$. If $y_j\in\pi_{2b_j-1}(X)\otimes\mathbb{Q})$ and $x_i\in\pi_{2a_i}(X)\otimes\mathbb{Q}$, $2b_j-1$ and $2a_i$ are called the degrees of y_j and x_j. $(b_1,\cdots,b_q)$ and $(a_1,\cdots,a_r)$ are respectively called *b-exponents* and *a-exponents* of X in [6]. The largest integer n_X such that $H_{n_X}(X;\mathbb{Q})\neq 0$ is called the formal dimension of X. Halperin showed the following:

Theorem 4.2 ([7, Theorem 3' and Corollary 2])**.**

1. *$\sum_{j=1}^{q}(2b_i-1)\leq 2n_X-1$ and $\sum_{i=1}^{r}2a_j\leq n_X$.*

2. $n_X = \sum_{j=1}^{q}(2b_j - 1) - \sum_{i=1}^{r}(2a_i - 1)$.
3. *Betti numbers* $\beta_i = \dim H_i(X;\mathbb{Q})$ *satisfy Poincaré duality;* $\beta_i = \beta_{n_X - i}$.
4. *In the case when* $\chi^\pi(X) = 0$, *i.e.,* $q = r$, *Poincaré polynomial of* X *is*

$$P_X(t) = \frac{\prod_{i=1}^{q}(1-t^{2b_i})}{\prod_{i=1}^{q}(1-t^{2a_i})}. \tag{4.1}$$

In particular, $\chi(X) = P_X(-1) = P_X(1) = \dim(H_*(X)\otimes\mathbb{Q}) = \frac{\prod_{i=1}^{q} b_i}{\prod_{i=1}^{q} a_i}$.

Note that $\chi(X) = \frac{\prod_{i=1}^{q} b_i}{\prod_{i=1}^{q} a_i}$ follows from

$$\frac{\prod_{i=1}^{q}(1-t^{2b_i})}{\prod_{i=1}^{q}(1-t^{2a_i})} = \frac{\prod_{i=1}^{q}(1-(t^2)^{b_i})}{\prod_{i=1}^{q}(1-(t^2)^{a_i})} = \frac{\prod_{i=1}^{q}(1+t^2+\cdots+(t^2)^{b_i-1})}{\prod_{i=1}^{q}(1+t^2+\cdots+(t^2)^{a_i-1})}.$$

Definition 4.1 ([6, Definition, pp.117–118])**.** *Let* $B = (b_1, b_2, \cdots, b_q)$ *and* $A = (a_1, a_2, \cdots, a_r)$ *be two finite sequences of positive integers.*

1. *We say that* $(B;A)$ *satisfies strong arithmetic condition (abbr. S.A.C.) if for every subsequence* A^* *of* A *of length* s $(1 \le s \le r)$ *there exists at least* s *elements* b_j*'s of* B *such that*

$$b_j = \sum_{a_i \in A^*} \gamma_{ij} a_i \tag{4.2}$$

where γ_{ij} *is a non-negative integer such that* $\sum_{a_i\in A^*}\gamma_{ij} \ge 2$.

2. *If* $\sum_{a_i\in A^*}\gamma_{ij} \ge 2$ *is not required, then we say that* (B, A) *satisfies arithmetic condition (abbr. A.C.).*

Thus, in both cases, it is necessary that $r \le q$.

In [6, Theorem 1, p.118] J. B. Friedlander and S. Halperin show the following characterization theorem about a pair $(B;A)$ satisfying S.A.C.

Theorem 4.3 (Friedlander–Halperin Theorem)**.** *Let* $B = (b_1, b_2, \cdots, b_q)$ *and* $A = (a_1, a_2, \cdots, a_r)$ *be a pair of sequences of positive intgers. The following conditions are equivalent:*

1. (B, A) *satisfies S.A.C.*
2. *The sequences* B *and* A *are respectively the b-exponents and a-exponents of a rationally elliptic space* X.

Moreover, if $b_i \ge 2$ *for all* i *and S.A.C. holds, then* X *may be chosen to be simply connected; if in addition* $q > r$, X *may be taken to be a closed manifold.*

Remark 4.2. $(b_1, b_2, \cdots, b_q)$ and $(a_1, a_2, \cdots, a_r)$ are respectively called *"odd" exponents* and *"even" exponents* of X in Félix–Halperin–Thomas's book [4].

In fact, from S.A.C., i.e., (4.2), we get the following result:

Lemma 4.1 ([6, 2.5. Lemma])**.** *If $B = (b_1, b_2, \cdots, b_q)$; $b_1 \geq b_2 \geq \cdots \geq b_q$, and $A = (a_1, a_2, \cdots, a_r)$; $a_1 \geq a_2 \geq \cdots \geq a_r$. If $(B; A)$ satisfies S.A.C, then $b_i \geq 2a_i$ for $1 \leq i \leq r$.*

Remark 4.3. Usually we consider the following order $b_1 \leq b_2 \leq \cdots \leq b_q$ for $B = (b_1, b_2, \cdots, b_q)$ and $a_1 \leq a_2 \leq \cdots \leq a_r$ for $A = (a_1, a_2, \cdots, a_r)$, but in order to prove the above lemma and also for the description of the statement of the lemma, the above descending order in the lemma is better.

Using Lemma 4.1 we can get the following formulas:

Corollary 4.1 ([6, 1.3. Corollary, p.118])**.**

1. $n_X \geq q + r = \dim(\pi_*(X) \otimes \mathbb{Q})$.
2. $n_X \geq \sum_{j=1}^{q} b_j$.
3. $2n_X - 1 \geq \sum_{j=1}^{q}(2b_j - 1)$.
4. $n_X \geq \sum_{i=1}^{r} 2a_i$.

Corollary 4.2 ([1, Proposition 2.1])**.** *If $q = r$, then the Hilali conjecture holds.*

Proof. $\dim(\pi_*(X) \otimes \mathbb{Q}) = 2q$. It follows from Theorem 4.2 (4) and Lemma 4.1 that

$$\dim(H_*(X;\mathbb{Q})) = \frac{\prod_{i=1}^{q} b_i}{\prod_{i=1}^{q} a_i} \geq \frac{\prod_{i=1}^{q} 2a_i}{\prod_{i=1}^{q} a_i} = 2^q, \tag{4.3}$$

which is [6, 2.6. Proposition (3)]. Since $2q \leq 2^q$ $(q \geq 1)$, we have

$$\dim(\pi_*(X) \otimes \mathbb{Q}) \leq \dim(H_*(X;\mathbb{Q})). \qquad \square$$

Next, we discuss the parity of $\varpi(X)$ and $\eta(X)$. We can show the following:

Lemma 4.2. *If $\varpi(X)$ is odd, then $\eta(X)$ is also odd.*

Proof. Suppose that $\varpi(X)$ is odd. Then, the homotopical Euler characteristic $\chi^\pi(X) = \varpi_{even}(X) - \varpi_{odd}(X) < 0$. Indeed, by the dichotomy, $\chi^\pi(X) \leq 0$. If $\chi^\pi(X) = \varpi_{even}(X) - \varpi_{odd}(X) = 0$, then $\varpi_{even}(X) = \varpi_{odd}(X)$, thus $\varpi(X) = \varpi_{even}(X) + \varpi_{odd}(X)$ is even, which is a contradiction. Now, it follows from Theorem 4.2 that $\chi^\pi(X) < 0$ implies that the Euler–Poincaré characteristic $\chi(X) = 0$, i.e., $\chi(X) = \chi_{\text{even}}(X) - \chi_{\text{odd}}(X) = 0$, thus $\chi_{\text{even}}(X) = \chi_{\text{odd}}(X)$. Hence $1 + \eta(X) = \chi_{\text{even}}(X) + \chi_{\text{odd}}(X)$ is even, thus $\eta(X)$ is also odd. $\square$

Remark 4.4. One might be tempted to expect that if $\varpi(X)$ is even, then $\eta(X)$ would be also even, but it is not the case. A very simple counterexample is $X = S^{2n+1} \times S^{2m+1}$. Then, $\varpi(X) = 2$, but $\eta(X) = 2^2 - 1 = 3$ is odd. In general, consider $X = S^{2n_1+1} \times \cdots \times S^{2n_k+1}$. Then, $\varpi(X) = k$ and $\eta(X) = 2^k - 1$, thus whether $\varpi(X) = k$ is even or odd, $\eta(X) = 2^k - 1$ is always odd. In fact, in the case when $\varpi(X)$ is even, $\eta(X)$ can be both even and odd. A typical example for this is the complex projective space $\mathbb{CP}^n$. $\varpi(\mathbb{CP}^n) = 2$ is even for any n, but the parity of $\eta(\mathbb{CP}^n) = n$ depends on the complex dimension n.

Corollary 4.3. *Suppose that the Hilali conjecture holds. If X is a rationally elliptic space such that $\dim(\pi_*(X) \otimes \mathbb{Q})$ is odd, then $\dim(\pi_*(X) \otimes \mathbb{Q}) < \dim H_*(X;\mathbb{Q})$.*

Proposition 4.1. *Let X be a rationally elliptic space. The parity of $\varpi(X)$ is the same as the parity of the formal dimension n_X, i.e., if $\varpi(X)$ is odd (resp., even), then its formal dimension n_X is odd (resp., even).*

Proof. Let $p := \varpi(X)$ and $e := \varpi_{even}(X)$. So, $\varpi_{odd}(X) = p - e$. Since $\chi^\pi(X) = \varpi_{even}(X) - \varpi_{odd}(X) \leq 0$, thus $e \leq p - e$. It follows from Theorem 4.2 (2) that if $e = 0$, then we have

$$n_X = \sum_{i=1}^{p}(2b_i - 1) = 2\sum_{i=1}^{p} b_i - p, \tag{4.4}$$

and if $e \geq 1$, then we have

$$n_X = \sum_{i=1}^{p-e}(2b_i - 1) - \sum_{j=1}^{e}(2a_j - 1),$$

which is

$$n_X = 2\sum_{i=1}^{p-e} b_i - (p-e) - 2\sum_{j=1}^{e} a_j + e = 2\left(\sum_{i=1}^{p-e} b_i - \sum_{j=1}^{e} a_j + e\right) - p. \tag{4.5}$$

Therefore, it follows from (4.4) and (4.5) that if $p = \varpi(X)$ is odd (resp., even), then n_X is odd (resp., even). $\square$

Remark 4.5. First we note that (4.5) can be also written as follows:

$$n_X = 2\left(\sum_{i=1}^{p-e} b_i - \sum_{j=1}^{e} a_j\right) - (p - 2e).$$

The parity of $p - 2e$, which is $-\chi^\pi(X) = \varpi_{odd}(X) - \varpi_{even}(X)$, is also the same as the parity of the formal dimension n_X. For example, in [13, §2] Nakamura and Yamaguchi call $(2a_1, 2a_2, \cdots, 2a_n : 2b_1 - 1, 2b_2 - 1, \cdots, 2b_{n+p} - 1)$ a homotopy rank type of X and all the *homotopy rank types* with the formal dimension ≤ 16 are listed in [13, §3]. In their list, if the formal dimension, denoted fd, is even (resp. odd), then p is even (resp. odd). Note that, their p is equal to our $p-2e$. Also note that clearly the parity of $p-2e$ is the same as the parity of $p = \varpi(X)$.

Finally, we discuss lower bounds of $\eta(X)$ for some cases. Before discussion, we recall Klaus–Kreck's rational Hurewicz theorem, which is a version stronger than the usual one:

Theorem 4.4 ([10, Theorem 1.1]). *Let X be a simply connected topological space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $1 < i < r$. Then, the Hurewicz map induces an isomorphism*

$$H : \pi_i(X) \otimes \mathbb{Q} \to H_i(X; \mathbb{Q})$$

for $1 < i < 2r-1$ and a surjection for $i = 2r-1$.

