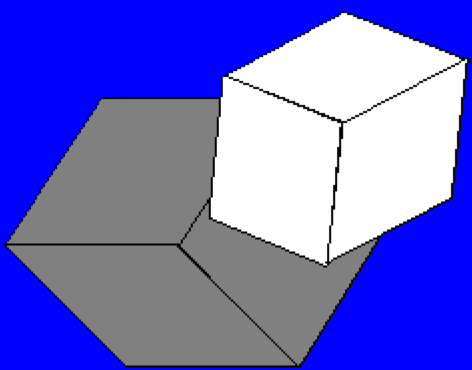
N° 50 – October 2023

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227



FORUM EDITRICE UNIVERSITARIA UDINESE
FARE srl

In memoriam of Professor **Ali Reza Ashrafi**

The ***Italian Journal of Pure and Applied Mathematics*** *(IJPAM)* cannot more take advantage of the precious collaboration of prof. Ali Reza Ashrafi, who has passed away this year.

The members of Editorial Board express their deep sorrow for this loss.

The Chief Editors and the IJPAM Core Team regret the loss of Prof. Ali Reza Ashrafi. He has been a great man of science.

All they who knew him will remember always his scientific value and his human qualities.

Piergiulio Corsini
Irina Cristea
Viloleta Fotea
Maria Antonietta Lepellere
Domenico Chillemi

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227

Web Site
http://ijpam.uniud.it/journal/home.html

Twitter
@ijpamitaly
https://twitter.com/ijpamitaly

FORUM EDITRICE UNIVERSITARIA UDINESE
FARE srl
Via Larga 38 - 33100 Udine
Tel: +39-0432-26001, Fax: +39-0432-296756
forum@forumeditrice.it

Table of contents

Editorial

On the occasion of the publication of the 50th volume of the Italian Journal of Pure and Applied Mathematics, the Editorial Core Team would like first to thank to the journal's founder and co-chief editor prof. Piergiulio Corsini for his immense devotion and extraordinary work and professionalism related to the leading of the journal. Secondly our gratitude goes to all members of the editorial board and the anonymous reviewers, who have dedicated a lot of time, expertise and energy for a qualitative and impartial peer-review process along 36 years of existence of the journal.

IJPAM was founded in 1987 by prof. Corsini, having initially the Italian title "Rivista Italiana di Matematica Pura ed Applicata", changed in the English one ten years later. Thanks to the financial support received (in the first decades of its existence) from the University of Udine, the journal had a continuous growth, having been exchanged with other 250 mathematical journals very well known all over the world. Starting with vol. 27 in 2010, the policy of the journal has changed and it became an online open access journal, publishing 2 issues per year. Since after 2010 exchanges lost their mission, due to the fact that the new online journal availability policy made it continually exchanged with every other journal or university, Exchanges pages will be removed from the online journal starting with number 50, and there will only remain a page in the web site to just keep the historical Exchanges information. Currently IJPAM is indexed in both Scopus and Web of Science (ESCI edition) data bases, getting its first impact factor in 2022. Since 2021 the journal has a new managing team: chief editors prof. Piergiulio Corsini and prof. Irina Cristea (University of Nova Gorica, Slovenia), vice-chief editors: prof. Violeta Fotea (University "Al. I. Cuza" Iasi, Romania) and prof. Maria Antonietta Lepellere (Udine University, Italy), forming the Editorial Core Team together with the managing board chief Domenico Chillemi (Former Executive Technical Specialist, Freelance IBM z Systems & Security Software).

Aiming to increase the quality of the journal, we continuously review the editorial team, by recruiting new members, who are active and renowned researchers in different (still not all of them) topics in pure and applied mathematics within the scope of the journal. In future the journal will publish also original research in mathematical education. IJPAM will remain an open access journal, where the publication fee (applied only to the accepted manuscripts that passed the blind peer-review process) will be calculated starting with 2024 by the formula 6€ $*$ n + 30€, where n represents the number of the pages of the manuscript prepared in the journal's format.

Editorial Core Team looks forward to serving the research math community by publishing in IJPAM high quality articles and overviews authored by talented researchers all over the world.

October, 2023

Editorial Core Team

Irina Cristea
Violeta Fotea
Maria Antonietta Lepellere
Domenico Chillemi

On the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)$-diaphony of the nets of type of Halton-Zaremba constructed over finite groups

Vesna Dimitrievska Ristovska*
Faculty of Computer Science and Engineering
University "SS. Cyril and Methodius"
16 Rugjer Boshkovikj str., 1000 Skopje
Macedonia
vesna.dimitrievska.ristovska@finki.ukim.mk

Vassil Grozdanov
Department of Mathematics
Faculty of Mathematics and Natural Sciences
South-West University "Neofit Rilski"
66 Ivan Mihailov str., 2700
Blagoevgrad
Bulgaria
vassgrozdanov@yahoo.com

Abstract. In the present paper, the authors introduce an arithmetic based on finite groups with respect to arbitrary bijections. This algebraic background is used to construct the function system $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}$ of the Walsh functions over the set $G_{\mathbf{b}}$ of groups with respect to the set $\varphi_{\mathbf{b}}$ of bijections. The developed algebraic base is also used to introduce a wide class of two-dimensional nets ${}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}$ of type of Halton-Zaremba. Four concrete nets of this class are constructed and graphically illustrated. The so-called $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony is applied as a appropriate tool for studying the nets of the introduced class. An exact formula for the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the nets of class ${}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}$ is presented. This formula allows us to show the influence of the vector α on the exact order of the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of these nets.

Keywords: $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$ diaphony, nets of type of Halton-Zaremba constructed over finite groups, exact formula, exact orders.

1. Introduction

Let $s \geq 1$ be a fixed integer, which will denote the dimension of the objects considered in the paper. We will remind the notion of uniformly distributed sequence. So, following Kuipers and Niederreiter [16] let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For an arbitrary integer $N \geq 1$ and a subinterval J of $[0,1)^s$ with a Lebesgue measure $\lambda_s(J)$ let us denote $A_N(\xi;J) = \#\{n : 0 \leq n \leq N-1, \mathbf{x}_n \in J\}$. The sequence ξ is called uniformly distributed in

∗. Corresponding author

$[0,1)^s$ if the limit equality $\lim_{N\to\infty} \frac{A_N(\xi;J)}{N} = \lambda_s(J)$ holds for each subinterval J of $[0,1)^s$.

The functions of some orthonormal function systems are used to solve many problems of the theory of the uniformly distributed sequences with very big success. We will remind the definitions of the functions of some of these classes.

For an arbitrary integer k and a real x the function $e_k : \mathbb{R} \to \mathbb{C}$ is defined as $e_k(x) = e^{2\pi \mathbf{i} kx}$. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ the $\mathbf{k}-$th multivariate trigonometric function $e_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as $e_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^s e_{k_j}(x_j)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$. The set $\mathcal{T}^s = \{e_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^s,\ \mathbf{x} \in [0,1)^s\}$ is called trigonometric function system.

Following Chrestenson [4] we will recall the constructive principle of the Walsh functions. Let $b \geq 2$ be a fixed integer. For an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the $b-$adic representation $k = \sum_{i=0}^{\nu} k_i b^i$ and $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where $k_i, x_i \in \{0, 1, \ldots, b-1\}$, $k_\nu \neq 0$ and for infinitely many values of i we have $x_i \neq b-1$, the $k-$th Walsh function ${}_b wal_k : [0,1) \to \mathbb{C}$ is defined as

$$ {}_b wal_k(x) = e^{\frac{2\pi \mathbf{i}}{b}(k_0 x_0 + \ldots + k_\nu x_\nu)}. $$

Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}-$th multivariate Walsh function ${}_b wal_{\mathbf{k}} : [0,1)^s \to \mathbb{C}$ is defined as

$$ {}_b wal_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_b wal_{k_j}(x_j), \quad \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s. $$

The set $\mathcal{W}(b) = \{{}_b wal_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ is called Walsh function system in base b. In the case when $b = 2$ the system $\mathcal{W}(2)$ is the original system of Walsh [22] functions.

The different kinds of the diaphony are numerical measures, which show the quality of the distribution of the points of sequences and nets. So, let $\xi_N = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$.

Firstly Zinterhof [25] introduced the notion of the so-called classical diaphony. So, the classical diaphony of the net ξ_N is defined as

$$ F(\mathcal{T}^s; \xi_N) = \left(\sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} R^{-2}(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} e_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}}, $$

where for each vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ the coefficient $R(\mathbf{k}) = \prod_{j=1}^s R(k_j)$ and for an arbitrary integer k

$$ R(k) = \begin{cases} 1, & \text{if } \ k = 0, \\ |k|, & \text{if } \ k \neq 0. \end{cases} $$

Hellekalek and Leeb [15] introduced the notion of the dyadic diaphony, which is based on using the original system $\mathcal{W}(2)$ of the Walsh function. Grozdanov and Stoilova [10] generalized the notion of the dyadic diaphony to the so-called $b-$adic diaphony. So, the $b-$adic diaphony of the net ξ_N is defined as

$$F(\mathcal{W}(b);\xi_N)=\left(\frac{1}{(b+1)^s-1}\sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}}\rho(\mathbf{k})\left|\frac{1}{N}\sum_{n=0}^{N-1}{}_bwal_{\mathbf{k}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k}=(k_1,\dots,k_s)\in\mathbb{N}_0^s$ the coefficient $\rho(\mathbf{k})=\prod_{j=1}^s\rho(k_j)$ and for an arbitrary non-negative integer k

$$\rho(k)=\begin{cases}1, & \text{if } \ k=0,\\ b^{-2g}, & \text{if } \ b^g\le k\le b^{g+1}, \quad g\ge 0, \quad g\in\mathbb{Z}.\end{cases}$$

In 1986 Proinov [18] established a general lower bound of the classical diaphony. So, for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$F(\mathcal{T}^s;\xi_N)>\alpha(s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{1}$$

holds, where $\alpha(s)$ is a positive constant depending only on the dimension s. For a dimension $s=1$ from the inequality (1) the result of Stegbuchner [20] is obtained

$$F(\mathcal{T}^s;\xi_N)\ge\frac{\pi}{\sqrt{3}}\cdot\frac{1}{N}.$$

To show the exactness of the lower bound (1) for a dimension $s=2$ we need to present the techniques of the construction of two classical two-dimensional nets. For this purpose, let $\nu>0$ be a fixed integer. For $0\le i\le b^\nu-1$ we denote $\eta_{b,\nu}(i)=\frac{i}{b^\nu}$. Following Van der Corput [21] and Halton [12] for an arbitrary integer i, $0\le i\le b^\nu-1$, with the $b-$adic representation $i=\sum_{j=0}^{\nu-1}i_jb^j$, where for $0\le j\le\nu-1$ $i_j\in\{0,1,\dots,b-1\}$, we put $p_{b,\nu}(i)=\sum_{j=0}^{\nu-1}i_jb^{-j-1}$. Roth [19] considered the two-dimensional net $R_{b,\nu}=\{(\eta_{b,\nu}(i),p_{b,\nu}(i)):0\le i\le b^\nu-1\}$, which now is called a net of Roth. The net $R_{b,\nu}$ is also known as two-dimensional Hammersley [14] point set.

In 1969, Halton and Zaremba [13] used the original net of Van der Corput $\{p_{2,\nu}(i)=0.i_0i_1\dots i_{\nu-1}:0\le i\le 2^\nu-1\ i_j\in\{0,1\}\}$ and change the digits i_j that stay in the even positions with the digit $1-i_j$. Let us for $0\le i\le 2^\nu-1$ signify $z_{2,\nu}(i)=0.(1-i_0)i_1(1-i_2)\dots$. The net $Z_{2,\nu}=\{(\eta_{2,\nu}(i),z_{2,\nu}(i)):0\le i\le 2^\nu-1\}$, which is called net of Halton-Zaremba is constructed.

In 1998 Xiao [24] proved that the classical diaphony of the net of Roth $R_{b,\nu}$ and the net of Halton-Zaremba $Z_{2,\nu}$ have an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$, where respectively $N=b^\nu$ and $N=2^\nu$.

Cristea and Pillichshammer [5] proved a general lower bound of the $b-$adic diaphony. So, for any net ξ_N composed of N points in $[0,1]$ the lower bound

$$F(\mathcal{W}(b);\xi_N)\ge C(b,s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{2}$$

holds, where $C(b,s)$ is a positive constant depending on the base b and the dimension s.

Grozdanov and Stoilova [11] proved the exactness of the lower bound (2) for dimension $s=2$. They proved that the $b-$adic diaphony of the net of Roth $R_{b,\nu}$ and the net of Halton-Zaremba $Z_{2,\nu}$ have an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$, where respectively $N=b^\nu$ and $N=2^\nu$.

The $b-$adic diaphony is closely related with the worst-case error of the quasi-Monte Carlo integration in appropriate Hilbert spaces. Aronszajn [1] introduced the notion of a reproducing kernel for Hilbert space. So, following this approach let $H_s(K)$ be a Hilbert space with a reproducing kernel $K:[0,1)^s\to\mathbb{C}$, an inner product $<\cdot,\cdot>_{H_s(K)}$ and a norm $||\cdot||_{H_s(K)}$. We are interested to approximate the multivariate integral

$$I_s(f)=\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x},\quad f\in H_s(K).$$

Let $N\geq 1$ be an arbitrary and fixed integer. We will approximate the integral $I_s(f)$ through quasi-Monte Carlo algorithm $Q_s(f;P_N)=\frac{1}{N}\sum_{n=0}^{N-1}f(\mathbf{x}_n)$, where $P_N=\{\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\}$ is a deterministic sample point set in $[0,1)^s$. The worst-case error of the integration in the space $H_s(K)$ by using the net P_N is defined as

$$e(H_s(K);P_N)=\sup_{f\in H_s(K),\ ||f||_{H_s(K)}\leq 1}|I_s(f)-Q_s(f;P_N)|.$$

Dick and Pillichshammer [6] used the Walsh functions as a tool for investigation of the worst-case error of the multivariate integration in Hilbert spaces. This error is presented in the terms of the reproducing kernel, which generates this space.

Likewise, Dick and Pillichshammer [7] introduced a special reproducing kernel Hilbert space and the worst-case error of the integration in this space and the $b-$adic diaphony of the net of the nodes of the integration are connected. In this sense, we see that the so-called low diaphony nets with very big success can be used in the practice of the quasi-Monte Carlo integration. This determines the interest to this class of nets.

The rest of the paper is organized in the following manner: In Section 2 the concept of the function system $\mathcal{W}_{G_\mathbf{b},\varphi_\mathbf{b}}$ is reminded. In Section 3 we introduce a class of nets ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ of type of Halton-Zaremba constructed over finite groups. By graphical illustrations, we show the distribution of four nets from this class. In Section 4 the concept of the $(\mathcal{W}_{G_\mathbf{b},\varphi};\alpha)-$diaphony is presented. In Section 5 an explicit formula for the $(\mathcal{W}_{G_\mathbf{b},\varphi};\alpha)-$diaphony of the nets ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ is presented. This formula allows us to show the influence of the vector α of exponential parameters to the exact orders of the considered diaphony of these nets. In Section 6 some preliminary results are presented. In Section 7 the main results of the paper are proved.

2. The function system $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}$

In 1996 Larcher, Niederreiter and W. Ch. Schmid [17] introduced the concept of the so-called Walsh function system over finite groups. So, the details are as follows: Let $m \geq 1$ be a given integer and let $\{b_1, b_2, \ldots, b_m : b_l \geq 2, 1 \leq l \leq m\}$ be a set of fixed integers. For $1 \leq l \leq m$ let $\mathbb{Z}_{b_l} = \{0, 1, \ldots, b_l - 1\}$ and the operation $\oplus_{b_l}$ be the addition modulus b_l of the elements of the set $\mathbb{Z}_{b_l}$. Then, $(\mathbb{Z}_{b_l}, \oplus_{b_l})$ is a discrete cyclic group of order b_l.

Let $G = \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_m}$ be the Cartesian product of the sets $\mathbb{Z}_{b_1}, \ldots, \mathbb{Z}_{b_m}$. For each pair $\mathbf{g} = (g_1, \ldots, g_m)$, $\mathbf{y} = (y_1, \ldots, y_m) \in G$ by using the group operations $\oplus_{b_1}, \ldots, \oplus_{b_s}$ let the operation $\oplus_G$ be defined as $\mathbf{g} \oplus_G \mathbf{y} = (g_1 \oplus_{b_1} y_1, \ldots, g_m \oplus_{b_m} y_m)$. Then, $(G, \oplus_G)$ is a finite group of order $b = b_1 b_2 \ldots b_m$. For the given elements $\mathbf{g}, \mathbf{y} \in G$ the character function on the group G is defined as

$$\chi_{\mathbf{g}}(\mathbf{y}) = \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{g_l y_l}{b_l}}.$$

Let us denote $\mathbb{Z}_b = \{0, 1, \ldots, b-1\}$ and let $\varphi : \mathbb{Z}_b \to G$ be an arbitrary bijection, which satisfies the condition $\varphi(0) = \mathbf{0}$.

Definition 1. *For an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the $b-$adic representations $k = \sum_{i=0}^{\nu} k_i b^i$ and $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where for $i \geq 0$ $k_i, x_i \in \{0, 1, \ldots, b-1\}$ $k_\nu \neq 0$ and for infinitely many values of i $x_i \neq b-1$, the function ${}_{G,\varphi}wal_k : [0,1) \to \mathbb{C}$ is defined as ${}_{G,\varphi}wal_k(x) = \prod_{i=0}^{\nu} \chi_{\varphi(k_i)}\left(\varphi(x_i)\right)$.*

The set $\mathcal{W}_{G,\varphi} = \{{}_{G,\varphi}wal_k(x) : k \in \mathbb{N}_0,\ x \in [0,1)\}$ is called Walsh function system over the group G with respect to the bijection φ.

Now, we will introduce the concept of the multidimensional function system of Walsh functions over finite groups. For this purpose, let $\mathbf{b} = (b_1, \ldots, b_s)$ be a vector of not necessarily distinct integers $b_j \geq 2$. For $1 \leq j \leq s$ let $(G_{b_j}, \oplus_{G_{b_j}})$ be an arbitrary group of order b_j constructed as above. Let us denote $\mathbb{Z}_{b_j} = \{0, 1, \ldots, b_j - 1\}$ and let $\varphi_{b_j} : \mathbb{Z}_{b_j} \to G_{b_j}$ be an arbitrary bijection, which satisfies the condition $\varphi_{b_j}(0) = \mathbf{0}$. Let $\mathcal{W}_{G_{b_j},\varphi_{b_j}} = \{{}_{G_{b_j},\varphi_{b_j}}wal_k(x) : k \in \mathbb{N}_0,\ x \in [0,1)\}$ be the corresponding Walsh function system over the group G_{b_j} with respect to the bijection φ_{b_j}.

By using the groups $G_{b_1}, \ldots, G_{b_s}$, the sets $\mathbb{Z}_{b_1}, \ldots, \mathbb{Z}_{b_s}$ and the bijections $\varphi_{b_1}, \ldots, \varphi_{b_s}$ let us introduce the next significations $G_{\mathbf{b}} = (G_{b_1}, \ldots, G_{b_s})$, $\mathbb{Z}_{\mathbf{b}} = (\mathbb{Z}_{b_1}, \ldots, \mathbb{Z}_{b_s})$ and $\varphi_{\mathbf{b}} = (\varphi_{b_1}, \ldots, \varphi_{b_s})$.

Let $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}} = \mathcal{W}_{G_{b_1},\varphi_{b_1}} \otimes \ldots \otimes \mathcal{W}_{G_{b_s},\varphi_{b_s}}$ be the tensor product of the function systems $\mathcal{W}_{G_{b_1},\varphi_{b_1}}, \ldots, \mathcal{W}_{G_{b_s},\varphi_{b_s}}$. This means that for an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}-$th Walsh function ${}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x})$ is defined as

$${}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j), \quad \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s.$$

We will call the set $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}} = \{_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s, \ \mathbf{x} \in [0,1)^s\}$ a multidimensional system of the Walsh functions over the set $G_{\mathbf{b}}$ of groups with respect to the set $\varphi_{\mathbf{b}}$ of bijections.

We will introduce some elements of the $\mathbf{b}-$adic arithmetic. By using the operation $\oplus_G$ over the group G and the bijection φ we will define the operation $\oplus_{\mathbb{Z}_b,\varphi} : \mathbb{Z}_b^2 \to \mathbb{Z}_b$ by the following manner: for arbitrary elements $u, v \in \mathbb{Z}_b$, we put $u \oplus_{\mathbb{Z}_b,\varphi} v = \varphi^{-1}(\varphi(u) \oplus_G \varphi(v))$. For an arbitrary element $u \in \mathbb{Z}_b$, let the element $\overline{u} \in \mathbb{Z}_b$ be such that $u \oplus_{\mathbb{Z}_b,\varphi} \overline{u} = 0$. We will prove that for arbitrary digits $p, q, r \in \mathbb{Z}_b$ the character function satisfies the equalities

(3) $$\chi_{\varphi(p)}(\varphi(q) \oplus_G \varphi(r)) = \chi_{\varphi(p)}(\varphi(q))\chi_{\varphi(p)}(\varphi(r))$$

and

(4) $$\chi_{\varphi(p)\oplus_G\varphi(q)}(\varphi(r)) = \chi_{\varphi(p)}(\varphi(r))\chi_{\varphi(q)}(\varphi(r)).$$

Let us signify $\varphi(p) = (p^{(1)}, \ldots, p^{(m)})$, $\varphi(q) = (q^{(1)}, \ldots, q^{(m)})$ and $\varphi(r) = (r^{(1)}, \ldots, r^{(m)})$. Hence, we obtain that

$$\chi_{\varphi(p)}(\varphi(q) \oplus_G \varphi(r)) = \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{p^{(l)}[q^{(l)}+r^{(l)} (mod \ b_l)]}{b_l}} = \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{p^{(l)}(q^{(l)}+r^{(l)})}{b_l}}$$
$$= \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{p^{(l)}q^{(l)}}{b_l}} \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{p^{(l)}r^{(l)}}{b_l}} = \chi_{\varphi(p)}(\varphi(q))\chi_{\varphi(p)}(\varphi(r))$$

and

$$\chi_{\varphi(p)\oplus_G\varphi(q)}(\varphi(r)) = \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{[p^{(l)}+q^{(l)} (mod \ b_l)]r^{(l)}}{b_l}} = \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{(p^{(l)}+q^{(l)})r^{(l)}}{b_l}}$$
$$= \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{p^{(l)}r^{(l)}}{b_l}} \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{q^{(l)}r^{(l)}}{b_l}} = \chi_{\varphi(p)}(\varphi(r))\chi_{\varphi(q)}(\varphi(r)).$$

For arbitrary reals $x, y \in [0,1)$ with the $b-$adic representations $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$ and $y = \sum_{i=0}^{\infty} y_i b^{-i-1}$, where for $i \geq 0$ $x_i, y_i \in \mathbb{Z}_b$ and for infinitely many values of i $x_i, y_i \neq b-1$, let us define the next operation

$$x \oplus_{\mathbb{Z}_b,\varphi}^{[0,1)} y = \left(\sum_{i=0}^{\infty} (x_i \oplus_{\mathbb{Z}_b,\varphi} y_i) b^{-i-1} \right) \ (mod\ 1).$$

We will prove that for an arbitrary integer $k \in \mathbb{N}_0$ and arbitrary reals $x, y \in [0,1)$ the equality holds

(5) $$_{G,\varphi}wal_k(x \oplus_{\mathbb{Z}_b,\varphi}^{[0,1)} y) = {}_{G,\varphi}wal_k(x)_{G,\varphi}wal_k(y).$$

Let k have the $b-$adic representation $k=\sum_{i=0}^{\nu} k_i b^i$, where for $0\le i\le \nu$ $k_i\in\{0,1,\dots,b-1\}$, x and y be as above. Then, by using the equality (3) we obtain that

$$
\begin{aligned}
&{}_{G,\varphi}wal_k(x\oplus_{\mathbb{Z}_b,\varphi}^{[0,1)} y)\\
&=\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(\varphi^{-1}(\varphi(x_i)\oplus_G \varphi(y_i))))=\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(x_i)\oplus_G\varphi(y_i))\\
&=\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(x_i))\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(y_i))={}_{G,\varphi}wal_k(x){}_{G,\varphi}wal_k(y).
\end{aligned}
$$

For arbitrary vectors $\mathbf{x}=(x_1,\dots,x_s)\in[0,1)^s$ and $\mathbf{y}=(y_1,\dots,y_s)\in[0,1)^s$ to define the operation $\mathbf{x}\oplus_{\mathbb{Z}_{\mathbf{b}},\varphi_{\mathbf{b}}}^{[0,1)^s}\mathbf{y}=(x_1\oplus_{\mathbb{Z}_{b_1},\varphi_{b_1}}^{[0,1)} y_1,\dots,x_s\oplus_{\mathbb{Z}_{b_s},\varphi_{b_s}}^{[0,1)} y_s)$. Then, the following equality holds

$$
{}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}\oplus_{\mathbb{Z}_{\mathbf{b}},\varphi_{\mathbf{b}}}^{[0,1)^s}\mathbf{y})={}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}){}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{y}),\ \forall\,\mathbf{k}\in\mathbb{N}_0^s. \tag{6}
$$

Let $\mathbf{k}=(k_1,\dots,k_s)\in\mathbb{N}_0^s$ be an arbitrary vector. Then, by using the equality (5) the following holds

$$
\begin{aligned}
&{}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}\oplus_{\mathbb{Z}_{\mathbf{b}},\varphi_{\mathbf{b}}}^{[0,1)^s}\mathbf{y})=\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j\oplus_{\mathbb{Z}_{b_j},\varphi_{b_j}}^{[0,1)} y_j)\\
&=\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j){}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(y_j)\\
&=\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j)\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(y_j)={}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}){}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{y}).
\end{aligned}
$$

3. Nets of type of Halton - Zaremba constructed over finite groups

To present the definition of the nets of type of Halton-Zaremba constructed over finite groups, we will apply the same algebraic basis, which we used to present the functions of the system $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}$. In this way, a process of a synchronization between the construction of the nets and the tool for their investigation will be realized.

For this purpose, let $b_1\ge 2$ and $b_2\ge 2$ be arbitrary and fixed bases and denote $\mathbf{b}=(b_1,b_2)$. Let $(\mathbb{Z}_{b_1},\oplus_{b_1})$ and $(\mathbb{Z}_{b_2},\oplus_{b_2})$ be the corresponding discrete cyclic groups of orders b_1 and b_2. Let $b=b_1b_2$ and as yet to define $G_b=\mathbb{Z}_{b_1}\times\mathbb{Z}_{b_2}$ and $\oplus_{G_b}=(\oplus_{b_1},\oplus_{b_2})$. Let $\mathbb{Z}_b=\{0,1,\dots,b-1\}$, $\varphi_1:\mathbb{Z}_b\to G_b$ and $\varphi_2:\mathbb{Z}_b\to G_b$ be two arbitrary bijections, which satisfy the conditions $\varphi_1(0)=\mathbf{0}$, $\varphi_2(0)=\mathbf{0}$ and denote $\varphi_b=(\varphi_1,\varphi_2)$. Let $\oplus_{\mathbb{Z}_b,\varphi_1}^{[0,1)}$ and $\oplus_{\mathbb{Z}_b,\varphi_2}^{[0,1)}$ be the operations over $[0,1)$, which correspond respectively to the bijections φ_1 and φ_2.

Let $\nu\ge 1$ be an arbitrary and fixed integer. Let $\kappa=0.\kappa_0\kappa_1\dots\kappa_{\nu-1}$ and $\mu=0.\mu_0\mu_1\dots\mu_{\nu-1}$ be arbitrary and fixed $b-$adic rational numbers. For $0\le$

$i \le b^\nu - 1$ let us denote $\eta_{b,\nu}(i) = \frac{i}{b^\nu}$ and $p_{b,\nu}(i)$ be the general term of the Van der Corput sequence. Let us define the $b-$adic rational numbers

$$_{G_b,\varphi_1}\xi^{\kappa}_{b,\nu}(i) = \eta_{b,\nu}(i) \oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_1} \kappa \text{ and } {}_{G_b,\varphi_2}\zeta^{\mu}_{b,\nu}(i) = p_{b,\nu}(i) \oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_2} \mu.$$

Dimitrievska Ristovska and Grozdanov [8] introduced the next class of two-dimensional nets:

Definition 2. *For an arbitrary integer $\nu \ge 1$ and for arbitrary fixed $b-$adic rational numbers κ and μ we define the net*

$$_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu} = \left\{ \left({}_{G_b,\varphi_1}\xi^{\kappa}_{b,\nu}(i), {}_{G_b,\varphi_2}\zeta^{\mu}_{b,\nu}(i) \right) : 0 \le i \le b^\nu - 1 \right\},$$

which we will call a net of type of Halton-Zaremba constructed over the group G_b with respect to the set φ_b, which corresponds to the parameters κ and μ in base b.

We will concrete the choice of the parameters κ and μ from Definition 2: Let us choose $\kappa = 0$. We will construct the digits of the parameters μ in the following manner: Let $p, q \in \mathbb{Z}_b$ be arbitrary and fixed digits. For $0 \le j \le \nu - 1$ we define the digits $\mu_j \in \mathbb{Z}_b$ as the solutions of the linear recurrence equation $\mu_j \equiv p \cdot j + q \ (mod\ b)$ and to put $\mu = 0.\mu_0\mu_1 \dots \mu_{\nu-1}$. For $0 \le i \le b^\nu - 1$ let us denote ${}_{G_b,\varphi_2}\zeta^{p,q}_{b,\nu}(i) = p_{b,\nu}(i) \oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_2} \mu$. In this case, we obtain the net ${}_{G_b,\varphi_2}Z^{p,q}_{b,\nu} = \left\{ (\eta_{b,\nu}(i), {}_{G_b,\varphi_2}\zeta^{p,q}_{b,\nu}(i)) : 0 \le i \le b^\nu - 1 \right\}$, which was introduced by Grozdanov [9].

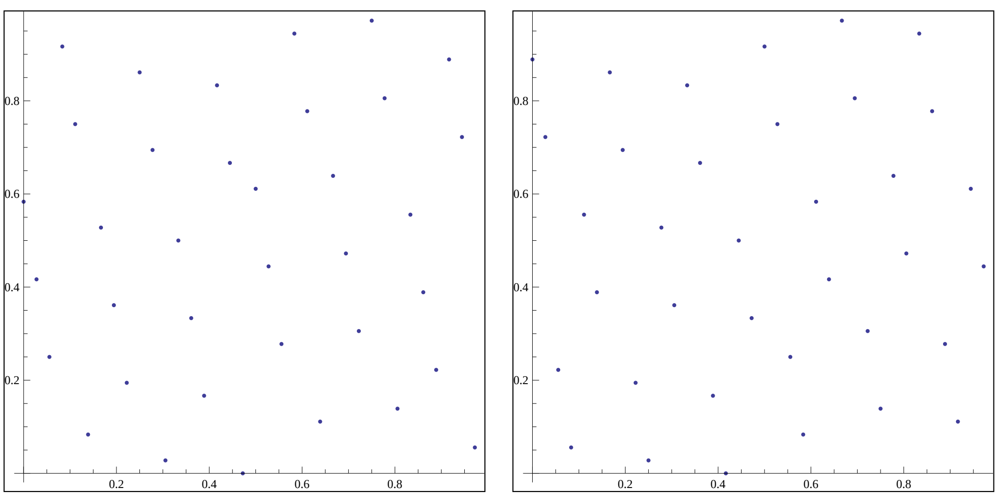
In the case when $G = \mathbb{Z}_b$ and $\varphi_2 = id$ is the identity of the set $\mathbb{Z}_b$ in itself, from the net ${}_{G_b,\varphi_2}Z^{p,q}_{b,\nu}$ we obtain the net ${}_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$, which was introduced by Grozdanov and Stoilova [11]. In the case when $p = 1$ and $q = 0$ from the net ${}_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$ we obtain the net ${}_{\mathbb{Z}_b,id}Z^{1,0}_{b,\nu}$, which was introduced by Warnock [23]. In the case when $b = 2$, $p = 1$ and $q = 1$ from the net ${}_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$ we obtain the net ${}_{\mathbb{Z}_2,id}Z^{1,1}_{2,\nu}$, which is the original net of Halton-Zaremba. In the case when $b = 2$, $p = 0$ and $q = 0$ from the net ${}_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$ we obtain the net ${}_{\mathbb{Z}_2,id}Z^{0,0}_{2,\nu}$, which is the original net of Roth [19].

We will construct and show the distributions of the points of four concrete nets ${}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}$ of type of Halton-Zaremba.

Example 1. The algebraic background of the first example is as follows: Let $m = 2$ and choose the bases $b_1 = 2$ and $b_2 = 3$. The discrete cyclic groups of orders b_1 and b_2 are $\mathbb{Z}_{b_1} = \{0, 1\}$ and $\mathbb{Z}_{b_2} = \{0, 1, 2\}$. We have that $b = 6$, the group G_b is $G_b = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ and $\mathbb{Z}_b = \{0, 1, 2, 3, 4, 5\}$. Let us select the bijections φ_1 and φ_2 as $\varphi_1(0) = (0, 0), \varphi_1(1) = (1, 0), \varphi_1(2) = (0, 2), \varphi_1(3) = (1, 2), \varphi_1(4) = (0, 1), \varphi_1(5) = (1, 1)$ and $\varphi_2(0) = (0, 0), \varphi_2(1) = (1, 2), \varphi_2(2) = (1, 0), \varphi_2(3) = (1, 1)$, $\varphi_2(4) = (0, 2), \varphi_2(5) = (0, 1)$. In addition, we choose the parameters $\nu = 2$, $\kappa = 0.42$ and $\mu = 0.15$. The points of the obtained net are:

$$
\begin{aligned}
&{}_{G_6,\varphi_6}Z_{6,2}^{\kappa,\mu} \\
&= \left\{ \left(\frac{26}{36},\frac{11}{36}\right), \left(\frac{27}{36},\frac{35}{36}\right), \left(\frac{28}{36},\frac{29}{36}\right), \left(\frac{29}{36},\frac{5}{36}\right), \left(\frac{24}{36},\frac{23}{36}\right), \left(\frac{25}{36},\frac{17}{36}\right), \right. \\
&\left(\frac{32}{36},\frac{8}{36}\right), \left(\frac{33}{36},\frac{32}{36}\right), \left(\frac{34}{36},\frac{26}{36}\right), \left(\frac{35}{36},\frac{2}{36}\right), \left(\frac{30}{36},\frac{20}{36}\right), \left(\frac{31}{36},\frac{14}{36}\right), \\
&\left(\frac{2}{36},\frac{9}{36}\right), \left(\frac{3}{36},\frac{33}{36}\right), \left(\frac{4}{36},\frac{27}{36}\right), \left(\frac{5}{36},\frac{3}{36}\right), \left(\frac{0}{36},\frac{21}{36}\right), \left(\frac{1}{36},\frac{15}{36}\right), \\
&\left(\frac{8}{36},\frac{7}{36}\right), \left(\frac{9}{36},\frac{31}{36}\right), \left(\frac{10}{36},\frac{25}{36}\right), \left(\frac{11}{36},\frac{1}{36}\right), \left(\frac{6}{36},\frac{19}{36}\right), \left(\frac{7}{36},\frac{13}{36}\right), \\
&\left(\frac{14}{36},\frac{6}{36}\right), \left(\frac{15}{36},\frac{30}{36}\right), \left(\frac{16}{36},\frac{24}{36}\right), \left(\frac{17}{36},\frac{0}{36}\right), \left(\frac{12}{36},\frac{18}{36}\right), \left(\frac{13}{36},\frac{12}{36}\right), \\
&\left. \left(\frac{20}{36},\frac{10}{36}\right), \left(\frac{21}{36},\frac{34}{36}\right), \left(\frac{22}{36},\frac{28}{36}\right), \left(\frac{23}{36},\frac{4}{36}\right), \left(\frac{18}{36},\frac{22}{36}\right), \left(\frac{19}{36},\frac{16}{36}\right) \right\}.
\end{aligned}
$$

The distribution of the points of the net ${}_{G_6,\varphi_6}Z_{6,2}^{\kappa,\mu}$ is shown in Figure 1a).



a) b)

Figure 1: Nets of Example 1 and 2 ($\nu = 2, b1 = 2, b2 = 3$, different bijections φ_1, φ_2)

Example 2. To construct the second net, we will use the same group G_b and parameters $\nu = 2$, $\kappa = 0.42$ and $\mu = 0.15$. Let us choose the bijections $\varphi_1(0) = (0,0), \varphi_1(1) = (1,1), \varphi_1(2) = (1,2), \varphi_1(3) = (0,2), \varphi_1(4) = (0,1), \varphi_1(5) = (1,0)$

and $\varphi_2(0) = (0,0)$, $\varphi_2(1) = (0,2), \varphi_2(2) = (1,1), \varphi_2(3) = (1,0), \varphi_2(4) = (1,2), \varphi_2(5) = (0,1)$. The distribution of the points of the obtained net is shown in Figure 1b).

Example 3. To construct the third net, we use the same group G_b and bijections φ_1 and φ_2 as in Example 1. We choose the parameters $\nu = 4$, $\kappa = 0.2112$ and $\mu = 0.1302$. The distribution of the points of the obtained net is shown in Figure 2.

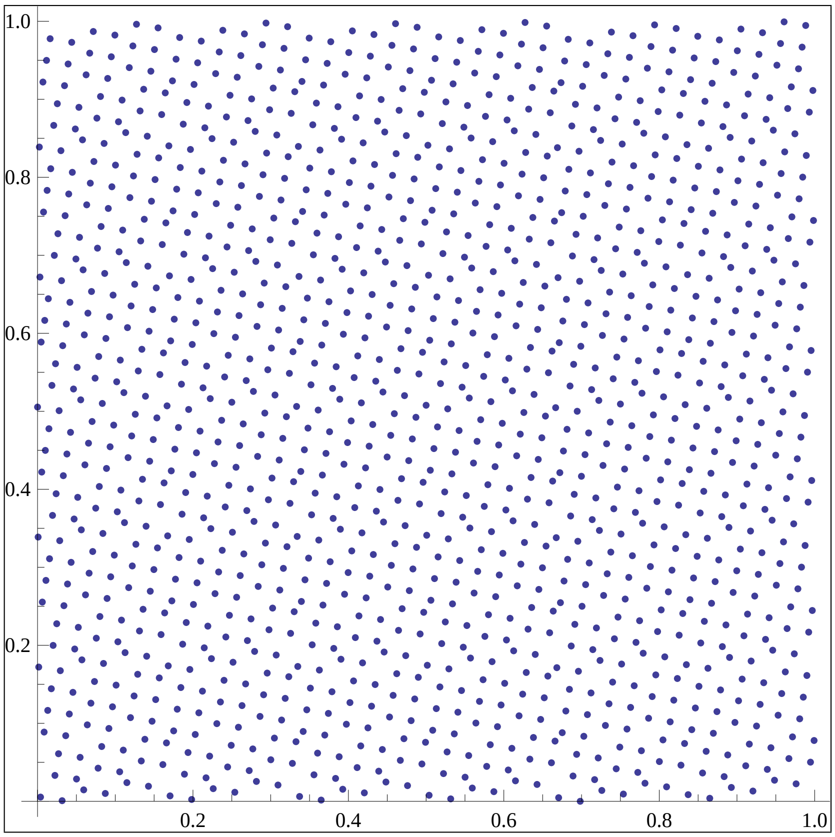


Figure 2: Net of Example 3: $\nu = 4, b1 = 2, b2 = 3$.

Example 4. The algebraic background of the fourth net is as follows: Let $m = 2$ and choose the bases $b_1 = 3$ and $b_2 = 4$. The discrete cyclic groups of orders b_1 and b_2 are $\mathbb{Z}_{b_1} = \{0,1,2\}$ and $\mathbb{Z}_{b_2} = \{0,1,2,3\}$. We have that $b = 12$, the group G_b is $G_b = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3)\}$ and $\mathbb{Z}_b = \{0,1,\ldots,11\}$. Let us select the bijections $\varphi_1(0) = (0,0)$, $\varphi_1(1) = (1,0)$, $\varphi_1(2) = (0,3)$, $\varphi_1(3) = (1,2)$, $\varphi_1(4) = (0,1)$, $\varphi_1(5) = (2,3)$, $\varphi_1(6) = (0,2)$, $\varphi_1(7) = (2,2)$, $\varphi_1(8) = (1,3)$, $\varphi_1(9) = (1,1)$, $\varphi_1(10) = (2,0)$, $\varphi_1(11) = (2,1)$ and $\varphi_2(0) = (0,0)$, $\varphi_2(1) = (1,3)$, $\varphi_2(2) = (1,0)$, $\varphi_2(3) = (2,1)$,

$\varphi_2(4) = (0,3)$, $\varphi_2(5) = (2,3)$, $\varphi_2(6) = (2,0)$, $\varphi_2(7) = (1,2)$, $\varphi_2(8) = (1,1)$, $\varphi_2(9) = (0,2)$, $\varphi_2(10) = (0,1)$, $\varphi_2(11) = (2,2)$. We choose the parameters $\nu = 2$, $\kappa = 0.42$ and $\mu = 0.15$. The distribution of the points of the obtained net is shown in Figure 3.

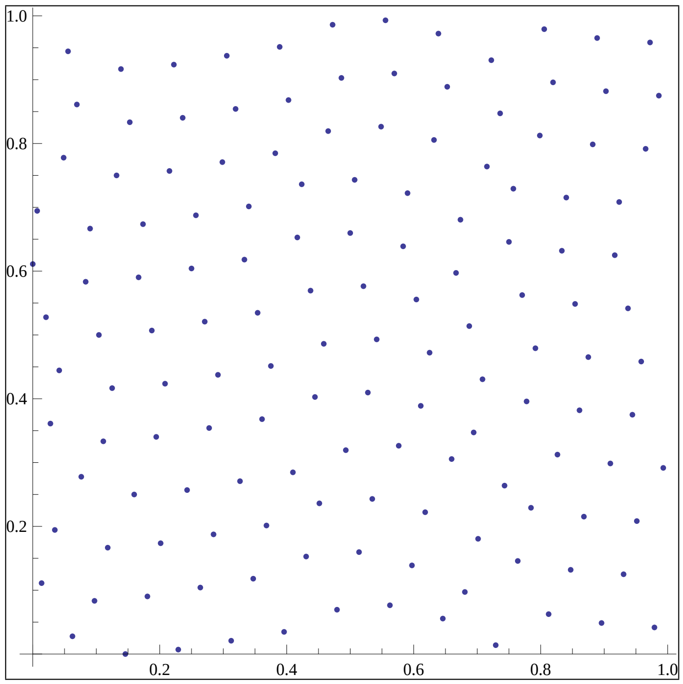


Figure 3: Net of Example 4: $m = 2, b1 = 3, b2 = 4$.

We will present the program code in the mathematical package *Mathematica*, which can compute the coordinates and visualize the points of an arbitrary net of type of Halton-Zaremba.

```
(*Program code for constructing nets *)
e = Input[e];m = Input[m]; (*vectors Eta and Mu*)
points = {};
b1 = Input[b1];b2 = Input[b2];
ni = Input[ni];b = b1*b2;
phi1 = Input[phi1];
phi2 = Input[phi2];
Do[i = IntegerDigits[i1, b]; k = ni - 1;
    While[k > 0,
        If[i1 < b^k, PrependTo[i, 0]]; k = k - 1];
    apc = ord = 0;
    Do[  cif1 = phi1[[i[[j]] + 1]];
         cif2 = phi1[[e[[j]] + 1]];
```

```
          cif = {Mod[cif1[[1]] + cif2[[1]], b1],
          Mod[cif1[[2]] + cif2[[2]], b2]};
          cifra = Position[phi1, cif][[1]][[1]] - 1;
          apc = apc + cifra/b^j;
          cif1 = phi2[[i[[ni - j + 1]] + 1]];
          cif2 = phi2[[m[[j]] + 1]];
          cif = {Mod[cif1[[1]] + cif2[[1]], b1],
          Mod[cif1[[2]] + cif2[[2]], b2]};
       cifra = Position[phi2, cif][[1]][[1]] - 1;
          ord = ord + cifra/b^j,
    {j, 1, ni}];
    AppendTo[points, {apc, ord}],
{i1, 0, b^ni - 1}];
ListPlot[points,AspectRatio->Automatic]
```

4. The $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony

In the previous section, we presented one wide class of two-dimensional nets constructed over finite groups with respect to arbitrary bijections. We need of appropriate analytical tool for studying the quality of the distribution of the points of these nets. In our case, it is important to realize a process of a synchronisation between the technique for construction of the nets and the tool for their investigation.

The different kinds of the diaphony are numerical measures for studying the irregularity of the distribution of sequences and nets. The construction of the diaphony is always connected with some complete orthonormal function system. Concrete for studying sequences and nets constructed over finite groups with respect to arbitrary bijections, the suitable version of the diaphony is the one, which is based on the system of Walsh functions constructed also over the same finite groups. For us, this is the motivation to use the so-called $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony as a tool for studying of the nets of the class ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$.

To define the concept of the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony we need to present some preliminary notations. Let the considered sets of bases and bijections be $\mathbf{b} = (b,\ldots,b)$ and $\varphi = (\varphi,\ldots,\varphi)$. Let $\mathcal{W}_{G_{\mathbf{b}},\varphi} = \{{}_{G_{\mathbf{b}},\varphi}wal_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ be the defined in previous section system of Walsh functions over the group $G_{\mathbf{b}}$ with respect to the bijection φ.

For arbitrary integers $b \geq 2$, $k \geq 0$ and a real $\alpha > 1$ we introduce the coefficient

$$\rho(\alpha;b;k) = \begin{cases} 1, & \text{if } k = 0, \\ b^{-\alpha\cdot g}, & \text{if } b^g \leq k < b^{g+1},\ g \geq 0,\ g \in \mathbb{Z}. \end{cases}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$, where for $1 \le j \le s$ $\alpha_j > 1$, be a given vector of real numbers. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we define the coefficient

$$R(\alpha; \mathbf{b}; \mathbf{k}) = \prod_{j=1}^{s} \rho(\alpha_j; b; k_j). \tag{7}$$

Let us signify $C(\alpha; \mathbf{b}) = \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} R(\alpha; \mathbf{b}; \mathbf{k})$. So, the equality holds

$$C(\alpha; \mathbf{b}) = -1 + \prod_{j=1}^{s} \left[1 + (b-1)\frac{b^{\alpha_j}}{b^{\alpha_j} - b}\right]. \tag{8}$$

The notion of $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony is a partial case of more general kind of the diaphony, called hybrid weighted diaphony, which was introduced by Baycheva and Grozdanov [2]. So, following this concept we will present the next definition:

Definition 3. *Let $\xi = (\mathbf{x}_n)_{n\ge 0}$ be an arbitrary sequence of points in $[0,1)^s$. For each integer $N \ge 1$ the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the first N elements of the sequence ξ is defined as*

$$F_N(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;\xi) = \left(\frac{1}{C(\alpha;\mathbf{b})} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} R(\alpha;\mathbf{b};\mathbf{k}) \left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{G_{\mathbf{b}},\varphi}wal_{\mathbf{k}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}},$$

where the coefficients $R(\alpha;\mathbf{b};\mathbf{k})$ and the constant $C(\alpha;\mathbf{b})$ are defined respectively by the equalities (7) *and* (8)*.*

Following Baycheva and Grozdanov [2], see also [3], it is a well known fact that the sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the next limit equality $\lim_{N\to\infty} F_N(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;\xi) = 0$ holds for each vector α, as above.

To the authors is unknown a lower bound of the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of an arbitrary net as the one presented in the equality (2) and which is related with the $b-$adic diaphony.

5. On the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the nets of type of Halton-Zaremba

In the next theorem we will give an explicit formula for the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of an arbitrary net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ of type of Halton-Zaremba.

Theorem 1. *Let ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ be an arbitrary net of type of Halton-Zaremba. For each integer $\nu \ge 1$ the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ satisfies the* *equality*

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})$$
$$= \frac{1}{C(\alpha;b)}\left\{\frac{(b-1)b^{\alpha_2}(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\sum_{g=0}^{\nu-1}b^{(\alpha_2-\alpha_1)g}\right.$$
$$\left.+(b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\left[1+(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\right]\frac{1}{b^{\alpha_1\nu}}+(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\right\},$$

where $C(\alpha;b)=\frac{(b-1)b^{\alpha_1}}{b^{\alpha_1}-b}+\frac{(b-1)b^{\alpha_2}}{b^{\alpha_2}-b}+(b-1)^2\frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}$.

Corollary 1. *Let the conditions of Theorem 1 be realized. Let us assume that* $\alpha_1=\alpha_2=\alpha>1$. *Then, the following statements follow:*

(i) *For each integer* $\nu>0$ *the* $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$*diaphony of the net* ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ *satisfies the equality*

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})=\frac{b^\alpha-1}{(b-1)\frac{b^\alpha}{b^\alpha-b}+2}\cdot\frac{\nu}{b^{\alpha\nu}}+\frac{1}{b^{\alpha\nu}};$$

(ii) *Let us signify* $N=b^\nu$. *Then, the limit equality holds*

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}}\frac{N^{\frac{\alpha}{2}}\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\sqrt{\log N}}=\sqrt{\frac{b^\alpha-1}{\left[(b-1)\frac{b^\alpha}{b^\alpha-b}+2\right]\log b}}.$$

(iii) *Let* $1<\alpha<2$. *Then, there exists a number* ε *such that* $0<\varepsilon<\frac{1}{2}$, *for which the inclusion* $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1-\varepsilon}}\right)$ *holds;*

(iv) *Let* $\alpha=2$. *Then, the inclusion* $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$ *holds;*

(v) *Let* $\alpha=2$. *Then, the limit equality holds*

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}}\frac{N\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\sqrt{\log N}}=\sqrt{\frac{b^2-1}{(b+2)\log b}}.$$

(vi) *Let* $\alpha>2$. *Then, there exists a positive number* ε *such that the inclusion holds*

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1+\varepsilon}}\right).$$

Corollary 2. *Let the conditions of Theorem 1 be realized. Let us assume that* $\alpha_1>\alpha_2>1$. *Then, the following statements follow:*

(i) For each integer $\nu > 0$ the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the net ${}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}$ satisfies the equality

$$\begin{aligned} & b^{\alpha_2\nu}\cdot F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}) \\ & = \frac{1}{C(\alpha;b)}\left\{(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\left[\frac{b^{\alpha_1}(b^{\alpha_2}-1)}{b^{\alpha_1}-b^{\alpha_2}}+1\right]\right. \\ & \left. + \left[\frac{(b-1)b^{\alpha_1+\alpha_2}(b\cdot b^{\alpha_1}+b^{\alpha_2}-b^{\alpha_1+\alpha_2}-b)}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)(b^{\alpha_1}-b^{\alpha_2})}+(b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\right]\frac{1}{b^{(\alpha_1-\alpha_2)\nu}}\right\}, \end{aligned}$$

where the constant $C(\alpha;b)$ was defined in the condition of Theorem 1;

(ii) Let us signify $N=b^\nu$. Then, the limit equality holds

$$\begin{aligned} & \lim_{\substack{\nu\to\infty \\ N=b^\nu}} N^{\frac{\alpha_2}{2}}\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}) \\ & = \sqrt{\frac{b^{\alpha_2}(b^{\alpha_1}-b)[b^{\alpha_1}-b^{\alpha_2}+b^{\alpha_1}(b^{\alpha_2}-1)]}{(b^{\alpha_1}-b^{\alpha_2})[b^{\alpha_1}(b^{\alpha_2}-b)+b^{\alpha_2}(b^{\alpha_1}-b)+(b-1)b^{\alpha_1+\alpha_2}]}}. \end{aligned}$$

(iii) Let $1<\alpha_2<2$. Then, there exists a number ε such that $0<\varepsilon<\frac{1}{2}$, for which the inclusion $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu})\in\mathcal{O}\left(\frac{1}{N^{1-\varepsilon}}\right)$ holds;

(iv) Let $\alpha_2=2$. Then, the inclusion $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu})\in\mathcal{O}\left(\frac{1}{N}\right)$ holds;

(v) Let $\alpha_2>2$. Then, there exists a number $\varepsilon>0$ such that the inclusion holds

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu})\in\mathcal{O}\left(\frac{1}{N^{1+\varepsilon}}\right).$$

The results of Theorem 1 and Corollaries 1 and 2 were announced by authors in [8]. Here we will develop the complete proofs of these statements.

6. Preliminary results

In this section, we will present some preliminary statements, which will be essentially used to prove the main results of the paper. The following lemmas hold:

Lemma 1. *Let $b\geq 2$ be a fixed integer, G_b be a finite group of order b and $\varphi:\mathbb{Z}_b\to G_b$ be an arbitrary bijection. For arbitrary integers $\nu>0$ and $k\geq 1$ we define the function*

$$\delta_{b^\nu}(k)=\begin{cases}1, & \text{if } k\equiv 0\ (mod\ b^\nu),\\ 0, & \text{if } k\not\equiv 0\ (mod\ b^\nu).\end{cases}$$

Then, the equalities hold

$$\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i))=\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(p_{b,\nu}(i))=b^\nu\cdot\delta_{b^\nu}(k).$$

Proof. For the integer k and an arbitrary integer i, $0 \le i \le b^\nu - 1$, we will use the $b-$adic representations $k = \sum_{j=0}^{\infty} k_j b^j$ and $i = \sum_{j=0}^{\nu-1} i_j b^j$. Then, we have that $\eta_{b,\nu}(i) = 0.i_{\nu-1}i_{\nu-2}\dots i_0$ and ${}_{G_{b,\varphi}}wal_k(\eta_{b,\nu}(i)) = \prod_{j=0}^{\nu-1} \chi_{\varphi(k_j)}(\varphi(i_{\nu-1-j}))$. Hence, we obtain that

$$\sum_{i=0}^{b^\nu-1} {}_{G_{b,\varphi}}wal_k(\eta_{b,\nu}(i)) = \sum_{i_{\nu-1}=0}^{b-1} \chi_{\varphi(k_0)}(\varphi(i_{\nu-1}))\dots\sum_{i_0=0}^{b-1}\chi_{\varphi(k_{\nu-1})}(\varphi(i_0)). \tag{9}$$

Let us assume that $k \equiv 0(mod\ b^\nu)$. Then, we have that $k_0 = k_1 = \dots = k_{\nu-1} = 0$ and from the equality (9) we obtain that $\sum_{i=0}^{b^\nu-1} {}_{G,\varphi}wal_k(\eta_{b,\nu}(i)) = b^\nu$.

Let us assume that $k \not\equiv 0(mod\ b^\nu)$. Then, there exists at least one index δ, $0 \le \delta \le \nu - 1$ such that $k_\delta \ne 0$. In this case, the corresponding sum $\sum_{i_{\nu-1-\delta}=0}^{b-1} \chi_{\varphi(k_\delta)}(\varphi(i_{\nu-1-\delta})) = 0$ and from the equality (9) we obtain that $\sum_{i=0}^{b^\nu-1} {}_{G,\varphi}wal_k(\eta_{b,\nu}(i)) = 0$.

The second equality of the statement of the Lemma can be proved by similar manner. $\square$

Lemma 2. *Let the conditions* $(C1)$ *and* $(C2)$ *be fulfilled. Then, the following holds:*

(i) *For arbitrary integers* $0 \le g \le g_1 \le \nu - 1$ *we define the set*

$$A(g_1;g) = \left\{ k_1 : k_1 = \sum_{j=g}^{g_1} k_j^{(1)} b^j, g \le j \le g_1, k_j^{(1)} \in \{0,1,\dots,b-1\} \text{ and } k_g^{(1)}, k_{g_1}^{(1)} \ne 0 \right\}.$$

For each integer $k_1 \in A(g_1;g)$ *we define the integer* $k_1^* = \sum_{j=g}^{g_1} \overline{k}_j^{(1)} b^{\nu-1-j}$. *Then, for all integers* $0 \le g_2 \le \nu - 1$ *and* $b^{g_2} \le k_2 \le b^{g_2+1} - 1$ *the equalities hold*

$$\sum_{i=0}^{b^\nu-1} {}_{G_{b,\varphi}}wal_{k_1}(\eta_{b,\nu}(i))\,{}_{G_{b,\varphi}}wal_{k_2}(p_{b,\nu}(i)) = \begin{cases} b^\nu, & \text{if } k_2 = k_1^*, \\ 0, & \text{if } k_2 \ne k_1^*. \end{cases}$$

In the case when $k_2 = k_1^*$, *we have that* $g_2 = \nu - 1 - g$;

(ii) *Let the integers* g_1 *and* g_2 *such that* $0 \le g_1 \le \nu - 1 < g_2$ *be arbitrary. An arbitrary integer* k_1 *such that* $b^{g_1} \le k_1 \le b^{g_1+1} - 1$ *we present in the form* $k_1 = \sum_{j=0}^{\nu-1} k_j^{(1)} b^j$. *An arbitrary integer* k_2 *such that* $b^{g_2} \le k_2 \le b^{g_2+1} - 1$ *we present in the form* $k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j$. *For each integer* k_1, *as above, we define the set*

$$A(k_1) = \left\{ k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j : k_0^{(2)} = \overline{k}_{\nu-1}^{(1)}, k_1^{(2)} = \overline{k}_{\nu-2}^{(1)}, \dots, k_{\nu-1}^{(2)} = \overline{k}_0^{(1)} \right.$$
$$\left. \text{and the digits } k_\nu^{(2)}, k_{\nu+1}^{(2)}, \dots, k_{g_2}^{(2)} \text{ are arbitrary} \right\}.$$

Then, the equalities hold

$$\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right| = \begin{cases} b^\nu, & \text{if } k_2 \in A(k_1),\\ 0, & \text{if } k_2 \notin A(k_1);\end{cases}$$

(*iii*) *Let the integers g_2 and g_1 such that $0 \le g_2 \le \nu-1 < g_1$ be arbitrary. An arbitrary integer k_2 such that $b^{g_2} \le k_2 \le b^{g_2+1}-1$ we present in the form $k_2 = \sum_{j=0}^{\nu-1} k_j^{(2)} b^j$. An arbitrary integer k_1 such that $b^{g_1} \le k_1 \le b^{g_1+1}-1$ we present in the form $k_1 = \sum_{j=0}^{g_1} k_j^{(1)} b^j$. For each integer k_2, as above, we define the set*

$$B(k_2) = \Bigg\{ k_1 = \sum_{j=0}^{g_1} k_j^{(1)} b^j : k_0^{(1)} = \overline{k}_{\nu-1}^{(2)}, k_1^{(1)} = \overline{k}_{\nu-2}^{(2)}, \dots, k_{\nu-1}^{(1)} = \overline{k}_0^{(2)}$$
$$\text{and the digits } k_\nu^{(1)}, k_{\nu+1}^{(1)}, \dots, k_{g_1}^{(1)} \text{ are arbitrary}\Bigg\}.$$

Then, the equalities hold

$$\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right| = \begin{cases} b^\nu, & \text{if } k_1 \in B(k_2),\\ 0, & \text{if } k_1 \notin B(k_2);\end{cases}$$

(*iv*) *Let the integers g_1 and g_2 such that $g_1 \ge \nu$ and $g_2 \ge \nu$ be arbitrary. Arbitrary integers k_1 and k_2 such that $b^{g_1} \le k_1 \le b^{g_1+1}-1$ and $b^{g_2} \le k_2 \le b^{g_2+1}-1$ we present in the form $k_1 = \sum_{j=0}^{g_1} k_j^{(1)} b^j$ and $k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j$. For each integer k_1, as above, we define the set*

$$C(k_1) = \Bigg\{ k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j : k_0^{(2)} = \overline{k}_{\nu-1}^{(1)}, k_1^{(2)} = \overline{k}_{\nu-2}^{(1)}, \dots, k_{\nu-1}^{(2)} = \overline{k}_0^{(1)}$$
$$\text{and the digits } k_\nu^{(2)}, k_{\nu+1}^{(2)}, \dots, k_{g_2}^{(2)} \text{ are arbitrary}\Bigg\}.$$

Then, the equalities hold

$$\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right| = \begin{cases} b^\nu, & \text{if } k_2 \in C(k_1),\\ 0, & \text{if } k_2 \notin C(k_1).\end{cases}$$

Proof. For an arbitrary integer i, $0 \le i \le b^\nu - 1$, with the $b-$adic representation $i = \sum_{j=0}^{\nu-1} i_j b^j$ we have that $\eta_{b,\nu}(i) = 0.i_{\nu-1}i_{\nu-2}\dots i_0$ and $p_{b,\nu}(i) = 0.i_0 i_1 \dots i_{\nu-1}$.

(i) For each integer $k_1 \in A(g_1; g)$ we have that

$$(10)\quad {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i)) = \prod_{j=g}^{g_1} \chi_{\varphi(k_j^{(1)})}(\varphi(i_{\nu-1-j})) = \prod_{j=\nu-1-g_1}^{\nu-1-g} \chi_{\varphi(k_{\nu-1-j}^{(1)})}(\varphi(i_j)).$$

Let an arbitrary integer k_2 such that $b^{g_2} \leq k_2 \leq b^{g_2+1}-1$ have the $b-$adic representation $k_2 = \sum_{j=0}^{\nu-1} k_j^{(2)} b^j$ with the assumption that for $g_2+1 \leq j \leq \nu-1$ the equalities $k_j^{(2)} = 0$ hold. Hence, we have that

$$
\begin{aligned}
&{}_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) = \prod_{j=0}^{\nu-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)) \\
&= \prod_{j=0}^{\nu-2-g_1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)). \prod_{j=\nu-1-g_1}^{\nu-1-g} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)). \prod_{j=\nu-g}^{\nu-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)).
\end{aligned} \tag{11}
$$

Then, from the equalities (4), (10) and (11) we obtain that

$$
\begin{aligned}
&\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) \\
&= \prod_{j=0}^{\nu-2-g_1} \sum_{i_j=0}^{b-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)) \\
&\times \prod_{j=\nu-1-g_1}^{\nu-1-g} \sum_{i_j=0}^{b-1} \chi_{\varphi(k_{\nu-1-j}^{(1)}) \oplus_{G_b} \varphi(k_j^{(2)})}(\varphi(i_j)) \prod_{j=\nu-g}^{\nu-1} \sum_{i_j=0}^{b-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)).
\end{aligned} \tag{12}
$$

Let us assume that $k_2 = k_1^*$. This means the following: For $0 \leq j \leq \nu-2-g_1$ we have that $k_j^{(2)} = 0$. For $\nu-1-g_1 \leq j \leq \nu-1-g$ we have that $k_j^{(2)} = \overline{k}_{\nu-1-j}^{(1)}$ and hence, for each i_j, $0 \leq i_j \leq b-1$, the equality $\chi_{\varphi(k_{\nu-1-j}^{(1)}) \oplus_{G_b} \varphi(k_j^{(2)})}(\varphi(i_j)) = 1$ holds. For $\nu-g \leq j \leq \nu-1$ we have that $k_j^{(2)} = 0$. Then, from the equality (12) we obtain that

$$
\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) = b^\nu.
$$

The condition $k_2 \neq k_1^*$ means that there exists at least one index δ, $0 \leq \delta \leq \nu-2-g_1$, such that $k_\delta^{(2)} \neq 0$, or there exists at least one index κ, $\nu-1-g_1 \leq \kappa \leq \nu-1-g$, such that $k_\kappa^{(2)} \neq \overline{k}_{\nu-1-\kappa}^{(1)}$, or there exists at least one index τ, $\nu-g \leq \tau \leq \nu-1$, such that $k_\tau^{(2)} \neq 0$. In the first case, the corresponding sum $\sum_{i_\delta=0}^{b-1} \chi_{\varphi(k_\delta^{(2)})}(\varphi(i_\delta)) = 0$, in the second case $\sum_{i_\kappa=0}^{b-1} \chi_{\varphi(k_{\nu-1-\kappa}^{(1)}) \oplus_{G_b} \varphi(k_\kappa^{(2)})}(\varphi(i_\kappa)) = 0$ and in the third case $\sum_{i_\tau=0}^{b-1} \chi_{\varphi(k_\tau^{(2)})}(\varphi(i_\tau)) = 0$. According to the equality (12), we obtain that $\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) = 0$.

The another statements of Lemma 2 can be proved by using similar techniques. □

7. Proofs of the main results

Proof of Theorem 1. According to Definition 3, and by using the equality (5) for the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the net ${}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}$, we have that

$$\begin{aligned}
&F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}) = \frac{1}{C(\alpha;b)}\sum_{(k_1,k_2)\in\mathbb{N}_0^2\setminus\{\mathbf{0}\}} R(\alpha;\mathbf{b};(k_1,k_2))\\
&\times |{}_{G_b,\varphi}wal_{k_1}(\kappa)|^2|{}_{G_b,\varphi}wal_{k_2}(\mu)|^2\\
&\times \left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))\,{}_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right|^2\\
&= \frac{1}{C(\alpha;b)}\left\{\sum_{k=1}^{\infty}\rho(\alpha_1;b;k)\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i))\right|^2\right.\\
&+\sum_{k=1}^{\infty}\rho(\alpha_2;b;k)\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(p_{b,\nu}(i))\right|^2\\
&+\left[\sum_{g_1=0}^{\nu-1}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=0}^{\nu-1}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}+\sum_{g_1=0}^{\nu-1}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=\nu}^{\infty}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}\right.\\
&+\left.\sum_{g_1=\nu}^{\infty}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=0}^{\nu-1}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}+\sum_{g_1=\nu}^{\infty}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=\nu}^{\infty}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}\right]\\
&\left.\times\ R(\alpha;\mathbf{b};(k_1,k_2))\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))\,{}_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right|^2\right\}
\end{aligned}$$

$$= \frac{1}{C(\alpha;b)}(\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6). \tag{13}$$

We will calculate the sums in the equality (13). For the sum Σ_1, we have the following: In Lemma 1 for each integer $k\geq 1$ was shown the exact value of the trigonometric sum $\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i))$. By using this result, we obtain that

$$\begin{aligned}
\Sigma_1 &= \sum_{k=1}^{\infty}\rho(\alpha_1;b;k)\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i))\right|^2 = \sum_{k=1}^{\infty}\rho(\alpha_1;b;k).\delta_{b^\nu}(k)\\
&= \sum_{\substack{k=1\\k\equiv 0(mod\ b^\nu)}}^{\infty}\rho(\alpha_1;b;k) = \sum_{k_1=1}^{\infty}\rho(\alpha_1;b;k_1b^\nu) = \sum_{g_1=0}^{\infty}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\rho(\alpha_1;b;k_1b^\nu)\\
&= \sum_{g_1=0}^{\infty}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}b^{-\alpha_1(g_1+\nu)} = b^{-\alpha_1\nu}\sum_{g_1=0}^{\infty}b^{-\alpha_1 g_1}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}1
\end{aligned}$$

$$= (b-1)b^{-\alpha_1\nu}\sum_{g_1=0}^{\infty} b^{(1-\alpha_1)g_1} = (b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\cdot\frac{1}{b^{\alpha_1\nu}}. \tag{14}$$

By using the same technique, we can prove that

$$\Sigma_2 = \sum_{k=1}^{\infty}\rho(\alpha_2;b;k)\left|\frac{1}{b^{\nu}}\sum_{i=0}^{b^{\nu}-1} {}_{G_{b,\varphi}}wal_k(p_{b,\nu}(i))\right|^2 = (b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}. \tag{15}$$

To calculate the sum Σ_3, we will use the introduced in Lemma 2 (i) sets $A(g_1;g)$ and obtain that

$$\Sigma_3 = \sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1}\sum_{k_1\in A(g_1;g)}\sum_{g_2=0}^{\nu-1} b^{-\alpha_2 g_2}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}$$
$$\times\left|\frac{1}{b^{\nu}}\sum_{i=0}^{b^{\nu}-1} {}_{G_{b,\varphi}}wal_{k_1}(\eta_{b,\nu}(i))\,{}_{G_{b,\varphi}}wal_{k_2}(p_{b,\nu}(i))\right|^2.$$

By using Lemma 2 (i), we have that only in the case when $g_2 = \nu-1-g$ and $k_2 = k_1^*$ the trigonometric sum $\sum_{i=0}^{b^{\nu}-1} {}_{G_{b,\varphi}}wal_{k_1}(\eta_{b,\nu}(i))\,{}_{G_{b,\varphi}}wal_{k_2}(p_{b,\nu}(i))$ has a value b^{ν} and in the another cases - a value 0. In this way, we obtain that

$$\Sigma_3 = \sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1}\sum_{k_1\in A(g_1;g)} b^{-\alpha_2(\nu-1-g)} = \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1} b^{\alpha_2 g}\sum_{k_1\in A(g_1;g)} 1.$$

For arbitrary integers $0\le g\le g_1\le\nu-1$ the set $A(g_1;g)$ has a cardinality

$$|A(g_1;g)| = \begin{cases}(b-1)^2 b^{g_1-g-1}, & \text{if } \le g\le g_1-1,\\ b-1, & \text{if } g=g_1.\end{cases}$$

According to the above two statements, for the sum Σ_3, we will use the following presentation

$$\Sigma_3 = \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left[\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1}\sum_{k\in A(g_1;g_1)} 1 + \sum_{g_1=1}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1-1} b^{\alpha_2 g}\sum_{k_1\in A(g_1;g)} 1\right]$$
$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left[(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\sum_{g=0}^{g_1-1} b^{(\alpha_2-1)g}\right]$$
$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\left[b^{(\alpha_2-1)g_1}-1\right]\right\}$$
$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\left[\sum_{g_1=1}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} - \sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\right]\right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}} \left\{ (b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\left[\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} - 1\right] \right.$$

$$\left. -\frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1} \right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}} \left\{ (b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} \right.$$

$$\left. -\frac{(b-1)^2}{b^{\alpha_2}-b} - \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1} \right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}} \left\{ \frac{(b-1)(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} - \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(1-\alpha_1)g_1} \right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}} \left\{ \frac{(b-1)(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2 b^{\alpha_1}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}\left[b^{(1-\alpha_1)\nu}-1\right] \right\}$$

$$= \frac{(b-1)b^{\alpha_2}(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g}$$

$$+\frac{(b-1)^2 b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}\cdot\frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}} - \frac{(b-1)^2 b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}\cdot\frac{1}{b^{\alpha_2\nu}}. \tag{16}$$

We will calculate the sum Σ_4. For this purpose, let the integers $0 \le g_1 \le \nu-1$, $b^{g_1} \le k_1 \le b^{g_1+1}-1$, and $g_2 \ge \nu$ be fixed. We will use the introduced in Lemma 2 (ii) sets $A(k_1)$. Hence for each integer $b^{g_2} \le k_2 \le b^{g_2+1}-1$ the modulus of the trigonometric sum $\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right|$ will accept a value b^ν exactly $(b-1)b^{g_2-\nu}$ times. This is based on the fact that the digits $k^{(2)}_\nu, k^{(2)}_{\nu+1}, \dots, k^{(2)}_{g_2}$ can be arbitrary. In this way, we obtain that

$$\Sigma_4 = \sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} b^{-\alpha_2 g_2}.(b-1)b^{g_2-\nu}$$

$$= \frac{(b-1)^2}{b^\nu}\sum_{g_1=0}^{\nu-1} b^{(1-\alpha_1)g_1}\sum_{g_2=\nu}^{\infty} b^{(1-\alpha_2)g_2}$$

$$= (b-1)^2\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^\nu}\cdot\frac{1}{b^{(\alpha_2-1)\nu}}\sum_{g_1=0}^{\nu-1} b^{(1-\alpha_1)g_1}$$

$$= (b-1)^2\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\left[\frac{b^{\alpha_1}}{b^{\alpha_1}-b} - \frac{b^{\alpha_1}}{b^{\alpha_1}-b}\cdot\frac{1}{b^{(\alpha_1-1)\nu}}\right]$$

$$= (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{\alpha_2\nu}}$$

$$- (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}}. \tag{17}$$

To calculate the sum Σ_5, we can use the same techniques as above and obtain that

$$\Sigma_5 = (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{\alpha_1\nu}}$$

$$- (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}}. \tag{18}$$

It is evident the symmetry between the results obtained in the equalities (17) and (18).

We will calculate the sum Σ_6. For this purpose, let the integers $g_1 \geq \nu$, $b^{g_1} \leq k_1 \leq b^{g_1+1}-1$, and $g_2 \geq \nu$ be fixed. We will use the introduced in Lemma 2 (iv) sets $C(k_1)$. Hence, for each integer $b^{g_2} \leq k_2 \leq b^{g_2+1}-1$ the modulus of the trigonometric sum $\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right|$ will accept a value b^ν exactly $(b-1)b^{g_2-\nu}$ times. This is based on the fact that the digits $k_\nu^{(2)}, k_{\nu+1}^{(2)}, \ldots, k_{g_2}^{(2)}$ can be arbitrary. In this way, we obtain that

$$\Sigma_6 = \sum_{g_1=\nu}^{\infty} b^{-\alpha_1 g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} b^{-\alpha_2 g_2}.(b-1)b^{g_2-\nu}$$

$$= (b-1)^2 \frac{1}{b^\nu} \sum_{g_1=\nu}^{\infty} b^{(1-\alpha_1)g_1} \sum_{g_2=\nu}^{\infty} b^{(1-\alpha_2)g_2}$$

$$= (b-1)^2 \frac{1}{b^\nu} \cdot \frac{b^{\alpha_1}}{b^{\alpha_1}-b} \cdot \frac{1}{b^{(\alpha_1-1)\nu}} \cdot \frac{b^{\alpha_2}}{b^{\alpha_2}-b} \cdot \frac{1}{b^{(\alpha_2-1)\nu}}$$

$$= (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}}. \tag{19}$$

From the equalities (13), (14), (15), (16), (17), (18) and (19) we obtain that the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$ diaphony of the net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ satisfies the equality

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) = \frac{1}{C(\alpha;b)} \left\{ \frac{(b-1)b^{\alpha_2}(b^{\alpha_2}-1)}{b^{\alpha_2}-b} \cdot \frac{1}{b^{\alpha_2\nu}} \sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g} \right.$$
$$\left. +(b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\left[1+(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\right]\frac{1}{b^{\alpha_1\nu}} + (b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b} \cdot \frac{1}{b^{\alpha_2\nu}} \right\}$$

with the introduced in the condition of the theorem constant $C(\alpha;b)$. Theorem 1 is finally proved.

Proof of Corollary 1. (i) According to Theorem 1, in the case when $\alpha_1 = \alpha_2 = \alpha$ we obtain that

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}) = \frac{b^\alpha - 1}{(b-1)\frac{b^\alpha}{b^\alpha - b} + 2} \cdot \frac{\nu}{b^{\alpha\nu}} + \frac{1}{b^{\alpha\nu}}.$$

(ii) From the above expression we obtain that

$$\frac{b^{\alpha\nu} \cdot F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu})}{\nu} = \frac{b^\alpha - 1}{(b-1)\frac{b^\alpha}{b^\alpha - b} + 2} + \frac{1}{\nu}$$

and hence, the limit equality holds

$$\lim_{\nu\to\infty} \frac{b^{\frac{\alpha}{2}\nu} \cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu})}{\sqrt{\nu}} = \sqrt{\frac{b^\alpha - 1}{(b-1)\frac{b^\alpha}{b^\alpha - b} + 2}}.$$

We put $N = b^\nu$ and find that $\nu = \frac{\log N}{\log b}$. From the above limit equality we obtain that

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}} \frac{N^{\frac{\alpha}{2}} \cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu})}{\sqrt{\log N}} = \sqrt{\frac{b^\alpha - 1}{\left[(b-1)\frac{b^\alpha}{b^\alpha - b} + 2\right]\log b}}. \tag{20}$$

(iii) Let us assume that $1 < \alpha < 2$. Then, there exists a number $0 < \varepsilon < \frac{1}{2}$ such that $\frac{\alpha}{2} = 1 - \varepsilon$. The equality (20) gives us that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}) \in \mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1-\varepsilon}}\right).$$

(iv) When $\alpha = 2$ the equality (20) shows that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}) \in \mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right).$$

(v) Let us in the equality (20) put $\alpha = 2$ and obtain the limit equality

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}} \frac{N \cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu})}{\sqrt{\log N}} = \sqrt{\frac{b^2 - 1}{(b+2)\log b}}.$$

(vi) Let us assume that $\alpha > 2$. Then, there exists a number $\varepsilon > 0$ that $\frac{\alpha}{2} = 1 + \varepsilon$. The equality (20) shows that the inclusion

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}) \in \mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1+\varepsilon}}\right)$$

holds.

Corollary 1 is finally proved.