Theorem 4.5. *For any rationally elliptic space X such that $1 \leq \varpi(X) \leq 4$, the Hilali conjecture holds. To be more precise,*

1. *If $\varpi(X) = 1$, then $\eta(X) = 1$, thus $1 = \varpi(X) < 1 + \eta(X) = 2$.*
2. *If $\varpi(X) = 2$, then $\eta(X) \geq 1$, thus $\varpi(X) \leq 1 + \eta(X)$.*
3. *If $\varpi(X) = 3$, then $\eta(X) \geq 3$, thus $\varpi(X) < 1 + \eta(X)$.*
4. *If $\varpi(X) = 4$, then $\eta(X) \geq 3$, thus $\varpi(X) \leq 1 + \eta(X)$.*

Proof. First we recall that

$$\varpi(X) = \varpi_{\text{even}}(X) + \varpi_{\text{odd}}(X) \quad \text{and} \quad \varpi_{\text{even}}(X) \leq \varpi_{\text{odd}}(X). \tag{4.6}$$

1. Let $\varpi(X) = 1$. It follows from (4.6) that $\varpi_{\text{even}}(X) = 0$ and $\varpi_{\text{odd}}(X) = 1$. Hence, $\pi_{2b-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ for some odd integer $2b-1$ ($b \geq 2$) (since X is simply connected) and $\pi_i(X) \otimes \mathbb{Q} \cong 0$ for $i \neq 2b-1$. It follows from Theorem 4.2 (2) that the formal dimension $n_X = 2b-1$ and $H_{2b-1}(X;\mathbb{Q}) \cong \mathbb{Q}$ and it also follows from the rational Hurewicz Theorem that $H_i(X;\mathbb{Q}) \cong 0$ for $i \neq 0, 2b-1$. Thus, $\eta(X) = 1$.
2. Let $\varpi(X) = 2$. It follows from (4.6) that we have two possibilities:
   (a) $\varpi_{\text{even}}(X) = 0$ and $\varpi_{\text{odd}}(X) = 2$. In this case we have

   $$\pi_{2b_1-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}, \quad \pi_{2b_2-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}, \text{ where } 2 \leq b_1 \leq b_2. \tag{4.7}$$

   Here, we are a bit sloppy. If $b_1 = b_2$, then the above (4.7) is really understood to mean the following:

   $$\pi_{2b_1-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q}.$$

It follows from Theorem 4.2 (2) that the formal dimension $n_X = 2b_1 - 1 + 2b_2 - 1 = 2(b_1 + b_2) - 2$, which is greater than $2b_2 - 1$ since $2b_1 - 1 \geq 0$ (in fact, $2b_1 - 1 \geq 3$.) It follows from the Hurewicz Theorem that $H_{2b_1-1}(X;\mathbb{Q}) \cong \mathbb{Q}$ if $b_1 < b_2$ and $H_{2b_1-1}(X;\mathbb{Q}) \cong \mathbb{Q}\oplus\mathbb{Q}$ if $b_1 = b_2$. In the case when $b_1 < b_2$, by the Poincaré duality (Theorem 4.2 (3)) we do have $H_{2b_2-1}(X;\mathbb{Q}) \cong \mathbb{Q}$ since $n_X - (2b_1 - 1) = 2b_2 - 1$. Hence, in any case we can see that $\eta(X) \geq 3$.

(b) $\varpi_{\text{even}}(X) = 1$ and $\varpi_{\text{odd}}(X) = 1$. It follows from Corollary 4.2 that the Hilali conjecture holds, thus we are done. However, in this paper we take a more direct approach in order to see more information about $\eta(X)$.

$$\pi_{2b-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q} \text{ and } \pi_{2a}(X) \otimes \mathbb{Q} \cong \mathbb{Q}, \text{ where } b \geq 2 \text{ and } a \geq 1.$$

The formal dimension $n_X = 2b-1-(2a-1) = 2b-2a$. It follows from Theorem 4.2 that we have $2b - 1 \leq 2(2b - 2a) - 1$ and $2a \leq 2b - 2a$, both of which is the same inequality $4a \leq 2b$, i.e., $2a \leq b$. Thus, we have $2a < 2b - 1$ since $b - 1 \geq 1$. Then, we can see that we have the following orders:

$$2a \leq b \leq 2b - 2a < 2b - 1.$$

Hence, we have $H_{2b-2a}(X;\mathbb{Q}) \cong \mathbb{Q}$ and it follows from the rational Hurewicz theorem that $H_{2a}(X;\mathbb{Q}) \cong \mathbb{Q}$.

i. If $b = 2a$, then $2b - 2a = 2a$. Thus, we have $\eta(X) = 1$.

ii. If $2a < b < 3a$, then $(2b - 2a) - 2a = 2b - 4a = 2(b - 2a) > 0$ and $2b - 4a < 2a$. Thus, we have the following orders:

$$2b - 4a < 2a < 2b - 2a.$$

Then, it follows from the Poincaré duality that $H_{2b-4a}(X;\mathbb{Q}) \cong Q$. However, since $\pi_i(X) \otimes \mathbb{Q} \cong 0$ for $2 \leq i2a$, the rational Hurewicz theorem implies that $H_{2b-4a}(X;\mathbb{Q}) \cong 0$. Therefore, the case $2a < b < 3a$ is *ruled out*. This is a stringent restriction due to the Poincaré duality.

iii. If $b = 3a$, then $2b - 2a = 4a$ and $2b - 4a = 2a$, thus we have $\eta(X) = 2$.

iv. If $b > 3a$, then we have the following orders:

$$2a < 2b - 4a < 2b - 2a.$$

Then, by the Poincaré duality $H_{2b-4a}(X;\mathbb{Q}) \cong \mathbb{Q}$, hence $\eta(X) \geq 3$.

In any case we have $\eta(X) \geq 1$, thus we have $\pi(X) \leq 1 + \eta(X)$.

3. Let $\varpi(X) = 3$. In this case we have the following two possibilities:

   (a) $\varpi_{\text{even}}(X) = 0$ and $\varpi_{\text{odd}}(X) = 3$. In this case we have

   (4.8) $\pi_{2b_i-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$, where $2 \leq b_1 \leq b_2 \leq b_3$

   As above, here we are a bit sloppy. E.g., if $b_1 = b_2$, then the above (4.8) is really understood to mean the following:

   $$\pi_{2b_1-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q} \oplus \mathbb{Q} \text{ and } \pi_{2b_3-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q}$$

   The formal dimension $n_X = 2(b_1 + b_2 + b_3) - 3$. If $b_1 < b_2$, then by the Poincaré duality $H_{2(b_2+b_3)-2}(X;\mathbb{Q}) \cong \mathbb{Q}$. Hence, $\eta(X) \geq 3$. If $b_1 = b_2$, then by the Poincaré duality $H_{2(b_1+b_3)-2}(X;\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$. Hence, $\eta(X) \geq 5$. In any case we have $\eta(X) \geq 3$.

   (b) $\varpi_{\text{even}}(X) = 1$ and $\varpi_{\text{odd}}(X) = 2$. In this case we have

   $$\pi_{2b_i-1}(X) \otimes \mathbb{Q} \cong \mathbb{Q} \text{ and } \pi_{2a}(X) \otimes \mathbb{Q} \cong \mathbb{Q},$$

   where $2 \leq b_1 \leq b_2$ and $a \geq 1$. The formal dimension $n_X = 2b_1 + 2b_2 - 2a - 1$. It follows from Theorem 4.2 that $4a \leq 2b_1 + 2b_2 - 1$, which is in fact $4a < 2b_1 + 2b_2 - 1$ since $4a$ is even and $2b_1 + 2b_2 - 1$ is odd. Hence, $4a \leq 2b_1 + 2b_2 - 1 - 1 = (2b_1 - 1) + (2b_2 - 1) \leq 2(2b_2 - 1)$ since $b_1 \leq b_2$. If $b_1 < b_2$, then $2a < 2b_2 - 1$. If $b_1 = b_2$, then $2a \leq 2b_1 - 1$. Thus, two possibilities: $2b_1 - 1 < 2a$ and $2a < 2b_1 - 1$. In any case by the Hurewicz theorem $H_{2a}(X;\mathbb{Q}) \cong \mathbb{Q}$ or $H_{2b_1-1}(X;\mathbb{Q}) \cong \mathbb{Q}$, and by the Poincareé duality we have $H_{2b_1+2b_2-1}(X;\mathbb{Q}) \cong \mathbb{Q}$ or $H_{2b_2-2a}(X;\mathbb{Q}) \cong \mathbb{Q}$. Hence, we can see $\eta(X) \geq 3$.

   In any case, we can see that $\eta(X) \geq 3$. The above argument is quite detailed. Here, is a very simpler argument, which is as follows. In both cases (a) and (b), we can see that $\eta(X) \geq 2$ since $\dim H_{2b_1-1}(X;\mathbb{Q}) \geq 1$ and $H_{n_X}(X;\mathbb{Q}) \cong \mathbb{Q}$ in the case (a) (note that $2b_1 - 1 < n_X$), and $\dim H_{2b_1-1}(X;\mathbb{Q}) = 1$ or $\dim H_{2a}(X;\mathbb{Q}) = 1$ and $H_{n_X}(X;\mathbb{Q}) \cong \mathbb{Q}$ in the case (b). Since $\eta(X)$ has to be odd, it follows that $\eta(X) \geq 3$.

4. $\varpi(X) = 4$. In this case we have the following cases

   (a) $\pi_{\text{even}}(X) = 0$ and $\pi_{\text{odd}}(X) = 4$: Consider the degrees:

   $$2b_1 - 1, 2b_2 - 1, 2b_3 - 1, 2b_4 - 1, \quad (2 \leq b_1 \leq b_2 \leq b_3 \leq b_4).$$

   $n_X = \sum(2b_i - 1)$. Since $\dim(\pi_{2b_1-1}(X) \otimes \mathbb{Q}) \geq 1$ (because $b_1 \leq b_2 \leq b_3 \leq b_4$), it follows from the Hurewicz theorem and the Poincaré duality that $\eta(X) \geq 1 + 1 + 1 = 3$. If $2b_i - 1 \leq 2(2b_1 - 2)$ holds, then the inequality $\eta(X) \geq 3$ can be sharpened to $\eta(X) \geq 2i + 1$, which follows from the rational Hurewicz Theorem. Here, we note that $i \geq 1$, since $2b_1 - 1 \leq 2(2b_1 - 2)$, i.e., $3 \leq 2b_1$, which holds since $b_1 \geq 2$. In any case, we have $\varpi(X) \leq 1 + \eta(X)$.

(b) $\varpi_{\text{even}}(X) = 1$ and $\varpi_{\text{odd}}(X) = 3$: Consider the degrees:

$$2b_1 - 1, 2b_2 - 1, 2b_3 - 1, 2a, \quad (2 \le b_1 \le b_2 \le b_3, a \ge 1).$$

$n_X = 2(b_1 + b_2 + b_3 - a) - 2$. Whether $2b_1 - 1 < 2a$ or $2a < 2b_1 - 1$, by the Hurewicz Theorem $\beta_{2b_1-1} \ge 1$ or $\beta_{2a} \ge 1$, thus by the Poincaré duality $\beta_{n_X-2b_1-1} \ge 1$ or $\beta_{n_X-2a} \ge 1$. Since $\beta_{n_X} = 1$, we have $\eta(X) \ge 3$. Therefore, $\varpi(X) \le 1 + \eta(X)$. Here, we need to be a bit careful about β_{n_X-2a}. We need to check whether $n_X - 2a = 2a$. Namely, if $2a < 2b_1 - 1$ and $n_X - 2a = 2a$, in which case we cannot use the trick of Poincaré duality, hence $\eta(X) \ge 2$ instead of $\eta(X) \ge 3$. However, we do have $n_X - 2a > 2a$. Indeed $n_X - 2a - 2a = (2b_1 - 1) + (2b_2 - 1) + (2b_3 - 1) - (2a - 1) - 2a - 2a = \{(2b_1 - 1) - 2a\} + \{(2b_2 - 1) - 2a\} + \{(2b_3 - 1) - 2a\} + 1 \ge 4$, because $2a < 2b_1 - 1$ and $b_1 \le b_2 \le b_3$.

(c) $\varpi_{\text{even}}(X) = 2$ and $\varpi_{\text{odd}}(X) = 2$: By Corollary 4.2 we do know that the Hilali conjecture holds, thus $\eta(X) \ge 3$. However, let us see this without using this corollary. Consider the degrees:

$$2b_1 - 1, 2b_2 - 1, 2a_1, 2a_2, \quad (2 \le b_1 \le b_2, 1 \le a_1 \le a_2).$$

It follows from Lemma 4.1 that the following cases are possible:

i. $2a_1 < 2b_1 - 1 < 2a_2 < 2b_2 - 1$.
ii. $2a_1 \le 2a_2 < 2b_1 - 1 \le 2b_2 - 1$.