Proof of Corollary 2. (i) The condition $\alpha_1 > \alpha_2$ allows us to calculate the value of the sum $\sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g}$. So, the equality holds

$$\sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g} = \frac{b^{\alpha_1}}{b^{\alpha_1}-b^{\alpha_2}} - \frac{b^{\alpha_1}}{b^{\alpha_1}-b^{\alpha_2}} \cdot \frac{1}{b^{(\alpha_1-\alpha_2)\nu}}.$$

According to Theorem 1, in the case when $\alpha_1 > \alpha_2$ the presentation holds

$$\begin{aligned} & b^{\alpha_2\nu} \cdot F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \\ & = \frac{1}{C(\alpha;b)} \left\{ (b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b} \left[\frac{b^{\alpha_1}(b^{\alpha_2}-1)}{b^{\alpha_1}-b^{\alpha_2}} + 1 \right] \right. \\ & \left. + \left[\frac{(b-1)b^{\alpha_1+\alpha_2}(b \cdot b^{\alpha_1} + b^{\alpha_2} - b^{\alpha_1+\alpha_2} - b)}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)(b^{\alpha_1}-b^{\alpha_2})} + (b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b} \right] \frac{1}{b^{(\alpha_1-\alpha_2)\nu}} \right\}. \end{aligned}$$

(ii) From the above equality we obtain the limit equality

$$\begin{aligned} & \lim_{\substack{\nu\to\infty \\ N=b^\nu}} N^{\frac{\alpha_2}{2}} \cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \\ & = \sqrt{\frac{b^{\alpha_2}(b^{\alpha_1}-b)[(b^{\alpha_1}-b^{\alpha_2}) + b^{\alpha_1}(b^{\alpha_2}-1)]}{(b^{\alpha_1}-b^{\alpha_2})[b^{\alpha_1}(b^{\alpha_2}-b) + b^{\alpha_2}(b^{\alpha_1}-b) + (b-1)b^{\alpha_1+\alpha_2}]}}. \end{aligned} \tag{21}$$

(iii) Let us assume that $1 < \alpha_2 < 2$. Then, there exists a number $0 < \varepsilon < \frac{1}{2}$ such that $\frac{\alpha_2}{2} = 1 - \varepsilon$. The equality (21) gives us that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{1}{N^{1-\varepsilon}}\right).$$

(iv) Let $\alpha = 2$. From the equality (21) we find that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{1}{N}\right).$$

(v) Let us assume that $\alpha_2 > 2$. Then, there exists a number $\varepsilon > 0$ such that $\frac{\alpha_2}{2} = 1 + \varepsilon$.

The equality (21) shows us that the inclusion

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{1}{N^{1+\varepsilon}}\right)$$

holds.

Corollary 2 is finally proved.

Acknowledgements

The authors desire to thank of the reviewer for his useful notes which contribute to improve the quality of the article.

The authors want to thank to Faculty of Computer Science and Engineering at the "S's. Cyril and Methodius University" in Skopje for financial support of our investigation.

References

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., 68 (1950), 337-404.

[2] S. G. Baycheva, V. S. Grozdanov, *The hybrid weighted diaphony*, Comp. Rendus Akad. Bulgare Sci., 68 (2015), 437-448.

[3] S. G. Baycheva, V. S. Grozdanov, *The hybrid weighted diaphony*, Lambert Academic Publishing, 2017.

[4] H. E. Chrestenson, *A class of generalized Walsh functions*, Pacific J. Math., 5 (1955), 17-31.

[5] L. L. Cristea, F. Pillichshammer, *A lower bound on the $b-$adic diaphony*, Ren. Math. Appl., 27 (2007), 147-153.

[6] J. Dick, F. Pillichshammer, *Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces*, J. Comlexity, 21 (2005), 149-195.

[7] J. Dick, F. Pillichshammer, *Diaphony, discrepancy, spectral test and worst-case error*, Math. Comput. Simulation, 70 (2005), 159-171.

[8] V. Dimitrievska Ristovska, V. Grozdanov, *On the $(\mathcal{W}_{G_b,\varphi};\alpha;\beta;\gamma)-$diaphony of the net of Zaremba-Halton over finite groups*, Comp. Rendus Akad. Bulgare Sci., 70 (2017), 341-352.

[9] V. S. Grozdanov, *The generalized b-adic diaphony of the net of Zaremba-Halton over finite abelian groups*, Comp. Rendus Akad. Bulgare Sci., 57 (2004), 43-48.

[10] V. S. Grozdanov, S. S. Stoilova, *On the theory of $b-$adic diaphony*, Comp. Rendus Akad. Bulgare Sci., 54 (2001), 31-34.

[11] V. S. Grozdanov, S. S. Stoilova, *On the $b-$adic diaphony of the Roth net and generalized Zaremba net*, Mathematica Balcanica, New Series, 17, (2003), 103-112.

[12] J. H. Halton, *On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals*, Numer Math., 2 (1960), 84-90.

[13] J. H. Halton, S. K. Zaremba, *The extreme and L^2 discrepancy of some plane sets*, Monatsh. Math., 73 (1969), 316-328.

[14] J. M. Hammarsley, Monte Carlo methods for solving multivariable problems, Ann. New Work Acad. Sci., 86 (1960), 844-874.

[15] P. Hellekalek, H. Leeb, *Dyadic diaphony*, Acta Arith. LXXX, 2 (1997), 187-196.

[16] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, John Wiley & Sons, New York, 1974.

[17] G. Larcher, H. Niederreiter, W. Ch. Schmid, *Digital nets and sequences constructed over finite rings and their applications to quasi-Monte Carlo integration*, Monatsh. Math., 121 (1996), 231-253.

[18] P. D. Proinov, *On irregularities of distribution*, Comp. Rendus Akad. Bulgare Sci., 39 (1986), 31-34.

[19] K. Roth, *On the irregularities of distribution*, Mathematika, 1 (1954), 73-79.

[20] H. B. Stegbuchner, Math. Inst. Univ. Salzburg, Heft, 3 (1980).

[21] J. G. Van der Corput, *Verteilungsfunktionen* I, Proc. Akad. Wetensch. Amsterdam, 38 (1935), 813-821.

[22] J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math., 45 (1923), 5-24.

[23] T. Warnock, *Computational investigations of low-discrepancy point sets*, in: Applications of Number Theory to Numerical Analysis, N. Y., 1972, 319-343.

[24] Yi-Jun Xiao, *On the diaphony and the star-diaphony of the Roth sequences and the Zaremba sequences*, 1998, preprint.

[25] P. Zinterhof, Ü*ber einige Abschätzungen bei der Approximation von Funktionen mit Gleichverteilungsmethoden*, S. B. Akad. Wiss., Math.-Naturw. Klasse, Abt. II, 185 (1976), 121-132.

Accepted: February 20, 2023

Some properties of regular topology on $C(X,Y)$

Mir Aaliya*
Department of Mathematics
Lovely Professional University
India
miraaliya212@gmail.com

Sanjay Mishra
Department of Mathematics
Lovely Professional University
India
drsanjaymishra1@gmail.com

Abstract. The recently introduced regular topology for the function space $C(X,Y)$ has been explored up to some metrizability and various countability and completeness properties. The main aim of this paper is to explore the regular topology on the function space $C(X,Y)$ in which we study submetrizability and extend various properties equivalent to the metrizability of the space $C_r(X,Y)$. We also study number of maps corresponding to the space $C_r(X,Y)$ and prove that the regular topology on the space $C(X,Y)$ is strong when X is taken discrete. Furthermore, we study various separation axioms on the space $C_r(X,Y)$, where we prove that the function space $C_r(X)$ is normal by taking X to be countable, compactly generated compact space and prove certain equivalent conditions to various separation axioms on the space $C_r(X,Y)$.

Keywords: function space, regular topology, G_δ set, submetrizability, induced map, pseudocompact, separation axioms.

1. Introduction

The function space $C(X,Y)$ symbolizes the space of continuous functions from a space X to a space Y. This space has been topologized in numerous ways and those topologies include the innate topologies such as point-open topology, compact-open topology and uniform topology. However, more stronger topologies than that of the uniform topology such as the fine topology (also known as m-topology) and the graph topology have also been studied. The fine topology on $C(X) = C(X,\mathbb{R})$ along with the topological properties was studied by Hewitt [4]. Moreover, the basis elements for fine topology on $C(X,Y)$ where X is a Tychonoff space and (Y,d) a metric space are of the fashion: $B(f,\epsilon) = \{g \in C(X,Y) | d(f(x), g(x)) < \epsilon(x), \forall x \in X\}$, where $f \in C(X,Y)$ and ϵ is a positive unit of the ring $C(X)$. Later, the topological properties corre-

∗. Corresponding author

sponding to this topology have also been discussed in [11]. The space $C(X, Y)$ equipped with fine topology is proved to be submetrizable in [11].

Iberklied et al. in [15] introduced a more stronger topology than the fine topology on the space $C(X)$ and named it as the regular topology or the r-topology. This topology was defined in a manner that the positive unit in the basis elements of fine topology is replaced by a positive regular element of the ring $C(X)$. That is the basis elements for the regular topology on the space $C(X)$ are of the fashion: $R(f, r) = \{g \in C(X) \colon |f(x) - g(x)| < r(x), \forall x \in coz(r)\}$, where $f \in C(X)$, r is a positive regular element (non-zero divisor) of the ring $C(X)$ and $coz(r) = \{x \in X \colon r(x) \neq 0\}$. The space $C(X)$ equipped with the regular topology is represented as $C_r(X)$. Afterwards, Azarpanah et al. in [5] investigated compactness, connectedness and countability of this topology on the space $C(X)$. However, no study has been done on the submetrizability, separation axioms with respect to the regular topology on $C(X)$ and no map has been studied corresponding to the regular topology on the space $C(X)$.

Later, Jindal et al. [1] explored this regular topology on a more general space $C(X, Y)$, where X is Tychonoff and Y is a metric space with non-trivial path. They used the same idea as before to define the basis element for the regular topology on $C(X, Y)$ as: $R(f, r) = \{g \in C(X, Y) | d(f(x), g(x)) < r(x), \forall x \in coz(r)\}$, where where $f \in C(X, Y)$, r is a positive regular element (non-zero divisor) of the ring $C(X)$. The space $C(X, Y)$ endowed with regular topology is represented as $C_r(X, Y)$. Moreover, they studied various topological properties like metrizability, countability and several completeness properties. Despite all, the submetrizability was not studied on the space $C_r(X, Y)$, no separation axiom has been investigated for the space $C_r(X, Y)$ and no map with respect to this topology was studied. However, the submetrizability property has been studied for various function space topologies in [12], [14], [2].

The main concern of our work is to investigate submetrizability for the function space $C_r(X, Y)$, to investigate certain separation axioms and various kinds of maps on the space $C_r(X, Y)$, where X is a Tychonoff space and Y a metric space with a non-trivial path. In the first section, we demonstrate that the space $C_r(X, Y)$ is submetrizabe along with some equivalent conditions to its submetrizability. Moreover, we stretch the listicle of equivalent properties to its metrizability by replacing the metric space Y with a normed linear space with supremum norm. With this, we also see how by taking Y as a normed linear space makes the function space $C_r(X, Y)$ into a topological group.

In the second section, we study various maps such as composition function, induced map and embedding with respect to the regular topology on $C(X, Y)$. Specifically, we show how one function space can be embedded into other and derive a necessary condition when the regular topology on $C(X, Y)$ can be categorized as a strong topology.

Finally, in last portion we examine several separation axioms for the space $C_r(X, Y)$ such as Hausdorffness and regularity and provide some equivalent characterizations with respect to other function space topologies.

Moreover, the conventions that we use throughout this paper are: The space X will always represent a Hausdorff completely regular space (we will acknnowledge if it has an extra structure). The set of positive regular elements(non-zero divisors) of the ring $C(X)$ is symbolized by $r^+(X)$ and the multiplicative units of the same ring are symbolized by $U^+(X)$. The function space $C(X)$ and $C(X, Y)$ equipped with the regular topology are represented as $C_r(X)$ and $C_r(X, Y)$, respectively. The operation $\leq$ is used to represent the strength of two comparative topologies, which means the one on LHS is weaker than the one on RHS.

2. Pre-requisites

Definition 2.1.

1. *Let $g \in C(X)$, then $Z(g) = \{x \in X : g(x) = 0\}$ denotes the zero set of g and $coz(g) = \{x \in X : g(x) \neq 0\}$, is the set-theoretic complement of $Z(g)$.*
2. *Topologically, the regular elements of the ring $C(X)$ are characterized as : Let $g \in C(X)$, then it is said to be the regular element of $C(X)$ if and only if $Int_X(Z(g)) = \phi$ if and only if $coz(g)$ is dense subset of X.*
3. *A space Z is said to be pseudocompact if $f(Z)$ is bounded subset of $\mathbb{R}$, $\forall$ $f \in C(X)$, that is, for every $f \in C(X)$ there exists a natural number N for which $|f(z)| \leq N$ $\forall$ $z \in Z$.*

Definition 2.2. *In [15], an almost P-space is defined as the space where each nonempty G_δ-set has a nonempty interior. Moreover, in terms of elements of the ring $C(X)$, a space X is said to be an almost P-space if the regular elements coincide with the multiplicative units of ring $C(X)$.*

Theorem 2.1 (Theorem 2.1, [1])**.** *A space X is said to be an almost P-space if it satisfies anyone of the following conditions :*

1. *Every non-empty zero set of X has a non-empty interior.*
2. *Every non-empty G_δ-set of X has a non-empty interior.*
3. *Every zero set in X is a regular-closed set.*
4. *Every G_δ-set has an interior dense in itself.*

Theorem 2.2 (Theorem 1.8, [15])**.** *For a space X, the following are equivalent:*

1. $C_r(X) = C_m(X)$.
2. *X is an almost P-space.*
3. $r^+(X) = U^+(X)$.

Theorem 2.3 (Theorem 1.9, [15])**.** *For a space X, the following are equivalent:*

1. $C_r(X) = C_u(X)$
2. *X is pseudocompact, almost P-space.*

3. Submetrizability

In this section, we are going to investigate when the space $C_r(X,Y)$ is submetrizable. Moreover, we discuss how the submetrizability of space $C_r(X)$ can be characterized in terms of other weaker properties.

Definition 3.1. *A completely regular Hausdorff space (X,τ) is called submetrizable if it admits a weaker metrizable topology, equivalently, if there exists a continuous injection $f\colon X \to Y$, where Y is a metric space.*

Theorem 3.1. *For a space X and a Tychonoff space Y, the space $C_r(X,Y)$ is Tychonoff.*

Proof. Suppose Y is a Tychonoff space, implies Y is uniformizable. Consequently, $C_r(X,Y)$ is uniformizable [1]. Which means $C_r(X,Y)$ is Tychonoff. □

Theorem 3.2. *For a space X and a metric space (Y,d), the space $C_r(X,Y)$ is always submetrizable.*

Proof. As we know that the regular topology on $C(X,Y)$ is stronger than the fine topology on it [1]. Consequently, we can write $C_d(X,Y) \leq C_r(X,Y)$, and since $C_d(X,Y)$ is always metrizable (Corollary 2.1, [11]). Therefore, the space $C_r(X,Y)$ is submetrizable. □

Definition 3.2 (Definition 2.2, [11])**.** *A topological space Y is called a space of countable pseudocharacter if every point in Y is a G_δ-set (countable intersection of open sets) in Y. Such spaces are also called as E_0-spaces. Moreover, in a submetrizable space, every point is a G_δ-set. So, the submetrizable spaces are E_0-spaces. The study regarding E_0-spaces and submetrizable spaces can be found in [3] and [6], respectively.*

Corollary 3.1. *The space $C_r(X,Y)$ is of countable pseudocharacter.*

Remark 3.1 (Remark 5.2 in [12])**.**

1. If a space is having G_δ-diagonal, that is for a space X, if the set $\{(x,x)\colon x \in X\}$ is a G_δ-set in the product space $X \times X$, then each element of X is a G_δ-set. Note that every metrizable space has a zero-set diagonal which implies it has a regular G_δ-diagonal implies it has a G_δ-diagonal. Consequently, every submetrizable space has a zero-set diagonal.

2. In submetrizable spaces, all compact sets, pseudocompact sets, countably compact sets and singleton sets are G_δ-sets.

Next, we see various properties which are equivalent to the submetrizability of space $C_r(X)$. The above remark leads us to the following theorem:

Theorem 3.3. *For a space X, we have the following equivalent properties:*

1. *$C_r(X)$ is submetrizable.*
2. *$C_r(X)$ has a zero set diagonal.*
3. *$C_r(X)$ has a regular G_δ-diagonal.*
4. *$C_r(X)$ has a G_δ-diagonal.*
5. *Each singleton set in $C(X)$ is G_δ in $C_r(X)$.*
6. *$\{0_X\}$ is a G_δ in $C_r(X)$.*
7. *X is separable*
8. *$C_p(X)$ is submetrizable.*

Proof. Since (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from the above discussion. (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are immediate.
(6) $\Rightarrow$ (7) Suppose $\{0_X\}$ is G_δ in $C_r(X)$, then there exists a countable family $\mathfrak{N}$ of open sets in $C_r(X)$ so that $\{0_X\} = \bigcap \mathfrak{N}$.

Now, assume that $\mathfrak{N}$ has elements of the form $B(f_1, r_1), \cdots \cap B(f_k, r_k) \cap B(0_X, r_m), \cdots \cap B(0_X, r_n)$, where $f_i \in C(X), r_j \in r^+(X), 0_X$ is a constant function and $1 < i < k$ and $1 < j < n$.

Now, for each $U = B(f_1, r_1), \cdots \cap B(f_k, r_k) \cap B(0_X, r_m), \cdots \cap B(0_X, r_n) \in \mathfrak{N}$, fix $x_j \in coz(r_j)$ and put $H(U) = \{y_1, \cdots y_m, x_1, \cdots, x_n\}$. Let $A = \{H(U) \colon U \in \mathfrak{N}\}$. Clearly, A is countable. Suppose $Cl(A) \neq X$, so $\exists\ x_0 \in X - Cl(A)$. Since X is a completely regular space so $\exists\ f \in C(X)$ such that $f(x_0) = 1, f(y) = 0 \forall y \in cl(A)$. This implies $f \in U$ for each $U \in \mathfrak{N}$. So, $f = 0_X$, but $f(x_0) = 1$. Thus, $cl(A) = X$. Hence, X is separable.

(7) $\Leftrightarrow$ (8) is well known.

(8) $\Rightarrow$ (1) Since $C_p(X) \leq C_r(X)$. □

In the next result, we stretch the list of equivalent characterizations of metrizability of $C_r(X,Y)$. Infact, we see how X being pseudocompact, almost P-space acts also as the necessary and sufficient condition for the space $C_r(X,Y)$ to be countably tight, radial and pseudoradial.

Theorem 3.4. *For a space X and a metric space (Y, d) with a non-trivial path, we have the following equivalent conditions:*

1. *X is pseudocompact, almost P-space.*
2. *$C_d(X,Y) = C_r(X,Y)$.*
3. *$C_r(X,Y)$ is metrizable.*
4. *$C_r(X,Y)$ is first countable.*
5. *$C_r(X,Y)$ is of pointwise countable type.*

6. *$C_r(X,Y)$ is an r-space.*
7. *$C_r(X,Y)$ is an M-space.*
8. *$C_r(X,Y)$ is an p-space.*
9. *$C_r(X,Y)$ is an q-space.*
10. *$C_r(X,Y)$ is a Frechet space.*
11. *$C_r(X,Y)$ is a Sequential space.*
12. *$C_r(X,Y)$ is a k-space.*
13. *$C_r(X,Y)$ is countably tight.*
14. *$C_r(X,Y)$ is radial.*
15. *$C_r(X,Y)$ is pseudoradial.*

Proof. The equivalent conditions from (1) upto (9) are true as proved in (Theorem 2.7, [1]).

And since (4) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are well known.

(12) $\Rightarrow$ (13) It supports because a regular k-space having points G_δ is countably tight. However, let's prove it by contradiction. Suppose a regular k-space Z with points G_δ is not countably tight, then there exists a subset S of Z in such manner that the set $H = \{\bar{P} \colon P \subseteq S$ and P is countable $\} \subsetneq \bar{S}$. Since H contains S and H is not closed. Therefore, there exists a compact subset C of Z in such a way that $H \cap C$ is not closed in C. In addition, every compact space where singleton sets are G_δ is first countable. Thus, there exists a sequence (x_n) in $H \cap C$ converging to some $x \in C \backslash H$.

Now, $\forall\, n \in N$, $\exists$ a countable $P_n \subseteq S$ so that $x_n \in \bar{P_n}$. Hence, $x \in \overline{\bigcup_{n\in N} P_n}$. Since $\bigcup_{n\in N} P_n$ is countable in S, $x \in H$. Which is a contradiction.

Now, (13) $\Rightarrow$ (1) Suppose X is not an almost P-space. Then, we can find a non-empty zero set say S in X which has empty interior. Let $r \in C(X)$ such that $Z(r) = S$. Since $Z(r) = Z(|r|)$, then we can assume $r \geq 0$. Consequently, $r \in r^+(X)$. As $C_r(X,Y)$ is countably tight, so we can consider a countable subset $\{g_n \colon n \in N\}$.

Now, choose $e \in Z(r)$. Since Y contains a non-trivial path, so we can find $t_0 \in Y \setminus \{g_n(e) \colon n \in N\}$. Let g_0 be a constant function in $C_r(X,Y)$ taking values t_0. Then, $R(g_0, r)$ is a non-empty open set in $C_r(X,Y)$ that does not intersect $\{g_n \colon n \in N\}$. Which is not true. Thus, X is an almost P-space.

Hence, by (Theorem 2.2, [1]), $C_f(X,Y) = C_r(X,Y)$. Thus, $C_f(X,Y)$ is also countably tight. But, the (Theorem 3.3, [8]) implies that X is pseudocompact. Which finishes the proof (13) $\Rightarrow$ (1).

Clearly, (10) $\Rightarrow$ (14) $\Rightarrow$ (15). We show that (15) $\Rightarrow$ (13) by contradiction. Consider a nonclosed subset N of $C_r(X,Y)$. Then, there exists a cardinal k

and a k-sequence in N, say $(g_\sigma)_{\sigma<k}$ in such a way that the sequence converges to some $g \in N$. We lay claim to the fact that there is an $\aleph_0$-subsequence that converges to g. If this is shown, it will declare that $C_r(X,Y)$ is a sequential space.

For every natural number n, we can choose an ordinal $\sigma_n < k$ so that $\sigma_n > \sigma_{n-1}$ and for every $\sigma_n < \tau < k, g_\tau \in B_g(g, 1/n)$. The sequence (σ_n) converges to k. Otherwise there is an ordinal $\tau < k$ such that $\sigma_n < \tau$ for each n, hence $g = g_\tau \in N$; a contradiction. Next, for any $r \in r^+(X)$, there is an ordinal σ such that for every $\sigma < \tau < k$, we have $g_\tau \in B_g(g, r)$. Since (σ_n) converges to k, there is an n such that $\sigma < \sigma_m < k, \forall m \geq n$. Hence, $g_{\sigma_m} \in B_g(g, r)$ for each $m \geq n$. Thus, $g_{\sigma_m} \forall m \geq n$ converges to g. □

Example 3.1. Let $X = [0, \omega_1)$ and $Y = \mathbb{R}$, the the space $C_r([0, \omega_1))$ is submetrizable. Since the space $[0, \omega_1)$ is countably compact [Example 2.2, [11]] implies X is pseudocompact. The space $C_f([0, \omega_1))$ is metrizable. Also the space $[0, \omega_1)$ is not an almost P-space. Therefore, we have $C_f([0, \omega_1)) \neq C_r([0, \omega_1))$. Hence, the space $C_r([0, \omega_1))$ is submetrizable.

Example 3.2. For a real line $\mathbb{R}$, let $\beta\mathbb{R}$ denotes its Stone-Cech compactification. Let $X = \beta\mathbb{R} - \mathbb{R}$, then X is an almost P-space [10] and since $\mathbb{R}$ is locally compact, so it is open in $\beta\mathbb{R}$, and $\beta\mathbb{R} - \mathbb{R}$ is therefore compact, thus pseudocompact. Then, we have $C_d(\beta\mathbb{R} - \mathbb{R}) = C_r(\beta\mathbb{R} - \mathbb{R})$, implies $C_r(\beta\mathbb{R} - \mathbb{R})$ is metrizable and hence submetrizable.

In the upcoming result, we see how by taking Y as a normed linear space with supremum norm, one can further stretch the list of characterizations equivalent to metrizability of the space $C_r(X,Y)$. Before that we require the below results to prove the main theorem.

Theorem 3.5. *For a space X and a normed linear space $(Y, \|.\|_\infty)$ with supremum norm, the function space $C_r(X,Y)$ is a topological group under pointwise addition.*

Proof. Clearly, under pointwise addition, $C_r(X,Y)$ is a group.

Now, it is sufficient to prove that the group operations are continuous. Suppose $s\colon C_r(X,Y) \times C_r(X,Y) \to C_r(X,Y)$ be defined as $s(g_1, g_2) = g_1 + g_2, \forall g_1, g_2 \in C(X,Y)$. Consider a basic neighborhood $B(g_1 + g_2, r)$ of $g_1 + g_2$ in $C_r(X,Y)$, where r is the regular element of ring $C(X)$. Take $\epsilon_1 = r(x)/3 = \epsilon_2, x \in coz(r)$, and observe the neighborhood $B(g_1, \epsilon_1) \times B(g_2, \epsilon_2)$ of (g_1, g_2) in $C_r(X,Y) \times C_r(X,Y)$. Suppose $(h_1, h_2) \in B(g_1, \epsilon_1) \times B(g_2, \epsilon_2)$. Then, for $x \in coz(r)$,

$$\begin{aligned} \|(g_1 + g_2)(x) - (h_1 + h_2)(x)\| &\leq \|g_1(x) - h_1(x)\| + \|g_2(x) - h_2(x)\| \\ &< \epsilon_1(x) + \epsilon_2(x) < r(x) \end{aligned}$$

Then, $s(B(g_1, \epsilon_1) \times B(g_2, \epsilon_2)) \subseteq B(g_1 + g_2, r)$. Therefore, s is continuous.

Now, let $I\colon C_r(X,Y) \to C_r(X,Y)$ defined by $I(f) = -f$ for any $f \in C(X,Y)$, where $(-f)(x) = -f(x) \in Y$. Observe the neighborhood $B(-f,r)$ of $-f$. Therefore, $I(B(f,r)) = B(-f,r)$. Thus, I is continuous. Hence, $C_r(X,Y)$ is a topological group. □

Since we have shown that the function space $C_r(X,Y)$ is topological group, for a space X and a normed linear space $(Y,\|.\|_\infty)$. Thus, it is a homogeneous space [11]. However, a space A is termed to be a homogeneous space if for each pair of points $a,b \in A$, there exists a homeomorphism of A onto itself that carries a to b. Further, to prove next result, we first require the following two lemmas:

Lemma 3.1 (Lemma 2.1, [11]). *Let D be a dense subset of a space X and $x \in D$. Then, x has a countable local π-base in D if and only if x has a countable local π-base in X.*

Lemma 3.2 (Lemma 2.3, [11]). *Let D be a dense subset of a space X and C be a compact subset D. Then, C has countable character in D if and only if C has countable character in X.*

Theorem 3.6. *For a space X and a normed linear space $(Y,\|.\|_\infty)$, the space $C_r(X,Y)$ has a countable π-character if and only if $C_r(X,Y)$ has a dense subspace having countable π-character.*

Proof. Consider a dense subspace C of $C_r(X,Y)$ having a countable π-character. Take $f \in C$ to be arbitrary. Because f has a countable local π-base in C, then by the (Lemma 3.1) f has a countable local π-base in $C_r(X,Y)$. Therefore, there exists a sequence $\{O_n\colon n \in \mathbb{N}\}$ of open sets in $C_r(X,Y)$ in such a manner that whenever O is an open set carrying f, $O_n \subseteq O$ for some n. Take an arbitrary $g \in C_r(X,Y)$. As $C_r(X,Y)$ is a homogeneous space, thus there exists a homeomorphism $h\colon C_r(X,Y) \to C_r(X,Y)$ defined by $h(f) = g$. Therefore, $\{h(O_n)\colon n \in \mathbb{N}\}$ is a sequence of open sets in $C_r(X,Y)$. Let P be an open set with $g \in P$. Therefore, $f \in h^{-1}(P)$ and there exists n such that $O_n \subseteq f^{-1}(P)$. As a consequence, g has a countable local π-base in $C_r(X,Y)$. Hence, $C_r(X,Y)$ has a countable π-character. Clearly, the converse follows. □

Theorem 3.7. *For a space X and a normed linear space $(Y,\|.\|_\infty)$, the space $C_r(X,Y)$ is of pointwise countable type if and only if $C_r(X,Y)$ has a dense subspace of pointwise countable type.*

Proof. Consider a dense subspace C of $C_r(X,Y)$ that is of pointwise countable type. Let $f \in C$ and $g \in C(X,Y)$. Since $C_r(X,Y)$ is homogeneous, so there exists a homeomorphism $H\colon C_r(X,Y) \to C_r(X,Y)$ so that $H(f) = g$. Since C is a dense subspace of $C_r(X,Y)$, so there exists a compact subset, say K so that $f \in K$ and is of pointwise countable character in C. Thus, by above (Lemma 3.2), K has countable character in $C_r(X,Y)$. Therefore, $H(K)$ is a compact subset of $C_r(X,Y)$ having countable character in $C_r(X,Y)$, and $g \in H(K)$. Hence, $C_r(X,Y)$ is of pointwise countable type. The converse is immediate. □

Theorem 3.8. *For a space X and a normed linear space $(Y, \|.\|_\infty)$, we have the following equivalences :*

1. *X is pseudocompact, almost P-space.*
2. *$C_d(X,Y) = C_r(X,Y)$.*
3. *$C_r(X,Y)$ is metrizable.*
4. *$C_r(X,Y)$ is of pointwise countable type.*
5. *$C_r(X,Y)$ has a dense subset which is of pointwise countable type.*
6. *$C_r(X,Y)$ is countably tight.*
7. *$C_r(X,Y)$ is first countable.*
8. *$C_r(X,Y)$ has a countable π-character.*
9. *$C_r(X,Y)$ has a dense subspace of countable π-character.*
10. *$C_r(X,Y)$ is normed linear space.*
11. *$C_r(X,Y)$ is topological vector space.*

Proof. The equivalences $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$ are true (Theorem 2.7, [1]).

$(4) \Leftrightarrow (5)$ is proved in above (Theorem 3.7).

$(1) \Leftrightarrow (6) \Leftrightarrow (7)$ are true as proved in (Theorem 3.4).

$(7) \Rightarrow (8)$. Since $C_r(X,Y)$ is a topological group and a topological group is first countable if and only if it has countable π-character.

$(8) \Leftrightarrow (9)$ is proved in above (Theorem 3.6).

$(1) \Rightarrow (10)$ Suppose X is pseudocompact and almost P-space then $C_r(X,Y) = C_d(X,Y)$ (Theorem 2.7, [1]). But when X is pseudocompact, then $C_d(X,Y)$ is a normed linear space under the supremum norm $||.||_\infty$ defined by $\|f\|_\infty = sup\{\|f(x)\| \colon x \in X\}$. Thus, the space $C_r(X,Y)$ is a normed linear space.

$(10) \Rightarrow (11)$ is immediate.

$(11) \Rightarrow (1)$ Suppose X is not an almost P-space, then there exists a non-empty zero set say A which has empty interior in X. Let $s \in C(X)$ be in such a way that $Z(s) = A$. As $Z(s) = Z(|s|)$, thus $s \in r^+(X)$. Without the loss of generality, we can assume s in such a way that there $\nexists$ any $\delta > 0$ so that $\delta < s(x), \forall x \in coz(s)$. Consider a non-zero element y_0 and define $f_{y_0} \colon X \to Y$ as $f_{y_0}(x) = y_0, \forall x \in X$. We prove that the scaler multiplication is not continuous at $(0, f_{y_0}) \in \mathbb{R} \times C_r(X,Y)$. Consider a basic neighborhood $B(0_X, s)$ of 0_X in $C_r(X,Y)$ where $0_X(x) = 0, \forall x \in X$.

Now, consider a basic neighborhood $(-\epsilon, \epsilon) \times B(f_{y_0}, r)$ of $(0, f_{y_0})$ in $\mathbb{R} \times C_r(X,Y)$, where $\epsilon > 0$ and $r \in r^+(X)$. Then, for any non-zero $\alpha \in (-\epsilon, \epsilon)$, αf_{y_0} does not belong to $B(0_X, s), \forall x \in coz(s)$. Because then $||\alpha f_{y_0}(x)\| = |\alpha| \|y_0|| < s(x), \forall x \in coz(s)$. But this contradicts our choice of $s \in r^+(X)$. So, if X is not an almost P-space, then $C_r(X,Y)$ is not a topological vector space. In other words, $C_r(X,Y)$ being topological vector space implies X is an almost P-space.

But X being almost P-space implies that $C_r(X,Y) = C_f(X,Y)$ (Theorem 2.2, [1]). Therefore, $C_f(X,Y)$ is a topological vector space. However, (Theorem 2.2, [11]) shows that $C_f(X,Y)$ is topological vector space if and only if X is pseudocompact. This finishes the proof that (11) ⇒ (1). □

4. Some special maps

In this section, we will be discussing various maps that can be drawn over or from the space $C_r(X,Y)$ which includes composition function, induced map and embedding. In function spaces, the function $i\colon Y \to C(X,Y)$ defined as $i(t) = c_t$, where c_t is a constant map is an injection [7]. However, in particular, the function $i\colon \mathbb{R} \to C(X,\mathbb{R})$ defined as $i(t) = c_t$, where $c_t\ \forall\ t \in \mathbb{R}$ is a constant map is an injection [7].

Definition 4.1 (Composition function)**.** *Suppose X, Y and $\mathbb{R}$ are spaces, a composition function $\phi\colon C_r(X,Y)\times C_r(Y,\mathbb{R}) \to C_r(X,\mathbb{R})$ is defined by $\phi(f,g) = g\circ f$, $f \in C_r(X,Y)$, $g \in C_r(Y,\mathbb{R})$*

Definition 4.2 (Induced map)**.** *Suppose $g \in C_r(Y,\mathbb{R})$, then an induced map $g_*\colon C_r(X,Y) \to C_r(X,\mathbb{R})$ is defined by $g_*(f) = \phi(f,g) = g\circ f$, $f \in C_r(X,Y)$. In particular, for $g \in C_r(X,Y)$, then an induced map for the function space $C(X)$ is defined as $g_*\colon C_r(Y) \to C_r(X)$ with $g_*(f) = \phi(f,g) = g\circ f$, $f \in C_r(Y)$.*

An induced map is formed by fixing one of the components of composition function. Note that the induced maps preserve composition as : $(g\circ f)_* = g_*\circ f_*$.

Theorem 4.1. *Let $g \in C_r(Y,\mathbb{R})$, then g is one-to-one if and only if $g_*: C_r(X,Y) \to C_r(X,\mathbb{R})$ is one-to-one.*

Proof. Let g is one-to-one. To prove $g_*\colon C_r(X,Y) \to C_r(X,\mathbb{R})$ is one-to-one. Let's consider f_1, $f_2 \in C_r(X,Y)$ and let $g_*(f_1) = g_*(f_2)$. This implies $\phi(f_1,g) = \phi(f_2,g)$. Which implies $g\circ f_1 = g\circ f_2$. Then, $g(f_1) = g(f_2)$. Implies $f_1 = f_2$. Therefore, $g_*\colon C_r(X,Y) \to C_r(X,\mathbb{R})$ is one-to-one.

Conversely, let g_* is one-to-one. To prove $g \in C(Y,\mathbb{R})$ is one-to-one. For this, consider $x_1, x_2 \in Y$ and let $g(x_1) = g(x_2)$. This implies $g_*(g(x_1)) = g_*(g(x_2))$. Which implies $\phi(g(x_1),g) = \phi(g(x_2),g)$. Then, $\phi(g,g) = \phi(g,g)$. Then, we can write $g^{-1}(g(x_1)) = g^{-1}(g(x_2))$. Implies $x_1 = x_2$. Therefore, g is one-to-one. □

Theorem 4.2. *Let $g \in C_r(Y,\mathbb{R})$ and $g_*\colon C_r(X,Y) \to C_r(X,\mathbb{R})$ is onto then g is onto.*

Proof. Let g_* is onto, then by definition there exists $f_1 \in C(X,\mathbb{R})$ such that $f_1 = g_*(g_1)$, $\forall\ g_1 \in C(X,Y)$. This implies $f_1 = \phi(g_1,g)$, which implies $f_1 = g\circ g_1$. Then, $f_1 = g(g_1)$. Thus, g is onto. □

Definition 4.3. *A function f from a non-empty set A to a topological space B is said to be an almost onto map if $f(A)$ is dense in B.*

Theorem 4.3 (Theorem 2.2.6 (a), [7])**.** *Let $g \in C(X,Y)$, then the induced map $g_*: C(Y) \to C(X)$ is one-one if and only if g is almost onto.*

Theorem 4.4. *For a Tychonoff space X and a metric space (Y,d), and let $g \in C_r(Y,\mathbb{R})$, then the induced map $g_*: C_r(X,Y) \to C_r(X,\mathbb{R})$ defined as $g_*(f) = \phi(f,g) = g \circ f$, $f \in C_r(X,Y)$ is continuous.*

Proof. Let $B(f,r)$ be a basic open subset of $C_r(X)$, where r is a non-negative regular element of the ring $C(X)$ and $B(f,r) = \{h \in C(X)\colon |f(x) - h(x)| < r(x), \forall x \in coz(r)\}$.

Now, we will show that $g_*^{-1}[B(f,r)]$ is open in $C_r(X,Y)$. So, for this, let $h \in g_*^{-1}[B(f,r)]$ and we will show it is an interior point of $g_*^{-1}[B(f,r)]$.

For every $x \in coz(r)$, we know from the definition that

$$|g(h(x)) - f(x)| < r(x) \Rightarrow g(h(x)) \in B_{r(x)}(f(x))$$

Since $B_{r(x)}(f(x)$ is open, we can thus find another regular element $\acute{r} \in C(X)$ so that

$$B_{\acute{r}(x)}(g(h(x))) \subseteq B_{r(x)}(f(x)) \tag{1}$$

Then, as g is continuous so by the continuity of g at x, $\exists\, \delta$ a non-negative regular element of ring $C(X)$ such that

$$\forall y \in coz(\delta)\colon d_Y(h(x),y) < \delta(x) \Rightarrow g(y) \in B_{\acute{r}(x)}(g(h(x))) \tag{2}$$

Now, if $\acute{h} \in B(h,\delta)$, from (2) we can conclude that

$$\forall x \in coz(\acute{r})\colon g(\acute{h}(x)) \in B_{\acute{r}(x)}(g(h(x)))$$

Thus, from (1) it is evident that $g_*(\acute{h}) \in B(f,r)$. Therefore, $B(h,\delta) \subseteq g_*^{-1}[B(f,r)]$ as required. □

Corollary 4.1. *For a space X, let $g \in C_r(X,Y)$ for some space Y, then the induced map $g_*: C_r(Y) \to C_r(X)$ is continuous.*

Theorem 4.5. *For a space X and a metric space (Y,d), the map $\phi\colon Y \to C_r(X,Y)$ where $\phi(y) = \bar{y}$ and $\bar{y}$ is a constant map in $C_r(X,Y)$, is an embedding.*

Proof. Since, ϕ is one-one and the basis elements for regular topology on $C(X,Y)$ are of the form $B(f,r)$ where $f \in C(X,Y)$, r is a non-negative regular element of the ring $C(X)$, and

$$B(f,r) = \{g \in C(X,Y)\colon d(f(x),g(x)) < r(x), \forall x \in coz(r)\}$$

Now, as ϕ maps $y \in Y$ to $\phi(y) \in C_r(X, Y)$ defined by $\phi(y)(x) = \bar{y}(x) \forall x \in X$ is continuous.

Suppose $y_n \to y_0$ in (Y, d), it is enough to show sequential continuity, as Y is a first countable space. Then, it is clear that $\phi(y_n) \to \phi(y_0)$ such that if $B(\phi(y_0), r)$ is a basic neighborhood of $\phi(y_0)$ then by convergence, there is some N such that $n \geq N$ implies $d(y_n, y_0) < r(x), \forall x \in coz(r)$. Then, also $n \geq N$ implies $\phi(y_n) \in B(\phi(y_0), r)$.

Thus, ϕ is an embedding and we have $\phi[B(y, r)] \cap \phi[Y] = B(\phi(y), r) \cap \phi[Y]$ so ϕ maps open sets to open sets in $\phi(y)$. □

Corollary 4.2. *For a space X and a real line $\mathbb{R}$, the map $\phi\colon \mathbb{R} \to C_r(X)$ where $\phi(y) = \bar{y}$ and $\bar{y}$ is a constant map in $C_r(X)$ is an embedding.*

Now, we provide a scenario in which a function space can be embedded into another function space with regular topology.

Theorem 4.6. *Suppose that the space Y is a continuous image of the space X. Then, $C_r(Y)$ can be embedded into $C_r(X)$.*

Proof. Let $s\colon X \to Y$ be a continuous surjection, i.e. s is a continuous function from X onto Y. Define the map $\psi\colon C_r(Y) \to C_r(X)$ by $\psi(f) = f \circ s$ for all $f \in C_r(Y)$. We show that ψ is a homeomorphism from $C_r(Y)$ into $C_r(X)$.

First we show ψ is a one-to-one map. Let $f, g \in C_r(Y)$ with $f \neq g$ such that $\psi(f) \neq \psi(g)$. Then, there exists $y \in Y\colon f(y) \neq g(y)$. Choose some $x \in X\colon s(x) = y$. Which means $f \circ s \neq g \circ s$. Implies that $f(s(x)) \neq g(s(x)) \Rightarrow f(y) \neq g(y)$.

Next, we show that ψ is continuous. Let $f \in C_r(Y)$ and $B(g, r_i) = \{q \in C_r(X)\colon |q(x_i) - g(x_i)| < r_i(x_i), x_i \in Coz(r_i)\}$, where $x_i \in X$ and $r_i \in r^+(X)$. Next, for each i, $f(s(x_i)) \in B(g, r_i)$.

Now, consider $R(h, l_i) = \{p \in C_r(Y)\colon |p(s(x_i)) - h(s(x_i))| < l_i(x_i), x_i \in Coz(l_i)\}$. Clearly $f \in R(h, l)$. It follows that $\psi R(h, l_i) \subset B(g, r_i)$. Since for each $p \in R(h, l_i)$, it is clear that $\psi(p) = p \circ s \in B(g, r_i)$.

Now, we prove that $\psi^{-1}\colon \psi(C_r(Y)) \to C_r(Y)$ is continuous. Let $\psi(f) = f \circ s \in \psi(C_r(Y)), f \in C_r(Y)$. Let G be an open set with $\psi^{-1}(f \circ s) = f \in G$ such that $G(g, r_i) = \{p \in C_r(Y)\colon |g(y_i) - p(y_i)| < r_i(y_i), y_i \in Coz(r_i)\}$. Choose $x_1, x_2, \ldots x_m$ such that $s(x_i) = y_i \forall i$. We have $f(s(x_i)) \in G(g, r_i) \forall i$. Define an open set $H(h, l_i) = \{q \in \psi(C_r(Y)) \subset C_r(X), \forall i$ such that $|h(x_i) - q(x_i)| < l_i(x_i)\}$. Clearly, $f \circ s \in H$. Note that $\psi^{-1}(H) \subset G$. To see this, let $p \circ s \in H$, where $p \in C_r(Y)$. Implies $p(s(x_i)) = p(y_i)$. It follows that ψ^{-1} is continuous. □

Now, we define restriction map. Suppose A is a subset of B, then the restriction map is defined as: $\pi_A\colon C(B) \to C(A)$ as $\pi_A(f) = f_{|A}$.

Theorem 4.7. *For an arbitrary subspace Y of a space X, the map $\pi_Y\colon C_r(X) \to C_r(Y)$ is continuous.*

Proof. Let $B(f,r) = \{g \in C(Y) \colon |f(y) - g(y)| < r(y), y \in coz(r)\}$ be an open set in $C_r(Y)$. We need to prove that $\pi_Y^{-1}(B(f,r))$ is open in $C_r(X)$. We have $\pi_Y^{-1}(B(f,r)) = \{g \in C(X) \colon |\pi_Y(g)(y) - f(y)| < r(y), y \in coz(r)\}$ $= \{g \in C(X) \colon |g_{|Y}(y) - f(y)| < r(y)\}$ which is open in $C_r(X)$. Hence, the map $\pi_Y \colon C_r(X) \to C_r(Y)$ is continuous. □

Theorem 4.8. *The map $\pi_Y \colon C_r(X) \to C_r(Y)$ is one-to-one if and only if Y is dense in X.*

Proof. Suppose Y is dense in X, we will show that $\pi_Y \colon C_r(X) \to C_r(Y)$ is one-to-one. Let $f, g \in C_r(X)$. Then, due to the continuity of these functions and $\bar{Y} = X$, it implies that if $f \neq g$ then $f_{|Y} \neq g_{|Y} \Rightarrow \pi_Y(f) \neq \pi_Y(g)$. Hence, π_Y is one-to-one.

Conversely, suppose that π_Y is one-to-one. We will show that Y is dense in X by contradiction. Assume that Y is not dense in X and let $f, g \in C_r(X)$. Then, $f \neq g$ does not imply that $f_{|Y} \neq g_{|Y}$. Thus, we can have $f_{|Y} = g_{|Y} \Rightarrow \pi_Y(f) = \pi_Y(g)$, which is a contradiction to π_Y being one-to-one. Hence, Y is dense in X. □

Theorem 4.9. *For a dense subspace Y of a space X, the map $\pi_Y \colon C_r(X) \to C_r(Y)$ is an embedding.*

Proof. Since the map π_Y is one-to-one and continuous. Then, we only need to prove that it is an open map onto $\pi_Y(C_r(X))$. For this let $B(f,r)$ be an open set in $C_r(X)$.

Now, we will show that $\pi_Y(B(f,r)) = B(f_{|Y}, r) \cap \pi_Y(C_r(X))$. Let $h \in \pi_Y(B(f,r))$, then by definition $|h(y) - \pi_Y(f)(y)| < r(y), y \in coz(r) \Rightarrow |h(y) - f_{|Y}(y)| < r(y)$. This implies $h \in B(f_{|Y}) \cap \pi_Y(C_r(X))$. Therefore, $\pi_Y(B(f,r)) \subset B(f_{|Y}, r) \cap \pi_Y(C_r(X))$.

Next, let $h \in B(f_Y, r) \cap \pi_Y(C_r(X))$. Then, $|h(y) - f_{|Y}(y)| < r(y), y \in coz(r)$ $\Rightarrow |h(y) - \pi_Y(f)(y)| < r(y)$. Therefore, $h \in \pi_Y(B(f,r))$ and thus $B(f_{|Y}, r) \cap \pi_Y(C_r(X)) \subset \pi_Y(B(f,r))$. Hence, π_Y is an embedding and $\pi_Y(C_r(X))$ can be treated as a subspace of $C_r(Y)$. □

Theorem 4.10. *For a space X, if Y is a subspace of X and $\pi_Y \colon C_r(X) \to C_r(Y)$ is defined as $\pi_Y(f) = f_{|Y}$. Then, $C_r(Y) = \pi_Y(C_r(X))$.*

Proof. Since $\pi_Y(C_r(X)) \subset C_r(Y)$, we will show that $C_r(Y) \subset \pi_Y(C_r(X))$. So, for this, let $g \in C_r(Y)$ and $B(g,r)$ be a basic neighborhood of g in $C_r(Y)$. Define a function $f \colon X \to \mathbb{R}$ as:

$$f(x) = \begin{cases} 0, & x \in X \ coz(r), \\ g(y), & x \in coz(r). \end{cases}$$

Consequently, $f \in C_r(X)$ and $\pi_Y(f) \in B(g,r)$. Thus, $C_r(Y) \subset \pi_Y(C_r(X))$. Hence, $C_r(Y) = \pi_Y(C_r(X))$. □

In the next result, we show that the regular topology on the space $C(X,Y)$ is strong based on the result that was investigated in [9] as: A topology on $C(X,Y)$ is said to be strong if and only if it makes the evaluation map $e\colon C(X,Y)$ $\times X \to Y$ as $(f,x) \mapsto f(x)$ continuous.

Theorem 4.11. *For a discrete space X and a metric space (Y,d), the regular topology on $C(X,Y)$ is strong.*

Proof. To prove that the regular topology on $C(X,Y)$ is strong, it is sufficient to prove that the evaluation map $e\colon C_r(X,Y)\times X \to Y$ defined as $(f,x)\mapsto f(x)$ is continuous.

Given a point (f,x) in $C_r(X,Y) \times X$ and an open set $B(f(x),\epsilon), \epsilon > 0$ about the image point $e(f,x) = f(x)$, we wish to find an open set about (f,x) that e maps into $B(f(x),\epsilon)$. Let $B(f,r)$ be an open set in $C_r(X,Y)$ such that $B(f,r) = \{g \in C(X,Y)\colon d(f(x),g(x)) < r(x), x \in coz(r)\}$. Since $coz(r)$ is dense in X and X has a discrete topology, then for all $x \in X$, there exists a neighborhood of x. As a consequence, there exists an open set say U in X such that $B(f,r)\times U$ is open in $C_r(X,Y)\times X$ that maps (f,x) to $f(x)$ in Y. Thus, if $(g,a) \in B(f,r)\times U$, then $e(g,a) = g(a)$. □

5. Separation axioms

In this section, we are going to discuss about various separation axioms corresponding to the function space $C_r(X,Y)$ such as Hausdorffness, regularity and normality.

Theorem 5.1. *For a space X, if Y is T_0 or T_1, then the space $C_r(X,Y)$ is T_0 or T_1, respectively.*

Proof. Suppose Y is T_0 or T_1. Then, the space Y^X is T_0 or T_1, respectively in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies $C_p(X,Y)$ is T_0 or T_1. As $C_p(X,Y) \leq C_r(X,Y)$ and hence $C_r(X,Y)$ is T_0 or T_1, respectively. □

Theorem 5.2. *For a space X, if Y is Hausdorff, then the space $C_r(X,Y)$ is also Hausdorff.*

Proof. Suppose Y is Hausdorff, then the space Y^X is Hausdorff in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies $C_p(X,Y)$ is Hausdorff. As $C_p(X,Y) \leq C_r(X,Y)$, hence $C_r(X,Y)$ is Hausdorff. □

Theorem 5.3. *For a space X, if Y is a completely regular space, then the space $C_r(X,Y)$ is also completely regular.*

Proof. Since every uniformizable space is completely regular. However, we can prove it as : Suppose Y is completely regular, then the space Y^X is completely regular in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies

$C_p(X,Y)$ is completely regular. As $C_p(X,Y) \leq C_r(X,Y)$, hence $C_r(X,Y)$ is completely regular. □

Theorem 5.4. *For a space X, if Y is a regular space, then the space $C_r(X,Y)$ is also regular.*

Proof. Suppose Y is regular, then the space Y^X is regular in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies $C_p(X,Y)$ is regular. As $C_p(X,Y) \leq C_r(X,Y)$ and hence $C_r(X,Y)$ is regular. □

Theorem 5.5. *For a pseudocompact and almost P-space X and a metric space (Y,d), the space $C_r(X,Y)$ is normal.*

Proof. Since the space $C_r(X,Y)$ is metrizable if and only if X is pseudocompact and almost P-space. Also we know that all metrizable spaces are normal (Theorem 3.20, [13]). Hence, the space $C_r(X,Y)$ is normal. □

Theorem 5.6. *For a countable, compactly generated, compact space X, the space $C_r(X)$ is normal.*

Proof. Suppose X is a compactly generated compact space, then $C_k(X) = C_r(X)$ and thus $C_r(X)$ is closed in $\mathbb{R}^X$. Since X is countable, and we know that $\mathbb{R}^X$ is normal if and only if X is countable. Thus, we get $\mathbb{R}^X$ is normal. However, $C_r(X)$ being closed subset of $\mathbb{R}^X$ is also normal. □

Corollary 5.1. *For a discrete space X, the space $C_r(X)$ is normal if and only if X is countable.*

Theorem 5.7. *For a pseudocompact and almost P-space X and a metric space (Y,d), the space $C_r(X,Y)$ is completely normal.*

Proof. Since the space $C_r(X,Y)$ is metrizable if and only if X is pseudocompact and almost P-space. Also, metrizable spaces are completely normal (Chapter 4, [13]). Hence, the space $C_r(X,Y)$ is completely normal. □

Theorem 5.8. *For a pseudocompact and almost P-space X, the space $C_r(X,Y)$ is perfectly normal Hausdorff.*

Proof. Since the space $C_r(X,Y)$ is metrizable if and only if X is pseudocompact and almost P-space. As we know that all metrizable spaces are perfectly normal Hausdorff. Hence, the proof. □

Corollary 5.2. *For a pseudocompact and almost P-space, the space $C_r(X,Y)$ is completely normal Hausdorff.*

Proof. All perfectly normal Hausdorff spaces are completely normal Hausdorff. □

Theorem 5.9. *For Tychonoff spaces X and Y, the space $C_r(X,Y)$ is regular Hausdorff and completely Hausdorff.*

Proof. Since the space $C_r(X,Y)$ is a Tychonoff space, so as every Tychonoff space is regular Hausdorff and completely Hausdorff. Which proves the theorem. □

Theorem 5.10. *For a space X and a metric space (Y,d), the following are equivalent:*

1. *Y is T_1 (respectively T_0);*
2. *$C_p(X,Y)$ is T_1 (respectively T_0);*
3. *$C_k(X,Y)$ is T_1 (respectively T_0);*
4. *$C_f(X,Y)$ is T_1 (respectively T_0);*
5. *$C_r(X,Y)$ is T_1 (respectively T_0).*

Proof. If Y is T_0, T_1, then Y^X with Tychonoff topology is T_0, T_1, respectively. Since $C_p(X,Y)$ is a subspace of Y^X is T_0, T_1, respectively. Moreover, $C_p(X,Y) \leq C_k(X,Y) \leq C_f(X,Y) \leq C_r(X,Y)$, then $C_k(X,Y)$, $C_f(X,Y)$ and $C_r(X,Y)$ are T_0, T_1, respectively.

(2) ⇒ (3) ⇒ (4) ⇒ (5) are immediate.

(5) ⇒ (1) Now, if $C_r(X,Y)$ is T_0 or T_1. Since $\phi\colon Y \to C_r(X,Y)$ is an embedding, and therefore Y can be treated as subspace. Consequently, Y is T_0, T_1, respectively. □

Theorem 5.11. *For a space X and a metric space (Y,d), the following are equivalent:*

1. *Y is T_2 (respectively $T_3, T_{3(1/2)}$);*
2. *$C_p(X,Y)$ is T_2 (respectively $T_3, T_{3(1/2)}$);*
3. *$C_k(X,Y)$ is T_2 (respectively $T_3, T_{3(1/2)}$);*
4. *$C_r(X,Y)$ is T_2 (respectively $T_3, T_{3(1/2)}$).*

Proof. If Y is T_2 (respectively, $T_3, T_{3(1/2)}$), then Y^X with Tychonoff topology is T_2 (respectively, $T_3, T_{3(1/2)}$). Since $C_p(X,Y)$ is a subspace of Y^X is T_2 (respectively, $T_3, T_{3(1/2)}$). Moreover, $C_p(X,Y) \leq C_k(X,Y) \leq C_r(X,Y)$, then $C_k(X,Y)$ and $C_r(X,Y)$ are T_2 (respectively, $T_3, T_{3(1/2)}$).

(2) ⇒ (3) is immediate.

(3) ⇒ (4) Suppose $C_k(X,Y)$ is T_2 (respectively, $T_3, T_{3(1/2)}$). Since $C_k(X,Y) \leq C_r(X,Y)$, then $C_r(X,Y)$ is T_2 (respectively, $T_3, T_{3(1/2)}$).

(4) ⇒ (1) Now, if $C_r(X,Y)$ is T_2 (respectively, $T_3, T_{3(1/2)}$). Since $\phi\colon Y \to C_r(X,Y)$ is an embedding, and therefore Y can be treated as subspace. Consequently, Y is T_2 (respectively, $T_3, T_{3(1/2)}$). □

References

[1] A. Jindal, V. Jindal, *The regular topology on* $C(X, Y)$, Acta Math. Hungar., 158 (2019), 1-16.

[2] A. Jindal, R. A. McCoy, S. Kundu, *The open-point and bi-point-open topologies on* $C(X)$*: submetrizability and cardinal functions*, Topol. Appl., 196 (2015), 229-240.

[3] C. E. Aull, *Some base axioms for topology involving enumerability, general topology and its relations to modern analysis and algebra,* III. (Proc. Conf., Kanpur-1968), Academia Prague, 1971.

[4] E. Hewitt, *Rings of real valued continuous functions i*, Trans. Am. Math. Soc., 64 (1948), 45-99

[5] F. Azarpanah, P. Paimann, A. R. Salehi, *Compactness, connectedness and countability properties of* $C(X)$ *with the* r*-topology*, Acta Math. Hungar., 146 (2015), 265-284.

[6] G. Gruenhage, *Generalized metric spaces-in handbook of set-theoretic topology*, Elsevier, North-Holland Amsterdam, 1984.

[7] I. Ntantu, R. A. McCoy, *Topological properties of spaces of continuous functions,* Lecture Notes Math., Springer, Verlag, Berlin, 1988.

[8] L. Hola, V. Jindal, *On graph and fine topology*, Topol. Proc., 45 (2016).

[9] M. Escardo, R. Heckmann, *Topologies on the spaces of continuous functions*, Topol. Proc., 2001, 545-564.

[10] R. Levy, *Almost p-spaces*, Can. Jour. Math., 2 (1977), 284-288.

[11] R. A. McCoy, S. Kundu, V. Jindal, *Function spaces with fine, graph and uniform topologies*, Springer, Gewerbrestrasse-11, Switzerland, 2018.

[12] S. Kundu, P. Garg, *The pseudocompact-open topology on* $C(X)$, Topolo. Proc., 30 (2006), 279-299.

[13] S. Willard, *General topology*, Addison-Wesley Publishing Co.,MA, London, Toronto, ON, 1970.

[14] V. Jindal, S. Kundu, *Topological and functional analytic properties of the compact* G_δ*-open topology on* $C(X)$, Topol. Appl., 174 (2014), 1-13.

[15] W. Iberkleid, R. L. Rodriguez, W. W. McGovern, *The regular topology on* $C(X)$, Comment. Math. Univ. Carolinae, 52 (2011), 445-461.

Accepted: June 10, 2022

Petrov-discontinuous Galerkin finite element method for solving diffusion-convection problems

Mohammed Waleed AbdulRidha
Department of Mathematics
College of Education for Pure Sciences
University of Basrah
Basrah
Iraq
mohammed.abdul_ridha@uobasrah.edu.iq

Hashim A. Kashkool
Department of Mathematics
College of Education for Pure Sciences
University of Basrah
Basrah
Iraq
hashim.kashkool@uobasrah.edu.iq

Ali Hasan Ali*
Department of Mathematics
College of Education for Pure Sciences
University of Basrah
Basrah
Iraq
and
College of Engineering Technology
National University of Science and Technology
Dhi Qar 64001
Iraq
and
Institute of Mathematics University of Debrecen
Pf. 400, H-4002 Debrecen
Hungary
ali.hasan@science.unideb.hu

Abstract. In this paper, we present a new modification of the discontinuous Galerkin Finite element method (DGFEM). The proposed modification is considered when the symmetric interior penalty Galerkin scheme involves only space variables by using the Petrov discontinuous Galerkin Finite element method (PDGFEM), while the time in the linear diffusion-convection problem remains continuous. We prove the properties of the bi-linear form (V-elliptic, continuity and stability), and we show that the error estimate is of second order with respect to the space. We also present some numerical

*. Corresponding author

experiments to validate the proposed method, and we simulate these peppermints to illustrate the theoretical results.
Keywords: linear diffusion-convection, Petrov-discontinuous, Galerkin finite element method, error estimate.

1. Introduction

We consider the problem mentioned in [1, 2, 3] of the diffusion-convection, $U \in Q_T \longrightarrow \mathbb{R}$, suth that $Q_T = \Omega \times (0,T)$:

$$U_t - \lambda \Delta U + \boldsymbol{b} \cdot \nabla U = f \quad \text{in} \quad Q_T, \tag{1.1}$$

$$U = U^D \quad \text{on} \quad \partial\Omega^D \times (0,T), \tag{1.2}$$

$$\lambda \frac{\partial U}{\partial n} = U^N \quad \text{on} \quad \partial\Omega^N \times (0,T), \tag{1.3}$$

$$U(x,0) = U^0(x), \quad \forall x \in \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ denotes a polygonal domain and $T > 0$.

Assume that $\partial\Omega = \partial\Omega^D \cup \partial\Omega^N$

$$\begin{array}{llll} \boldsymbol{b} \cdot n & \leq & 0 & on \quad \partial\Omega^D, \\ \boldsymbol{b} \cdot n & \geq & 0 & on \quad \partial\Omega^N, \quad ; \quad \forall t \in [0,T]. \end{array} \tag{1.5}$$

Here n is the unit outer normal to the boundary $\partial\Omega$ of Ω, the inflow boundary is $\partial\Omega^D$, and the outflow boundary is $\partial\Omega^N$.

Assumptions:

a) $U_t \in L^2(Q_T)$, $U, U^0 \in L^2(\Omega)$,

b)$f \in C([0,T]; L^2(\Omega))$,

c) U^D is the trace of some $U \in C([0,T]; H^1(\Omega)) \cap L^\infty(Q_T)$ on $\partial\Omega^D \times (0,T)$,

d) $U^N \in C([0,T]; L^2(\partial\Omega^N))$,

e) $|K| =$ is the area of $K \in T_h$,

f) $\sigma = \frac{\sigma^0}{|E|^{\beta_0}}$, $\beta_0 \geq (d-1)^{-1}$, $\sigma^0 > 0$.

This problem consists mainly of two components: the terms of diffusion with the coefficient of diffusion and the terms of convection with the field of convection velocity. When using the Galerkin Finite Element Method (GFEM) to solve one-sided turbulent convection problems, the approximate solutions show pseudo-oscillation, i.e. $\forall h > 0$, $\frac{\lambda}{|b|h} << 1$, this condition can occur as any combination of weak diffusion (small), strong convection (large), alternatively, as a result of a large domain, the last case accurate frequently in geophysical applications. Several approaches have been intensively researched to eliminate such a downside adding stabilization terms to the problem's formulation is a common concept. This is done predominantly by stabilized processes such as upwinding methods [4, 5], Petrov-Galerkin approach [6, 7], nonlinear diffusivity method [8, 9, 10], Weak Galerkin method [11, 12] and oscillation theory [13, 14, 15].

Researchers devised a new approach to address these problems in the 1970s called the Discontinues Galerkin finite element method (DGFEM). Without having any consistency criteria, the DGFEM approach approximates the approximate limits of the ideal grid solution on finite elements. The DGFEM utilizes the same function space as the finite volume method (FVM) and continuous finite element method (FEM), but also with relaxed continuity at inter-element borders, and may be thought of as a hybrid of the two. The convection component dominates over diffusion when $\lambda < h$, where h is mesh size, and the usual Galerkin finite element technique generates an oscillating solution that is not near to the exact solution ([16]). The PDGFEM is an improvement and provident of DGFEM.In DGFEM, the shape function and trail function are in the same field, but in PDGFEM, the test function space differs from the trial function space. In this paper, we shall show and analyze the PDGFEM in the case of the SIPG for the linear diffusion-convection problem.$V-$elliptic,continuity, stability, and convergence were demonstrated in thesemi-discrete PDGFEM. We found the L^2-error and H^1-error of PDGFEM and DGFEM for solving a linear diffusion-convection problem to discuss the approximation between the L^2-error and the order of error. The following is how this paper is structured. In the section 2, we have shown the discretization. The variation formulation of PDGFEM and the semi-discrete PDGFEM are presented in the section 3. In the section 4, we proved the properties of the bilinear form and stability. The error estimate is presented in the section 5. In the section 6, we showed numerical results to confirm the theoretical results. Finally, the conclusions are shown in the section 6.

2. The discretization

Let T_h $(h > 0)$ represent a limited number of closed triangles with mutually disjoint interiors divided by $\overline{\Omega}$ (the domain closure Ω).A triangulation of Ω is what we'll call T_h . The conforming qualities of T_h that are employed in the FEM are denoted by T_h. That suggests that we recall what are known as "hanging nodes". Neighbors are two elements $K^i, K^j \in \mathrm{T}_h$ that share a non-empty open portion of their sides. If we provide $\partial K^1 \cap \partial K^2$ to has $(d-1)$ a positive dimensional measure, suppose that $E \in K$ is the edge of K if it is a maximum connected open subset either of $K^1 \cap K^2$, where K^1 is a neighbor of K^2or a subset of $\partial K \cap \partial\Omega$.The term $\partial\mathrm{T}_h$ refers to the system of all sides of all elements $K \in \mathrm{T}_h$. In addition, all inner and border edges are specified in [17] by

$$\begin{aligned}
\partial\mathrm{T}_h^I &= \{E \subset \Omega, E \in \partial\mathrm{T}_h\}, \\
\partial\mathrm{T}_h^B &= \{E \subset \partial\Omega, E \in \partial\mathrm{T}_h\}, \\
\Gamma^D &= \{E \subset \partial\Omega^D, E \in \partial\mathrm{T}_h^B\}, \\
\Gamma^N &= \{E \subset \partial\Omega^N, E \in \partial\mathrm{T}_h^B\}.
\end{aligned}$$

Obviously $\partial \mathrm{T}_h = \partial \mathrm{T}_h^I \cup \partial \mathrm{T}_h^B$ for $\varphi \in H^1(\Omega, \mathrm{T}_h)$, $\partial \mathrm{T}_h^B = \Gamma^D \cup \Gamma^N$ for each $E \in \partial \mathrm{T}_h$.

Each edge $E \in K$ has elements on both sides, and they are called outside and inside elements, respectively, with arbitrary constants. The assessment of a function v in the inside of E is defined as $\forall x \in E; v^-(x) = v(x)|_{inside}$ where $v^-(x) = \lim_{\epsilon \longrightarrow 0}(x - \epsilon); \epsilon > 0$, and the external or the outside elements are defined as $\forall x \in E; v^+(x) = v(x)|_{outside}$ where $v^+(x) = \lim_{\epsilon \longrightarrow 0}(x + \epsilon); \epsilon > 0$.

On the side E, the function v is discontinuous. The discontinuity size must be quantified. Let us define $[v](x) = -(v^-(x) - v^+(x))$ as the function v jumping on the side E for each $x \in E$. On the discontinuity side E, a function v is undefined, and the average v is used to close this gap in the definition. For each $x \in E$, let it be $v(x) = \frac{(v^+(x)+v^-(x))}{2}$, defined as the average of function v on side E.

2.1 Broken Sobolev spaces

Discontinuous approximations are used in the DGFEM. This is why, for each r∈ℕ, the so-called broken Sobolev space is defined over triangulation T_h:

$$H^r(\Omega, \mathrm{T}_h) = \{\forall K \in \mathrm{T}_h; v \in L^2(\Omega); v|_K \in H^r(K)\}.$$

The norm of $v \in H^r(\Omega, \mathrm{T}_h)$ is defined

$$||U||_{H^r(\Omega,\mathrm{T}_h)} = (\sum_{K \in \mathrm{T}_h} ||U||^2_{H^r(\Omega)})^{1/2},$$

and semi-norm $|U|_{H^r(\Omega,\mathrm{T}_h)} = (\sum_{K\in\mathrm{T}_h} |U|^2_{H^r(\Omega)})^{1/2}$. Assume that $l \geq 0$ is a positive integer. Piecewise polynomial functions with discontinuous coefficients have a space represented by

$$S_h = \{\forall K \in \mathrm{T}_h; v \in L^2(\Omega); v|_K \in P_l(K)\},$$

where P_l (K) represents the space occupied by all degree $\leq$ polynomials on K. The number$l$ represents the degree of polynomial approximation ([18]). Obviously, $\mathrm{S_h \subset H^r (\Omega, T_h)}$.

Let ϑ be trial space and $\emptyset$ be a test space

$$\begin{aligned} \vartheta &= H^r(\Omega, \mathrm{T}_h), \\ \emptyset &= \{w : w = v + \delta \boldsymbol{b} \cdot \nabla v; v \in \vartheta\}, \end{aligned}$$

and dim ϑ = dim $\emptyset$.

We defined PDGFE space

$$\begin{aligned} \vartheta_h &= S_h, \\ \emptyset_h &= \{w : w = v + \delta \boldsymbol{b} \cdot \nabla v; v \in \vartheta_h\}, \end{aligned}$$

where δ denotes a constant stability parameter in Q_T. It can be selected as [19],

$$\delta \equiv \begin{cases} \eta h, & \text{if} \quad \lambda < h \\ 0, & \text{if} \quad \lambda \geq h \end{cases}; 0 < \eta < \frac{1}{4}.$$

3. The variation formulation of PDGFEM

By multiplying equation (1.1) by the test function w, we can get $U \in \vartheta$ in the SIPG form of the PDGFEM approximation:

$$\begin{aligned}
&(U_t, w) + \sum_{K\in \mathrm{T}_h} \lambda(\nabla U, \nabla w)_K - \sum_{E\in\partial \mathrm{T}_h} \int (\{\lambda \nabla U \cdot n\}[w] - \varepsilon[U]\{\lambda\nabla w \cdot n\})ds \\
&\quad + \sum_{E\in\partial \mathrm{T}_h} \int (\{|\boldsymbol{b}\cdot n|U\}[w])ds + \sigma \sum_{E\in\partial \mathrm{T}_h} \int [U][w]ds - \sum_{K\in \mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U, w)_K \\
&\quad = (f,w) + \sum_{E\in\Gamma^N} \int U^N w ds + \sum_{E\in\Gamma^D} \int \lambda\nabla w \cdot n U^D ds - \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n| U^D w ds \\
&\quad - \sigma \sum_{E\in\Gamma^D} \int U^D w ds; \qquad \forall w \in \emptyset.
\end{aligned}$$

Since $\varepsilon = -1$(SIPG) ([1]) and $w = v + \delta\boldsymbol{b}\cdot\nabla v$ then

$$\begin{aligned}
&(U_t, v+\delta\boldsymbol{b}\cdot\nabla v) + \sum_{K\in \mathrm{T}_h} \lambda(\nabla U, \nabla(v + \delta\boldsymbol{b}\cdot\nabla v))_K \\
&- \sum_{E\in\partial \mathrm{T}_h} \int (\{\lambda\nabla U\cdot n\}[v+\delta\boldsymbol{b}\cdot\nabla v] \\
&+ \sum_{E\in\partial \mathrm{T}_h} \int (\{|\boldsymbol{b}\cdot n|U\}[v+\delta\boldsymbol{b}\cdot\nabla v])ds + [U]\{\lambda\nabla(v+\delta\boldsymbol{b}\cdot\nabla v)\cdot n\})ds \\
&- \sum_{K\in \mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U, v+\delta\boldsymbol{b}\cdot\nabla v)_K + \sigma \sum_{E\in\partial \mathrm{T}_h} \int [U][v+\delta\boldsymbol{b}\cdot\nabla v]ds \\
&= (f, v+\delta\boldsymbol{b}\cdot\nabla v) + \sum_{E\in\Gamma^N} \int U^N(v+\delta\boldsymbol{b}\cdot\nabla v)ds \\
&+ \sum_{E\in\Gamma^D} \int \lambda\nabla(v+\delta\boldsymbol{b}\cdot\nabla v)\cdot n U^D ds - \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n|U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds \\
&- \sigma \sum_{E\in\Gamma^D} \int U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds, \qquad \forall v \in \vartheta.
\end{aligned} \tag{3.1}$$

The variation formulation of PDGFEM is find $U \in \vartheta \ni$

$$\begin{aligned}
(U_t, v) + (U_t, \delta\boldsymbol{b}\cdot\nabla v) + a_{PD}(U,v) &= (f, \delta\boldsymbol{b}\cdot\nabla v) + (f, v) \\
&- \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n|U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds \\
&+ \sum_{E\in\Gamma^N} \int U^N(v+\delta\boldsymbol{b}\cdot\nabla v)ds - \sigma \sum_{E\in\Gamma^D} \int U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds \\
&+ \sum_{E\in\Gamma^D} \int \lambda\nabla(v+\delta\boldsymbol{b}\cdot\nabla v)\cdot n U^D, \quad \forall v \in \vartheta,
\end{aligned} \tag{3.2}$$

where

$$a_{PD}(U,v) = \sum_{K\in \mathrm{T}_h} \lambda(\nabla U, \nabla v)_K - \sum_{E\in\partial \mathrm{T}_h} \int ([U]\{\lambda\nabla v\cdot n\} + \{\lambda \nabla U\cdot n\}[v])ds$$
$$- \sum_{K\in \mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U, v+\delta\boldsymbol{b}\cdot\nabla v)_K + \sum_{E\in\partial \mathrm{T}_h} \int (\{|\boldsymbol{b}\cdot n|U\}[v])ds$$
$$+\sigma \sum_{E\in\partial \mathrm{T}_h} \int [U][v]ds. \tag{3.3}$$

3.1 The semi-discrete PDGFEM

The semi-discrete solution: find $U_h \in \vartheta_h, \forall v\in \vartheta_h$, such that:

$$(U_{h,t}, v) + a_{PD}\,(U_h, v) + (U_{h,t}, \delta\boldsymbol{b}\cdot\nabla v) = (f,v) + (f, \delta\boldsymbol{b}\cdot\nabla v)$$
$$+ \sum_{E\in\Gamma^N} \int U^N(v+\delta\boldsymbol{b}\cdot\nabla v)ds + \sum_{E\in\Gamma^D} \int \lambda\nabla(v+\delta\boldsymbol{b}\cdot\nabla v)\cdot nU^D$$
$$- \sum_{E\in\Gamma^D} \int |\boldsymbol{b}\cdot n|U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds - \sigma \sum_{E\in\Gamma^D} \int U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds, \tag{3.4}$$

where

$$a_{PD}(U_h,v) = \sum_{K\in \mathrm{T}_h} \lambda(\nabla U_h, \nabla v)_K - \sum_{E\in\partial \mathrm{T}_h} \int (\{\lambda\nabla U_h\cdot n\}[v] + [U_h]\{\lambda \nabla v\cdot n\})ds$$
$$+ \sum_{E\in\partial \mathrm{T}_h} \int (\{|b\cdot n|U_h\}[v])ds - \sum_{K\in \mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U_h, v+\delta\boldsymbol{b}\cdot\nabla v)_K$$
$$+\sigma \sum_{E\in\partial \mathrm{T}_h} \int [U_h][v]ds. \tag{3.5}$$

4. The properties of $a_{PD}(U,v)$aPD and stability

In this section, we prove some impotent lemmas for the bilinear form ($V-$elliptic, continuous) and stability.

Lemma 4.1 ($V-$elliptic)**.** *Assume the penalty σ is large enough, and there is a positive constant α independent of h, $\beta_0 \geq (d-1)^{-1}$ such that*

$$a_{PD}(U,U) \geq \ \alpha\ \|U\|^2_{H^1(\mathrm{T}_h)}, \tag{4.1}$$

where

$$\|U\|_{H^1(\mathrm{T}_h)} = \left(\sum_{K\in \mathrm{T}_h} \left\|\lambda^{\frac{1}{2}}\nabla U\right\|^2_{L^2(K)} + \left(\sum_{E\in\partial \mathrm{T}_h} \int \sigma^{-1}(\{\lambda\nabla U\cdot n\})^2 ds \right)^{\frac{1}{2}} \right)^2$$

$$+\sum_{E\in\partial \mathrm{T}_h}\sigma\|[U]\|^2_{L^2(E)}+\|U\|^2_{H^1(\ \mathrm{T}_h)}+\sum_{K\in\mathrm{T}_h}\|\boldsymbol{b}\cdot\nabla U\|^2_{L^2(\ \mathrm{K})}$$

$$+\left(\left(\left(\sum_{E\in\partial \mathrm{T}_h}\int\sigma^1[U]^2ds\right)^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}.$$

Proof. In the equation (3.3), put $v=U$

$$\begin{aligned}a_{PD}(U,U) &= \sum_{K\in\mathrm{T}_h}\lambda(\nabla U,\nabla U)_K-\sum_{E\in\partial\mathrm{T}_h}\int(\{\lambda\nabla U\cdot n\}[U]\\ &+ \sum_{E\in\partial\mathrm{T}_h}\int(\{|\boldsymbol{b}\cdot n|U\}[U])ds+[U]\{\lambda\nabla U\cdot n\})ds\\ &+ \sigma\sum_{E\in\partial\mathrm{T}_h}\int[U][U]ds-\sum_{K\in\mathrm{T}_h}(\boldsymbol{b}\cdot\nabla U,U+\delta\boldsymbol{b}\cdot\nabla U)_K+.\end{aligned}\tag{4.2}$$

From [1]

$$a_{PD}(U,U)=\sum_{K\in\mathrm{T}_h}\left\|\lambda^{\frac{1}{2}}\nabla U\right\|^2_{L^2(K)}+\frac{\beta}{2}\left(\left(\sum_{E\in\partial\mathrm{T}_h}\int\sigma^{-1}(\{\lambda\nabla U\cdot n\})^2ds\right)^{\frac{1}{2}}\right)^2$$

$$+\frac{2}{\beta}\left(\left(\sum_{E\in\partial T_h}\int\sigma^1[U]^2ds\right)^{\frac{1}{2}}\right)^2+\varrho\|U\|^2_{H^1(T_h)}+\sigma^2G_t\|U\|^2_{H^1(\ \mathrm{T}_h)}$$

$$+\frac{\beta}{2}\sum_{E\in\partial T_h}\sigma\|[U]\|^2_{L^2(E)}+\frac{\omega^2}{2\beta}\|U\|^2_{H^1(\mathrm{T}_h)}+\sum_{K\in\mathrm{T}_h}\delta\|\boldsymbol{b}\cdot\nabla U\|^2_{L^2(\mathrm{K})},$$

$$a_{PD}(U,U)\geq g\left(\sum_{K\in\mathrm{T}_h}\left\|\lambda^{\frac{1}{2}}\nabla U\right\|^2_{L^2(K)}+\left(\left(\sum_{E\in\partial\mathrm{T}_h}\int\sigma^{-1}(\{\lambda\nabla U\cdot n\})^2ds\right)^{\frac{1}{2}}\right)^2\right.$$

$$+\sum_{E\in\partial T_h}\sigma\|[U]\|^2_{L^2(E)}+\|U\|^2_{H^1(\mathrm{T}_h)}+\sum_{K\in\mathbb{T}_h}\|\boldsymbol{b}\cdot\nabla U\|^2_{L^2(\ \mathrm{K})}$$

$$\left.+\left(\left(\sum_{E\in\partial T_h}\int\sigma^1[U]^2ds\right)^{\frac{1}{2}}\right)^2\right)+\varrho\|U\|^2_{H^1(\mathrm{T}_h)}+\sigma^2G_t\|U\|^2_{H^1(\mathrm{T}_h)},$$

where $g=\min(\frac{\beta}{2},\frac{\omega^2}{2\beta},\ 1,\frac{2}{\beta},\delta)$,

$$a_{PD}(U,U)\geq g\|U\|^2_{H^{1'}(\mathrm{T}_h)}+q\|U\|^2_{H^{1'}(\mathrm{T}_h)}$$

then

$$a_{PD}(U,U) \geq \alpha \ \|U\|^2_{H^1(\mathrm{T}_h)},$$

where $q \leq \left(\varrho + \sigma^2 G_t\right)$, and $\alpha \leq (g+q)$. □

Lemma 4.2 (countinuity)**.** *If U is the solution of equation (3.2), and $v \in \vartheta$ is the test function, then $a_{PD}(U,v)$ is continuous if κ is nonnegative,such that:*

$$\|a_{PD}(U,v)\| \leq \kappa \|U\|_{H^1(\mathrm{T}_h)} \ |v\|_{H^1(\mathrm{T}_h)} \ , \qquad \forall \ U, v \in \ \vartheta.$$

Proof. From the equation(3.3) we have

$$\begin{aligned}|a_{PD}(U,v)| =| \ \sigma \sum_{E\in\partial\mathrm{T}_h} \int [U][v]ds - \sum_{E\in\partial\mathrm{T}_h} \int (\{\lambda\nabla U\cdot n\}[v] + [U]\{\lambda\nabla v\cdot n\})ds \\ - \sum_{K\in\mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U, v+\delta\boldsymbol{b}\cdot\nabla v)_K + \sum_{E\in\partial\mathrm{T}_h} \int (\{|\boldsymbol{b}\cdot n|U\}[v])ds \\ + \sum_{K\in\mathrm{T}_h} \lambda(\nabla U,\nabla v)_K \ |,\end{aligned}$$

$$\begin{aligned}|a_{PD}(U,v)| \leq \sigma \sum_{E\in\partial\mathrm{T}_h} \int |[U][v]|ds - \sum_{E\in\partial\mathrm{T}_h} \int |[U]\{\lambda\nabla v\cdot n\} + \{\lambda\nabla U\cdot n\}[v]|ds \\ + \sum_{E\in\partial\mathrm{T}_h} \int |(\{|\boldsymbol{b}\cdot n|U\}[v])|ds - \sum_{K\in\mathrm{T}_h} |(\boldsymbol{b}\cdot\nabla U, v+\delta\boldsymbol{b}\cdot\nabla v)_K| \\ + \sum_{K\in\mathrm{T}_h} |\lambda(\nabla U,\nabla v)_K| = \sum_{i=1}^{6} B_i. \qquad (4.3)\end{aligned}$$

From [1], we get

$$\begin{aligned}|a(U,v)| \leq \varsigma ||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} + 2|\lambda|\sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} \\ + \sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} + \sigma^2 G_t^2||U||_{L^2(\mathrm{T}_h)}||v||_{L^2(\mathrm{T}_h)}. \qquad (4.4)\end{aligned}$$

To estimate B_5

$$\begin{aligned}B_5 = \sum_{K\in\mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U, \delta\boldsymbol{b}\cdot\nabla v)_K \leq \sum_{K\in\mathrm{T}_h} |\delta|_{L_\infty}|\boldsymbol{b}\cdot\nabla U|_{L^2(K)}|\boldsymbol{b}\cdot\nabla v|_{L^2(K)} \\ \leq \sum_{K\in\mathrm{T}_h} |\delta|_{L_\infty} \left|\boldsymbol{b}^2\right|_{L^\infty} |\nabla U|_{L^2(K)}|\nabla v|_{L^2(K)} = \Lambda \sum_{K\in\mathrm{T}_h} \|U\|_{H^1(K)}\|v\|_{H^1(K)} \\ = \Lambda\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)}, \qquad (4.5)\end{aligned}$$

where $\Lambda = |\delta|_{L_\infty}\left|\boldsymbol{b}^2\right|_{L^\infty}$.

Substituting (4.4) and (4.5) in(4.3) we get,

$$\begin{aligned}\|a_{PD}(U,v)\| &\leq \Lambda\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)} + |a(U,v)| = \varsigma||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} \\ &+ 2|\lambda|\sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} + \sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} \\ &+ \sigma^2 G_t^2||U||_{L^2(\mathrm{T}_h)}||v||_{L^2(\mathrm{T}_h)} + \Lambda\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)} \\ &= (\varsigma + 2|\lambda|\sigma G_t^2 + \sigma G_t^2 + \sigma^2 G_t^2 + \Lambda)\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)} \\ &\leq \kappa\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)}, \end{aligned} \tag{4.6}$$

where $\kappa \geq (\varsigma + 2\,|\lambda|\,\sigma {G_t}^2 + \sigma {G_t}^2 + \sigma^2 {G_t}^2 + \Lambda)$. □

Lemma 4.3 (stability)**.** *There are a set of variables* ξ, A, $\varpi > 0$ *that are independent of* h *and are as follows:*

$$\|U_h(t)\|^2_{L^2(\Omega)} + \Upsilon\,\|U_{h,t}\|^2_{L^2(0,T;L^2(\Omega))} + \xi\,\|U_h\|^2_{L^2(0,T;H^1(T_h))} \leq \varpi\left(\|f\|^2_{L^2(0,T;L^2(\mathrm{T}_h))}\right.$$
$$\left. + \left\|U_h(0)\|^2_{L^2(\Omega)}\right) + \varpi\sum_{E\in\partial\mathrm{T}_h}\left(\|U_N\|^2_{L^2(0,T;L^2(\Gamma^N))} + \|U_D\|^2_{L^2(0,T;L^2(\Gamma^D))}\right).$$

Proof. Let $v = U_h$ in equation (3.4), we obtain

$$\begin{aligned}&(U_{h,t},U_h) + (U_{h,t},\delta\boldsymbol{b}\cdot\nabla U_h) + a_{PD}\left(U_h,U_h\right) = \left(f,U_h\right) + \left(f,\delta\boldsymbol{b}\cdot\nabla U_h\right) \\ &+ \sum_{E\in\Gamma^N}\int U^N\left(U_h + \delta\boldsymbol{b}\cdot\nabla U_h\right)ds + \sum_{E\in\Gamma^D}\int \lambda\nabla\left(U_h + \delta\boldsymbol{b}\cdot\nabla U_h\right)\cdot nU^D \\ &- \sum_{E\in\Gamma^D}\int|\boldsymbol{b}\cdot n|U^D(U_h+\delta\boldsymbol{b}\cdot\nabla U_h)ds - \sigma\sum_{E\in\Gamma^D}\int U^D(U_h+\delta\boldsymbol{b}\cdot\nabla U_h)ds.\end{aligned} \tag{4.7}$$

From Lemma (4.1), we have

$$(U_{h,t},U_h) + a_{PD}\left(U_h,U_h\right) \geq \frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)} + \alpha\,\|U_h\|^2_{H^1(\mathrm{T}_h)}. \tag{4.8}$$

By Young's-inequality and Cauchy [18], we get

$$\begin{aligned}(U_{h,t},\delta\boldsymbol{b}\cdot\nabla U_h) &\leq \|U_{h,t}\|_{L^2(\Omega)}\,\|\delta\boldsymbol{b}\cdot\nabla U_h\|_{L^2(\;\mathrm{T}_h)} \\ &\leq \frac{\beta}{2}\,\|U_{h,t}\|^2_{L^2(\Omega)} + \frac{1}{2\beta}\delta\,\|\boldsymbol{b}\cdot\nabla U_h\|^2_{L^2(\mathrm{T}_h)} \\ &\leq \Upsilon\left(\|U_h\|^2_{H^1(\mathrm{T}_h)} + \|U_{h,t}\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{4.9}$$

By the Young's-inequality and using Cauchy inequality of equation (4.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)} + \Upsilon\left(\|U_{h,t}\|^2_{L^2(\Omega)} + \|U_h\|^2_{H^1(\mathrm{T}_h)}\right) + \alpha\|U_h\|^2_{H^1(\mathrm{T}_h)}\\
&\le C\left(\|f\|^2_{L^2(\mathrm{T}_h)} + \|U_h\|_{L^2(\mathrm{T}_h)}\right) + \Upsilon\left(\|f\|^2_{L^2(\mathrm{T}_h)} + \|U_h\|^2_{H^1(\mathrm{T}_h)}\right)\\
&+ 2C\sum_{E\in\Gamma^D}\left(\|U_h\|^2_{H^1(K)} + \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right) + 2C\sum_{E\in\Gamma^N}\left(\|U_h\|^2_{H^1(K)} + \left\|U^N\right\|^2_{L^2(\Gamma^N)}\right)\\
&+ 2C\sum_{E\in\Gamma^D}\left(\|U_h\|^2_{H^1(K)} + \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right) + 2C\sum_{E\in\Gamma^D}\left(\|U_h\|^2_{H^1(K)} + \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)} + \Upsilon\|U_{h,t}\|^2_{L^2(\Omega)} + (\alpha-9C)\|U_h\|^2_{H^1(\mathrm{T}_h)} \le (C+\Upsilon)\|f\|^2_{L^2(\mathrm{T}_h)}\\
&+ 2C\sum_{E\in\partial T_h}(3\|U^D\|^2_{L^2(\Gamma^D)} + \|U^N\|^2_{L^2(\Gamma^N)}).
\end{aligned} \tag{4.10}$$

By integrating the equation (4.10) both sides from 0 to t, we get,

$$\begin{aligned}
&\|U_h(t)\|^2_{L^2(\Omega)|} - \|U_h(0)\|^2_{L^2(\Omega)} + \Upsilon\int_0^t \|U_{h,t}\|^2_{L^2(\Omega)} + \xi\int_0^t \|U_h\|^2_{H^1(\mathrm{T}_h)}\\
&\le \mathrm{A}\int_0^t \|f\|^2_{L^2(\mathrm{T}_h)} + 2C\sum_{E\in\partial\mathrm{T}_h}\left(\int_0^t \left\|U^N\right\|^2_{L^2(\Gamma^N)} + 3\int_0^t \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right),
\end{aligned}$$

where $\xi \le (\alpha - 9C)$ and $\mathrm{A} = (C+\Upsilon)$, we obtain

$$\begin{aligned}
&\|U_h(t)\|^2_{L^2(\Omega)} + \Upsilon\|U_{h,t}\Big\|^2_{L^2(0,T;L^2(\Omega))} + \xi\Big\| U_h \Big\|^2_{L^2(0,T;H^1(\mathrm{T}_h))}\\
&\le A\| f\|^2_{L^2(0,T;L^2(\mathrm{T}_h))} + \|U_h(0)\|^2_{L^2(\Omega)}\\
&+ 2C\sum_{E\in\partial\mathrm{T}_h}\left(\left\|U^N\right\|^2_{L^2(0,T;L^2(\Gamma^N))} + 3\left\|U^D\right\|^2_{L^2(0,T;L^2(\Gamma^D))}\right),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\|U_h(t)\|^2_{L^2(\Omega)} + \Upsilon\|U_{h,t}\|^2_{L^2(0,T;L^2(\Omega))} + \xi\|U_h\|^2_{L^2(0,T;H^1(\mathrm{T}_h))}\\
&\le \varpi\left(\|f\|^2_{L^2(0,T;L^2(\mathrm{T}_h))} + \|U_h(0)\|^2_{L^2(\Omega)}\right)\\
&+ \varpi\sum_{E\in\partial\mathrm{T}_h}\left(\left\|U^N\right\|^2_{L^2(0,T;L^2(\Gamma^N))} + \left\|U^D\right\|^2_{L^2(0,T;L^2(\Gamma^D))}\right),
\end{aligned} \tag{4.12}$$

where $\varpi \ge 6C$. □

5. The error estimate

This section shows the semi-discrete PDGFEM error estimates in the SIPG case. The L^2-error will be used to estimate the $U - U_h$ error.

Theorem 5.1. *Let U represent the solution of equation (3.2), $U_h \in \vartheta_h$ represent the approximate solution of equation (3.4) and $U \in L^2(H^1(\Omega))$, $U_t \in L^2\left(0,\ T;\ H^1(\Omega)\right)$ and σ is large enough, then C is a constant such that:*

$$\|U-U_h\|_{L^2(\Omega)} \leq \mathrm{c}h^2\|U\|_{L^2(H^1)} + \sqrt{\frac{\beta}{2}}\mathrm{c}h^2\left(\|U_t\|_{L^2(0,T;H^1)} + \|U\|_{L^2(0,T;H^1)}\right).$$

Proof. Let ΠU be the interpolate of U, and $e = U - U_h = (U - \Pi U) + (\Pi U - U_h) = \Theta - \Xi$,So

$$\|U-U_h\|_{L^2(\Omega)} \leq \quad \|\Theta\|_{L^2(\Omega)} + \|\Xi\|_{L^2(\Omega)}. \tag{5.1}$$

From [3]

$$\|\Theta\|_{L^2(\Omega)} \leq ch^2\|U\|_{L^2(H^1)}. \tag{5.2}$$

Now,

$$\begin{aligned}(U_t, w) + a_{PD}(U, w) = (f, w) + \sum_{E\in\Gamma^N}\int U^N w ds + \sum_{E\in\Gamma^D}\int \lambda\nabla w\cdot nU^D ds\\ -\sum_{E\in\Gamma^D}\int |\boldsymbol{b}\cdot n|U^D w ds - \sigma\sum_{E\in\Gamma^D}\int U^D w ds,\ \ \forall w\in\emptyset,\end{aligned} \tag{5.3}$$

$$\begin{aligned}(U_{h,t}, w) + a_{PD}(U_h, w) = (f, w) + \sum_{E\in\Gamma^N}\int U^N w ds + \sum_{E\in\Gamma^D}\int \lambda\nabla w\cdot nU^D ds\\ -\sum_{E\in\Gamma^D}\int |\boldsymbol{b}\cdot n|U^D w ds - \sigma\sum_{E\in\Gamma^D}\int U^D w ds,\ \ \forall w\in\emptyset_h.\end{aligned} \tag{5.4}$$

Subtracting (5.3) from (5.4), we obtain,

$$\begin{aligned}&((U-U_h)_t\ , w) + a_{PD}(U-U_h, w) = ((\Theta-\Xi)_t\ , w)\\ &+ a_{PD}(\Theta-\Xi,\ w) = 0,\ \forall w\in\emptyset_h.\end{aligned} \tag{5.5}$$

Then

$$(\Theta_t, w) + a_{PD}(\Theta, w) = (\Xi_t, w) + a_{PD}(\Xi, w). \tag{5.6}$$

Let $w = \Xi$, we have,

$$(\Theta_t,\ \Xi) + a_{PD}(\Theta,\ \Xi) = (\Xi_t,\ \Xi) + a_{PD}(\Xi,\ \Xi)\ \ . \tag{5.7}$$

From Lemma 4.1, we have,

$$(\Xi_t,\ \Xi) + a_{PD}(\Xi,\ \Xi) \geq \frac{1}{2}\frac{d}{dt}\parallel\Xi\|^2_{L^2(\Omega)} + \alpha\ \|\Xi\|^2_{L^2(\mathrm{T}_h)}. \tag{5.8}$$

By Young inequality and Schwartz [18], we have,

$$(\Theta_t,\ \Xi) \leq \frac{\beta}{2} c^2 h^4 U_t{}^2_{L^2(H^1)} + \frac{1}{2\beta} \|\Xi\|^2_{L^2(\mathrm{T}_h)}. \tag{5.9}$$

From Lemma 4.2, we obtain,

$$\begin{aligned} a_{PD}(\Theta, \Xi) &\leq \kappa \|\Theta\|_{L^2(\mathrm{T}_h)} \|\Xi\|_{L^2(\mathrm{T}_h)} \\ &\leq \frac{\beta}{2} \|\Theta\|_{L^2(\mathrm{T}_h)} + \frac{\kappa^2}{2\beta} \|\Xi\|_{L^2(\ \mathrm{T}_h)} \\ &\leq \frac{\beta}{2} c^2 h^4 \|U\|^2_{L^2(H^1)} + \frac{\kappa^2}{2\beta} \|\Xi\|_{L^2(\mathrm{T}_h)}. \end{aligned} \tag{5.10}$$

Substituting (5.8),(5.9) and (5.10) in (5.7), we have,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Xi\|^2_{L^2(\Omega)} + \alpha\|\Xi\|^2_{L^2(\mathrm{T}_h)} \leq \frac{\beta}{2} c^2 h^4 \|U_t\|^2_{L^2(H^1)} + \frac{1}{2\beta}\|\Xi\|^2_{L^2(\mathrm{T}_h)} \\ + \frac{\beta}{2} c^2 h^4 \|U\|^2_{L^2(H^1)} + \frac{\kappa^2}{2\beta}\|\Xi\|_{L^2(\mathrm{T}_h)}. \end{aligned} \tag{5.11}$$

Then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \parallel \Xi\|^2_{L^2(\Omega)} + \left(\alpha - \frac{1}{2\beta} - \frac{\kappa^2}{2\beta}\right) \|\Xi\|^2_{L^2(\mathrm{T}_h)} \\ &\leq \frac{\beta}{2} c^2 h^4 \left(||U_t||^2_{L^2(H^1)} + \parallel U\|^2_{L^2(H^1)} \right). \end{aligned} \tag{5.12}$$

Then

$$\frac{1}{2}\frac{d}{dt} \parallel \Xi\|^2_{L^2(\Omega)} + C_1 \|\Xi\|^2_{L^2(\mathrm{T}_h)} \leq \frac{\beta}{2} c^2 h^4 \left(||U_t||^2_{L^2(H^1)} + \parallel U\|^2_{L^2(H^1)} \right), \tag{5.13}$$

where $C_1 \leq (\alpha - \frac{1}{2\beta} - \frac{\kappa^2}{2\beta})$.

We can get the following result by integrating two sides of the equation (5.13) from 0 to t:

$$\parallel \Xi\,(t) \,\|^2_{L^2(\Omega)} - \parallel \Xi^0 \,\|^2_{L^2(\Omega)} \leq \frac{\beta}{2} c^2 h^4 \int_0^t \left(\parallel U_t \,\|^2_{L^2(H^1)} + \parallel U\|^2_{L^2(H^1)} \right). \tag{5.14}$$

Since $\Xi^0 = 0$, then

$$\|\Xi\,(t)\,\|^2{}_{L^2(\Omega)} \leq \frac{\beta}{2} c^2 h^4 \left(\|U_t\|^2{}_{L^2(0,T;H^1)} + \|U\|^2{}_{L^2(0,T;H^1)} \right). \tag{5.15}$$

Then

$$\parallel \Xi\|_{L^2(\Omega)} \leq \sqrt{\frac{\beta}{2}} c h^2 \left(\|U_t\|_{L^2(0,T;H^1)} + \|U\|_{L^2(0,T;H^1)} \right). \tag{5.16}$$

Substituting equations (5.2) and (5.16) in (5.1), we have,

$$\|U - U_h\|_{L^2(\Omega)} \leq ch^2 \|U\|_{L^2(H^1)} + \sqrt{\frac{\beta}{2}} ch^2 \left(\|U\|_{L^2(0,T;H^1)} + \|U_t\|_{L^2(0,T;H^1)} \right). \tag{5.17}$$

Then

$$\|U - U_h\|_{L^2(\Omega)} \leq Ch^2 \left(||U||_{L^2(H^1)} + (||U_t||_{L^2(0,\ T;\ H^1(\Omega))} + ||U||_{L^2(0,T;H^1(K)}) \right).$$

where $C \geq c + c\sqrt{\frac{\beta}{2}}$. □

6. Numerical results

In this section, we find the error $U - U_h$ of L^2−error and H^1−error of the semi-discrete PDGFEM and DGFEM in the SIPG case by using Matlab software. The problem of diffusion-convection is as follows:

$$U_t - \lambda \Delta U + \boldsymbol{b} \cdot \nabla U = f, \quad \text{in } \Omega \times \text{J}. \tag{6.1}$$

A homogeneous Dirichlet border condition and a homogeneous beginning condition were used.The analytical solution to this problem is:

$$U(x, y, t) = e^{-t} \sin(\pi x) \sin(\pi y).$$

Suppose that $\Omega = [0,1] \times [0,1]$, $\boldsymbol{b} = [0,1]$, as the time interval $J = (0,1)$, $\sigma = 2782$, and f it is calculated by inserting the real solution into the left side of the equation (6.1). The square domain is evenly partitioned into N × N sub-squares by $\Omega = (0,1) \times (0,1)$. For triangular meshes, set h = 1/N($N = 4, 8, 16, 32, 64$) as the mesh size. The numerical error outcomes and degree of convergence for DGFEM when $\delta = 0$ in Table 1 and convergence rate in Figure 1, the results of the numerical error and degree of convergence for PDGFEM when δ = h/6 in Table 2 and convergence rate in Figure 2. In DGFEM, we can note that the numerical solution is not compatible with the precise solution (see Figure 3), but in PDGFEM, we note that the numerical solution is compatible with the precise solution (see Figure 4).

Table 1: Numerical results for $\lambda = 0.001$ in DGFEM.

h	H^1-error	H^1-order	L^2-error	L^2-order
1/4	0.4075	0	0.1473	0
1/8	0.3227	0.3367	0.0557	1.4042
1/16	0.2373	0.4434	0.0191	1.5416
1/32	0.1734	0.4528	0.0066	1.5261
1/64	0.1283	0.4349	0.0024	1.4978

Table 2: Numerical results for $\lambda = 0.001$ in PDGFEM.

h	H^1-error	H^1-order	L^2-error	L^2-order
1/4	0.1992	0	0.0736	0
1/8	0.1021	0.9644	0.0196	1.9074
1/16	0.0533	0.9382	0.0048	2.0310
1/32	0.0293	0.8631	0.0012	2.0272
1/64	0.0158	0.8906	0.0003	2.0017

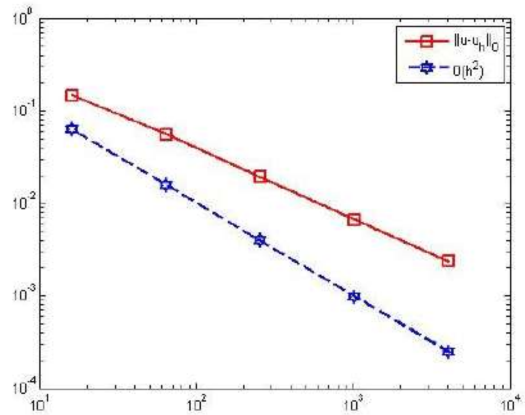


Figure 1: Convergence rate in DGFEM for $\lambda = 0.001$ in L^2norm.

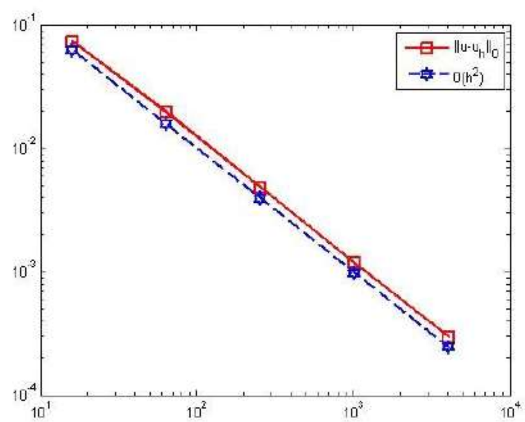
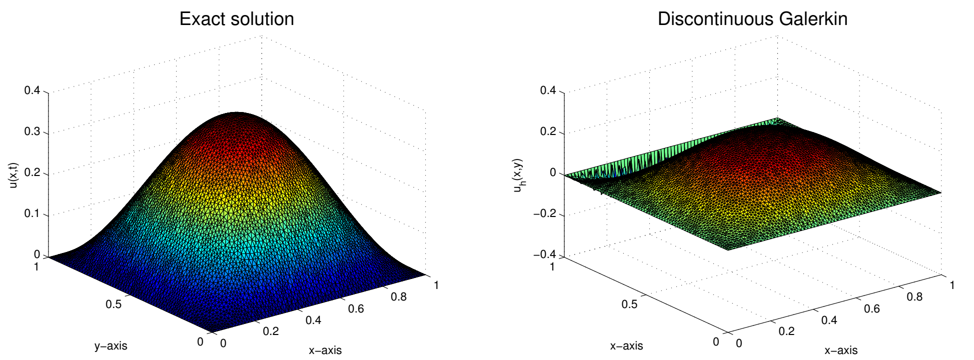
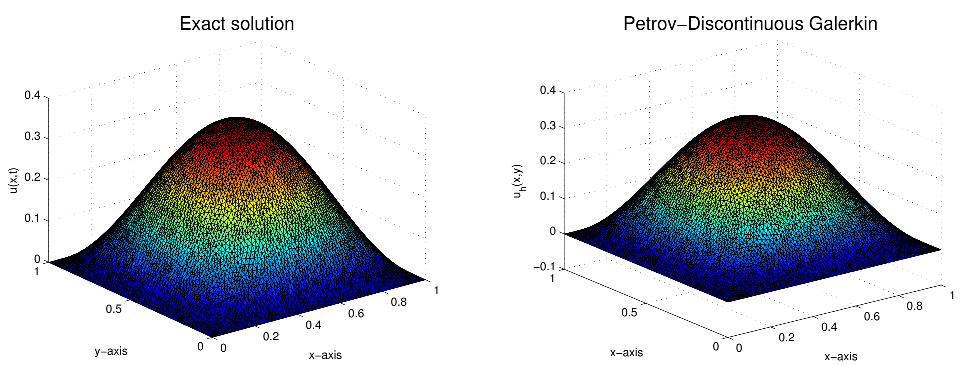


Figure 2: Convergence rate in PDGFEM for $\lambda = 0.001$ in L^2-norm.



(a) (b)

Figure 3: (a) The exact solution with $\lambda = 0.001$ and $h = 1/64$. (b) The numerical solution of DGFEM with $\lambda = 0.001$ and $h = 1/64$.



(a) (b)

Figure 4: (a) The exact solution with $\lambda = 0.001$ and $h = 1/64$. (b) The numerical solution of PDGFEM with $\lambda = 0.001$ and $h = 1/64$.

Conclusion

Throughout this current work, we have proved the continuity and $V-$elliptic properties of $a_{PD}(U, v)$ and the stability in PDGFEM. In addition, we demonstrated a theoretical analysis that shows how the PDGFEM is convergent of order $O(h^2)$. Moreover, depending on the comparison of Table 1 and Figure 1 for the DGFEM with Table 2 and Figure 2 for the PDGFEM, we stated that the numerical results of the PDGFEM showed improvement and regularity when compared to the numerical results of the DGFEM. Finally, when we smoothed the network with $n = 64$, we found that the numerical results in DGFEM are oscillated as shown in the Figure 3, but the numerical results in PDGFEM were appropriately approximated as well as free from oscillation as in the Figure 4.

References

[1] M. W. AbdulRidha, H. A. Kashkool, *The error analysis for the discontinuous galerkin finite element method of the convection- diffusion problem,* Journal of Basrah Researches (Sciences), 45 (2019).

[2] M. W. AbdulRidha, H. A. Kashkool, *Space-time petrov- discontinuous galerkin finite element method for solving linear convection-diffusion problems,* Journal of Physics: Conference Series, IOP Publishing, 2322 (2022), 012007.

[3] H. A. Kashkool, Y. H. Salim, G. A. Muften, *Error estimate of the discontinuous galerkin finite element method for convection-diffusion problems,* Basrah Journal of Science, 34(A(2)) (2016), 55-72.

[4] A. K. Hashim, *Upwind type finite element method for nonlinear convection-diffusion problem and applicationto numerical reservoir simulation,* Nankai Universty, China, 2002.

[5] A. H. Ali, A. S. Jaber, M. T. Yaseen, M. Rasheed, O. Bazighifan, T. A. Nofal, *A comparison of finite difference and finite volume methods with numerical simulations: burgers equation model,* Complexity, 2022.

[6] D. Broersen, R. Stevenson. *A robust Petrov-Galerkin discretisation of convection-diffusion equations,* Computers & Mathematics with Applications, 68 (2014), 1605-1618.

[7] J. Chan, J. A. Evans, W. Qiu, *A dual petrov-galerkin finite element method for the convection-diffusion equation,* Computers & Mathematics with Applications, 68 (2014), 1513-1529.

[8] G. A. Meften, A. H. Ali, K. S. Al-Ghafri, J. Awrejcewicz, O. Bazighifan, *Nonlinear stability and linear instability of double-diffusive convection in a rotating with ltne effects and symmetric properties: Brinkmannforchheimer model,* Symmetry, 14 (2022), 565.

[9] A. H. Ali, G. A. Meften, O. Bazighifan, M. Iqbal, S. Elaskar, J. Awrejcewicz, *A study of continuous dependence and symmetric properties of double diffusive convection: Forchheimer model,* Symmetry, 14 (2022), 682.

[10] B. Fiorina, S. K. Lele, *An artificial nonlinear diffusivity method for supersonic reacting flows with shocks,* Journal of Computational Physics, 222 (2007), 246-264.

[11] M. S. Cheichan, H. A. Kashkool, F. Gao, *A weak galerkin finite element method for solving nonlinear convection-diffusion problems in two dimensions,* Applied Mathematics and Computation, 354 (2019), 149-163.

[12] F. Gao, S. Zhang, P. Zhu, *Modified weak galerkin method with weakly imposed boundary condition for convection-dominated diffusion equations,* Applied Numerical Mathematics, 157 (2020), 490-504.

[13] O. Bazighifan, A. H. Ali, F. Mofarreh, Y. N. Raffoul, *Extended approach to the asymptotic behavior and symmetric solutions of advanced differential equations,* Symmetry, 14 (2022), 686.

[14] B. Almarri, A. H. Ali, K. S. Al-Ghafri, A. Almutairi, O. Bazighifan, J. Awrejcewicz, *Symmetric and non-oscillatory characteristics of the neutral differential equations solutions related to p-laplacian operators,* Symmetry, 14 (2022), 566.

[15] B. Almarri, A. H. Ali, A. M. Lopes, O. Bazighifan, *Nonlinear differential equations with distributed delay: some new oscillatory solutions,* Mathematics, 10 (2022), 995.

[16] C. Johnson, *Numerical solution of partial differential equations by the finite element method,* Courier Corporation, 2012.

[17] M. A. Saum, *Adaptive discontinuous galerkin finite element methods for second and fourth order elliptic partial differential equations*, 2006.

[18] B. Cockburn, G. E. Karniadakis, C.-W. Shu, *Discontinuous Galerkin methods: theory, computation and applications,* volume 11, Springer Science & Business Media, 2012.

[19] A. J. Perella, *A class of Petrov-Galerkin finite element methods for the numerical solution of the stationary convection-diffusion equation,* PhD thesis, Durham University, 1996.

Accepted: June 10, 2022

On structures of rough topological spaces based on neighborhood systems

Radwan Abu-Gdairi
Department of Mathematics
Faculty of Science
Zarqa university
Zarqa 13132
Jordan
rgdairi@zu.edu.jo

Abstract. Keeping in view the generalized approximation space, the goal of this paper is to suggest and investigate four different styles for approximating rough sets. The proposed approximations are based on various general topologies. In fact, we first generalize the notion of the initial-neighborhood and thus we construct four different topologies generated from these neighborhoods. The relationships between the new neighborhoods (respectively, topologies) and the previous are studied. Comparisons of the degrees of different accuracy of the presented approximations are investigated. The essential characteristics of these operators are obtained.

Keywords: initial-neighborhoods, rough sets, topology.

1. Introduction

The number of research articles published has been rapidly increasing, particularly in Topology and its applications. Several proposals were made for using mathematical methodologies and relevant formulas to solve real-world problems in order to assist decision-makers in making the best decisions possible to deal with unpredictability in challenges (see [1, 4, 5, 6, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 28, 30, 31, 36]). In 1982, Pawlak [25] proposed rough set theory as a new mathematical technique or set of simple tools for dealing with ambiguity in knowledge-based systems and data dissection. This theory has a wide range of applications, including process control, economics, medical diagnosis, and others (see [5, 6, 10, 13, 14, 18, 19, 20, 21, 22, 23, 28, 30, 31]). To extend the field of application for this theory, many papers were published (see [1]-[18], [27]-[28], [32]-[36]).

The novel notion "the $\mathcal{J}$-neighborhood space" (in short, $\mathcal{J}-NS$) was suggested by Abd El-Monsef et al. [1] as a general frame of neighborhood space. In fact, they hosted a structure for extending Pawlak's approach [25, 26] and some of the other generalizations. As a result, they devised various rough approximations to fulfill all properties of the rough sets without any constraints.

These methods paved the way for more topological applications of rough sets, as well as assisting in the formalization of many real-worlds applications.

The involvement of this article is to suggest a generalization for the idea of “initial-neighborhood” given by El-Sayed et al. [18]. It must be mentioned that the concept of ”initial-neighborhood” was proposed by another notion (namely, ”subset neighborhood”) by Al-shami and Ciucci [9] in 2022 as an extension of the concept of initial-neighborhood. Hence, we produce four topologies and then we investigate the relationships among these topologies and the previous ones [1, 18]. Accordingly, we achieve four techniques to find the approximations of rough sets. Comparisons of the degrees of different accuracy of the presented approximations are investigated. Therefore, we ascertain that the recommended ways are extra precise than the others.

The present manuscript is prepared as follows: In section 2, we outline the main ideas about the $\mathcal{J} - NS$ cited in [1] and the basic properties of the initial-neighborhood [18]. Section 3 is devoted to introducing and studying new generalizations to the concept of “initial-neighborhood”. We define three different types of initial neighborhoods and compare them with the previous one [18]. Moreover, using Theorem 1 in [1], we purpose a new method to generate four different topologies induced by the new neighborhoods. A comparison between these topologies and the previous one is investigated. Finally, in section 4, we use these new topologies to generate new generalizations to Pawlak rough sets and study their properties. We compare the suggested approaches with the previous one [1, 18] and verify that these techniques are more perfect than other approaches.

2. Preliminaries

The central ideas about $\mathcal{J} - NS$ cited in [1] and properties of the initial-neighborhood [18] are provided in the present part.

Definition 2.1 ([1]). Suppose that $\mathcal{U}$ be a non-empty finite set and $\mathcal{R}$ be a binary relation on it. Therefore, we define a $\mathcal{J}$-neighborhood of $x \in \mathcal{U}$, denoted by $\mathcal{N}_{\mathcal{J}}(x), \mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$ as follows:

(i) r-neighborhood: $\mathcal{N}_r(x) = \{\mathcal{Y} \in \mathcal{U} : x\mathcal{R}y\}$.

(ii) $\upharpoonleft$-neighborhood: $\mathcal{N}_{\upharpoonleft}(x) = \{\mathcal{Y} \in \mathcal{U} : y\mathcal{R}x\}$.

(iii) $\curlywedge$-neighborhood: $\mathcal{N}_{\curlywedge}(x) = \mathcal{N}_r(x) \cap \mathcal{N}_{\upharpoonleft}(x)$.

(iv) $\curlyvee$-neighborhood: $\mathcal{N}_{\curlyvee}(x) = \mathcal{N}_r(x) \cup \mathcal{N}_{\upharpoonleft}(x)$.

Definition 2.2 ([1]). Consider $\mathcal{R}$ be a binary relation on $\mathcal{U}$ and $\xi_{\mathcal{J}} : \mathcal{U} \to \mathcal{P}(\mathcal{U})$ represents a map that gives for every x in $\mathcal{U}$ its $\mathcal{J}$-neighborhood $\mathcal{N}_{\mathcal{J}}(x)$. Thus, triple $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ is said to be a $\mathcal{J}$-neighborhood space (in briefly, $\mathcal{J} - NS$).

Theorem 2.3 ([1])**.** *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$*, then for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$ *the collection*

$$\mathcal{T}_{\mathcal{J}} = \{\mathcal{M} \subseteq \mathcal{U} : \forall \mathrm{m} \in \mathcal{M}, \mathcal{N}_{\mathcal{J}}(\mathrm{m}) \subseteq \mathcal{M}\}$$

represents a topology on $\mathcal{U}$*.*

Definition 2.4 ([1])**.** Consider $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$. The subset $\mathcal{M} \subseteq \mathcal{U}$ is said to be an "$\mathcal{J}$-open set" if $\mathcal{M} \in \mathcal{T}_{\mathcal{J}}$, its complement is an "$\mathcal{J}$-closed set". The family $\mathcal{F}_{\mathcal{J}}$ of all $\mathcal{J}$-closed sets of a $\mathcal{J} - NS$ is defined by $\mathcal{F}_{\mathcal{J}} = \{F \subseteq \mathcal{U} : F^c \in \mathcal{T}_{\mathcal{J}}\}$.

Definition 2.5 ([1])**.** Suppose that $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$ and $\mathcal{M} \subseteq \mathcal{U}$. The "$\mathcal{J}$-lower" (respectively, "$\mathcal{J}$-upper") approximation of $\mathcal{M}$ is provided by

$$\underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \cup\{G \in \mathcal{T}_{\mathcal{J}} : G \subseteq \mathcal{M}\} = int_{\mathcal{J}}(\mathcal{M})$$

(respectively, $\overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \cap\{H \in \mathcal{F}_{\mathcal{J}} : \mathcal{M} \subseteq H\} = cl_{\mathcal{J}}(\mathcal{M})$), where $int_{\mathcal{J}}(\mathcal{M})$ (respectively, $cl_{\mathcal{J}}(\mathcal{M})$) is the $\mathcal{J}$-interior of $\mathcal{M}$ (respectively, $\mathcal{J}$-closure of $\mathcal{M}$).

Definition 2.6 ([1])**.** Let $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$ and $\mathcal{M} \subseteq \mathcal{U}$. Then, for each $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$, the subset $\mathcal{M}$ is called "$\mathcal{J}$-exact" set if $\underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \mathcal{M}$. Else, it is "$\mathcal{J}$-rough".

Definition 2.7 ([1])**.** Consider $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ to be a $\mathcal{J} - NS$ and $\mathcal{M} \subseteq \mathcal{U}$. The "$\mathcal{J}$-boundary", "$\mathcal{J}$-positive" and "$\mathcal{J}$-negative" regions of $\mathcal{M}$ are defined respectively by $\mathcal{B}_{\mathcal{J}}(\mathcal{M}) = \overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) - \underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})$, $POS_{\mathcal{J}}(\mathcal{M})= \underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})$ and $NEG_{\mathcal{J}}(\mathcal{M}) =U - \overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})$.

The "$\mathcal{J}$-accuracy" of $\mathcal{J}$-approximations of $\mathcal{M} \subseteq \mathcal{U}$ is given as follows: $\delta_{\mathcal{J}}(\mathcal{M}) = \frac{|\underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})|}{|\overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})|}$, where $|\overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})| \neq 0$. Clearly, $0 \leq \delta_{\mathcal{J}}(\mathcal{M}) \leq 1$ and if $\delta_{\mathcal{J}}(\mathcal{M}) = 1$, then $\mathcal{M}$ is a $\mathcal{J} - exact$ set. Else,it is $\mathcal{J}$-rough.

3. Topologies generated from neighborhoods

The main ideas of this part is to generalize the concept of "initial-neighborhood [18]" and thus we produce four different topologies from these neighborhoods.

Definition 3.1. For a binary relation $\mathcal{R}$ on $\mathcal{U}$, we define the following neighborhoods of $x \in \mathcal{U}$:

(*i*) r-initial neighborhood [18]: $\mathcal{N}_r^i(x) = \{\mathcal{Y} \in \mathcal{U} : \mathcal{N}_r(x) \subseteq \mathcal{N}_r(\mathcal{Y})\}$;

(*ii*) $\upharpoonleft$-initial neighborhood: $\mathcal{N}_{\upharpoonleft}^i(x) = \{\mathcal{Y} \in \mathcal{U} : \mathcal{N}_{\upharpoonleft}(x) \subseteq \mathcal{N}_{\upharpoonleft}(\mathcal{Y})\}$;

(*iii*) $\curlywedge$-initial neighborhood: $\mathcal{N}_{\curlywedge}^i(x) = \mathcal{N}_r^i(x) \cap \mathcal{N}_{\upharpoonleft}^i(x)$;

(*iv*) $\curlyvee$-initial neighborhood: $\mathcal{N}_{\curlyvee}^i(x) = \mathcal{N}_r^i(x) \cup \mathcal{N}_{\upharpoonleft}^i(x)$.

The next lemmas give the main properties of the above neighborhoods.

Lemma 3.2. *If* $\mathcal{R}$ *is a binary relation on* $\mathcal{U}$*. Then, for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$*:*

(*i*) $x \in \mathcal{N}^i_{\mathcal{J}}(x)$.

(*ii*) $\mathcal{N}^i_{\mathcal{J}}(x) \neq \varphi$.

(*iii*) *If* $\mathcal{Y} \in \mathcal{N}^i_{\mathcal{J}}(x)$, *then* $\mathcal{N}^i_{\mathcal{J}}(y) \subseteq \mathcal{N}^i_{\mathcal{J}}(x)$, *for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge\}$.

Proof. Firstly, the proof of (i) and (ii) is obvious by Definition 3.1.
(iii) According to Definition 3.1, if $\mathcal{Y} \in \mathcal{N}^i_{\mathcal{J}}(x)$. Then

$$\mathcal{N}_r(x) \subseteq \mathcal{N}_r(y) \tag{1}$$

Now, let $\mathcal{Z} \in \mathcal{N}^i_{\mathcal{J}}(\mathcal{Y})$. Then $\mathcal{N}_r(\mathcal{Y}) \subseteq \mathcal{N}_r(\mathcal{Z})$. Consequently, by(1), $\mathcal{N}_r(x) \subseteq \mathcal{N}_r(\mathcal{Z})$ and this implies $\mathcal{Z} \in \mathcal{N}^i_{\mathcal{J}}(x)$. Consequently, $\mathcal{N}^i_{\mathcal{J}}(\mathcal{Y}) \subseteq \mathcal{N}^i_{\mathcal{J}}(x)$. □

Lemma 3.3. *If* $\mathcal{R}$ *is a binary relation on* $\mathcal{U}$. *Then,* $\forall x \in \mathcal{U}$:

(*i*) $\mathcal{N}^i_{\curlywedge}(x) \subseteq \mathcal{N}^i_r(x) \subseteq \mathcal{N}^i_{\curlyvee}(x)$.

(*ii*) $\mathcal{N}^i_{\curlywedge}(x) \subseteq \mathcal{N}^i_{\upharpoonleft}(x) \subseteq \mathcal{N}^i_{\curlyvee}(x)$.

Proof. Straightforward. □

The relationships between the initial-neighborhoods and $\mathcal{J}$-neighborhoods are given by the next lemma.

Lemma 3.4. *Suppose that* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *represents a* $\mathcal{J} - NS$. *If* $\mathcal{R}$ *is a reflexive and symmetric relation. Then,* $\forall x \in \mathcal{U}$, $\mathcal{N}^i_{\mathcal{J}}(x) \subseteq \mathcal{N}_{\mathcal{J}}(x)$.

Proof. Let $\mathcal{Y} \in \mathcal{N}^i_{\mathcal{J}}(x)$, then $\mathcal{N}_{\mathcal{J}}(x) \subseteq \mathcal{N}_{\mathcal{J}}(\mathcal{Y})$. But, $\mathcal{R}$ is a reflexive relation which implies $x \subseteq \mathcal{N}_{\mathcal{J}}(x)$ and thus $x \subseteq \mathcal{N}_{\mathcal{J}}(\mathcal{Y})$. Since $\mathcal{R}$ is a symmetric relation, then $\mathcal{Y} \subseteq \mathcal{N}_{\mathcal{J}}(x)$. Therefore, $\mathcal{N}^i_{\mathcal{J}}(x) \subseteq \mathcal{N}_{\mathcal{J}}(x)$,$\forall x \in \mathcal{U}$. □

The following result (depends on Theorem 2.3.) discusses an exciting technique to create different topologies using the above neighborhoods.

Theorem 3.5. *Let* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *be a* $\mathcal{J} - NS$. *Then, for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$, *the collection* $\mathcal{T}^i_{\mathcal{J}} = \{\mathcal{M} \subseteq \mathcal{U} : \forall \mathrm{m} \in \mathcal{M}, \mathcal{N}^i_{\mathcal{J}}(\mathcal{M}) \subseteq \mathcal{M}\}$ *is a topology on* $\mathcal{U}$.

Proof.

($T1$) Clearly, $\mathcal{U}$ and φ belong to $\mathcal{T}^i_{\mathcal{J}}$.

($T2$) Let$\{A_n : n \in N\}$ be a family of members in $\mathcal{T}^i_{\mathcal{J}}$ and $p \in U_n A_n$. Then there exists $n_0 \in N$ such that $P \in A_{n_0}$. Thus $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_{n_0}$ this implies $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq U_n A_n$. Therefore, $U_n A_n \in \mathcal{N}^i_{\mathcal{J}}(p)$.

($T3$) Let $A_1, A_2 \in \mathcal{N}^i_{\mathcal{J}}$ and $p \in A_1 \cap A_2$. Then $p \in A_1$ and $p \in A_2$ which implies $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_1$ and $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_2$. Thus $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_1 \cap A_2$ and hence $A_1 \cap A_2 \in \mathcal{T}^i_{\mathcal{J}}$.

From (T1), (T2)and (T3) $\mathcal{T}^i_{\mathcal{J}}$ forms a topology on $\mathcal{U}$. □

The next proposition gives the relationships among different topologies $\mathcal{T}^i_{\mathcal{J}}$.

Proposition 3.6. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *be a* $\mathcal{J} - NS$. *Then:*

(*i*) $\mathcal{T}^i_{\curlyvee} \subseteq \mathcal{T}^i_r \subseteq \mathcal{N}^i_{\curlywedge}$.

(*ii*) $\mathcal{T}^i_{\curlyvee} \subseteq \mathcal{T}^i_{\upharpoonleft} \subseteq \mathcal{T}^i_{\curlywedge}$.

Proof. By Lemma 3.3, the proof is obvious. □

Example 3.7 demonstrates that the opposite of Proposition 3.6 is not correct in general.

Example 3.7. Suppose that $\mathcal{R} = \{(a,a),(a,d),(b,a),(b,c),(c,c),(c,d),(d,a)\}$ be a relation on $\mathcal{U} = \{a,b,c,d\}$. Accordingly, we obtain $\mathcal{N}_r(a) = \{a,d\}$, $\mathcal{N}_r(b) = \{a,c\}$, $\mathcal{N}_r(c) = \{c,d\}$, and $\mathcal{N}_r(d) = \{a\}$.

$$\mathcal{N}_{\upharpoonleft}(a) = \{a,b,d\}, \mathcal{N}_{\upharpoonleft}(b) = \varphi, \mathcal{N}_{\upharpoonleft}(c) = \{b,c\}, \mathcal{N}_{\upharpoonleft}(d) = \{a,c\},$$
$$\mathcal{N}_{\curlywedge}(a) = \{a,d\}, \mathcal{N}_{\curlywedge}(b) = \varphi, \mathcal{N}_{\curlywedge}(c) = \{c\}, \mathcal{N}_{\curlywedge}(d) = \{a\},$$
$$\mathcal{N}_{\curlyvee}(a) = \{a,b,d\}, \mathcal{N}_{\curlyvee}(b) = \{a,c\}, \mathcal{N}_{\curlyvee}(c) = \{b,c,d\}, \mathcal{N}_{\curlyvee}(d) = \{a,c\}.$$

Therefore, we obtain $\mathcal{N}^i_r(a) = \{a\}$, $\mathcal{N}^i_r(b) = \{b\}$, $\mathcal{N}^i_r(c) = \{c\}$, and $\mathcal{N}^i_r(d) = \{a,b,d\}$

$$\mathcal{N}^i_{\upharpoonleft}(a) = \{a\}, \mathcal{N}^i_{\upharpoonleft}(b) = \mathcal{U}, \mathcal{N}^i_{\upharpoonleft}(c) = \{c\}, \mathcal{N}^i_{\upharpoonleft}(d) = \{d\},$$
$$\mathcal{N}^i_{\curlywedge}(a) = \{a\}, \mathcal{N}^i_{\curlywedge}(b) = \{b\}, \mathcal{N}^i_{\curlywedge}(c) = \{c\}, \mathcal{N}^i_{\curlywedge}(d) = \{d\},$$
$$\mathcal{N}^i_{\curlyvee}(a) = \{a\}, \mathcal{N}^i_{\curlyvee}(b) = \mathcal{U}, \mathcal{N}^i_{\curlyvee}(c) = \{c\}, \mathcal{N}^i_{\curlyvee}(d) = \{a,b,d\}.$$

Consequently, we generate the following topologies:

$$\mathcal{T}^i_r = \{\mathcal{U}, \varphi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\},$$
$$\mathcal{T}^i_{\upharpoonleft} = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{c,d\}, \{a,c,d\}\},$$
$$\mathcal{T}^i_{\curlywedge} = \mathcal{P}(\mathcal{U}), \mathcal{T}^i_{\curlyvee} = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}.$$

The subsequent proposition gives the connections amongst the topologies $\mathcal{T}^i_{\mathcal{J}}$ and $\mathcal{T}_{\mathcal{J}}$.

Proposition 3.8. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$ *such that* $\mathcal{R}$ *is a reflexive and symmetric relation. Then, for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\} : \mathcal{T}_{\mathcal{J}} \subseteq \mathcal{T}^i_{\mathcal{J}}$.

Proof. By Lemma 3.4, the proof is clear. □

Remark 3.9. For any a $\mathcal{J} - NS(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$, Example 3.7 shows the following:

(i) The topologies $\mathcal{T}^i_{\mathcal{J}}$ and $\mathcal{T}_{\mathcal{J}}$ are independent in general case.

(ii) The topologies $\mathcal{T}^i_r$ and $\mathcal{T}^i_{\upharpoonleft}$ are independent in general case.

(iii) The property (iii) in Lemma 3.2 is not true for case $j = \curlyvee$.

Example 3.10 proves that the opposite of Proposition 3.8 is not correct generally.

Example 3.10. Let $\mathcal{U} = \{a, b, c, d\}$ and $\mathcal{R} = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, b), (c, c), (d, d)\}$ be a reflexive and symmetric relation on $\mathcal{U}$. Thus, we compute the topologies $\mathcal{T}^i_{\mathcal{J}}$, and $\mathcal{T}_{\mathcal{J}}$ in the case of $\mathcal{J} = r$, and the others similarly $\mathcal{T}_r = \{\mathcal{U}, \varphi, \{d\}, \{a, b, c\}\}$, and $\mathcal{T}^i_r = \{\mathcal{U}, \varphi, \{b\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.

Diagram 1 summarize the relationships among different topologies such that $\mathcal{R}$ represents a reflexive and symmetric relation.

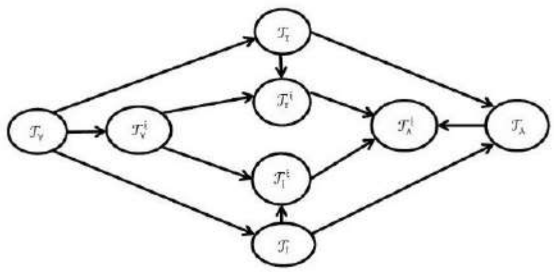


Diagram 1 The relationships among different topologies

4. Rough approximations based on topological structures

In this part, we present four new approximations called $\mathcal{J}$-initial lower and $\mathcal{J}$-initial upper approximations, which we use to define new regions and accuracy measures of a set using the interior and closure of the topologies $\mathcal{T}^i_{\mathcal{J}}$, for each $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$. We show that these methods yield the best approximations and the highest accuracy measures. There are illustrative examples provided.

Definition 4.1. Suppose that $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$ and $A \subseteq \mathcal{U}$. Therefore, A is called an $\mathcal{J}$-initial open set if $A \subseteq \mathcal{T}^i_{\mathcal{J}}$, and its complement is called an $\mathcal{J}$-initial closed set. The family $\mathcal{F}^i_{\mathcal{J}}$ of all $\mathcal{J}$-initial closed sets is defined by: $\mathcal{F}^i_{\mathcal{J}} = \{\mathcal{F} \subseteq \mathcal{U} : \mathcal{F}^c \in \mathcal{T}^i_{\mathcal{J}}\}$. Moreover, we define the following:

(i) The $\mathcal{J}$-*initial interior* of $A \subseteq \mathcal{U}$ is: $int^i_{\mathcal{J}}(A) = \cup\{G \in \mathcal{T}^i_{\mathcal{J}} : G \subseteq A\}$.

(ii) The $\mathcal{J}$-*initial closure* of $A \subseteq \mathcal{U}$ is: $cl^i_{\mathcal{J}}(A) = \cap\{H \in \mathcal{F}^i_{\mathcal{J}} : A \subseteq H\}$.

Definition 4.2. Let $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$. Then, we define $\mathcal{J}$-*initial lower* and $\mathcal{J}$-*initial upper* approximations of A respectively as follows: $\underline{R}^i_{\mathcal{J}}(A) = int^i_{\mathcal{J}}(A)$, and $\overline{R}^i_{\mathcal{J}}(A) = cl^i_{\mathcal{J}}(A)$.

Definition 4.3. Suppose that $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$. The $\mathcal{J}$-*initial boundary*, $\mathcal{J}$-*initial positive* and $\mathcal{J}$-*initial negative* regions of $A \subseteq \mathcal{U}$ are given, respectively, by $B^i_{\mathcal{J}}(A) = \overline{R}^i_{\mathcal{J}}(A) - \underline{R}^i_{\mathcal{J}}(A)$, $POS^i_{\mathcal{J}}(A) = \underline{R}^i_{\mathcal{J}}(A)$ and $NEG^i_{\mathcal{J}}(A) = \mathcal{U} - \overline{R}^i_{\mathcal{J}}(A)$.

Moreover, the $\mathcal{J}$-*initial accuracy* of the $\mathcal{J}$-*initial approximations* of $A \subseteq \mathcal{U}$ is defined by $\alpha^i_{\mathcal{J}}(A) = \frac{|\underline{R}^i_{\mathcal{J}}(A)|}{|\overline{R}^i_{\mathcal{J}}(A)|}$, where $|\overline{R}^i_{\mathcal{J}}(A)| \neq 0$. It is clear that, $0 \leq \alpha^i_{\mathcal{J}}(A) \leq 1$. In addition, A is called an $\mathcal{J}$-*initial* definable ($\mathcal{J}$-*initial exact*) set if $\alpha^i_{\mathcal{J}}(A) = 1$, and it is called $\mathcal{J}$-*initial rough* if $\alpha^i_{\mathcal{J}}(A) \neq 1$.

Example 4.4. By using Example 3.7, we get the following:

$$\begin{aligned}
&\mathcal{T}^i_r = \{\mathcal{U}, \varphi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\},\\
&\mathcal{F}^i_r = \{\mathcal{U}, \varphi, \{c\}, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\},\\
&\mathcal{T}^i_\rceil = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{c,d\}, \{a,c,d\}\},\\
&\mathcal{F}^i_\rceil = \{\mathcal{U}, \varphi, \{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\},\\
&\mathcal{T}^i_\curlywedge = \mathcal{F}^i_\curlywedge = \mathcal{P}(\mathcal{U}), \mathcal{T}^i_\curlyvee = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{a,c\}\},\\
&\mathcal{F}^i_\curlyvee = \{\mathcal{U}, \varphi, \{b,d\}, \{a,b,d\}, \{b,c,d\}\}.
\end{aligned}$$

Thus, we can get Tables 1 and 2 that give the $\mathcal{J}$-*initial lower*, $\mathcal{J}$-*initial upper* approximations and the $\mathcal{J}$-*initial accuracy* of $\mathcal{J}$-initial approximations of all subsets of $\mathcal{U}$:

Table 1: Comparison among different types of $\mathcal{J}$-*initial approximations*

$\mathcal{P}(\mathcal{U})$	$\mathcal{T}^i_r$		$\mathcal{T}^i_\rceil$		$\mathcal{T}^i_\curlywedge$		$\mathcal{T}^i_\curlyvee$	
	$\underline{\mathcal{R}}^i_r(A)$	$\overline{\mathcal{R}}^i_r(A)$	$\underline{\mathcal{R}}^i_\rceil(A)$	$\overline{\mathcal{R}}^i_\rceil(A)$	$\underline{\mathcal{R}}^i_\curlywedge(A)$	$\overline{\mathcal{R}}^i_\curlywedge(A)$	$\underline{\mathcal{R}}^i_\curlyvee(A)$	$\overline{\mathcal{R}}^i_\curlyvee(A)$
$\{a\}$	$\{a\}$	$\{a,d\}$	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b,d\}$
$\{b\}$	$\{b\}$	$\{b,d\}$	φ	$\{b\}$	$\{b\}$	$\{b\}$	φ	$\{b,d\}$
$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{b,c,d\}$
$\{d\}$	φ	$\{d\}$	$\{d\}$	$\{b,d\}$	$\{d\}$	$\{d\}$	φ	$\{b,d\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a\}$	$\{a,b,d\}$
$\{a,c\}$	$\{a,c\}$	$\{a,c,d\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\mathcal{U}$
$\{a,d\}$	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$	$\{a,b,d\}$
$\{b,c\}$	$\{b,c\}$	$\{b,c,d\}$	$\{c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{c\}$	$\{b,c,d\}$
$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	φ	$\{b,d\}$
$\{c,d\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{b,c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{c\}$	$\{b,c,d\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\mathcal{U}$	$\{c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,c\}$	$\mathcal{U}$
$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b,d\}$
$\{a,c,d\}$	$\{a,c\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\mathcal{U}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c\}$	$\mathcal{U}$
$\{b,c,d\}$	$\{b,c\}$	$\{b,c,d\}$	$\{c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{c\}$	$\{b,c,d\}$
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$

Table 2: Comparison among different types of $\mathcal{J}$-*initial accuracy*

$\mathcal{P}(\mathcal{U})$	$\alpha_r^i(A)$	$\alpha_\uparrow^i(A)$	$\alpha_\curlywedge^i(A)$	$\alpha_\curlyvee^i(A)$
$\{a\}$	1/2	1/2	1	1/3
$\{b\}$	1/2	0	1	0
$\{c\}$	1	1/2	1	1/3
$\{d\}$	0	1/2	1	0
$\{a,b\}$	2/3	1/2	1	1/3
$\{a,c\}$	2/3	2/3	1	1/2
$\{a,d\}$	1/2	2/3	1	1/3
$\{b,c\}$	2/3	1/2	1	1/3
$\{b,d\}$	1/2	1/2	1	0
$\{c,d\}$	1/2	2/3	1	1/3
$\{a,b,c\}$	3/4	1/3	1	1/2
$\{a,b,d\}$	1	2/3	1	1/3
$\{a,c,d\}$	2/3	3/4	1	1/2
$\{b,c,d\}$	2/3	2/3	1	1/3
$\mathcal{U}$	1	1	1	1

Remark 4.5. According to Tables 1 and 2 of Example 4.4, we conclude that by using different types of $\mathcal{T}_\mathcal{J}^i$ in constructing the approximations of sets, the best of them is that given by $\mathcal{T}_\curlywedge^i$ since $\alpha_\curlyvee^i(A) \leq \alpha_r^i(A) \leq \alpha_\curlywedge^i(A)$ and $\alpha_\curlyvee^i(A) \leq \alpha_\uparrow^i(A) \leq \alpha_\curlywedge^i(A)$. In addition, these approaches are more accurate than the previous one in [18].

Some properties of the $\mathcal{J}$-*initial approximations* are provided in the next result. Moreover, it represents one of the distinctions between our approaches and other generalizations such as [1, 12-16, 21, 22, 25-28, and 33-36].

Proof. Suppose that $(\mathcal{U}, \mathcal{R}, \xi_\mathcal{J})$ be a $\mathcal{J}-NS$ and $A,\ B \subseteq \mathcal{U}$. Thus:

(1) $\underline{R}_\mathcal{J}^i(A) \subseteq A \subseteq \overline{R}_\mathcal{J}^i(A)$.
(2) $\underline{R}_\mathcal{J}^i(\mathcal{U}) = \overline{R}_\mathcal{J}^i(\mathcal{U}) = \mathcal{U}$, and $\underline{R}_\mathcal{J}^i(\varphi) = \overline{R}_\mathcal{J}^i(\varphi) = \varphi$.
(3) $\overline{R}_\mathcal{J}^i(A \cup B) = \overline{R}_\mathcal{J}^i(A) \cup \overline{R}_\mathcal{J}(B)$.
(4) $\underline{R}_\mathcal{J}^i(A \cap B) = \underline{R}_\mathcal{J}^i(A) \cap \underline{R}_\mathcal{J}^i(B)$.
(5) If $A \subseteq B$, then $\underline{R}_\mathcal{J}^i(A) \subseteq \underline{R}_\mathcal{J}^i(B)$.
(6) If $A \subseteq B$, then $\overline{R}_\mathcal{J}^i(A) \subseteq \overline{R}_\mathcal{J}^i(B)$.
(7) $\underline{R}_\mathcal{J}^i(A \cup B) \supseteq \underline{R}_\mathcal{J}^i(A) \cup \underline{R}_\mathcal{J}^i(B)$.
(8) $\overline{R}_\mathcal{J}^i(A \cap B) \subseteq \overline{R}_\mathcal{J}^i(A) \cap \overline{R}_\mathcal{J}^i(B)$.
(9) $\underline{R}_\mathcal{J}^i(A) = [\overline{R}_\mathcal{J}^i(A^c)]^c$, A^c is the complement of A.
(10) $\overline{R}_\mathcal{J}^i(A) = [\underline{R}_\mathcal{J}^i(A^c)]^c$.

(11) $\underline{R}^i_{\mathcal{J}}(\underline{R}^i_{\mathcal{J}}(A) = \underline{R}^i_{\mathcal{J}}(A)$.
(12) $\overline{R}^i_{\mathcal{J}}(\overline{R}^i_{\mathcal{J}}(A)) = \overline{R}^i_{\mathcal{J}}(A)$. □

Proof. The proof is directly simple by applying the properties of interior $int^i_{\mathcal{J}}$ and closure $cl^i_{\mathcal{J}}$. □

The subsequent results illustrate the relationships among the suggested approximations ($\mathcal{J}$-*initial approximations*).

Proposition 4.6. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$ *and* $\mathrm{A} \subseteq \mathcal{U}$. *Then:*
(1) $\underline{R}^i_{\curlyvee}(\mathrm{A}) \subseteq \underline{R}^i_{r}(\mathrm{A}) \subseteq \underline{R}^i_{\curlywedge}(\mathrm{A})$.
(2) $\underline{R}^i_{\curlyvee}(\mathrm{A}) \subseteq \underline{R}^i_{\upharpoonleft}(\mathrm{A}) \subseteq \underline{R}^i_{\curlywedge}(\mathrm{A})$.
(3) $\overline{R}^i_{\curlywedge}(\mathrm{A}) \subseteq \overline{R}^i_{r}(\mathrm{A}) \subseteq \overline{R}^i_{\curlyvee}(\mathrm{A})$.
(4) $\overline{R}^i_{\curlywedge}(\mathrm{A}) \subseteq \overline{R}^i_{\upharpoonleft}(\mathrm{A}) \subseteq \overline{R}^i_{\curlyvee}(\mathrm{A})$.

Proof. By using Proposition 3.6, the proof is obvious. □

Corollary 4.7. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$ *and* $\mathrm{A} \subseteq \mathcal{U}$. *Then:*
(1) $\underline{B}^i_{\curlywedge}(\mathrm{A}) \subseteq \underline{B}^i_{r}(\mathrm{A}) \subseteq \underline{B}^i_{\curlyvee}(\mathrm{A})$.
(2) $\underline{B}^i_{\curlywedge}(\mathrm{A}) \subseteq \underline{B}^i_{\upharpoonleft}(\mathrm{A}) \subseteq \underline{B}^i_{\curlyvee}(\mathrm{A})$.
(3) $\alpha^i_{\curlyvee}(\mathrm{A}) \leq \alpha^i_{r}(\mathrm{A}) \leq \alpha^i_{\curlywedge}(\mathrm{A})$.
(4) $\alpha^i_{\curlyvee}(\mathrm{A}) \leq \alpha^i_{\upharpoonleft}(\mathrm{A}) \leq \alpha^i_{\curlywedge}(\mathrm{A})$.
(5) *The subset* A *is an* $\curlyvee$*-initial exact set* $\Rightarrow$ A *is* r*-initial exact* $\Rightarrow$ A *is* $\curlywedge$*-initial exact.*
(6) *The subset* A *is an* $\curlyvee$*-initial exact set* $\Rightarrow$ A *is* $\upharpoonleft$*-initial exact* $\Rightarrow$ A *is* $\curlywedge$*-initial exact.*

Remark 4.8. The converse of the above results is not true in general as illustrated in Example 4.4.

The following results introduce comparisons between the proposed approximations ($\mathcal{J}$-*initial approximations*) and the previous approximations ($\mathcal{J}$-*initial approximations* [1]).

Theorem 4.9. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J}$-NS *and* $\mathrm{A} \subseteq \mathcal{U}$ *such that* $\mathcal{R}$ *is a reflexive and symmetric relation on* $\mathcal{U}$. *Then, for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$*:*
(1) $\underline{\mathcal{R}}_{\mathcal{J}}(\mathrm{A}) \subseteq \underline{\mathcal{R}}^i_{\mathcal{J}}(\mathrm{A})$.
(2) $\overline{\mathcal{R}}^i_{\mathcal{J}}(\mathrm{A}) \subseteq \overline{\mathcal{R}}_{\mathcal{J}}(\mathrm{A})$.

Proof. We shall prove the first statement and the other similarly.

Let $x \in \underline{\mathcal{R}}_{\mathcal{J}}(A)$, then $\exists G \in \mathcal{T}_{\mathcal{J}}$ such that $x \in G \subseteq A$. But, from Proposition 3.8, $\mathcal{T}_{\mathcal{J}} \subseteq \mathcal{T}^i_{\mathcal{J}}$. Therefore, $G \in \mathcal{T}^i_{\mathcal{J}}$ such that $x \in G \subseteq A$ which implies $x \in \mathcal{R}^i_{\mathcal{J}}(A)$. □

Corollary 4.10. *Let* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *be a* $\mathcal{J}$-NS . *Then:*

(1) $\mathcal{B}^i_{\mathcal{J}}(\mathrm{A}) \subseteq \mathcal{B}_{\mathcal{J}}(\mathrm{A})$.

(2) $\alpha_{\mathcal{J}}(\mathrm{A}) \leq \alpha^i_{\mathcal{J}}(\mathrm{A})$.

(3) *The subset* A *is an* $\mathcal{J}$-exact *set if it is* $\mathcal{J}$-initial exact.

Remark 4.11. The inverse of the above results is not true in general as illustrated by Example 4.12.

Example 4.12. Consider Example 3.10, we compare between the $\mathcal{J}$-*approximations* and $\mathcal{J}$-*initial approximations* in the case of $\mathcal{J} = r$ and the others similarly.

First, the topologies $\mathcal{T}^i_{\mathcal{J}}$ and $\mathcal{T}_{\mathcal{J}}$ in the case of $\mathcal{J} = r$ are:

$\mathcal{T}_r = \mathcal{F}_r = \{\mathcal{U}, \varphi, \{d\}, \{a, b, c\}\}$,

$\mathcal{T}^i_r = \{\mathcal{U}, \varphi, \{b\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$ and

$\mathcal{F}^i_r = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}\}$.

Therefore, we get Table 3 which represents a comparison between the r-accuracy of $\mathcal{J}$- *approximations* and r-initial accuracy of r-initial approximations of all subsets of $\mathcal{U}$.

Table 3: Comparison between r-accuracies and r-initial accuracies

$\mathcal{P}(\mathcal{U})$	$\alpha_r(A)$	$\alpha^i_r(A)$
$\{a\}$	0	0
$\{b\}$	0	1/3
$\{c\}$	0	0
$\{d\}$	1	1
$\{a, b\}$	0	2/3
$\{a, c\}$	0	0
$\{a, d\}$	1/4	1/2
$\{b, c\}$	0	2/3
$\{b, d\}$	1/4	1/2
$\{c, d\}$	1/4	1/2
$\{a, b, c\}$	1	1
$\{a, b, d\}$	1/4	3/4
$\{a, c, d\}$	1/4	1/3
$\{b, c, d\}$	1/4	3/4
$\mathcal{U}$	1	1

Remark 4.13. According to Table 3 of Example 4.12, we notice that r-initial approximations are more accurate than r-approximations of sets since $\alpha_r(A) \leq \alpha^i_r(A)$. Therefore, we can say that the proposed approximations $\mathcal{J}$-*initial approximations* represent golden tools in removing the vagueness of sets. For example, in Table 3, the subset $A = \{b, c\}$ its r-approximations are $\underline{R}_r(A) = \varphi$ and $\overline{R}_r(A) = \{a, b, c\}$ which implies $B_r(A) = \{a, b, c\}$ and $\alpha_r(A) = 0$ and this means

that A is a r-rough set. Moreover, the r-positive region of A is $POS_r(A) = \varphi$ although A consist of two elements which is a contradiction to the knowledge of Example 4.12. On the other hand, we find r-initial approximations of A are $\underline{R}^i_r(A) = \{b, c\}$ and $\overline{R}^i_r(A) = \{a, b, c\}$ that is the r-initial positive region of A is $POS^i_r(A) = A$ and $\alpha^i_r(A) = 2/3$.

Conclusion

The present paper is devoted to introducing and studying new generalizations to the concept of "initial-neighborhood". We defined three different types and compare them with the previous one [18]. Moreover, using Theorem 1 in [1], we purposed a new method to generate four different topologies induced by the new neighborhoods. A comparison between these topologies and the previous one was investigated. Finally, we used these new topologies to generate new generalizations to Pawlak rough sets and study their properties. We compared the suggested approaches with the previous one [1, 18] and proved that these methods are more accurate than other methods. Theorem 3.5 gives an easy method to generate these topologies directly from relations without using sub-base or base. We believe that the using of this technique is easier in application fields and useful for applying many topological concepts in future studies.

Acknowledgements

The authors thank the referees and editor for their valuable comments and recommendations that helped in improving this paper.

References

[1] M. E. Abd El-Monsef, O.A. Embaby, M.K. El-Bably, *Comparison between rough set approximations based on different topologies*, Journal of Granular Computing, Rough Sets, and Intelligent Systems, 3 (2014), 292-305.

[2] M.E. Abd El-Monsef, M.A. EL-Gayar, R.M. Aqeel, *On relationships between revised rough fuzzy approximation operators and fuzzy topological spaces*, International Journal of Granular Computing, Rough Sets, and Intelligent Systems, 3 (2014), 257–271.

[3] M.E. Abd El-Monsef, M.A. EL-Gayar, R.M. Aqeel, *A comparison of three types of rough fuzzy sets based on two universal sets,* International Journal of Machine Learning and Cybernetics, 8 (2017), 343–353.

[4] E.A. Abo-Tabl, M.K. El-Bably, *Rough topological structure based on reflexivity with some applications*, AIMS Mathematics, 7 (2022), 9911-99.

[5] R.A. Abu-Gdairi, M.A. El-Gayar, T.M. Al-shami, A.S. Nawar, M.K. El-Bably, *Some topological approaches for generalized rough sets and their decision-making applications*, Symmetry, 14 (2022).

[6] R.A. Abu-Gdairi, M.A. El-Gayar, M.K. El-Bably, K.K. Fleifel, *Two views for generalized rough sets with applications*, Mathematics, 18 (2021), 2275.

[7] A.A. Allam, M.Y. Bakeir, E.A. Abo-Tabl, *New approach for basic rough set concepts, in: rough sets, fuzzy sets, data mining, and granular computing,* Lecture Notes in Artificial Intelligence 3641, D. Slezak, G. Wang, M. Szczuka, I. Dntsch, Y. Yao (Eds.), Springer Verlag GmbH, Regina, (2005), 64-73.

[8] M.I. Ali, M.K. El-Bably, E.A. Abo-Tabl, *Topological approach to generalized soft rough sets via near concepts*, Soft Computing, 26 (2022), 499-509.

[9] T.M. Al-shami, D. Ciucci, *Subset neighborhood rough sets*, Knowledge-Based Systems, 237 (2022).

[10] T.M. Al-shami, B.A. Asaad, M.A. El-Gayar, *Various types of supra pre-compact and supra pre-Lindelöf spaces*, Missouri Journal of Mathematical Sciences, 32 (2020), 1-20.

[11] M.K. El-Bably, M. El-Sayed, *Three methods to generalize Pawlak approximations via simply open concepts with economic applications,* Soft Computing, (2022).

[12] M.K. El-Bably, M.K. El-Bably, K.K. Fleifel, O.A. Embaby, *Topological approaches to rough approximations based on closure operators*, Granular Computing, 7 (2022), 1-14.

[13] M.K. El-Bably, T.M. Al-shami, *Different kinds of generalized rough sets based on neighborhoods with a medical application*, International Journal of Biomathematics, 14 (2021), 2150086.

[14] M.K. El-Bably, E.A. Abo-Tabl, *A topological reduction for predicting of a lung cancer disease based on generalized rough sets,* Journal of Intelligent and Fuzzy Systems, 41 (2021), 335-346.

[15] M.K. El-Bably, A.A. El Atik, *Soft β-rough sets and its application to determine COVID-19,* Turkish Journal of Mathematics, 45(2021), 1133-1148.

[16] M.K. El-Bably, M.I. Ali, E.A. Abo-Tabl, *New topological approaches to generalized soft rough approximations with medical applications,* Journal of Mathematics, 2021 (2021), Article ID 2559495.

[17] M.A. El-Gayar, A.A. El Atik, *Topological models of rough sets and decision making of COVID-19*, Complexity, Volume 2022, and Article ID 2989236.

[18] M. El Sayed, M.A. El Safty, M.K. El-Bably, *Topological approach for decision-making of COVID-19 infection via a nano-topology model,* AIMS Mathematics, 6 (2021), 7872-7894.

[19] A. Galton, *A generalized topological view of motion in discrete space,* Theoretical Computer Sciences, 305 (2003), 111-134.