$n_X = 2(b_1 + b_2) - 2(a_1 + a_2)$. Then, in which case is it possible that $n_X - 2a_1 = 2a_1$? $n_X - 2a_1 - 2a_1 = 2(b_1 + b_2) - 2(a_1 + a_2) - 4a_1 = 2(b_1 - 3a_1) + 2(b_2 - a_2) \ge 2(2a_1 - 3a_1) + 2(2a_2 - a_2) = -2a_1 + 2a_2 = 2(a_2 - a_1)$. So, when $a_1 = a_2$, $b_1 = 2a_1$ and $b_2 = 2a_2$, we do have $n_X - 2a_1 = 2a_1$. In this case, surely we have $\eta(X) \ge 2 + 1 = 3$, since $H_{2a_1}(X;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$ and $H_{n_X}(X;\mathbb{Q}) = \mathbb{Q}$. Otherwise we have $n_X - 2a_1 > 0$ and $n_X - 2a_1 \ne 2a_1$. In this case we also have $\eta(X) \ge 3$, since $H_{2a_1}(X;\mathbb{Q}) = \mathbb{Q}$, $H_{n_X-2a_1}(X;\mathbb{Q}) = \mathbb{Q}$ by the Poincaré duality and $H_{n_X}(X;\mathbb{Q}) = \mathbb{Q}$. In any case we do have $\eta(X) \ge 3$.

□

Remark 4.6. Let $\varpi(X) = 5$. Then, it follows from Lemma 4.2 that $\eta(X)$ is odd. By an analysis as above, we see that $\eta(X) \ge 2$, hence $\eta(X) \ge 3$ since $\eta(X)$ is odd. If $\eta(X) \ge 5$, then the Hilali conjecture holds. If not, there would exist a counterexample such that $\varpi(X) = 5$ and $\eta(X) = 3$, i.e., $5 = \dim(\pi_*(X) \otimes \mathbb{Q}) > \dim(H_*(X;\mathbb{Q}) = 1 + \eta(X) = 4$. It follows from [1] that the formal dimension of such a counterexample is greater than or equal to 21.

Proposition 4.2. *Let $\varpi(X) = 2m+1$ such that $\varpi_{\text{even}}(X) = 0$ and $\varpi_{\text{odd}}(X) = 2m+1$. Let the degrees be*

$$2b_1 - 1, \cdots, 2b_m - 1, \cdots, 2b_{2m+1} - 1,$$

where $b_1 \leq b_2 \leq \cdots \leq b_{2m+1}$*. If* $2b_m - 1 \leq 2(2b_1 - 2)$*, then we have* $\eta(X) \geq 2m+1$*. In particular the Hilali conjecture holds for such a space* X*.*

Proof. This simply follows from Klaus–Kreck's rational Hurewicz theorem, i.e., we have the Hurewicz homomorphism $\pi_i(X) \otimes \mathbb{Q} \cong H_i(X;\mathbb{Q})$ for $1 \leq i \leq 2(2b_1 - 2)$. Since the formal dimension $n_X = \sum_j^{2m+1}(2b_j - 1)$, by the Poincaré duality we see that $\eta(X) \geq 2m+1$, since $n_X - (2b_j - 1)$ is even for any b_j, thus $n_X - (2b_j - 1)$ cannot be equal to any odd integer $2b_k - 1$ $(k = 1, 2, \cdots, m)$, therefore $H_{n_X-(2b_j-1)}(X;\mathbb{Q}) = \mathbb{Q}$ for $k = 1, 2, \cdots, m$. □

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A method for solving quadratic equations in real quaternion algebra by using Scilab software

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Abstract. In this paper, we present some numerical applications for the equation $x^2 + ax + b = 0$, where a, b are two quaternionic elements in $\mathbb{H}(\alpha, \beta)$. $\mathbb{H}(\alpha, \beta)$ represents the algebra of real quaternions with parameterized coefficients by α and β. The algebra of real quaternions is an extension of complex numbers and is represented by algebraic objects called quaternions. These quaternions are composed of four components: a real part and three imaginary components. In general, $\mathbb{H}(\alpha, \beta)$ indicates a family of parameterized quaternion algebras, in which the specific values of α and β determine the specific properties and structure of the quaternion algebra. Based on well-known solving methods, we have developed a new numerical algorithm that solves the equation for any quaternions a and b in any algebra $\mathbb{H}(\alpha, \beta)$.

Keywords: quaternion, quadratic formula, solving polynomial equation.
MSC 2020: 17A35, 17A45, 15A18

Introduction

Quaternions are a number system first introduced in 1843 by Irish mathematician Sir William Rowan Hamilton. Hamilton was seeking a way to extend the complex numbers to three dimensions and realized that he could do so by adding an additional imaginary unit.

Quaternions are different from complex numbers in that they are non-commutative. Quaternions have found many practical applications in fields such as computer graphics, physics, and engineering. For instance, they are used in computer graphics to represent 3D rotations and orientations, and in aerospace engineering to model spacecraft altitude and control systems.

Quaternions are essential in control systems for guiding aircraft and rockets: each quaternion has an axis indicating the direction and a magnitude determining the size of the rotation. Instead of representing an orientation change

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through three separate rotations, quaternions use a single rotation to achieve the same transformation.

Despite their usefulness, quaternions are not as widely used as complex numbers, largely due to their non-commutative nature. However, they remain an important topic in mathematics and physics, and continue to be studied and applied in various fields to this day. ([1], [4], [6], [10], [13])

We will numerically solve the monic quadratic equation with quaternion coefficients in the algebra $\mathbb{H}(\alpha, \beta)$ using Scilab, a free and open-source software for numerical computation.

We chose to use the Scilab software to numerically solve the monic quadratic equations with quaternionic coefficients in the algebra $\mathbb{H}(\alpha, \beta)$ because Scilab is a free and open-source software, making it accessible and usable by a large number of users. Additionally, this software allows us to customize and adapt it to the specific needs and requirements of our problem. Scilab is renowned for its powerful functionality in numerical computation. It offers a wide range of mathematical and algebraic functions, including an integrated solver for polynomial equations. The built-in polynomial equation solver in Scilab provides us with the necessary tools to efficiently solve the monic quadratic equation with quaternionic coefficients. Scilab, such as Matlab, which is more widely known, has a user-friendly and intuitive interface, facilitating ease of use and navigation within the software. The programming is very intuitive and doesn't require definition of any parameters, so the main focus remains the mathematical modeling of the equations and the algorithm. This decision allows us to obtain precise and efficient results in studying and applying our new findings in quaternion algebra.

The aim of the paper is to present an innovative, efficient, and accurate method for the numerical solution of monic quadratic equations in the algebra of real quaternions using the Scilab software. We develop a new algorithm that solves these equations for any quaternionic coefficients in any algebra $\mathbb{H}(\alpha, \beta)$. Our ultimate goal is to contribute to the development and application of this knowledge in various fields such as computer graphics, physics, and engineering, opening up new research and application perspectives for quaternions and monic quadratic equations with quaternionic coefficients.

1. Preliminaries

The quadratic equation has been explored in the context of Hamilton quaternions in the works [11], [13]. In [11], the equation $x^2 + bx + c = 0$ is analyzed and explicit formulas for its roots are obtained. These formulas were subsequently used in the classification of quaternionic Möbius transformations [14], [2]. In Hamilton quaternions, every nonzero element can be inverted, while in $\mathbb{H}(\alpha, \beta)$ there exist split quaternions that cannot be inverted. In an algebraic system, finding the roots of a quadratic equation is always connected to the factorization of a quadratic polynomial [12]. In the case of real numbers ($\mathbb{R}$) and complex numbers ($\mathbb{C}$), the two problems are identical. However, in noncommutative algebra, these two problems are interconnected. Scharler et al. [15] analyzed the factorizability of a quadratic split quaternion polynomial, revealing certain information about the roots of a split quaternionic quadratic equation.

In a publication from 2022, [7] exploring algebras derived from the Cayley-Dickson process presents challenges in achieving desirable properties due to computational complexities. Hence, the discovery of identities within these algebras it gains meaning, helping to acquire new properties and making calculations easier. To this end, the study introduces several fresh identities and properties within the algebras derived from the Cayley-Dickson process. Furthermore, when certain elements serve as coefficients, quadratic equations in real division quaternion algebra can be solved, showcasing the authors ability to provide direct solutions without relying on specialized software.

In the paper [3], the author specifically focuses on deriving explicit formulas for the roots of the quadratic equation $x^2 + bx + c = 0$ where b and c are split quaternions ($\mathbb{H}_S$).

The same subject can be found in [1], where quadratic formulas for generalized quaternions are studied. It focuses on obtaining explicit formulas for the roots of quadratic equations in this specific context of generalized quaternions.

Let $\mathbb{H}(\alpha, \beta)$ be the generalized quaternion algebra over an arbitrary field $\mathbb{K}$, that is the algebra of the elements of the form $q = q_1 + q_2e_1 + q_3e_2 + q_4e_3$ where $q_i \in \mathbb{K}$, $i \in \{1, 2, 3, 4\}$, and the basis elements $\{1, e_1, e_2, e_3\}$ satisfy the following multiplication table:

(1)

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	α	e_3	αe_2
e_2	e_2	$-e_3$	β	$-\beta e_1$
e_3	e_3	$-\alpha e_2$	βe_1	$-\alpha\beta$

The conjugate of a quaternion is obtained by changing the sign of the imaginary part: $\overline{q} = q_1 - q_2e_1 - q_3e_2 - q_4e_3$, where $q = q_1 + q_2e_1 + q_3e_2 + q_4e_3$.

The norm of a quaternion is defined as the sum of the squares of its components, for this case, the norm is:

$$\boldsymbol{n}\,(q) = q \cdot \overline{q} = ||q||^2 = q_1^2 - \alpha q_2^2 - \beta q_3^2 + \alpha\beta q_4^2.$$

If for $x \in \mathbb{H}\,(\alpha, \beta)$, the relation $n(x) = 0$ implies $x = 0$, then the algebra $\mathbb{H}\,(\alpha, \beta)$ is called a division algebra, otherwise the quaternion algebra is called a split algebra. (see [4])

If α and β are negative real numbers, it becomes a division algebra, therefore the norm will be different from zero. The role of α and β is to parameterize the coefficients of the quaternion algebra $\mathbb{H}(\alpha, \beta)$. These values determine the specific properties and structure of the quaternion algebra. In the multiplication table given in equation (1), α and β appear as parameters that determine the specific structure and properties of the quaternion algebra $\mathbb{H}(\alpha, \beta)$.

The role of the norm is to provide a measure of the size of a quaternion in the algebra $\mathbb{H}(\alpha, \beta)$. The norm expression involves the coefficients q_1, q_2, q_3, q_4, and the parameters α and β. The norm plays a crucial role in determining whether the algebra $\mathbb{H}(\alpha, \beta)$ is a division algebra or a split algebra, based on whether the norm is nonzero or zero, respectively.

Split quaternions form an algebraic structure and are linear combinations with real coefficients. Every quaternion can be written as a linear combination of the elements 1, e_1, e_2, and e_3, where e_1, e_2, and e_3 are the imaginary units that satisfy the relations $e_1^2 = \alpha$, $e_2^2 = \beta$, and $e_3^2 = -\alpha\beta$.

We will now present some of the most important properties and relations of quaternions, which play a fundamental role in various fields such as physics, engineering, computer science, and applied mathematics:

- The addition is done component-wise:
  $a = a_1 \cdot 1 + a_2 e_1 + a_3 e_2 + a_4 e_3$,
  $b = b_1 \cdot 1 + b_2 e_1 + b_3 e_2 + b_4 e_3$,
  $\Rightarrow a + b = (a_1 + b_1) \cdot 1 + (a_2 + b_2)e_1 + (a_3 + b_3)e_2 + (a_4 + b_4)e_3$.
- Quaternion multiplication is not commutative:
  $a \cdot b = (a_1b_1 + \alpha a_2b_2 + \beta a_3b_3 - \alpha\beta a_4b_4) + e_1(a_1b_2 + a_2b_1 - \beta a_3b_4 + \beta a_4b_3) + e_2(a_1b_3 + \alpha a_2b_4 + a_3b_1 - \alpha a_4b_2) + e_3(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)$
  $b \cdot a = (a_1b_1 + \alpha a_2b_2 + \beta a_3b_3 - \alpha\beta a_4b_4) + e_1(a_2b_1 + a_1b_2 - \beta a_4b_3 + \beta a_3b_4) + e_2(a_3b_1 + \alpha a_4b_2 + a_1b_3 - \alpha a_2b_4) + e_3(a_4b_1 + a_3b_2 - a_2b_3 + a_1b_4)$
  $\Rightarrow a \cdot b \neq b \cdot a$.
- Quaternions are associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c$.
- The trace of the element q:
  $$t(q) = q + \overline{q}.$$
- The multiplication of a quaternion by a scalar:
  $$\alpha \cdot q = \alpha \cdot (q_1 + q_2e_1 + q_3e_2 + q_4e_3) = (\alpha \cdot q_1) + (\alpha \cdot q_2) \cdot e_1 + (\alpha \cdot q_3) \cdot e_2 + (\alpha \cdot q_4) \cdot e_3.$$
- The inverse of a non-zero quaternion q is given by
  $$q^{-1} = \frac{\overline{q}}{||q||^2} = \frac{q_1 - q_2e_1 - q_3e_2 - q_4e_3}{q_1^2 - \alpha q_2^2 - \beta q_3^2 + \alpha\beta q_4^2}.$$
- The dot product of two quaternions can be defined as $q \cdot r = (qr + rq)/2$.