[20] J. Kortelainen, *On the relationship between modified sets, topological spaces and rough sets*, Fuzzy Sets and Systems, 61 (1994), 91–95.

[21] C. Largeron, S. Bonnevay, *A pretopological approach for structural analysis*, Information Sciences, 144 (2002), 185–196.

[22] E.F. Lashin, A.M. Kozae, A.A. Abo Khadra, T. Medhat, *Rough set theory for topological spaces,* International Journal of Approximate Reasoning, 40 (2005), 35-43.

[23] T.Y. Lin, *Granular computing on binary relations I: Data mining and neighborhood systems, II: Rough set representations and belief functions,* in: Rough Sets in Knowledge Discovery 1, Polkowski, L., and Skowron, A. (Eds.), Physica¬Verlag, Heidelberg, (1998), 107-140.

[24] H. Lu, A.M. Khalil, W. Alharbi, M.A. El-Gayar, *A new type of generalized picture fuzzy soft set and its application in decision making,* Journal of Intelligent and Fuzzy Systems, 40 (2021), 12459-12475.

[25] Z. Pawlak, *Rough sets,* Int. J. of Information and Computer Sciences, 11 (1982), 341–356.

[26] Z. Pawlak, *Rough sets theoretical aspects of reasoning about data,* vol. 9, Kluwer- Academic Publishers, Dordrecht, 1991.

[27] Z. Pawlak, A. Skowron, *Rough sets: some extensions,* Information Sciences, 177 (2007), 28–40.

[28] L. Polkowski, *Metric spaces of topological rough sets from countable knowledge bases*, in: Slowinski, R., and Stefanowski, J.(Eds.), Foundations of Computing and Decision Sciences, 18 (3–4) special issue (1993), 293–306.

[29] A. Skowron, J. Stepaniuk, *Tolerance approximation spaces*, Fundamenta Informaticae, 27 (1996), 245–253.

[30] B.M.R. Stadler, P.F. Stadler, *Generalized topological spaces in evolutionary theory and combinatorial chemistry*, J. Chem. Inf. Comp. Sci., 42 (2002), 577–585.

[31] B.M.R. Stadler, P.F. Stadler, *The topology of evolutionary Biology*, Ciobanu, G., and Rozenberg, G. (Eds.): Modeling in Molecular Biology, Springer Verlag, Natural Computing Series, (2004), 267-286.

[32] C. Wang, C. Wu, D. Chen, *A systematic study on attribute reduction with rough sets based on general binary relations*, Information Sciences, 178 (2008), 2237–2261.

[33] Y.Y. Yao, *Two views of the theory of rough sets in finite universes,* International Journal of Approximate Reasoning, 15 (1996), 291–317.

[34] Y.Y. Yao, *Generalized rough set models*, in: Rough Sets in knowledge Discovery 1, Polkowski, L. and Skowron, A. (Eds.), Physica Verlag, Heidelberg, (1998), 286-318.

[35] Y.Y. Yao, *Relational interpretations of neighborhood operators and rough set approximation operators,* Information Sciences, 111 (1998), 239–259.

[36] W. Zhu, *Topological approaches to covering rough sets*, Information Sciences, 177 (2007), 1499–1508.

Accepted: June 9, 2022

Some separation axioms via nano S_β-open sets in nano topological spaces

Nehmat K. Ahmed
Department of Mathematics
College of Education
Salahaddin University
Erbil, 44001
Iraq
nehmat.ahmed@su.edu.krd

Osama T. Pirbal*
Department of Mathematics
College of Education
Salahaddin University
Erbil, 44001
Iraq
osama.pirbal@su.edu.krd

Abstract. In this present study, we shed light on some separation axioms via nano S_β-open sets including nano S_β-regular, S_β-normal, $S_\beta - S_0$ and $S_\beta - S_1$ axioms in nano topological spaces where nano S_β-open set is defined and related to nano semi-open and nano β-closed sets. Here, we implement each axiom on the family of all nano S_β-open sets according to upper and lower approximations in which there exist exactly six families of nano S_β-open sets. This research work brings out some interesting results such as it is shown that in which condition a nano topological space is always nano S_β-normal space where upper and lower approximations are leading conditions. In addition, the relationship among those axioms is also considered.

Keywords: nano S_β-open sets, nano S_β-regular, nano S_β-normal, nano $S_\beta - S_0$, nano $S_\beta - S_1$.

1. Introduction

The concept of nano topological space is introduced by Thivagar and Richard [2] with respect to a subset X of U as the universe. Then, some types of nano open sets are defined and introduced such as nano semi-open sets, nano α-open sets and nano pre-open sets in [2] and nano *rare* sets by Thivagar et al., [7]. After that, nano β-open sets are introduced by Revathy and Ilango [3]. By using nano semi-open sets with nano β-open sets, nano S_β-open sets are introduced by Pirbal and Ahmed [4]. Moreover, regarding the structure of nano S_β-open sets, nano S_C-open sets defined by Pirbal and Ahmed [10]. The authors of [4] studied connectedness by using nano S_β-open sets in [11]. In addition, some separation

∗. Corresponding author

axioms via nano β-open sets studied by Ghosh [8] and almost nano regular space by David et al., [9]. So, in this study, since separation axioms are the main tool to distinguish two points, two sets or a point with a set topologically, it was such an inspiration for the authors to introduce some separation axioms such as nano S_β-regular, S_β-normal, $S_\beta - S_0$ and $S_\beta - S_1$ axioms in nano topological spaces. Then, each axiom is applied on the family of all nano S_β-open sets in terms of upper and lower approximations.

2. Preliminaries

Definition 2.1 ([1]). Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U called as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U,\ R)$ is said to be the approximation space. Let $X \subseteq U$:

1. The lower approximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U} \{R(x); R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U} \{R(x); R(x) \cap X \neq \phi\}$.
3. The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2 ([2]). Let U be the universe and R be an equivalence relation on U and $\tau_R(X) = \{\phi,\ U,\ L_R(X),\ U_R(X),\ B_R(X)\}$ where $X \subseteq U$. Then, $\tau_R(X)$ satisfies the followings axioms:

1. U and $\phi \in \tau_R(X)$;
2. the union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$;
3. the intersection of elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology on U and called the nano topology on U with respect to X. We call $(U,\ \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the nano dual topology of $\tau_R(X)$.

Definition 2.3. Let $(U,\ \tau_R(X))$ be a nano topological space and $A \subseteq U$. The set A is said to be:

1. Nano semi-open [2], if $A \subseteq ncl\,(nint\,(A))$.
2. Nano β-open (nano semi pre-open) [3], if $A \subseteq ncl\,(nint\,(ncl\,(A)))$.
3. Nano S_β-open [4], if A is nano semi-open and $A = \cup\{F_\alpha;\ F_\alpha$ nano β-closed sets$\}$.

The set of all nano semi-open, nano β-open and nano S_β-open sets denoted by $nSO\,(U,\ X)$, $n\beta O\,(U,\ X)$ and $nS_\beta O\,(U,\ X)$.

Theorem 2.4 ([5]). *Let A be any subset of a nano topological space $(U, \tau_R(X))$, then:*

1. $nS_\beta int(A) = \cup\{G : G$ *is* nS_β*-open and* $G \subseteq A\}$;
2. $nS_\beta cl(A) = \cap\{F : F$ *is* nS_β*-closed and* $A \subseteq F\}$;

Theorem 2.5 ([4]). *If $U_R\,(X) = U$ and $L_R\,(X) = \phi$ in a nano topological space $(U,\ \tau_R\,(X))$, then $nS_\beta O\,(U, X) = \{U,\ \phi\}$.*

Theorem 2.6 ([4]). *If $U_R\,(X) = U$ and $L_R\,(X) \neq \phi$ in a nano topological space $(U,\ \tau_R\,(X))$, then $\tau_R\,(X) = \tau_R^{S_\beta}\,(X)$.*

Theorem 2.7 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) = L_R\,(X) = \{x\}$, $x \in U$, then $nS_\beta O\,(U, X) = \{\phi,\ U\}$.*

Theorem 2.8 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) = L_R(X) \neq U$ and $U_R\,(X)$ contains more than one element of U, then the set of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$.*

Theorem 2.9 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) \neq U$, $L_R\,(X) = \phi$ and $U_R\,(X)$ contains more than one element of U, then the set of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$.*

Theorem 2.10 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) \neq L_R(X)$ where $U_R(X) \neq U$ and $L_R\,(X) \neq \phi$, then ϕ, $L_R\,(X)$, $B_R\,(X)$, $L_R\,(X) \cup B$, $B_R\,(X) \cup B$ and any set containing $U_R\,(X)$ where $B \subseteq [U_R\,(X)]^c$ are the only nS_β-open sets in U.*

Theorem 2.11 ([5]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) = L_R(X) \neq U$ and $U_R\,(X)$ contains more than one element of U, then for any non-empty subset A of U:*

$$nS_\beta cl\,(A) = \begin{cases} A, & if\ A \subset [U_R\,(X)]^c \\ U, & otherwise \end{cases}.$$

Theorem 2.12. *Let $(U,\ \tau_R\,(X))$ be a nano topological space. The only nS_β-clopen subset of U are ϕ and U if:*

1. *$U_R(X) = U$ and $L_R(X) = \phi$;*

2. *$U_R(X) = L_R(X) = \{x\}$, $x \in U$;*

3. *$U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U;*

4. *$U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U.*

Proof. Obvious. □

Theorem 2.13. *If $U_R(X) = U$ and $L_R(X) \neq \phi$ in a nano topological space $(U,\ \tau_R(X))$, then*

$$\left[\tau_{R}^{S_\beta}(X)\right] = \left[\tau_{R}^{S_\beta}(X)\right]^c$$

.

Proof. Obvious. □

Theorem 2.14. *Let $(U,\ \tau_R(X))$ be a nano topological space. If $U_R(X) \neq L_R(X)$, where $U_R(X) \neq U$ and $L_R(X) \neq \phi$, then $L_R(X)$, $B_R(X)$, $L_R(X) \cup B$ and $B_R(X) \cup B$ are non-empty proper nS_β-clopen in U where $B \subseteq [U_R(X)]^c$.*

Proof. By Theorem 2.10, $L_R(X)$, $B_R(X)$, $L_R(X) \cup B$ *and* $B_R(X) \cup B$ are non-empty proper nS_β-open set in U where $B \subseteq [U_R(X)]^c$. We have to show that they are also nS_β-closed in U.

Now, $nS_\beta cl(L_R(X)) = nS_\beta cl([B_R(X) \cup B]^c)$ where $B = [U_R(X)]^c$, but $[B_R(X) \cup B]^c$ is nS_β-closed, so $nS_\beta cl([B_R(X) \cup B]^c) = [B_R(X) \cup B]^c = L_R(X)$. Also, $nS_\beta cl(B_R(X)) = nS_\beta cl([L_R(X) \cup B]^c)$, where $B \subseteq [U_R(X)]^c$, but $[L_R(X) \cup B]^c$ is nS_β-closed, so $nS_\beta cl([L_R(X) \cup B]^c) = [L_R(X) \cup B]^c = B_R(X)$.

Also, $nS_\beta cl(L_R(X) \cup B) = nS_\beta cl([B_R(X)]^C) = [B_R(X)]^C = L_R(X) \cup B$ and $nS_\beta cl(B_R(X) \cup B) = nS_\beta cl([L_R(X)]^C) = [L_R(X)]^C = B_R(X) \cup B$, where $B \subset [U_R(X)]^c$. Hence, $L_R(X)$, $B_R(X)$, $L_R(X) \cup B$ and $B_R(X) \cup B$ are nS_β-clopen where $B \subseteq [U_R(X)]^c$. □

3. Nano S_β-regular spaces

Definition 3.1. A nano topological space $(U,\ \tau_R(X))$ is said to be nS_β-regular if for each $x \in U$ and each nano closed set A such that $x \notin$A, there exist two nS_β-open sets G and H such that $x \in G, A \subseteq H$ and $G \cap H = \phi$.

Remark 3.2. Nano indiscrete topological space is nS_β-regular space.

Theorem 3.3. *Let $(U,\ \tau_R(X))$ be a nano topological space. Then, U is a nS_β-regular space if and only if for each $x \in U$ and each nano open set G containing x, there exist a nS_β-open set V containing x such that $x \in V \subseteq nS_\beta cl(V) \subseteq G$.*

Proof. Let G be a nano open set and $x \in G$. Then, $U - \mathrm{G}$ is nano closed set such that $x \notin U - G$. By nS_β-regularity of U, there are nS_β-open sets M and W such that $x \in M$, $U - G \subseteq \mathrm{W}$ and $M \cap \mathrm{W} = \phi$. Therefore, $x \in M \subseteq U - \mathrm{W} \subseteq G$, Hence, $x \in M \subseteq nS_\beta cl\,(M) \subseteq nS_\beta cl\,(U - \mathrm{W}) = U - \mathrm{W} \subseteq G$. Thus, $nS_\beta cl(M) \subseteq U - \mathrm{W} \subseteq G$. Conversely, let F be nano closed set in U and let $x \notin F$. Then, $U - F$ is a nano open set and $x \in U - F$. By assumption, there exist a nS_β-open set H such that $x \in H$ and $nS_\beta cl(H) \subseteq U - F$. Define $K = U - nS_\beta cl(H)$. Then, $K \in nS_\beta O(U, X)$ and $H \subseteq nS_\beta cl\,(H)$, then $H \cap K = H \cap (U - nS_\beta cl(H)) = \phi$ $(U - nS_\beta cl(H) \subseteq U - H)$. Thus, for $x \notin F$, $\exists$ disjoint nS_β-open sets H and K such that $x \in H$ and $F \subseteq K$. Hence, U is a nS_β-regular space. □

Theorem 3.4. *Let $(U,\ \tau_R\,(X))$ be a nano topological space, then U is nS_β-regular if:*

1. *$U_R\,(X) = U$ and $L_R\,(X) = \phi$;*
2. *$U_R\,(X) = U$ and $L_R(X) \neq \phi$;*
3. *$U_R\,(X) \neq L_R(X)$, where $U_R(X) \neq U$ and $L_R\,(X) \neq \phi$.*

Proof.

1. By Theorem 2.5, $\tau_R\,(X) = nS_\beta O\,(U, X) = \{\phi,\, U\}$. Hence, U is nS_β-regular.

2. By Theorem 2.6, $\tau_R\,(X) = nS_\beta O\,(U, X) = \{\phi,\, U,\, L_R\,(X)\,,\, B_R\,(X)\}$. Let $x \in U$, since $L_R\,(X) \cap B_R\,(X) = \phi$ and $L_R\,(X) \cup B_R\,(X) = U$, then either $x \in L_R\,(X)$ or $x \in B_R\,(X)$. Also, $L_R\,(X) = [B_R\,(X)]^c$. Let say $x \in L_R\,(X)$, then $x \in L_R\,(X) \subseteq nS_\beta cl(L_R\,(X)) = [B_R\,(X)]^c \subseteq L_R\,(X)$. If $x \in B_R(X)$, then $x \in B_R\,(X) \subseteq nS_\beta cl(B_R\,(X)) = [L_R\,(X)]^c \subseteq B_R\,(X)$. Hence, U is nS_β-regular.

3. Let $x \in U_R(X)$, then either $x \in L_R\,(X)$ or $x \in B_R\,(X)$. Since by Theorem 2.14 $L_R\,(X)$ and $B_R(X)$ are nS_β-clopen in U, then let $x \in U_R(X)$:

   If $x \in L_R\,(X)$, then
   $$x \in L_R\,(X) \subseteq nS_\beta cl\,(L_R\,(X)) \subseteq \begin{cases} U_R(X) \\ L_R(X) \end{cases}.$$

   If $x \in B_R(X)$, then
   $$x \in B_R\,(X) \subseteq nS_\beta cl\,(B_R\,(X)) \subseteq \begin{cases} U_R(X) \\ L_R(X) \end{cases}.$$

   If $x \notin U_R(X)$, then the only nano open set containing x is U.

   Therefore, U is nS_β-regular space. □

Remark 3.5. Let $(U,\ \tau_R(X))$ be a nano topological space, then U is not nS_β-regular if:

1. $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U. Since $\tau_R(X) = \{\phi,\ U,\ U_R(X)\}$ and by Theorem 2.8, ϕ and those subsets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U. Let $x \in U_R(X)$, then there is no nS_β-open set V such that $x \in V \subseteq nS_\beta cl(V) \subseteq U_R(X)$, since by Theorem 2.11, $nS_\beta cl(V) = U$. Hence, U is not nS_β-regular space.

2. $U_R(X) \neq U,\ L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U. Since $\tau_R(X) = \{\phi,\ U,\ U_R(X)\}$ and by Theorem 2.9, ϕ and those subsets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U. Let $x \in U_R(X)$, then there is no nS_β-open set V such that $x \in V \subseteq nS_\beta cl(V) \subseteq U_R(X)$, since by Theorem 2.11, $nS_\beta cl(V) = U$. Hence, U is not nS_β-regular space.

3. $U_R(X) = L_R(X) \neq U$ and $U_R(X) = \{x\}$, $x \in U$. Since $\tau_R(X) = \{\phi,\ U, \{x\}\}$ and by Theorem 2.7, $nS_\beta O(U, X) = \{\phi,\ U\}$. Then, there is no nS_β-open set V such that $x \in V \subseteq nS_\beta cl(V) \subseteq \{x\}$. Hence, U is not nS_β-regular space.

4. Nano S_β-normal spaces

Definition 4.1. A nano topological space U is said to be nS_β-normal if for any disjoint nano closed sets A, B of U, there exist nS_β-open sets G and H such that $A \subseteq G, B \subseteq H$ and $G \cap H = \phi$.

Theorem 4.2. *A topological space U is nS_β-normal if and only if for each nano closed set F in U and nano open set G containing F, there is an nS_β-open set H such that $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$.*

Proof. Suppose that G is nano open set containing F, then $U - G$ and F are disjoint nano closed sets in U. Since U is nS_β-normal, so there exist nS_β-open sets H and V such that $F \subseteq H, U - G \subseteq V$ and $H \cap V = \phi$. Hence, $F \subseteq H \subseteq nS_\beta cl(H) \subseteq nS_\beta cl(U - V) = U - V \subseteq G$, or $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$.

Conversely, assume that for any nano-closed F and nano open set G containing F, there exists a nS_β-open set H such that $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$. Let F and K be disjoint nS_β-closed sets in U. So $F \cap K = \phi$ then $F \subseteq U - K$.As F is a nS_β-closed set and $U - K$ is a nS_β-open set, by assumption $\exists$ nS_β-open sets H in U such that, $F \subseteq H \subseteq nS_\beta cl(H) \subseteq U - K$. We get $K \subseteq U - nS_\beta cl(H)$. Define $G = U - nS_\beta cl(H)$. Thus $\exists\ G, H \in nS_\beta O(U, X)$ such that $F \subseteq H, K \subseteq G$ and $H \cap G = \phi$. Hence, U is a nS_β-normal space. □

Theorem 4.3. *Let $(U,\ \tau_R(X))$ be a nano topological space, then U is nS_β-normal if:*

1. *$U_R(X) = U$ and $L_R(X) = \phi$;*

2. *$U_R(X)=U$ and $L_R(X)\neq\phi$;*
3. *$U_R(X)=L_R(X)=\{x\}$, $x\in U$;*
4. *$U_R(X)=L_R(X)\neq U$ and $U_R(X)$ contains more than one element of U;*
5. *$U_R(X)\neq U$, $L_R(X)=\phi$ and $U_R(X)$ contains more than one element of U;*
6. *$U_R(X)\neq L_R(X)$, where $U_R(X)\neq U$ and $L_R(X)\neq\phi$.*

Proof.

1. Since $\tau_R(X)=\{\phi, U\}$. By Theorem 2.5, $\tau_R(X)=nS_\beta O(U,X)=\{\phi, U\}$. Hence, U is nS_β-normal space.
2. Since $\tau_R(X)=\{\phi, U, L_R(X), B_R(X)\}$. By Theorem 2.6, $\tau_R(X)=nS_\beta O(U,X)=\{\phi, U, L_R(X), B_R(X)\}$. Since $L_R(X)\cap B_R(X)=\phi$, $L_R(X)\cup B_R(X)=U$ and $L_R(X)=[B_R(X)]^c$. Hence, U is nS_β-normal space.
3. Since $\tau_R(X)=\{\phi, U,\{x\}\}$ and by Theorem 2.7, $nS_\beta O(U,X)=\{\phi, U\}$. Then, it is clear that U is nS_β-normal space.
4. Since $\tau_R(X)=\{\phi, U, U_R(X)\}$, by Theorem 2.8, ϕ, U and those sets A for which $U_R(X)\subseteq A$ are nS_β-open sets in U. In this case, 'ϕ with $[U_R(X)]^c$' and 'ϕ with U' are the disjoint nano closed sets in U. For 'ϕ with $[U_R(X)]^c$, $\phi\subseteq\phi$ and $[U_R(X)]^c\subseteq U$ and for 'ϕ and U' the result is clear. Hence, U is nS_β-normal.
5. Similar to part (i).
6. Since the only disjoint nano closed sets are ϕ and U. Therefore, U is nS_β-normal space. □

5. Nano S_β-S$_0$ and S_β-S$_1$ spaces

Definition 5.1. A nano topological space $(U, \tau_R(X))$ is called nS_β-S_0 if for every non-empty nS_β- open set A, $A\subseteq nS_\beta cl(\{x\})$, $\forall x\in A$.

Definition 5.2. A topological space $(U, \tau_R(X))$ is called nS_β-S_1 if for any distinct points $x,y\in U$ with $nS_\beta cl(\{x\})\neq nS_\beta cl(\{y\})$, there exist non-empty disjoint nS_β-open sets G and H such that, $G\subseteq nS_\beta cl(\{x\})$ and $H\subseteq nS_\beta cl(\{y\})$.

Theorem 5.3. *Let $(U, \tau_R(X))$ be a nano topological space. Then, U is nS_β-S_0 space if:*

1. *$U_R(X)=U$ and $L_R(X)=\phi$;*

2. *$U_R(X) = L_R(X) \neq U$ and $U_R(X) = \{x\}$, $x \in U$;*
3. *$U_R(X) = U$ and $L_R(X) \neq \phi$;*
4. *$U_R(X) \neq U$ and $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U;*
5. *$U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U.*

Proof.

1. Since $nS_\beta O(U, X) = \{\phi, U\}$. Then, only non-empty nS_β-open subset is U and $U = nS_\beta cl(\{x\})$, $\forall x \in U$. Hence, U is nS_β-S_0 space.
2. Since $nS_\beta O(U, X) = \{\phi, U\}$. Then, only non-empty nS_β-open subset is U and $U \subseteq nS_\beta cl(\{x\})$, $\forall x \in U$. Hence, U is nS_β-S_0 space.
3. Since $nS_\beta O(U, X) = \{\phi, U, L_R(X), B_R(X)\}$, then

   $$L_R(X) \subseteq nS_\beta cl(\{x\}) = L_R(X), \forall x \in L_R(X)$$

   and similarly for $B_R(X)$. Hence, U is nS_β-S_0.
4. Since ϕ, U and those sets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U, but $nS_\beta cl(\{x\}) = U$, $\forall x \in U_R(X)$, then $U_R(X) \subseteq U$. Hence, U is nS_β-S_0 space.
5. Since ϕ, U and those sets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U, but $nS_\beta cl(\{x\}) = U$, $\forall x \in U_R(X)$, then $U_R(X) \subseteq U$. Hence, U is nS_β-S_0 space. □

Remark 5.4. Let $(U, \tau_R(X))$ be a nano topological space. Then, U is not nS_β-S_0 space if $U_R(X) \neq L_R(X)$ where $U_R(X) \neq U$ and $L_R(X) \neq \phi$. Since $U_R(X) \in nS_\beta O(U, X)$ but $U_R(X) \nsubseteq nS_\beta cl(\{x\})$ for any $x \in U_R(X)$. Hence, U is not nS_β-S_0 space.

Theorem 5.5. *Let $(U, \tau_R(X))$ be a nano topological space. Then, U is nS_β-S_1 space if:*

1. *$U_R(X) = U$ and $L_R(X) = \phi$;*
2. *$U_R(X) = L_R(X) = \{x\}$, $x \in U$;*
3. *$U_R(X) = U$ and $L_R(X) \neq \phi$.*

Proof.

1. Obvious.
2. Obvious.

3. Since $nS_\beta O(U,X)=\{\phi, U, L_R(X), B_R(X)\}$ and $L_R(X)\cap B_R(X)=\phi$ also $L_R(X)\subseteq nS_\beta cl(\{x\})=L_R(X), \forall x\in L_R(X)$ and $B_R(X)\subseteq nS_\beta cl(\{x\})=B_R(X), \forall x\in B_R(X)$, then for any $x\in L_R(X)$ and $y\in B_R(X)$, $nS_\beta cl(\{x\})\neq nS_\beta cl(\{y\})$ and $L_R(X)\subseteq nS_\beta cl(\{x\})$ and $B_R(X)\subseteq nS_\beta cl(\{y\})$. Hence, U is nS_β-S_1 space. □

Remark 5.6. Let $(U, \tau_R(X))$ be a nano topological space. Then, U is not nS_β-S_1 space if:

1. $U_R(X)\neq U$ and $L_R(X)=\phi$ and $U_R(X)$ contains more than one element of U.

   Since for any distinct points $x,y\in U$ with $nS_\beta cl(\{x\})\neq nS_\beta cl(\{y\})$, there is no non-empty disjoint nS_β-open sets G and H such that, $G\subseteq nS_\beta cl(\{x\})$ and $H\subseteq nS_\beta cl(\{y\})$, since every nS_β-open set containing $U_R(X)$. Hence, U is not nS_β-S_1 space.

2. Similar to part (i).

3. $U_R(X)\neq L_R(X)$ where $U_R(X)\neq U$ and $L_R(X)\neq\phi$. Since any non-empty proper subset A of U with less than one element of U is nS_β-open set and its complement is singleton nS_β-closed, then $nS_\beta cl(\{x\})=\{x\}$, for any $x\in[U_R(X)]^c$. Also, $nS_\beta cl(\{y\})=L_R(X)$, for any $y\in L_R(X)$, then $nS_\beta cl(\{x\})\neq nS_\beta cl(\{y\})$, but there is no non-empty nS_β-open set such that $G\subseteq nS_\beta cl(\{x\})$. Hence, U is not nS_β-S_1 space.

Theorem 5.7. *Every nS_β-S_1 space is nS_β-S_0.*

Proof. The proof follows form Theorem 5.3 and Theorem 5.5. □

The converse of above theorem need not to be true, as it shown by the following example.

Example 5.8. Let $U=\{a,b,c\}$ with $U/R=\{\{a,b\},\{c\}\}$ and $X=\{a,b\}$. Then, $\tau_R(X)=nS_\beta O(U,X)=\{\phi, U, \{a,b\}\}$. Then, $nS_\beta cl(\{a\})=U$ and $nS_\beta cl(\{c\})=\{c\}$ which there is no non-empty disjoint nS_β-open sets G and H containing $nS_\beta cl(\{a\})$ and $nS_\beta cl(\{c\})$ respectively. Hence, U is not nS_β-S_1 space.

6. Conclusion

In this paper, we have introduced the concepts of nano S_β-regular, S_β-normal, $S_\beta-S_0$ and $S_\beta-S_1$ axioms in nano topological spaces. According to the family of all nano S_β-open sets, the axioms are studied and the relationship among the axioms presented in the table below. For instance, we can see that every nS_β-regular space is nS_β-normal but the converse is proved that is not true in three cases.

Family of nS_β-open sets in term of upper and lower approximations if:	nS_β-regular	nS_β-normal	$nS_\beta - S_0$	$nS_\beta - S_1$
$U_R(X) = U$ and $L_R(X) = \phi$	1	1	1	1
$U_R(X) = U$ and $L_R(X) \neq \phi$	1	1	1	1
$U_R(X) = L_R(X) = \{x\}$, $x \in U$	0	1	1	1
$U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U.	0	1	1	0
$U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U.	0	1	1	0
$U_R(X) \neq U$, $L_R(X) \neq \phi$ and $U_R(X) \neq L_R(X)$	1	1	0	0

References

[1] Z. Pawlak, *Rough sets*, Int. J. Comput. Inf. Sci., 11 (1982), 341-356.

[2] M. L. Thivagar, C. Richard, *On nano forms of weakly open sets*, Int. J. Math. Stats. Inv., 1 (2013), 31-37.

[3] A. Revathy, G. Ilango, *On nano β-open sets*, Int. J. Eng. Contemp. Math. Sci., 1 (2015), 1-6.

[4] O. T. Pirbal, N. K. Ahmed, *On nano S_β-open sets in nano topological spaces*, Gen. Lett. Math., 12 (2022), 23-30.

[5] N. K. Ahmed, O. T. Pirbal, *Nano S_β-operators and nano S_β-continuity in nano topological spaces*, Duhok Univ. J., (2022), 26 (2023), 1-11.

[6] M. L. Thivagar, C. Richard, *Note on nano topological spaces*, communicated, 2013.

[7] M. L. Thivagar, S. Jafari, V. S. Devi., *On new class of contra continuity in nano topology*, Italian J. Pure. Appl. Math., 41 (2017), 1-12.

[8] M. K. Ghosh, *Separation axioms and graphs of functions in nano topological spaces via nano β-open sets*, Annals Pure. Appl. Math., 14 (2017), 213-223.

[9] S. A. David, M. Rameshpandi, R. Premkumar, *Almost nano regular spaces*, Adv. Math. Sci. J., 9 (2020), 5631-5636.

[10] N. K. Ahmed, O. T. Pirbal, *Nano S_C-open sets in nano topological spaces*, Ibn Al-Haitham J. Pure. Appl. Sci., 36 (2023), 306-313.

[11] N. K. Ahmed, O. T. Pirbal, *Nano S_β-connectedness in nano topological spaces*, Al-Mustansiriyah J. Sci., 34 (2023), 87-94.

Accepted: January 27, 2023

On graded weakly classical 2-absorbing submodules of graded modules over graded commutative rings

Shatha Alghueiri
Department of Mathematics and Statistics
Jordan University of Science and Technology
P.O.Box 3030, Irbid 22110
Jordan
soalghueiri@just.edu.jo

Khaldoun Al-Zoubi[∗]
Department of Mathematics and Statistics
Jordan University of Science and Technology
P.O.Box 3030, Irbid 22110
Jordan
kfzoubi@just.edu.jo

Abstract. In this paper, we introduce the concept of graded weakly classical 2-absorbing submodule as a generalization of a graded classical 2-absorbing submodule. We give a number of results concerning this class of graded submodules and their homogeneous components.

Keywords: graded weakly classical 2-absorbing submodule, graded classical 2-absorbing submodule, graded 2-absorbing submodule.

1. Introduction and preliminaries

Throughout this paper all rings with identity and all modules are unitary.

Refai and Al-Zoubi in [23] introduced the concept of graded primary ideal. The concept of graded 2-absorbing ideal was introduced and studied by Al-Zoubi, Abu-Dawwas and Ceken in [5]. The concept of graded prime submodule was introduced and studied by many authors, see for example [2, 3, 12, 13, 15, 22]. The concept of graded classical prime submodules as a generalization of graded prime submodules was introduced in [17] and studied in [11]. The concept of graded weakly classical prime submodules, generalizations of graded classical prime submodules, was introduced by Abu-Dawwas and Al-Zoubi in [1]. The concept of graded 2-absorbing submodule, generalizations of graded prime submodule, was introduced by Al-Zoubi and Abu-Dawwas in [4] and studied in [8, 9]. Then, many generalizations of graded 2-absorbing submodules were studied such as graded 2-absorbing primary (see [16]), graded weakly 2-absorbing primary (see [7]) and graded 2-absorbing I_e-prime submodules (see [14]).

∗. Corresponding author

Recently, Al-Zoubi and Al-Azaizeh, in [6] introduced the concept of graded classical 2-absorbing submodules over a graded commutative ring as a new generalization of graded 2-absorbing submodules.

Here, we introduce the concept of graded weakly classical 2-absorbing submodule as a new generalization of graded classical 2-absorbing submodule on the one hand and a generalization of a graded weakly classical prime submodule on other hand.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [18, 19, 20, 21] for these basic properties and more information on graded rings and modules. Let G be a group with identity element e. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_h of R indexed by the elements $h \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The non-zero elements of R_g are said to be homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{h\in G} R_h$. If $r \in R$, then r can be written uniquely as $\sum_{g\in G} r_g$, where r_g is called a homogeneous component of r in R_g. Moreover, R_e is a subring of R and $1 \in R_e$ (see [21]). Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \sum_{h\in G}(I \cap R_h) := \sum_{h\in G} I_h$ (see [21]). Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. A left R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Similarly, if an element of M belongs to $\cup_{g\in G} M_h = h(M)$, then it is called a homogeneous. Note that M_g is an R_e-module for every $g \in G$. Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. A submodule N of M is said to be a graded submodule of M if $N = \oplus_{g\in G}(N \cap M_g) := \oplus_{g\in G} N_g$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g := (M_g + N)/N$ for $g \in G$.

2. Graded weakly classical 2-absorbing submodules

Definition 2.1. Let R be a G-graded ring, M a graded R-module, N a proper graded submodule of M and $g \in G$.

(i) We say that N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g if $N_g \neq M_g$; and whenever $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$ with $0 \neq r_e s_e t_e m_g \in N_g$, then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$.

(ii) We say that N is a graded weakly classical 2-absorbing submodule of M if $r_h, s_\alpha, t_\beta \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_h s_\alpha t_\beta m_\lambda \in N$, then either $r_h s_\alpha m_\lambda \in N$ or $r_h t_\beta m_\lambda \in N$ or $s_\alpha t_\beta m_\lambda \in N$.

Clearly, every graded classical 2-absorbing submodule is a graded weakly classical 2-absorbing. However, since $\{0\}$ is always a graded weakly classical 2-absorbing submodule (by defintion), a graded weakly classical 2-absorbing submodule need not be a graded classical 2-absorbing submodule.

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded weakly classical 2-absorbing submodule of M, then for each $g \in G$ with $N_g \neq M_g$, N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g.*

Proof. Suppose that N is a graded weakly classical 2-absorbing submodule of M and $g \in G$ with $N_g \neq M_g$. Now, assume that $0 \neq r_e s_e t_e m_g \in N_g$ where $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$. Then, $0 \neq r_e s_e t_e m_g \in N$. Since N is a graded weakly classical 2-absorbing submodule of M, either $r_e s_e m_g \in N$ or $r_e t_e m_g \in N$ or $s_e t_e m_g \in N$. But $N_g = N \cap M_g$, so we get that either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. Hence, N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g. □

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. A proper submodule N_g of the R_e-module M_g is said to be a classical g-2-absorbing submodule if whenever $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$ with $r_e s_e t_e m_g \in N_g$, then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$ (see [6]).

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $g \in G$. If N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g, then either N_g is a classical g-2-absorbing submodule of the R_e-module M_g or $(N_g :_{R_e} M_g)^3 N_g = 0$.*

Proof. Suppose that $(N_g :_{R_e} M_g)^3 N_g \neq 0$. Let $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$ such that $r_e s_e t_e m_g \in N_g$. If $r_e s_e t_e m_g \neq 0$, then we get the result as N_g is a weakly classical g-2-absorbing of M_g. So, we assume $r_e s_e t_e m_g = 0$. Now, if $r_e s_e t_e N_g \neq 0$, then there exists $n_{1_g} \in N_g$ such that $r_e s_e t_e n_{1_g} \neq 0$, so $0 \neq r_e s_e t_e (m_g + n_{1_g}) \in N_g$ which yields either $r_e s_e (m_g + n_{1_g}) \in N_g$ or $r_e t_e (m_g + n_{1_g}) \in N_g$ or $s_e t_e (m_g + n_{1_g}) \in N_g$ and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. So, we can assume that $r_e s_e t_e N_g = 0$. Now, if $r_e s_e (N_g :_{R_e} M_g) m_g \neq 0$, then there exists $t_{1_e} \in (N_g :_{R_e} M_g)$ such that $r_e s_e t_{1_e} m_g \neq 0$. Thus, $0 \neq r_e s_e (t_e + t_{1_e}) m_g \in N_g$ and then either $r_e s_e m_g \in N_g$ or $r_e (t_e + t_{1_e}) m_g \in N_g$ or $s_e (t_e + t_{1_e}) m_g \in N_g$ which follows either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. We can assume that $r_e s_e (N_g :_{R_e} M_g) m_g = 0$, $r_e t_e (N_g :_{R_e} M_g) m_g = 0$ and $s_e t_e (N_g :_{R_e} M_g) m_g = 0$. Now, if $r_e (N_g :_{R_e} M_g)^2 m_g \neq 0$, then there exist $s_{2_e}, t_{2_e} \in (N_g :_{R_e} M_g)$ such that $r_e s_{2_e} t_{2_e} m_g \neq 0$. Thus, by our assumptions we get $0 \neq r_e (s_e + s_{2_e})(t_e + t_{2_e}) m_g \in N_g$ which gives either $r_e (s_e + s_{2_e}) m_g \in N_g$ or $r_e (t_e + t_{2_e}) m_g \in N_g$ or $(s_e + s_{2_e})(t_e + t_{2_e}) m_g \in N_g$, and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. So, we assume that $r_e (N_g :_{R_e} M_g)^2 m_g = 0$, $s_e (N_g :_{R_e} M_g)^2 m_g = 0$ and $t_e (N_g :_{R_e} M_g)^2 m_g = 0$. Now, if $r_e s_e (N_g :_{R_e} M_g) N_g \neq 0$, then there exist $t_{3_e} \in (N_g :_{R_e} M_g)$ and $n_{2_g} \in N_g$ such that $r_e s_e t_{3_e} n_{2_g} \neq 0$. Hence, by our assumptions we get $0 \neq r_e s_e (t_e + t_{3_e})(m_g + n_{2_g}) \in N_g$ and then either $r_e s_e (m_g + n_{2_g}) \in N_g$ or $r_e (t_e + t_{3_e})(m_g + n_{2_g}) \in N_g$ or $s_e (t_e + t_{3_e})(m_g + n_{2_g}) \in N_g$ which yields either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. We assume that $r_e s_e (N_g :_{R_e} M_g) N_g = 0$, $r_e t_e (N_g :_{R_e} M_g) N_g = 0$ and $s_e t_e (N_g :_{R_e} M_g) N_g = 0$.

Now, if $r_e(N_g :_{R_e} M_g)^2 N_g \neq 0$, then there exist $s_{4_e}, t_{4_e} \in (N_g :_{R_e} M_g)$ and $n_{3_g} \in N_g$ such that $r_e s_{4_e} t_{4_e} n_{3_g} \neq 0$. Thus, by assumptions, $0 \neq r_e(s_e + s_{4_e})(t_e + t_{4_e})(m_g + n_{3_g}) \in N_g$, then either $r_e(s_e + s_{4_e})(m_g + n_{3_g}) \in N_g$ or $r_e(t_e + t_{4_e})(m_g + n_{3_g}) \in N_g$ or $(s_e + s_{4_e})(t_e + t_{4_e})(m_g + n_{3_g}) \in N_g$, and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. So, we can assume that $r_e(N_g :_{R_e} M_g)^2 N_g = 0$, $s_e(N_g :_{R_e} M_g)^2 N_g = 0$ and $t_e(N_g :_{R_e} M_g)^2 N_g = 0$. Since $(N_g :_{R_e} M_g)^3 N_g \neq 0$, there exist $r_{5_e}, s_{5_e}, t_{5_e} \in (N_g :_{R_e} M_g)$ and $n_{4_g} \in N_g$ such that $r_{5_e} s_{5_e} t_{5_e} n_{4_g} \neq 0$. Hence, by our assumptions we get $0 \neq (r_e + r_{5_e})(s_e + s_{5_e})(t_e + t_{5_e})(m_g + n_{4_g}) \in N_g$ which follows that either $(r_e + r_{5_e})(s_e + s_{5_e})(m_g + n_{4_g}) \in N_g$ or $(r_e + r_{5_e})(t_e + t_{5_e})(m_g + n_{4_g}) \in N_g$ or $(s_e + s_{5_e})(t_e + t_{5_e})(m_g + n_{4_g}) \in N_g$, and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. Therefore, N_g is a classical g-2-absorbing submodule of the R_e-module M_g. □

Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be a graded weakly classical prime submodule if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h m_\lambda \in N$, then either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ (see [1]).

It is easy to see that every graded weakly classical prime submodule is a graded weakly classical 2-absorbing. The following example shows that the converse is not true in general.

Example 2.4. Let $G = \mathbb{Z}_2$, then $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodule $N = 4\mathbb{Z}$ of M. Then, N is not a graded weakly classical prime submodule of M since $0 \neq 2 \cdot 2 \cdot 3 \in N$ but $2 \cdot 3 \notin N$. However, easy computations show that N is a graded weakly classical 2-absorbing submodule of M.

Theorem 2.5. *Let R be a G-graded ring, M a graded R-module and N and K be two graded submodules of M with $N \subsetneq K$. If N is a graded weakly classical 2-absorbing submodule of M, then N is a graded weakly classical 2-absorbing submodule of K.*

Proof. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in K \cap h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in N$, then either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$ as N is a graded weakly classical 2-absorbing submodule of M. So, we get the result. □

The following example shows that a graded submodule of a graded weakly classical 2-absorbing submodule need not be a graded weakly classical 2-absorbing.

Example 2.6. Let $G = \mathbb{Z}_2$, then $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodules $N = 4\mathbb{Z}$ and $K = 16\mathbb{Z} \subseteq N$ of M. It is easy to see that N is a graded weakly classical 2-absorbing submodule of M but K is not a graded weakly classical 2-absorbing since $0 \neq 2 \cdot 2 \cdot 2 \cdot 2 \in K$ and $2 \cdot 2 \cdot 2 \notin K$.

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and N and K be two proper graded R-submodules of M such that $K \subseteq N$. Then, the following statements hold:*

(i) *If N is a graded weakly classical 2-absorbing submodule of M, then N/K is a graded weakly classical 2-absorbing submodule of M/K.*

(ii) *If K is a graded weakly classical 2-absorbing submodule of M and N/K is a graded weakly classical 2-absorbing submodule of M/K, then N is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda + K \in h(M/K)$ such that $0_{M/K} \neq r_g s_h t_\alpha m_\lambda + K \in N/K$. Hence, $0_M \neq r_g s_h t_\alpha m_\lambda \in N$ which implies that either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$ and then either $r_g s_h m_\lambda + K \in N/K$ or $r_g t_\alpha m_\lambda + K \in N/K$ or $s_h t_\alpha m_\lambda + K \in N/K$. Therefore, N/K is a graded weakly classical 2-absorbing submodule of M/K.

(ii) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0_M \neq r_g s_h t_\alpha m_\lambda \in N$. Now, if $0_M \neq r_g s_h t_\alpha m_\lambda \in K$, then either $r_g s_h m_\lambda \in K \subseteq N$ or $r_g t_\alpha m_\lambda \in K \subseteq N$ or $s_h t_\alpha m_\lambda \in K \subseteq N$. Otherwise, we get $0_{M/K} \neq r_g s_h t_\alpha m_\lambda + K \in N/K$ and then either $r_g s_h m_\lambda + K \in N/K$ or $r_g t_\alpha m_\lambda + K \in N/K$ or $s_h t_\alpha m_\lambda + K \in N/K$. Thus, either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. Therefore, N is a graded weakly classical 2-absorbing submodule of M. □

The following example shows that the intersection of two graded weakly classical 2-absorbing submodules need not be a graded weakly classical 2-absorbing submodule.

Example 2.8. Let $G = \mathbb{Z}_2$. Then, $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodules $N = 4\mathbb{Z}$ and $K = 9\mathbb{Z}$ of M. It is easy to see that N and K are graded weakly classical 2-absorbing submodules of M. But $N \cap K = 36\mathbb{Z}$ is not a graded weakly classical 2-absorbing submodule of M, since $0 \neq 2 \cdot 2 \cdot 3 \cdot 3 \in 36\mathbb{Z}$ and neither $2 \cdot 2 \cdot 3 \in 36\mathbb{Z}$ nor $2 \cdot 3 \cdot 3 \in 36\mathbb{Z}$.

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module and N and K be two graded submodules of M. If N and K are graded weakly classical prime submodules of M, then $N \cap K$ is a graded weakly classical 2-absorbing submodule of M.*

Proof. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in N \cap K$. Hence, $0 \neq r_g s_h t_\alpha m_\lambda \in N$ and $0 \neq r_g s_h t_\alpha m_\lambda \in K$. This yields that either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $t_\alpha m_\lambda \in N$ and either $r_g m_\lambda \in K$ or $s_h m_\lambda \in K$ or $t_\alpha m_\lambda \in K$ as N and K are graded classical prime submodules of M. Assume, without loss of generality, $r_g m_\lambda \in N$ and $s_h m_\lambda \in K$. Thus, $r_g s_h m_\lambda \in N \cap K$. Therefore, $N \cap K$ is a graded weakly classical 2-absorbing submodule of M. □

Let M and M' be two graded R-modules. A homomorphism of graded R-modules $f: M \to M'$ is a homomorphism of R-modules which satisfies $f(M_g) \subseteq M'_g$ for every $g \in G$ (see [21]).

Theorem 2.10. *Let R be a G-graded ring, M and M' be two graded R-modules and $f: M \to M'$ be a graded homomorphism.*

(i) *If f is a graded epimorphism and N is a graded weakly classical 2-absorbing submodule of M with $ker(f) \subseteq N$, then $f(N)$ is a graded weakly classical 2-absorbing submodule of M'.*

(ii) *If f is a graded isomorphism and N' is a graded weakly classical 2-absorbing submodule of M', then $f^{-1}(N')$ is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Clearly, $f(N)$ is a proper graded submodule of M'. Now, let $r_g, s_h, t_\alpha \in h(R)$ and $m'_\lambda \in h(M')$ such that $0 \neq r_g s_h t_\alpha m'_\lambda \in f(N)$. Since f is a graded epimorphism, there exists $m_\lambda \in h(M)$ such that $f(m_\lambda) = m'_\lambda$. Hence, $0 \neq r_g s_h t_\alpha m'_\lambda = f(r_g s_h t_\alpha m_\lambda) \in f(N)$ and then there exists $n \in N \cap h(M)$ such that $f(r_g s_h t_\alpha m_\lambda) = f(n)$ which yields that $r_g s_h t_\alpha m_\lambda - n \in ker(f) \subseteq N$, so $0 \neq r_g s_h t_\alpha m_\lambda \in N$. Thus, as N is a graded weakly classical 2-absorbing submodule of M we get either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. So, either $r_g s_h m'_\lambda \in f(N)$ or $r_g t_\alpha m'_\lambda \in f(N)$ or $s_h t_\alpha m'_\lambda \in f(N)$. Therefore, $f(N)$ is a graded weakly classical 2-absorbing submodule of M'.

(ii) It is easy to see that $f^{-1}(N')$ is a proper graded submodule of M. Now, let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in f^{-1}(N')$. Thus, $0 \neq r_g s_h t_\alpha f(m_\lambda) \in N'$ and then either $r_g s_h f(m_\lambda) \in N'$ or $r_g t_\alpha f(m_\lambda) \in N'$ or $s_h t_\alpha f(m_\lambda) \in N'$ as N' is a graded weakly classical 2-absorbing submodule of M'. Hence, either $r_g s_h m_\lambda \in f^{-1}(N')$ or $r_g t_\alpha m_\lambda \in f^{-1}(N')$ or $s_h t_\alpha m_\lambda \in f^{-1}(N')$. Therefore, $f^{-1}(N')$ is a graded weakly classical 2-absorbing submodule of M. □

Recall from [4] that a proper graded submodule N of a graded R-module M is said to be a graded weakly 2-absorbing submodule of M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g s_h \in (N :_R M)$ or $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$.

Theorem 2.11. *Let R be a G-graded ring, M a graded gr-cyclic R-module and N a proper graded submodule of M. If N is a graded weakly classical 2-absorbing submodule of M, then N is a graded weakly 2-absorbing submodule of M.*

Proof. Since M is a gr-cyclic, there exists $m_{\lambda_1} \in h(M)$ such that $M = Rm_{\lambda_1}$. Now, let $r_g, s_h \in h(R)$ and $m_{\lambda_2} \in h(M)$ with $0 \neq r_g s_h m_{\lambda_2} \in N$. Hence, there exists $t_\alpha \in h(R)$ such that $0 \neq r_g s_h m_{\lambda_2} = r_g s_h t_\alpha m_{\lambda_1} \in N$. This yields that either $r_g m_{\lambda_2} = r_g t_\alpha m_{\lambda_1} \in N$ or $s_h m_{\lambda_2} = s_h t_\alpha m_{\lambda_1} \in N$ or $r_g s_h \in (N :_R m_{\lambda_1}) = (N :_R M)$ as N is a graded weakly classical 2-absorbing submodule of M. Therefore, N is a graded weakly 2-absorbing submodule of M. □

Recall from [5] that a proper graded ideal I of R is said to be a graded weakly 2-absorbing ideal of R if whenever $r_g, s_h, t_\alpha \in h(R)$ with $0 \neq r_g s_h t_\alpha \in I$, then $r_g s_h \in I$ or $r_g t_\alpha \in I$ or $s_h t_\alpha \in I$.

Theorem 2.12. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M.*

(i) *If N is a graded weakly classical 2-absorbing submodule of M and $m_\lambda \in h(M)\backslash N$ with $Ann_R(m_\lambda) = \{0\}$, then $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R.*

(ii) *If $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R for each $m_\lambda \in h(M)\backslash N$, then N is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Let $m_\lambda \in h(M)\backslash N$, so $(N :_R m_\lambda)$ is a proper graded ideal of R. Now, let $r_g, s_h, t_\alpha \in h(R)$ with $0 \neq r_g s_h t_\alpha \in (N :_R m_\lambda)$. Since $Ann_R(m_\lambda) = \{0\}$, $0 \neq r_g s_h t_\alpha m_\lambda \in N$. Hence, we get either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$ as N is a graded weakly classical 2-absorbing submodule of M. This yields that either $r_g s_h \in (N :_R m_\lambda)$ or $r_g t_\alpha \in (N :_R m_\lambda)$ or $s_h t_\alpha \in (N :_R m_\lambda)$. Therefore, $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R.

(ii) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in N$. If $m_\lambda \in N$, the we get the result. So, we assume $m_\lambda \notin N$, then $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R. Hence, $0 \neq r_g s_h t_\alpha \in (N :_R m_\lambda)$ which yields that $r_g s_h \in (N :_R m_\lambda)$ or $r_g t_\alpha \in (N :_R m_\lambda)$ or $s_h t_\alpha \in (N :_R m_\lambda)$ and then either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. Therefore, N is a graded weakly classical 2-absorbing submodule of M. □

A graded zero-divisor on a graded R-module M is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $rm = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$.

The following result studies the behavior of graded weakly classical 2-absorbing submodules under localization.

Theorem 2.13. *Let R be a G-graded ring, M a graded R-module, $S \subseteq h(R)$ a multiplication closed subset of R and N a graded submodule of M. Then, the following statements hold.*

(i) *If N is a graded weakly classical 2-absorbing submodule of M and $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$.*

(ii) *If $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$ such that $S \cap G\text{-}Zdv_R(N) = \emptyset$ and $S \cap G\text{-}Zdv_R(M/N) = \emptyset$, then N is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Suppose that N is a graded weakly classical 2-absorbing submodule of M. Since $(N :_R M) \cap S = \emptyset$, $S^{-1}N$ is a proper graded submodule of

$S^{-1}M$. Now, let $\frac{r_g}{s_1}, \frac{s_h}{s_2}, \frac{t_\alpha}{s_3} \in h(S^{-1}R)$ and $\frac{m_\lambda}{s_4} \in h(S^{-1}M)$ such that $0_{S^{-1}M} \neq \frac{r_g}{s_1}\frac{s_h}{s_2}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{r_g s_h t_\alpha m_\lambda}{s_1 s_2 s_3 s_4} \in S^{-1}N$. Hence, there exists $s_5 \in S$ such that $s_5 r_g s_h t_\alpha m_\lambda \in N$. If $s_5 r_g s_h t_\alpha m_\lambda = 0_M$, then $\frac{r_g}{s_1}\frac{s_h}{s_2}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{s_5 r_g s_h t_\alpha m_\lambda}{s_5 s_1 s_2 s_3 s_4} = 0_{S^{-1}M}$, a contradiction. So, $0_M \neq s_5 r_g s_h t_\alpha m_\lambda \in N$. This yields that either $s_5 r_g s_h m_\lambda \in N$ or $s_5 r_g t_\alpha m_\lambda \in N$ or $s_5 s_h t_\alpha m_\lambda \in N$. Thus, either $\frac{r_g}{s_1}\frac{s_h}{s_2}\frac{m_\lambda}{s_4} = \frac{s_5 r_g s_h m_\lambda}{s_5 s_1 s_2 s_4} \in S^{-1}N$ or $\frac{r_g}{s_1}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{s_5 r_g t_\alpha m_\lambda}{s_5 s_1 s_3 s_4} \in S^{-1}N$ or $\frac{s_h}{s_2}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{s_5 s_h t_\alpha m_\lambda}{s_5 s_2 s_3 s_4} \in S^{-1}N$. Therefore, $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$.

(ii) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0_M \neq r_g s_h t_\alpha m_\lambda \in N$. Hence, $\frac{r_g}{1}\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$. If $\frac{r_g}{1}\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} = 0_{S^{-1}M}$, then there exists $s \in S$ with $s r_g s_h t_\alpha m_\lambda = 0_M$, but $S \cap G\text{-}Zdv_R(N) = \emptyset$, a contradiction. So, $0_{S^{-1}M} \neq \frac{r_g}{1}\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$. Thus, either $\frac{r_g}{1}\frac{s_h}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{r_g}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$ as $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$. If $\frac{r_g}{1}\frac{s_h}{1}\frac{m_\lambda}{1} \in S^{-1}N$, then there exists $s \in S$ with $s r_g s_h m_\lambda \in N$ and this follows that $r_g s_h m_\lambda \in N$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. Similarly, if either $\frac{r_g}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$, then $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. Therefore, N is a graded weakly classical 2-absorbing submodule of M. □

Theorem 2.14. *Let R be a G-graded ring, M_1 and M_2 be two graded R-modules and N_1 and N_2 be two proper graded submodules of M_1 and M_2, respectively. Let $M = M_1 \times M_2$. Then, the following statements hold.*

(i) *N_1 is a graded weakly classical 2-absorbing submodule of M_1 and for each $r_g, s_h, t_\alpha \in h(R)$ and $m_{1_\lambda} \in h(M_1)$ with $r_g s_h t_\alpha m_{1_\lambda} = 0$, $r_g s_h m_{1_\lambda} \notin N_1$, $r_g t_\alpha m_{1_\lambda} \notin N_1$ and $s_h t_\alpha m_{1_\lambda} \notin N_1$, implies $r_g s_h t_\alpha \in Ann_R(M_{2_\lambda})$ if and only if $N_1 \times M_2$ is a graded weakly classical 2-absorbing submodule of M.*

(ii) *N_2 is a graded weakly classical 2-absorbing submodule of M_2 and for each $r_g, s_h, t_\alpha \in h(R)$ and $m_{2_\lambda} \in h(M_2)$ with $r_g s_h t_\alpha m_{2_\lambda} = 0, r_g s_h m_{2_\lambda} \notin N_2$, $r_g t_\alpha m_{2_\lambda} \notin N_2$ and $s_h t_\alpha m_{2_\lambda} \notin N_2$, implies $r_g s_h t_\alpha \in Ann_R(M_{1_\lambda})$ if and only if $M_1 \times N_2$ is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Suppose that $N_1 \times M_2$ is a graded weakly classical 2-absorbing submodule of M. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_{1_\lambda} \in h(M_1)$ such that $0 \neq r_g s_h t_\alpha m_{1_\lambda} \in N_1$. Hence, $(0,0) \neq r_g s_h t_\alpha (m_{1_\lambda}, 0) \in N_1 \times M_2$ and then either $r_g s_h (m_{1_\lambda}, 0) \in N_1 \times M_2$ or $r_g t_\alpha (m_{1_\lambda}, 0) \in N_1 \times M_2$ or $s_h t_\alpha (m_{1_\lambda}, 0) \in N_1 \times M_2$, and so either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$ or $s_h t_\alpha m_{1_\lambda} \in N_1$. Thus, N_1 is a graded weakly classical 2-absorbing submodule of M_1. Now, let $r_g, s_h, t_\alpha \in h(R)$ and $m_{1_\lambda} \in h(M_1)$ such that $r_g s_h t_\alpha m_{1_\lambda} = 0$ and neither $r_g s_h m_{1_\lambda} \in N_1$ nor $r_g t_\alpha m_{1_\lambda} \in N_1$ nor $s_h t_\alpha m_{1_\lambda} \in N_1$. And assume $r_g s_h t_\alpha \notin Ann_R(M_{2_\lambda})$, then there exists $m_{2_\lambda} \in M_{2_\lambda}$ such that $r_g s_h t_\alpha m_{2_\lambda} \neq 0$. Thus, $(0,0) \neq r_g s_h t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$, which yields either $r_g s_h (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $r_g t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $s_h t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ and then either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$ or $s_h t_\alpha m_{1_\lambda} \in N_1$, a contradiction. Therefore, $r_g s_h t_\alpha \in Ann_R(M_{2_\lambda})$. Conversely, let $r_g, s_h, t_\alpha \in h(R)$ and $(m_{1_\lambda}, m_{2_\lambda}) \in h(M)$ such that $(0,0) \neq r_g s_h t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$. If $0 \neq r_g s_h t_\alpha m_{1_\lambda} \in N_1$, then either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$

or $s_h t_\alpha m_{1_\lambda} \in N_1$ as N_1 is a graded weakly classical 2-absorbing submodule of M_1, so either $r_g s_h(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $r_g t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $s_h t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$. Now, if $r_g s_h t_\alpha m_{1_\lambda} = 0$, then $r_g s_h t_\alpha m_{2_\lambda} \neq 0$ and so $r_g s_h t_\alpha \notin Ann_R(M_{2_\lambda})$. Thus, either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$ or $s_h t_\alpha m_{1_\lambda} \in N_1$ and then either $r_g s_h(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $r_g t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $s_h t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a graded weakly classical 2-absorbing submodule of M.

(*ii*) The proof is similar to that in part (*i*). □

Theorem 2.15. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If N is a graded weakly classical 2-absorbing submodule of M, then for each $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$, then $(N :_R r_g s_h t_\alpha m_\lambda) = (0 :_R r_g s_h t_\alpha m_\lambda) \cup (N :_R r_g s_h m_\lambda) \cup (N :_R r_g t_\alpha m_\lambda) \cup (N :_R s_h t_\alpha m_\lambda)$.*

Proof. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$. It is easy to see that $(0 :_R r_g s_h t_\alpha m_\lambda) \cup (N :_R r_g s_h m_\lambda) \cup (N :_R r_g t_\alpha m_\lambda) \cup (N :_R s_h t_\alpha m_\lambda) \subseteq (N :_R r_g s_h t_\alpha m_\lambda)$. Now, let $l_\beta \in (N :_R r_g s_h t_\alpha m_\lambda) \cap h(R)$, then $l_\beta r_g s_h t_\alpha m_\lambda \in N$. If $l_\beta r_g s_h t_\alpha m_\lambda = 0$, then $l_\beta \in (0 :_R r_g s_h t_\alpha m_\lambda)$. If $0 \neq l_\beta r_g s_h t_\alpha m_\lambda \in N$, then either $l_\beta r_g s_h m_\lambda \in N$ or $l_\beta r_g t_\alpha m_\lambda \in N$ or $l_\beta s_h t_\alpha m_\lambda \in N$. Thus, either $l_\beta \in (N :_R r_g s_h m_\lambda)$ or $l_\beta \in (N :_R r_g t_\alpha m_\lambda)$ or $l_\beta \in (N :_R s_h t_\alpha m_\lambda)$. Hence, we get the result. □

Theorem 2.16. *Let R_i be a G-graded ring and M_i a graded R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$ and $g \in G$ with $M_{2_g} \neq 0$. Suppose that $N = N_1 \times M_2$ is a proper graded submodule of M. Then, the following statements are equivalent:*

(i) *N_{1_g} is a classical g-2-absorbing submodule of an R_{1_e}-module M_{1_g}.*

(ii) *N_g is a classical g-2-absorbing submodule of an R_e-module M_g.*

(iii) *N_g is a weakly classical g-2-absorbing submodule of an R_e-module M_g.*

Proof. (*i*) ⇒ (*ii*) Let $(r_{1_e}, r_{2_e}), (s_{1_e}, s_{2_e}), (t_{1_e}, t_{2_e}) \in R_e$ and $(m_{1_g}, m_{2_g}) \in M_g$ such that $(r_{1_e}, r_{2_e})(s_{1_e}, s_{2_e})(t_{1_e}, t_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. Then, $r_{1_e} s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$, so we get either $r_{1_e} s_{1_e} m_{1_g} \in N_{1_g}$ or $r_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$or $s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$ as N_{1_g} is a classical g-2-absorbing submodule of M_{1_g}.

Hence, either $(r_{1_e}, r_{2_e})(s_{1_e}, s_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(r_{1_e}, r_{2_e})(t_{1_e}, t_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(s_{1_e}, s_{2_e})(t_{1_e}, t_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. Therefore, N_g is a classical g-2-absorbing submodule of M_g.

(*ii*) ⇒ (*iii*) It is easy to see that every classical g-2-absorbing submodule is a weakly classical g-2-absorbing.

(*iii*) ⇒ (*i*) Let $r_{1_e}, s_{1_e}, t_{1_e} \in R_{1_e}$ and $m_{1_g} \in M_{1_g}$ such that $r_{1_e} s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$. Hence, for any $0 \neq m_{2_g} \in M_{2_g}$, we get $0 \neq (r_{1_e}, 1_{2_e})(s_{1_e}, 1_{2_e})(t_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. So, either $(r_{1_e}, 1_{2_e})(s_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(r_{1_e}, 1_{2_e})(t_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(s_{1_e}, 1_{2_e})(t_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. Then, either $r_{1_e} s_{1_e} m_{1_g} \in N_{1_g}$ or $r_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$ or $s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$. Therefore, N_{1_g} is a classical g-2-absorbing submodule of an R_{1_e}-module M_{1_g}. □

Theorem 2.17. *Let R_i be a G-graded ring, M_i a graded R_i-module and N_i a proper graded submodule of M_i, for $i = 1, 2$. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$, $N = N_1 \times N_2$ and $g \in G$. If N_g is a weakly classical g-2-absorbing submodule of an R_e-module M_g and $N_{2_g} \neq M_{2_g}$, then N_{1_g} is a weakly classical g-prime submodule of an R_{1_e}-module M_{1_g}.*

Proof. Let $r_{1_e}, s_{1_e} \in R_{1_e}$ and $m_{1_g} \in M_{1_g}$ such that $0 \neq r_{1_e}s_{1_e}m_{1_g} \in N_{1_g}$. Since $N_{2_g} \neq M_{2_g}$, there exists $m_{2_g} \in M_{2_g} \backslash N_{2_g}$. Hence,

$$(0_{1_g}, 0_{2_g}) \neq (r_{1_e}, 1_{2_e})(s_{1_e}, 1_{2_e})(1_{1_e}, 0_{2_e})(m_{1_g}, m_{2_g}) = (r_{1_e}s_{1_e}m_{1_g}, 0_{2_g}) \in N_g.$$

This implies that either $(r_{1_e}, 1_{2_e})(1_{1_e}, 0_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or

$$(s_{1_e}, 1_{2_e})(1_{1_e}, 0_{2_e})(m_{1_g}, m_{2_g}) \in N_g$$

as N_g is a weakly classical g-2-absorbing submodule of M_g and $m_{2_g} \notin N_{2_g}$. Thus, either $r_{1_e}m_{1_g} \in N_{1_g}$ or $s_{1_e}m_{1_g} \in N_{1_g}$. Therefore, N_{1_g} is a weakly classical g-prime submodule of an R_{1_e}-module M_{1_g}. □

Acknowledgments

The authors wish to thank sincerely the referees for their valuable comments and suggestions.

References

[1] R. Abu-Dawwas, K. Al-Zoubi, *On graded weakly classical prime submodules*, Iran. J. Math. Sci. Inform., 12 (2017), 153-161.

[2] K. Al-Zoubi, *Some properties of graded 2-prime submodules*, Asian-Eur. J. Math., 8 (2015), 1550016-1-1550016-5.

[3] K. Al-Zoubi, R. Abu-Dawwas, *On graded quasi-prime submodules*, Kyungpook Math. J., 55 (2015), 259-266.

[4] K. Al-Zoubi, R. Abu-Dawwas, *On graded 2-absorbing and weakly graded 2-absorbing submodules*, J. Math. Sci. Adv. Appl., 28 (2014), 45-60.

[5] K. Al-Zoubi, R. Abu-Dawwas, S. Çeken, *On graded 2-absorbing and graded weakly 2-absorbing ideals*, Hacet. J. Math. Stat., 48 (2019), 724-731.

[6] K. Al-Zoubi, M. Al-Azaizeh, *On graded classical 2-absorbing submodules of graded modules over graded commutative rings*, Rend. Istit. Mat. Univ. Trieste, 50 (2018), 37-46.

[7] K. Al-Zoubi, M. Al-Azaizeh, *On graded weakly 2-absorbing primary submodules*, Vietnam J. Math., 47 (2019), 297-307.

[8] K. Al-Zoubi, M. Al-Azaizeh, *Some properties of graded 2-absorbing and graded weakly 2-absorbing submodules*, J. Nonlinear Sci. Appl., 12 (2019), 503-508.

[9] K. Al-Zoubi, I. Al-Ayyoub and M. Al-Dolat, *On graded 2-absorbing compactly packed modules*, Adv. Stud. Contemp. Math. (Kyungshang), 28 (2018), 479-486.

[10] K. Al-Zoubi, A. Al-Qderat, *Some properties of graded comultiplication modules*, Open Math., 15 (2017), 187-192.

[11] K. Al-Zoubi, M. Jaradat, R. Abu-Dawwas, *On graded classical prime and graded prime submodules*, Bull. Iranian Math. Soc., 41 (2015), 217-225.

[12] K. Al-Zoubi and F. Qarqaz, *An Intersection condition for graded prime submodules in Gr-multiplication modules*, Math. Reports, 20 (2018), 329-336.

[13] K. Al-Zoubi and B. Rabab'a, *Some properties of graded prime and graded weakly prime submodules*, Far East J. Math. Sci. (FJMS), 102 (2017), 1571-1829.

[14] S. Alghueiri and K. Al-Zoubi, *On graded 2-absorbing I_e-prime submodules of graded modules over graded commutative rings*, AIMS Math., 5 (2020), 7623-7630.

[15] S. E. Atani, *On graded prime submodules*, Chiang Mai J. Sci., 33 (2006), 3-7.

[16] E. Y. Celikel,*On graded 2-absorbing primary submodules*, Int. J. Pure Appl. Math., 109 (2016), 869-879.

[17] A. Y. Darani, S. Motmaen, *Zariski topology on the spectrum of graded classical prime submodules*, Appl. Gen. Topol., 14 (2013), 159-169.

[18] R. Hazrat, *Graded rings and graded grothendieck groups*, Cambridge University Press, Cambridge, 2016.

[19] C. Nastasescu, F. Van Oystaeyen, *Graded and filtered rings and modules*, Lecture notes in mathematics 758, Berlin-New York: Springer-Verlag, 1982.

[20] C. Nastasescu, F. Van Oystaeyen, *Graded ring theory*, Mathematical Library 28, North Holand, Amsterdam, 1982.

[21] C. Nastasescu, F. Van Oystaeyen, *Methods of graded rings*, LNM 1836. Berlin-Heidelberg: Springer-Verlag, 2004.

[22] K. H. Oral, U. Tekir, A. G. Agargun, *On graded prime and primary submodules*, Turk. J. Math., 35 (2011), 159-167.

[23] M. Refai, K. Al-Zoubi, *On graded primary ideals*, Turk. J. Math. 28 (2004), 217-229.

Accepted: November 5, 2022

Strong modular product and complete fuzzy graphs

Talal Al-Hawary
Mathematics Department
Yarmouk University
Irbid
Jordan
talalhawary@yahoo.com

Abstract. In this paper, we provide an improvement of the modular product of fuzzy graphs defined by [16] in 2015, which we call strong modular product. We give sufficient conditions for the strong modular product of two fuzzy graphs to be complete and we show that if the strong modular product of two fuzzy graphs is complete, then at least one factor is a complete fuzzy graph. Moreover, we give necessary and sufficient conditions for the strong modular product of two balanced fuzzy graphs to be balanced.
Keywords: fuzzy graph, complete fuzzy graph, strong modular product, balanced fuzzy graph.

1. Introduction

Graph theory applications in system analysis, operations research and economics are very important. Since the appearance of graph problems are somtimes not known beyond doubt, it is nice to deal with them via fuzzy logic. The concept of fuzzy relation was introduced by Zadeh [23] in his landmark paper "Fuzzy sets" in 1965. Fuzzy graph and several fuzzy graph concepts were introduced by Rosenfeld [21] in 1975. Lately, fuzzy graph theory is having more and more applications in real time modeling in which the level of information immanent in the system changes.

Mordeson and Peng [17] defined the concept of complement of fuzzy graph and studied some operations on fuzzy graphs. In [22], modified the definition of complement of a fuzzy graph so that the complement of the complement is the original fuzzy graph, which agrees with the classical graph case. Moreover several properties of self-complementary fuzzy graphs and the complement of some operations of fuzzy graphs that were introduced in [17] were studied. For more on the previous notions and the following ones, one can see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22].

A fuzzy subset of a non-empty set V is a function $\sigma : V \to [0,1]$ and a fuzzy relation μ on σ is a fuzzy subset of $V \times V$. All throughout this paper, we assume that V is finite, σ is reflexive and μ is symmetric.

Definition 1.1. *[21] A* fuzzy graph $G : (\sigma, \mu)$ *where σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(x, y) \le \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$,* *where* $\wedge$ *stands for minimum. The underlying crisp graph of* G *is denoted by* $G^* : (\sigma^*, \mu^*)$ *where* $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$ *and* $\mu^* = \sup p(\mu) = \{(x, y) \in V \times V : \mu(x, y) > 0\}$.$H = (\sigma', \mu')$ *is a* fuzzy subgraph *of* G *if there exists* $X \subseteq V$ *such that,* $\sigma' : X \to [0, 1]$ *is a fuzzy subset and* $\mu' : X \times X \to [0, 1]$ *is a fuzzy relation on* σ' *such that* $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ *for all* $x, y \in X$.

Definition 1.2 ([20]). *A fuzzy graph* $G : (\sigma, \mu)$ *is* complete *if* $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ *for all* $x, y \in V$.

Next, we recall the following two results from [22].

Lemma 1.1. *Let* $G : (\sigma, \mu)$ *be a self-complemetary fuzzy graph. Then* $\sum_{x,y \in V} \mu(x, y) = (1/2) \sum_{x,y \in V} (\sigma(x) \wedge \sigma(y))$

Lemma 1.2. *Let* $G : (\sigma, \mu)$ *be a fuzzy graph satisfying* $\mu(x, y) = (1/2)(\sigma(x) \wedge \sigma(y))$ *for all* $x, y \in V$.*Then* G *is self-complemetary.*

Definition 1.3 ([15]). *Two fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *with crisp graph* $G_1^* : (V_1, E_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *with crisp graph* $G_2^* : (V_2, E_2)$ *are* isomorphic *if there exists a bijection* $h : V_1 \to V_2$ *such that* $\sigma_1(x) = \sigma_2(h(x))$ *and* $\mu_1(x, y) = \mu_2(h(x), h(y))$ *for all* $x, y \in V_1$.

Lemma 1.3 ([18]). *Any two isomorphic fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *satisfy* $\sum_{x \in V_1} \sigma_1(x) = \sum_{x \in V_2} \sigma_2(x)$ *and*

$$\sum_{x,y \in V_1} \mu_1(x, y) = \sum_{x,y \in V_2} \mu_2(x, y).$$

Definition 1.4 ([5]). *The* density *of a fuzzy graph* $G : (\sigma, \mu)$ *is*

$$D(G) = 2(\sum_{u,v \in V} \mu(u, v))/(\sum_{u,v \in V} (\sigma(u) \wedge \sigma(v))).$$

G *is* balanced *if.* $D(H) \leq D(G)$ *for all fuzzy non-empty subgraphs* H *of* G.

Theorem 1.1 ([5]). *A complete fuzzy graph is balanced.*

A new operation on fuzzy graphs is next recalled:

Definition 1.5 ([16]). *The* modular product *of two fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *with crisp graph* $G_1^* : (V_1, E_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *with crisp graph* $G_2^* : (V_2, E_2)$ *is defined to be the fuzzy graph* $G_1 \circledcirc G_2 : (\sigma_1 \circledcirc \sigma_2, \mu_1 \circledcirc \mu_2)$ *with crisp graph* $G^* : (V_1 \times V_2, E)$ *where*

$$E = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2\},$$

$(\sigma_1 \circledcirc \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, *for all* $(u, v) \in V_1 \times V_2$ *and* $(\mu_1 \circledcirc \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$ *when* $u_1u_2 \in E_1, v_1v_2 \in E_2$, $(\mu_1 \circledcirc \mu_2)((u_1, v_1)(u_2, v_2)) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$ *when* $u_1u_2 \notin E_1, v_1v_2 \notin E_2$.

In [16], it was proved thet the modular product of two strong fuzzy graphs is a strong fuzzy graph. Clearly, the modular product of two complete fuzzy graphs need not be a complete fuzzy graph as $(\mu_1 \circledcirc \mu_2)((u_1,v_1)(u_2,v_2))$ is not defined above for the case $u_1 = u_2$ or $v_1 = v_2$.

In Section 2 of this paper, we provide an improvement of the modular product of fuzzy graphs defined by [16], which we call strong modular product. We give sufficient conditions for the strong modular product of two fuzzy graphs to be complete and we show that if the strong modular product is complete, then at least one factor is a complete fuzzy graph. Section 3 is divoted to give necessary and sufficient conditions for the strong modular product of two fuzzy balanced graphs to be balanced.

2. Strong modular product of fuzzy graphs

It clear that the modular product of two complete fuzzy graphs need not be complete, see the example in Figure 4.1 in [16]. Next, we modify the above definition so that the preceding property holds.

Definition 2.1. *The* strong modular product *of two fuzzy graphs* $G_1 : (\sigma_1, \mu_1)$ *with crisp graph* $G_1^* : (V_1, E_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *with crisp graph* $G_2^* : (V_2, E_2)$ *is defined to be the fuzzy graph* $G_1 \boxplus G_2 : (\sigma_1 \boxplus \sigma_2, \mu_1 \boxplus \mu_2)$ *with crisp graph* $G^* : (V_1 \times V_2, E)$ *where*

$$E = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2\},$$

$(\sigma_1 \boxplus \sigma_2)(u,v) = \sigma_1(u) \wedge \sigma_2(v),$ *for all* $(u,v) \in V_1 \times V_2$ *and*

$$(\mu_1 \boxplus \mu_2)((u_1,v_1)(u_2,v_2)) = \begin{cases} \mu_1(u_1u_2) \wedge \mu_2(v_1v_2), & u_1u_2 \in E_1, v_1v_2 \in E_2 \\ \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2), & u_1u_2 \notin E_1, v_1v_2 \notin E_2 \\ \sigma_1(u_1) \wedge \mu_2(v_1v_2), & u_1 = u_2, v_1v_2 \in E_2 \\ \sigma_2(v_1) \wedge \mu_1(u_1u_2), & u_1u_2 \in E_1, v_1 = v_2. \end{cases}$$

Next, we show that the above definition is well-defined.

Theorem 2.1. *The strong modular product of two fuzzy graphs is a fuzzy graph.*

Proof. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$, respectively. Since Case 1 and Case 2 are proved in [16] and as Case 3 is similar to Case 4, we only prove Case 3.
Case 3. If $u_1 = u_2, v_1v_2 \in E_2$, then as G_2 is a fuzzy graph

$$\begin{aligned}(\mu_1 \boxplus \mu_2)((u_1,v_1)(u_2,v_2)) &= \sigma_1(u_1) \wedge \mu_2(v_1v_2) \\ &\leq \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2).\end{aligned}$$

Thus

$$(\mu_1 \boxplus \mu_2)((u_1,v_1)(u_2,v_2)) \leq \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$$
$$((\sigma_1 \boxplus \sigma_2)(u_1,v_1)) \wedge ((\sigma_1 \boxplus \sigma_2)(u_2,v_2)). \quad \square$$

Next, we show that the strong modular product of two complete fuzzy graphs are again a complete fuzzy graph.

Theorem 2.2. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $G_1 \boxplus G_2$ is a complete fuzzy graph.*

Proof. If $(u_1, v_1)(u_2, v_2) \in E$, then we have the following cases:
Case 1. $u_1u_2 \in E_1, v_1v_2 \in E_2$.
Case 2. $u_1u_2 \notin E_1, v_1v_2 \notin E_2$.
Case 3. $u_1 = u_2, v_1v_2 \in E_2$.
Case 4. $u_1u_2 \in E_1, v_1 = v_2$.

Cases 1 and 2 follow from the proof of Theorem 4.2 in [16]. Case 3 and Case 4 are similar, so we only prove Case 3.
Case 3. Since G_2 is complete,

$$\begin{aligned}(\mu_1 \boxplus \mu_2)((u_1,v_1)(u_2,v_2)) &= \sigma_1(u_1) \wedge \mu_2(v_1v_2)\\ &= \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)\\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)\\ &= (\sigma_1 \boxplus \sigma_2)((u_1,v_1)) \wedge (\sigma_1 \boxplus \sigma_2)((u_2,v_2)).\end{aligned}$$

Hence, $G_1 \boxplus G_2$ is complete. $\square$

Corollary 2.1. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete (strong) fuzzy graphs, then $G_1 \boxplus G_2$ is a strong fuzzy graph.*

An interesting property of complement is given next.

Theorem 2.3. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are complete fuzzy graphs, then $\overline{G_1 \boxplus G_2} \simeq \overline{G_1} \boxplus \overline{G_2}$.*

Proof. Let $G : (\sigma, \overline{\mu}) = \overline{G_1 \boxplus G_2}$, $\overline{\mu} = \overline{\mu_1 \boxplus \mu_2}$, $\overline{G^*} = (V, \overline{E})$, $\overline{G_1} : (\sigma_1, \overline{\mu_1})$, $\overline{G_1^*} = (V_1, \overline{E_1})$, $\overline{G_2} : (\sigma_2, \overline{\mu_2})$, $\overline{G_2^*} = (V_2, \overline{E_2})$ and $\overline{G_1} \boxplus \overline{G_2} : (\sigma_1 \boxplus \sigma_2, \overline{\mu_1} \boxplus \overline{\mu_2})$. We only need to show $\overline{\mu_1 \boxplus \mu_2} = \overline{\mu_1} \boxplus \overline{\mu_2}$. For any arc e joining nodes of V,we have the following cases:
Case 1. If $u_1u_2 \in E_1, v_1v_2 \in E_2$, then as G is complete by Theorem 2.2, $\overline{\mu}(e) = 0$. On the other hand, $(\overline{\mu_1} \boxplus \overline{\mu_2})(e) = 0$ since $u_1u_2 \notin \overline{E_1}$ and $v_1v_2 \notin \overline{E_2}$.
Case 2. If $u_1u_2 \notin E_1, v_1v_2 \notin E_2$, then is case is not possible to occur as both G_1 and G_2 are complete.
Case 3. $e = (u, v_1)(u, v_2)$ where $v_1v_2 \in E_2$.Then as G is complete by Theorem 2.2, $\overline{\mu}(e) = 0$. On the other hand, $(\overline{\mu_1} \boxplus \overline{\mu_2})(e) = 0$ since $v_1v_2 \notin \overline{E_2}$.
Case 4. Similar proof to Case 3.

In all cases $\overline{\mu_1 \boxplus \mu_2} = \overline{\mu_1} \boxplus \overline{\mu_2}$ and therefore, $\overline{G_1 \boxplus G_2} \simeq \overline{G_1} \boxplus \overline{G_2}$. $\square$

Next, we show that if the strong modular product of two fuzzy graphs is complete, then at least one of the two fuzzy graphs must be complete.

Theorem 2.4. *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are fuzzy graphs such that $G_1 \boxplus G_2$ is complete, then at least G_1 or G_2 must be complete.*

Proof. Suppose to the contrary that both G_1 and G_2 are not complete. Then there exists at least one $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$ such that $\mu_1(u_1u_2) < \sigma_1(u_1) \wedge \sigma_1(u_2))$ and $\mu_2(v_1v_2) < \sigma_2(v_1) \wedge \sigma_2(v_2))$ then, we have the following cases:
Case 1. If $u_1u_2 \in E_1, v_1v_2 \in E_2$, then $(\mu_1 \boxplus \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)$ and as $G_1 \boxplus G_2$ is complete,

$$\begin{aligned}(\mu_1 \boxplus \mu_2)((u_1, v_1)(u_2, v_2)) &= (\sigma_1 \boxplus \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \boxplus \sigma_2)((u_2, v_2)) \\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &> \mu_1(u_1u_2) \wedge \mu_2(v_1v_2),\end{aligned}$$

which is a contradiction.
Case 2. If $u_1u_2 \notin E_1, v_1v_2 \notin E_2$, then $(\mu_1 \boxplus \mu_2)((u_1, v_1)(u_2, v_2)) = \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2)$ and as $G_1 \boxplus G_2$ is complete,

$$\begin{aligned}(\mu_1 \boxplus \mu_2)((u_1, v_1)(u_2, v_2)) &= (\sigma_1 \boxplus \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \boxplus \sigma_2)((u_2, v_2)) \\ &= \sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &= \mu_1(u_1u_2) \wedge \mu_2(v_1v_2),\end{aligned}$$

which is a contradiction.
Case 3. If $u_1 = u_2, v_1v_2 \in E_2$, then $(\mu_1 \boxplus \mu_2)((u_1, v_1)(u_2, v_2)) = \sigma_1(u_1) \wedge \mu_2(v_1v_2)$ and as $G_1 \boxplus G_2$ is complete,

$$\begin{aligned}(\mu_1 \boxplus \mu_2)((u_1, v_1)(u_2, v_2)) &= (\sigma_1 \boxplus \sigma_2)((u_1, v_1)) \wedge (\sigma_1 \boxplus \sigma_2)((u_2, v_2)) \\ &= \sigma_1(u_1) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2) \\ &> \mu_1(u_1u_2) \wedge \mu_2(v_1v_2),\end{aligned}$$

thus $G_1 \boxplus G_2$ is not complete.
Case 4. If $u_1u_2 \in E_1, v_1 = v_2$, the proof is similar to Case 3. □

3. Blanced notion virsus strong modular product

We begin this section by proving the following lemma that we use to give necessary and sufficient conditions for the strong modular product of two balanced fuzzy graphs to be balanced.

Lemma 3.1. *Let G_1 and G_2 be fuzzy graphs. Then $D(G_i) \leq D(G_1 \boxplus G_2)$ for $i = 1, 2$ if and only if $D(G_1) = D(G_2) = D(G_1 \boxplus G_2)$.*

Proof. If $D(G_i) \leq D(G_1 \boxplus G_2)$ for $i = 1, 2$, then

$$D(G_1) = 2\Big(\sum_{u_1,u_2\in V_1} \mu_1(u_1u_2)\Big)/\Big(\sum_{u_1,u_2\in V_1} (\sigma_1(u_1) \wedge \sigma_1(u_2))\Big)$$

$$\geq 2\Big(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1(u_1u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2)\Big)/\Big(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1(u_1)\wedge\sigma_1(u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2))\Big)$$

$$= 2\Big(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1(u_1u_2) \wedge \mu_2(v_1v_2)\Big)/\Big(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1(u_1) \wedge \sigma_1(u_2) \wedge \sigma_2(v_1) \wedge \sigma_2(v_2))\Big)$$

$$= 2\Big(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} \mu_1 \boxplus \mu_2((u_1, v_1)(u_2, v_2))\Big)/\Big(\sum_{\substack{u_1,u_2\in V_1\\ v_1,v_2\in V_2}} (\sigma_1 \boxplus \sigma_2((u_1, v_1)(u_2, v_2))\Big)$$

$$= D(G_1 \boxplus G_2).$$

Hence, in all cases $D(G_1) \geq D(G_1 \boxplus G_2)$ and thus $D(G_1) = D(G_1 \boxplus G_2)$. Similarly, $D(G_2) = D(G_1 \boxplus G_2)$. Therefore, $D(G_1) = D(G_2) = D(G_1 \boxplus G_2)$. □

Theorem 3.1. *Let G_1 and G_2 be balanced fuzzy graphs. Then $G_1 \boxplus G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \boxplus G_2)$.*

Proof. If $G_1 \boxplus G_2$ is balanced, then $D(G_i) \leq D(G_1 \boxplus G_2)$ for $i = 1, 2$ and by Lemma 3.1, $D(G_1) = D(G_2) = D(G_1 \boxplus G_2)$.

Conversely, if $D(G_1) = D(G_2) = D(G_1 \boxplus G_2)$ and H is a fuzzy subgraph of $G_1 \boxplus G_2$, then there exist fuzzy subgraphs H_1 of G_1 and H_2 of G_2. As G_1 and G_2 are balanced and $D(G_1) = D(G_2) = n_1/r_1$, then $D(H_1) = a_1/b_1 \leq n_1/r_1$ and $D(H_2) = a_2/b_2 \leq n_1/r_1$. Thus $a_1r_1 + a_2r_1 \leq b_1n_1 + b_2n_1$ and hence $D(H) \leq (a_1 + a_2)/(b_1 + b_2) \leq n_1/r_1 = D(G_1 \boxplus G_2)$. Therefore, $G_1 \boxplus G_2$ is balanced. □

References

[1] T. Al-Hawary, *Matroid's filterbase*, Indian J. Math., 60 (2018), 301-310.

[2] T. Al-Hawary, B. Hourani, *On intuitionistic product fuzzy graphs*, Ital. J. Pure Appl. Math., 38 (2017), 113-126.

[3] T. Al-Hawary, *β-greedoids*, Matematicki Vesnik, 66 (2014), 343-350.

[4] T. Al-Hawary, S. Al-Shalaldeh, M. Akram, *Certain matrices and energies of fuzzy graphs*, TWMS J. Pure App. Math., 14 (2023), 50-68.

[5] T. Al-Hawary, *Complete fuzzy graphs*, Inter. J. Math. Combin., 4 (2011), 26-34.

[6] T. Al-Hawary, *Complete Hamacher fuzzy graphs*, J. Appl. Math. Informatics, 40 (2022), 1043-1052.

[7] T. Al-Hawary, *Maximal Strong product and balanced fuzzy graphs*, to appear in J. Appl. Math. Informatics.

[8] T. Al-Hawary, Mohammed Hashim, *Semi-fuzzy graphs*, to appear in Boletim da Sociedade Paranaense de Matematica.

[9] T. Al-Hawary, *Fuzzy flats*, Indian J. Mathematics, 55 (2013), 223-236.

[10] T. Al-Hawary, *Operations on greedoids*, Mathematica, 64 (2022), 3-8.

[11] T. Al-Hawary, B. Hourani, *On intuitionistic product fuzzy graphs*, Ital. J. Pure Appl. Math., 38 (2016), 113-126.

[12] T. Al-Hawary, *On modular flats and pushouts of matroids*, Ital. J. Pure Appl. Math., 43 (2020), 237-241.

[13] T. Al-Hawary, *Density Results for Perfectly Regular and Perfectly Edge-regular Fuzzy Graphs*, J. Disc. Math. Scie. Cryptography, 2 (2022), 1-10.

[14] M. Akram, D. Saleem, T. Al-Hawary, *Spherical fuzzy graphs with application to decision-making*, Math. and Comput. Appls., 25 (2020), 8-40.

[15] K. R. Bhutani, *On automorphism of fuzzy graphs*, Pattern Recognition Letter, 9 (1989), 159-162.

[16] S. Dogra, *Different types of product of fuzzy graphs*, Prog. Nonlin. Dyn. Chaos, 3 (2015), 41-56.

[17] J. N. Mordeson, C. S. Peng, *Operations on fuzzy graphs*, Information Sciences, 79 (1994), 381-384.

[18] A. Nagoor Gani, J. Malarvizhi, *Isomorphism on fuzzy graphs*, Int. J. Comp. and Math. Sci., 2 (2008), 190-196.

[19] A. Nagoor Gani, J. Malarvizhi, *Isomorphism properties on strong fuzzy graphs*, Int. J. Algorithms, Comp. and Math., 2 (2009), 39-47.

[20] A. Nagoor Gani, K. Radha, *On regular fuzzy graphs*, J. Physical Sciences, 12 (2008), 33-40.

[21] A. Rosenfeld, Fuzzy graphs, in L.A. Zadeh, K.S. Fu, K. Tanaka and M. Shirmura (Eds.), *Fuzzy and their applications to cognitive and decision processes*, Academic Press, New York, 1975, 77-95.

[22] M.S. Sunitha, A. V. Kumar, *Complements of fuzzy graphs*, Indian J. Pure Appl. Math., 33 (2002), 1451-1464.

[23] L.A. Zadeh, *Fuzzy sets*, Inform. Control., 8 (1965), 338-353.

Accepted: June 17, 2022

Chain dot product graph of a commutative ring

Basem Alkhamaiseh
Department of Mathematics
Faculty of Science
Yarmouk University
Irbid
Jordan
basem.m@yu.edu.jo

Abstract. In this article, we generalized the concepts of total dot product graph (the chain zero-divisor dot product), which were investigated in 2015 by A. Badawi, to what we call chain total dot product graph $CTD(R)$ (the chain zero-divisor dot product graph $CZD(R)$). We give some basic graph properties for the graphs $CTD(R)$ and $CZD(R)$ such as connectedness, diameter and the girth.

Keywords: zero-divisor graph, dot product zero-divisor graph, diameter, girth.

1. Introduction

Graph theory has recently become a significant tool for studying the structure of rings, in addition to being a beautiful and sophisticated theory in its own right. As a result, several writers explore the relationship between rings and graph theory. see for example [3, 5, 4].

Throughout this article, let A be a commutative ring with nonzero identity 1, for the natural number n, let $R = A \times A \times \cdots \times A(n - times)$. Badawi in [2] presented the total and the zero-divisor dot product graphs associated to the ring A, where the total dot product graph, denoted by $TD(R)$, is the graph with vertex set $R^* = R \backslash \{(0, 0, \cdots, 0)\}$, and two vertices x, y are adjacent if $x.y = 0 \in A$ (the normal dot product between x and y is zero). Also the zero-divisor dot product graph, denoted by $ZD(R)$, is the induced subgraph of the total dot product graph $TD(R)$ with vertex set $Z(R)^* = Z(R) \backslash \{(0, 0, \cdots, 0)\}$.

In this article, we generalized these concepts by developing the concept of the dot product. Let $A_1, A_2, \ldots, A_n$ be commutative rings with nonzero identity 1, such that $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$. Let $R = A_1 \times A_2 \times \ldots \times A_n$, then the generalized dot product between $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is $x.y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in A_n$.

Now, we introduce our generalization. Let A be a commutative ring with nonzero identity 1, $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \cdots \times A[\alpha_1, \alpha_2, \ldots \alpha_n]$, where $A[\alpha_1, \alpha_2, \ldots \alpha_k]$ is a ring with elements of the form $x = x_{k1} + x_{k2}\alpha_1 + x_{k3}\alpha_2 + \cdots + x_{kk}\alpha_k$ such that $\alpha_i\alpha_j = 0$ for $1 \leq i, j \leq k$, with the operations

Addition: $(x_{k1}+x_{k2}\alpha_1+x_{k3}\alpha_2+\cdots+x_{kk}\alpha_k)+(y_{k1}+y_{k2}\alpha_1+y_{k3}\alpha_2+\cdots+y_{kk}\alpha_k)=(x_{k1}+y_{k1})+(x_{k2}+y_{k2})\alpha_1+(x_{k3}+y_{k3})\alpha_2+\cdots+(x_{kk}+y_{kk})\alpha_k$, and

Multiplication: $(x_{k1}+x_{k2}\alpha_1+x_{k3}\alpha_2+\cdots+x_{kk}\alpha_k)(y_{k1}+y_{k2}\alpha_1+y_{k3}\alpha_2+\cdots+y_{kk}\alpha_k)=x_{k1}y_{k1}+(x_{k1}y_{k2}+x_{k2}y_{k1})\alpha_1+(x_{k1}y_{k3}+x_{k3}y_{k1})\alpha_2+\cdots+(x_{k1}y_{kk}+x_{kk}y_{k1})\alpha_k$.

The chain dot product graph, denoted by $CTD(R)$ is a graph with a vertex set $R^*=R\backslash\{(0,0,\cdots,0)\}$, and two vertices x,y are adjacent if $x.y=0\in A$ (the generalized dot product between x and y is 0). Similarly, as above, the chain zero-divisor dot product graph, denoted by $CZD(R)$, is the induced subgraph of the chain total dot product graph $CTD(R)$ with a vertex set $Z(R)^*=Z(R)\backslash\{(0,0,\cdots,0)\}$ (the nonzero zero-divisors of R).

For undefined notation or terminology consult [6] for graph theory and [7] for ring theory.

2. Some basic properties of $CTD(R)$ and $CZD(R)$

In this section, we will study some properties of $CTD(R)$ and $CZD(R)$, such as connectedness, diameter and girth.

We start by defining the $k-th$ neighborhood for the vertex x.

Definition 2.1. *Let G be a finite simple graph, and x be any vertex in G and let k be any nonnegative integer. Then, the $k-th$ neighborhood for the vertex x, denoted by $N^k(x)$, is defined as*

$$\begin{aligned}
N^0(x) &= \{x\},\\
N^1(x) &= N(x), \textit{the usual neighborhood of } x.\\
&\vdots\\
\textit{for } k &\geq 1\\
N^k(x) &= \left\{ y\in V(G)\backslash\bigcup_{j=1}^{k-1}N^j(x) : z \ \ \textit{is adjacent to } y\textit{, for any } z\in N^{k-1}(x)\right\}\\
&\quad \textit{where } V(G) \textit{ is the vertex set of the graph } G.
\end{aligned}$$

The definition of $N^k(x)$ makes it obvious that there is a path of length k, between the vertex x and any vertex in $N^k(x)$.

Lemma 2.1. *Let G be a finite simple graph, and x,y be two distinct vertices. Then, there is a path between x and y if and only if there exist two non negative integers n,m such that $N^n(x)$ and $N^m(x)$ are not disjoint sets.*

Proof. Suppose that $x-a_1-a_2-\cdots-a_t-y$ is a path between x and y. Then, $a_1\in N^1(x)\cap N^t(y)$. Conversely, assume that $N^n(x)$ and $N^m(x)$ are not disjoint sets, for some non negative integers n,m. Hence, $N^n(x)$ and $N^m(x)$ have at least one vertex in common , say z. Thus, and since $z\in N^n(x)$, there is a path between the vertex x and z, say $x - c_1 - c_2 - \cdots - c_n - z$. Similarly, and since $z \in N^m(y)$, there is a path between the vertex y and z, say $z - d_1 - d_2 - \cdots - d_m - y$. Therefore, $x - c_1 - c_2 - \cdots - c_n - z - d_1 - d_2 - \cdots - d_m - y$. □

The following theorem describes when $CTD(R)$ is disconnected.

Theorem 2.1. *If A is an integral domain and $R = A \times A[\alpha]$, then $CTD(R)$ is disconnected.*

Proof. Let $B = \{(a,a), (-a,a), (a,-a) : a \in A^*\}$ and let $x \in B$. Suppose that $y \in R^*$, that is $y = (y_{11}, y_{21} + y_{22}\alpha)$, such that $x.y = 0$. Since A is an integral domain, one can deduce $y \in B$ (in general, $N^n(y) \subseteq B$ for any *positive* integer n)

Let $M = \{(a, b\alpha) : a \in A^*$ and $b \in A\} \cup \{(0, a + b\alpha) : a, b \in A$ not both zero$)\}$ and let $m \in M$. Suppose that $m.r = 0$ for some $r \in R^*$. Again, since A is an integral domain, we deduce that $r \in M$ (in general, $N^m(r) \subseteq M$ for any *positive* integer m). It is clear that B and M are disjoint sets.

We claim here that the sets B and M are disconnected in the graph $CTD(R)$. To see this, suppose the contrary. If $x \in M$ and $y \in B$ and there is a path between x and y in the graph $CTD(R)$, then by Lemma 2.1 there exist two non negative integers n, m such that $N^n(x) \cap N^m(y)$ is nonempty, which is a conradiction, since $N^n(x) \cap N^m(y) \subseteq B \cap M$. Thus, the graph $CTD(R)$ is disconnected. □

The following theorem establishes the necessary conditions for the chain zero-divisor dot product graph $CZD(R)$ to be equal to the known zero-divisor graph $\Gamma(R)$.

Theorem 2.2. *Let A be a ring, $2 \leq n < \infty$, and $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$. Then, $CZD(R) = \Gamma(R)$ if and only if $n = 2$ and A is an integral domain.*

Proof. Suppose that A is an integral domain and $R = A \times A[\alpha]$. Then, $Z(R) = \{(a, b\alpha) : a \in A^*$ and $b \in A\} \cup \{(0, a + b\alpha) : a, b \in A\ \}$. Let $x, y \in Z^*(R)$ such that $x.y = 0$. Hence, we have three cases to consider, which are $x = (x_{11}, x_{22}\alpha)$ and $y = (y_{11}, y_{22}\alpha)$, $x = (x_{11}, x_{22}\alpha)$ and $y = (0, y_{21} + y_{22}\alpha)$ or $x = (0, x_{21} + x_{22}\alpha)$ and $y = (0, y_{21} + y_{22}\alpha)$. In all three cases it is clear that $x.y = 0$ if and only if $xy = (0,0)$. Hence, $CZD(R) = \Gamma(R)$.

Conversely, suppose that $CZD(R) = \Gamma(R)$. Assume that $n \geq 3$, then there exist $x = (0, \alpha_1, \alpha_1, 0, \ldots, 0)$, $y = (0, 1, -1, 0, \ldots, 0) \in Z^*(R)$, with $x.y = 0$, but $xy \neq (0, 0, 0, \ldots, 0)$. Thus, $x - y$ is an edge of $CZD(R)$ that is not an edge of $\Gamma(R)$, a contradiction. Thus, $n = 2$. Now, if A is not an integral domain, then there are $a, b \in A^*$ such that $ab = 0$. Hence, $x = (1, a)$, $y = (a, -1 + b\alpha) \in Z^*(R)$, and $x.y = 0$, but $xy \neq (0, 0)$. Again, $x - y$ is an edge of $CZD(R)$ that is not an edge of $\Gamma(R)$, a contradiction. Thus, A must be an integral domain. □

Corollary 2.1. *Let A be an integral domain. If $R = A \times A[\alpha]$, then $CZD(R)$ is connected with $diam(CZD(R)) = 3$.*

Proof. Since A is an integral domain, the vertex set of $CZD(R)$ can be divided into three disjoint sets $X = \{(a, b\alpha) : a \in A^*$ and $b \in A\}$, $Y = \{(0, a + b\alpha) : a \in A^*$ and $b \in A\}$ and $Z = \{(0, b\alpha) : b \in A^*\}$. It is clear that X, Y are independent sets (that is any two vertices in X or Y are not adjacent). Also, Z forms a complete subgraph of $CZD(R)$. Now, by Theorem 2.2 and since X is an independent set, we deduce that $CZD(R)$ is connected with $2 \leq diam(CZD(R)) \leq 3$. Now, let $x = (1, \alpha)$ and $y = (0, 1+\alpha)$. Then, $x.y \neq 0$. Let $t = (t_{11}, t_{21}+t_{22}\alpha) \in Z^*(R)$ such that $x.t = t.y = 0$. Then, we conclude that $t = (0, 0)$ which is a contradiction. Thus, $d_{cz}(x, y) = 3$. Hence, $diam(CZD(R)) = 3$.

Or (Another Proof) By Theorem (2.2) and since R is nonreduced ring and the zero divisors of R does not form an ideal, then by [1], $diam(CZD(R)) = 3$. □

Theorem 2.3. *Let A be a ring that is not an integral domain, and let $R = A \times A[\alpha]$. Then:*

1. *$CTD(R)$ is connected with $diam(CTD(R)) = 3$.*
2. *$CZD(R)$ is connected with $diam(CZD(R)) = 3$.*

Proof. 1) Let $x = (x_{11}, x_{21} + x_{22}\alpha)$, $y = (y_{11}, y_{21} + y_{22}\alpha) \in R^*$, where $x \neq y$, and assume that $x.y \neq 0$. Since A is not an integral domain, there are $a, b \in A^*$ (not necessarily distinct) such that $ab = 0$. Let $w = (ax_{21}, -ax_{11} + ax_{22}\alpha)$ and $v = (by_{21}, -by_{11} + by_{22}\alpha)$. Note that $w, v \in Z(R)$. It is clear that $x.w = w.v = v.y = 0$. Since $x.y \neq 0$, $w \neq y$ and $v \neq x$. Now, there are two cases:

Case 1. Suppose that $w \neq (0, 0)$ and $v \neq (0, 0)$. If $x.v = 0$ or $y.w = 0$, then $x - v - y$ or $x - w - y$ is a path of length 2 in $CTD(R)$ from x to y. But, if $x.v \neq 0$ or $y.w \neq 0$, then x, w, v and y are distinct and $x - w - v - y$ is a path of length 3 in $CTD(R)$ from x to y.

Case 2. Suppose that $w = (0, 0)$ and $v = (0, 0)$. If $w = (0, 0)$, then replace w by $(a, a) \in Z^*(R)$, and hence $x.w = (x_{11}, x_{21} + x_{22}\alpha).(a, a) = (ax_{11} + ax_{21}) + ax_{22}\alpha = 0$. Again, if $v = (0, 0)$, then replace v by $(b, b) \in Z^*(R)$, and hence, $y.v = 0$. Thus, as we have done, we can redefine w and v so that $w, v \in Z^*(R)$ and $x.w = w.v = v.y = 0$. Hence, as in the earlier argument, we can conclude that there is a path of length at most 3 in $CTD(R)$ from x to y.

Thus, $CTD(R)$ is connected with $d_{CT}(x, y) \leq 3$, for every $x, y \in R^*$. Now, let $x = (1, 1)$ and $y = (1, 0)$. It is clear that, $x.y \neq 0$. Let $t = (t_{11}, t_{21} + t_{22}\alpha) \in R^*$ such that $x.t = t.y = 0$. Then, $t_{11} = t_{21} = t_{22} = 0$, so $t = (0, 0)$ a contradiction. Therefore, $d_{CT}(x, y) = 3$, and hence, $diam(CTD(R)) = 3$. □

Theorem 2.4. *Let A be a ring , $4 \leq n < \infty$, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$. Then, $CTD(R)$ is connected with $diam(CTD(R)) = 2$.*

Proof. Let $x = (x_{11}, x_{21} + x_{22}\alpha_1, x_{31} + x_{32}\alpha_1 + x_{33}\alpha_2, \ldots, x_{n1} + \sum_{i=1}^{n} x_{ni}\alpha_{i-1})$, $y = (y_{11}, y_{21} + y_{22}\alpha_1, y_{31} + y_{32}\alpha_1 + y_{33}\alpha_2, \ldots, y_{n1} + \sum_{i=1}^{n} y_{ni}\alpha_{i-1}) \in R^*$, and suppose that $x.y \neq 0$. Then, let $M = \{j : x_{ji} = y_{ji} = 0, 1 \leq j \leq n$ and $1 \leq i \leq j\}$. Now, we have two cases:

Case 1. Suppose that M is not empty set. Then, choose $k \in M$, and let $w = (w_{11}, w_{21} + w_{22}\alpha_1, w_{31} + w_{32}\alpha_1 + w_{33}\alpha_2, \ldots, \sum_{i=1}^{n} w_{ni}\alpha_{i-1}) \in R^*$, where

$$w_{ij} = \begin{cases} 1, & j = k \text{ and } i = 1, \\ 0, & j = k \text{ and } 1 < i \leq j, \\ 0, & j \neq k. \end{cases}$$

Then, $x - w - y$ is a path of length 2 in $CTD(R)$ from x to y.

Case 2. Suppose that M is empty set. Then, let $f(x) = \min\{j : x_{j1} \neq 0, 2 \leq j \leq n\}$ and $f(y) = \min\{j : y_{j1} \neq 0, 2 \leq j \leq n\}$. Since M is empty set, we deduce that $f(x) = 2$ or $f(y) = 2$, without loss of generality, assume that $f(x) = 2$. Let $v = (0, (x_{31}y_{41} - x_{41}y_{31})\alpha_1, (x_{41}y_{21} - x_{21}y_{41})\alpha_1, (x_{21}y_{31} - x_{31}y_{21})\alpha_1, 0, \ldots, 0)$. Now, we have two subcases:

Subcase 2.1. Suppose that $v \neq (0, 0, \ldots, 0)$. Then, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$. Hence, $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y.

Subcase 2.2. Suppose that $v = (0, 0, \ldots, 0)$. Then, $x_{21}y_{31} - x_{31}y_{21} = 0$. Let $w = (0, -x_{31}\alpha_1, x_{21}\alpha_1, 0, \ldots, 0)$ Since $x_{21} \neq 0$, $w \in R^*$. Hence, $x.w = -x_{31}x_{21} + x_{21}x_{31} = 0$ and $w.y = -x_{31}y_{21} + x_{21}y_{31} = 0$. Since $x.w = w.y = 0$, and $x.y \neq 0$, $x \neq w$ and $y \neq w$. Thus, $x - w - y$ is a path of length 2 in $CTD(R)$ from x to y. Hence, $CTD(R)$ is connected with $diam(CTD(R)) = 2$. □

Theorem 2.5. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2]$. Then, $CTD(R)$ is connected with $diam(CTD(R)) = 2$.*

Proof. Let $x = (x_{11}, x_{21}+x_{22}\alpha_1, x_{31}+x_{32}\alpha_1+x_{33}\alpha_2)$, $y = (y_{11}, y_{21}+y_{22}\alpha_1, y_{31}+y_{32}\alpha_1 + y_{33}\alpha_2) \in R^*$, and suppose that $x.y \neq 0$. Then, let $M = \{j : x_{j1} = y_{j1} = 0, 1 \leq j \leq 3\}$. Now, we have two cases:

Case 1. Suppose that M is not empty set. Then, choose $k \in M$, and let $z =$, where

$$z = \begin{cases} (1, 0, 0), & \text{if } k = 1 \\ (0, \alpha_1, 0), & \text{if } k = 2 \in R^*. \\ (0, 0, \alpha_1), & \text{if } k = 3 \end{cases}$$

Then, $x - z - y$ is a path of length 2 in $CTD(R)$ from x to y.

Case 2. Suppose that M is an empty set. Then, define $f(x) = \min\{j : x_{j1} \neq 0, 2 \leq j \leq 3\}$ and $f(y) = \min\{j : y_{j1} \neq 0, 2 \leq j \leq 3\}$. Since M is an empty set, we deduce that $f(x) = 2$ or $f(y) = 2$, without loss of generality, assume that $f(x) = 2$, that is $x_{21} \neq 0$. Now, we have three subcases:

Subcase 2.1. Suppose that $x_{31} \neq 0$, $y_{21} = 0$. If $y_{31}x_{21} \neq 0$, then select $v_1 = (0, x_{31}\alpha_1, -x_{21}\alpha_1)$, $v_2 = (0, \alpha_1, 0) \in R^*$. Thus, $x.v_1 = v_1.v_2 = v_2.y = 0$. Since $x.y \neq 0$, $x.v_2 \neq 0$, $y.v_1 \neq 0$, $x \neq v_1$ and $y \neq v_2$. Hence, $x - v_1 - v_2 - y$ is a path of length 3 in $CTD(R)$ from x to y. If $y_{31}x_{21} = 0$, then select $v = (0, x_{31}\alpha_1, -x_{21}\alpha_1) \in R^*$. So, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$. Hence, $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y.

Subcase 2.2. Suppose that $x_{31} = 0$, $y_{21} = 0$. If $y_{31} \neq 0$, then select $v_1 = (0, 0, \alpha_1)$, $v_2 = (0, \alpha_1, 0) \in R^*$. Then, $x.v_1 = v_1.v_2 = v_2.y = 0$. Since $x.y \neq 0$, $x.v_2 \neq 0$, $y.v_1 \neq 0$, $x \neq v_1$ and $y \neq v_2$. Hence, $x - v_1 - v_2 - y$ is a path of length 3 in $CTD(R)$ from x to y. If $y_{31} = 0$, then select $v = (0, 0, \alpha_1) \in R^*$. So, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$. Hence, $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y.

Subcase 2.3. Suppose that $x_{31} \neq 0$, $y_{21} \neq 0$. If $x_{21}y_{31} - x_{31}y_{21} = 0$, then select $v = (0, x_{31}\alpha_1, -x_{21}\alpha_1) \in R^*$. So, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$, we have $x - v - y$ a path of length 2 in $CTD(R)$ from x to y. If $x_{21}y_{31} - x_{31}y_{21} \neq 0$, then select $v_1 = (0, x_{31}\alpha_1, -x_{21}\alpha_1)$, $v_2 = (0, y_{31}\alpha_1, -y_{21}\alpha_1) \in R^*$. Since $x.y \neq 0$, $x.v_2 \neq 0$, $y.v_1 \neq 0$, $x \neq v_1$ and $y \neq v_2$, we have $x - v_1 - v_2 - y$ a path of length 3 in $CTD(R)$ from x to y.

Therefore, by the previous cases we deduce that $diam(CTD(R)) \leq 3$. Now, let $x = (1, \alpha_1, 1 + \alpha_1 + \alpha_2)$ and $y = (1, 1 + \alpha_1, \alpha_1 + \alpha_2)$. Suppose there exists $(v_{11}, v_{21} + v_{22}\alpha_1, v_{31} + v_{32}\alpha_1 + v_{33}\alpha_2) \in R^*$ such that $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y. Since $x.v = v.y = 0$, we have the following equations

$$\begin{aligned} v_{11} + v_{31} &= 0 \\ v_{21} + v_{32} + v_{31} &= 0 \\ v_{33} + v_{31} &= 0 \end{aligned}$$

$$\begin{aligned} v_{11} + v_{21} &= 0 \\ v_{21} + v_{22} + v_{31} &= 0 \\ v_{31} &= 0 \end{aligned}$$

Solving these equations produces that $v = (0, 0, 0)$ which is a contradiction. Thus, $d_{CT}(x, y) = 3$, and hence, $diam(CTD(R)) = 3$. □

Theorem 2.6. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2]$.If A is an integral domain, then $CZD(R)$ is connected with $diam(CZD(R)) = 3$.*

Proof. Every path in $\Gamma(R)$ is also a path in $CZD(R)$. Now, since $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$ by [3], we conclude that $CZD(R)$ is connected with $diam(CZD(R)) \leq diam(\Gamma(R))$. Thus, $diam(CZD(R)) \leq 3$. Let $x = (1, -1, 0)$, $y = (1, 0, -1) \in Z(R)^*$. It is clear that $x.y = 1 \neq 0$. Hence, $1 < d_{CZ}(x, y) \leq 3$. Suppose that $d_{CZ}(x, y) = 2$. Then, there is $w = (w_{11}, w_{21} +$

$w_{22}\alpha_1, w_{31} + w_{32}\alpha_1 + w_{33}\alpha_2) \in Z(R)^*$ (Since A is an integral domain $w_{11}, w_{21} or$ w_{31} must be zero) such that $x.w = w.y = 0$. By direct calculations, we deduce that $w = (0, 0, 0)$ which is a contradiction. Hence, $d_{CZ}(x, y) = 3$. Therefore, $diam(CZD(R)) = 3$. □

Theorem 2.7. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$.*

(1) If $|A| > 2$ and $2 \leq n < \infty$, then $gr(CTD(R)) = gr(CZD(R)) = 3$.

(2) If A is isomorphic to $\mathbb{Z}_2$, and $3 \leq n < \infty$, then $gr(CTD(R)) = gr(CZD(R)) = 3$.

(3) If A is isomorphic to $\mathbb{Z}_2$, and $n = 2$ then $gr(CZD(R)) = \infty$.

Proof. (1) Since $|A| > 2$, there is $a \in A\backslash\{0, 1\}$. Let $x = (1, 0, \ldots, 0)$, $y = (0, \alpha_1, \ldots, 0)$, and $z = (0, a\alpha_1, \ldots, 0)$. Then, $x - y - z - x$ is a cycle of length 3.

(2) Let $x = (1, 0, 0, \ldots, 0)$, $y = (0, 1, 0, \ldots, 0)$, and $z = (0, 0, 1, 0 \ldots, 0)$. Then, $x - y - z - x$ is a cycle of length 3.

(3) Clear. □

According to the previous results, one can conclude the following corollaries.

Corollary 2.2. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ (with $2 \leq n < \infty$). Then, the following are equivalent:*

(1) $gr(CTD(R)) = 3$.

(2) $gr(CZD(R)) = 3$.

(3) $|A| > 2$ or A is isomorphic to $\mathbb{Z}_2$, and $3 \leq n$.

Proof. Obvious, by Theorem 2.7. □

Corollary 2.3. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ (with $2 \leq n < \infty$). Then, the following are equivalent:*

(1) $gr(CZD(R)) = \infty$.

(2) A is isomorphic to $\mathbb{Z}_2$, and $n = 2$.

Proof. Obvious, by Theorem 2.7. □

Corollary 2.4. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ (with $2 \leq n < \infty$). Then, the following are equivalent:*

(1) $CZD(R) = \Gamma(R)$.

(2) $CTD(R)$ is disconnected.

(3) A is an integral domain and $n = 2$.

3. Conclusion

Let A be a commutative ring with nonzero identity 1. for the natural number n, we use the ring $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \cdots \times A[\alpha_1, \alpha_2, \ldots \alpha_n]$ to construct what we call the chain total dot product graph (the chain zero-divisor dot product graph), denoted by $CTD(R)$ ($CZD(R)$). These two graphs are considered to be a generalization of the total and the zero-divisor dot product graphs in [2]. In this article, we studied some basic graph properties for the graphs $CTD(R)$ and $CZD(R)$ such as connectedness, diameter and the girth. Many graph properties, such as the graph's core, center, and median, as well as planarity, can be explored in the future for the graphs $CTD(R)$ and $CZD(R)$.

References

[1] T. G. Lucas, *The diameter of a zero-divisor graph*, J. Algebra, 301 (2006), 3533-3558.

[2] Badawi Ayman, *On the dot product graph of a commutative ring*, Comm. Algebra, 34 (2015), 43-50.

[3] D.F. Anderson, P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (1999), 434-447.

[4] S.P. Redmond, *On zero-divisor graphs of small finite commutative rings*, Discrete Math., 307 (2007), 1155-1166.

[5] I. Beck, *Coloring of commutative rings*, J. Algebra, 116 (1988), 208-226.

[6] R. Diestel, *Graph theory*, Springer, New York, 1997.

[7] J.A. Huckaba, *Commutative rings with zero divisors*, Marcel Dekker, New York/Basil, 1988.

Accepted: June 9, 2022

Projection graphs of rings and near-rings

Teresa Arockiamary S.
Department of Mathematics
Stella Maris College
Chennai-600086, Tamil Nadu
India
drtessys70@gmail.com

Meera C.
Department of Mathematics
Bharathi Women's College (Autonomous)
Chennai-600108, Tamil Nadu
India
eya278@gmail.com

Santhi V.[*]
Department of Mathematics
Presidency College (Autonomous)
Chennai-600005, Tamil Nadu
India
santhivaiyapuri2019@gmail.com

Abstract. Association of graphs with algebraic structures facilitates the process of understanding the properties of algebraic structures through graphs. In this paper, projection graph $P(R)$ of a ring R is introduced as an undirected graph, whose vertices are the nonzero elements of R and any two distinct vertices x and y are adjacent if and only if their product is equal to either x or y. The projection graph $P(N)$ of a near-ring N is also defined in the same way. It is proved that $P(R)$ is a star graph if and only if R has no nonzero zero-divisors. A method of finding adjacent vertices with the help of annihilators is developed. The projection graphs of certain classes of rings are found to be bipartite and $P(R)$ is proved to be weakly pancyclic when R is a local ring with ascending chain condition on the annihilator ideals of its elements. $P(\mathbb{Z}_n)$ are constructed for certain values of n and their properties are studied. Moreover, $P(N)$ is shown as a complete graph when N is either a constant near-ring or an almost trivial near-ring.

Keywords: commutative rings, annihilator, near-ring, independent set, clique, planar graph.

∗. Corresponding author

1. Introduction

There are many graphs associated to rings and the other algebraic structures such as groups, semigroups, semirings, near-rings, ternary rings, modules etc. to understand the properties of algebraic structures via graphs and vice versa.

The idea of associating a graph to a commutative ring R was introduced by Beck [11] in 1988. He defined a graph with the vertex set as the set of all elements of R and two distinct vertices x and y are adjacent if and only if $xy = 0$ and mainly studied about coloring of the graph. In 1993, Anderson and Naseer [5] determined all finite commutative rings with chromatic number 4. Anderson and Livingston [6] in 1999, redefined Beck's graph by taking $ZD^*(R)$, the set of nonzero zero-divisors of R, as the vertex set and named the graph of R as zero-divisor graph denoted by $\Gamma(R)$. They proved that the zero-divisor graph of a commutative ring R is complete if and only if either $R \cong \mathbb{Z}_2^2$ or $xy = 0$ for all $x, y \in ZD(R)$, the set of zero-divisors of R.

Afkhami and Khashyarmanesh [1] introduced cozero-divisor graph $\Gamma'(R)$ of a commutative ring R. The vertex set of $\Gamma'(R)$ is $W^*(R)$, the set of nonzero nonunits of R and $a, b \in W^*(R)$ are adjacent if and only if $a \notin bR$ and $b \notin aR$. They studied $\Gamma'(R)$ and its complement $\overline{\Gamma'(R)}$ in [2]. In particular, they characterized all commutative rings whose cozero-divisor graphs are double-star, unicyclic, a star, or a forest. Further, Akbari et al. [3] continued the study of cozero-divisor graphs of commutative rings and proved that if $\Gamma'(R)$ is a forest, then $\Gamma'(R)$ is a union of isolated vertices or a star.

The concept of annihilator graph was introduced in 2014 by Badawi [9]. The annihilator graph of a commutative ring R is the simple graph denoted by $AG(R)$, whose vertex set is $ZD^*(R)$ and two distinct vertices x and y are adjacent if and only if $Ann(xy) \neq Ann(x) \cup Ann(y)$, where $Ann(x) = \{y \in R \mid xy = 0\}$. If R is a commutative ring with more than 2 nonzero zero-divisors, then $AG(R)$ is proved to be connected and $diam(AG(R)) \leq 2$. More results on $AG(R)$ can be found in the survey article [10].

Teresa Arockiamary et al. [18] defined annihilator 3-uniform hypergraph $AH_3(N)$ of a right ternary near-ring (RTNR) N. Let $(N, +, [\ ])$ be an RTNR. Then, $AH_3(N)$ is defined as the 3-uniform hypergraph whose vertex set is the set of all elements of N having nontrivial annihilators and three distinct vertices x, y and z are adjacent whenever the intersection of their annihilators is not $\{0\}$, where the annihilator of x is given by $(0 : x) = \cap_{s \in N}(0 : x)_s$ and $(0 : x)_s = \{t \in N \mid [t\ s\ x] = 0\}$. $AH_3(N)$ is shown to be an empty hypergraph if N is a constant RTNR, and $AH_3(N)$ is trivial when N is a zero-symmetric integral RTNR.

Motivated by the results established in [6], [9], [10] and [18], the projection graphs of rings and near-rings are introduced in this article. Throughout, this article R is considered as a nonnil unital commutative ring unless otherwise mentioned. The induced subgraph of $P(R)$ on $R \setminus \{0, 1\}$ is denoted by $P_1(R)$. Also, $U(R)$ denotes the set of all units of R.

Let R be a commutative ring. Then, the vertex set of $P(R)$ is R^*, the set of all nonzero elements of R and $x, y \in R^*$ are adjacent if and only if the product xy in R equals either x or y. It is observed that $x, y \in W^*(R)$ are adjacent in $P(R)$ implies x, y are adjacent in $\overline{\Gamma'(R)}$ and therefore the induced subgraph of $P(R)$ on $W^*(R)$ is a subgraph of $\overline{\Gamma'(R)}$. It is proved that $P(R)$ is a connected graph with diameter at most 2. Let $|R| > 4$. Then, it is seen that $P_1(R)$ is nontrivial if and only if R has nonzero zero-divisors. Also $P(R)$ is a star if and only if R is a field. The girth of $P(R)$ is either 3 or ∞.

A method of finding adjacent vertices using concept of annihilators is given and it is illustrated for $R = \mathbb{Z} \times \mathbb{Z}$. $Reg(R)\backslash\{1\}$, $Nil(R)\backslash\{0\}$ are found independent sets, where $Reg(R)$ is the set of all regular elements of R and $Nil(R)$ is the set of all nilpotent elements of R. If R is presimplifiable ring which is not a domain, then it is proved that $P_1(R)$ is bipartite. $P(R)$ is shown to be weakly pancyclic when R is a local ring, which is not a domain, with ascending chain condition on the annihilator ideals of elements of R. The projection graphs of finite isomorphic rings are proved to be isomorphic. It is also shown that $P(R)$ is complete if and only if either $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_4$. Some of the graph properties of $P(\mathbb{Z}_n)$ are verified for $n = 2q, 2^k$, q is prime and $k \geq 1$.

Let N be a near-ring. Then, the projection graph $P(N)$ of N is defined in the same way as that of a ring. It is shown that if N is either a constant near-ring or an almost trivial near-ring, then $P(N)$ is a complete graph. Also $P(N)$ is complete if N is a Boolean near-ring which is subdirectly irreducible.

2. Preliminaries

In this section the basic definitions along with the results relevant to this paper, related to rings ([8], [4], [14]), near-rings ([15], [16], [17]) and graphs ([12]) are given. Let R be a commutative ring with unity. Then, an element $x \in R$ is called *Von Neumann regular* if $x = ax^2$ for some $a \in R$. R is called (i) *Boolean* if every $x \in R$ is idempotent (ii) a *quasilocal ring* if R has finitely many maximal ideals. (iii) a *local ring* if R has a unique maximal ideal. (iv) [4] a *presimplifiable ring* if, for any $a, b \in R$, $a = ab$ implies either $a = 0$ or $b \in U(R)$. (v) a *domain-like ring* if $ZD(R) \subseteq Nil(R)$, where $Nil(R)$ equals the set of all nitpotent elements of R. (vi) a *nil ring* if every element in R is nilpotent. It is known that quasilocal rings are presimplifiable rings.

Lemma 2.1 ([14]). *If R is nil, then $xy \neq y$ for all $x, y \in R^*$.*

Lemma 2.2 ([4]). *If R is a commutative ring, then the following are equivalent:*

(i) *R is presimplifiable;*

(ii) *$ZD(R) \subseteq J(R)$;*

(iii) *$ZD(R) \subseteq \{1 - u \mid u \in U(R)\}$, where $J(R)$ denotes the Jacobson radical and $J(R)$ equals the intersection of all maximal ideals of R.*

Definition 2.1 ([15]). *A right near-ring N is an algebraic system with two binary operations $+$ and $\cdot$ satisfying the following conditions:*

(i) $(N, +)$ *is a group (not necessarily abelian);*

(ii) $(N, \cdot)$ *is a semigroup;*

(iii) $(x + y)z = xz + yz$ *for every* $x, y, z \in N$.

If $N = N_0 = \{x \in N | x0 = 0\}$, then N is called a *zero-symmetric near-ring.* If $N = N_c = \{x \in N | x0 = x\} = \{x \in N | xy = x$ for every $y \in N\}$, then N is called a *constant near-ring.* A *near-field* is a near-ring, in which there is a multiplicative identity and every non-zero element has a multiplicative inverse. Also by *Pierce Decomposition*, $(N, +) = N_0 + N_c$ and $N_0 \cap N_c = \{0\}$.

Definition 2.2 ([16]). *A near-ring N is called an almost trivial near-ring if for all* $x, y \in N$, $xy = \begin{cases} x & \text{if } y \notin N_c \\ 0 & \text{if } y \in N_c \end{cases}$.

Lemma 2.3 ([16]). *If N is a subdirectly irreducible Boolean near-ring, then N is an almost trivial near-ring.*

A pair $G = (V, E)$ is an *undirected graph* if V is the set of vertices and E is set of edges $\overline{xy}$, where $x, y \in V$ and $x \neq y$. If $x \in V$, then $N_G(x) = \{y \in V \mid \overline{xy} \in E, x \neq y\}$. The *girth* of G is the length of shortest cycle in G and if G has no cycles, then the girth of G is defined to be infinite. G is called *weakly pancyclic* if it contains cycles of all lengths between its girth and the longest cycle. The sequence of degrees of vertices in G arranged in a non decreasing order is called the *degree sequence* of G.

3. Projection graphs of rings

Definition 3.1. *Let $(R, +, \cdot)$ be a ring. Then, the projection graph of R, denoted by $P(R)$, is defined as an undirected graph whose vertex set is the set of all nonzero elements of R and two distinct vertices x and y are adjacent whenever the product $x \cdot y$ equals either x or y. That is, $P(R) = (V, E)$, where $V = R^*$ and $E = \{\overline{xy} \mid x \cdot y = x$ or $y, x \neq y\}$. For the sake of convenience, $x \cdot y$ is simply written as xy.*

Example 3.1. It is evident that the projection graph of $2\mathbb{Z}$ is an empty graph. The projection graphs of the rings $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_2^3$, $\mathbb{Z}_{12}$ and $\mathbb{Z}_3^2$ are shown in Figure 1, Figure 2, Figure 3, Figure 4, Figure 5 and Figure 6, respectively. Note that, $P(\mathbb{Z}_4)$ is a complete graph and $P(\mathbb{Z}_5)$ is a star. In $P(\mathbb{Z}_2^3)$, ijk stands for (i, j, k), where $i, j, k \in \mathbb{Z}_2$. In $P(\mathbb{Z}_3^2)$, ij stands for (i, j), where $i, j \in \mathbb{Z}_3$.

Proposition 3.1. *Let R be a commutative ring with nonzero identity. Then, $P(R)$ is a connected graph with diameter at most 2.*

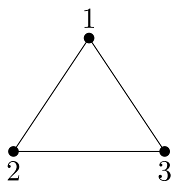


Figure 1: $P(\mathbb{Z}_4)$

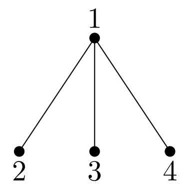


Figure 2: $P(\mathbb{Z}_5)$

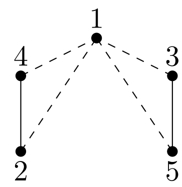


Figure 3: $P(\mathbb{Z}_6)$

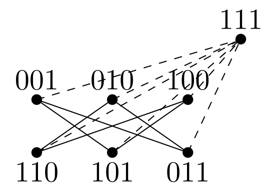


Figure 4: $P(\mathbb{Z}_2^3)$

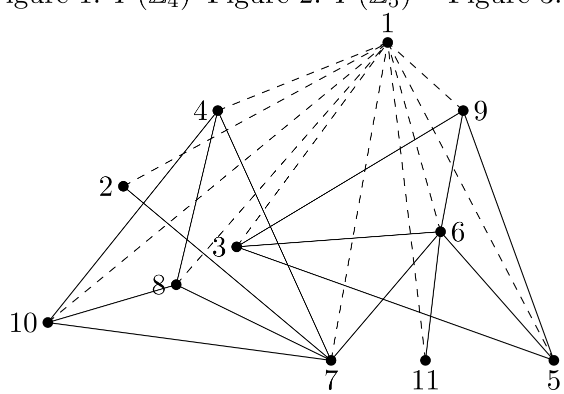


Figure 5: $P(\mathbb{Z}_{12})$

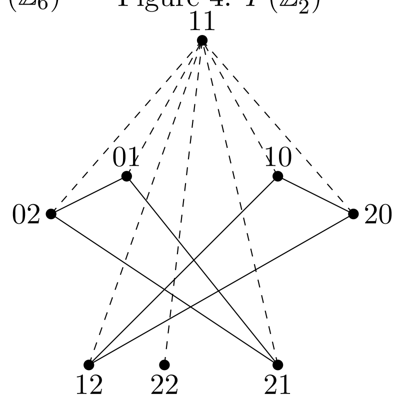


Figure 6: $P(\mathbb{Z}_3^2)$

Proof. Note that, $P(R)$ is nontrivial since $\overline{1x}$ is an edge for every $x \in R^*\backslash\{1\}$. Let x, $y \in R^*$. If $\overline{xy}$ is an edge, then the distance between x and y is 1. If $\overline{xy}$ is not an edge, then $x - 1 - y$ is a path between x and y. Thus, $P(R)$ is connected and the distance between x and y is at the most 2, which proves the proposition. $\square$

Remark 3.1. Notice that the removal of 1 from the vertex set may result in disconnection of $P(R)$. For example, $P_1(\mathbb{Z}_5)$, $P_1(\mathbb{Z}_6)$ and $P_1(\mathbb{Z}_3^2)$ are disconnected. Also it is observed that $P_1(R)$ is disconnected for the Boolean ring $R = \mathbb{Z}_2^2$.

Let R be a commutative ring with nonzero identity. If $x, y \in ZD^*(R)$ are adjacent in $\Gamma(R)$, then x, y are not adjacent in $P(R)$. However, $P_1(R)$ is nontrivial if and only if R has nonzero zero-divisor, which is proved in this section.

Proposition 3.2. *If $x, y \in R^*\backslash\{1\}$ are distinct elements such that $x + y \neq 1$, then the following assertions hold in $P_1(R)$:*

(*i*) *If $xy = 0$, then $1 - y \in N_{P_1(R)}(x)$ and $1 - x \in N_{P_1(R)}(y)$.*

(*ii*) *If x is adjacent to y, then $1 - x \in N_{P_1(R)}(1 - y)$.*

Proof. (i) If $xy = 0$, then $x(1 - y) = x$ and $(1 - x)y = y$, where $1 - x, 1 - y$ are in $R^*\backslash\{1, x, y\}$, proving (i).

(ii) If x is adjacent to y, then either $xy = x$ or $xy = y$.

If $xy = x$, then $(1-x)(1-y) = 1-y$. Similarly, if $xy = y$, then $(1-x)(1-y) = 1-x$, where $1-x, 1-y \in R^*\backslash\{1, x, y\}$, proving (ii). □

Proposition 3.3. *If R is a Boolean ring with more than 4 elements and $x, y \in R^*\backslash\{1\}$, then the following assertions hold in $P_1(R)$:*

(i) If $xy = 0$ and $x + y \neq 1$, then $x - (x+y) - y$ is a path between x and y.

(ii) If $xy = 0$ and $x + y = 1$, then there is no $z \in R^*\backslash\{1\}$ such that $x - z - y$ is a path between x and y.

(iii) If x and y are adjacent and $x + y \neq 1$, then either $x + y \in N_{P_1(R)}(x)$ or $x + y \in N_{P_1(R)}(y)$, but not both.

(iv) If $xy \neq 0$ and x, y are not adjacent, then $x - xy - y$ is a path between x and y.

Proof. (i) If $xy = 0$ and $x + y \neq 1$, then $x(x+y) = x$ and $(x+y)y = y$, where $x + y \in R^*\backslash\{1, x, y\}$, proving (i).

(ii) Suppose $xy = 0$ and $x + y = 1$.

Let $z \in R^*\backslash\{1\}$ be adjacent to x. Then, either $xz = x$ or $xz = z$.

Case (a). Suppose $xz = x$. Then, zy is neither z nor y. For, if $zy = z$, then $x = xz = xzy = 0$, a contradiction to the choice of x. If $zy = y$, then $1 = x + y = xz + zy = z(x+y) = z$, a contradiction to the choice of z.

Case (b). Suppose $xz = z$. Then, zy is neither z nor y. For, if $zy = z$, then $z = (x+y)z = xz + yz = z + z = 0$, a contradiction to the choice of z. If $zy = y$, then $y = zy = xzy = 0$, a contradiction to the choice of y.

Hence, z is not adjacent to y in both the cases, which completes the proof of (ii).

(iii) Suppose x, y are adjacent and $x + y \neq 1$. Then, either $xy = x$ or $xy = y$. If $xy = x$, then $x(x+y) = x^2 + xy = x + x = 0$, since R is of characteristic 2. Also $(x+y)y = xy + y^2 = x + y$. Hence, $x + y \notin N_{P_1(R)}(x)$, whereas $x + y \in N_{P_1(R)}(y)$.

Similarly, if $xy = y$, then it can be seen that $x + y \in N_{P_1(R)}(x)$ and $x + y \notin N_{P_1(R)}(y)$.

(iv) If $xy \neq 0$ and x, y are not adjacent, then $x(xy) = xy$ and $(xy)y = xy$, where $xy \in R^*\backslash\{1, x, y\}$, proving (vi). □

Proposition 3.4. *If $P_1(R)$ is nontrivial, then R has nonzero zero-divisor.*

Proof. Suppose $x,\ y \in R^*\backslash\{1\}$ and $\overline{xy}$ is an edge. Then, either $xy = x$ or $xy = y$. If $xy = x$, then $x(1-y) = 0$, which shows that x is a nonzero zero-divisor. Similarly, if $xy = y$, then y is nonzero zero-divisor. □

Remark 3.2. If $e \in R$ is a nontrivial idempotent, then $1 - e$ is also a nontrivial idempotent and the principal ideal generated by e has at least two elements, namely 0 and e. Also eR has more than 2 elements only if $|R| \geq 6$.

Proposition 3.5. *If $e \in R$ is a nontrivial idempotent, then* (*i*) *e is adjacent to every element in $eR\backslash\{0,e\}$.*

(*ii*) *no element in $eR\backslash\{0\}$ is adjacent to an element in $(1-e)R\backslash\{0\}$.*

Proof. Suppose $e \in R$ is a nontrivial idempotent.

(i) Let $x \in eR\backslash\{0,e\}$. Then, $x = er$ for some $r \in R^*\backslash\{1\}$ and hence $ex = e(er) = er = x$, which shows that e is adjacent to x.

(ii) Let $x \in eR\backslash\{0\}$ and $y \in (1-e)R\backslash\{0\}$. Then, $x = er$ and $y = (1-e)s$, for some r, s in R^* and therefore $xy = 0$ since $e(1-e) = 0$. Hence, x and y are not adjacent. □

Proposition 3.6. *Let $e \in R$ be a nontrivial idempotent. If the principal ideal generated by e is of size two, then either $\overline{ex} \in E$ or $\overline{(1-e)x} \in E$, for every $x \in R^*\backslash\{1,e,1-e\}$.*

Proof. Suppose $|eR| = 2$. Then, er is either 0 or e for every r in R.

Let $A_1(e) = \{r \in R^* | er = e\}$ and $A_1'(e) = \{r \in R^* | er = 0\}$. Then, $R^* = A_1(e) \cup A_1'(e)$, where $1, e \in A_1(e)$ and $1-e \in A_1'(e)$.

Let $x \in R^*\backslash\{1,e,1-e\}$. If $x \in A_1(e)$, then $ex = e$, which implies $\overline{ex} \in E$. If $x \in A_1'(e)$, then $(1-e)x = x$, which implies $\overline{(1-e)x} \in E$. □

Proposition 3.7. *Let R be a commutative ring with nonzero identity such that $|R| > 4$. Then, $P_1(R)$ is nontrivial if and only if R has a nonzero zero-divisor.*

Proof. By Proposition 3.4, it is enough to prove that $P_1(R)$ is nontrivial if R has nonzero zero-divisor.

Let $x \in R$ be nonzero zero-divisor. Then, there exists $y \in R^*$ such that $xy = 0$.

Suppose $1-y \neq x$. Then $x(1-y) = x - xy = x$ and so $\overline{x(1-y)}$ is an edge, where $x, 1-y \in R^*\backslash\{1\}$. Suppose $1-y = x$. Then, x is a nontrivial idempotent. Now, consider the cases:

(i) $|xR| = 2$ (ii) $|xR| > 2$.

If $|xR| = 2$, then $xR = \{0,x\}$ and therefore there exists $r \in R^*\backslash\{1\}$ such that $xr = x$, which implies $\overline{xr} \in E$, where $x, r \in R^*\backslash\{1\}$.

If $|xR| > 2$, then by Proposition 3.5(i), there exists $y \in xR\backslash\{0,x\}$ such that $\overline{xy} \in E$, where $x, y \in R^*\backslash\{1\}$. □

Corollary 3.1. *Let R be a ring with $|R| > 4$. Then, $P(R)$ is a star if and only if R satisfies any one of the following equivalent conditions:*

(*i*) *$P_1(R)$ is trivial.*

(*ii*) *R has no nonzero zero-divisor.*

(*iii*) *Every element in R^* has trivial annihilator.*

Proof. $P_1(R)$ is trivial if and only if $E = \{\overline{x1} | x \in R^*\backslash\{1\}\}$. Therefore, $P(R)$ is a star if and only if $P_1(R)$ is trivial.

(i) $\Leftrightarrow$ (ii) follows from the above proposition.

(ii) $\Leftrightarrow$ (iii) follows from the definition of annihilator. □

Corollary 3.2. *Let R be a ring with $|R| > 4$. Then, $P(R)$ is a star if and only if R is a field.*

Proposition 3.8. *Let R be a ring with $|R| > 4$. Then, the girth of $P(R)$ is either 3 or ∞.*

Proof. If R has no nonzero zero-divisors, then $P(R)$ is a star by Corollary 3.1 and hence the girth is ∞.

If R has nonzero zero-divisor, then $P_1(R)$ is nontrivial by Proposition 3.7.

Let $\overline{xy} \in E$, where $x, y \in R^*\backslash\{1\}$. Then, $1 - x - y - 1$ forms a cycle and hence the girth is 3. □

For any ring R, write $V = R^* = \{1\} \cup (Reg(R)\backslash\{1\}) \cup (ZD(R)\backslash\{0\})$, where $Reg(R) = \{x \in R^* | x \notin ZD(R)\}$. Then, $N_{P(R)}(1) = R^*\backslash\{1\}$ and for every $x \in R^*\backslash\{1\}$, $N_{P(R)}(x) = \{y \in R^* | xy = x \text{ or } xy = y, y \neq x\}$. Now, for every $x \in R^*\backslash\{1\}$, write $A_1(x) = \{y \in R^* | xy = x\}$ and $A_2(x) = \{y \in R^* | xy = y\}$. Then, it is observed that $x = xy = xy^2 = \ldots = xy^k = \ldots$ holds if $y \in A_1(x)$ and $y = xy = x^2y = \ldots = x^ky = \ldots$ holds if $y \in A_2(x)$. Thus, $N_{P(R)}(x)$ contains an infinite number of elements if any one of the above sequences does not terminate.

Proposition 3.9. *Let $x \in R^*\backslash\{1\}$. Then, the following assertions hold:*

(i) $A_1(x) \cap A_2(x) = \{x\}$ *if and only if x is an idempotent.*

(ii) $A_1(x) = Ann(x) + 1$; $A_2(x) = Ann(1 - x)\backslash\{0\}$.

Proof. (i) Suppose $x \in R^*\backslash\{1\}$ is an idempotent element. Then, $x^2 = x$ and so $x \in A_1(x) \cap A_2(x)$. Also, $y \in A_1(x) \cap A_2(x)$ implies $y = xy = x$ and hence $A_1(x) \cap A_2(x) = \{x\}$.

Conversely, suppose $A_1(x) \cap A_2(x) = \{x\}$. Then, $xx = x$, which proves (i).

(ii) By the definition of $A_1(x)$, $y \in A_1(x) \Leftrightarrow xy = x \Leftrightarrow x(y - 1) = 0 \Leftrightarrow y - 1 \in Ann(x)$.

Now, $y - 1 \in Ann(x) \Leftrightarrow y \in Ann(x) + 1$. For, if $y - 1 \in Ann(x)$, then $y = (y - 1) + 1 \in Ann(x) + 1$. Also if $y \in Ann(x) + 1$, then $y = z + 1$, for some $z \in Ann(x)$, which implies $y - 1 = z \in Ann(x)$. Hence, $A_1(x) = Ann(x) + 1$. By the definition of $A_2(x)$, $y \in A_2(x) \Leftrightarrow y \neq 0$ and $xy = y \Leftrightarrow y \neq 0$ and $y(1 - x) = 0 \Leftrightarrow y \in Ann(1 - x)\backslash\{0\}$ and hence $A_2(x) = Ann(1 - x)\backslash\{0\}$. □

Proposition 3.10. *If $x \in Reg(R)\backslash\{1\}$, then $N_{P(R)}(x) \subseteq (ZD(R)\backslash\{0\}) \cup \{1\}$.*

Proof. Let $x \in Reg(R)\backslash\{1\}$ and $y \in N_{P(R)}(x)$. Then, $xy = x$ or $xy = y$.

If $xy = x$, then $x(y-1) = 0$, which implies $y = 1$ by the hypothesis.

If $xy = y$, then $(x-1)y = 0$, which implies $y \in ZD(R)\backslash\{0\}$, completing the proof. □

Corollary 3.3. *$Reg(R)\backslash\{1\}$ is an independent set.*

Proof. Let $x \in Reg(R)\backslash\{1\}$ and $y \in N_{P(R)}(x)$. Then, $y \notin Reg(R)\backslash\{1\}$ from the above proposition. Hence, $Reg(R)\backslash\{1\}$ is independent. □

Remark 3.3. If R is finite, then $V = R^* = \{1\} \cup (U(R)\backslash\{1\}) \cup (ZD(R)\backslash\{0\})$. Hence, $U(R)\backslash\{1\}$ is independent by the above corollary.

Theorem 3.1. *For any $x \in R^*\backslash\{1\}$, the following assertions hold, in which $\mathbb{E}$ denotes the set of all nontrivial idempotents in R:*

(i) $N_{P(R)}(x) = \{1\} \cup (Ann(1-x)\backslash\{0\})$ *if* $x \in Reg(R)\backslash\{1\}$.

(ii) $N_{P(R)}(x) = ((Ann(x)+1) \cup Ann(1-x))\backslash\{0\}$ *if* $x \in ZD(R)\backslash\{0\}$ *and* $x \notin \mathbb{E}$.

(iii) $N_{P(R)}(x) = ((Ann(x)+1) \cup Ann(1-x))\backslash\{0, x\}$ *if* $x \in ZD(R)\backslash\{0\}$ *and* $x \in \mathbb{E}$.

Proof. Let $x \in R^*\backslash\{1\}$. Then, by the definitions of $A_1(x)$ and $A_2(x)$ and Proposition 3.9(ii), $N_{P(R)}(x) = A_1(x) \cup A_2(x) = (Ann(x)+1) \cup (Ann(1-x)\backslash\{0\})$.

(i) If $x \in Reg(R)\backslash\{1\}$, then $Ann(x) = \{0\}$. Hence, $N_{P(R)}(x) = \{1\} \cup (Ann(1-x)\backslash\{0\})$.

(ii) If $x \in ZD(R)\backslash\{0\}$ and $x \notin \mathbb{E}$, then $N_{P(R)}(x) = (Ann(x)+1) \cup (Ann(1-x)\backslash\{0\})$.

(iii) If $x \in ZD(R)\backslash\{0\}$ and $x \in \mathbb{E}$, then $N_{P(R)}(x) = ((Ann(x)+1) \cup Ann(1-x))\backslash\{0, x\}$, by Proposition 3.9(i). □

Proposition 3.11. *If $x \in R^*\backslash\{1\}$ is not a zero-divisor, then $N_{P(R)}(x)\backslash\{1\}$ together with 0 forms an ideal.*

Proof. If x is not a zero-divisor, then by Theorem 3.1(i), $(N_{P(R)}(x)\backslash\{1\}) \cup \{0\} = Ann(1-x)$, which is an ideal. □

Illustration 3.1. Consider $R = \mathbb{Z} \times \mathbb{Z}$, where $ZD(R) = (\mathbb{Z} \times \{0\}) \cup (\{0\} \times \mathbb{Z})$ and $Reg(R) = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} | m, n \neq 0\}$.

If $x = (1, 1)$, then $N_{P(R)}(x) = R^*\backslash\{(1, 1)\}$.

If $x = (m, n) \in Reg(R)\backslash\{(1, 1)\}$, then $N_{P(R)}(x) = (\{0\} \times \mathbb{Z}^*) \cup \{(1, 1)\}$ if $m \neq 1, n = 1$, $N_{P(R)}(x) = (\mathbb{Z}^* \times \{0\}) \cup \{(1, 1)\}$ if $m = 1, n \neq 1$, $N_{P(R)}(x) = \{(1, 1)\}$ if $m, n \neq 1$. Thus, $Reg(R)\backslash\{(1, 1)\}$ is independent.

If $x = (m, n) \in ZD(R)\backslash\{(0, 0)\}$, then $N_{P(R)}(0, 1) = (\mathbb{Z} \times \{1\}) \cup (\{0\} \times \mathbb{Z}^*)\backslash\{(0, 1)\}$, $N_{P(R)}(1, 0) = (\{1\} \times \mathbb{Z}) \cup (\mathbb{Z}^* \times \{0\})\backslash\{(1, 0)\}$.

$N_{P(R)}(x) = \mathbb{Z} \times \{1\}$ if $m = 0, n \neq 1$, $N_{P(R)}(x) = \{1\} \times \mathbb{Z}$ if $m \neq 1, n = 0$.

Note that, $(0, 1)$ and $(1, 0)$ are the nontrivial idempotents in R.

Proposition 3.12. *Let $e \in R$ be a nontrivial idempotent. Then*

(i) $N_{P(R)}(e) = (((1-e)R+1) \cup eR)\backslash\{0,e\}$.

(ii) *Every element in $eR\backslash\{0\}$ is adjacent to every element in $(1-e)R+1$.*

(iii) *For every $x \in eR\backslash\{0,e\}$ and $y \in ((1-e)R+1)\backslash\{e\}$, $e-x-y-e$ forms a cycle.*

Proof. (i) If $e \in R$ is a nontrivial idempotent, then by Theorem 3.1(iii), $N_{P(R)}(e) = ((Ann(e)+1) \cup Ann(1-e))\backslash\{0,e\}$.

Now, if $r \in Ann(e)$, then $re = 0$, which implies $r = r1 = r((1-e)+e) = r(1-e) \in (1-e)R$. Also, $r \in (1-e)R$ implies $r \in Ann(e)$. Hence, $Ann(e) = (1-e)R$.

Similarly, it can be proved that $Ann(1-e) = eR$. Thus, $N_{P(R)}(e) = (((1-e)R+1) \cup eR)\backslash\{0,e\}$.

(ii) Let $x \in eR\backslash\{0\}$ and $y \in (1-e)R+1$. Then, $x \in Ann(1-e)\backslash\{0\}$, which implies $xe = x$ and there exists $z \in Ann(e)$ such that $y = z+1$.

Now, $xy = x(z+1) = xe(z+1) = x$. Hence, $\overline{xy} \in E$, proving (ii).

(iii) Let $x \in eR\backslash\{0,e\}$ and $y \in ((1-e)R+1)\backslash\{e\}$. Then, $\overline{ex}, \overline{ye} \in E$ by (i) and $\overline{xy} \in E$ by (ii). Hence, $e-x-y-e$ forms a cycle. □

Proposition 3.13. *Let $e \in R$ be a nontrivial idempotent such that both of eR and $(1-e)R+1$ contain more than 2 elements. Then, the following assertions hold in $P_1(R)$:*

(i) *$P_1(R)$ contains $K_{i,j}$, where $i = |eR|-2$ and $j = |(1-e)R+1|-2$.*

(ii) *$P_1(R)$ is not planar if both of eR and $(1-e)R+1$ contain more than 5 elements.*

Proof. (i) Let $V_1 = eR\backslash\{0,e\}$ and $V_2 = ((1-e)R+1)\backslash\{1,e\}$. Then, for any $x \in V_1$ and $y \in V_2$, $\overline{xy} \in E$ by Proposition 3.12(ii), proving (i).

(ii) Clearly, $P_1(R)$ contains $K_{3,3}$ if both of eR and $(1-e)R+1$ have more than 5 elements by (i). Hence, $P_1(R)$ is not a planar graph. □

Proposition 3.14. *The following assertions hold in $P(R)$:*

(i) *If $x \in R^*$ is a nilpotent element, then there exists an integer $k \geq 2$ such that x^i is adjacent to $1-x^{k-i}$ for every $1 \leq i \leq k-1$.*

(ii) *If $x \in R^*$ is a nilpotent element, then $N_{P(R)}(x)$ is a multiplicatively closed set of the form $I+1$ for an ideal I of R.*

(iii) *$Nil(R)\backslash\{0\}$ is an independent set.*

Proof. (i) If $x \in R^*$ is a nilpotent element, then there exists an integer $k \geq 2$ such that $x^k = 0$ and $x^i \neq 0$ for $1 \leq i \leq k-1$. Hence, $x^i(1 - x^{k-i}) = x^i$, which implies that x^i is adjacent to $1 - x^{k-i}$ for all $1 \leq i \leq k - i$.

(ii) Let $x \in R^*$ be a nilpotent element and k be the least positive integer such that $x^k = 0$. Then, it can be seen that $(1-x)(1+x+x^2+\ldots+x^{k-1}) = 1$ and so $1-x$ is a unit. Hence, by Theorem 3.1(ii), $N_{P(R)}(x) = Ann(x)+1$. Thus, by taking $I = Ann(x)$, $N_{P(R)}(x) = I + 1$, which is a multiplicatively closed set.

(iii) Let $x, y \in Nil(R)\backslash\{0\}$ and k and l be the least positive integers such that $x^k = 0 = y^l$.

Suppose, $\overline{xy} \in E$. Then, either $xy = x$ or $xy = y$.

If $xy = x$, then $x = xy = xy^2 = \ldots = xy^k$, a contradiction to the choice of x.

Similarly, $xy = y$ implies $y = x^l y$, a contradiction to the choice of y. Hence, $\overline{xy} \notin E$.

□

Example 3.2. In $R = \frac{\mathbb{Z}_2[x]}{(x^3)}, Nil(R)\backslash\{0\} = \{[x], [x^2], [x^2+x]\}$, which is an independent set.

Remark 3.4. If R is a domainlike ring, then every zero-divisor is a nilpotent and hence the set of nonzero zero-divisors in R is independent.

Proposition 3.15. *If R is not a domain, then $P_1(R)$ is bipartite when R has any one of the following equivalent conditions:*

(i) Every nonunit is a nilpotent.

(ii) R has a unique prime ideal.

(iii) $\frac{R}{Nil(R)}$ is a field.

Proof. Suppose that every nonunit in R is a nilpotent. Then, $R^*\backslash\{1\} = (Nil(R)\backslash\{0\}) \cup (U(R)\backslash\{1\})$, in which $Nil(R)\backslash\{0\}$ and $U(R)\backslash\{1\}$ are independet sets. Hence, any edge $\overline{xy}$ with $x, y \in R^*\backslash\{1\}$ has one end in $Nil(R)\backslash\{1\}$ and the other end in $U(R)\backslash\{1\}$. Thus, $Nil(R)\backslash\{1\}$ and $U(R)\backslash\{1\}$ form a bipartition for $P_1(R)$, as required.

As it is known that (i) ⇔ (ii) ⇔ (iii), the proposition follows. □

Proposition 3.16. *If R is a ring which is not domain, then $P_1(R)$ is bipartite when R has any one of the following equivalent conditions:*

(i) R is presimplifiable.

(ii) $ZD(R) \subseteq J(R)$.

(iii) $ZD(R) \subseteq \{1 - u | u \in U(R)\}$.

Proof. By Lemma 2.2, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Suppose that R is presimplifiable.

Let $\overline{xy}$ be any edge with $x, y \in R^*\backslash\{1\}$. Then, $xy = x$ or $xy = y$. Now, consider the following cases:

(i) $x, y \in U(R)\backslash\{1\}$ (ii) $x, y \in W^*(R)$ (iii) $x \in U(R)\backslash\{1\}$ and $y \in W^*(R)$.

Since $U(R)\backslash\{1\}$ is independent case (i) is not possible. Also, since R is presimplifiable and x, y are nonzero elements, if $xy = x$, then $y \in U(R)$. Similarly, if $xy = y$, then $x \in U(R)$, which shows that case (ii) is also not possible.

Hence, the only possible choice is case (iii). That is, $x \in U(R)\backslash\{1\}$, $y \in W^*(R)$. Thus, $U(R)\backslash\{1\}$ and $W^*(R)$ form a bipartition for $P_1(R)$, as desired. □

Corollary 3.4. *If R is a local ring, which is not a domain, then $P_1(R)$ is bipartite.*

Proof. As R is local, it is presimplifiable and hence the proof follows from Proposition 3.16. □

Proposition 3.17. *Let R be a local ring, which is not a domain.*

If $x, y \in R^*\backslash\{1\}$ and $Ann(x) \cap Ann(y) \neq \{0\}$, then there exists a path $x - u - y$ with $u \in U(R)\backslash\{1\}$.