These are just some of the many important relations and properties of quaternions. All these properties make quaternions a powerful tool in mathematics and practical applications.

2. Known results

In [16] and [17], to find the root of the equation $f(x_t) = 0$, the Newton-Raphson method relies on the Taylor series expansion of the function around the estimate x_i to find a better estimate x_{i+1}:

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \mathcal{O}(h^2),$$

where x_{i+1} is the estimate of the root after iteration $i+1$ and x_i is the estimate at iteration i. $\mathcal{O}(h^2)$ means the order of error of the Taylor series around the point x_i. Assuming $f(x_{i+1}) = 0$ and rearranging:

$$x_{i+1} \approx x_i - \frac{f(x_i)}{f'(x_i)}.$$

The procedure is as follows. Setting an initial guess x_0, a tolerance ε_s, and a maximum number of iterations N:

At iteration i, calculate $x_i \approx x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$ and ε_r. If $\varepsilon_r \leq \varepsilon_s$ or if $i \geq N$, stop the procedure. Otherwise, repeat.

In [10], the authors present specific formulas to solve the monic quadratic equation $x^2 + bx + c = 0$ with $b, c \in \mathbb{H}(\alpha, \beta)$, where $\alpha = -1$, $\beta = -1$, the real division algebra, according to the multiplication table presented in (1). In the following we present the results we will use in developing our solutions, and a proof of lemma 2:

Lemma 2.1 ([10], Lemma 2.1)**.** *Let $A, B, C \in \mathbf{R}$ with the following properties: $C \neq 0$, $A < 0$ implies $A^2 < 4B$.*

Then the equation of order 3:

$$y^3 + 2Ay^2 + (A^2 - 4B)y - C^2 = 0 \tag{2}$$

has exactly one positive solution y.

Lemma 2.2 ([10], Lemma 2.2)**.** *Let $A, B, C \in R$ such that: $B \geq 0$ and $A < 0$ implies $A^2 < 4B$ then the real system:*

$$\begin{cases} Y^2 - (A + W^2)Y + B = 0, \\ W^3 + (A - 2Y)W + C = 0 \end{cases} \tag{3}$$

has at most two solutions (W, Y) with $W \in \mathbf{R}$ and $Y \geq 0$ as follows:

(i) $W = 0$, $Y = \frac{A \pm \sqrt{A^2 - 4B}}{2}$ *provided that $C = 0$, $A^2 \geq 4B$;*

(ii) $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$ *provided that $C = 0$, $A^2 < 4B$.*

(iii) $W = \pm\sqrt{z}$, $Y = \frac{W^3 + AW + C}{2W}$ *provided that $C \neq 0$ and z is the unique positive solution of the real polynomial:*

$$z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0.$$

Proof. Let $A, B, C \in \mathbf{R}$ such that $B \geq 0$ and $A < 0 \implies A^2 < 4B$.
We want to show that the real system has at most two solutions (W, Y) with $W \in \mathbf{R}$ and $Y \geq 0$ as follows:

(i) $W = 0$, $Y = \frac{A \pm \sqrt{A^2-4B}}{2}$ provided that $C = 0$, $A^2 \geq 4B$;

(ii) $W = \pm\sqrt{2\sqrt{B} - A}, Y = \sqrt{B}$ provided that $C = 0$, $A^2 < 4B$;

(iii) $W = \pm\sqrt{z}, Y = \frac{W^3+AW+C}{2W}$ provided that $C \neq 0$ and z is the unique positive solution of the real polynomial:

$$z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0.$$

From Lemma 2.1, we know that the polynomial $z^3+2Az^2+(A^2-4B)z-C^2 = 0$ has exactly one positive solution z when $C \neq 0$.

For the cases (i) and (ii), when $C = 0$, the first equation becomes a quadratic equation in Y. If $A^2 \geq 4B$, there are two real solutions for Y, and if $A^2 < 4B$, there is one real solution for Y. Since $W = 0$, these solutions correspond to the cases 1. and 2. in the lemma.

The case (iii), when $C \neq 0$, we can express Y as a function of W using the second equation: $Y = \frac{W^3+AW+C}{2W}$. Substituting this expression for Y in the first equation, we obtain a polynomial equation in W^2 of degree 3. Since z is the unique positive solution of this polynomial, there are two solutions for W: $W = \pm\sqrt{z}$. These solutions correspond to the case 3. in the lemma.

In conclusion, the real system (3) has at most two solutions (W, Y) with $W \in \mathbf{R}$ and $Y \geq 0$ as described in the lemma. $\square$

Theorem 2.3 ([10], Theorem 2.3)**.** *The solution of the quadratic equation $x^2 + bx + c = 0$ can be obtained in the following way:*

Case 1. If $b, c \in \mathbf{R}$ and $b^2 < 4c$ then:

$$x = \frac{1}{2}(-b + e \cdot e_1 + f \cdot e_2 + g \cdot e_3), \tag{4}$$

where $e^2 + f^2 + g^2 = 4c - b^2$ where $e, f, g \in \mathbf{R}$.

Case 2. If $b, c \in \mathbf{R}$ and $b^2 \geq 4c$ then:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \tag{5}$$

Case 3. If $b \in \mathbf{R}$, $c \notin \mathbf{R}$ then:

$$x = \frac{-b}{2} \pm \frac{m}{2} \mp \frac{c_1}{m} \cdot e_1 \mp \frac{c_2}{m} \cdot e_2 \mp \frac{c_3}{m} \cdot e_3, \tag{6}$$

where $c = c_0 + c_1 \cdot e_1 + c_2 \cdot e_2 + c_3 \cdot e_3$, and

$$m = \sqrt{\frac{b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)}}{2}}. \tag{7}$$

Case 4. If $b \notin \mathbf{R}$ then:

$$x = \frac{(-Re(b))}{2} - (b' + W)^{-1}(c' - Y), \tag{8}$$

where $b' = b - Re(b) = Im(b)$, $c' = c - (Re(b)/2)(b - (Re(b))/2)$, where (W, Y) are chosen in the following way:

(*i*) $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$ *provided that* $C = 0$, $A^2 \geq 4B$*;*

(*ii*) $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$ *provided that* $C = 0$, $A^2 < 4B$*;*

(*iii*) $W = \pm\sqrt{z}$, $Y = (W^3 + AW + C)/2W$ *provided that* $C \neq 0$ *and z is the unique positive solution of the equation:*

$$z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0,$$

where $A = |b'|^2 + 2Re(c')$, $B = |c'|^2$ *and* $C = 2Re(\overline{b'}c')$.

Corollary 2.4 ([10], Corollary 2.4)**.** *The equation has an infinity of solutions if $b, c \in \mathbf{R}$ and $b^2 < 4c$.*

Corollary 2.5 ([10], Corollary 2.6)**.** *The equation has an unique solution if and only if:*

1. $b, c \in \mathbf{R}$ *and* $b^2 - 4c = 0$*;*
2. $b \notin \mathbf{R}$ *and* $C = 0 = A^2 - 4B$.

Corollary 2.6. *If the quadratic equation $x^2 + bx + c = 0$ has real coefficients b and c, and $b^2 < 4c$, then the solution of the equation can be expressed as $x = \frac{1}{2}(-b + e \cdot e_1 + f \cdot e_2 + g \cdot e_3)$, where $e^2 + f^2 + g^2 = 4c - b^2$ and $e, f, g \in \mathbf{R}$.*

Corollary 2.7. *If the quadratic equation $x^2 + bx + c = 0$ has real coefficients b and c, and $b^2 \geq 4c$, then the solutions of the equation are $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$.*

Corollary 2.8. *If b and c are the coefficients of the quadratic equation $x^2 + bx + c = 0$, such that $b \notin \mathbf{R}$, then the solution of the equation can be expressed as:*

$$x = \frac{(-Re(b))}{2} - (b' + W)^{-1}(c' - Y),$$

where $b' = b - Re(b) = Im(b)$, $c' = c - (Re(b)/2)(b - (Re(b))/2)$, and (W, Y) are chosen such that:

- $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$ *if* $C = 0$ *and* $A^2 \geq 4B$*;*
- $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$ *if* $C = 0$ *and* $A^2 < 4B$*;*
- $W = \pm\sqrt{z}$, $Y = (W^3 + AW + C)/2W$ *if* $C \neq 0$ *and z is the unique positive solution of the equation* $z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0$, *where* $A = |b'|^2 + 2Re(c')$, $B = |c'|^2$ *and* $C = 2Re(\overline{b'}c')$.

3. The solutions of the second-degree equation in real quaternions

It is important to mention that the algebra $\mathbb{H}(\alpha, \beta)$ is a mathematical construction, and its properties can vary depending on the values chosen for α and β. When we take negative values for α and β in the algebra $\mathbb{H}(\alpha, \beta)$), it becomes a division algebra. This means that every nonzero element in the algebra can be inverted. Multiplication and inversion of elements can be performed using the specific rules of this algebra.

Therefore, for the algebra $\mathbb{H}(\alpha, \beta)$, we will take negative values for α and β, thus making it a division algebra, and the norm will be nonzero. If the values of α and β are positive, we no longer have a division algebra because the norm is zero.

Next, we will describe the solution of a monic quadratic equation in the algebra of real quaternions. This statement provides an explicit formula for finding the solutions of the equation and explains how to perform the necessary calculations. It presents the general formula for the solution of the monic quadratic equation, where the equation's coefficients are represented as real quaternions, and the solution is a linear combination of the imaginary units of the quaternions. This formula is presented in a detailed manner, specifying the values of each component of the solution in terms of the coefficients and other terms involved in the equation.

Proposition 3.1. *Let $b = b_0+b_1\cdot e_1+b_2\cdot e_2+b_3\cdot e_3$ and $c = c_0+c_1\cdot e_1+c_2\cdot e_2+c_3\cdot e_3$ where b, c are two quaternionic elements in $\mathbb{H}(\alpha, \beta)$ and knowing W and Y of the Theorem 2.3 the solution of the second degree equation $x^2 + bx + c = 0$ is of the form*

$$x = x_1 + x_2e_1 + x_3e_2 + x_4e_3, \tag{9}$$

where:

$$\begin{aligned}
x_1 &= -t-[Wc_1-YW-b_2c_2\alpha-b_3c_3\beta+b_4c_4\alpha\beta-t(Wt-b_2^2\alpha-b_3^2\beta+b_4^2\alpha\beta)]/m,\\
x_2 &= (Wc_2 - b_2c_1 + b_2Y + b_3c_4\beta - b_4c_3\beta - tb_2(W - t))/m,\\
x_3 &= (Wc_3 - b_2c_4\alpha - b_3c_1 + b_3Y + b_4c_2\alpha - tb_3(W - t))/m,\\
x_4 &= (Wc_4 - b_2c_3 + b_3c_2 + b_4c_1 + b_4Y - tb_4(W - t))/m
\end{aligned}$$

with $t = \frac{b_1}{2}$ and

$$m = W^2 - \alpha b_2^2 - \beta b_3^2 + \alpha\beta b_4^2.$$

Proof. Let $b = b_1 + b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3$ and $c = c_1 + c_2 \cdot e_1 + c_3 \cdot e_2 + c_4 \cdot e_3$. for this case, the norm is:

$$\boldsymbol{n}(a) = a\bar{a} = a_1^2 - \alpha a_2^2 - \beta a_3^2 + \alpha\beta a_4^2.$$

We compute the necessary elements for applying the theorem: $Re(b) = b_1$. Therefore,

$$b' = b - Re(b) = Im(b) = b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3$$

and

$$\begin{aligned} c' &= c - (Re(b)/2)(b - (Re(b))/2) \\ &= c_1 + c_2 \cdot e_1 + c_3 \cdot e_2 + c_4 \cdot e_3 - \frac{b_1}{2}\left(b_1 + b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3 - \frac{b_1}{2}\right) \\ &= \left(c_1 - \frac{b_1^2}{2} + \frac{b_1^2}{4}\right) + \left(c_2 - \frac{b_1 b_2}{2}\right) e_1 + \left(c_3 - \frac{b_1 b_3}{2}\right) e_2 + \left(c_4 - \frac{b_1 b_4}{2}\right) e_3 \\ &= \left(c_1 - \frac{b_1^2}{4}\right) + \left(c_2 - \frac{b_1 b_2}{2}\right) e_1 + \left(c_3 - \frac{b_1 b_3}{2}\right) e_2 + \left(c_4 - \frac{b_1 b_4}{2}\right) e_3. \end{aligned}$$