Proof. Since R is local, it has a unique maximal ideal $\mathcal{M}$, say.

Let $x, y \in R^*\backslash\{1\}$ and $t(\neq 0) \in Ann(x) \cap Ann(y)$. Then, $tx = ty = 0$, which implies $(1-t)x = x$ and $(1-t)y = y$.

Hence, as $1 - t \in R^*\backslash\{1, x, y\}$, $x - (1-t) - y$ is a path between x and y. Now, it is claimed that $1-t$ is a unit. Suppose $1-t$ is not a unit. Then, it must be in a maximal ideal. Now, both $t, 1-t \in \mathcal{M}$, which is closed under addition.

Hence, $1 \in \mathcal{M}$, showing that $\mathcal{M} = R$, a contradiction to the fact that $\mathcal{M}$ is a proper ideal. Thus, the claim is proved. □

Proposition 3.18. *Let R be a local ring, which is not a domain, and R has ascending chain condition(ACC) on ideals of the form $Ann(x)$, $x \in R$. Then, the following assertions hold:*

(*i*) *$P(R)$ contains cycles of lengths j, $3 \leq j \leq 2k+1$, where k is the number of nontrivial annihilators in R.*

(*ii*) *$P(R)$ is weakly pancyclic.*

Proof. Since the ideals $Ann(x), x \in R$ satisfy ACC, there exist $x_1, \ldots, x_k, x_{k+1} \ldots$ in R such that $Ann(x_1) \subset Ann(x_2) \subset \ldots \subset Ann(x_k) = Ann(x_{k+1}) = \ldots$ for some positive integer k.

(i) Let $y_i \in Ann(x_i)\backslash Ann(x_{i-1})$ for every $1 \leq i \leq k$. Then, $x_i y_i = x_{i+1} y_i = 0$, which implies $x_i(1 - y_i) = x_i$ and $x_{i+1}(1 - y_i) = x_{i+1}$, where $1 - y_i \in R^*\backslash\{1, x_i, x_{i+1}\}$. Hence, $x_i - (1 - y_i) - x_{i+1}$ is a path as in Proposition 3.17.

Thus, each one of the following is a cycle: $1-x_1-(1-y_1)-1$, (a cycle of length 3), $1-x_1-(1-y_1)-x_2-1$, (a cycle of length 4), $1-x_1-(1-y_1)-x_2-(1-y_2)-1$, (a cycle of length 5) and so on, proving (i).

(ii) $P(R)$ is weakly pancyclic by (i) and the definition of weakly pancyclic graph. □

The proof of the following proposition is omitted as it is trivial from the natural product defined in a quotient ring.

Proposition 3.19. *Let I be a nontrivial ideal in R. If x, y are adjacent in $P(R)$, then $x+I$ and $y+I$ are adjacent in $P(\frac{R}{I})$, where $\frac{R}{I}$ denotes the quotient ring.*

The following proposition shows that the projection graphs of finite isomorphic rings are isomorphic.

Proposition 3.20. *Let R and S be finite rings such that $R \cong S$. Then, $P(R) \cong P(S)$.*

Proof. By the hypothesis, there exists a one-one, onto ring homomorphism ϕ between R and S. Let ϕ^* be the restriction of ϕ to R^*. Then, ϕ^* is a one-one, onto function. As $|R^*| = |S^*|$, $|V(P(R))| = |V(P(S))|$, where $V(P(R))$ and $V(P(S))$ denote the sets of vertices of R and S respectively.

Let $x, y \in V(P(R))$ such that x and y are adjacent. Then, $xy = x$ or $xy = y$. If $xy = x$, then $\phi^*(xy) = \phi^*(x)$, which implies $\phi^*(x)\phi^*(y) = \phi^*(x)$. Therefore, $\phi^*(x)$ is adjacent to $\phi^*(y)$ in $P(S)$.

A similar argument holds for the case, where $xy = y$, proving that ϕ^* preserves the adjacency between vertices. Thus, $P(R) \cong P(S)$. □

Example 3.3. Let $R = \frac{\mathbb{Z}_2[x]}{(x^2)}$; $S = \frac{\mathbb{Z}_2[x]}{(x^2+1)}$. Then, $R \cong S$ and $P(R) \cong P(S)$.

Remark 3.5. The converse of the above proposition need not be true. For, if $R = \mathbb{Z}_4$ and $S = \frac{\mathbb{Z}_2[x]}{(x^2)}$, then $P(R) \cong P(S)$ and $R \ncong S$.

Proposition 3.21. *P(R) is not complete in each of the following cases:*

(*i*) *R has nontrivial idempotent elements.*

(*ii*) $|(U(R)| \geq 3$.

Proof. (i) If R has nontrivial idempotent element e, then $P(R)$ is not complete since e and $1-e$ are not adjacent.

(ii) If there are more than three units, then $P(R)$ is not complete since $U(R)\backslash\{1\}$ is independent. □

Proposition 3.22. *Let R be finite. Then, $P(R)$ is complete if and only if either $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_4$.*

Proof. It is known that $P(\mathbb{Z}_3)$ and $P(\mathbb{Z}_4)$ are complete. Hence, if $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_4$, then $P(R)$ is complete by Proposition 3.20.

Conversely, suppose that $P(R)$ is complete. Then, $|U(R)| \leq 2$ and R has no nontrivial idempotents by the above proposition.

Let $R = \{0, 1, u\} \cup ZD(R)$, where $u \neq 1$ is a unit. Then, it is claimed that $|ZD(R)| \leq 1$.

Suppose $x, y \in ZD(R)$ be distinct nonzero zero-divisors. Then, $xy = x$ or $xy = y$ by the hypothesis.

If $xy = x$, then $(x+u)y = xy+uy = x+y$ since $xu = x$ by the completeness. But, $(x+u)y = x+u$ or $(x+u)y = y$ since $P(R)$ is complete.

If $(x+u)y = x+u$, then from the previous step, $x+u = x+y$ which implies $y = u$, a contradiction to the choice of y. Therefore, $(x+u)y = y$, which implies $x = 0$.

By a similar argument, it can be shown that if $xy = y$, then $y = 0$. Hence, there can be at the most one nonzero zero-divisor. Thus, $|R| \leq 4$.

If $|R| = 3$, then $R \cong \mathbb{Z}_3$.

If $|R| = 4$, then $R \cong \mathbb{Z}_4$, since R is the unital commutative ring of cardinality 4 with no nontrivial idempotents, which completes the proof. □

Proposition 3.23. *If $P(R)$ is not a star, then there exists $x \in R^*\backslash\{1\}$ such that either xR or $(1-x)R$ has a nonzero annihilating ideal.*

Proof. If $P(R)$ is not a star, then there exists $\overline{xy} \in E$, for some $x, y \in R^*\backslash\{1\}$, which implies that either $y \in (Ann(x)+1)\backslash\{1\}$ or $y \in Ann(1-x)\backslash\{0\}$ by Theorem 3.1.

If $y \in (Ann(x)+1)\backslash\{1\}$, then there exists a nozero $z \in Ann(x)$ such that $y = z+1$ and $(y-1)xr = zxr = 0$ for every r in R, showing that $Ann(xR) \neq \{0\}$.

If $y \in Ann(1-x)\backslash\{0\}$, then $(1-x)y = 0$ and therefore $(1-x)yr = 0$ for every $r \in R$. Hence, $Ann((1-x)R) \neq \{0\}$. This completes the proof. □

Proposition 3.24. *If $x, y \in R^*$ are adjacent, then either $xR \subseteq yR$ or $yR \subseteq xR$.*

Proof. Suppose $x, y \in R^*$ and $\overline{xy} \in E$. Then, either $xy = x$ or $xy = y$.

Consider the following possible cases:

(i) $x, y \in U(R)$ (ii) $x \in U(R)$ and $y \notin U(R)$ (iii) $x, y \notin U(R)$.

Case (i) If $x, y \in U(R)$, then $xR = yR = R$.

Case (ii) If $x \in U(R)$ and $y \notin U(R)$, then $xR = R$ and so $yR \subseteq xR$.

Case (iii) Let $x, y \notin U(R)$. If $xy = x$, then $z \in xR$ implies $z = xr$ for some $r \in R$. Therefore, $z = (xy)r = y(xr) \in yR$ and so $xR \subseteq yR$.

Similarly, if $xy = y$, then it can be shown that $yR \subseteq xR$, which completes the proof. □

4. Projection graphs of $\mathbb{Z}_n$

In this section, $\mathbb{Z}_n$, $n \geq 3$, is considered and $P(\mathbb{Z}_n)$ is studied. It is observed that the vertex set V of $P(\mathbb{Z}_n)$ is given by $V = \mathbb{Z}_n^* = U(\mathbb{Z}_n) \cup (ZD(\mathbb{Z}_n)\backslash\{0\})$ and $|V| = n - 1$.

Proposition 4.1. *Let $n \geq 3$. Then:*

(i) $P(\mathbb{Z}_n)$ is complete if and only if $n = 3, 4$.

(ii) $P(\mathbb{Z}_n)$ is a star if and only if n is a prime.

Proof. (i) The proof follows from Proposition 3.22.

(ii) $\mathbb{Z}_n$ has no zero-divisors if and only if n is a prime. Hence, (ii) follows from Corollary 3.1. □

Proposition 4.2. $diam(P(\mathbb{Z}_n)) = \begin{cases} 1, & if\, n = 3, 4 \\ 2, & otherwise. \end{cases}$

Proof. By Proposition 4.1(i), it is clear that the diameter of $P(\mathbb{Z}_n)$ is 1 if and only if $n = 3, 4$. Hence, by Proposition 3.1, the diameter of $P(\mathbb{Z}_n)$ is 2 if $n \geq 5$. □

Proposition 4.3. $girth(P(\mathbb{Z}_n)) = \begin{cases} \infty, & if\, n \;\; is \;\; prime \\ 3, & otherwise. \end{cases}$

Proof. By Proposition 4.1(ii), it is clear that the girth of $P(\mathbb{Z}_n)$ is ∞ if and only if n is a prime. Hence, if n is not a prime, then the girth of $P(\mathbb{Z}_n)$ is 3 by Proposition 3.8. □

Remark 4.1. Note that, $\mathbb{Z}_n$ has nontrivial idempotent, if and only if $x^2 \equiv x \ mod\ n$ for some $1 < x < n$ if and only if n divides $x(1 - x)$ if and only if n has at least two nontrivial divisors.

Proposition 4.4. *Let $x, y \in \mathbb{Z}_n^*$. Then:*

(i) $Ann(x) = Ann(c)$ if $(x, n) = c$.

(ii) $Ann(x) = \{0\}$ if and only if $x \in U(\mathbb{Z}_n)$.

(iii) $Ann(x) = Ann(y)$ if and only if $(x, n) = (y, n)$.

(iv) If $(x, n) = x$, then $Ann(x) = k\mathbb{Z}_n$, where $k = \frac{n}{x}$ and $|k\mathbb{Z}_n| = x$.

(v) $Ann(e) = (1 - e)\mathbb{Z}_n$ and $Ann(1 - e) = e\mathbb{Z}_n$, where e is a nontrivial idempotent.

Proof. (i) Suppose $(x,n)=c$. Then, there exist integers k and l,m such that $x=kc$ and $c=lx+mn$.

Now, $Ann(x)\subseteq Ann(c)$. For, $t\in Ann(x)\Rightarrow tx=0\Rightarrow tlx=0\Rightarrow tc=0\Rightarrow t\in Ann(c)$.

Also, $Ann(c)\subseteq Ann(x)$, since $t\in Ann(c)\Rightarrow tc=0\Rightarrow tkc=0\Rightarrow tx=0\Rightarrow t\in Ann(x)$, which proves (i).

(ii) If $x\in U(\mathbb{Z}_n)$, then $Ann(x)=\{t\in\mathbb{Z}_n|tx=0\}=\{0\}$. Conversely, suppose $x\notin U(\mathbb{Z}_n)$. If $x=0$, then $Ann(x)=\mathbb{Z}_n$.

If $x\neq 0$, then there exists $y\in\mathbb{Z}_n^*$ such that $xy=0$, which implies $Ann(x)\neq\{0\}$.

(iii) The proof of (iii) follows from (i).

(iv) As $Ann(x)$ is an ideal and every ideal in $\mathbb{Z}_n$ is principal, $Ann(x)=a\mathbb{Z}_n$ for some $a\in\mathbb{Z}_n$.

If $(x,n)=x$, then there exists an integer k such that $kx=n$, which implies $k\in Ann(x)$ and hence $k\mathbb{Z}_n\subseteq Ann(x)$. Also, $t\in Ann(x)\Rightarrow tx=0\Rightarrow tx=ln$, for some $l\in\mathbb{Z}_n\Rightarrow t=kl\in k\mathbb{Z}_n$. Hence, $Ann(x)\subseteq k\mathbb{Z}_n$ and $|k\mathbb{Z}_n|=x$, proving (iv).

(v) Assertion (v) follows from the proof of Proposition 3.12 (i). □

Proposition 4.5. *Let s,t be two distinct factors of n. Then:*

(i) $Ann(s)\neq Ann(t)$

(ii) $Ann(s)\subset Ann(t)$, *whenever* $s\mid t$.

(iii) $Ann(s)\cap Ann(t)=\{0\}$ *if and only if* $(s,t)=1$.

Proof. (i) Note that, $(s,n)=s$ and $(t,n)=t$. Therefore, from Proposition 4.4(iv), $Ann(s)=k\mathbb{Z}_n$ and $Ann(t)=l\mathbb{Z}_n$, where $k=\frac{n}{s}$, $l=\frac{n}{t}$. Hence, $Ann(s)\neq Ann(t)$, since $k\neq l$.

(ii) If $s\mid t$, then $sk=t$ for some integer k and therefore $r\in Ann(s)\Rightarrow rs=0\Rightarrow krs=0\Rightarrow tr=0\Rightarrow r\in Ann(t)$. Hence, $Ann(s)\subset Ann(t)$, since $|Ann(s)|=s<t=|Ann(t)|$.

(iii) Suppose $(s,t)=1$. Then, there exist integers k and l such that $ks+lt=1$. Hence, if $r\in Ann(s)\cap Ann(t)$, then $r=rks+rlt$ and so $r=0$.

Conversely, suppose $(s,t)=r\neq 1$. Then, $r\mid s$ and $r\mid t$ and hence by (ii), $Ann(s)\cap Ann(t)\supset Ann(r)\neq\{0\}$. □

Definition 4.1. *Define a relation $\sim$ on $\mathbb{Z}_n^*$ by $x\sim y$ if and only if $Ann(x)=Ann(y)$ for every $x,y\in\mathbb{Z}_n^*$.*

Remark 4.2. The relation $\sim$ defined above on $\mathbb{Z}_n^*$ is an equivalence relation. Hence, if $x\in\mathbb{Z}_n^*$ and $[x]_\sim$ denotes the equivalence class of x, then by Proposition 4.4(iii), $[x]_\sim=\{y\in\mathbb{Z}_n^*|Ann(y)=Ann(x)\}=\{y\in\mathbb{Z}_n^*|(y,n)=(x,n)\}$.

Proposition 4.6. *Using the above notations, the following statements are true:*

(i) $[1]_\sim = U(\mathbb{Z}_n)$; $|[1]_\sim| = \phi(n)$.

(ii) $[1]_\sim \backslash \{1\}$ *is an independent set of size* $\phi(n) - 1$.

(iii) *If* $d \mid n$, *then* $[d]_\sim = \{y \in \mathbb{Z}_n^* | (y, n) = d\}$.

(iv) $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{(x,n)\neq 1}[x]_\sim = \cup_{d|n, d\neq 1}[d]_\sim$.

Proof. (i) By using Remark 4.2, $[1]_\sim = \{y \in \mathbb{Z}_n^* | Ann(y) = \{0\}\} = \{y \in \mathbb{Z}_n^* | (y, n) = 1\} = U(\mathbb{Z}_n)$ and hence $|[1]_\sim| = \phi(n)$.

(ii) The proof follows from Corollary 3.3 using (i).

(iii) Let $d \mid n$. Then, $(d, n) = d$ and hence $[d]_\sim = \{y \in \mathbb{Z}_n^* | Ann(y) = Ann(d)\} = \{y \in \mathbb{Z}_n^* | (y, n) = d\}$.

(iv) From Remark 4.2, $\mathbb{Z}_n^* = [1]_\sim \cup (\cup_{x\in\mathbb{Z}_n^*\backslash\{1\}}[x]_\sim)$ and hence $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{(x,n)\neq 1}[x]_\sim = \cup_{d|n,d\neq 1}[d]_\sim$, by (iii). □

Proposition 4.7. *Let* $n = p^k$, *for some* $k \geq 2$. *Then, the following assertions hold:*

(i) $ZD(\mathbb{Z}_n)\backslash\{0\}$ *is an independent set.*

(ii) $P_1(\mathbb{Z}_n)$ *is bipartite.*

(iii) $P(\mathbb{Z}_n)$ *is weakly pancyclic.*

Proof. If $n = p^k$, then $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{i=1}^{k-1}[p^i]_\sim$, where $[p^i]_\sim = \{y \in \mathbb{Z}_n^* | (y, n) = p^i\}$, by Proposition 4.6(iv).

(i) It is claimed that $ZD(\mathbb{Z}_n) = Nil(\mathbb{Z}_n)$. For, if $x \in ZD(\mathbb{Z}_n)\backslash\{0\}$, then $x \in [p^i]_\sim$, for some i, which implies $x = tp^i$ for some integer t. Hence, $x^{k-i} = 0$ and thus x is a nilpotent element, proving the claim.

Hence, $ZD(\mathbb{Z}_n)\backslash\{0\} = Nil(\mathbb{Z}_n)\backslash\{0\}$, which is independent by 3.14(iii).

(ii) From the proof of (i), it is noted that the set of all nonunits is equal to $Nil(\mathbb{Z}_n)$, which is the unique maximal ideal. Hence, $\mathbb{Z}_n$ is local and thus $P_1(\mathbb{Z}_n)$ is bipartite by Corollary 3.4.

(iii) It is claimed that the ideals of the form $Ann(x)$, $x \in \mathbb{Z}_n$, have ACC.

If $x \in \mathbb{Z}_n^*$, then either $x \in U(\mathbb{Z}_n)$ or $x \in [p^i]_\sim = \{t \in \mathbb{Z}_n^* | Ann(t) = Ann(p^i)\}$, for some i. If $x \in U(\mathbb{Z}_n)$, then $Ann(x) = \{0\}$.

Also, by Proposition 4.5 (ii), $Ann(p) \subset Ann(p^2) \subset \ldots \subset Ann(p^{k-1})$, proving the claim. Thus, $P(\mathbb{Z}_n)$ is weakly pancyclic by Proposition 3.18. □

Proposition 4.8. *If* $n = 2^k$, *for some* $k \geq 2$, *then the following assertions hold:*

(i) $|U(\mathbb{Z}_n)\backslash\{1\}| = |ZD(\mathbb{Z}_n)\backslash\{0\}| = \frac{n}{2} - 1$, $U(\mathbb{Z}_n) = [1]_\sim = \{2j + 1 \in \mathbb{Z}_n^* | j \in \mathbb{Z}_n\}$.

(ii) *If* $x \in [2^i]_\sim$ *and* $x + u = 1$, *then* $deg(x) = deg(u) = 2^i$, *for* $1 \leq i \leq k - 1$.

(iii) *The degree sequence is given by* $(2^{(a_1)}, 2^{2^{(a_2)}}, \ldots, 2^{k-1^{(a_{k-1})}}, n - 2^{(1)})$, *where* (a_i) *denotes the multiplicity and* $(a_i) = 2|[2^i]_\sim|$ *for* $1 \leq i \leq k - 1$.

Proof. (i) $|U(\mathbb{Z}_n)| = \phi(n) = 2^k - 2^{k-1} = n - \frac{n}{2} = \frac{n}{2}$.

Hence, $|U(\mathbb{Z}_n)\backslash\{1\}| = |ZD(\mathbb{Z}_n)\backslash\{0\}| = \frac{n}{2} - 1$. Also, $U(\mathbb{Z}_n) = [1]_\sim = \{y \in \mathbb{Z}_n^* | (y, 2^k) = 1\} = \{2j+1 \in \mathbb{Z}_n^* | j \in \mathbb{Z}_n\}$.

(ii) Let $x \in [2^i]_\sim$ and $x + u = 1$. Then, $u = 1 - x \in U(\mathbb{Z}_n)\backslash\{1\}$ since x is nilpotent from 4.7(i). Therefore, by Theorem 3.1(i), $N_{P(R)}(u) = \{1\} \cup (Ann(1-u)\backslash\{0\}) = \{1\} \cup (Ann(x)\backslash\{0\}) = \{1\} \cup (Ann(2^i)\backslash\{0\}) = \{1\} \cup (2^{k-i}\mathbb{Z}_n\backslash\{0\})$ and so $|N_{P(R)}(u)| = 2^i$. Thus, $deg(u) = 2^i$. Also, $N_{P(\mathbb{Z}_n)}(x) = Ann(2^i) + 1 = 2^{k-i}\mathbb{Z}_n + 1$ and so $|N_{P(\mathbb{Z}_n)}(x)| = |2^{k-i}\mathbb{Z}_n| = 2^i$. Thus, $deg(x) = 2^i$. From the above discussion, it is clear that $deg(u) = deg(x) = 2^i$.

(iii) Note that, $\mathbb{Z}_n^* = \{1\} \cup (U(\mathbb{Z}_n)\backslash\{1\}) \cup (ZD(\mathbb{Z}_n)\backslash\{0\})$, where $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{i=1}^{k-1}[2^i]_\sim$.

As the degree of 1 is $n-2$ and for every $x \in ZD(\mathbb{Z}_n)\backslash\{0\}$, there is a unique $u \in U(\mathbb{Z}_n)\backslash\{1\}$ such that $x + u = 1$, (iii) follows from (ii). □

Proposition 4.7 and Proposition 4.8 are illustrated in Figure 7 and Table 1 for $n = 32$.

Illustration 4.1. Consider $\mathbb{Z}_{32}$, where $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{i=1}^{4}[2^i]_\sim$ and $U(\mathbb{Z}_n) = [1]_\sim = \{y \in \mathbb{Z}_n^* | (y, 2^5) = 1\} = \{1, 3, 5, \ldots, 31\}$.

i	$\{x \mid x \in [2^i]_\sim\}$	$Ann(2^i) = k\mathbb{Z}_n, k = \frac{n}{2^i}$	$u = 1 - x$	$deg(x) = deg(u)$
1	$\{2, 6, \ldots, 30\}$	$\{0, 16\}$	$\{31, 27, \ldots, 3\}$	2
2	$\{4, 12, 20, 28\}$	$\{0, 8, 16, 24\}$	{29,21,13,5}	4
3	$\{8, 24\}$	$\{0.4, 8, \ldots, 28\}$	$\{25, 9\}$	8
4	$\{16\}$	$\{0.2, 4, \ldots, 30\}$	$\{17\}$	16

Table 1: $\mathbb{Z}_{32}$

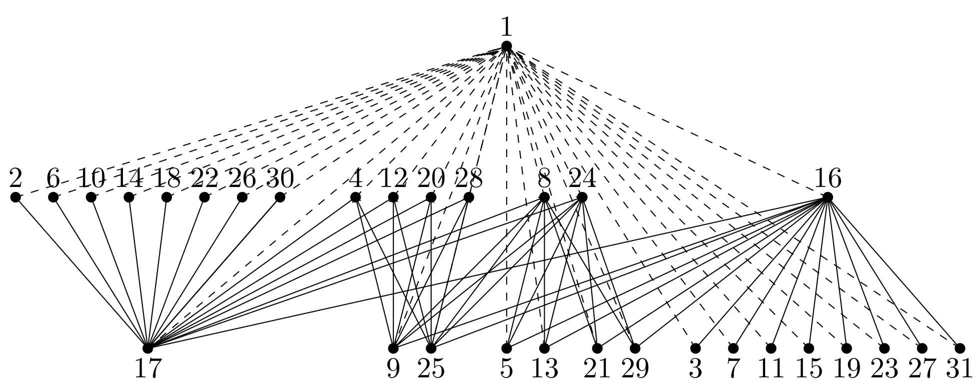


Figure 7: $P(\mathbb{Z}_{32})$

Proposition 4.9. *Let $n = 2q$. Then, the following assertions hold:*

(i) $ZD(\mathbb{Z}_n) = \{2, 4, \ldots, 2q-2\} \cup \{q\}$, $U(\mathbb{Z}_n) = \{1, 3, 5, \ldots, 2q-1\} \backslash \{q\}$.

(ii) $q, q+1$ *are the nontrivial idempotents.*

(iii) $N_{P(\mathbb{Z}_n)}(q) = \{1, 3, \ldots, 2q-1\} \backslash \{q\}$, $N_{P(\mathbb{Z}_n)}(q+1) = \{2, 4, \ldots, 2q-2\} \backslash \{q+1\}$.

(iv) $N_{P(\mathbb{Z}_n)}(x) = \{1, q\}$ *if* $x \in [2]_{\sim} \backslash \{q+1\}$, $N_{P(\mathbb{Z}_n)}(x) = \{1, q+1\}$ *if* $x \in U(\mathbb{Z}_n) \backslash \{1\}$.

(v) $deg(x) = \begin{cases} n-2, & \text{if } x = 1 \\ q-1, & \text{if } x = q, q+1 \\ 2, & \text{otherwise.} \end{cases}$

(vi) *The number of triangles in* $P(\mathbb{Z}_n)$ *is* $2q-4$.

(vii) $P(\mathbb{Z}_n)$ *is the union of two copies of triangular book*

(viii) $|E| = 4q-6$.

(ix) $P(\mathbb{Z}_n)$ *is planar.*

(x) $P_1(\mathbb{Z}_n)$ *is disconnected.*

Proof. (i) By Proposition 4.6(iv), $ZD(\mathbb{Z}_n) \backslash \{0\} = [2]_{\sim} \cup [q]_{\sim}$, where $[2]_{\sim} = \{y \in \mathbb{Z}_n^* | (y, n) = 2\} = \{2, 4, \ldots, 2q-2\}$ and $[q]_{\sim} = \{y \in \mathbb{Z}_n^* | (y, n) = q\} = \{q\}$. Hence, $U(\mathbb{Z}_n) = \mathbb{Z}_n \backslash ZD(\mathbb{Z}_n) = \{1, 3, 5, \ldots, 2q-1\} \backslash \{q\}$.

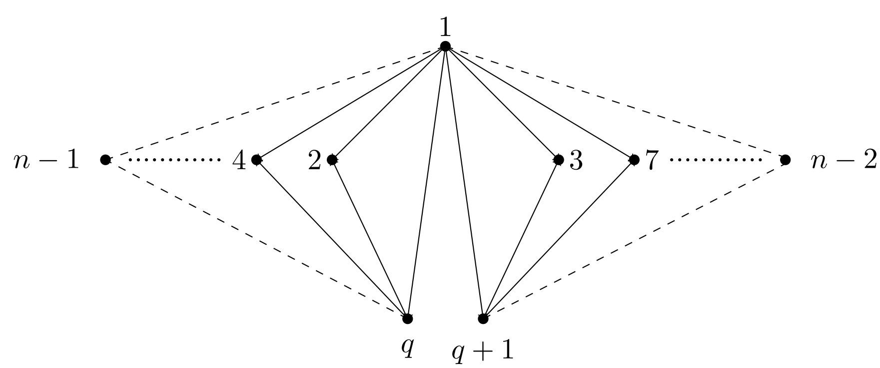


Figure 8: $P(\mathbb{Z}_{2q})$

(ii) Since q is odd, $q(q+1) \equiv 0 \ mod \ 2q$ and hence q and $q+1$ are the idempotents.

(iii) Note that, $1 - q = q + 1$ and as in the proof of Proposition 3.12, $Ann(1 - q) = Ann(q + 1) = q\mathbb{Z}_n = \{0, q\}$. Also, $Ann(q) = 2\mathbb{Z}_n$ by Proposition 4.4(iv). Hence, by using Theorem 3.1(iii), $N_{P(\mathbb{Z}_n)}(q) = ((Ann(q) + 1) \cup Ann(1 - q))\backslash\{0, q\} = ((2\mathbb{Z}_n + 1) \cup q\mathbb{Z}_n)\backslash\{0, q\} = \{1, 3, 5, \ldots, 2q - 1\}\backslash\{q\}$.

Similarly, $N_{P(\mathbb{Z}_n)}(q+1) = ((Ann(q+1)+1) \cup Ann(q))\backslash\{0, q+1\} = ((q\mathbb{Z}_n + 1) \cup 2\mathbb{Z}_n)\backslash\{0, q + 1\} = \{1, 2, 4, \ldots, 2q - 2\}\backslash\{q + 1\}$.

(iv) If $x \in [2]_\sim\backslash\{q+1\}$, then $N_{P(\mathbb{Z}_n)}(x) = ((Ann(x)+1)\cup Ann(1-x))\backslash\{0\}$ by Theorem 3.1(ii) $= (Ann(2)+1)$ by the definition of $\sim = q\mathbb{Z}_n + 1$ by Proposition 3.1(iv) $= \{1, q + 1\}$. Also, since $|q\mathbb{Z}_n| = 2$, by Proposition 3.6, either $\overline{qx} \in E$ or $\overline{(1 - q)x} \in E$, for every $x \in \mathbb{Z}_n^*\backslash\{1, q, 1 - q\}$. But, $\mathbb{Z}_n^*\backslash\{1, q, 1 - q\} = ([2]_\sim\backslash\{q + 1\}) \cup ([1]_\sim\backslash\{1\})$, where $[1]_\sim\backslash\{1\} = U(\mathbb{Z}_n)\backslash\{1\}$.

Hence, for $x \in U(\mathbb{Z}_n)\backslash\{1\}$, $N_{P(\mathbb{Z}_n)}(x) = \{1, q\}$.

(v) The proof of (v) follows from (iii) and (iv).

(vi) From (iv), it can be seen that $1 - x - (q + 1) - 1$ form triangles, which share $\overline{(q + 1)1}$ in common for every $x \in U(\mathbb{Z}_n)\backslash\{1\}$.

Similarly, $1 - q - y - 1$ form triangles, which share $\overline{1q}$ in common for every $y \in [2]_\sim\backslash\{q + 1\}$, as drawn in Figure 8.

Hence, the number of triangles $= |U(\mathbb{Z}_n)\backslash\{1\}| + |[2]_\sim\backslash\{q + 1\}| = 2(q - 1) = 2q - 4$. (vii) From Figure, it is clear that $P(\mathbb{Z}_n)$ is the union of two copies of triangular book.

(viii) As each triangle in one page of the triangular book counts two edges excluding the common edge, $|E| = (2(2q - 4)) + 2 = 4q - 6$.

(ix) Obviously, $P(\mathbb{Z}_n)$ is planar.

(x) $P(\mathbb{Z}_n)$ is disconnected if 1 is removed. Hence, $P_1(\mathbb{Z}_n)$ is disconnected. □

5. Projection graphs of near-rings

In this section, the projection graph $P(N)$ of a near-ring N is defined as the same as that of a ring and the properties of $P(N)$ are discussed. Throughout, this section N denotes a right near-ring with at least 3 elements.

Proposition 5.1. *If N is a near-field, then $P(N)$ is a star.*

Proof. Let N be a near-field and 1 be the multiplicative identity. Then, $\overline{x1} \in E$ since the equation $x1 = x$ holds in N, for every $x \in N^*$. If $\overline{xy} \in E$, then either $xy = x$ or $xy = y$, which implies $x = 1$ or $y = 1$ as every nonzero element in N has multiplicative inverse. Hence, $E = \{\overline{x1} \mid x \in N^*\}$. Thus, $P(N)$ is a star. □

Proposition 5.2. *If N is a near-ring, then the following hold in $P(N)$:*

(*i*) *Every nonzero element in N is adjacent to every element in its constant part.*

(*ii*) *The subgraph induced on the constant part forms a clique.*

Proof. The proof follows from the definition of constant part of N. □

Corollary 5.1. *If N is a constant near-ring, then $P(N)$ is complete.*

Proof. If N is a constant near-ring, then $N = N_c$ and hence $P(N)$ is complete, by Proposition 5.2(ii). □

Remark 5.1. The converse of the above proposition need not be true. For, consider $N = (D_8, +, \cdot)$, where $(D_8, +)$ is the dihedral group and $\cdot$ is defined by $x \cdot y = \begin{cases} x, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$ Clearly, N is a near-ring, which is not constant and $P(N)$ is complete.

Theorem 5.1. *If N is an almost trivial near-ring, then $P(N)$ is complete.*

Proof. Suppose N is an almost trivial near-ring, then $xy = \begin{cases} x, & \text{if } y \notin N_c \\ 0, & \text{if } y \in N_c \end{cases}$, for every x, $y \in N$.

Let $x, y \in N^*$. Then, by Pierce decomposition, $x = x_0 + x_c$ and $y = y_0 + y_c$, where x_0 and y_0 are the zero-symmetric parts and x_c and y_c are the constant parts of x and y, respectively.

Now, consider the following possible cases:

(i) $x, y \in N_0$ (ii) $x, y \in N_c$ (iii) $x \in N_0$ and $y \in N_c$ (iv) $x, y \notin N_0 \cup N_c$.

It is claimed that $\overline{xy} \in E$. For,

(i) If $x, y \in N_0$, then $x = x_0$ and $x_c = 0$. Therefore, $xy = x$.

(ii) If $x, y \in N_c$, then $x = x_c$ and $x_0 = 0$. Therefore, $xy = x_c = x$.

(iii) If $x \in N_0$ and $y \in N_c$, then $y = y_c$ and $y_0 = 0$. So, $yx = y$.

(iv) If $x, y \notin N_0 \cup N_c$, then $x = x_0 + x_c$, $y = y_0 + y_c$, where $x_0, y_0 \in N_0 \backslash \{0\}$ and x_c, $y_c \in N_c \backslash \{0\}$. Hence, $xy = (x_0 + x_c)(y_0 + y_c) = x_0(y_0 + y_c) + x_c(y_0 + y_c) = x_0 + x_c = x$.

Hence, the claim is proved. □

Proposition 5.3. *If N is a Boolean near-ring, which is subdirectly irreducible, then $P(N)$ is complete.*

Proof. The proof follows from Lemma 2.3 and Theorem 5.1. □

6. Conclusion

In this paper, the projection graphs $P(R)$ of a ring R and $P(N)$ of a near-ring N are introduced and their graph properties are studied. A method of finding adjacent vertices in $P(R)$, using annihilators is provided. Certain algebraic properties of rings are observed through their projection graphs. This paper may be extended by considering substructures of rings and near-rings and more algebraic properties can be obtained through their projection graphs.

References

[1] M. Afkhami, K. Khashyarmanesh, *The cozero-divisor graph of a commutative ring*, South-East Asian Bull. Math., 35 (2011), 753–762.

[2] M. Afkhami, K. Khashyarmanesh, *On the cozero-divisor graphs of commutative rings and their complements*, Bull. Malays. Math. Sci. Soc., 35 (2012), 935– 944.

[3] S. Akbari, F. Alizadeh, S. Khojasteh, *Some results on cozero-divisor graph of a commutative ring*, J. Algebra Appl., 13 (2014).

[4] D.D. Anderson, M. Axtell, S.J. Foreman, J. Stickles, *When are associates unit multiples?*, Rocky Mountain J. Math., 34 (2004), 811-828.

[5] D. D. Anderson, M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, 159 (1993), 500–514.

[6] D. F. Anderson, P.S. Livingston, *The zero–divisor graph of a commutative ring*, J. Algebra, 217 (1999), 434–447.

[7] D.F. Anderson, A. Badawi, *On the zero-divisor graph of a ring*, Comm. Algebra, 36 (2008), 3073-3092.

[8] M. F. Atiyah, I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company, 1969.

[9] A. Badawi, *On the annihilator graph of a commutative ring*, Comm. Algebra, 42 (2014), 108–121.

[10] A. Badawi, *Recent results on annihilator graph of a commutative ring: A survey. In nearrings, nearfields, and related topics,* edited by K. Prasad et al, (2017), New Jersey: World Scientific, 170–185.

[11] Beck, *Coloring of commutative rings*, J. Algebra, 116 (1988), 208-226.

[12] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, 62 (419), 1978.

[13] G. A. Cannon, K. M. Neuerburg, S. P. Redmond, *Zero-divisor graphs of near– rings and semigroups, Near rings and Near fields*, Dordrecht: Springer, 2005, 189–200.

[14] M. Holloway, *Some characterizations of finite commutative nil rings*, Palestine Journal of Mathematics, 2 (2013), 6-8.

[15] G. Pilz, *Near rings*, Mathematic studies 23, North Holland Publishing Company, 1983.

[16] K. Pushpalatha, *On subdirectly irreducible boolean near rings*, Int.l J. of Pure and Applied Mathematics I, 113 (2017), 272-281.

[17] Bh. Satyanarayana, Kuncham Syamprasad, *Near rings, fuzzy ideals, and graph theory*, CRC Press, 2013.

[18] S. Teresa Arockiamary, C. Meera and V. Santhi, *Annihilator 3-uniform hypergraphs of right ternary near-rings*, South East Asian J. of Mathematics and Mathematical Sciences, 17 (2021), 251-264.

Accepted: June 9, 2022

Characterization of generalized n-semiderivations of 3-prime near rings and their structure

Asma Ali*
Department of Mathematics
Aligarh Muslim University
Aligarh-202002
India
asma_ali2@rediffmail.com

A. Mamouni
Moulay Ismail University
Faculty of Sciences and Technology
Errachidia
Morocco
a.mamouni.fste@gmail.com

Inzamam Ul Huque
Department of Mathematics
Chandigarh University
Mohali
Punjab
India
inzamam.e15870@cumail.in

Abstract. Let N be a near ring and n be a fixed positive integer. An n-additive (additive in each argument) mapping $F : \underbrace{N\times N\times \ldots \times N}_{n-times} \to N$ is said to be a permuting generalized n-semiderivation on a near ring N if there exists an n-semiderivation $d : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ associated with a map $g : N \to N$ such that the relation $F(x_1x_1', x_2, \ldots, x_n)=F(x_1, x_2, \ldots, x_n)x_1'+g(x_1)d(x_1', x_2, \ldots, x_n)=d(x_1, x_2, \ldots, x_n)g(x_1')$ $+x_1F(x_1', x_2, \ldots, x_n)$ and $g(F(x_1, x_2, \ldots, x_n)) = F(g(x_1), g(x_2), \ldots, g(x_n))$ hold, for all $x_1, x_1', x_2, ..., x_n \in N$. The purpose of the present paper is to prove some commutativity theorems in case of a semigroup ideal of a 3-prime near ring admitting a generalized n-semiderivation, thereby extending some known results of derivations, semiderivations and generalized derivations.

Keywords: 3-prime near-rings, n-semiderivations, generalized n-semiderivations, semigroup ideals.

*. Corresponding author

1. Introduction

A left near ring N is a triplet $(N,+,\cdot)$, where $+$ and $\cdot$ are two binary operations such that (i) $(N,+)$ is a group (not necessarily abelian), (ii) $(N,\cdot)$ is a semigroup, and (iii) $x\cdot(y+z)=x\cdot y+x\cdot z$, for all $x,y,z\in N$. Analogously, if instead of (iii), N satisfies the right distributive law, then N is said to be a right near ring. The most natural example of a non-commutative left near ring is the set of all identity preserving mappings acting from right of an additive group G (not necessarily abelian) into itself with pointwise addition and composition of mappings as multiplication. If these mappings act from left on G, then we get a non-commutative right near ring (For more examples, we can refer Pilz [2]). Throughout the paper, N represents a zero-symmetric left near ring with multiplicative centre $Z(N)$ and for any pair of elements $x,y\in N$, the symbols $[x,y]$ and $(x\circ y)$ denote the Lie Product $xy-yx$ and Jordan product $xy+yx$. A near ring N is called zero-symmetric if $0x=0$, for all $x\in N$ (recall that left distributivity yields that $x0=0$). A near ring N is said to be 3-prime if $xNy=\{0\}$ for $x,y\in N$ implies that $x=0$ or $y=0$. A near ring N is called 2-torsion free if $(N,+)$ has no element of order 2. A nonempty subset U of N is called a semigroup right (resp. semigroup left) ideal of N if $UN\subseteq U$ (resp. $NU\subseteq U$) and if U is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. Let $n\geq 2$ be a fixed positive integer and $N^n=\underbrace{N\times N\times\ldots\times N}_{n-times}$. A map $\Delta:N^n\to N$ is said to be permuting on a near ring N if the relation $\Delta(x_1,x_2,\ldots,x_n)=\Delta(x_{\pi(1)},x_{\pi(2)},\ldots,x_{\pi(n)})$ holds, for all $x_i\in N$, $i=1,2,\ldots,n$ and for every permutation $\pi\in S_n$, where S_n is the permutation group on $\{1,2,\ldots,n\}$. An additive mapping $F:N\to N$ is said to be a right (resp. left) generalized derivation with associated derivation d if $F(xy)=F(x)y+xd(y)$ (resp. $F(xy)=d(x)y+xF(y)$), for all $x,y\in N$ and F is said to be a generalized derivation with associated derivation d on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation d.

Ozturk et. al. [6] and Park et. al. [5] studied bi-derivations and tri-derivations in near rings. A symmetric bi-additive mapping $d:N\times N\to N$ (i.e., additive in both arguments) is said to be a symmetric bi-derivation on N if $d(xy,z)=d(x,z)y+xd(y,z)$ holds, for all $x,y,z\in N$. A permuting tri-additive mapping $d:N\times N\times N\to N$ is said to be a permuting tri-derivation on N if

$$d(xw,y,z)=d(x,y,z)w+xd(w,y,z)$$

is fulfilled, for all $w,x,y,z\in N$. Muthana [7] defined bimultipliers in rings as follows: A biadditive (additive in both arguments) mapping $B:R\times R\to R$ is called a left (resp. right) bimultiplier on a ring R if $B(xy,z)=B(x,z)y$ (resp. $B(xy,z)=xB(y,z)$) holds, for all $x,y,z\in R$. Motivated by this definition we define an n-additive mapping $F:\underbrace{N\times N\times ...\times N}_{n-times}\to N$ is called a left (resp.

right) n-multiplier on a near ring N if $F(x_1x_1', x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)x_1'$ (resp. $F(x_1x_1', x_2, \ldots, x_n) = x_1F(x_1', x_2, \ldots, x_n)$), for all $x_1, x_1', x_2, \ldots, x_n \in N$. Very recently Asma et. al. [1] defined semiderivations in near rings. An additive mapping $d : N \to N$ is said to be a semiderivation on a near ring N if there exists a mapping $g : N \to N$ such that $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$, for all $x, y \in N$. Let n be a fixed positive integer. An n-additive (i.e., additive in each argument) mapping $d : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ is said to be an n-semiderivation on a near ring N if there exists a mapping $g : N \to N$ such that the relations

$$\begin{aligned}
d(x_1x_1', x_2, \ldots, x_n) &= d(x_1, x_2, \ldots, x_n)g(x_1') + x_1d(x_1', x_2, \ldots, x_n)\\
&= d(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\\
d(x_1, x_2x_2', \ldots, x_n) &= d(x_1, x_2, \ldots, x_n)g(x_2') + x_2d(x_1, x_2', \ldots, x_n)\\
&= d(x_1, x_2, \ldots, x_n)x_2' + g(x_2)d(x_1, x_2', \ldots, x_n)\\
&\vdots\\
d(x_1, x_2, \ldots, x_nx_n') &= d(x_1, x_2, \ldots, x_n)g(x_n') + x_nd(x_1, x_2, \ldots, x_n')\\
&= d(x_1, x_2, \ldots, x_n)x_n' + g(x_n)d(x_1, x_2, \ldots, x_n')
\end{aligned}$$

and $g(d(x_1, x_2, \ldots, x_n)) = d(g(x_1), g(x_2), \ldots, g(x_n))$ hold, for all $x_i, x_i' \in N$ for $i = 1, 2, \ldots, n$. An n-additive (i.e., additive in each argument) mapping $F : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ is said to be a generalized n-semiderivation on N if there exists an n-semiderivation $d : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ associated with a map $g : N \to N$ such that the relations

$$\begin{aligned}
F(x_1x_1', x_2, \ldots, x_n) &= F(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\\
&= d(x_1, x_2, \ldots, x_n)g(x_1') + x_1F(x_1', x_2, \ldots, x_n)\\
F(x_1, x_2x_2', \ldots, x_n) &= F(x_1, x_2, \ldots, x_n)x_2' + g(x_2)d(x_1, x_2', \ldots, x_n)\\
&= d(x_1, x_2, \ldots, x_n)g(x_2') + x_2F(x_1, x_2', \ldots, x_n)\\
&\vdots\\
F(x_1, x_2, \ldots, x_nx_n') &= F(x_1, x_2, \ldots, x_n)x_n' + g(x_n)d(x_1, x_2, \ldots, x_n')\\
&= d(x_1, x_2, \ldots, x_n)g(x_n') + x_nF(x_1, x_2, \ldots, x_n')
\end{aligned}$$

and $g(F(x_1, x_2, \ldots, x_n)) = F(g(x_1), g(x_2), \ldots, g(x_n))$ hold, for all $x_i, x_i' \in N$ for $i = 1, 2, \ldots, n$. All n-semiderivations are generalized n-semiderivations. Moreover, if g is the identity map on N, then all generalized n-semiderivations are merely generalized n-derivations, the notion of generalized n-semiderivation generalizes that of generalized n-derivation. Moreover, generalization is not trivial, as the following example shows:

Example 1. Let S be a commutative near ring. Consider

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\}.$$

Then N is a zero-symmetric left near ring with respect to matrix addition and matrix multiplication. Define mappings $F, d : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ by

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & z_1 z_2 ... z_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & x_1 x_2 ... x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and a map $g : N \to N$ by

$$g \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily verified that F is a generalized n-semiderivation associated with an n-semiderivation d and a map g associated with d on N.

Example 2. Let S be a commutative near ring. Consider

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid 0, x, y, z \in S \right\}.$$

Then N is a zero-symmetric left near ring with respect to matrix addition and matrix multiplication. Define mappings $F, d : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ by

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix} \right) = \begin{pmatrix} 0 & x_1 x_2 ... x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & z_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & 0 & z_n \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & y_1 z_2 ... z_n \\ 0 & 0 & 0 \\ 0 & 0 & z_1 z_2 ... z_n \end{pmatrix}$$

and a map $g : N \to N$ by

$$g\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that F is a generalized n-semiderivation associated with an n-semiderivation d and a map g associated with d on N. However, F is not a generalized n-derivation on N.

2. Preliminary results

We begin with several Lemmas, most of which have been proved elsewhere.

Lemma 2.1 ([3, Lemma 1.2 and Lemma 1.3])**.** *Let N be 3-prime near ring.*

(*i*) *If $z \in Z(N)\backslash\{0\}$, then z is not a zero divisor.*

(*ii*) *If $Z(N)\backslash\{0\}$ contains an element z for which $z + z \in Z(N)$, then $(N, +)$ is abelian.*

(*iii*) *If $Z(N)\backslash\{0\}$ and x is an element of N for which $xz \in Z(N)$, then $x \in Z(N)$.*

Lemma 2.2 ([3, Lemma 1.3 and Lemma 1.4])**.** *Let N be 3-prime near ring and U be a nonzero semigroup ideal of N.*

(*i*) *If $x \in N$ and $xU = \{0\}$ or $Ux = \{0\}$, then $x = 0$.*

(*ii*) *If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*

(*iii*) *If $x \in N$ centralizes U, then $x \in Z(N)$.*

Lemma 2.3 ([3, Lemma 1.5])**.** *If N is a 3-prime near ring and $Z(N)$ contains a nonzero semigroup left ideal or a nonzero semigroup right ideal, then N is a commutative ring.*

Lemma 2.4. *Let N be a 3-prime near ring and d be a nonzero n-semiderivation of N associated with a map g. If $U_1, U_2, \dots, U_n$ are nonzero semigroup ideals of N, then $d(U_1, U_2, \dots, U_n) \neq \{0\}$.*

Proof. Suppose that $d(U_1, U_2, \dots, U_n) = \{0\}$. Then

(1) $\quad d(x_1, x_2, \dots, x_n) = 0, \;$ for all $\; x_1 \in U_1, x_2 \in U_2, \dots, x_n \in U_n$

Replacing x_1 by x_1r_1 for $r_1 \in N$ in (1) and using it, we have

$$x_1d(r_1, x_2, \dots, x_n) = 0.$$

By Lemma 2.2(i), we obtain

$$d(r_1, x_2, \ldots, x_n) = 0. \tag{2}$$

Now, substituting x_2r_2 for x_2, where $r_2 \in N$ in (2), we get $d(r_1, r_2, \ldots, x_n) = 0$. Proceeding inductively as above, we conclude that $d(r_1, r_2, \ldots, r_n) = 0$, for all $r_1, r_2, \ldots, r_n \in N$. This shows that $d(N, N, \ldots, N) = \{0\}$, leading to a contradiction as d is a nonzero n-semiderivation. Therefore, $d(U_1, U_2, \ldots, U_n) \neq \{0\}$. □

Lemma 2.5. *Let N be a 3-prime near ring. Then F is a generalized n-semiderivation associated with an n-semiderivation d and a map g associated with d of N if and only if*

$$F(x_1x_1', x_2, \ldots, x_n) = g(x_1)d(x_1', x_2, \ldots, x_n) + F(x_1, x_2, \ldots, x_n)x_1',$$

for all $x_1, x_1', x_2, \ldots, x_n \in N$.

Proof. We have

$$\begin{aligned} &F(x_1(x_1' + x_1'), x_2, \ldots, x_n) \\ &= F(x_1, x_2, \ldots, x_n)(x_1' + x_1') + g(x_1)d(x_1' + x_1', x_2, \ldots, x_n) \\ &= F(x_1, x_2, \ldots, x_n)x_1' + F(x_1, x_2, \ldots, x_n)x_1' \\ &+ g(x_1)d(x_1', x_2, \ldots, x_n) + g(x_1)d(x_1', x_2, \ldots, x_n) \end{aligned} \tag{3}$$

and

$$\begin{aligned} F(x_1x_1' + x_1x_1', x_2, \ldots, x_n) &= F(x_1x_1', x_2, \ldots, x_n) + F(x_1x_1', x_2, \ldots, x_n) \\ &= F(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n) \\ &+ F(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n). \end{aligned} \tag{4}$$

Comparing (3) and (4), we get

$$\begin{aligned} &F(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n) \\ &\qquad = g(x_1)d(x_1', x_2, \ldots, x_n) + F(x_1, x_2, \ldots, x_n)x_1'. \end{aligned}$$

This implies that

$$F(x_1x_1', x_2, \ldots, x_n) = g(x_1)d(x_1', x_2, \ldots, x_n) + F(x_1, x_2, \ldots, x_n)x_1'.$$

Converse can be proved in a similar way. □

Lemma 2.6. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. If N admits a generalized n-semiderivation F associated with an n-semiderivation d and a map g associated with d such that $g(U_1) = U_1$ and $U_1 \cap Z(N) \neq \{0\}$, then $F(Z(N), U_2, U_3, \ldots, U_n) \subseteq Z(N)$.*

Proof. If $z \in U_1 \cap Z(N)$, then

$$F(zx_1, x_2, \ldots, x_n) = F(x_1 z, x_2, \ldots, x_n), \quad \text{for all } x_i \in U_i \quad \text{for } i = 1, 2, \ldots, n.$$

Using Lemma 2.5, we have

$$g(z)d(x_1, x_2, \ldots, x_n) + F(z, x_2, \ldots, x_n)x_1 = d(x_1, x_2, \ldots, x_n)g(z) + x_1 F(z, x_2, \ldots, x_n).$$

Since $g(U_1) = U_1$, so replacing $g(z)$ by arbitrary element $z' \in U_1 \cap Z(N)$, we get

$$z'd(x_1, x_2, \ldots, x_n) + F(z, x_2, \ldots, x_n)x_1 = d(x_1, x_2, \ldots, x_n)z' + x_1 F(z, x_2, \ldots, x_n).$$

This implies that $F(z, x_2, \ldots, x_n)x_1 = x_1 F(z, x_2, \ldots, x_n)$, for all $z \in U_1 \cap Z(N), x_i \in U_i$ for $i = 1, 2, \ldots, n$. Now, replacing x_1 by $x_1 r$, where $r \in N$ in the last expression and using it again, we obtain $x_1[F(z, x_2, \ldots, x_n), r] = 0$, for all $x_i \in U_i, r \in N$ for $i = 1, 2, \ldots, n$. By Lemma 2.2(i), we find that $[F(z, x_2, \ldots, x_n), r] = 0$. Hence, $F(Z(N), U_2, U_3, \ldots, U_n) \subseteq Z(N)$. □

Lemma 2.7. *Let N be a 3-prime near ring admitting an n-semiderivation d associated with a map g such that $g(x_1 x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in N$, then N satisfies the following partial distributive law:*

$$\{d(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\}y = d(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1', x_2, \ldots, x_n)y,$$

for all $x_1, x_1', x_2, \ldots, x_n, y \in N$.

Proof. For all $x_1, x_1', x_2, \ldots, x_n, y \in N$, we have

$$\begin{aligned} d((x_1 x_1')y, x_2, \ldots, x_n) &= d(x_1 x_1', x_2, \ldots, x_n)y + g(x_1 x_1')d(y, x_2, \ldots, x_n) \\ &= \{d(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\}y \\ &\quad + g(x_1)g(x_1')d(y, x_2, \ldots, x_n). \end{aligned} \tag{5}$$

On the other hand

$$\begin{aligned} d(x_1(x_1'y), x_2, \ldots, x_n) &= d(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1'y, x_2, \ldots, x_n) \\ &= d(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)\{d(x_1', x_2, \ldots, x_n)y \\ &\quad + g(x_1')d(y, x_2, \ldots, x_n)\}, \\ d(x_1(x_1'y), x_2, \ldots, x_n) &= d(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1', x_2, \ldots, x_n)y \\ &\quad + g(x_1)g(x_1')d(y, x_2, \ldots, x_n). \end{aligned} \tag{6}$$

From (5) and (6), we get

$$\{d(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\}y = d(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1', x_2, \ldots, x_n)y. \square$$

Lemma 2.8. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. Let d be a nonzero n-semiderivation of N associated with a map g such that $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in U_1$. If $x \in N$ and $d(U_1, U_2, \ldots, U_n)x = \{0\}$ (or $xd(U_1, U_2, \ldots, U_n) = \{0\}$), then $x = 0$.*

Proof. By hypothesis,

$$d(x_1, x_2, \ldots, x_n)x = 0, \text{ for all } x_i \in U_i; 1 \leq i \leq n, x \in N. \tag{7}$$

Replacing x_1 by r_1x_1 for $r_1 \in N$ in (7), we get

$$\{d(r_1, x_2, \ldots, x_n)x_1 + g(r_1)d(x_1, x_2, \ldots, x_n)\}x = 0.$$

Using Lemma 2.7 and (7), we get $d(r_1, x_2, \ldots, x_n)U_1x = \{0\}$. By Lemma 2.2(ii), we have either $d(r_1, x_2, \ldots, x_n) = 0$ or $x = 0$. If $d(r_1, x_2, \ldots, x_n) = 0$, for all $r_1 \in N$, $x_2 \in U_2, \ldots, x_n \in U_n$, then proceeding as in the proof of Lemma 2.4, we can show that $d(N, N, \ldots, N) = \{0\}$, leading to a contradiction. Therefore, $x = 0$.

A similar argument using above, handles the case $xd(x_1, x_2, \ldots, x_n)=\{0\}$. □

Lemma 2.9. *Let N be a 3-prime near ring admitting a generalized n-semiderivation F associated with an n-semiderivation d and an onto map g associated with d such that $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in N$. Then N satisfies the following partial distributive laws:*

(i) $\{F(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\}y = F(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1', x_2, \ldots, x_n)y.$

(ii) $\{d(x_1, x_2, \ldots, x_n)g(x_1') + x_1F(x_1', x_2, \ldots, x_n)\}y = d(x_1, x_2, \ldots, x_n)g(x_1')y + x_1F(x_1', x_2, \ldots, x_n)y,$

for all $x_1, x_1', x_2, \ldots, x_n, y \in N$.

Proof. For all $x_1, x_1', x_2, \ldots, x_n, y \in N$, we have

$$\begin{aligned} F((x_1x_1')y, x_2, \ldots, x_n) &= F(x_1x_1', x_2, \ldots, x_n)y + g(x_1x_1')d(y, x_2, \ldots, x_n) \\ &= \{F(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\}y \\ &+ g(x_1)g(x_1')d(y, x_2, \ldots, x_n). \end{aligned} \tag{8}$$

On the other hand

$$\begin{aligned} F(x_1(x_1'y), x_2, \ldots, x_n) &= F(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1'y, x_2, \ldots, x_n) \\ &= F(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)\{d(x_1', x_2, \ldots, x_n)y \\ &+ g(x_1')d(y, x_2, \ldots, x_n)\}, \end{aligned}$$

$$\begin{aligned} F(x_1(x_1'y), x_2, \ldots, x_n) &= F(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1', x_2, \ldots, x_n)y \\ &+ g(x_1)g(x_1')d(y, x_2, \ldots, x_n). \end{aligned} \tag{9}$$

From (8) and (9), we get

$$\{F(x_1, x_2, \ldots, x_n)x_1' + g(x_1)d(x_1', x_2, \ldots, x_n)\}y \\ = F(x_1, x_2, \ldots, x_n)x_1'y + g(x_1)d(x_1', x_2, \ldots, x_n)y,$$

for all $x_1, x_1', x_2, \ldots, x_n, y \in N$.

Arguing in the similar manner, we can prove the result for case (ii). □

Lemma 2.10. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. If F is a nonzero generalized n-semiderivation on N associated with an n-semiderivation d and a map g associated with d such that $g(U_1) = U_1$, then $F(U_1, U_2, \ldots, U_n) \neq \{0\}$.*

Proof. Suppose that

$$(10) \quad F(x_1, x_2, \ldots, x_n) = 0, \text{ for all } x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n.$$

Substituting $x_1 r_1$ in place of x_1, where $r_1 \in N$ in (10), we have

$$F(x_1, x_2, \ldots, x_n)r_1 + g(x_1)d(r_1, x_2, \ldots, x_n) = 0.$$

Using (10) and since $g(U_1) = U_1$, so replacing $g(x_1)$ by an arbitrary element x_1', we get

$$x_1'd(r_1, x_2, \ldots, x_n) = 0, \text{ for all } x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, r_1 \in N.$$

It follows by Lemma 2.2(i) that $d(r_1, x_2, \ldots, x_n) = 0$, for all $x_2 \in U_2, \ldots, x_n \in U_n, r_1 \in N$. Arguing in the similar manner as in Lemma 2.4, we obtain $d = 0$. Therefore, we have $F(r_1x_1, x_2, \ldots, x_n) = F(r_1, x_2, \ldots, x_n)x_1 = 0$, for all $x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, r_1 \in N$, and another appeal to Lemma 2.2(i) gives $F = 0$, which is a contradiction. □

Lemma 2.11. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. If N admits a nonzero generalized n-semiderivation F associated with an n-semiderivation d and a map g associated with d such that $g(U_1) = U_1$ and $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in U_1$. If $a \in N$ and $aF(U_1, U_2, \ldots, U_n) = \{0\}$ (or $F(U_1, U_2, \ldots, U_n)a = \{0\}$), then $a = 0$.*

Proof. Suppose that

$$(11) \quad aF(x_1, x_2, \ldots, x_n) = 0, \quad \text{for all } x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, a \in N.$$

Replacing x_1 by x_1x_1' in (11) for $x_1' \in U_1$, we get

$$aF(x_1, x_2, \ldots, x_n)x_1' + ag(x_1)d(x_1', x_2, \ldots, x_n) = 0.$$

This implies that $aU_1d(x_1, x_2, \ldots, x_n) = \{0\}$. In view of Lemma 2.2(ii), we obtain either $d(U_1, U_2, \ldots, U_n) = \{0\}$ or $a = 0$, for all $a \in N$.

If $d(U_1, U_2, \ldots, U_n) = \{0\}$, then $aF(x_1x_1', x_2, \ldots, x_n) = ax_1F(x_1', x_2, \ldots, x_n) = 0$, for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, a \in N$. Therefore, it follows by Lemma 2.2(ii) and Lemma 2.10 that $a = 0$.

Suppose that $F(U_1, U_2, ...U_n)a = \{0\}$. Then,

$$F(x_1, x_2, \ldots, x_n)a = 0, \text{ for all } x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, a \in N. \tag{12}$$

Replacing x_1 by x_1x_1' in (12), where $x_1' \in U_1$, we get

$$(d(x_1, x_2, \ldots, x_n)g(x_1') + x_1F(x_1', x_2, \ldots, x_n))a = 0.$$

Using Lemma 2.9(i), we get

$$d(x_1, x_2, \ldots, x_n)g(x_1')a + x_1F(x_1', x_2, \ldots, x_n)a = 0.$$

This implies that $d(x_1, x_2, \ldots, x_n)g(x_1')a = 0$, for all $x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, a \in N$. Replacing $g(x_1')$ by an arbitrary element $x_1'' \in U_1$ in the last expression and applying Lemma 2.2(ii), we find that $d(U_1, U_2, ...U_n) = \{0\}$ or $a = 0$, for all $a \in N$.

If $d(U_1, U_2, ...U_n) = \{0\}$, then $F(x_1x_1', x_2, \ldots, x_n)a = F(x_1, x_2, \ldots, x_n)x_1'a = 0$, for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, a \in N$. Therefore, it follows by Lemma 2.2(ii) and Lemma 2.10 that $a = 0$. □

3. Main results

Theorem 3.1. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Let F_1 and F_2 be any two generalized n-semiderivations associated with n-semiderivations d_1 and d_2 respectively and a map g associated with d_1 and d_2 such that $g(U_1) = U_1$. If $[F_1(U_1, U_2, \ldots, U_n), F_2(U_1, U_2, \ldots, U_n)] = \{0\}$, then at least one of F_1 and F_2 is trivial or $(N, +)$ is an abelian group.*

Proof. Suppose that $x \in N$ is such that

$$[x, F_2(U_1, U_2, \ldots, U_n)] = [x + x, F_2(U_1, U_2, \ldots, U_n)] = 0.$$

For all $x_1, x_1' \in U_1$ such that $x_1 + x_1' \in U_1$,

$$[x + x, F_2(x_1 + x_1', x_2, \ldots, x_n)] = 0.$$

This implies that

$$(x + x)F_2(x_1 + x_1', x_2, \ldots, x_n) = F_2(x_1 + x_1', x_2, \ldots, x_n)(x + x),$$
$$(x + x)F_2(x_1, x_2, \ldots, x_n) + (x + x)F_2(x_1', x_2, \ldots, x_n)$$
$$= F_2(x_1 + x_1', x_2, \ldots, x_n)x + F_2(x_1 + x_1', x_2, \ldots, x_n)x,$$
$$F_2(x_1, x_2, \ldots, x_n)(x + x) + F_2(x_1', x_2, \ldots, x_n)(x + x)$$
$$= xF_2(x_1 + x_1', x_2, \ldots, x_n) + xF_2(x_1 + x_1', x_2, \ldots, x_n),$$
$$F_2(x_1, x_2, \ldots, x_n)x + F_2(x_1, x_2, \ldots, x_n)x + F_2(x_1', x_2, \ldots, x_n)x + F_2(x_1', x_2, \ldots, x_n)x$$
$$= xF_2(x_1, x_2, \ldots, x_n) + xF_2(x_1', x_2, \ldots, x_n) + xF_2(x_1, x_2, \ldots, x_n)$$
$$+ xF_2(x_1', x_2, \ldots, x_n),$$

which reduces to $xF_2((x_1, x_1'), x_2, \ldots, x_n) = 0$, for all $x_2 \in U_2, \ldots, x_n \in U_n, x \in N$, where (x_1, x_1') is the additive commutator $(x_1 + x_1' - x_1 - x_1')$.

If $r, s \in U_1$, we have $rs \in U_1$ and $rs + rs = r(s + s) \in U_1$ and since $[F_1(U_1, U_2, \ldots, U_n), F_2(U_1, U_2, \ldots, U_n)] = \{0\}$, taking $x = F_1(rs, x_2', \ldots, x_n')$, where $r, s \in U_1, x_2' \in U_2, \ldots, x_n' \in U_n$ gives

$$\begin{aligned}&[F_1(rs, x_2', \ldots, x_n'), F_2(U_1, U_2, \ldots, U_n)] = \{0\}\\ &\quad = [F_1(rs, x_2', \ldots, x_n') + F_1(rs, x_2', \ldots, x_n'), F_2(U_1, U_2, \ldots, U_n)].\end{aligned}$$

Arguing in the similar manner as above, we get

$$F_1(U_1^2, U_2, \ldots, U_n)F_2(x_1 + x_1' - x_1 - x_1', x_2, \ldots, x_n) = \{0\}.$$

Since U_1^2 is a semigroup ideal, Lemma 2.11 gives

$$F_2(x_1 + x_1' - x_1 - x_1', x_2, \ldots, x_n) = 0, \tag{13}$$

for all $x_1, x_1' \in U_1$ such that $x_1 + x_1' \in U_1$. Now, take $x_1 = rx'$ and $x_1' = ry'$ for $r \in U_1$ and $x', y' \in N$, so that x_1, x_1' and $x_1 + x_1' = rx' + ry' = r(x' + y') \in U_1$. It follows from relation (13) that

$$F_2(rx' + ry' - rx' - ry', x_2, \ldots, x_n) = 0, \text{ for all } r \in U_1, x', y' \in N.$$

Replacing r by rw, $w \in U_1$ we get $F_2(U_1, U_2, \ldots, U_n)U_1(x' + y' - x' - y') = \{0\}$, for all $x', y' \in N$ and by Lemma 2.2(ii) either $F_2(U_1, U_2, \ldots, U_n) = \{0\}$ or $x' + y' - x' - y' = 0$, for all $x', y' \in N$. If $F_2(U_1, U_2, \ldots, U_n) = \{0\}$, then proceeding as in Lemma 2.10, we find $F_2 = 0$ and the second case implies that $(N, +)$ is an abelian group. Similarly if we consider

$$[F_1(U_1, U_2, \ldots, U_n), x] = [F_1(U_1, U_2, \ldots, U_n), x + x] = 0$$

and proceeding as above, we can find either $F_1 = 0$ or $(N, +)$ is an abelian group. □

Theorem 3.2. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Let F be a generalized n-semiderivation associated with an n-semiderivation d and a map g associated with d such that $g(U_1) = U_1$ and $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in U_1$. If $F(U_1, U_2, \ldots, U_n) \subseteq Z(N)$, then $F = 0$ or N is a commutative ring.*

Proof. For all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, we get

$$F(x_1x_1', x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)g(x_1') + x_1F(x_1', x_2, \ldots, x_n) \in Z(N). \tag{14}$$

Now, commuting (14) with the element x_1, we get

$$\begin{aligned}&(d(x_1, x_2, \ldots, x_n)g(x_1') + x_1F(x_1', x_2, \ldots, x_n))x_1\\ &\qquad = x_1(d(x_1, x_2, \ldots, x_n)g(x_1') + x_1F(x_1', x_2, \ldots, x_n)).\end{aligned}$$

Using the hypothesis and Lemma 2.9(ii), we have

$$d(x_1, x_2, \ldots, x_n)g(x_1')x_1 + x_1x_1F(x_1', x_2, \ldots, x_n) = x_1d(x_1, x_2, \ldots, x_n)g(x_1') + x_1x_1F(x_1', x_2, \ldots, x_n).$$

This implies that,

$$d(x_1, x_2, \ldots, x_n)x_1'x_1 = x_1d(x_1, x_2, \ldots, x_n)x_1'. \tag{15}$$

Replacing x_1' by $x_1'r$ for $r \in N$ in (22) and using it again, we get

$$d(x_1, x_2, \ldots, x_n)x_1'[x_1, r] = 0, \text{ for all } x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n, r \in N.$$

By Lemma 2.2(ii), either $d(x_1, x_2, \ldots, x_n) = 0$, for all $x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$ or $U_1 \subseteq Z(N)$. If $d(x_1, x_2, \ldots, x_n) = 0$, for all $x_1, \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, then

$$F(x_1x_1', x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)x_1' \in Z(N).$$

This implies that $F(x_1, x_2, \ldots, x_n)x_1's = sF(x_1, x_2, \ldots, x_n)x_1'$, for all $x_1, x_1' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, and $s \in N$. Replacing x_1' by $x_1'x_1''$, for all $x_1'' \in U_1$ in above expression and using it again, we find that

$$F(x_1, x_2, \ldots, x_n)U_1[x_1'', s] = \{0\}.$$

By Lemma 2.2(ii), we have $F(x_1, x_2, \ldots, x_n) = 0$, for all $x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$ or $U_1 \subseteq Z(N)$. If $F(x_1, x_2, \ldots, x_n) = 0$, for all $x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, then proceeding as in Lemma 2.10, we can get $F = 0$ on N. In later case $U_1 \subseteq Z(N)$ implies that N is a commutative ring by Lemma 2.3. □

Theorem 3.3. *Let N be a 2-torsion free 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Suppose that N admits a nonzero generalized n-semiderivation F associated with an n-semiderivations d and a map g associated with d such that $g(U_1) = U_1$ and $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in U_1$. If $[F(U_1, U_2, \ldots, U_n), F(U_1, U_2, \ldots, U_n)] = \{0\}$, then F maps U^n into $Z(N)$ or F is an n-multiplier on N.*

Proof. By hypothesis, for all $x_1, y_1 \in U_1, x_2, y_2 \in U_2, \ldots, x_n, y_n \in U_n$,

$$F(x_1, x_2, \ldots, x_n)F(y_1, y_2, \ldots, y_n) = F(y_1, y_2, \ldots, y_n)F(x_1, x_2, \ldots, x_n). \tag{16}$$

Replacing y_1 by $F(z_1, z_2, \ldots, z_n)y_1$ in (16), where $z_1 \in U_1, z_2 \in U_2, \ldots, z_n \in U_n$, we get

$$F(x_1, x_2, \ldots, x_n)F(F(z_1, z_2, \ldots, z_n)y_1, y_2, \ldots, y_n) = F(F(z_1, z_2, \ldots, z_n)y_1, y_2, \ldots, y_n)F(x_1, x_2, \ldots, x_n),$$

$$\begin{aligned}&F(x_1,x_2,\ldots,x_n)\{d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)g(y_1)\\&+F(z_1,z_2,\ldots,z_n)F(y_1,y_2,\ldots,y_n)\}\\&=\{d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)g(y_1)\\&+F(z_1,z_2,\ldots,z_n)F(y_1,y_2,\ldots,y_n)\}F(x_1,x_2,\ldots,x_n).\end{aligned}$$

By Lemma 2.9(ii), we have

$$\begin{aligned}&F(x_1,x_2,\ldots,x_n)d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)g(y_1)\\&+F(x_1,x_2,\ldots,x_n)F(z_1,z_2,\ldots,z_n)F(y_1,y_2,\ldots,y_n)\\&=d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)g(y_1)F(x_1,x_2,\ldots,x_n)\\&+F(z_1,z_2,\ldots,z_n)F(y_1,y_2,\ldots,y_n)F(x_1,x_2,\ldots,x_n).\end{aligned}$$

This implies that

$$\begin{aligned}&F(x_1,x_2,\ldots,x_n)d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)y_1\\&=d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)y_1F(x_1,x_2,\ldots,x_n).\end{aligned}\tag{17}$$

Replacing y_1 by y_1t, for all $t\in N$ and using (17), we obtain

$$\begin{aligned}&F(x_1,x_2,\ldots,x_n)d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)y_1t\\&=F(x_1,x_2,\ldots,x_n)y_1td(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n),\end{aligned}$$

which reduces to,

$$d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)U_1[F(x_1,x_2,\ldots,x_n),t]=\{0\}.$$

By Lemma 2.2(ii), we get$[F(x_1,x_2,\ldots,x_n),t]=0$, for all $x_1\in U_1,x_2\in U_2,\ldots,x_n\in U_n,t\in N$ or $d(F(z_1,z_2,\ldots,z_n),y_2,\ldots,y_n)=0$, for all $z_1\in U_1,y_2,z_2\in U_2,\ldots,y_n,z_n\in U_n$. In the first case $F(U_1,U_2,\ldots,U_n)\subseteq Z(N)$ shows that F maps U^n into $Z(N)$, the centre of N. Let us assume that $d(F(U_1,U_2,\ldots,U_n),U_2,\ldots,U_n)=\{0\}$, then

$$\begin{aligned}0&=d(F(y_1y_1',y_2,\ldots,y_n),y_2,\ldots,y_n)\\&=d\{(F(y_1,y_2,\ldots,y_n)y_1'+g(y_1)d(y_1',y_2,\ldots,y_n)),y_2,\ldots,y_n\}\\&=d((F(y_1,y_2,\ldots,y_n)y_1',y_2,\ldots,y_n)+d(y_1d(y_1',y_2,\ldots,y_n),y_2,\ldots,y_n)\\&=F(y_1,y_2,\ldots,y_n)d(y_1',y_2,\ldots,y_n)+d(y_1,y_2,\ldots,y_n)d(y_1',y_2,\ldots,y_n)\\&+y_1d(d(y_1',y_2,\ldots,y_n),y_2,\ldots,y_n)\text{ for all }y_1,y_1'\in U_1,y_2\in U_2,\ldots,y_n\in U_n.\end{aligned}$$

Now, replacing y_1 by y_1z_1, for all $z_1\in U_1$, we have

$$\begin{aligned}\{d(y_1,y_2,\ldots,y_n)z_1+y_1F(z_1,y_2,\ldots,y_n)\}&d(y_1',y_2,\ldots,y_n)\\&+\{d(y_1,y_2,\ldots,y_n)z_1+y_1d(z_1,y_2,\ldots,y_n)\}d(y_1',y_2,\ldots,y_n)\\&+y_1z_1d(d(y_1',y_2,\ldots,y_n),y_2,\ldots,y_n)\}=0,\end{aligned}$$

$$2d(y_1,y_2,\ldots,y_n)z_1d(y_1',y_2,\ldots,y_n)+y_1\{F(z_1,y_2,\ldots,y_n)d(y_1',y_2,\ldots,y_n) \\ +d(z_1,y_2,\ldots,y_n)d(y_1',y_2,\ldots,y_n)+z_1d(d(y_1',y_2,\ldots,y_n),y_2,\ldots,y_n)\}=0,$$

which implies that

$$2d(y_1,y_2,\ldots,y_n)z_1d(y_1',y_2,\ldots,y_n)=0 \text{ for all } y_1,y_1',z_1\in U_1,y_2\in U_2,\ldots,y_n\in U_n.$$

Since N is 2-torsion free, we get

$$d(y_1,y_2,\ldots,y_n)U_1d(y_1',y_2,\ldots,y_n)=\{0\} \text{ for all } y_1,y_1'\in U_1,y_2\in U_2,\ldots,y_n\in U_n.$$

Thus, we obtain $d(U_1,U_2,\ldots,U_n)=\{0\}$. Arguing as above, we conclude that F is an n-multiplier on N. □

Theorem 3.4. *Let N be a 3-prime near ring and $U_1,U_2,\ldots,U_n$ be nonzero semigroup ideals of N. Suppose that N admits a generalized n-semiderivation F associated with an n-semiderivation d and an additive map g associated with d such that $g(U_1)=U_1$ and $g(x_1x_1')=g(x_1)g(x_1')$, for all $x_1,x_1'\in U_1$. If $F([x,y],x_2,\ldots,x_n)=\pm[x,y]$, for all $x,y\in U_1,x_2\in U_2,\ldots,x_n\in U_n$, then F is an n-multiplier or N is a commutative ring.*

Proof. By hypothesis

$$F([x,y],x_2,\ldots,x_n)=\pm[x,y], \text{ for all } x,y\in U_1,x_2\in U_2,\ldots,x_n\in U_n. \tag{18}$$

Replacing y by xy in (18) and using $[x,xy]=x[x,y]$, we get

$$F(x[x,y],x_2,\ldots,x_n)=\pm x[x,y], \\ d(x,x_2,\ldots,x_n)g([x,y])+xF([x,y],x_2,\ldots,x_n)=\pm x[x,y].$$

Using (18), we get

$$d(x,x_2,\ldots,x_n)g([x,y])=0, \text{ for all } x,y\in U_1,x_2\in U_2,\ldots,x_n\in U_n. \tag{19}$$

This implies that

$$d(x,x_2,\ldots,x_n)g(x)g(y)=d(x,x_2,\ldots,x_n)g(y)g(x).$$

Replacing y by yz in the above expression and using it again, we arrive at

$$d(x,x_2,\ldots,x_n)g(y)[g(x),g(z)]=0.$$

Since $g(U_1)=U_1$, substituting arbitrary elements x',y' and z' of U_1 in place of $g(x),g(y)$ and $g(z)$ respectively, we obtain

$$d(x,x_2,\ldots,u_n)U_1[x',z']=\{0\}, \text{ for all } x,x',z'\in U_1,x_2\in U_2,\ldots,x_n\in U_n.$$

By Lemma 2.2(ii), we have either $d(x, x_2, \ldots, x_n) = 0$ or $[x', z'] = 0$, for all $x, x', z' \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. If $d(x, x_2, \ldots, x_n) = 0$, then proceeding as in Lemma 2.4, we can find $d = 0$ on N. Therefore,

$$F(x_1x_1', x_2, \ldots, x_n) = x_1F(x_1', x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)x_1',$$

for all $x_1, x_1', x_2, \ldots, x_n \in N$ and hence F is an n-multiplier on N. In later case, we have $[x', z'] = 0$, i.e., $x'z' = z'x'$. Replacing z' by $z'r$ and using it again, we find that $z'[x', r] = 0$, i.e., $U_1[x', r] = \{0\}$, for all $x' \in U_1, r \in N$. By an application of Lemma 2.2(i) and Lemma 2.3, N is a commutative ring. □

Theorem 3.5. *Let N be a 2-torsion free 3-prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. Suppose that N admits a generalized n-semiderivation F associated with an n-semiderivation d and an additive map g associated with d such that $g(U_1) = U_1$ and $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in U_1$. If $F(x \circ y, x_2, \ldots, x_n) = 0$, for all $x, y \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, then $F = 0$.*

Proof. By hypothesis

(20) $$F(x \circ y, x_2, \ldots, x_n) = 0, \text{ for all } x, y \in U_1, x_2 \in U_2, \ldots, x_n \in U_n.$$

Replacing y by xy in (20), we get

$$d(x, x_2, \ldots, x_n)g(x \circ y) + xF(x \circ y, x_2, \ldots, x_n) = 0.$$

Using (20), we get

(21) $$d(x, x_2, \ldots, x_n)g(x \circ y) = 0, \text{ for all } x, y \in U_1, x_2 \in U_2, \ldots, x_n \in U_n.$$

Since g is additive and $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in U_1$, then (21) can be written as

$$d(x, x_2, \ldots, x_n)g(x)g(y) = -d(x, x_2, \ldots, x_n)g(y)g(x).$$

Replacing y by yz in the above expression and using it again, we arrive at

$$d(x, x_2, \ldots, x_n)g(y)g(-x)g(z) = d(x, x_2, \ldots, x_n)g(y)g(z)g(-x),$$

which implies that

$$d(x, x_2, \ldots, x_n)g(y)[g(-x), g(z)] = 0, \text{ for all } x, y, z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n.$$

Putting $-x$ in place of x in the last expression, we obtain

$$d(-x, x_2, \ldots, x_n)g(y)[g(x), g(z)] = 0, \text{ for all } x, y, z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n.$$

Now, replacing $g(x)$, $g(y)$ and $g(z)$ by arbitrary elements x', y' and z' of U_1 and applying Lemma 2.2(ii), we get either $d(-x, x_2, \ldots, x_n) = 0$ or $[x', z'] =$

0, for all $x, y, z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Since d is n-additive, then $d(-x, x_2, \ldots, x_n) = 0$ implies that $d(x, x_2, \ldots, x_n) = 0$. Hence, we have either $d(x, x_2, \ldots, x_n) = 0$ or $[x', z'] = 0$, for all $x, y, z \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Arguing in the similar manner as in Theorem 3.4, we get F is an n-multiplier or N is commutative.

If N is commutative, then the hypothesis becomes

$$0 = F(x \circ y, x_2, \ldots, x_n) = 2F(xy, x_2, \ldots, x_n).$$

Since N is 2-torsion free, we get

$$F(xy, x_2, \ldots, x_n) = 0. \tag{22}$$

Replacing y by yz in (22), we obtain

$$F(xy, x_2, \ldots, x_n)z + g(xy)d(z, x_2, \ldots, x_n) = 0,$$
$$g(x)g(y)d(z, x_2, \ldots, x_n) = 0.$$

Since $g(U_1) = U_1$, then by Lemma 2.2(ii), we have $d(z, x_2, \ldots, x_n) = 0$, so Lemma 2.4 forces that $d = 0$, thus F is an n-multiplier and (22) becomes $F(x, x_2, \ldots, x_n)y = 0$ and Lemma 2.10 forces that $F = 0$.

If F is an n-multiplier, then replacing y by xy in (20), we obtain

$$F(x, x_2, \ldots, x_n)(x \circ y) = 0.$$

By using same argument as above, we get

$$F(x, x_2, \ldots, x_n)U_1[x, z] = 0.$$

By Lemma 2.2(ii), we get $x \in Z(N)$ or $F(x, x_2, \ldots, x_n) = 0$. If $x \in Z(N)$, then the hypothesis becomes $2F(xy, u_2, u_3, \ldots, u_n) = 0$. By 2-torsion freeness of N, we find that $F(x, x_2, ...u_n)y = 0$, thus in all the cases we arrive at $F(x, x_2, \ldots, x_n) = 0$ and Lemma 2.10 forces that $F = 0$. □

Theorem 3.6. *Let N be a 2-torsion free 3-prime near ring; $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N and an additive map g such that $g(U_1) = U_1$ and $g(x_1x_1') = g(x_1)g(x_1')$, for all $x_1, x_1' \in U_1$. There is no generalized n-semiderivation F associated with an n-semiderivation d and g such that $F(x \circ y, x_2, \ldots, x_n) = \pm(x \circ y)$, for all $x, y \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$.*

Proof. Suppose that there exists F such that

$$F(x \circ y, x_2, \ldots, x_n) = \pm(x \circ y) \text{ for all } x, y \in U_1, x_2 \in U_2, \ldots, x_n \in U_n. \tag{23}$$

Substituting xy for y in (23), we get

$$F(x(x \circ y), x_2, \ldots, x_n) = \pm x(x \circ y).$$

This implies that

$$d(x, x_2, \ldots, x_n)g(x \circ y) + xF((x \circ y), x_2, \ldots, x_n) = \pm x(x \circ y).$$

Using (23), we get $d(x, x_2, \ldots, x_n)g(x \circ y) = 0$. Arguing in the similar manner as in Theorem 3.4 and Theorem 3.5, we get N is commutative or F is an n-multiplier.

If N is commutative, then the hypothesis becomes $2F(xy, x_2, \ldots, x_n) = 2xy$ that is $F(xy, x_2, \ldots, x_n) = xy$ this yields that $d = 0$ and replacing x_2 by x_2x_2' and x_2x_2'', where $x_2' \neq x_2''$ and comparing the result, we arrive at

$$(x_2' - x_2'')(x \circ y) = 0$$

This leads to $N = (0)$, a contradiction.

If F is an n-multiplier, then reasoning as above we arrive at $N = (0)$, a contradiction, so we obtain the required result. □

Theorem 3.7. *Let N be a prime near ring and $U_1, U_2, \ldots, U_n$ be nonzero semigroup ideals of N. Suppose that N admits a generalized n-semiderivation F associated with a map $d : \underbrace{N \times N \times \ldots \times N}_{n-times} \to N$ and a map g such that $g(U_1) = U_1$ and $U_1 \cap Z(N) \neq \{0\}$. If $F([x_1, y_1], x_2, \ldots, x_n) = \pm[F(x_1, x_2, \ldots, x_n), y_1]$, for all $x_1, y_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, then F is commuting on U_1.*

Proof. By hypothesis

$$F([x_1, y_1], x_2, \ldots, x_n) = [F(x_1, x_2, \ldots, x_n), y_1] \tag{24}$$

Replacing y_1 by x_1y_1 in (24), we have

$$d(x_1, x_2, ...x_n)g([x_1, y_1]) + x_1F([x_1, y_1], x_2, \ldots, x_n) = [F(x_1, x_2, \ldots, x_n), x_1y_1],$$
$$d(x_1, x_2, ...x_n)g([x_1, y_1]) + x_1[F(x_1, x_2, \ldots, x_n), y_1] = [F(x_1, x_2, \ldots, x_n), x_1y_1],$$
$$\begin{aligned} d(x_1, x_2, ...x_n)g([x_1, y_1]) &+ x_1F(x_1, x_2, \ldots, x_n)y_1 - x_1y_1F(x_1, x_2, \ldots, x_n) \\ &= F(x_1, x_2, \ldots, x_n)x_1y_1 - x_1y_1F(x_1, x_2, \ldots, x_n). \end{aligned}$$

If we choose $y_1 \in U_1 \cap Z(N)$, then above relation yields that $x_1F(x_1, x_2, \ldots, x_n)y_1 = F(x_1, x_2, \ldots, x_n)x_1y_1$. This implies that $y_1[F(x_1, x_2, \ldots, x_n), x_1] = 0$ and by Lemma 2.2(i), we find $[F(x_1, x_2, \ldots, x_n), x_1] = 0$. Hence, F is commuting on U_1. In the similar manner we can prove the result for $F([x_1, y_1], x_2, \ldots, x_n) = -[F(x_1, x_2, \ldots, x_n), y_1]$, for all $x_1, y_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. □

Theorem 3.8. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N. Suppose that N admits a generalized n-semiderivation F associated with a map $d : \underbrace{N \times N \times \ldots \times N}_{n-times} \to N$ and a map g such that $g(U_1) = U_1$ and $U_1 \cap Z(N) \neq \{0\}$. If $F([x_1, y_1], x_2, \ldots, x_n) = \pm[x_1, F(y_1, x_2, \ldots, x_n)]$, for all $x_1, y_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$, then F is commuting on U_1.*

Proof. By hypothesis

$$F([x_1,y_1],x_2,\ldots,x_n)=[x_1,F(y_1,x_2,\ldots,x_n)] \tag{25}$$

Replacing x_1 by y_1x_1 in (25), we get

$$d(y_1,x_2,...x_n)g([x_1,y_1])+y_1F([x_1,y_1],x_2,\ldots,x_n)=[y_1x_1,F(x_1,x_2,\ldots,x_n)],$$
$$d(y_1,x_2,...x_n)g([x_1,y_1])+y_1[x_1,F(y_1,x_2,\ldots,x_n)]=[y_1x_1,F(x_1,x_2,\ldots,x_n)],$$
$$d(y_1,x_2,...x_n)g([x_1,y_1])+y_1x_1F(y_1,x_2,\ldots,x_n)-y_1F(y_1,x_2,\ldots,x_n)x_1$$
$$=y_1x_1F(x_1,x_2,\ldots,x_n)-F(x_1,x_2,\ldots,x_n)y_1x_1$$

If we choose $x_1\in U_1\cap Z(N)$, then above relation yields that $y_1F(y_1,x_2,\ldots,x_n)x_1$ $=F(x_1,x_2,\ldots,x_n)y_1x_1$. This implies that $x_1[F(y_1,x_2,\ldots,x_n),y_1]=0$ and by Lemma 2.2(i), we find $[F(y_1,x_2,\ldots,x_n),y_1]=0$. Hence F is commuting on U_1. In the similar manner we can prove the result for $F([x_1,y_1],x_2,\ldots,x_n)=-[x_1,F(y_1,x_2,\ldots,x_n)]$, for all $x_1,y_1\in U_1,x_2\in U_2,\ldots,x_n\in U_n$, then F is commuting on U_1. □

Theorem 3.9. *Let N be a 3-prime near ring and $U_1,U_2,\ldots,U_n$ be nonzero semigroup ideals of N. Suppose that N admits a nonzero generalized n-semiderivation F associated with an n-semiderivation d on N and a map g such that $g(U_1)=U_1$ and $d(Z(N),U_2,\ldots,U_n)\neq\{0\}$. If $[F(x_1,x_2,\ldots,x_n),F(y_1,y_2,\ldots,y_n)]=0$, for all $x_1,y_1\in U_1,x_2\in U_2,\ldots,x_n\in U_n$, then N is a commutative ring.*

Proof. Let $z\in Z(N)$ and $d(z,y_2,\ldots,y_n)\neq 0$. Then by hypothesis

$$F(x_1,x_2,\ldots,x_n)F(y_1z,y_2,\ldots,y_n)=F(y_1z,y_2,\ldots,y_n)F(x_1,x_2,\ldots,x_n),$$
$$F(x_1,x_2,\ldots,x_n)F(y_1,y_2,\ldots,y_n)z+F(x_1,x_2,\ldots,x_n)g(y_1)d(z,y_2,\ldots,y_n)$$
$$=F(y_1,y_2,\ldots,y_n)zF(x_1,x_2,\ldots,x_n)$$
$$+g(y_1)d(z,y_2,\ldots,y_n)F(x_1,x_2,\ldots,x_n).$$

This implies that,

$$F(x_1,x_2,\ldots,x_n)g(y_1)d(z,y_2,\ldots,y_n)=g(y_1)d(z,y_2,\ldots,y_n)F(x_1,x_2,\ldots,x_n).$$

By hypothesis, we find $d(z,y_2,\ldots,y_n)[F(x_1,x_2,\ldots,x_n),g(y_1)]=0$. By Lemma 2.1(i), we get $[F(x_1,x_2,\ldots,x_n),y_1]=0$. Replacing y_1 by y_1r for $r\in N$, we have

$$y_1[F(x_1,x_2,\ldots,x_n),r]=0.$$

By Lemma 2.2(ii), we obtain

$$[F(x_1,x_2,\ldots,x_n),r]=0,\ \text{ for all }\ x_1\in U_1,x_2\in U_2,\ldots,x_n\in U_n,r\in N.$$

Therefore, $F(U_1,U_2,\ldots,U_n)\subseteq Z(N)$ and hence N is a commutative ring by Theorem 3.2. □

Theorem 3.10. *Suppose that N is a prime near ring; $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N and $V_1, V_2, \ldots, V_n$ are nonempty subsets of N.*

If F is a generalized n-semiderivation acts as a left multiplier such that $F(x_1y_1, x_2, \ldots, x_n) = F(y_1x_1, x_2, \ldots, x_n)$, for all $y_1 \in V_1, x_1 \in U_1, x_2 \in U_2..., x_n \in U_n$, then $F(V_1, V_2, \ldots, V_n) = \{0\}$ or $V_1 \subseteq Z(N)$.

Proof. By hypothesis, for all $y_1 \in V_1, x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$,

$$F(x_1y_1, x_2, \ldots, x_n) = F(y_1x_1, x_2, \ldots, x_n). \tag{26}$$

Replacing x_1 by y_1x_1 in (26), we get

$$F(y_1, x_2, \ldots, x_n)x_1y_1 = F(y_1, x_2, \ldots, x_n)y_1x_1. \tag{27}$$

Replacing x_1 by x_1x_1', for all $x_1' \in U_1$ in (27), we have

$$F(y_1, x_2, \ldots, x_n)x_1x_1'y_1 = F(y_1, x_2, \ldots, x_n)x_1y_1x_1',$$

which implies that,

$$F(y_1, x_2, \ldots, x_n)U_1[x_1', y_1] = \{0\}.$$

By Lemma 2.2(ii), we have $F(y_1, x_2, \ldots, x_n)$=0, for all $y_1 \in V_1, x_2 \in U_2, \ldots, x_n \in U_n$ or y_1 centralizes U_1. In first case, replacing x_2 by y_2x_2, for all $y_2 \in V_2$, we find that $F(y_1, y_2, \ldots, x_n)x_2 = 0$ and again by Lemma 2.2(i), we get $F(y_1, y_2, \ldots, x_n) = 0$. Proceeding inductively, we obtain $F(y_1, y_2, \ldots, y_n) = 0$, for all $y_1 \in V_1, y_2 \in V_2, \ldots, y_n \in V_n$, which completes the proof. □

Theorem 3.11. *Let N be a 3-prime near ring and $U_1, U_2, \ldots, U_n$ are nonempty subsets of N and $V_1, V_2, \ldots, V_n$ are nonzero semigroup ideals of N. Suppose that N admits a generalized n-semiderivation F associated with an n- semiderivation d and an additive map g such that $g(V_1) = V_1$. If $F(x_1y_1, y_2, \ldots, y_n) = F(y_1x_1, y_2, \ldots, y_n)$, for all $x_1 \in U_1, y_1 \in V_1, y_2 \in V_2, \ldots, y_n \in V_n$, then $D(U_1, U_2, \ldots, U_n) = \{0\}$ or $U_1 \subseteq Z(N)$.*

Proof. By hypothesis, for all $x_1 \in U_1, y_1 \in V_1, y_2 \in V_2, \ldots, y_n \in V_n$,

$$F(x_1y_1, y_2, \ldots, y_n) = F(y_1x_1, y_2, \ldots, y_n). \tag{28}$$

Replacing y_1 by x_1y_1 in (28), we have

$$\begin{aligned} &d(x_1, y_2, \ldots, y_n)g(x_1y_1) + x_1F(x_1y_1, y_2, \ldots, y_n)\\ &\qquad = d(x_1, y_2, \ldots, y_n)g(y_1x_1) + x_1F(y_1x_1, y_2, \ldots, y_n),\\ &d(x_1, y_2, \ldots, y_n)g(x_1y_1) + x_1F(x_1y_1, y_2, \ldots, y_n)\\ &\qquad = d(x_1, y_2, \ldots, y_n)g(y_1x_1) + x_1F(x_1y_1, y_2, \ldots, y_n). \end{aligned}$$

This implies that,

$$d(x_1, y_2, \ldots, y_n)g(x_1y_1 - y_1x_1) = 0.$$

Since g is additive and $g(V_1) = V_1$, we have

$$d(x_1, y_2, \ldots, y_n)[x_1, y_1]{=}0, \quad \text{for all} \;\; x_1{\in}U_1, y_1{\in}V_1, y_2{\in}V_2, \ldots, y_n \in V_n. \tag{29}$$

Replacing y_1 by $y_1 r$, for all $r \in N$ in (29) and using (29), we find

$$d(x_1, y_2, \ldots, y_n)y_1[x_1, r] = 0.$$

By Lemma 2.2(ii), we get $d(x_1, y_2, \ldots, y_n) = 0$, for all $x_1 \in U_1, y_2 \in V_2, \ldots, y_n \in V_n$ or $U_1 \subseteq Z(N)$. In first case, replacing y_2 by x_2y_2, for all $x_2 \in U_2$, we conclude that

$$d(x_1, x_2, \ldots, y_n)y_2 + g(x_2)d(x_1, y_2, \ldots, y_n) = 0.$$

The last expression yields that $d(x_1, x_2, \ldots, y_n) = 0$, for all $x_1 \in U_1, x_2 \in U_2, \ldots, y_n \in V_n$. Proceeding inductively, we obtain $d(x_1, x_2, \ldots, x_n) = 0$, for all $x_1 \in U_1, x_2 \in U_2, \ldots, x_n \in U_n$. Hence, $d(U_1, U_2, \ldots, U_n) = \{0\}$ or $U_1 \subseteq Z(N)$. $\square$

The following example demonstrates that the primeness hypothesis in Theorems 3.2, 3.4 to 3.11 is not superfluous.

Example 3. Let S be a commutative near ring. Consider

$$N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid 0, x, y, z \in S \right\} \quad \text{and} \quad U = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid 0, x, y \in S \right\}.$$

It can be easily seen that N is a non prime zero-symmetric left near ring with respect to matrix addition and matrix multiplication and U is a nonzero semigroup ideal of N. Define mappings $F, d : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ by

$$F\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & z_1z_2...z_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & 0 \\ 0 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & 0 \\ 0 & z_2 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & 0 \\ 0 & z_n & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & y_1y_2...y_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define a map $g : N \to N$ by

$$g \begin{pmatrix} ccc0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we choose $U_1 = U_2 = \cdots = U_n = U$, then it is easy to check that F is a nonzero generalized n-semiderivation associated with a nonzero n-semiderivation d and a map g associated with d on N satisfying the following conditions:

(i) $F(U_1, U_2, \ldots, U_n) \subseteq Z(N)$, (ii) $F([x_1, y_1], x_2, \ldots, x_n) = \pm[x_1, y_1]$,

(iii) $F(x_1 \circ y_1, x_2, \ldots, x_n) = 0$, (iv) $F(x_1 \circ y_1, x_2, \ldots, x_n) = \pm(x_1 \circ y_1)$,

(v) $F([x_1, y_1], x_2, \ldots, x_n) = \pm[F(x_1, x_2, \ldots, x_n), y_1]$,

(vi) $F([x_1, y_1], x_2, \ldots, x_n) = \pm[x_1, F(y_1, x_2, \ldots, x_n)]$,

(vii) $[F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)] = 0$,

for all $x_1, y_1 \in U_1, x_2, y_2 \in U_2, \ldots, x_n, y_n \in U_n$. However, N is not commutative.

Example 4. Let $N_1 = (\mathbb{C}, +, \cdot)$ be the ring of complex numbers with respect to the usual addition and multiplication of complex numbers and $N_2 = (\mathbb{C}, +, \star)$, where $\mathbb{C}$ is the set of complex numbers, $+$ is the usual addition of complex numbers and $\star$ is defined by $x \star y = \mid x \mid \cdot y$, for all $x, y \in \mathbb{C}$. Then it is easy to see that N_2 is a zero-symmetric left near ring. Now, consider the set $S = N_1 \times N_2$, which is a non-commutative zero-symmetric left near ring with respect to the componentwise addition and multiplication. Suppose that

$$N = \left\{ \begin{pmatrix} (0,0) & (x,x') & (y,y') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z,z') & (0,0) \end{pmatrix} \mid (x,x'), (y,y'), (z,z'), (0,0) \in S \right\}.$$

Then N is a zero-symmetric left near ring with respect to matrix addition and matrix multiplication but N is not 3-prime. Let

$$U = \left\{ \begin{pmatrix} (0,0) & (x,x') & (y,y') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix} \mid (x,x'), (y,y'), (0,0) \in S \right\},$$

which is a nonzero semigroup ideal of N.

Define mappings $F, d : \underbrace{N \times N \times ... \times N}_{n-times} \to N$ by

$$F\left(\begin{pmatrix} (0,0) & (x_1,x_1') & (y_1,y_1') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z_1,z_1') & (0,0) \end{pmatrix}, \begin{pmatrix} (0,0) & (x_2,x_2') & (y_2,y_2') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z_2,z_2') & (0,0) \end{pmatrix}, \ldots, \right.$$
$$\left. \begin{pmatrix} (0,0) & (x_n,x_n') & (y_n,y_n') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z_n,z_n') & (0,0) \end{pmatrix} \right) = \begin{pmatrix} (0,0) & (\bar{y_1}\bar{y_2}...\bar{y_n},0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix},$$

$$d\left(\begin{pmatrix} (0,0) & (x_1,x_1') & (y_1,y_1') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z_1,z_1') & (0,0) \end{pmatrix}, \begin{pmatrix} (0,0) & (x_2,x_2') & (y_2,y_2') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z_2,z_2') & (0,0) \end{pmatrix}, \ldots, \right.$$

$$\begin{pmatrix} (0,0) & (x_n,x'_n) & (y_n,y'_n) \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z_n,z'_n) & (0,0) \end{pmatrix} \Bigg) = \begin{pmatrix} (0,0) & (y_1 y_2 ... y_n, 0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix}$$

and a map $g : N \to N$ by

$$g \begin{pmatrix} (0,0) & (x,x') & (y,y') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (z,z') & (0,0) \end{pmatrix} = \begin{pmatrix} (0,0) & (x,x') & (y,y') \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0') & (0,0) \end{pmatrix},$$

where $\bar{y_1}, \bar{y_2}, \ldots, \bar{y_n}$ are the complex conjugates of $y_1, y_2, \ldots, y_n$ respectively. If we choose $U_1 = U_2 = \cdots = U_n = U$, then it is verified that F is a generalized n-semiderivation associated with an n-semiderivation d and a map g associated with d on N satisfying the following conditions:

(i) $F(U_1, U_2, \ldots, U_n) \subseteq Z(N)$, (ii) $F([x_1, y_1], x_2, \ldots, x_n) = \pm[x_1, y_1]$,

(iii) $F(x_1 \circ y_1, x_2, \ldots, x_n) = 0$, (iv) $F(x_1 \circ y_1, x_2, \ldots, x_n) = \pm(x_1 \circ y_1)$,

(v) $F([x_1, y_1], x_2, \ldots, x_n) = \pm[F(x_1, x_2, \ldots, x_n), y_1]$,

(vi) $F([x_1, y_1], x_2, \ldots, x_n) = \pm[x_1, F(y_1, x_2, \ldots, x_n)]$,

(vii) $[F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)] = 0$,

for all $x_1, y_1 \in U_1, x_2, y_2 \in U_2, \ldots, x_n, y_n \in U_n$.

But, N is not commutative.

Open problem

(i) However, one can construct a natural example of a non-commutative near ring satisfying the hypothesis of the above theorems. (ii) Our hypothesis are dealt with the prime near rings. For further research, one can discuss the commutativity of semiprime near rings which is an interesting work in future.

Acknowledgement

The authors are very thankful to the referees for their valuable suggestions and comments.

References

[1] A. Ali, H. E. Bell, R. Rani, P. Miyan, *On semiderivations on prime near rings*, Southeast Asian Bull. Math., 40 (2016), 321-327.

[2] G. F. Pilz, *Near-rings. The theory and its applications*, 2nd edition, North-Holland: Amsterdam, The Netherlands; New York, NY, USA, Volume 23, 1983.

[3] H. E. Bell, *On derivations in near-rings II*, Kluwer Academic Publishers Netherlands, 1997, 191-197.

[4] K. H. Park, *On prime and semiprime rings with symmetric n-derivations*, J. Chungcheong Math. Soc., 22 (2009), 451-458.

[5] K. H. Park, Y. S. Jung, *On permuting 3-derivations and commutativity in prime near rings*, Commun. Korean Math. Soc., 25 (2010), DOI 10. 4134/CKMS. 2010. 25. 1001.

[6] M. A. Ozturk, Y. B. Jun, *On trace of symmetric bi-derivations in near rings*, Inter. J. Pure and Appl. Math., 17 (2004), 95-102.

[7] N. Muthana, *Rings endowed with biderivations and other biadditive mappings*, Ph.D. Dissertation, Girls College of Education Jeddah, Saudi Arabia, (2005).

Accepted: February 02, 2021

Subspace diskcyclic tuples of operators on Banach spaces

Nareen Bamerni
Department of Mathematics
University of Duhok
Kurdistan Region
Iraq
nareen_bamerni@yahoo.com

Abstract. In this paper, we study subspace diskcyclic and subspace-disk transitive tuples of operators. We give some characterizations of these tuples. Also, we give a set of sufficient conditions for a tuple to be subspace-diskcyclic. We find a relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators. Finally, we show that if a tuple of operators is subspace-diskcyclic, then not every operator in the tuple has to be subspace-diskcyclic.

Keywords: subspace-diskcyclic operators, tuple of operators.

1. Introduction

A bounded linear operator T on a separable Banach space X is hypercyclic if there is a vector $x \in X$ such that $Orb(T, x) = \{T^n x : n \geq 0\}$ is dense in X, such a vector x is called hypercyclic for T. The first example of a hypercyclic operator on a Banach space was constructed by Rolewicz in [12]. He showed that if B is the backward shift on $\ell^p(\mathbb{N})$ then λB is hypercyclic if and only if $|\lambda| > 1$.

In 1974, Hilden and Wallen [6] defined the supercyclicity concept. An operator T is called supercyclic if there is a vector x such that the scaled orbit $\mathbb{C}Orb(T, x)$ is dense in X. The notion of a diskcyclic operator was introduced by Zeana [17]. An operator T is called diskcyclic if there is a vector $x \in X$ such that the disk orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \leq 1, n \in \mathbb{N}\}$ is dense in X, such a vector x is called diskcyclic for T. For more information about diskcyclic operators, the reader may refer to [3] [1] [17].

In 2011, Madore and Martínez-Avendaño [9] considered the density of the orbit in a non-trivial subspace instead of the whole space, this phenomenon is called the subspace-hypercyclicity. An operator is called $\mathcal{M}$-hypercyclic or subspace-hypercyclic for a subspace $\mathcal{M}$ of X if there exists a vector such that the intersection of its orbit and $\mathcal{M}$ is dense in $\mathcal{M}$. For more information on subspace-hypercyclicity, one may refer to [7], [8] and [11].

In [14] Xian-Feng et al. defined the subspace-supercyclic operator as follows: An operator is called $\mathcal{M}$-supercyclic or subspace-supercyclic for a subspace $\mathcal{M}$

of X if there exists a vector such that the intersection of its scaled orbit and $\mathcal{M}$ is dense in $\mathcal{M}$.

Also, Bamerni and Kılıçman [15] introduced the subspace-diskcyclicity concept in a Banach space X that is the disk orbit of an operator T is dense in a subspace of X.

Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of continuous linear operators on a Banach space X and $\mathcal{F} = \left\{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} : k_i \geq 0, 1 \leq i \leq n\right\}$ be the semigroup generated by $\mathcal{T}$, then $\mathcal{T}$ is called hypercyclic if there is $x \in X$ such that $Orb(\mathcal{T}, x) = \{Tx : T \in \mathcal{F}\}$ is dense in X ([5]).

A tuple $\mathcal{T}$ is supercyclic if there exists $x \in X$ such that $\mathbb{C}Orb(\mathcal{T}, x) = \{\alpha Tx : T \in \mathcal{F}, \alpha \in \mathbb{C}\}$ is dense in X ([13]).

For subspaces, Moosapoor [10] defined subspace-hypercyclic tuples of operators as follows: A tuple $\mathcal{T}$ is subspace-hypercyclic for a subspace $\mathcal{M}$ if there exists a vector $x \in X$ such that $Orb(\mathcal{T}, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$. By the same way, a tuple $\mathcal{T}$ is subspace-suercyclic for a subspace $\mathcal{M}$ if there exists a vector $x \in X$ such that $\mathbb{C}Orb(\mathcal{T}, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$ ([16]).

Both subspace-hypercyclic and subspace-suercyclic tuples were studied in details; therefore, in this paper, we study some properties of subspace-diskcyclic tuples. In particular, we give an equivalent assertion to subspace- diskcyclic tuple which is called subspace-disk transitive tuple. Also, we give some sufficient conditions for a tuple to be subspace-diskcyclic which we call subspace-diskcyclic tuple criterion. We find a relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators. Finally, we show that if a tuple of operators is subspace-diskcyclic, then not every operator in the tuple has to be subspace-diskcyclic.

2. Main results

In this section, we characterize the equivalent conditions for a tuple of operators to be subspace-disk transitive. We provide some sufficient conditions for a tuple to be subspace-diskcyclic which is called subspace-diskcyclic tuple criterion. Also, we study the diskcyclicity of tuples of direct sum of operators.

In what follows, we let $\mathbb{U} = \{\alpha \in \mathbb{C} : |\alpha| < 1\}$ and $\mathbb{D}C(\mathcal{T}, \mathcal{M})$ be the set of all $\mathcal{M}$-diskcyclic vectors for the tuple $\mathcal{T}$, that is

$$\mathbb{D}C(\mathcal{T}, \mathcal{M}) = \{x \in X : \mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M} \text{ is dense in } \mathcal{M}\}.$$

Definition 2.1. *If $\mathcal{T} = (T_1, \ldots, T_n)$ is a tuple on a Banach space X, $\mathcal{F} = \{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} : k_i \geq 0, 1 \leq i \leq n\}$ and $\mathcal{M}$ be a closed subspace of X then the tuple $\mathcal{T}$ is called subspace-diskcyclic for $\mathcal{M}$ (or $\mathcal{M}$-diskcyclic) if there exists a vector $x \in X$ such that*

$$\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M} = \{\alpha Tx : T \in \mathcal{F}, \alpha \in \mathbb{D}\} \cap \mathcal{M}$$

is dense in $\mathcal{M}$.

It is clear from the above definition, that every subspace-hypercyclic tuple is subspace-diskcyclic which in turn is subspace-supercyclic.

Definition 2.2. *If $\mathcal{T} = (T_1, \ldots, T_n)$ is a tuple on a Banach space X, $\mathcal{F} = \{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} : k_i \geq 0, 1 \leq i \leq n\}$ and $\mathcal{M}$ be a closed subspace of X then the tuple $\mathcal{T}$ is called subspace-disk transitive (or $\mathcal{M}$-disk transitive) if for any two nonempty sets U and V in $\mathcal{M}$, there exists $\alpha \in \mathbb{U}$ and some positive integers k_i, $1 \leq i \leq n$ such that $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha U) \cap V$ contains a relatively open nonempty subset G of $\mathcal{M}$.*

We give the following example of a subspace-diskcyclic tuple.

Example 2.1. Suppose that T is a diskcyclic operator on a Banach space X and I is the identity operator. Then, it is easy to show that the tuple $\mathcal{T} = (T \oplus I, I \oplus T)$ is both $\mathcal{M}$-diskcyclic and $\mathcal{N}$-diskcyclic where $\mathcal{M} = X \oplus \{0\}$ and $\mathcal{N} = \{0\} \oplus X$ since both $T \oplus I$ and $I \oplus T$ are subspace-diskcyclic operators [15, Example 2.2.].

The following example shows that not every subspace-diskcyclic tuples is diskcyclic.

Example 2.2. Let $\mathcal{T} = (\alpha B \oplus I, \beta B \oplus I)$ be a 2-tuple where α, β are complex numbers with modulus greater than 1, I is the identity operator and B is the backward shift on the sequence space $\ell^2(\mathbb{N})$. Since αB is diskcyclic [3] then it has a diskcyclic vector, say x. Therefore, the tuple $\mathcal{T}$ has an $\mathcal{M}$-diskcyclic vector $(x, 0)$ for the subspace $\mathcal{M} = \ell^2(\mathbb{N}) \oplus \{0\}$. However, the tuple $\mathcal{T}$ is not diskcyclic since $\alpha B \oplus I$ is not diskcyclic operator.

The following proposition gives an equivalent assertion to subspace- disk transitive tuple.

Proposition 2.1. *Let M be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a tuple of operators. Then, the following statements are equivalent.*

1. *The tuple $\mathcal{T}$ is $\mathcal{M}$-disk transitive,*
2. *For any two relatively open subsets U and V of $\mathcal{M}$ there exist $\alpha \in \mathbb{U}^C$ and some positive integers k_i, $1 \leq i \leq n$ such that $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha U) \cap V \neq \phi$ and $T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}(\mathcal{M}) \subset \mathcal{M}$.*
3. *For any two relatively open subsets U and V of $\mathcal{M}$ there exists $\alpha \in \mathbb{U}^C$ and some positive integers k_i, $1 \leq i \leq n$ such that $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha U) \cap V$ is non-empty open set in $\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2): Let U and V be two relatively open subsets of $\mathcal{M}$. By the statement (1), there exist $\alpha \in \mathbb{U}^C$, some positive integers k_i, $1 \leq i \leq n$ and an open set G in $\mathcal{M}$ such that $G \subset T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha U) \cap V$. It follows that

$$T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha U) \cap V \neq \phi. \tag{1}$$

Since $G \subset T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha U)$, it follows that $\frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(G) \subset U \subset \mathcal{M}$. Let $x \in \mathcal{M}$ and $x_0 \in G$. Then, there exists $r \in \mathbb{N}$ such that $(x_0 + rx) \in G$. Then, we get

$$\begin{aligned} \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x_0 + \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} rx &= \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(x_0 + rx) \\ &\in \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(G) \subset \mathcal{M}. \end{aligned}$$

Since $x_0 \in G$ then $\frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x_0 \in \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(G) \subset \mathcal{M}$, it follows that $\frac{r}{\alpha} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \in \mathcal{M}$ and so $T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \subset \mathcal{M}$, i.e,

$$T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(\mathcal{M}) \subset \mathcal{M}. \tag{2}$$

The proof follows by (1) and (2).

(2)$\Rightarrow$ (3): The restriction function $T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$, then $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha U) \cap \mathcal{M}$ is open in $\mathcal{M}$ for any open set U of $\mathcal{M}$. Since $V \subset \mathcal{M}$ is open, it follows that $T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n}(\alpha U) \cap V$ is an open set in $\mathcal{M}$.

(3)$\Rightarrow$ (1) is trivial. □

The next theorem shows that every subspace-disk transitive tuple is subspace-diskcyclic for the same subspace. First, we need the following lemma.

Lemma 2.1. *Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be $\mathcal{M}$-diskcyclic tuple. Then, there exists $k_j \in \mathbb{N}$, $1 \le j \le n$ such that*

$$\mathbb{D}C(\mathcal{T}, \mathcal{M}) = \bigcap_{i \in \mathbb{N}} \Big(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha B_i) \Big).$$

where $\{B_i : i \in \mathbb{N}\}$ is a countable open basis for the relative topology of a subspace $\mathcal{M}$.

Proof. We have $x \in \mathbb{D}C(\mathcal{T}, \mathcal{M})$ if and only if

$$\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M} = \{\alpha T x : T \in \mathcal{F}, \alpha \in \mathbb{D}\} \cap \mathcal{M}$$

is dense in $\mathcal{M}$ if and only if for each $i > 0$, there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $k_j \in \mathbb{N}$, $1 \le j \le n$ such that $\alpha T_1^{k_1} T_2^{k_2} \dots T_n^{k_n x} \in B_i$ if and only if

$$x \in \bigcap_{i \in \mathbb{N}} \Big(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha B_i) \Big).$$

Theorem 2.1. *Let $\mathcal{M}$ be a subspace of a Banach space X and $\mathcal{T}=(T_1, T_2, \dots, T_n)$ be a tuple of operators. Suppose that $\mathcal{T}$ is $\mathcal{M}$-disk transitive tuple. Then,*

$$\bigcap_{i \in \mathbb{N}} \Big(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha B_i) \Big).$$

is dense in $\mathcal{M}$.

Proof. Since T is $\mathcal{M}$-transitive, then by Proposition 2.1, for each $i, j \in \mathbb{N}$, there exist $k_{i,j}^{(r)} \in \mathbb{N}$, $1 \le r \le n$ and $\alpha_{i,j} \in \mathbb{U}^C$ such that

$$T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j$$

is nonempty open in $\mathcal{M}$. Suppose that

$$A_i = \bigcup_{j \in \mathbb{N}} \left(T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j \right),$$

for all $i \in \mathbb{N}$. Then, A_i is nonempty and open in $\mathcal{M}$ since it is a countable union of open sets in $\mathcal{M}$. Furthermore, each A_i is dense in $\mathcal{M}$ since it intersects each B_j. By the Baire category theorem, we get

$$\bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left(T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j \right)$$

is a dense set in $\mathcal{M}$. Clearly,

$$\bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left(T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j \right) \subset \bigcap_{i \in \mathbb{N}} (\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1} (\alpha B_i)) \cap \mathcal{M}.$$

It follows that $\bigcap_{i \in \mathbb{N}} \left(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1} (\alpha B_i) \right) \cap \mathcal{M}$ is desne in $\mathcal{M}$. The proof is completed. □

Corollary 2.1. *If $\mathcal{T}$ is an $\mathcal{M}$-disk transitive tuple, then $\mathcal{T}$ is $\mathcal{M}$-diskcyclic.*

Proof. The proof follows by Proposition 2.1 and Theorem 2.1. □

Theorem 2.2 ($\mathcal{M}$-Diskcyclic Tuple Criterion)**.** *Let M be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be a tuple of operators. Suppose that for each $1 \le i \le n$, $\left\langle r_k^{(i)} \right\rangle_{k \in \mathbb{N}}$ is an increasing sequence of positive integers and $D_1, D_2 \in \mathcal{M}$ are two dense sets in $\mathcal{M}$ such that*

1. *For every $y \in D_2$, there is a sequence $\langle x_k \rangle_{k \in \mathbb{N}}$ in $\mathcal{M}$ such that $\|x_k\| \to 0$ and $T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x_k \to y$ as $k \to \infty$,*
2. *$\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\| \|x_k\| \to 0$, for all $x \in D_1$ as $k \to \infty$,*
3. *$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} \mathcal{M} \subseteq \mathcal{M}$, for all $k \in \mathbb{N}$.*

Then, $\mathcal{T}$ is said to be satisfied $\mathcal{M}$-diskcyclic criterion, and $\mathcal{T}$ is an $\mathcal{M}$-diskcyclic tuple.

Proof. Let U_1 and U_2 be two relatively open sets in $\mathcal{M}$. Then, we can find $x \in D_1 \cap U_1$ and $y \in D_2 \cap U_2$ since both D_1 and D_2 are dense in $\mathcal{M}$. It follows from the condition (2) that there exists a sequence of non-zero scalars $\langle \lambda_k \rangle_{k \in \mathbb{N}}$ such that $\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x \to 0$ and $\lambda_k^{-1} x_k \to 0$. Suppose that $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x\|$ and $\|x_k\|$ are not both zero. Then, we have the following cases:

1. if $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x\| = 0$, set $\lambda_k = 2^k \|x_k\|$.

Then, $\mathcal{T}$ turns to be $\mathcal{M}$-hypercyclic tuple [4, Theorem 2.4.] and thus $\mathcal{M}$-diskcyclic.

2. if $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x\|\|x_k\| \neq 0$, set $\lambda_k = \|x_k\|^{\frac{1}{2}} \|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x\|^{-\frac{1}{2}}$,
3. if $\|x_k\| = 0$, set $\lambda_k = 2^{-k} \|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x\|^{-1}$.

For these two cases if $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x\| \to 0$, then $\mathcal{T}$ is $\mathcal{M}$-hypercyclic tuple and so $\mathcal{M}$-diskcyclic. Otherwise, it follows easily that $|\lambda_k| \leq 1$, for all $k \in \mathbb{N}$. Set $z = x + {\lambda_k}^{-1} x_k$ for a large enough k. Since $x \in U_1 \subset \mathcal{M}$ and ${\lambda_k}^{-1} x_k \in \mathcal{M}$, then $z \in \mathcal{M}$. Since

$$\|z - x\| \to 0,$$

it follows that $z \in U_1$.

Now, since

$$\begin{aligned} &\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} z \\ &= \lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x + T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x_k \end{aligned}$$

then, by using the condition (3), we get

$$\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} z \text{ and } T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x_k$$

belong to $\mathcal{M}$ and so $\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x \in \mathcal{M}$.

Moreover, since $T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} x_k \to y$ for a large enough k, then

$$\left\| \lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} z - y \right\| \to 0.$$

Thus, $\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \ldots T_n^{r_k^{(n)}} z \in U_2$. It follows that there exists $k \in \mathbb{N}$ such that

$$U_1 \cap T_n^{-r_k^{(n)}} T_{n-1}^{-r_k^{(n-1)}} \ldots T_1^{-r_k^{(1)}} \left(\lambda_k^{-1} U_2\right) \neq \phi.$$

By Proposition 2.1 and Corollary 2.1, T is an $\mathcal{M}$-diskcyclic tuple. □

The following theorem gives the relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators.

Proposition 2.2. *Let $\mathcal{M}$ be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a tuple. Then, $\mathcal{T}$ satisfies subspace-diskcyclic criterion if and only if the tuple $\mathcal{S} = (T_1 \oplus T_1, T_2 \oplus T_2, \ldots, T_n \oplus T_n)$ satisfies subspace-diskcyclic criterion.*

With out loss of generality, we suppose that $\mathcal{S} = (T_1 \oplus T_1, T_2 \oplus T_2)$ and then the general case follows by the same way.

For the "if" part, let $\mathcal{M}$ be a closed subspace of X such that $\mathcal{S}$ satisfies $\mathcal{M} \oplus \mathcal{M}$-diskcyclic criterion. Let D_1 and D_2 be dense sets in $\mathcal{M}$ then $W = D_1 \oplus D_2$ is dense in $\mathcal{M} \oplus \mathcal{M}$. Let $x \in D_1$ and $y \in D_2$, then $(x, y) \in W$. By hypothesis, there exist two increasing sequence of positive integers $\left\langle r_k^{(i)} \right\rangle_{k \in \mathbb{N}}$ for $i = 1, 2$ and a sequence $\langle (x_k, y_k) \rangle_{k \in \mathbb{N}}$ in $\mathcal{M} \oplus \mathcal{M}$ such that $\|(x_k\, y_k)\| \to (0, 0)$ and $(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (x_k, y_k) \to (x, y)$ as $k \to \infty$. which means that $(T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} x_k, T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} y_k) \to (x, y)$. It follows that for each $y \in D_2$ there is a sequence $\langle y_k \rangle_{k \in \mathbb{N}} \to 0$ in $\mathcal{M}$ such that

$$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} y_k \to y. \tag{3}$$

By hypothesis, we have $\|(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (x, y)\| \, \|(x_k, y_k)\| \to (0, 0)$. Then, for all $x \in D_1$ it easy follows that

$$\left\| T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} x \right\| \|y_k\| \to 0. \tag{4}$$

Also, since $(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (\mathcal{M} \oplus \mathcal{M}) \subseteq (\mathcal{M} \oplus \mathcal{M})$, then

$$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} (\mathcal{M}) \subseteq \mathcal{M}. \tag{5}$$

From (3), (4) and (5), the tuple $\mathcal{T} = (T_1, T_2)$ satisfies diskcyclic criterion.

For the "only if" part, since $T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \mathcal{M} \subseteq \mathcal{M}$, for all $k \in \mathbb{N}$, then,

$$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \mathcal{M} \oplus T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \mathcal{M} \subseteq \mathcal{M} \oplus \mathcal{M}.$$

So,

$$(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (\mathcal{M} \oplus \mathcal{M}) \subseteq \mathcal{M} \oplus \mathcal{M}.$$

The remainder of the proof follows easily from [13, Corollary 1]. □

Proposition 2.3. *Let $\mathcal{M}$ be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a tuple of operators. If the semigroup $\mathcal{F}$ contains an $\mathcal{M}$-diskcyclic operator, then $\mathcal{T}$ is $\mathcal{M}$-diskcyclic tuple.*

Proof. Suppose that T is an $\mathcal{M}$-diskcyclic operator in $\mathcal{F}$, then

$$\mathcal{M} = \overline{\mathbb{D}Orb(T, x) \cap \mathcal{M}} \subseteq \overline{\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M}} \subseteq \mathcal{M}.$$

It follows that $\overline{\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M}} = \mathcal{M}$ and so $\mathcal{T}$ is $\mathcal{M}$-diskcyclic tuple. □

The following example gives a tuple of operators which is $\mathcal{M}$-diskcyclic, however, not every operator in the tuple is $\mathcal{M}$-diskcyclic.

Example 2.3. Let $T_1, T_2 \in B(\ell^2(\mathbb{Z}))$ be bilateral forward weighted shifts with the weight sequences w_n, k_n respectively, where

$$w_n = \begin{cases} \frac{1}{3} & \text{if } n \geq 0 \\ \frac{1}{2} & \text{if } n < 0 \end{cases} \quad \text{and} \quad k_n = \begin{cases} 4 & \text{if } n \geq 0 \\ 5 & \text{if } n < 0 \end{cases}$$

and let $\mathcal{M}$ be the subspace of $\ell^2(\mathbb{Z})$ consisting of all sequences with zeroes on the even entries; that is,

$$\mathcal{M} = \left\{ \{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) : a_{2n} = 0, n \in \mathbb{Z} \right\},$$

then by [2, Theorem 3.6] T_1 is not $\mathcal{M}$-diskcyclic but T_2 is $\mathcal{M}$-diskcyclic. However, the tuple $\mathcal{T} = (T_1, T_2)$ is $\mathcal{M}$-diskcyclic by Proposition 2.3.

3. Conclusion

We studied both subspace-diskcyclic and subspace-disk transitive tuples. We provided some sufficient conditions for a tuple to be subspace-diskcyclic which is called subspace-diskcyclic tuple criterion. Then, we found a relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators. By giving an example, we showed that if a tuple is subspace-diskcyclic, then there may be a non-diskcyclic operator in that tuple.

References

[1] N. Bamerni, A. Kılıçman, *Operators with diskcyclic vectors subspaces*, Journal of Taibah University for Science, 9 (2015), 414-419.

[2] N. Bamerni, A. Kılıçman, *On subspace-disk transitivity of bilateral weighted shifts*, Malaysian Journal of Science, 34 (2015), 208-213.

[3] N. Bamerni and A. Kılıçman, M.S.M. Noorani, *A review of some works in the theory of diskcyclic operators*, Bull. Malays. Math. Sci. Soc., 39 (2016), 723-739.

[4] E. Fathi and B. Yousefi, *Subspace cyclicity and hypercyclicity for tuples of operators*, Southeast Asian Bulletin of Mathematics, 42 (2018),183-194.

[5] N. S. Feldman, *Hypercyclic tuples of operators and somewhere dense orbits*, Journal of Mathematical Analysis and Applications, 346 (2008), 82-98.

[6] H. Hilden, L. Wallen, *Some cyclic and non-cyclic vectors of certain operators*, Indiana Univ. Math. J., 23 (1974), 557-565.

[7] R.R. Jiménez-Munguía, R. A. Martínez-Avendaño, A. Peris, *Some questions about subspace hypercyclic operators*, J. Math. Anal. Appl., 408 (2013), 209-212.

[8] C. Le, *On subspace-hypercyclic operators*, Proc. Amer. Math. Soc., 139 (2011), 2847-2852.

[9] B.F. Madore, R.A. Martínez-Avendaño, *Subspace hypercyclicity*, J. Math. Anal. Appl., 393 (2011), 502-511.

[10] M. Moosapoor, *Subspace-hypercyclic tuples of operators*, International Journal of Pure and Applied Mathematics, 107 (2016), 17-22.

[11] H. Rezaei, *Notes on subspace-hypercyclic operators*, J. Math. Anal. Appl., 397 (2013), 428-433.

[12] S. Rolewicz, *On orbits of elements*, Stud. Math., 32 (1969), 17-22.

[13] R. Soltani, K. Hedayatian and B. K. Robati, *On supercyclicity of tuples of operators*, Bulletin of the Malaysian Mathematical Sciences Society, 38 (2015), 1507-1516.

[14] Z. Xian-Feng, S. Yong-Lu, Z. Yun-Hua, *Subspace-supercyclicity and common subspace-supercyclic vectors*, Journal of East China Normal University., 1 (2012), 106-112.

[15] N. Bamerni, A. Kılıçman, *On subspaces diskcyclicity*, Arab Journal of Mathematical Sciences, 23 (2015), 133-140.

[16] B. Yousefi and E. Fathi, *Subspace supercyclicity of tuples of operators*, International Journal of Pure and Applied Mathematics, 101 (2015), 421-424.

[17] J. Zeana, *Cyclic phenomena of operators on Hilbert space*, Thesis, University of Baghdad, (2002).

Accepted: December 14, 2022

Nonlinear mappings preserving the kernel or range of skew product of operators

H. Benbouziane*
Faculty of Sciences DharMahraz Fes
University Sidi Mohammed Ben Abdellah
Fez
Morocco
hassane.benbouziane@usmba.ac.ma

M. Ech-Chérif El Kettani
Faculty of Sciences DharMahraz Fes
University Sidi Mohammed Ben Abdellah
Fez
Morocco
mostapha.echcherifelkettani@usmba.ac.ma

Ahmedou Mohamed Vadel
Faculty of Sciences DharMahraz Fes
University Sidi Mohammed Ben Abdellah
Fez
Morocco
fadel.abdellahi@yahoo.fr

Abstract. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operator on H. We characterise surjective maps $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, such that $F(\phi(A) \diamond \phi(B)) = F(A \diamond B)$, for all $A, B \in \mathcal{B}(H)$, where $F(A)$ denotes any of $R(A)$ or $N(A)$ and $A \diamond B$ denotes any binary operations A^*B, AB^*A for all $A, B \in \mathcal{B}(\mathcal{H})$.

Keywords: nonlinear preservers problem, kernel range operator, skew product.

1. Introduction and preliminaries

Throughout this note, $\mathcal{H}$ will denote a Hilbert space over the complex field $\mathbb{C}$ and $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$ with unit I. For $A \in \mathcal{B}(\mathcal{H})$ denoted by $R(A)$ the range of A, $N(A)$ its kernel and A^* its adjoint. The hyper-range of $A \in \mathcal{B}(X)$ is defined by $\mathcal{R}^\infty(A) := \bigcap_{n\in\mathbb{N}} R(A^n)$.

For any $x, f \in \mathcal{H}$, as usual, we denote $x \otimes f$ the rank at most one operator defined by $(x \otimes f)(y) = f(y)x =< y, f > x$, for every $y \in \mathcal{H}$. The set of all rank one operators is denoted by $\mathcal{F}_1(\mathcal{H})$. Fix an arbitrary orthogonal basis $\{e_i\}_{i\in\Gamma}$ of $\mathcal{H}$. For $x \in \mathcal{H}$, write $x = \sum_{i\in\Gamma} \lambda_i e_i$, and define the conjugate operator $J : \mathcal{H} \to \mathcal{H}$ by $Jx = \overline{x} = \sum_{i\in\Gamma} \overline{\lambda_i} e_i$.

*. Corresponding author

The study of maps on operator algebras preserving certain properties is a topic which attracts much attention of many authors, see for example [3, 6, 7, 9, 10, 12, 13], and the references therein. In this direction, in the last decades, a great activity has occurred in characterising maps preserving a certain property of the product or triple product (see [1, 2, 4, 6, 11]). In [2], the authors determine the form of surjective maps on $\mathcal{B}(\mathcal{H})$ which satisfies $F(\phi(A)\diamond\phi(B)) = F(A\diamond B)$ for all $A, B \in \mathcal{B}(H)$ where $F(A)$ denotes any of $R(A)$ or $N(A)$ and $A\diamond B$ denotes any binary operations: the usual product AB and triple product ABA for all $A, B \in \mathcal{B}(\mathcal{X})$. They also cover the main results of [12] by characterizing the maps that satisfy $N(\phi(A) - \phi(B)) = N(A - B)$ (or $R(\phi(A) - \phi(B)) = R(A - B)$).

As a continuation, in this direction, we propose to determine the forms of all surjective maps $\phi : \mathcal{B}(H) \to \mathcal{B}(\mathcal{H})$ which satisfy one of the following preserving properties:

- $N(\phi(A)\phi(B)^*\phi(A)) = N(AB^*A)$;
- $N(\phi(A)^*\phi(B)) = N(A^*B)$;
- $R(\phi(A)\phi(B)^*\phi(A)) = R(AB^*A)$;
- $R(\phi(A)^*\phi(B)) = R(A^*B)$,

for all $A, B \in \mathcal{B}(H)$.

2. Preliminaries

In this section, we collect some lemmas that will be used in the proof of our main results. The first one gives the range and kernel of rank one operators.

Lemma 2.1. *Let $x, f \in \mathcal{H}$ nonzeros vectors. We have*

1. *$R(x \otimes f) = span\{x\}$ and $N(x \otimes f) = \{f\}^{\perp}$.*
2. *If $f(x) = 1$, then $N(I - x \otimes f) = R(x \otimes f) = span\{x\}$ and $R(I - x \otimes f) = N(x \otimes f) = \{f\}^{\perp}$.*
3. *If $f(x) \neq 0$ then $\mathcal{R}^{\infty}(x \otimes f) = R(x \otimes f) = span\{x\}$.*

Proof. See, for example, [8, Lemma 2.1]. □

The second, quoted from [4], characterizes maps preserving zero skew products of operators in both directions.

Lemma 2.2. *Let $\mathcal{H}$ be a complex Hilbert space with dim $\mathcal{H} \geq 3$. Suppose $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective map such that*

$$A^*B = 0 \Leftrightarrow \phi(A)^*\phi(B) = 0 \text{ for all } A, B \in \mathcal{B}(\mathcal{H}). \tag{1}$$

Then, ϕ preserves rank one operators in both directions and $\phi(0) = 0$. Moreover, there exist unitary $U \in \mathcal{B}(\mathcal{H})$ and a map $h : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that:

$$\phi(x \otimes f) = Ux \otimes h(x, f), \text{ for all } x, f \in \mathcal{H},$$

or

$$\phi(x \otimes f) = UJx \otimes h(x, f), \text{ for all } x, f \in \mathcal{H}.$$

Proof. See, [4, Theorem 2.1]. □

The following lemma determines the structure of surjective maps preserving the zero skew triple product of operators.

Lemma 2.3. *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 3$. Suppose that $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective map. Then, ϕ satisfies*

$$AB^*A = 0 \Leftrightarrow \phi(A)\phi(B)^*\phi(A) = 0, \text{ for all } A, B \in \mathcal{B}(\mathcal{H}), \tag{2}$$

if and only if there exist unitary linear or conjugate linear operators U, V on $\mathcal{H}$ and functional $h : \mathcal{H} \to \mathbb{C} \setminus \{0\}$ such that ϕ is of one of the forms:

$$\phi(A) = h(A)UAV, \text{ for all } A \in \mathcal{B}(\mathcal{H}),$$

or

$$\phi(A) = h(A)UA^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}).$$

Proof. See, [11, Corollary 3.5]. □

We end this section by stating and proving the following lemma which will be used later.

Lemma 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent.*

1. *$N(R^*A) = N(R^*B)$ for all rank one operators R.*
2. *$R(A^*R) = R(B^*R)$ for all rank one operators R.*
3. *$A = cB$ for a nonzero scalar $c \in \mathbb{C}$.*

Proof. It's easy to check that (3) implies (1) and (3) implies (2).

1 ⇒ 3): Assume that $N(R^*A) = N(R^*B)$ for all rank one operators R. Let $R = x \otimes f$ be a rank one operator where $x, f \in \mathcal{H}$. By hypothesis we have

$$\begin{aligned} N(R^*A) = N(R^*B) \quad &\Longleftrightarrow \quad N((A^*R)^*) = N((B^*R)^*) \\ &\Longleftrightarrow \quad R(A^*R)^\perp = R(B^*R)^\perp. \end{aligned}$$

Which implies that $span\{A^*x\}^\perp = span\{B^*x\}^\perp$.

Since $span\{A^*x\}$ and $span\{B^*x\}$ are closed subspaces, we deduce that $span\{A^*x\} = span\{B^*x\}$. Therefore, $A^*x = c_x B^*x$, where $c_x \in \mathbb{C}$ is a scalar depending to x.

Now, to complete the proof, it is suffice to show that $N(A^*) = N(B^*)$. Indeed, suppose that there is $g \in \mathcal{H}$ such that $A^*g = 0$ and $B^*g \neq 0$. Then, there is a non zero vector $x \in \mathcal{H}$ such that $< x.B^*g >= 1$.

Note that, $(x \otimes g)B(x) = x \otimes B^*g(x) =< x, B^*g > x = x \neq 0$. Then, $x \notin N((x \otimes g)B)$. But $x \in N((x \otimes g)A)$ because $(x \otimes g)A(x) = (x \otimes A^*g)(x) = 0$. Which contradict the hypothesis.

$2 \Rightarrow 3$) let x be a non zero vector in $\mathcal{H}$. By hypothesis, we have

$$R(A^*x \otimes x) = R(B^*x \otimes x).$$

Which implies that $span\{A^*x\} = span\{B^*x\}$. We can show, by the same method as above, that $N(A^*) = N(B^*)$. Therefore, A^* and B^* are linearly dependent. Thus, A and B are linearly dependent, as desired. □

3. Nonlinear maps preserving the kernel

We begin this section with the following result which characterizes surjective maps that preserve the kernel of triple skew product of operators.

Theorem 3.1. *Let $\mathcal{H}$ be a complex Hilbert space with dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$N(\phi(A)\phi(B)^*\phi(A)) = N(AB^*A), \text{ for all } A, B \in \mathcal{B}(\mathcal{H}), \tag{3}$$

if and only if there exist $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K} \setminus \{0\}$ and U unitairy operator in $\mathcal{B}(\mathcal{H})$ such that $\phi(A) = \varphi(A)UA$ for all $A \in \mathcal{B}(\mathcal{H})$.

Proof. The necessarily condition is easily verified. Conversely, assume that ϕ satisfies the equation (3). In particular,

$$N(\phi(A)\phi(B)^*\phi(A)) = \mathcal{H} \iff N(AB^*A) = \mathcal{H}, \text{ for all } A, B \in \mathcal{B}(\mathcal{H}).$$

Then, ϕ satisfies the equation (2). Since ϕ is surjective, by Lemma 2.3, there exist unitary linear or conjugate linear operators U, V on $\mathcal{H}$ and functional $h : \mathcal{H} \to \mathbb{C} \setminus \{0\}$ such that ϕ is of one of the forms:

$$\phi(A) = h(A)UAV, \text{ for all } A \in \mathcal{B}(\mathcal{H}), \tag{4}$$

or

$$\phi(A) = h(A)UA^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}). \tag{5}$$

We shall show that ϕ can not take the form (5). Assume for the sak of contradiction that ϕ takes a such form, and let us first show that V is a scalar operator. It suffices to prove that V^* is a scalar operator. To do that, assume, on the contrary, that there exists a non zero vector $x \in \mathcal{H}$ such that V^*x and

x are linearly independent. We could find $f \in \mathcal{N}$ such that $< x, f >= 1$ and $< V^*x.f >= 0$. For any $B \in \mathcal{B}(\mathcal{H})$, we have $\phi(I) = h(I)UV$. Then

$$N(B^*) = N(\phi(I)\phi(B^*)^*\phi(I)) = N(UBV), \text{ for all } B \in \mathcal{B}(\mathcal{H}). \tag{6}$$

According to the lemma 2.1 and applying (6) to $B = I - x \otimes f$, we obtain

$$\begin{aligned} span\{f\} &= N((I - x \otimes f)^*) \\ &= N(U(I - x \otimes f)V) \\ &= N(UV - Ux \otimes V^*f) \\ &= N(I - V^*x \otimes V^*f) \end{aligned}$$

Since $< V^*f.V^*x >=< f.x >= 1$, then by Lemma 2.1, $span\{f\} = span\{V^*x\}$. Therefore, $f = \lambda V^*x$, for some non zero $\lambda \in \mathcal{H}$.

This shows that $< V^*x.f >= \lambda \parallel f \parallel^2 \neq 0$, which is a contradiction. Hence, V is a scalar operator and $\phi(A) = \varphi(A)UA$, where φ is a scalar function $\mathcal{B}(\mathcal{H}) \to \mathbb{K}^*$. Since U is injective, (6) becomes

$$N(B^*) = N(B), \text{ for all } B \in \mathcal{B}(\mathcal{H}). \tag{7}$$

On the other hand, we can find $z_1, z_2 \in \mathcal{H}$ such that z_1, z_2 are linearly independent and $< z_1, z_2 >= 1$. Applying (7) to $B = I - z_1 \otimes z_2$ we obtain

$$\begin{aligned} span\{z_1\} &= N(I - z_1 \otimes z_2) \\ &= N((I - z_1 \otimes z_2)^*) = N(I - z_2 \otimes z_1) \\ &= span\{z_2\}. \end{aligned}$$

This contadiction shows that ϕ takes the formes (4).

Now, let $x, f \in \mathcal{H}$ such that $< x, f >= 1$. For $B = I - x \otimes f$, from (3) and Lemma 2.1, we have

$$\begin{aligned} span\{f\} &= N((I - x \otimes f)^*) \\ &= N(B^*) = N(UB^*V) \\ &= N(U(I - f \otimes x)V) \\ &= N(UV - Uf \otimes V^*x) \\ &= N(UV(I - V^*f \otimes V^*x)) \\ &= N((I - V^*f \otimes V^*x)) \\ &= span\{V^*f\}. \end{aligned}$$

Therefore, V^* is a scalar operator and V is also. Which proves that $\phi(A) = \varphi(A)UA$, for all $A \in \mathcal{B}(\mathcal{H})$, with $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K}^*$ is a scalar function. This completes the proof. □

The following theorem characterizes surjective maps that preserve the kernel of skew product of operators.

Theorem 3.2. *Let $\mathcal{H}$ be a complex Hilbert space with dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$N(\phi(A)^*\phi(B)) = N(A^*B), \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}), \tag{8}$$

if and only if there exists $c \in \mathbb{K} \setminus \{0\}$ and and unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$\phi(A) = cUA, \quad \forall A \in \mathcal{B}(\mathcal{H}). \tag{9}$$

Proof. The "if" part is easily verified. We, therefore, will only deal with the "only if" part. So, assume that ϕ is a surjective map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ satisfying (8). In particular,

$$N(\phi(A)^*\phi(B)) = \mathcal{H} \iff N(A^*B) = \mathcal{H}, \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}).$$

This entails that ϕ satisfies the equation (1). since ϕ is surjective, by Lemma 2.2, there exist unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a map $h : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that:

$$\phi(x \otimes f) = Ux \otimes h(x, f), \text{ for all } x, f \in \mathcal{H}, \tag{10}$$

or

$$\phi(x \otimes f) = UJx \otimes h(x, f), \text{ for all } x, f \in \mathcal{H}. \tag{11}$$

Let $f, x \in \mathcal{H}$ and put $g = h(x, f)$. If (10) holds, then

$$\begin{aligned} \{f\}^\perp &= N((x \otimes f)^*(x \otimes f)) \\ &= N((\phi(x \otimes f))^*(\phi(x \otimes f))) \\ &= N((Ux \otimes g)^*(Ux \otimes g)) \\ &= \{g\}^\perp. \end{aligned}$$

So, there exists $\lambda \in \mathcal{H}$ such that $g = \lambda f$.

If (11) holds, with no extra effort, we get the same result. Therefore, for every $R \in \mathcal{F}_1$ we obtain

$$\phi(R) = \lambda V R, \tag{12}$$

or

$$\phi(R) = \lambda V J R. \tag{13}$$

Let $A \in \mathcal{B}(\mathcal{H})$ and $R \in \mathcal{F}_1$. If (12) holds, then

$$N(R^*A) = N(\phi(R)^*\phi(A)) = N(R^*V^*\phi(A)).$$

Therefore, by Lemma 2.4, there exists non zero scalar $c \in \mathcal{H}$ such that $\phi(A) = cVA$ or

$$N(R^*A) = N(\phi(R)^*\phi(A)) = N(R^*J^*V^*\phi(A)).$$

Then, $\phi(A) = cVJA$, for some non zero scalar $c \in \mathcal{H}$.

Now, assume that V is unitary. Take an orthonormal basis $\{e_i\}_{i\in\Gamma}$ of $\mathcal{H}$ and define the conjugate operator $J : \mathcal{H} \to \mathcal{H}$ by $Jx = \overline{x} = \sum_{i\in\Gamma} \overline{\lambda_i} e_i$. Then, J is conjugate unitary. Let $U = VJ$ then U is unitary (see, [5, Claim 3 in Theorem 5.1]). We conclude that $\phi(A) = cUA$, for all $A \in \mathcal{B}(\mathcal{H})$ with U is unitary, the proof is complete. □

4. Nonlinear maps preserving the range

The first theorem in this section characterizes surjective maps that preserve the Range of triple skew product of operators.

Theorem 4.1. *Let $\mathcal{H}$ be a real or complex Hilbert space of dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$R(\phi(A)\phi(B)^*\phi(A)) = R(AB^*A), \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}), \tag{14}$$

if and only if there exists $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K} \setminus \{0\}$ and V unitairy in $\mathcal{B}(\mathcal{H})$ such that $\phi(A) = \varphi(A)AV$, for all $A \in \mathcal{B}(\mathcal{H})$.

Proof. The necessary condition is easily verified since the operator V is surjective. Conversely, assume that ϕ is a surjective map satisfying (14). Then

$$R(\phi(A)\phi(B)^*\phi(A)) = \mathcal{H} \iff R(AB^*A) = \mathcal{H}, \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}).$$

Which shows that ϕ satisfying the equation (1). It follows, by Lemma 2.3, that there exist unitary linear or conjugate linear operators U, V on $\mathcal{H}$ and functional $h : \mathcal{H} \to \mathbb{C} \setminus \{0\}$ such that ϕ is of one of the forms:

$$\phi(A) = h(A)UAV, \text{ for all } A \in \mathcal{B}(\mathcal{H}), \tag{15}$$

or

$$\phi(A) = h(A)UA^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}). \tag{16}$$

Similarly to the proof of Theorem 2.1, let us first show that ϕ can not take the second form. Assume, to the contrary, that ϕ takes a such form. Let x be a non zero vector in $\mathcal{H}$. By (14) and Lemma 2.3, we have

$$\begin{aligned} span\{x\} &= R((x \otimes x)^*) \\ &= R(U(x \otimes x)U^*UV) \\ &= R(Ux \otimes x)U^*) \\ &= R(Ux \otimes Ux) \\ &= span\{Ux\}. \end{aligned}$$

Which proves that U is a scalar operator. Thus,

$$\phi(A) = h(A)A^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}).$$

In particular, for $A = x \otimes y$ where x and y are linearly independent, we obtain

$$span\{y\} = R(B^*) = R(\phi(I)\phi(B)^*\phi(I)) = R(BV) = R(B) = span\{x\},$$

which is a contradiction. We conclude that ϕ takes the form (15).

To finish the proof, it remains to show that U is a scalar operator. Indeed, for any nonzero vector $x \in \mathcal{H}$ we have

$$\begin{aligned} span\{x\} &= R((x \otimes x)^*) \\ &= R(\phi(I)\phi((x \otimes x))^*\phi(I)) \\ &= R(U(x \otimes x)^*V) \\ &= R(Ux \otimes x) \\ &= span\{Ux\}. \end{aligned}$$

This proves that Ux and x are linearly dependent, for all $x \in \mathcal{H}$. Therefore, there is a non zero scalar C such that $U = cI$. The proof is complete. □

By replacing the range of operator by the hyper-range of operator in the previous theorem we get the following result.

Theorem 4.2. *Let $\mathcal{H}$ be a real or complex Hilbert space of dimension ≥ 3.*
A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies

$$\mathcal{R}^\infty(\phi(A)\phi(B)^*\phi(A)) = \mathcal{R}^\infty(AB^*A), \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}), \tag{17}$$

if and only if, there exists $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K} \setminus \{0\}$ and V unitary in $\mathcal{B}(\mathcal{H})$ such that $\phi(A) = \varphi(A)AV$, for all $A \in \mathcal{B}(\mathcal{H})$.

We end this paper by the following result which characterizes surjective maps that preserve the Range of skew product of operators.

Theorem 4.3. *Let $\mathcal{H}$ be a complex Hilbert space of dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$R(\phi(A)^*\phi(B)) = R(A^*B), \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}), \tag{18}$$

if and only if there exists $c \in \mathbb{K} \setminus \{0\}$ and and unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$\phi(A) = cUA, \quad \text{for all } A \in \mathfrak{B}(H). \tag{19}$$

Proof. The necessarily condition is easily verified since the operators U is surjective. Conversely, assume that ϕ is a surjective additive map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ satisfying (18). In particular,

$$R(\phi(A)^*\phi(B)) = \{0\} \iff R(A^*B) = \{0\}, \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}).$$

This implies that ϕ satisfies the equation (1). By following the same approach of the proof of Theorem 3.2, we obtain

$$\phi(R) = \lambda UR \textit{ or } \phi(R) = \lambda UJR, \text{ for every } R \in \mathcal{F}_1.$$

By the same reasoning and by applying Lemma 2.4, the map ϕ has the desired form. □

References

[1] H. Benbouziane, Y. Bouramdane, M. Ech-Cherif El Kettani, A. Lahssaini, *Nonlinear maps preserving condition spectrum of jordan skew triple product of opertors*, Opertors and Matrices, 12 (2018), 933-9242.

[2] H. Benbouziane, Y. Bouramdane, M. Ech-Cherif El Kettani, *Nonlinear maps preserving certain subspaces*, Proyecciones Journal of Mathematics, 38 (2019), 163-174.

[3] J. Cui, J. Hou, *Additive maps on standard operator algebras preserving invertibilities or zero divisors*, Linear Algebra Appl., 359 (2003), 219-233.

[4] J. Cui, C.K. Li, N.S. Sze, *Unitary similarity invariant function preservers of skew products of operators*, J. Math. Anal. Appl., 454 (2017), 716-729.

[5] J. Cui, C.K. Li, Y.-T. Poon, *Pseudospectra of special operators and pseudospectrum preservers*, J. Math. Anal. Appl., 419 (2014), 1261–1273.

[6] G. Dobovišek, B. Kuzma, G. Lešnjak, C.K. Li, T. Petek, *Mappings that preserve pairs of operators with zero triple Jordan product*, Linear Algebra Appl., 426 (2007), 255-279.

[7] G. Dolinar, S.P. Du, J.C. Hou, P. Legiša, *General preservers of invariant subspace lattices*, Linear Algebra Appl., 429 (2008), 100-109.

[8] M. Elhodaibi, A. Jaatit, *On additive maps preserving the hyper-range or hper-kernel of operators*, Int. Math. Forum, 7 (2012), 1223-1231.

[9] L. Fang, G. Ji, Y. Pang, *Maps preserving the idempotency of products of operators*, Linear Algebra Appl., 426 (2007), 40-52.

[10] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineaire Substitusionen*, Sitzungsber, Deutsch. Akad. Wiss. Berling, (1897), 171-172.

[11] J. Hou, K. He, X. Zhang, *Nonlinear maps preserving numerical radius of indefinite skew products of operators*, Linear Algebra Appl., 430 (2009), 2240-2253.

[12] M. Oudghiri, *Additive mappings preserving the kernel or the range of operators*, Extracta Math., 24 (2009), 251-258.

[13] P. Semrl, *Two characterizations of automorphisms on $\mathcal{B}(X)$,* Studia Math., 150 (1993), 143-149.

Accepted: May 6, 2022

On the application of M-projective curvature tensor in general relativity

Kaushik Chattopadhyay*
Applied Science Section
RCC Institute of Information Technology
Kolkata-15
India
kjc.says@gmail.com

Arindam Bhattacharyya
Department of Mathematics
Jadavpur University
Kolkata-700032
India
bhattachar1968@yahoo.co.in

Dipankar Debnath
Bamanpukur High School (H.S)
Nabadwip
India
dipankardebnath123@hotmail.com

Abstract. In this paper the application of the M-projective curvature tensor in the general theory of relativity has been studied. Firstly, we have proved that an M-projectively flat quasi-Einstein spacetime is of a special class with respect to an associated symmetric tensor field, followed by the theorem that a spacetime with vanishing M-projective curvature tensor is a spacetime of quasi-constant curvature. Then we have proved that an M-projectively flat quasi-Einstein spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ξ. In the next section we have proved that an M-projectively flat Ricci semi-symmetric quasi-Einstein spacetime satisfying a definite condition is an $N(\frac{2l-m}{6})$-quasi Einstein spacetime. In the last section, we have firstly proved that an M-projectively flat perfect fluid spacetime with torse-forming vector field ξ satisfying Einstein field equation with cosmological constant represents an inflation, then we have found out the curvature of such spacetime, followed by proving the theorem that the spacetime also becomes semi-symmetric under these conditions. Lastly, we have found out the square of the length of the Ricci tensor in this type of spacetime and also proved that if an M-projectively flat perfect fluid spacetime satisfying Einstein field equation with cosmological constant, with torse-forming vector field ξ admits a symmetric $(0,2)$ tensor α parallel to ∇ then either $\lambda = \frac{k}{2}(p - \sigma)$ or α is a constant multiple of g.

Keywords: M-projective curvature tensor, Riemannian curvature tensor, torse-forming vector field, Einstein equation.