Using all the above and $C = 2Re(\overline{b'}c')$, we find

$$\begin{aligned} C = 2Re((-b_2 \cdot e_1 - b_3 \cdot e_2 - b_4 \cdot e_3) \cdot ((c_1 - \frac{b_1^2}{4}) + (c_2 - \frac{b_1 b_2}{2})e_1 \\ + (c_3 - \frac{b_1 b_3}{2})e_2 + (c_4 - \frac{b_1 b_4}{2})e_3)). \end{aligned}$$

The real part is obtained only by multiplying terms of the same kind, therefore we obtain:

$$C = -2b_2 c_2 \alpha + b_1 b_2^2 \alpha - 2b_3 c_3 \beta + b_1 b_3^2 \beta + 2b_4 c_4 \alpha\beta - b_1 b_4^2 \alpha\beta$$

and $A = |b'|^2 + 2Re(c') = (-\alpha b_2^2 - \beta b_2^3 + \alpha\beta b_4^2) + 2(c_1 - \frac{b_1^2}{4})$. Then $A = -\alpha b_2^2 - \beta b_2^3 + \alpha\beta b_4^2 + 2c_1 - \frac{b_1^2}{2}$

Computing $B = |c'|^2$ we get

$$B = \left(c_1 - \frac{b_1^2}{4}\right)^2 - \alpha\left(c_2 - \frac{b_1 b_2}{2}\right)^2 - \beta\left(c_3 - \frac{b_1 b_3}{2}\right)^2 + \alpha\beta\left(c_4 - \frac{b_1 b_4}{2}\right)^2.$$

We denote $\frac{b_1}{2} = t$ and obtain:

$$B = (c_1 - t^2)^2 - \alpha(c_2 - tb_2)^2 - \beta(c_3 - tb_3)^2 + \alpha\beta(c_4 - tb_4)^2.$$

We compute W and Y according to the cases of the theorem. By denoting $m = |b' + W| = W^2 - \alpha b_2^2 - \beta b_3^2 + \alpha\beta b_4^2$ and cu $t = b_1/2$, we apply equation (8) and we find

$$\begin{aligned} x_1 &= -t - (Wc_1 - YW - b_2 c_2 \alpha - b_3 c_3 \beta + b_4 c_4 \alpha\beta \\ &\quad - t(Wt - b_2^2\alpha - b_3^2\beta + b_4^2\alpha\beta))/m, \\ x_2 &= (Wc_2 - b_2 c_1 + b_2 Y + b_3 c_4 \beta - b_4 c_3 \beta - tb_2(W - t))/m, \\ x_3 &= (Wc_3 - b_2 c_4 \alpha - b_3 c_1 + b_3 Y + b_4 c_2 \alpha - tb_3(W - t))/m, \\ x_4 &= (Wc_4 - b_2 c_3 + b_3 c_2 + b_4 c_1 + b_4 Y - tb_4(W - t))/m. \end{aligned}$$

We obtain the solution as

$$x = x_1 + x_2 e_1 + x_3 e_2 + x_4 e_3.$$

□

4. Numerical applications and examples

For the implementation of numerical applications, let's consider the general case of $\mathbb{H}(\alpha, \beta)$, $b = b_1 + b_2 \cdot e_1 + b_3 \cdot e_2 + b_4 \cdot e_3$ and $c = c_1 + c_2 \cdot e_1 + c_3 \cdot e_2 + c_4 \cdot e_3$. Using Proposition 4.1, we present the algorithm from the table 1. The algorithm described has been implemented in Scilab 6.1.1. To verify our computations, we apply all the formulas, on some remarkable examples.

Steps		
1.		Input α, β, b, c
2.		Compute C, A, B
3.		Identify case
4.	If case 1: $C = 0, A \geq 4B$	Compute $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$
	If case 2: $C = 0, A^2 < 4B$	Compute $W = \pm\sqrt{2\sqrt{B} - A}$, $Y = \sqrt{B}$
	If case 3: $C \neq 0$	Solve the polynomial equations $z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0$ and find the positive root.
5.		Compute solutions using formula (9).

Table 1: Algorithm for computing the solutions of the quadratic equation.

Example 4.1 ([10], Example 2.12)**.** Consider the quadratic equation $x^2 + xe_1 + (1 + e_2) = 0$, i.e., $b = e_1$ and $c = 1 + e_2$. This belongs to Case 4 in Theorem 2.3. Then $b' = e_1$ and $c' = 1 + e_2$. Moreover, $A = 3$, $B = 2$, $C = 0$. It is Subcase 1 in Case 4. Hence, $W = 0$ and $Y = 2$ or $Y = 1$. Consequently, the two solutions are $x_1 = -e_1 + e_3$ and $x_2 = e_3$. For $\alpha = -1, \beta = -1$, the solution is:

$$
\begin{aligned}
C &= 0.000000, \\
A &= 3.000000, \\
B &= 2.000000, \\
Y_1 &= 2.000000, \\
Y_2 &= 1.000000,
\end{aligned}
$$

$$x_1 = -0.000000 - 1.000000e_1 - 0.000000e_2 + 1.000000e_3,$$
$$x_2 = -0.000000 - 0.000000e_1 - 0.000000e_2 + 1.000000e_3.$$

Example 4.2. ([10], Example 2.13) Consider the quadratic equation $x^2 + xe_1 + e_2 = 0$, i.e., $b = e_1$ and $c = e_2$. This belongs to Case 4 in Theorem 2.3. Then $b' = e_1$ and $c' = e_2$. Moreover, $A = 1$, $B = 1$, $C = 0$. It is Subcase 2 in Case 4. Hence, $W = +1$ or -1 and $Y = 1$. Consequently, the two solutions are $x_1 = (e_1 + 1)^{-1}(1 - e_2) = (1/2)(1 - e_1 - e_2 + e_3)$ and $x_2 = (e_1 - 1)^{-1}(1 - e_2) = (1/2)(-1 - e_1 + e_2 + e_3)$. For $\alpha = -1, \beta = -1$, the solution of the program:

$$C = 0.000000,$$
$$A = 1.000000,$$
$$B = 1.000000,$$
$$x_1 = 0.500000 - 0.500000e_1 - 0.500000e_2 + 0.500000e_3,$$
$$x_2 = -0.500000 - 0.500000e_1 + 0.500000e_2 + 0.500000e_3.$$

Example 4.3. ([10], Example 2.14) Consider the quadratic equation $x^2 + xe_1 + (1 + e_1 + e_2) = 0$, i.e., $b = e_1$ and $c = 1 + e_1 + e_2$. This belongs to Case 4 in Theorem 2.3. Then $b' = e_1$ and $c' = 1 + e_1 + e_2$. Moreover, $A = 3$, $B = 3$, $C = 2$. It is Subcase 3 in Case 4. Now the unique positive roots of $z^3 + 6z^2 - 3z - 4$ is 1, and hence,$W = 1$ and $Y = 3$ or $W = -1$ and $Y = 1$. Consequently, the two solutions are $x_1 = (1/2)(1 - 3e_1 - e_2 + e_3)$ and $x_2 = (1/2)(-1 + e_1 + e_2 + e_3)$. For $\alpha = -1, \beta = -1$, the solution of the program:

$$C = 2.000000,$$
$$A = 3.000000,$$
$$B = 3.000000,$$
$$x_1 = 0.500000 - 1.500000e_1 - 0.500000e_2 + 0.500000e_3,$$
$$x_2 = -0.500000 + 0.500000e_1 + 0.500000e_2 + 0.500000e_3.$$

The results obtained in Examples 5.1-5.3 are exactly the ones obtain by direct computation by the authors in [10].

In the following, we will present a few examples using the results presented above and also calculate the solutions of the equations using the described algorithm, for different values of α and β.

Example 4.4. Next, we aim to find the solution of the equation $x^2 + bx + c = 0$ in the case where b and c are quaternions:

$$b = 5 \cdot 1 + 6 \cdot e_1 + 7 \cdot e_2 + 8 \cdot e_3$$

and

$$c = 2 \cdot 1 + 3 \cdot e_1 + 4 \cdot e_2 + 5 \cdot e_3.$$

For $\alpha = -1, \beta = -1$, we can compute $b' = b - Re(b) = 6e_1 + 7e_2 + 8e_3$ and

$$c' = c - \frac{1}{2}Re(b)\left(b - \frac{1}{2}Re(b)\right),$$
$$c' = \left(2 - \frac{25}{2} + \frac{25}{4}\right)1 + (3-15)\,e_1 + \left(4 - \frac{35}{2}\right)e_2 + (5-20)\,e_3.$$

Then

$$c' = -\frac{17}{4} - 12e_1 - \frac{27}{2}e_2 - 15e_3.$$

Consequently,

$$A = |b'|^2 + 2Re(c') = 6^2 + 7^2 + 8^2 + 2\left(-\frac{17}{4}\right) = 140,5,$$
$$B = |c'|^2 = \left(\frac{-17}{4}\right)^2 + 12^2 + \left(\frac{27}{2}\right)^2 + (15)^2 = 569,3125,$$
$$C = 2Re(\overline{b'}c') = -573.$$

We can check that $A^2 \geq 4B$, so we can use case 4. Using the formulas in case 4, the next step is to find the values of (W, Y) using one of the three situations described in the formula from case 4. Since $C \neq 0$, we will use situation 3 $z^3 + 2Az^2 + (A^2 - 4B)z - C^2 = 0$.

To find the unique positive solution z, we will use the Newton-Raphson method. In this case, we have:

$$f(z) = z^3 + 2Az^2 + (A^2 - 4B)z - C^2,$$
$$f'(z) = 3z^2 + 4Az + (A^2 - 4B).$$

The analytical method to find the solutions of the equation is given by choosing $z_0 = 1$ and applying the Newton-Raphson formula. We can obtain successive values for z as the fixed number given by:

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} = 1 - \frac{f(1)}{f'(1)},$$
$$z_2 = z_1 - \frac{f(z_1)}{f'(z_1)},$$
$$z_3 = z_2 - \frac{f(z_2)}{f'(z_2)},$$
$$z_4 = z_3 - \frac{f(z_3)}{f'(z_3)}.$$

Computing by this formula we use decimal fractions with many decimals, therefore we used the Scilab solver:

$$p = -328329 + 17463x + 281x^2 + x^3.$$

By using of the solver in Scilab, we obtain: $W_1 = \pm 3.871934$, and using a numerical application, we obtain:

$$\begin{aligned}
&C = -573.000000,\\
&A = 140.500000,\\
&B = 569.312500,\\
&x_1 = -0.564033 + 0.008853e_1 + 0.306465e_2 - 0.017904e_3,\\
&x_2 = -4.435967 - 5.972266e_1 - 6.647896e_2 - 7.945509e_3.
\end{aligned}$$

For $\alpha = -2, \beta = -3$, the solution is

$$\begin{aligned}
&C = -2295.000000,\\
&A = 594.500000,\\
&B = 2202.812500,\\
&W = \pm 3.813764,\\
&x_1 = -0.593118 + 0.012038e_1 + 0.168839e_2 - 0.004699e_3,\\
&x_2 = -4.406882 - 5.982890e_1 - 6.819067e_2 - 7.985585e_3.
\end{aligned}$$

Example 4.5 ([7])**.** We aim to solve the following equation: $x^2 + (2 + 3e_1 + 4e_2 + 5e_3)x + (4 - 5e_1 - 6e_2 - 7e_3) = 0$. For $\alpha = -1, \beta = -1$, we write:

$$(a+be_1+ce_2+de_3)^2+(2+3e_1+4e_2+5e_3)(a+be_1+ce_2+de_3)+(4-5e_1-6e_2-7e_3) = 0.$$

We expand this equation and group the terms based on the quaternionic units:

$$\begin{aligned}
&(a^2 - b^2 - c^2 - d^2 + 2a - 3b - 4c - 5d + 4) + (2ab + 3a + 2b - 5c + 4d - 5)e_1\\
&+ (2ac + 4a + 5b + 2c - 3d - 6)e_2 + (2ad + 5a - 4b + 3c + 2d - 7)e_3 = 0.
\end{aligned}$$

Thus, we can obtain a system of linear equations with 4 equations and 4 unknowns:

$$\begin{cases}
a^2 - b^2 - c^2 - d^2 + 2a - 3b - 4c - 5d + 4 = 0,\\
2ab + 3a + 2b - 5c + 4d - 5 = 0,\\
2ac + 4a + 5b + 2c - 3d - 6 = 0,\\
2ad + 5a - 4b + 3c + 2d - 7 = 0.
\end{cases}$$

Solving this system of equations can provide us with the quaternionic solutions to the initial equation. Unfortunately, this system does not seem to have a simple and analytical solution, but we can try to solve it numerically or look for a specialized method for solving quaternionic equations.

Using the algorithm, we found the following results:

$$\begin{aligned}
&C = -248.000000,\\
&A = 56.000000,\\
&B = 317.000000,\\
&x_1 = 0.988335 + 0.435138e_1 - 0.199557e_2 + 0.624407e_3,\\
&x_2 = -2.988335 - 3.374360e_1 - 5.198324e_2 - 5.563629e_3.
\end{aligned}$$

For $\alpha = -2.35, \beta = -100$, the solution of the equations is

$$x_1 = 1.416406 + 0.030602e_1 - 0.009466e_2 + 0.006083e_3,$$
$$x_2 = -3.416406 - 2.977407e_1 - 4.019286e_2 - 5.005551e_3.$$

Moreover, $C = -36312.800000$, $A = 7502.150000$, $B = 43999.400000$.

Example 4.6. Next, we aim to find the solution of the equation in the case where b and c are quaternions: $b = 1.25 + 0.2e_1 - 0.31e_2 - 0.69e_3$ and $c = -1 + 0.56e_1 - 2.35e_2 - 4.56e_2$. Then, the equations is

$$x^2 + (1.25 + 0.2e_1 - 0.31e_2 - 0.69e_3)x - 1 + 0.56e_1 - 2.35e_2 - 4.56e_2 = 0.$$

Using the program, for $\alpha = -1, \beta = -1$, we found the following results:

$$C = 7.208550,$$
$$A = -2.169050,$$
$$B = 23.819054,$$
$$W = \pm 3.485216,$$
$$x_1 = 1.117608 + -0.251329e_1 + 0.667362e_2 + 1.505501e_3,$$
$$x_2 = -2.367608 + 0.018740e_1 - 0.560890e_2 - 0.861963e_3.$$

For $\alpha = -6, \beta = -8.5$, the solution is

$$C = 302.988862,$$
$$A = 22.556700,$$
$$B = 911.964612,$$
$$W = \pm 7.155732,$$
$$x_1 = 2.952866 - 0.219073e_1 + 0.340546e_2 + 0.917961e_3,$$
$$x_2 = -4.202866 - 0.027102e_1 - 0.234104e_2 - 0.235706e_3.$$

Example 4.7. Next, we aim to calculate by using of the program an example where $C = 0$:

Find the solutions of the equation: $x^2 + (e_1 + e_2 + e_3)x + (-3e_1 - 4e_2 + 7e_3) = 0$.

We can see that $b = e_1 + e_2 + e_3 \notin \mathbb{R}$, so we need to use the formula from case 4. Firstly, we will calculate the values of b', c', A, B, and C:

$$b' = b - \operatorname{Re}(b) = e_1 + e_2 + e_3,$$
$$c' = c - \frac{\operatorname{Re}(b)}{2}(b - \frac{\operatorname{Re}(b)}{2}) = -3e_1 - 4e_2 + 7e_3,$$
$$A = |b'|^2 + 2\operatorname{Re}(c') = 3,$$
$$B = |c'|^2 = 74,$$
$$C = 2\operatorname{Re}(\overline{b'}c') = 0.$$

The next step is to find the values of (W, Y) using one of the three situations described in the formula from case 4. Since $C = 0$ and $A^2 < 4B$. Now we can calculate (W, Y): $W = \pm\sqrt{2\sqrt{B} - A} = \pm 3,7689057476$ and $Y = \sqrt{B} = 8,602325267$.

By using of the program, we have found the following results:

$$\begin{aligned}
&C = 0.000000,\\
&A = 3.000000,\\
&B = 74.000000,\\
&W = \pm 3.768906,\\
&Y = 8.602325,\\
&x_1 = 1.884453 + 0.796552e_1 + 0.608748e_2 - 2.091566e_3,\\
&x_2 = -1.884453 - 0.517828e_1 - 1.143758e_2 + 0.975319e_3.
\end{aligned}$$

The same equation can be solved for $\alpha = -6$ and $\beta = -9$. In this case, $C \neq 0$. We get

$$\begin{aligned}
&C = 2088.000000,\\
&A = 528.000000,\\
&B = 2844.000000,\\
&W = \pm 3.919010,\\
&x_1 = 1.959505 - 0.537980e_1 - 1.973780e_2 - 3.017625e_3,\\
&x_2 = -1.959505 + 0.399290e_1 - 0.070390e_2 + 0.024986e_3.
\end{aligned}$$

Example 4.8. Next, we intend to use the program to calculate an example where C=0:

Let's find the solutions of the equation: $x^2 + (e_1 + e_2 + e_3)x + (-e_1 + e_3) = 0$.

We can see that $b = e_1 + e_2 + e_3 \notin \mathbb{R}$, so we need to use the formula from case 4.

Firstly, we will calculate the values of b', c', A, B and C:

$$\begin{aligned}
&b' = b - \mathrm{Re}(b) = e_1 + e_2 + e_3,\\
&c' = c - \frac{\mathrm{Re}(b)}{2}(b - \frac{\mathrm{Re}(b)}{2}) = -e_1 + e_3,\\
&A = |b'|^2 + 2\,\mathrm{Re}(c') = 3,\\
&B = |c'|^2 = 2,\\
&C = 2\,\mathrm{Re}(\overline{b'}c') = 0.
\end{aligned}$$

The next step is to find the values of (W, Y) using one of the three situations described in the formula of case 4. Since $C = 0$ and $A^2 \geq 4B$, we will use situation 1, $W = 0$, $Y = (A \pm \sqrt{A^2 - 4B})/2$ result $Y_1 = 2$,$Y_2 = 1$.

Calculating with the numerical application, we get:

$$\begin{aligned}
&C = 0.000000,\\
&A = 3.000000,\\
&B = 2.000000,\\
&Y_1 = 2.000000,\\
&Y_2 = 1.000000,\\
&x_1 = -0.000000 - 0.333333e_1 - 0.666667e_2 - 0.333333e_3,\\
&x_2 = -0.000000 - 0.000000e_1 - 0.333333e_2 - 0.000000e_3.
\end{aligned}$$

For $\alpha = -100, \beta = -100$, we get $C \neq 0$, like in the other example, and the solution is

$$\begin{aligned}
&C = 19800.000000,\\
&A = 10200.000000,\\
&B = 10100.000000,\\
&W = \pm 1.940836,\\
&x_1 = 0.970418 - 0.989912e_1 - 0.999903e_2 - 0.999995e_3,\\
&x_2 = -0.970418 + 0.009513e_1 - 0.000097e_2 + 0.000191e_3.
\end{aligned}$$

Example 4.9. ([7]) Let f_n be the Fibonacci sequence define as $f_0 = 0, f_1 = 1$ and $f_k = f_{k-1} + f_{k-2}$. We define the quaternion $F_n = f_n + f_{n+1}e_1 + f_{n+2}e_2 + f_{n+3}e_3$.

Consider the monic quadratic equation $x^2 + F_n x + F_m = 0$. We use the same algorithm for solving the equation.

For $n = 3, m = 3$, case discussed in ([7]), we obtain $F_3 = 2 + 3e_1 + 5e_2 + 8e_3$ and the equation $x^2 + (2 + 3e_1 + 5e_2 + 8e_3)x + (2 + 3e_1 + 5e_2 + 8e_3) = 0$.

Solving the equations for $\alpha = -1, \beta = -1$, and we get

$$\begin{aligned}
&C = 0.000000,\\
&A = 100.000000,\\
&B = 1.000000,\\
&Y_1 = 99.989999,\\
&Y_2 = 0.010001,\\
&x_1 = -1.000000 - 3.030306e_1 - 4.560714e_2 - 8.080816e_3,\\
&x_2 = -1.000000 + 0.030306e_1 + 0.540306e_2 + 0.080816e_3.
\end{aligned}$$

Solving the equations for $\alpha = -6.3, \beta = -5.25$, and we get

$$\begin{aligned}
&C = 0.000000,\\
&A = 2306.750000,
\end{aligned}$$

$$
\begin{aligned}
&B = 1.000000,\\
&Y_1 = 2306.749566,\\
&Y_2 = 0.000434,\\
&x_1 = -1.000000 + -3.001301e_1 - 4.870961e_2 - 8.003470e_3,\\
&x_2 = -1.000000 + 0.001301e_1 + 0.133376e_2 + 0.003470e_3.
\end{aligned}
$$

For $n = 5, m = 10$ we obtain $F_5 = 5 + 8e_1 + 13e_2 + 21e_3$, and $F_{10} = 55 + 89e_1 + 144e_2 + 233e_3$. Thus, the equation in this case is $x^2 + F_5 x + F_{10} = 0$. Then, the solution for $\alpha = -1, \beta = -1$ found by the algorithm is

$$
\begin{aligned}
&C = 11584.000000,\\
&A = 771.500000,\\
&B = 52150.062500,\\
&W = \pm 13.722364,\\
&x_1 = 4.361182 - 9.008123e_1 - 10.308573e_2 - 23.657396e_3,\\
&x_2 = -9.361182 + 1.019720e_1 + 5.966780e_2 + 2.645800e_3.
\end{aligned}
$$

For $\alpha = -6.3, \beta = -5.25$ the solution provided by the algorithm is

$$
\begin{aligned}
&C = -272916.525000,\\
&A = 15974.025000,\\
&B = 1175231.943750,\\
&W = \pm 16.934907,\\
&x_1 = 5.967453 - 8.058866e_1 - 11.642625e_2 - 21.158442e_3,\\
&x_2 = -10.967453 + 0.062114e_1 + 1.552659e_2 + 0.157823e_3.
\end{aligned}
$$

Example 4.10. Let p_n be the Pell sequence define as $p_0 = 0, p_1 = 1$ and $p_k = 2p_{k-1} + p_{k-2}$. Consider the quaternions $P_n = p_n + p_{n+1}e_1 + p_{n+2}e_2 + p_{n+3}e_3$. We solve the monic quadratic equation $x^2 + P_n x + P_m = 0$. For $n = 3, m = 3$, we get $P_3 = 3 + 7e_1 + 17e_2 + 41e_3$ and the equation is $x^2 + (3 + 7e_1 + 17e_2 + 41e_3)x + 3 + 7e_1 + 17e_2 + 41e_3 = 0$.

Solving the equations for $\alpha = -1, \beta = -1$ using the algorithm we obtain

$$
\begin{aligned}
&C = -2019.000000,\\
&A = 2020.500000,\\
&B = 505.312500,\\
&W = \pm 0.999011,\\
&x_1 = -1.000494 + 0.003464e_1 + 0.292570e_2 + 0.020287e_3,\\
&x_2 = -1.999506 - 7.003464e_1 - 16.724253e_2 - 41.020287e_3.
\end{aligned}
$$

For $\alpha = -7, \beta = -6$ the solutions are

$$C = -72679.000000,$$
$$A = 72680.500000,$$
$$B = 18170.312500,$$
$$W = \pm 0.999972,$$
$$x_1 = -1.000014 + 0.000096e_1 + 0.055517e_2 + 0.000564e_3,$$
$$x_2 = -1.999986 - 7.000096e_1 - 16.944950e_2 - 41.000564e_3.$$

For $n = 12, m = 19$, the quaternions are $P_{12} = 8119 + 19601e_1 + 47321e_2 + 114243e_3$ and $P_{19} = 3880899 + 9369319e_1 + 22619537e_2 + 54608393e_3$. The equations is $x^2 + P_{12}x + P_{19} = 0$. Solving for $\alpha = -1, \beta = -1$, we get

$$C = -112279524556439.000000,$$
$$A = 15649742008.500000,$$
$$B = 201223166914529952.000000,$$
$$W = \pm 7162.787683,$$
$$x_1 = -478.106158 + 0.284778e_1 + 136.813987e_2 + 1.659808e_3,$$
$$x_2 = -7640.893842 - 19601.284778e_1 - 47185.561043e_2 - 114244.659808e_3.$$

For $\alpha = -7, \beta = -6$ the solutions are

$$C = -4041981872234103.000000,$$
$$A = 564261307428.500000,$$
$$B = 7238333535486963712.000000,$$
$$W = \pm 7162.990016,$$
$$x_1 = -478.004992 + 0.007936e_1 + 26.572949e_2 + 0.046254e_3,$$
$$x_2 = -7640.995008 - 19601.007936e_1 - 47294.465369e_2 - 114243.046254e_3.$$

Example 4.11. Consider now the Lucas number sequences define as $l_0 = 2, l_1 = 1$ and $l_n = l_{n-1} + l_{n-2}$. We define the quaternion $L_n = l_n + l_{n+1}e_1 + l_{n+2}e_2 + l_{n+3}e_3$. We solve the monic quadratic equation $x^2 + L_n x + L_m = 0$. For $n = 3, m = 8$, the quaternions are $L_3 = 4 + 7e_1 + 11e_2 + 18e_3$ and $L_8 = 47 + 76e_1 + 123e_2 + 199e_3$.

Solving the equation $x^2 + L_3 x + L_8 = 0$ for $\alpha = -1, \beta = -1$, we get

$$C = 8958.000000,$$
$$A = 580.000000,$$
$$B = 42463.000000,$$
$$W = \pm 13.777285,$$
$$x_1 = 4.888642 - 8.113676e_1 - 8.726917e_2 - 20.805123e_3,$$
$$x_2 = -8.888642 + 1.040556e_1 + 5.802217e_2 + 2.878243e_3.$$

On the other hand, for $\alpha = -3, \beta = -10$, the solution are

$$\begin{aligned}
C &= 200864.000000,\\
A &= 11163.000000,\\
B &= 912461.000000,\\
W &= \pm 17.747518,\\
x_1 &= 6.873759 - 7.095766e_1 - 10.394477e_2 - 18.193193e_3,\\
x_2 &= -10.873759 + 0.051875e_1 + 0.848658e_2 + 0.197582e_3.
\end{aligned}$$

For $n = 11, m = 14$, $L_{11} = 199 + 322e_1 + 521e_2 + 843e_3$ and $L_{14} = 843 + 1364e_1 + 2207e_2 + 3571e_3$.

Solving the equation $x^2 + L_{11}x + L_{14} = 0$ for $\alpha = -1, \beta = -1$, we get

$$\begin{aligned}
C &= -4638388100.000000,\\
A &= 24326817.500000,\\
B &= 221017587902.562500,\\
W &= \pm 190.527592,\\
x_1 &= -4.236204 + 0.000233e_1 + 0.283353e_2 + 0.000622e_3,\\
x_2 &= -194.763796 - 322.000241e_1 - 520.717414e_2 - 843.000621e_3.
\end{aligned}$$

Finally, we solve the same equation for $\alpha = -1.236, \beta = -10.023$, the solution are

$$\begin{aligned}
C &= -2220150838.889460,\\
A &= 11634516.036772,\\
B &= 105832184609.751312,\\
W &= \pm 190.527281,\\
x_1 &= -4.236359 + 0.000487e_1 + 0.243975e_2 + 0.001297e_3,\\
x_2 &= -194.763641 - 322.000504e_1 - 520.757626e_2 - 843.001295e_3.
\end{aligned}$$

Conclusion

In this article, we have provided an algorithm in Scilab which allows us to find solutions for the monic quadratic equation $x^2 + bx + c = 0$, with $b, c \in \mathbb{H}(\alpha, \beta)$.

In Theorem 2.3, the authors offer solutions for all cases of the monic equation $x^2 + bx + c = 0$. We are interested only in cases 3 and 4 of the theorem. The article presents several equations solved using the algorithm, implemented in Scilab. By assigning specific values to the two quaternions, b and c, in the form of $b = b_1 + b_2e_1 + b_3e_2 + b_4e_3$ and $c = c_1 + c_2e_1 + c_3e_2 + c_4e_3$, and utilizing the formulas provided in the article, we perform the following calculations: Compute the values of A, B, and C: A is determined by evaluating the expression $A = |b'|^2 + 2Re(c')$, where $b' = b - Re(b)$ and $c' = c - (Re(b)/2)(b - (Re(b))/2)$. B is computed as $B = |c'|^2$. C is obtained by calculating $C = 2Re(\overline{b'}c')$. Identify the case we are in, based on the four cases specified in the theorem. Proceeding with the determined case, we find the two solutions of the monic quadratic equation, $x^2 + bx + c = 0$, using the appropriate formulas presented in the article. That this detailed procedure allows us to obtain precise and accurate solutions for the given quadratic equation in the context of the algebra of real quaternions.

The algorithm can solve monic quadratic equations for any base that respects the multiplication table of quaternions.

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On nearly CAP-embedded second maximal subgroups of Sylow p-subgroups of finite groups

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Abstract. A subgroup D of a group G is called a CAP-embedded subgroup of G, if for each prime p dividing the order of D, there exists a CAP-subgroup K of G such that a Sylow p-subgroup of D is also a Sylow p-subgroup of K. Later, we have generalized CAP-embedded subgroup to nearly CAP-embedded subgroup. A subgroup H of a group G is said to be nearly CAP-embedded in G if there is a subnormal subgroup T of G and a CAP-embedded subgroup H_{ce} of G contained in H such that G is equal to HT and the intersection of H and T is contained in H_{ce}. The main purpose of this paper is to study the p-nilpotentcy of a group which every second maximal subgroup of its Sylow p-subgroups is nearly CAP-embedded and some new results are obtained.
Keywords: nearly CAP-embedded subgroup, p-nilpotency, finite group.
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1. Introduction

In this paper, all groups are finite and G stands for a finite group. Let $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $\mathcal{F}$ denote a formation, $\mathcal{N}_p$ the class of all p-nilpotent groups, and let us denote

$$G^{\mathcal{F}} = \cap\{N \trianglelefteq G \big| G/N \in \mathcal{F}\}$$

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as the $\mathcal{F}$-residual of G. "H $Char$ G" means that H is a characteristic subgroup of G, "the group G is A_4-free" means that there are no subgroups in G for which A_4 is an isomorphic image. The other notations and terminology are standard (see, [8]).

The study of the embedding properties of subgroups of finite groups is one of the most fruitful research areas in the group theory. Many researchers use the embedding property of second maximal subgroups to describe the structure of supersolvable groups, solvable groups, p-solvable groups and other concrete groups. Given a group G, a subgroup K of G is called a second maximal subgroup if there exists a maximal subgroup M of G such that K is a maximal subgroup of M. One of the most classical results in this context is due to B. Huppert. He proved in [9] that if every second maximal subgroup of a group G is normal in G, then G is supersoluble, and if moreover the order of G is divisible by at least three distinct primes, then G is nilpotent. Agrawal in [1] generalized Huppert's result under the weaker hypothesis of permutability. A sharper insight regarding the groups such that every second maximal subgroup of G is normal in G is done by Li Shirong in [10].

Later, many authors investigated the influence of the embedding properties of second maximal subgroups of a Sylow subgroup on the structure of finite groups. For example, Adolfo et al. in [2] obtained the completely classification of finite groups in which the second maximal subgroups of the Sylow p-subgroups, p is a fixed prime, cover or avoid the chief factors of some of its chief series. Qiu et al. in [11] got the structure of finite groups in which the second maximal subgroups of the Sylow p-subgroups, p is a fixed prime, satisfy the partial Π-property (see, [3, Section 7]). Guo and Shum also in [7] proved the following result. Let G be a group and p the smallest prime number dividing the order of G. If all second maximal subgroup of every Sylow p-subgroup of G are c-normal in G and G is A_4-free, then G is p-nilpotent. In [5], Guo and Guo introduced the notion of CAP-embedded subgroup. A subgroup H of a group G is said to have the CAP-embedded property in G or is called a CAP-embedded subgroup of G if, for each prime p dividing the order of H, there exists a CAP-subgroup K of G such that a Sylow p-subgroup of H is also a Sylow p-subgroup of K. And they obtained the same conclusion in the case where c-normality is replaced by CAP-embedded property.

In order to generalize the c-normality and CAP-embedded property, Xu and Chen in [14] proposed the defintion of nearly CAP-embedded subgroup. A subgroup H of a group G is said to be nearly CAP-embedded in G if there is a subnormal subgroup T of G and a CAP-embedded subgroup H_{ce} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ce}$. Clearly a c-normal subgroup or CAP-embedded subgroup must be a nearly CAP-embedded subgroup. But the converse is not true in general.

Example 1.1. Let A_4 be the alternative group of degree 4 and $D = \langle d \rangle$ be a cyclic group of order 2. Let $G = D \times A_4$. Then, $A_4 = [K_4]C_3$ where $K_4 = \langle a, b \rangle$

is the Klein Four Group with generators a and b of order 2 and C_3 is the cyclic group of order 3. Take $H = \langle ad \rangle$ to be the cyclic subgroup of order 2 of G. Then $G = HA_4$ and $H \cap A_4 = 1$. By definition, H is nearly CAP-embedded in G. However, H is not a CAP-embedded subgroup of G as it neither covers nor avoids $(D \times K_4)/D$ and there is no the subgroup of order 6 containing H covers or avoids $(D \times K_4)/D$.

In this note, we investigate the structure of the groups in which some second maximal subgroups of a Sylow subgroup satisfy the nearly CAP-embedded property. Our results are as follows:

Theorem 1.1. *Suppose that N is a normal subgroup of a group G such that G/N is p-nilpotent and P is a Sylow p-subgroup of N, where p is the smallest prime divisor of $|G|$. If G is A_4-free and every second maximal subgroups of P is nearly CAP-embedded in G, then G is p-nilpotent.*

Theorem 1.2. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and N a normal subgroup of a group G such that $G/N \in \mathcal{F}$. Suppose that G is A_4-free. If, for every prime p dividing the order of N and $P \in Syl_p(N)$, every second maximal subgroup of P is nearly CAP-embedded in G, then G belongs to $\mathcal{F}$.*

Remark 1.1. The hypothesis that p is the smallest prime divisor of $|G|$ in Theorem 1.1 is essential. For example, consider an elementary abelian group $U = \langle a, b | a^5 = b^5 = 1, ab = ba \rangle$ of order 25. Let α be an automorphism of U of order 3 such that $a^\alpha = b, b^\alpha = a^{-1}b^{-1}$. Let $V = \langle c, d \rangle$ be a copy of U and $G = [U \times V]\langle \alpha \rangle$. For any subgroup H of G of order 25, there exists a minimal normal subgroup K such that $H \cap K = 1$ (for details, see [11, Example 1.5]), then H satisfies the nearly CAP-embedded property in G. However, G is not p-nilpotent.

The assumption that G is A_4-free in Theorem 1.1 and Theorem 1.2 can not be removed. In fact, let $G = A_4$, then the second maximal subgroup of a Sylow 2-subgroup of G is trivial, of course, it satisfies the nearly CAP-embedded property in G, but A_4 is neither a 2-nilpotent group nor a Sylow tower group.

2. Preliminary results

For convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 ([5, Lemma 1])**.** *Suppose that U is CAP-embedded in a group G and $N \trianglelefteq G$. Then, UN/N is CAP-embedded in G/N.*

Lemma 2.2 ([14, Lemma 2.8])**.** *Let U be a nearly CAP-embedded subgroup and N a normal subgroup of a group G. Then*

(1) *If $N \leq U$, then U/N is nearly CAP-embedded in G/N.*

(2) *If* $(|U|, |N|) = 1$*, then* UN/N *is nearly* CAP*-embedded in* G/N*.*

Lemma 2.3 ([12, Lemma 1.6])**.** *Let* P *be a nilpotent normal subgroup of a group* G*. If* $P \cap \Phi(G) = 1$*, then* P *is the direct product of some minimal normal subgroups of* G*.*

Lemma 2.4 ([15, Lemma 2.6])**.** *Let* G *be an* A_4*-free group,* $p = min\pi(G)$*, and let* N *be a normal subgroup of* G *such that* G/N *is* p*-nilpotent. If* $p^3 \nmid |N|$*, then* G *is* p*-nilpotent.*

Lemma 2.5 ([14, Theorem 3.1])**.** *Let* G *be a group,* N *a normal subgroup of* G *such that* G/N *is* p*-nilpotent and* P *a Sylow* p*-subgroup of* N*, where* $p \in \pi(G)$ *with* $(|G|, p-1) = 1$*. If all maximal subgroups of* P *are nearly* CAP*-embedded in* G*, then* G *is* p*-nilpotent.*

Lemma 2.6 ([4, Lemma 2.1])**.** *Let* H *be a subgroups of a group* G*. Let* $1 < \cdots < N < \cdots < M < \cdots < G$ *be a normal series. If* H *covers (avoid)* M/N*, then* H *covers (avoid) any quotient factor between* M *and* N *of any refinement of the normal series.*

Lemma 2.7. *Let* N *be a normal subgroup of a group* G *and* V *a nearly* CAP*-embedded subgroup of* G*. If* $V \leq N$*, then* V *is nearly* CAP*-embedded in* N*.*

Proof. By the hypothesis, there is a subnormal subgroup T of G and a CAP-embedded subgroup V_{ce} of G contained in V such that $G = VT$ and $V \cap T \leq V_{ce}$, For each prime p dividing the order of V_{ce}, there exists a CAP-subgroup K of G such that a Sylow p-subgroup $(V_{ce})_p$ of V_{ce} is also a Sylow p-subgroup K_p of K. Clearly, $T \cap N$ is subnormal in N, $V(T \cap N) = N$, $V \cap T \cap N = V \cap T \leq V_{ce}$, and by the Lemma 2.6, $K \cap N$ is a CAP-subgroup of N and $(V_{ce})_p = (V_{ce})_p \cap N = K_p \cap N = (K \cap N)_p$. Hence, V is nearly CAP-embedded in N. □

3. Proofs

Now, we prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Assume that the result is false. Let G be a minimal counterexample with least $|N| + |G|$.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-nilpotent and $L \nleq \Phi(G)$.

Let L be a minimal normal subgroup of G contained in N. Consider the factor group $\overline{G} = G/N$. Clearly, $\overline{G}/\overline{N} \cong G/N$ is p-nilpotent and $\overline{P} = PL/L$ is a Sylow p-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Now let $\overline{P_1} = P_1L/L$ be a second maximal subgroup of $\overline{P}$. We may assume that P_1 is a second maximal subgroup of P. Then, $P_1 \cap L = P \cap L$ is a Sylow p-subgroup of L. By the hypothesis, there is a subnormal subgroup B and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1B$ and $P_1 \cap B \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. We have $P_1L \cap BL = (P_1L \cap B)L$. Let $\pi(G) = \{p_1, p_2, \cdots p_n\}$, where $p_1 = p$, and B_{p_i} be a Sylow p_i-subgroup of B $(i = 2, \cdots, n)$. Then, B_{p_i} is also a Sylow p_i-subgroup of G, hence $B_{p_i} \cap N$ is a Sylow p_i-subgroup of N $(i = 2, \cdots, n)$. Write $V = \langle L \cap B_{p_2}, \cdots, L \cap B_{p_n} \rangle$, then $V \leq B$. Note that $(|L : P_1 \cap L|, |L : V|) = 1$, $L = (P_1 \cap L)V$, thus $P_1L \cap BL = (P_1L \cap B)L = (P_1V \cap B)L = (P_1 \cap B)VL = (P_1 \cap B)L$. By Lemma 2.1, we get $(P_1L/L) \cap (BL/L) = (P_1 \cap B)L/L \leq (P_1)_{ce}L/L \in Syl_p(KL/L)$. Therefore, $\overline{P_1}$ is nearly CAP-embedded in $\overline{G}$. The choice of G implies that $\overline{G}$ is p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G contained in N and $L \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

If $E = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/E$. Clearly, $\overline{G}/\overline{N} \cong G/NE$ is p-nilpotent because G/N is, where $\overline{N} = NE/E$. Let $\overline{P_1} = P_1E/E$ be a second maximal subgroup of PE/E. We may assume that P_1 is a second maximal subgroup of P. Since P_1 is nearly CAP-embedded in G, P_1E/E is nearly CAP-embedded in G/E by Lemma 2.2 (2). The minimality of G yields that G is p-nilpotent, therefore G is p-nilpotent, a contradiction.

(3) $O_p(N) = 1$ and so L is not p-nilpotent.

If $O_p(N) \neq 1$, then by (1), $L \leq O_p(N)$ and there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. By (1) and Lemma 2.4, we get $|L| \geq p^3$. So we may choose a second maximal subgroup P_1 of P containing M_p, where $M_p \in Syl_p(M)$. Because P_1 is a nearly CAP-embedded subgroup of G, there is a subnormal subgroup T of G and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \cap (P_1)_{ce} \in Syl_p(L)$, and so $L \leq P_1$, thus $P = LM_p = LP_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|P \cap T \cap L| \leq p^2$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$. Hence, G is p-nilpotent, a contradiction.

If L is p-nilpotent, then $L_{p'}$ $Char$ $L \trianglelefteq N$, so $L_{p'} \leq O_{p'}(N) \leq O_{p'}(G) = 1$ by (2). Thus L is a p-group, $L \leq O_p(N) = 1$, a contradiction. Hence, (3) holds.

(4) The final contradiction.

If $P \leq L$, then $P \in Syl_p(L)$. By (3) and Lemma 2.4, $|P| > p^2$. For every second maximal subgroup P_3 of P, P_3 is nearly CAP-embedded in G. So, there is a subnormal subgroup T_1 of G and a CAP-embedded subgroup $(P_3)_{ce}$ contained in P_3 of G such that $G = P_3T$ and $P_3 \cap T \leq (P_3)_{ce} \in Syl_p(A)$, where A is a CAP subgroup of G. Clearly $AL \neq A$ and $(P_3)_{ce} \leq L \cap A = 1$, then $p^3 \nmid |T|$, so T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then, $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$. Hence, G is p-nilpotent, a contradiction. Therefore, $P \nleq L$. If $P \cap L \leq \Phi(P)$, then L is p-nilpotent by Tate's theorem [8, IV, Th 4.7], contrary to (3). Consequently, $P \cap L \nleq \Phi(P)$.

Let P_1 be a maximal subgroup of P containing $L \cap P$. Clearly, $L \cap P \nleq \Phi(P_1)$. Hence, there exists a maximal subgroup P_2 of P_1 such that $P_1 = (L \cap P)P_2$. Let T be a subnormal supplement of P_2 in G, we have $P_2 \cap T \leq (P_2)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_2)_{ce} \in Syl_p(K)$ that $P_2 \cap K = (P_2)_{ce}$, then $P_2 \cap L \in Syl_p(L)$. Thus $L \cap P = L \cap P_2 \leq K \cap P_2 = (P_2)_{ce} \leq P_2$. We obtain $P_1 = (L \cap P)P_2 = P_2$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_2 \cap T \cap L = 1$. Consequently, $|P \cap T \cap L| \leq p^2$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.4. Hence, G is p-nilpotent by the subnormality of T, a contradiction.

This completes the proof. □

Corollary 3.1. *Let G be an A_4-free group, P a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. If G is not p-nilpotent, then there is a second maximal subgroup of $P \cap G^{\mathcal{N}_p}$ which is not nearly CAP-embedded in G.*

Corollary 3.2. *Let G be an A_4-free group. If, for every prime p dividing the order of G and $P \in Syl_p(G)$, every second maximal subgroup of P is nearly CAP-embedded in G, then G is a Sylow tower group of supersolvable type.*

Similarly, we have the following results.

Theorem 3.1. *Let N be a normal subgroup of a group G such that G/N is p-nilpotent and let P be a Sylow p-subgroup of N, where p is a prime divisor of $|G|$ with $(|G|, p^2 - 1) = 1$. If every second maximal subgroup of P is nearly CAP-embedded in G, then G is p-nilpotent.*

Proof of Theorem 1.2. By Lemma 2.7 and Corollary 3.2, we use induction on $|G|$ to see that N is a Sylow tower group of supersolvable type. Let r be the largest prime number in $\pi(N)$ and $R \in Syl_p(N)$. Then, R is normal in G and $(G/R)/(N/R) \cong G/N$ is a Sylow tower group of supersolvable type. By induction, $G/R \in \mathcal{F}$. Let q be the largest prime divisor of $|G|$ and Q a Sylow q-subgroup of G. Then, $RQ \trianglelefteq G$. If $q = r$, then G has the Sylow tower property, as desired. Hence, we may assume that $r < q$.

Case 1. $RQ < G$. In this case, we will show $G_1 = RQ$ is p-nilpotent. By the hypothesis and Lemma 2.7, G_1 is A_4-free and every second maximal subgroup R_1 of R is nearly CAP-embedded in G_1. Then, by Theorem 1.1, $Q\ Char\ RQ$ and so $Q \trianglelefteq G$. Now, consider $(G/Q, NQ/Q)$. Then, $G/Q \in \mathcal{F}$ by induction and Lemma 2.2. Thus $G \in \mathcal{F}$, as desired again.

Case 2. $G = RQ$. Let L be a minimal normal subgroup of G with $L \leq R$. Then, the quotient group G/L satisfies the hypothesis. By induction, we see that G/L is a Sylow tower group of supersolvable type. Since the class of all Sylow tower groups is a saturated formation, we have $L \nsubseteq \Phi(G)$ and L is the unique minimal normal subgroup of G which is contained in R. Therefore, $L = F(R) = R$ by Lemma 2.3. In particular, R is an abelian group. If R is a cyclic subgroup of order r, then $r < q$ implies that $G = R \times Q$. Of course, $G \in \mathcal{F}$, which completes the proof. Hence, we may assume that $|R| \geq r^2$. Let R_1 be a 2-maximal subgroup of R. By the hypothesis, R_1 is nearly CAP-embedded in G. By the proof of the step (3) in theorem 1.1, we have R is an elementary abelian group of order r^2. Now, any element g of Q induces an automorphism σ of R. When $|R| = r^2$, we know that $|Aut(R)| = (r+1)r(r-1)^2$. If $r = 2$ and some $\sigma \neq 1$, then the order of σ must be 3 as $r < q$. Thus the subgroup $R\langle g \rangle$ is not A_4-free, contrary to the hypothesis. Hence, all $\sigma = 1$, i.e., $G = R \times Q$, completing the proof. The remainder is to consider the case when $r > 2$. Noticing that $r+1$ is not a prime, so we have all $\sigma = 1$ and $G = R \times Q$, hence $G \in \mathcal{F}$. The proof is now completed. □

Corollary 3.3. *Let G be a group of odd order, N a normal subgroup of G such that G/N is a Sylow tower group of supersolvable type. If, for every prime p dividing the order of N and $P \in Sylp(N)$, every second maximal subgroup of P is nearly CAP-embedded in G, then G is a Sylow tower group of supersolvable type.*

4. Some applications

Obviously, c-normal subgroups, CAP-subgroups and CAP-embedded subgroups are nearly CAP-embedded subgroups, a lot of results can be obtained according to our theorems.

By Theorem 1.1, Theorem 1.2 and Theorem 3.1, we have:

Corollary 4.1 ([7, Theorem 3.2])**.** *Let p is the smallest prime divisor of $|G|$, P is a Sylow p-subgroup of G. If every second maximal subgroups of P is c-normal in G and G is A_4-free, then G is p-nilpotent.*

Corollary 4.2 ([13, Theorem 4.2])**.** *Let G be a finite group and let p be the smallest prime divisor of $|G|$. Assume that G is A_4-free and every second maximal subgroup of the Sylow p-subgroup of G is c-normal in G. Then, $G/O_p(G)$ is p-nilpotent.*

Corollary 4.3 ([6, Theorem 3.11])**.** *Let H be a normal subgroup of a group G and p the smallest prime number dividing the order of H. If all 2-maximal subgroups of every Sylow p-subgroup of H are CAP-subgroups of G and G is A_4-free, then H is p-nilpotent.*

Corollary 4.4 ([6, Corollary 3.13])**.** *Let H be a normal subgroup of a group G. If G is A_4-free and all 2-maximal subgroups of every Sylow subgroup of H are CAP-subgroups of G, then H is a Sylow tower group of supersolvable type.*

Corollary 4.5 ([5, Theorem 3.3])**.** *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If G is A_4-free, and there exists a Sylow p-subgroup P of*

H such that every 2-maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.

Corollary 4.6 ([5, Corollary 3.4]). *Let p be a prime dividing the order of the group G with $(|G|, p^2 - 1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.*

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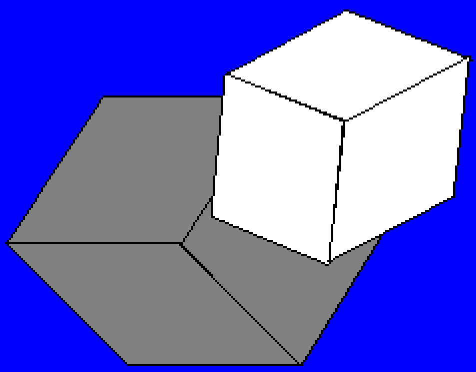

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