*. Corresponding author

1. Introduction

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor S of type $(0,2)$ is non-zero and proportional to the metric tensor. Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [4]. Also in Riemannian geometry as well as in general relativity theory, the Einstein manifold plays a very important role. Chaki and Maity [18] generalised the concept of Einstein manifold and introduced the notion of quasi-Einstein manifold. According to them, a Riemannian or semi-Riemannian manifold is said to be a quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the condition

$$S(U,V) = lg(U,V) + mA(U)A(V), \tag{1}$$

where l and m are two real-valued scalar functions where $m \neq 0$ and A is a non-zero 1-form equivalent to the unit vector field ξ, i.e. $g(U,\xi) = A(U)$, $g(\xi,\xi) = 1$. If $m = 0$ then the manifold becomes Einstein. Quasi-Einstein manifolds are denoted by $(QE)_n$, where n is the dimension of the manifold. There are many examples of quasi-Einstein manifolds, like the Robertson-Walker spacetime is a quasi-Einstein manifold. Also, quasi-Einstein manifolds can be taken as a model of perfect fluid spacetime in general relativity. The importance of quasi-Einstein spacetimes lies in the fact that 4-dimensional semi-Riemannian manifolds are related to study of general relativistic fluid spacetimes, where the unit vector field ξ is taken as timelike velocity vector field, that is, $g(\xi,\xi) = -1$. In the recent papers [1], [23], the application of quasi-Einstein spacetime and generalised quasi-Einstein spacetime in general relativity have been studied. Many more works have been done in the spacetime of general relativity [2], [16], [25], [26], [29], [30], [31]. Let (M_n, g) be an n-dimensional differentiable manifold of class C^∞ with the metric tensor g and the Riemannian connection ∇. In 1971 G. P. Pokhariyal and R. S. Mishra ([12]) defined the M-projective curvature tensor as follows

$$\begin{aligned}\tilde{P}(U,V)W = R(U,V)W - \frac{1}{2(n-1)}[S(V,W)U - S(U,W)V \\ + g(V,W)QU - g(U,W)QV],\end{aligned} \tag{2}$$

where R and S are the curvature tensor and the Ricci tensor of M_n, respectively. Such a tensor field $\tilde{P}$ is known as the M-projective curvature tensor. Some authors studied the properties and applications of this tensor [11], [15], [20] and [21]. In 2010, S. K. Chaubey and R. H. Ojha investigated the M-projective curvature tensor of a Kenmotsu manifold [24]. The concept of perfect fluid spacetime arose while discussing the structure of this universe. Perfect fluids are often used in the general relativity to model the idealised distribution of matter, such as the interior of a star or isotropic pressure. The energy-momentum tensor

$\tilde{T}$ of a perfect fluid spacetime is given by the following equation

$$\tilde{T}(U,V) = pg(U,V) + (\sigma + p)A(U)A(V), \tag{3}$$

where σ is the energy-density and p is the isotropic pressure, A is defined earlier and the unit vector field ξ is timelike, i.e. $g(\xi,\xi) = -1$. Einstein field equation with cosmological constant([7]) is given by

$$S(U,V) - \frac{\tilde{r}}{2}g(U,V) + \lambda g(U,V) = k\tilde{T}(U,V), \tag{4}$$

where, S is the Ricci tensor, $\tilde{r}$ is the scalar curvature of the spacetime while λ, k are the cosmological constant and the gravitational constant respectively. It's used to describe the dark energy of this universe in modern cosmology, which is responsible for the possible acceleration of this universe. The equations (3) and (4) together give

$$S(U,V) = (\frac{\tilde{r}}{2} - \lambda + pk)g(U,V) + k(\sigma + p)A(U)A(V). \tag{5}$$

Comparing to the equation (1) we can say the tensor of the equation (5) represents the tensor of a quasi-Einstein manifold. The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by

$$\begin{aligned} &N(k) : p \to N_p(k) = \\ &\{W \in T_p(M) : R(U,V)W = k[g(V,W)U - g(U,W)V]\}, \end{aligned} \tag{6}$$

for all $U, V \in T_pM$, where k is a smooth function. For a quasi-Einstein manifold M, if the generator ξ belongs to some $N(k)$, then M is said to be $N(k)$-quasi-Einstein manifold [19] . Özgür and Tripathi proved that for an n-dimensional $N(k)$-quasi Einstein manifold [9], $k = \frac{l+m}{n-1}$, where l and m are the respective scalar functions and n is the dimension of the manifold. In this paper we have first derived some theorems on M-projectively flat spacetimes. After that we have introduced the concept of Ricci semi-symmetric spacetime with vanishing M-projective curvature tensor. Lastly we introduced the concept of torse-forming vector field in this spacetime and derived some theorems on it, thereby finding the curvature of the spacetime and finding the square of the length of the Ricci tensor for this spacetime with torse-forming vector field.

2. Preliminaries

Consider a quasi-Einstein spacetime with associated scalars l, m and associated 1-form A. Then by (1), we have

$$r = 4l - m, \tag{7}$$

where r is a scalar curvature of the spacetime. If ξ is a unit timelike vector field, then $g(\xi,\xi) = -1$. Again from the equation (1), we have

$$S(\xi,\xi) = m - l, \tag{8}$$

For all vector fields U and V we have the following equation,

$$g(QU, V) = S(U, V), \tag{9}$$

where Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S. From the equation (5) and (9) we can get

$$QU = (\frac{\tilde{r}}{2} - \lambda + pk)U + k(\sigma + p)A(U)\xi. \tag{10}$$

If the unit timelike vector field ξ is a torse-forming vector field ([5], [6]) then it satisfies the following equation,

$$\nabla_U \xi = U + A(U)\xi. \tag{11}$$

In [28] Venkatesha and H. A. Kumara proved that:

Theorem 2.1. *On a perfect fluid spacetime with torse-forming vector field ξ, the following relation holds*

$$(\nabla_U A)(V) = g(U, V) + A(U)A(V). \tag{12}$$

Considering a frame field and taking a contraction over U and V from the equation (5) we get,

$$\tilde{r} = 4\lambda + k(\sigma - 3p). \tag{13}$$

3. M-projectively flat quasi-Einstein spacetime

In this section we consider a quasi-Einstein spacetime with vanishing M-projective curvature tensor. If a spacetime with dimension $n = 4$ is M-projectively flat then from the equation (2) we have

$$R(U, V)W = \frac{1}{6}[S(V, W)U - S(U, W)V + g(V, W)QU - g(U, W)QV]. \tag{14}$$

Using the equation (9) in the equation (1) we get

$$QU = lU + mA(U)\xi. \tag{15}$$

Using the equations (1), (15) and taking the inner product with T from (14) we get

$$\begin{aligned}\tilde{R}(U, V, W, T) = {} & \frac{l}{3}[g(V, W)g(U, T) - g(U, W)g(V, T)] \\ & + \frac{m}{6}[g(U, T)A(V)A(W) + g(V, W)A(U)A(T) \\ & - g(V, T)A(U)A(W) - g(U, W)A(V)A(T)].\end{aligned} \tag{16}$$

Now, taking

$$D(U,V) = \sqrt{\frac{l}{3}}g(U,V) + \frac{m}{2\sqrt{3l}}A(U)A(V), \tag{17}$$

from the equation (16) we have

$$\tilde{R}(U,V,W,T) = D(V,W)D(U,T) - D(U,W)D(V,T). \tag{18}$$

It is known that an n-dimensional Riemannian or semi-Riemannian manifold whose curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition (18), is called a special manifold with the associated symmetric tensor D and is denoted by the symbol $\psi(D)_n$, where D is a symmetric tensor field of type $(0,2)$. Recently, these types of manifolds are studied in [10] and [27]. With the use of the equations (17) and (18) we can state the following theorem:

Theorem 3.1. *An M-projectively flat quasi-Einstein spacetime is $\psi(D)_4$, where D is the associated symmetric tensor field.*

In [8], B.Y. Chen and K. Yano introduced the concept of quasi-constant curvature. A manifold is said to be a manifold of quasi-constant curvature if it satisfies the following condition

$$\begin{aligned} \tilde{R}(U,V,W,T) = {} & p[g(V,W)g(U,T) - g(U,W)g(V,T)] \\ & + q[g(U,T)\eta(V)\eta(W) - g(V,T)\eta(U)\eta(W) \\ & + g(V,W)\eta(U)\eta(T) - g(U,W)\eta(V\eta(T)], \end{aligned} \tag{19}$$

where $\tilde{R}$ is the scalar curvature of type (0, 4), p and q are scalar functions while $g(U,\nu) = \eta(U)$, ν is the unit vector field, η is the respective 1-form and $g(\nu,\nu) = 1$. Thus, in the view of (16) and (19) we state the following theorem:

Theorem 3.2. *A spacetime with vanishing M-projective curvature tensor is a spacetime of quasi-constant curvature.*

Now, let us consider the space $\xi^{\perp} = \{X : g(X,\xi) = 0, \forall\ X \in \chi(M)\}$. Let U, V, $W \in \xi^{\perp}$, then the equation (16) will imply

$$R(U,V)W = \frac{l}{3}[g(V,W)U - g(U,W)V]. \tag{20}$$

So, we can state the following theorem:

Theorem 3.3. *An M-projectively flat quasi-Einstein spacetime becomes an $N(\frac{l}{3})$-quasi Einstein spacetime provided U, V, $W \in \xi^{\perp}$, ξ is a unit timelike vector field and l is a non-zero real-valued scalar function.*

We also derive the following corollary:

Corollary 3.1. *An M-projectively flat quasi-Einstein spacetime satisfies the following results,*

$$(i) R(U,\xi)V = \frac{m-2l}{6} g(U,V)\xi,$$
$$(ii) R(U,\xi)\xi = \frac{m-2l}{6} U, \tag{21}$$

where U, $V \in \xi^{\perp}$, ξ is a unit timelike vector field and l, m are two non-zero real-valued scalar functions.

Lorentzian manifolds are extremely important in applications to general relativity. Lorentzian manifolds are of signature $(3,1)$ or, equivalently, $(1,3)$. A Lorentzian manifold is called infinitesimally spatially isotropic ([13]) relative to a unit timelike vector field ξ if its curvature tensor R satisfies the relation

$$R(X,Y)Z = \alpha[g(Y,Z)X - g(X,Z)Y], \tag{22}$$

for all X, Y, $Z \in \xi^{\perp}$ and $R(X,\xi)\xi = \beta X$ for all $X \in \xi^{\perp}$, α and β are two non-zero real-valued functions. From the equation (20) and the result (ii) of corollary (14) it is obvious that the manifold is infinitesimally spatially isotropic. Thus, we can state the following theorem:

Theorem 3.4. *An M-projectively flat quasi-Einstein spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.*

4. M-projectively flat Ricci semi-symmetric quasi-Einstein spacetime

In this section we consider a quasi-Einstein spacetime which is Ricci semi-symmetric. An n-dimensional semi-Riemannian manifold is said to be Ricci semi-symmetric if the tensor $R.S$ and the Tachibana tensor $\tilde{Q}(g,S)$ are linearly dependent, i.e.,

$$R(U,V)\cdot S(W,T) = F_S \tilde{Q}(g,S)(W,T;U,V) \tag{23}$$

holds on U_S where $U_S = \{x \in M : S \neq \frac{r}{n} g \ at \ x\}$ and F_S is a scalar function on U_S. Now, we know that

$$R(U,V)\cdot S(W,T) = -S(R(U,V)W,T) - S(W,R(U,V)T), \tag{24}$$

using the equation (23) we have

$$F_S \tilde{Q}(g,S)(W,T;U,V) = -S(R(U,V)W,T) - S(W,R(U,V)T). \tag{25}$$

We also know that

$$(U \wedge_g V)W = g(V,W)U - g(U,W)V. \tag{26}$$

Now, if it is a Ricci semi-symmetric quasi-Einstein spacetime then using the equations (24), (25) and (26) we get

$$\begin{aligned} S(R(U,V)W,T) + S(W,R(U,V)T) = F_S[&g(V,W)S(U,T) - g(U,W)S(V,T) \\ &+ g(V,T)S(W,U) - g(U,T)S(V,W)]. \end{aligned} \tag{27}$$

Since we know $\tilde{R}(U,V,W,T) = -\tilde{R}(U,V,T,W)$, thus using the equation (1) in the equation (27) we obtain

$$\begin{aligned} &A(R(U,V)W)A(T) + A(W)A(R(U,V)T) \\ &= F_S[g(V,W)A(U)A(T) - g(U,W)A(V)A(T) \\ &+g(V,T)A(W)A(U) - g(U,T)A(V)A(W)], \end{aligned} \tag{28}$$

putting $T = \xi$ in the equation (28) and applying the result $g(R(U,V)\xi,\xi) = g(R(\xi,\xi)U,V)$ we get,

$$A(R(U,V)W) = F_S[g(V,W)A(U) - g(U,W)A(V)], \tag{29}$$

applying the equation (16) from (29) we get,

$$(F_S - \frac{2l-m}{6})[g(V,W)A(U) - g(U,W)A(V)] = 0. \tag{30}$$

So, if $g(V,W)A(U) - g(U,W)A(V) \neq 0$ then

$$F_S = \frac{2l-m}{6}, \tag{31}$$

thus using the equations (29) and (31) we get,

$$R(U,V)W = \frac{2l-m}{6}[g(V,W)U - g(U,W)V], \tag{32}$$

from the equations (6) and (32) we observe that the spacetime becomes an $N(\frac{2l-m}{6})$-quasi Einstein spacetime provided $g(V,W)A(U) - g(U,W)A(V) \neq 0$. This leads us to the next theorem:

Theorem 4.1. *An M-projectively flat Ricci semi-symmetric quasi-Einstein spacetime with $g(V,W)A(U) - g(U,W)A(V) \neq 0$ is an $N(\frac{2l-m}{6})$-quasi Einstein spacetime, where l and m are two non-zero real valued scalar functions.*

5. M-projectively flat perfect fluid spacetime with torse-forming vector field

If a manifold is M-projectively flat then using the divergence ∇ to both the sides of the equation (14) we get

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = 0, \tag{33}$$

using the equation (9) we get,

$$g((\nabla_U Q)V - (\nabla_V Q)U, W) = 0. \tag{34}$$

From the equation (13) since we observe $\tilde{r}$ is a constant thus using the equation (10) we get

$$k(\sigma + p)[(\nabla_U A)(V)\xi + A(V)\nabla_U \xi - (\nabla_V A)(U)\xi - A(U)\nabla_V \xi] = 0, \tag{35}$$

using the equations (11) and (12) we get

$$k(\sigma + p)[g(V, \xi)U - g(U, \xi)V] = 0, \tag{36}$$

since k is the gravitational constant hence $k \neq 0$. Thus, $g(V, \xi)U - g(U, \xi)V \neq 0$ implies

$$\sigma + p = 0, \tag{37}$$

which means either $\sigma = p = 0$ (empty spacetime) or the perfect fluid satisfies the vacuum-like equation of state. This allows us to derive the following theorem:

Theorem 5.1. *An M-projectively flat perfect fluid spacetime with torse-forming vector field ξ satisfying Einstein field equation with cosmological constant is either an empty spacetime or satisfies the vacuumlike equation of state, provided $g(V, \xi)U - g(U, \xi)V \neq 0$.*

Now, $\sigma + p = 0$ means the fluid behaves as a cosmological constant [14]. This is also termed as Phantom Barrier [22]. Now, in cosmology we know such a choice $\sigma = -p$ leads to rapid expansion of the spacetime which is now termed as inflation [17], [3]. So, we obtain the following theorem:

Theorem 5.2. *An M-projectively flat perfect fluid spacetime with torse-forming vector field ξ satisfying Einstein field equation with cosmological constant represents an inflation.*

Now, putting $\sigma + p = 0$ from the equation (5) we get,

$$S(U, V) = [\lambda + \frac{k}{2}(\sigma - p)]g(U, V), \tag{38}$$

thus, the equation (10) becomes

$$QU = [\lambda + \frac{k}{2}(\sigma - p)]U. \tag{39}$$

Using the equations (38) and (39) in the equation (14) we get

$$R(U, V)W = \left\{\frac{2\lambda + k(\sigma - p)}{6}\right\}[g(V, W)U - g(U, W)V]. \tag{40}$$

Hence, we can state the following theorem:

Theorem 5.3. *An M-Projectively flat perfect fluid spacetime with torse-forming vector field ξ, satisfying Einstein field equation with cosmological constant is of constant curvature* $\frac{2\lambda+k(\sigma-p)}{6}$.

Consequently we obtain the following theorem as:

Theorem 5.4. *An M-projectively flat perfect fluid spacetime with torse-forming vector field ξ satisfying Einstein field equation with cosmological constant is an Einstein spacetime.*

From the equation (40) we easily obtain

$$\begin{aligned} &(R(U,V)\cdot\tilde{R})(X,Y,Z,W) = -\tilde{R}(R(U,V)X,Y,Z,W) \\ &-\tilde{R}(X,R(U,V)Y,Z,W) \\ &-\tilde{R}(X,Y,R(U,V)Z,W) - \tilde{R}(X,Y,Z,R(U,V)W) = 0, \end{aligned} \tag{41}$$

which implies the manifold is semi-symmetric. Hence, we obtain the following theorem:

Theorem 5.5. *An M-projectively flat perfect fluid spacetime with torse-forming vector field ξ satisfying Einstein field equation with cosmological constant is a semi-symmetric spacetime.*

Replacing U by QU from the equation (38) we get

$$S(QU,V) = [\lambda + \frac{k}{2}(\sigma - p)]g(QU,V). \tag{42}$$

Using the equation (38)which becomes

$$S(QU,V) = [\lambda + \frac{k}{2}(\sigma - p)]S(U,V) = [\lambda + \frac{k}{2}(\sigma - p)]^2 g(U,V). \tag{43}$$

Considering a frame field and taking a contraction over U and V from the equation (43) we get

$$||Q||^2 = 4[\lambda + \frac{k}{2}(\sigma - p)]^2 = [2\lambda + k(\sigma - p)]^2. \tag{44}$$

Hence, we can state the following theorem:

Theorem 5.6. *The square of the length of the Ricci tensor of an M-projectively flat perfect fluid spacetime with torse-forming vector field ξ satisfying Einstein field equation with cosmological constant is* $[2\lambda + k(\sigma - p)]^2$.

The Ricci identity is given by

$$\nabla^2_{U,V}\alpha(X,Y) - \nabla^2_{V,U}\alpha(X,Y) = \alpha(R(U,V)X,Y) + \alpha(X,R(U,V)Y), \tag{45}$$

where α is a symmetric $(0,2)$ tensor. Now, if α is parallel to ∇ then

$$\nabla\alpha = 0, \tag{46}$$

which further implies

$$\nabla^2\alpha = 0. \tag{47}$$

Thus, from the equation (45) we get

$$\alpha(R(U,V)X,Y) + \alpha(X,R(U,V)Y) = 0. \tag{48}$$

Thus, from the equation (40) we get

$$\begin{aligned}&\{\frac{2\lambda + k(\sigma - p)}{6}\}[g(V,X)\alpha(U,Y) - g(U,X)\alpha(V,Y)\\ &+g(V,Y)\alpha(U,X) - g(U,Y)\alpha(V,X)] = 0.\end{aligned} \tag{49}$$

Putting $X = Y = V = \xi$ in the equation (49) we get,

$$-\{\frac{2\lambda + k(\sigma - p)}{3}\}[\alpha(U,\xi) + A(U)\alpha(\xi,\xi)] = 0, \tag{50}$$

which means either $\lambda = \frac{k}{2}(p - \sigma)$ or

$$\alpha(U,\xi) = -A(U)\alpha(\xi,\xi). \tag{51}$$

Now, taking the derivative of $\alpha(\xi,\xi)$ with respect to V and using the equations (11) and (51) we get

$$V(\alpha(\xi,\xi)) = 0. \tag{52}$$

Taking the derivative of the equation (51) with respect to V and using the equation (52) we get

$$V(\alpha(U,\xi)) = -\alpha(\xi,\xi)V(g(U,\xi)). \tag{53}$$

Since α is parallel with respect to ∇ thus using the equation (11) from the equation (53) we get

$$\alpha(U,V) = -\alpha(\xi,\xi)g(U,V). \tag{54}$$

Therefore we obtain the following theorem as:

Theorem 5.7. *If an M-projectively flat perfect fluid spacetime satisfying Einstein field equation with cosmological constant, with torse-forming vector field ξ admits a symmetric $(0,2)$ tensor α parallel to ∇ then either $\lambda = \frac{k}{2}(p - \sigma)$ or α is a constant multiple of g.*

Acknowledgement:

We are thankful to the editor for his/her valuable suggestions for the improvement of the paper.

References

[1] A.A. Shaikh, D.Y. Yoon, S.K. Hui, *On quasi-Einstein spacetime*, Tsukuba J. Math., 33 (2009), 305-326.

[2] A. Bhattacharyya, T. De, D. Debnath, *Space time with generalized covariant recurrent energy momentum tensor*, Tamsui Oxford Journal of Mathematical Sciences, 25 (2009), 269-276.

[3] A.H. Guth, *Inflationary universe: a possible solution to the horizon and flatness problems*, Carnegie Observatories Astrophysics Series, vol. 2, Measuring and Modeling the Universe, Cambridge Univ. Press, Cambridge.

[4] A.L. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb., 3. Folge, Bd. 10. Berlin, Heidelberg, New York, Springer-Verlag, 1987.

[5] A.M. Blaga, *Solitons and geometrical structures in a perfect fluid spacetime*, Zbl 1439.53024 Rocky Mt. J. Math., 50 (2020), 41-53.

[6] A M. Blaga, M. Crasmareanu, *Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry*, Zbl 07380073, Filomat 31 (2017), 499-504.

[7] B. O'Neill, *Semi-Riemannian geometry and application to relativity*, Pure and Applied Mathematics, Academic Press, Inc. New York, 1983.

[8] B.Y. Chen, K. Yano, *Hypersurfaces of a conformally flat space*, Tensor (N.S.), 26 (1972), 315-321.

[9] C. Özgür, M.M. Tripathi, *On the concircular curvature tensor of an $N(k)$-Einstein manifold*, Math. Pannon., 18 (2007), 95-100.

[10] D.G. Prakasha, B.S. Hadimani, *η-Ricci soliton on para-Sasakian manifolds*, J. Geom., 108 (2017), 383-392.

[11] F.O. Zengin, *M-projectively flat spacetimes*, Math. Rep., 4 (2012), 363-370.

[12] G.P. Pokhariyal, R.S. Mishra, *Curvature tensors and their relativistic significance*, Yokohama Math. J., 19 (1971), 97-103.

[13] H. Karcher, *Infinitesimale charakterisierung von Friedmann-Univeren*, Arch. Math., 38 (1982), 58-64.

[14] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Monogr. on Math. Phys. Cambridge Univ. Press, 2nd edition, Cambridge, 2003.

[15] K. Chattopadhyay, A. Bhattacharyya, D. Debnath, *A study of spacetimes with vanishing M-projective curvature tensor*, Journal of the Tensor Society of India, 12 (2018), 23-31.

[16] K. Chattopadhyay, N. Bhunia, A. Bhattacharyya, *On Ricci-symmetric mixed-generalized quasi-Einstein spacetime*, Bull. Cal. Math. Soc., 110 (2018), 513-524.

[17] L. Amendola, S. Tsujikawa, *Dark energy: theory and observations*, Cambridge Univ. Press, Cambridge, 2010.

[18] M.C. Chaki, R.K. Maity, *On quasi-Einstein manifolds*, Publ. Math. Debrecen, 57 (2000), 297-306.

[19] M.M. Tripathi, J.S. Kim, *On $N(k)$-Einstein manifolds*, Commun. Korean Math. Soc., 22 (2007), 411-417.

[20] R.H. Ojha, *A note on the M-projective curvature tensor*, Indian J. Pure Appl. Math., 8 (1975), 1531-1534.

[21] R.H. Ojha, *M-projectively flat Sasakian manifolds*, Indian J. Pure Appl. Math., 17 (1986), 481-484.

[22] S. Chakraborty, N. Mazumder, R. Biswas, *Cosmological evolution across phantom crossing and the nature of the horizon*, Astrophys. Space Sci. Libr., 334 (2011), 183-186.

[23] S. Güler, S.A. Demirbağ, *A study of generalized quasi-Einstein spacetime with application in general relativity*, International Journal of Theoretical Physics, 55 (2016), 548-562.

[24] S.K. Chaubey, R.H. Ojha, *On the M-projective curvature tensor of a Kenmotsu manifold*, Diff. Geom. Dyn. Syst., 12 (2010), 52-60.

[25] S. Mallick, Y.J. Suh, U.C. De, *A spacetime with pseudo-projective curvature tensor*, J. Math. Phys., 57 (2016), Paper ID 062501.

[26] S. Ray-Guha, *On perfect fluid pseudo Ricci symmetric spacetime*, Tensor (N.S.), 67 (2006), 101-107.

[27] Venkatesha, D.M. Naik, *Certain results on K-para contact and para Sasakian manifold*, J. Geom., 108 (2017), 939-952.

[28] Venkatesha, H.A. Kumara, *Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field*, Afr. Mat., 2019.

[29] Z. Ahsan, M. Ali, *Curvature tensor for the spacetime of general relativity*, Int. J. of Geom. Methods Mod. Phys., 14 (2017), 1750078 (13 pages).

[30] Z. Ahsan, M. Ali, *Quasi-conformal curvature tensor for the spacetime of general relativity*, Palestine Journal of Mathematics, 4 (2015), 233-240.

[31] Z. Ahsan, S.A. Siddiqui, *Concircular curvature tensor and fluid spacetimes*, Internat. J. Theoret. Phys., 48 (2009), 3202-3212.

Accepted: October 29, 2021

Torsion section of elliptic curves over quadratic extensions of $\mathbb{Q}$

Zakariae Cheddour
zakariae.cheddour@usmba.ac.ma

Abdelhakim Chillali*
abdelhakim.chillali@usmba.ac.ma

A. Mouhib
University of Sidi Mohamed Ben Abdellah-USMBA, LSI, FP
MPI Department
BP. 1223, Taza
Morocco
ali.mouhib@usmba.ac.ma

Abstract. In this paper, we will study and determine all possible torsion sections of elliptic curves that can appear on quadratic extensions of the set of rational numbers endowed by the usual addition and a non-standard way of multiplication.
Keywords: elliptic curves, torsion section, quadratic extension.

1. Introduction

Let E be an elliptic curve over $\mathbb{Q}$. By the Mordell-Weil theorem, the group $E(\mathbb{Q})$ of rational points on E is a finitely generated abelian group. Therefore, it is the product of the torsion group and $r \geq 0$ copies of an infinite cyclic group:

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \times \mathbb{Z}^r.$$

By Mazur's theorem [5], we know that $E(\mathbb{Q})_{tors}$ is one of the following 15 groups:

$$\begin{cases} \mathbb{Z}/n\mathbb{Z}, & \text{with } 1 \leq n \leq 10 \ \ or \ \ n = 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}, & \text{with } 1 \leq m \leq 4. \end{cases}$$

Subsequently, S. Kamienny, F. Najman [3] and M. A. Kenku, F. Momose [4] have worked on the possible torsion groups which can appear on quadratic extensions of $\mathbb{Q}$. In [3, 4] we find that on a quadratic extension K of $\mathbb{Q}$, we have that $E(K)_{tors}$ is isomorphic to one of the following groups 26 :

$$\begin{cases} \mathbb{Z}/m\mathbb{Z}, & \text{with } 1 \leq m \leq 18 \ , \ m \neq 17, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}, & \text{with } 1 \leq m \leq 6, \\ \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3m\mathbb{Z}, & \text{with } 1 \leq m \leq 2, \\ \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}. & \end{cases}$$

*. Corresponding author

Note that $E(K)_{tors}$ is finite over a quadratic numbers field because of S. Kamienny theorem [2]. In particular, F. Najman [6, 7] has classified all possible torsion subgroups on cyclotomic quadratic extensions. Similarly, K. Sarma and A. Saikia [8] determined the possible torsion subgroups on the other imaginary quadratic fields of class 1.

In this paper, we will define a non-standard way of multiplying elements in the quadratic extension of the set of rational numbers, denoted by $\mathbb{Q}[\lambda]$ with $\lambda = \sqrt{d}$ and d is a square-free integer. Also, we will study and determine all possible torsion sections of elliptic curves given by a Weierstrass equation $Y^{*2} * Z = X^{*3} + a * X * Z^2 + b * Z^3$ that can appear on $\mathbb{Q}[\lambda]$, where $\mathbb{Q}[\lambda]$ endowed by the usual addition and the new product law defined as follows, so for $X = x_0 + x_1\lambda$ and $Y = y_0 + y_1\lambda$, where x_0, x_1, y_0 and $y_1 \in \mathbb{Q}$, we have

$$X + Y = (x_0 + y_0) + (x_1 + y_1)\lambda$$

and

$$X * Y = x_0y_0 + (x_0y_1 + y_0x_1 + x_1y_1)\lambda.$$

Note that, if X and Y are two elements of $\mathbb{Q}$, then the product law $*$ is the usual product law over $\mathbb{Q}$.

In a later work, we will use these results to study the classification of the torsion section of elliptic curves on imaginary (real) multiquadratic extensions of the set of rational numbers. Furthermore, we will use these results to give a new encryption scheme... In what follows, we will use the following notation:

- For $X \in \mathbb{Q}[\lambda]$, we have $X^{*n} = \underbrace{X * X * ... * X}_{n\ times}$,
- $\Im_{a,b}$ for an elliptic curve over the ring $(\mathbb{Q}[\lambda], +, *)$ given by a Weierstrass equation $Y^{*2} * Z = X^{*3} + a * X * Z^{*2} + b * Z^{*3}$, with $a, b \in \mathbb{Q}[\lambda]$ and such that the discriminant $D = 4a^{*3} + 27b^{*2}$ is invertible in $\mathbb{Q}[\lambda]$,
- $Tor(\Im_{a,b})$ for the torsion section of $\Im_{a,b}$.

In this article, we study the mentioned elliptic curve, and we prove the following theorem,

Theorem 1.1. *With the same notation as above, let $\Im_{a,b}$ be an elliptic curve defined over $\mathbb{Q}[\lambda]$. So,*

$$Tor(\Im_{a,b}, \mathbb{Q}[\lambda]) \simeq \begin{cases} \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, & n, m = 1, 2, ..., 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, & 1 \leq n \leq 4, m = 1, 2, ..., 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}, & 1 \leq n, m \leq 4. \end{cases}$$

2. The ring $(\mathbb{Q}[\lambda], +, *)$

In this section, we will give some results concerning the ring $\mathbb{Q}[\lambda]$, which are useful for the rest of this article. So, let X, Y and Z be elements of $\mathbb{Q}[\lambda]$ where $X = x_0 + x_1\lambda$, $Y = y_0 + y_1\lambda$ and $Z = z_0 + z_1\lambda$.

Lemma 2.1. *The set $\mathbb{Q}[\lambda]$ together with addition "$+$" and multiplication "$*$" is a finitely generated unitary commutative ring.*

Proof. By construction we have * is a commutative law.

We shall prove that $X * (Y * Z) = (X * Y) * Z$ so,

$$\begin{aligned} X * (Y * Z) &= X * (y_0z_0 + (y_0z_1 + z_0y_1 + y_1z_1)\lambda) \\ &= x_0y_0z_0 + (x_0[y_0z_1 + z_0y_1 + y_1z_1] + x_1y_0z_0 \\ &+ x_1[y_0z_1 + z_0y_1 + y_1z_1])\lambda \\ &= x_0y_0z_0 + (x_0y_0z_1 + x_0z_0y_1 + x_0y_1z_1 + x_1y_0z_0 + x_1y_0z_1 \\ &+ x_1z_0y_1 + x_1y_1z_1)\lambda \end{aligned}$$

on the other hand we have

$$\begin{aligned} (X * Y) * Z &= (x_0y_0 + (x_0y_1 + y_0x_1 + x_1y_1)\lambda) * Z \\ &= x_0y_0z_0 + (x_0y_0z_1 + [x_0y_1 + y_0x_1 + x_1y_1]z_0 \\ &+ [x_0y_1 + y_0x_1 + x_1y_1]z_1)\lambda \\ &= x_0y_0z_0 + (x_0y_0z_1 + x_0z_0y_1 + x_0y_1z_1 + x_1y_0z_0 + x_1y_0z_1 \\ &+ x_1z_0y_1 + x_1y_1z_1)\lambda \end{aligned}$$

hence $*$ is associative.

$*$ is distributive with respect to the law $+$

$$\begin{aligned} X * (Y + Z) &= X * (y_0 + z_0 + (y_1 + z_1)\lambda) \\ &= x_0(y_0 + z_0) + (x_0[y_1 + z_1] + x_1[y_0 + z_0] + x_1[y_1 + z_1])\lambda \\ &= x_0y_0 + x_0z_0 + (x_0y_1 + x_0z_1 + x_1y_0 + x_1z_0 + x_1y_1 + x_1z_1)\lambda \\ &= [x_0y_0+(x_0y_1+x_1y_0+x_1y_1)t]+[x_0z_0+(x_0z_1+x_1z_0+x_1z_1)\lambda] \\ &= X * Y + X * Z. \end{aligned}$$

□

Corollary 2.1. *$\mathbb{Q}[\lambda]$ is a vector space over $\mathbb{Q}$ of dimension 2, and $(1, \lambda)$ is its basis.*

The next proposition characterize the set $\mathbb{Q}[\lambda]^\times$ of invertible elements in $\mathbb{Q}[\lambda]$.

Proposition 2.1. *Let $X = x_0 + x_1\lambda \in \mathbb{Q}[\lambda]$, then $X \in \mathbb{Q}[\lambda]^\times$ if and only if $x_0 \neq 0$ and $x_0+x_1 \neq 0$. The inverse is given by: $X^{-1} = x_0^{-1} - x_1x_0^{-1}(x_0+x_1)^{-1}\lambda$.*

Proof. Let X be an invertible element of $\mathbb{Q}[\lambda]$, then there exist Y in $\mathbb{Q}[\lambda]$ such that $X * Y = 1$ so, $x_0 y_0 = 1$ and $x_0 y_1 + y_0 x_1 + x_1 y_1 = 0$, then we have $y_0 = x_0^{-1}$ and $y_1 = -(x_0 + x_1)^{-1} x_0^{-1} x_1$. So,

$$Y = X^{-1} = x_0^{-1} - (x_0 + x_1)^{-1} x_0^{-1} x_1 \lambda.$$

Hence, X is invertible if and only if $x_0 \neq 0$ and $x_0 + x_1 \neq 0$. □

Corollary 2.2. *The non invertible elements of $\mathbb{Q}[\lambda]$ are those elements of the form $a\lambda$ and $b - b\lambda$, where $a, b \in \mathbb{Q}$.*

Proposition 2.2.

- $\mathbb{Q}[\lambda]$ *is not a local ring,*
- $\mathbb{Q}[\lambda]$ *is not an integral domain.*

Proof. We use the fact that a ring R is a local ring if and only if all elements of R that are not units form an ideal. So, put $I = \{b - b\lambda \mid b \in \mathbb{Q}\} \cup \lambda\mathbb{Q}$ the set of non-invertible elements of $\mathbb{Q}[\lambda]$. We shall prove that I is not an ideal, this turns out to prove that $\{b - b\lambda | b \in \mathbb{Q}\} \cap \lambda\mathbb{Q} = \{0\}$. So, let $X \in \{b - b\lambda | b \in \mathbb{Q}\} \cap \lambda\mathbb{Q}$ then $X = b - b\lambda = a\lambda$ where $a, b \in \mathbb{Q}$, it follows that $X = 0$.

For the 2nd point it is enough to take $X = \lambda$ and $Y = 1 - \lambda$, for which we have $X * Y = 0$. □

In what follows, we denote by $\widehat{\mathbb{Q}[\lambda]}$, the set of integral elements of $\mathbb{Q}[\lambda]$ over $\mathbb{Z}$. That is, $b \in \widehat{\mathbb{Q}[\lambda]}$ if and only if b is a root of a monic polynomial over $\mathbb{Z}$.

The following theorem characterizes the set $\widehat{\mathbb{Q}[\lambda]}$,

Theorem 2.1. $\widehat{\mathbb{Q}[\lambda]} = \mathbb{Z}[\lambda]$.

Proof. Let $A = e + f\lambda \in \mathbb{Z}[\lambda]$, then $A^{*2} = e^2 + (2ef + f^2)\lambda$, it follows that $A^{*2} = e^2 + 2e(A - e) + f(A - e)$, then A is a root of $P(X) = X^{*2} - e^2 - 2e(X - e) - f(X - e)$ over $\mathbb{Z}$. So, we have A is an integral element over $\mathbb{Z}$, then $\mathbb{Z}[\lambda] \subset \widehat{\mathbb{Q}[\lambda]}$.

On the other hand, let $A = e + f\lambda \in \widehat{\mathbb{Q}[\lambda]}$, so there exists a monic polynomial $P(X) = X^{*n} + a_1 X^{*n-1} + ... + a_n$ over $\mathbb{Z}[X]$ such that $P(A) = 0$, then $P(e + f\lambda) = (e + f\lambda)^{*n} + a_1(e + f\lambda)^{*n-1} + ... + a_n = 0$. Since $\lambda^{*m} = \lambda$ for all $m \in \mathbb{N} - \{0\}$ it follows that $P(e + f\lambda) = e^n + Q_1(e) + \lambda(f^n + Q_2(e, f)) = 0$ with Q_1, Q_2 are two polynomials respectively belonging in $\mathbb{Z}[X]$ and $\mathbb{Z}[X, Y]$ such that $deg(Q_1(X)) < n$ and $deg(Q_2(e, Y)) < n$. So, $e^n + Q_1(e) = 0$ and $f^n + Q_2(e, f) = 0$ then:

- we have $T_1(X) = X^n + Q_1(X)$ is a monic polynomial over $\mathbb{Z}[X]$ and since $T_1(e) = 0$ it follows that e is an integral element over $\mathbb{Z}$. Hence, $e \in \mathbb{Z}$.
- on the other hand, since $e \in \mathbb{Z}$ we have $T_2(X) = X^n + Q_2(e, X)$ is a monic polynomial over $\mathbb{Z}[X]$ and since $T_2(f) = 0$ it follows that f is an integral element over $\mathbb{Z}$. Hence, $f \in \mathbb{Z}$. □

3. Elliptic curves over $\mathbb{Q}[\lambda]$

Definition 3.1. *An elliptic curve over a commutative ring R is a group scheme (a group object in the category of schemes) over $Spec(R)$ (the prime spectrum of R) that is a relative 1-dimensional, smooth, proper curve over R. For more background information about group schemes, consult [10] for an introduction to affine group schemes.*

Proposition 3.1 ([9])**.** *let R be a ring in which 6 is invertible, let a and b be two elements of R such that $4a^3 + 27b^2$ is invertible in R, the elliptic curve E of equation*

$$Y^2Z = X^3 + aXZ^2 + bZ^3$$

has a unique group scheme structure on $Spec(R)$ whose neutral element is $O = [0 : 1 : 0]$.

Remark 1. According to the previous proposition we can consider the elliptic curve $\Im_{a,b}$ on the ring $\mathbb{Q}[\lambda]$ giving by the weirstrass equation $Y^{*2} = X^{*3} + a * X + b$, where $(a = a_0 + a_1\lambda\ ,\ b = b_0 + b_1\lambda) \in (\mathbb{Q}[\lambda])^2$ and $4a^{*3} + 27b^{*2}$ is invertible in $\mathbb{Q}[\lambda]$.

In what follows, we consider $\Im_0$ and $\Im_1$ two restriction of $\Im_{a,b}$ over $\mathbb{Q}$, defined as follows

$$\Im_0 = \{[X : Y : Z] \in P^2(\mathbb{Q}) | Y^2Z = X^3 + a_0XZ^2 + b_0Z^3\}$$

and

$$\Im_1 = \{[X : Y : Z] \in P^2(\mathbb{Q}) | Y^2Z = X^3 + (a_0 + a_1)XZ^2 + (b_0 + b_1)Z^3\}$$

such that $4a_0^3 + 27b_0^2 \neq 0$ and $4(a_0 + a_1)^3 + 27(b_0 + b_1)^2 \neq 0$. Suppose that we have $\Im_{a,b}$ is an elliptic curve over $\mathbb{Q}[\lambda]$, so we have the following lemmas,

Lemma 3.1. *$\Im_0$ is an elliptic curve over $\mathbb{Q}$.*

Proof. To prove this result we shall prove that $4a_0^3 + 27b_0^2 \neq 0$ if $\triangle = 4a^{*3} + 27b^{*2}$ is invertible in $\mathbb{Q}[\lambda]$. So, $\triangle = 4a^{*3} + 27b^{*2}$, where $a^{*3} = a_0^3 + [(a_0 + a_1)^3 - a_0^3]\lambda$ and $b^{*2} = b_0^2 + [b_1^2 + 2b_0b_1]\lambda$ to simplify the notation put, $a^{*3} = a_0^3 + Q_1\lambda$ and $b^{*2} = b_0^2 + Q_2\lambda$, so we have $\triangle = 4(a_0^3 + Q_1\lambda) + 27(b_0^2 + Q_2\lambda)$ then $\triangle = 4a_0^3 + 27b_0^2 + [4Q_1 + 27Q_2]\lambda$ and since $\triangle$ is invertible it follows from the Proposition 2.1 that $4a_0^3 + 27b_0^2 \neq 0$. □

Lemma 3.2. *$\Im_1$ is an elliptic curve over $\mathbb{Q}$.*

Proof. To prove this result we shall prove that $4[a_0 + a_1]^3 + 27[b_0 + b_1]^2 \neq 0$ if $\triangle = 4a^{*3} + 27b^{*2}$ is invertible in $\mathbb{Q}[\lambda]$. From above we have $\triangle = 4a_0^3 + 27b_0^2 + [4[a_0 + a_1]^3 - 4a_0^3 + 27[b_0 + b_1]^2 - 27b_0^2]\lambda$ and since $\triangle$ is invertible it follows from the Proposition 2.1 that $4[a_0 + a_1]^3 + 27[b_0 + b_1]^2 \neq 0$. □

Theorem 3.1. *$\Im_i$ are elliptic curves over $\mathbb{Q}$ for $i = 0, 1$ if and only if $\Im_{a,b}$ is an elliptic curve over $\mathbb{Q}[\lambda]$.*

Proof. Suppose that $\Im_0$ and $\Im_1$ are elliptic curves, then we have $4a_0^3+27b_0^2 \neq 0$ and $4[a_0 + a_1]^3 + 27[b_0 + b_1]^2 \neq 0$, and from the Proposition 2.1, it follows that $\triangle = 4a_0^3+27b_0^2+[4[a_0+a_1]^3-4a_0^3+27[b_0+b_1]^2-27b_0^2]\lambda$ is invertible over $\mathbb{Q}[\lambda]$.

To show the opposite direction, we use the lemmas 3.1 and 3.2. □

4. The torsion section of elliptic curves over $\mathbb{Q}[\lambda]$

In this section we will give the possible structure of the torsion section of an elliptic curve defined over the ring $\mathbb{Q}[\lambda]$.

Theorem 4.1. *Let $\Im_{a,b}$ be an elliptic curve over $\mathbb{Q}[\lambda]$. So,*

$$Tor(\Im_{a,b}, \mathbb{Q}[\lambda]) \simeq \begin{cases} \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, & n, m = 1, 2, ..., 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, & 1 \leq n \leq 4, m = 1, 2, ..., 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}, & 1 \leq n, m \leq 4. \end{cases}$$

To prove this result we will define a relation between $\Im_{a,b}$ and $\Im_0 \times \Im_1$.

Lemma 4.1. *Let $X = x_0 + x_1\lambda$, $Y = y_0 + y_1\lambda$, $Z = z_0 + z_1\lambda$, $a = a_0 + a_1\lambda$ and $b = b_0 + b_1\lambda$ be elements of $\mathbb{Q}[\lambda]$, then we have $[X : Y : Z]$ is in $P^2(\mathbb{Q}[\lambda])$, if and only if $[x_0 : y_0 : z_0] \in P^2(\mathbb{Q})$, and $[x_0 + x_1 : y_0 + y_1 : z_0 + z_1] \in P^2(\mathbb{Q})$.*

Proof. Suppose that $[X : Y : Z] \in P^2(\mathbb{Q}[\lambda])$, then there exist $(U, V, W) \in (\mathbb{Q}[\lambda])^3$ such that $U * X + V * Y + W * Z = 1$. So, $[u_0x_0 + (u_0x_1 + u_1x_0 + u_1x_1)\lambda] + [v_0y_0 + (v_0y_1 + v_1y_0 + v_1y_1)\lambda] + [w_0z_0 + (w_0z_1 + w_1z_0 + w_1z_1)\lambda] = 1$, then $u_0x_0 + v_0y_0 + w_0z_0 = 1$ and $u_0x_1 + u_1x_0 + u_1x_1 + v_0y_1 + v_1y_0 + v_1y_1 + w_0z_1 + w_1z_0 + w_1z_1 = 0$.

It follows that $(u_0 + u_1)(x_0 + x_1) + (v_0 + v_1)(y_0 + y_1) + (w_0 + w_1)(z_0 + z_1) - (u_0x_0 + v_0y_0 + w_0z_0) = 0$, since $u_0x_0 + v_0y_0 + w_0z_0 = 1$ we have

$$\begin{cases} u_0x_0 + v_0y_0 + w_0z_0 = 1, \\ (u_0 + u_1)(x_0 + x_1) + (v_0 + v_1)(y_0 + y_1) + (w_0 + w_1)(z_0 + z_1) = 1. \end{cases}$$

So, $(x_0, y_0, z_0) \neq (0, 0, 0)$ and $(x_0 + x_1, y_0 + y_1, z_0 + z_1) \neq (0, 0, 0)$, which proves that $[x_0 : y_0 : z_0]$ and $[x_0 + x_1 : y_0 + y_1 : z_0 + z_1]$ are in $P^2(\mathbb{Q})$.

Conversely, let $[x_0 : y_0 : z_0]$, $[x_0 + x_1 : y_0 + y_1 : z_0 + z_1] \in P^2(\mathbb{Q})$. Suppose that $y_0 \neq 0$, then we distinguish between two case of $y_0 + y_1$:

- $y_0 + y_1 \neq 0$: then Y is invertible in $\mathbb{Q}[\lambda]$, so $[X : Y : Z] \in \mathrm{P}^2(\mathbb{Q}[\lambda])$.
- $y_0 + y_1 = 0$: then $x_0 + x_1 \neq 0$ or $z_0 + z_1 \neq 0$. So, without loss of generality, suppose that $x_0 + x_1 \neq 0$ then $Y + \lambda * X \in (\mathbb{Q}[\lambda])^\times$. Hence, $[X : Y : Z] \in P^2(\mathbb{Q}[\lambda])$.

We follow the same proof if $x_0 \neq 0$ or $z_0 \neq 0$. $\square$

Lemma 4.2. *With the same notation as above, we have* $[X : Y : Z]$ *is in* $\Im_{a,b}$ *if and only if* $[x_0 : y_0 : z_0] \in \Im_0$ *and* $[x_0 + x_1 : y_0 + y_1 : z_0 + z_1] \in \Im_1$.

Proof. From the previous lemma we have $[X : Y : Z]$ is in $P^2(\mathbb{Q}[\lambda])$, if and only if $[x_0 : y_0 : z_0] \in P^2(\mathbb{Q})$, and $[x_0 + x_1 : y_0 + y_1 : z_0 + z_1] \in P^2(\mathbb{Q})$.

On the other hand, it remains to show that $[X : Y : Z]$ is a solution of $Y^{*2} * Z = X^{*3} + a * X * Z^{*2} + b * Z^{*3}$ if and only if $[x_0 : y_0 : z_0]$ is a solution of $Y^2Z = X^3 + a_0XZ^2 + b_0Z^3$ and $[x_0 + x_1 : y_0 + y_1 : z_0 + z_1]$ is a solution of $Y^2Z = X^3 + (a_0 + a_1)XZ^2 + (b_0 + b_1)Z^3$.

So, with the same notation as above, we have:

- $Y^{*2} * Z = y_0^2z_0 + ((y_0 + y_1)^2(z_0 + z_1) - y_0^2z_0)\lambda$,
- $X^{*3} = x_0^3 + ((x_0 + x_1)^3 - x_0^3)\lambda$,
- $a * X * Z^{*2} = a_0x_0z_0 + ((a_0 + a_1)(x_0 + x_1)(z_0 + z_1)^2 - a_0x_0z_0)\lambda$
- $b * Z^{*3} = b_0z_0^3 + ((b_0 + b_1)(z_0 + z_1)^3 - b_0z_0^3)\lambda$.

We deduce from the Proposition 2.1 that $Y^{*2} * Z = X^{*3} + a * X * Z^{*2} + b * Z^{*3}$ if and only if $y_0^2z_0 = x_0^3 + a_0x_0z_0^2 + b_0z_0^3$ and $(y_0 + y_1)^2(z_0 + z_1) = (x_0 + x_1)^3 + (a_0 + a_1)(x_0 + x_1)(z_0 + z_1)^2 + (b_0 + b_1)(z_0 + z_1)^3$, hence the result. $\square$

In the following theorem, we will define a bijective application that allows us to connect the curve $\Im_{a,b}$ with the elliptic curves $\Im_0$ and $\Im_1$,

Theorem 4.2. *The mapping*

$$\begin{array}{ccc} \Im_{a,b} & \xrightarrow{\wp} & \Im_0 \times \Im_1 \\ [X : Y : Z] & \longmapsto & ([x_0 : y_0 : z_0], [x_0 + x_1 : y_0 + y_1 : z_0 + z_1]) \end{array}$$

is a bijection.

Proof. From lemma 4.2 it follows that $\wp$ is well defined.

$\wp$ is a surjective map:

Let $[x_0 : y_0 : z_0] \in \Im_0$ and $[x_1 : y_1 : z_1] \in \Im_1$ then

$$[x_0 + (x_1 - x_0)\lambda : y_0 + (y_1 - y_0)\lambda : z_0 + (z_1 - z_0)\lambda] \in \Im_{a,b}$$

so, we have:

$$\begin{aligned} &\wp([x_0 + (x_1 - x_0)\lambda : y_0 + (y_1 - y_0)\lambda : z_0 + (z_1 - z_0)\lambda]) \\ &= ([x_0 : y_0 : z_0], [x_0 + (x_1 - x_0) : y_0 + (y_1 - y_0) : z_0 + (z_1 - z_0)]) \\ &= ([x_0 : y_0 : z_0], [x_1 : y_1 : z_1]), \end{aligned}$$

hence $\wp$ is a surjective mapping.

$\wp$ is injective, for that lets $[X : Y : Z]$ and $[X' : Y' : Z']$ in $E_{a,b}$, where $X = x_0 + x_1\lambda$, $Y = y_0 + y_1\lambda$, $Z = z_0 + z_1\lambda$, $X' = x'_0 + x'_1\lambda$, $Y' = y'_0 + y'_1\lambda$ and $Z' = z'_0 + z'_1\lambda$. So, if $[x_0 : y_0 : z_0] = [x'_0 : y'_0 : z'_0]$ and $[x_0 + x_1 : y_0 + y_1 : z_0 + z_1] = [x'_0 + x'_1 : y'_0 + y'_1 : z'_0 + z'_1]$ then there exist β_0, $\beta 1 \in \mathbb{Q}^\times$ such that $x_0 = \beta_0 x'_0$, $y_0 = \beta_0 y'_0$, $z_0 = \beta_0 z'$ and $x_0 + x_1 = \beta_1(x'_0 + x'_1)$, $y_0 + y_1 = \beta_1(y'_0 + y'_1)$, $z_0 + z_1 = \beta_1(z'_0 + z'_1)$. Consider $\beta = \beta_0 + (\beta_1 - \beta_0)\lambda$, it follows that

$$\begin{cases} x_0 = \beta_0 x'_0, \\ y_0 = \beta_0 y'_0, \\ z_0 = \beta_0 z'_0 \end{cases}$$

and

$$\begin{cases} x_1 = \beta_1 x'_1 + x'_0(\beta_1 - \beta_0), \\ y_1 = \beta_1 y'_1 + y'_0(\beta_1 - \beta_0), \\ z_1 = \beta_1 z'_1 + z'_0(\beta_1 - \beta_0). \end{cases}$$

So, we have $X = \beta * X'$, $Y = \beta * Y'$, $Z = \beta * Z'$ and $\beta \in \mathbb{Q}[\lambda]^\times$ then $[X : Y : Z] = [X' : Y' : Z']$. Hence, $\wp$ is a bijection. We can show that the mapping $\wp^{-1}$ defined by:

$$\begin{aligned} &\wp^{-1}([x_0 : y_0 : z_0], [x_1 : y_1 : z_1]) \\ &= [x_0 + (x_1 - x_0)\lambda : y_0 + (y_1 - y_0)\lambda : z_0 + (z_1 - z_0)\lambda] \end{aligned}$$

is the inverse of $\wp$. □

4.1 The group law $\star$ over $\Im_{a,b}$

To define the group law $\star$ over $\Im_{a,b}$, we use the explicit formulas in the article [1] [pages : 236-238], and since $\wp$ is bijection we can define $\star$ as follows $P \star Q = \wp^{-1}(\wp(P) + \wp(Q))$ for P, $Q \in \Im_{a,b}$.

Corollary 4.1. *The mapping*

$$\begin{array}{lcl} (\Im_{a,b}, \star) & \xrightarrow{\wp} & (\Im_0 \times \Im_1, +) \\ {[X : Y : Z]} & \longmapsto & ([x_0 : y_0 : z_0], [x_0 + x_1 : y_0 + y_1 : z_0 + z_1]) \end{array}$$

is an isomorphism of groups.

Proof. From the previous theorem we have $\wp$ is a bijection and according to the construction of the group law over $\Im_{a,b}$ we have $\wp([X : Y : Z] \star [X' : Y' : Z']) = \wp([X : Y : Z]) + \wp([X' : Y' : Z'])$. So, $\wp$ is an isomorphism of groups. □

Proposition 4.1. *Let $P = [X : Y : Z] \in \Im_{a,b}$ such that $X = x_0 + x_1\lambda$, $Y = y_0 + y_1\lambda$ and $Z = z_0 + z_1\lambda$, so $P \in Tor(\Im_{a,b})$ if and only if $P_0 \in Tor(\Im_0)$ and $P_1 \in Tor(\Im_1)$, where $P_0 = [x_0 : y_0 : z_0]$ and $P_1 = [x_0 + x_1 : y_0 + y_1 : z_0 + z_1]$.*

Proof. Let $P \in Tor(\Im_{a,b})$ then there exist an integer m such that $mP = P\star...\star P = O$, so $\wp^{-1}(\wp(P)+...+\wp(P)) = O$ we obtain $(P_0, P_1)+...+(P_0, P_1) = \wp(O) = (O_0, O_1)$, then $mP_0 = O_0$ and $mP_1 = O_1$, hence $P_0 \in Tor(\Im_0)$ and $P_1 \in Tor(\Im_1)$. On the other hand, if there exist an integers m, n such that $mP_0 = O_0$ and $nP_1 = O_1$, we have $mnP = \wp^{-1}((P_0, P_1) + ... + (P_0, P_1) = \wp^{-1}((mnP_0, mnP_1)) = \wp^{-1}(O_0 \times O_1) = O$. □

Corollary 4.2. *With the same notation as above we have* $\wp(Tor(\Im_{a,b})) = Tor(\Im_0) \times Tor(\Im_1)$.

Proposition 4.2. *According to the above we have* $Tor(\Im_{a,b}) \simeq Tor(\Im_0) \times Tor(\Im_1)$.

Proof. Put

$$\begin{array}{ccc} Tor(\Im_{a,b}) & \overset{\wp/Tor(\Im_{a,b})}{\longrightarrow} & Tor(\Im_0) \times Tor(\Im_1) \\ P & \longmapsto & \wp(P). \end{array}$$

the $\wp$-restriction on the torsion section of $\Im_{a,b}$. From the theorem 4.2 and the previous lemmas we have $\wp/_{Tor(\Im_{a,b})}$ is an isomorphism of groups, hence the result. □

Proof of Theorem 4.1. *From the previous proposition we have* $Tor(\Im_{a,b}) \simeq Tor(\Im_0) \times Tor(\Im_1)$*, and from the Mazur's theorem [6] we deduce the result.* □

Example 1. Let λ be a root of the polynomial $P(X) = X^2 + 2$, let $a = -676 + 648\lambda$ and $b = 13662 - 4968\lambda$ two elements in $\mathbb{Q}[\lambda]$. So, let $\Im_{a,b}$ the Elliptic curve defined by $Y^{*2} * Z = X^{*3} + a * X * Z^{*2} + b * Z^{*3}$ over $\mathbb{Q}[\lambda]$. We consider $\Im_0$ and $\Im_1$ two restriction of $\Im_{a,b}$ over $\mathbb{Q}$, defined as follows $\Im_0 = \{[X : Y : Z] \in P^2(\mathbb{Q}) | Y^2 Z = X^3 - 675XZ^2 + 13662Z^3\}$ and $\Im_1 = \{[X : Y : Z] \in P^2(\mathbb{Q}) | Y^2 Z = X^3 - 27XZ^2 + 8694Z^3\}$. So, using the magma calculator, we find that

	$\triangle$	j	$\Im_i(\mathbb{Q})_{tor}$	Generator of $\Im_i(\mathbb{Q})_{tor}$
$\Im_0$	-2.14	$-\frac{5^6}{2.14}$	$\mathbb{Z}_6$	$(1, -2)$
$\Im_1$	-15	$-\frac{1}{15}$	$\mathbb{Z}_4$	$(15, 108)$

Hence,

	$\triangle$	j	$\Im_{a,b}(\mathbb{Q}[\lambda])_{tor}$	Generator of $\Im_{a,b}(\mathbb{Q}[\lambda])_{tor}$
$\Im_{a,b}$	$-2.14 + 13\lambda$	$\frac{-5^7.3+23.10189\lambda}{2^2.3.5.7}$	$\mathbb{Z}_4 \times \mathbb{Z}_6$	$(1 + 14\lambda, -2 + 110\lambda)$

5. Conclusion

In this paper, we have study an elliptic curve $\Im_{a,b}$ given by a Weierstrass equation $Y^{*2} * Z = X^{*3} + a * X * Z^2 + b * Z^3$ over $(\mathbb{Q}[\lambda], +, *)$ and determine all possible torsion sections of this elliptic curve. So,

$$Tor(\Im_{a,b}, \mathbb{Q}[\lambda]) \simeq \begin{cases} \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, & n, m = 1, 2, ..., 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, & 1 \le n \le 4, m = 1, 2, ..., 10, 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}, & 1 \le n, m \le 4. \end{cases}$$

In later work we will explain how our methods and results can be used to give a new encryption scheme. We expect that these methods and results can be used in many other settings.

Acknowledgments

We thank the referee by your suggestions.

References

[1] W. Bosma, H. W. Lenstra, *Complete system of two addition laws for elliptic curves*, Journal of Number Theory, 53 (1995), 229-240.

[2] S. Kamienny, *Torsion points on elliptic curves and q-coefficients of modular forms*, Invent. Math., 109 (1991), 221-229.

[3] S. Kamienny, F. Najman, *Torsion groups of elliptic curves over quadratic fields*, Acta Arith., 152 (2012), 291-305.

[4] M. A. Kenku, F. Momose, *Torsion points on elliptic curves defined over quadratic fields,* Nagoya Math. J., 109 (1988), 125-149.

[5] B. Mazur, *Modular curves and Eisenstein ideal*, IHES Publ. Math., 47 (1977), 33-186.

[6] F. Najmam, *Torsion of elliptic curves over quadratic cyclotomic field*, Math. J. Okayama Univ., 53 (2011), 75-82.

[7] F. Najmam, *Complete classification of torsion of elliptic curves over quadratic cyclotomic field*, J. Num. Th., 130 (2010), 1964-1968.

[8] N. K. Sarma, A. Saikia, *Torsion of elliptic curves over quadratic fields of class number 1*, Rocky Mountain Journal of Mathematics, 48 (2018).

[9] M. Virat, *Courbe elliptique sur un annean et applications cryptographiques*, Nice-Sophia Antipolis, (Thèse du Doctoral), 2009.

[10] W. C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, New York-Berlin: Springer-Verlag, 66 (1979).

Accepted: May 16, 2022

On k-perfect polynomials over $\mathbb{F}_2$

Haissam Chehade*
The International University of Beirut
School of Arts and Sciences
Department of Mathematics and Physics
Lebanon
haissam.chehade@liu.edu.lb

Yousuf Alkhezi
The Public Authority for Applied Education and Training
College of Basic Education
Department of Mathematics
Kuwait
ya.alkhezi@paaet.edu.kw

Wiam Zeid
Lebanese International University
School of Arts and Sciences
Department of Mathematics and Physics
Lebanon
wiam.zeid@liu.edu.lb

Abstract. A polynomial A is called $k-$perfect over the finite field $\mathbb{F}_2$ if the sum of the k^{th} powers of all distinct divisors of A equals A^k, where k is a positive integer. We show that a $k-$perfect polynomial A over $\mathbb{F}_2$ must be even when $k = 2^n$, n is a non-negative integer, and we characterize all 2^n-perfect polynomials over $\mathbb{F}_2$ that are of the form $x^a(x+1)^b \prod_{i=1}^{r} P_i^{h_i}$, where each P_i is a Mersenne prime and a, b and h_i are positive integers.

Keywords: sum of divisors, multiplicative function, polynomials, finite fields, characteristic 2.

1. Introduction

Let n be a positive integer and let $\sigma(n)$ denote the sum of positive divisors of the integer n. We call the number n a $k-$super perfect number if $\sigma^k(n) = \underbrace{\sigma(\sigma(...(\sigma}_{k-\text{times}}(n)))) = 2n$. When $k = 1$, n is called a perfect number. An integer $M = 2^p - 1$, where p is a prime number, is called a Mersenne number. It is also well known that an even integer n is perfect if and only if $n = M(M+1)/2$ for some Mersenne prime number M. Suryanarayana [11] considered $k-$super perfect numbers in the case $k = 2$. Numbers of the form 2^{p-1} (p is prime) are

∗. Corresponding author

2-super perfect if $2^{p-1}-1$ is a Mersenne prime. It is not known if there are odd $k-$super perfect numbers.

Researchers also studied the arithmetic function $\sigma_k(n)$ that finds the sum of the kth powers of the positive divisors of n. Recently, Luca and Ferdinands [10] showed that $\sigma_k(n)$ is divisible by n for infinitely many n when $k \geq 2$. Cai et al. [1] proved that if $n = 2^{a-1}p$ divides $\sigma_3(n)$, where $a > 1$ is an integer and p is an odd prime, then n is an even perfect number. Also, they proved that the converse is true when $n \neq 28$. Jiang [9] made an improvement to the result of Cai et al. They showed that $n = 2^{a-1}p^{b-1}$ divides $\sigma_3(n)$, where $a, b > 1$ are integers and p is an odd prime, if and only if n is an even perfect number other than 28. Chu [3] found a relation between an even perfect number n and $\sigma_k(n)$. He generalized the work of Cai et al. as given in the following theorem.

Theorem 1.1. *Let $k > 2$ be a prime such that $2^k - 1$ is a Mersenne prime. If $n = 2^{a-1}p$, where $a > 1$ and $p < 3{\cdot}2^{a-1} - 1$ is an odd prime. Then n divides $\sigma_k(n)$ if and only if n is an even perfect number other than $2^{k-1}(2^k - 1)$.*

Chu also generalized the work of Jiang as follows.

Theorem 1.2. *If $n = 2^{a-1}p^{b-1}$, where $a, b > 1$ and $p < 3{\cdot}2^{a-1} - 1$ is an odd prime. Then n divides $\sigma_5(n)$ if and only if n is an even perfect number other than 496.*

Chu conjectured if $k > 2$ is a prime such that $2^k - 1$ is a Mersenne prime and if $n = 2^{a-1}p^{b-1}$, where $a, b > 1$ and $p < 3.2^{a-1} - 1$ is an odd prime, then n divides $\sigma_k(n)$ if and only if n is an even perfect number other than $2^{k-1}(2^k - 1)$.

The present paper gives a polynomial analogue of the arithmetic function $\sigma_k(n)$. Let k be a positive integer and let A be a nonzero polynomial defined over the prime field $\mathbb{F}_2$. We denote by $\sigma_k(A)$ the sum of the k^{th} powers of the distinct divisors B of A. That is,

$$\sigma_k(A) = \sum_{B|A} B^k.$$

If $A \in \mathbb{F}_2[x]$ has the canonical decomposition $\prod_{i=1}^{r} P_i^{\alpha_i}$ where the primes $P_i \in \mathbb{F}_2[x]$ are distinct and $\alpha_i > 0$, then

$$\sigma_k(A) = \prod_{i=1}^{r} \frac{P_i^{k(\alpha_i+1)} - 1}{P_i^k - 1}.$$

In the case where $k = 1$, σ_k becomes the well-known σ function. For example, if $A = x(x+1)^2(x^2+x+1) \in \mathbb{F}_2[x]$ then

$$\begin{aligned}\sigma(A) &= \sum_{B|A} B \\ &= 1 + x + (x+1) + (x+1)^2 + (x^2+x+1) + x(x+1) + x(x+1)^2\end{aligned}$$

$$+ x(x^2+x+1) + (x+1)(x^2+x+1) + (x+1)^2(x^2+x+1)$$
$$+ x(x+1)(x^2+x+1) + x(x+1)^2(x^2+x+1)$$
$$= x(x+1)^2(x^2+x+1)$$

and

$$\sigma_4(A) = \sum_{B|A} B^4 = x^4(x+1)^8(x^2+x+1)^4.$$

Note that the function σ_k is multiplicative over $\mathbb{F}_2$.

Notation 1.1. *We use the following notations throughout the paper.*

- $\deg(A)$ *denotes the degree of the polynomial* A.
- $\overline{A}$ *is the polynomial obtained from* A *with* x *replaced by* $x+1$, *that is* $\overline{A}(x) = A(x+1)$.
- A^* *is the inverse of the polynomial* A *with* $\deg(A) = m$, *in this sense* $A^*(x) = x^m A(\frac{1}{x})$.
- P *and* Q *are distinct irreducible odd polynomials.*

A nonzero polynomial A defined over $\mathbb{F}_2$ is an even polynomial if it has a linear factor in $\mathbb{F}_2[x]$ else it is an odd polynomial. A polynomial T of the form $1 + x^a(x+1)^b$ with $\gcd(a,b) = 1$ is called a Mersenne polynomial, see [6]. The first five Mersenne polynomials over $\mathbb{F}_2$ are: $T_1 = 1+x+x^2$, $T_2 = 1+x+x^3$, $T_3 = 1+x^2+x^3, T_4 = 1+x+x^2+x^3+x^4$, $T_5 = 1+x^3+x^4$. Note that all these polynomials are irreducible, so we call them Mersenne primes.

The next definition is the main object of this study in which we introduce a new concept of $k-$perfect polynomials over $\mathbb{F}_2$.

Definition 1.1. *Let k be a positive integer. A polynomial A is called a $k-$perfect polynomial over $\mathbb{F}_2$ if $\sigma_k(A) = A^k$.*

A $1-$perfect polynomial A over $\mathbb{F}_2$ is a perfect polynomial, so we are interested in studying the case when $k > 1$. The polynomial $B = x(x+1)^2(x^2+x+1)$ is a $4-$perfect polynomial in $\mathbb{F}_2[x]$. Note that B is a perfect polynomial over $\mathbb{F}_2$. A natural question arise: Is there a relation between perfect polynomials and $k-$perfect polynomials in $\mathbb{F}_2[x]$? In Section 3, we answer this question and we find a relation between the sum of the divisors function $\sigma(A)$ and the sum of the powers of the divisors function $\sigma_k(A)$, $k > 1$, of the polynomial A over the finite field $\mathbb{F}_2$. We show that there are no odd 2^n-perfect polynomials over $\mathbb{F}_2$ and we characterize all even 2^n-perfect polynomials over $\mathbb{F}_2$ that have the form $x^a(x+1)^b \prod_{i=1}^{r} P_i^{h_i}$, where each P_i is a Mersenne prime and a, b and h_i are positive integers.

Our main result is given in the following theorem:

Theorem 1.3. *Let $a, b, t, h_i \in \mathbb{N}$ and let P_i be a Mersenne prime in $\mathbb{F}_2[x]$. Then, $A = x^a(x+1)^b \prod_{i=1}^{r} P_i^{h_i}$ is a 2^n-perfect polynomial over $\mathbb{F}_2$ for some $n \in \mathbb{N}$ if and only if $A \in \{x^{2^t-1}(x+1)^{2^t-1}, x^2(x+1)T_1, x(x+1)^2T_1, x^3(x+1)^4T_5, x^4(x+1)^3T_4, x^4(x+1)^4T_4T_5, x^6(x+1)^3T_2T_3, x^3(x+1)^6T_2T_3, x^6(x+1)^4T_2T_3T_5, x^4(x+1)^6T_2T_3T_5\}$.*

2. Preliminaries

The notion of perfect polynomials over $\mathbb{F}_2$ was introduced first by Canaday [2]. A polynomial A is perfect if $\sigma(A) = A$. Let $\omega(A)$ be the number of distinct irreducible polynomials that divide A. Canaday studied the case of even perfect polynomials with $\omega(A) \leq 3$. In the recent years, Gallardo and Rahavandrainy [4, 6, 7] showed the non-existence of odd perfect polynomials over $\mathbb{F}_2$ with either $\omega(A) = 3$ or with $\omega(A) \leq 9$ in the case where all the exponents of the irreducible factors of A are equal to 2. If the nonconstant polynomial A in $\mathbb{F}_2[x]$ is perfect, then $\omega(A) \geq 2$ (see [4], Lemma 2.3). Moreover, Canaday [2] showed that the only even perfect polynomials over $\mathbb{F}_2$ with exactly two prime divisors are $x^{2^n-1}(x+1)^{2^n-1}$ for some positive integers n.

It is well known that an even perfect number is exactly divisible by two distinct prime numbers but a non-trivial even perfect polynomial $A \in \mathbb{F}_2[x]$ may be divisible by more than 2 distinct primes as Gallardo and Rahavandrainy [6] gave some results with $\omega(A) \leq 5$. Although they did not give a general form of such polynomials in terms of Mersenne primes but all the non-trivial even perfect polynomials they found, with only two exceptions, have Mersenne primes as odd divisors.

The following two lemmas are useful.

Lemma 2.1 (Lemma 2.3 in [6]). *If $A{=}A_1A_2$ is perfect over $\mathbb{F}_2$ and if* $\gcd(A_1, A_2) = 1$*, then A_1 is perfect if and only if A_2 is perfect.*

Lemma 2.2 (Lemma 2.4 in [6]). *If A is perfect over $\mathbb{F}_2$, then the polynomial $\overline{A}$ is also perfect over $\mathbb{F}_2$*

In [5], Gallardo and Rahavandrainy gave a complete list for all even perfect polynomials with at most 5 irreducible factors as given in the following lemma.

Lemma 2.3. *The complete list of all even perfect polynomials over $\mathbb{F}_2$ with $\omega(A) \leq 5$ is:*

$\omega(A)$	A
0	0
1	1
2	$(x^2+x)^{2^n-1}$
3	$A_1 = x^2(x+1)T_1, A_2 = \overline{A_1}(x), A_3 = x^3(x+1)^4T_5, A_4(x) = \overline{A_3}$
4	$A_5 = x^2(x+1)(x^4+x+1)T_1^2, A_6 = \overline{A_5},$
	$A_7 = x^4(x+1)^4T_4T_5, A_8 = x^6(x+1)^3T_2T_3, C_9(x) = \overline{A_8}$
5	$A_{10} = x^6(x+1)^4T_2T_3T_5,\ A_{11} = \overline{A_{10}}.$

Lemma 2.4 (Proposition 5.1 in [6])**.** *If P is an odd irreducible polynomial in $\mathbb{F}_2[x]$, then $x(x+1)$ divides $\sigma(P^{2m-1})$ for $m \in \mathbb{N}$.*

The following lemma shows a nice relation between $\sigma_k(A)$ and $(\sigma(A))^k$ when A has exactly one prime factor.

Lemma 2.5. *Let $A = P^\alpha \in \mathbb{F}_2[x]$ with $\alpha \geq 1$. Then $\sigma_k(A) = \sigma(A)^k$ if and only if $k = 2^n$.*

Proof.

$$\sigma_{2^n}(A) = 1 + P^{2^n} + ... + P^{2^n\alpha} = (1 + P + ... + P^\alpha)^{2^n} = (\sigma(A))^{2^n}.$$

For the sufficient condition, the proof is done by contrapositive. Let $k = 2^n u$, $u > 1$ is odd, then $(\sigma(A))^k = (\sigma(A))^{2^n u} = (1 + P + ... + P^\alpha)^{2^n u} = (1 + P^{2^n} + ... + P^{2^n\alpha})^u \neq (1 + P^{2^n u} + ... + P^{2^n u\alpha}) = \sigma_k(A)$. □

Corollary 2.1. *Let $A = \prod_{i=1}^r P_i^{\alpha_i} \in \mathbb{F}_2[x]$, then $\sigma_{2^n}(A) = (\sigma(A))^{2^n}$.*

Lemma 2.6. *Let $A = P^\alpha \in \mathbb{F}_2[x]$ be an irreducible polynomial and $\alpha \geq 1$. Then A is not a factor of $\sigma_k(A)$.*

Proof. Assume that A divides $\sigma_k(A)$, then there exists a nonconstant $B \in \mathbb{F}_2[x]$ such that $\sigma_k(A) = AB$ with $\deg(B) < \deg(A^k)$. So, $1+P^k+...+P^{k(\alpha-1)}+P^{k\alpha} = P^\alpha B$ and $P\left(P^{k-1} + ... + P^{k(\alpha-1)-1} + P^{\alpha-1}(P^k + B)\right) = 1$. Hence, $P = 1$ and this contradicts the fact that P is prime in $\mathbb{F}_2[x]$. □

Lemma 2.7 (Lemma 2.6 in [8])**.** *Let m be a positive integer and let T be a Mersenne prime in $\mathbb{F}_2[x]$, then $\sigma(x^{2m})$ and $\sigma(T^{2m})$ are both odd and squarefree.*

Lemma 2.8. *If m and k are positive integers, then $\sigma_k(P^{2m-1})$ is divisible by $x(x+1)$.*

Proof. Let $2m = 2^h s$, where s is odd and $h \geq 1$. Then,

$$\begin{aligned}\sigma_k(P^{2m-1}) &= 1 + P^k + ... + P^{k\left(2^h s-1\right)} \\ &= (1 + P^k)^{2^h-1}\left(1 + P^k + ... + P^{k(s-1)}\right)^{2^h}\end{aligned}$$

But $x(x+1)$ divides $1 + P^k$, P is odd. This completes the proof. □

Lemma 2.9. *If m and k are positive integers, then $\sigma_k(P^{2m})$ is not divisible by $x(x+1)$.*

Proof. $\sigma_k(P^{2m})=1+P^k+...+P^{2km}$. So, $\sigma_k(P^{2m})(0)=1+\underbrace{P^k(0)+...+P^{2km}(0)}_{2m-\text{times}} = 1$ and x is not factor of $\sigma_k(P^{2m})$. Also, $\sigma_k(P^{2m})(1) = 1$ and hence $\sigma_k(P^{2m})$ is not divisible by $x + 1$. The proof is now complete. □

Next we give some properties when $k = 2$.

Lemma 2.10. *Let t be a positive integer, then $\sigma_2(x^{3.2^{t-1}-1}) = (1+x)^{2^t-2}T_1^{2^t}$.*

Proof. We use induction. For $t = 1$, we have $\sigma_2(x^2) = (1+x+x^2)^2 = T_1^2$. Hence, the statement is true for $t = 1$. Now assume it is true for t, so

$$\begin{aligned}\sigma_2\left(x^{3.2^t-1}\right) &= \left(1+x+...+x^{3.2^{t-1}-1}+x^{3.2^{t-1}}\left(1+x+...+x^{3.2^{t-1}-1}\right)\right)^2\\ &= \left(1+x+...+x^{3.2^{t-1}-1}\right)^2\left(1+x^{3.2^{t-1}}\right)^2\\ &= \sigma_2\left(x^{3.2^{t-1}-1}\right)\left(1+x^3\right)^{2^t}\\ &= (1+x)^{2^t-2}T_1^{2^t}\left((1+x)\,T_1\right)^{2^t}\\ &= (1+x)^{2^{t+1}-2}T_1^{2^{t+1}}.\end{aligned}$$

We are done. □

Lemma 2.11. *Let t be a positive integer, then $\sigma_2((1+x)^{3.2^{t-1}-1}) = x^{2^t-2}T_1^{2^t}$.*

Lemma 2.12. *Let t be a positive integer, then $\sigma_2(T_1^{2^t-1}) = (x^2+x)^{2(2^t-1)}$.*

Proof. For $t = 1$, we have $\sigma_2(T_1) = (1+T_1)^2 = (x^2+x)^2$. Hence, the statement is true for $t = 1$. Now assume $\sigma_2(T_1^{2^t-1}) = (x^2+x)^{2(2^t-1)}$. And,

$$\begin{aligned}\sigma_2\left(T_1^{2^{t+1}-1}\right) &= \left(1+T_1+...+T_1^{2^t-1}+T_1^{2^t}\left(1+T_1+...+T_1^{2^t-1}\right)\right)^2\\ &= \left(1+T_1+...+T_1^{2^t-1}\right)^2\left(1+T_1^{2^t}\right)^2\\ &= \sigma_2\left(T_1^{2^t-1}\right)(1+T_1)^{2^{t+1}}\\ &= (x^2+x)^{2(2^t-1)}\left(x^2+x\right)^{2^{t+1}}\\ &= (x^2+x)^{2(2^{t+1}-1)}.\end{aligned}$$

The proof is complete. □

The following lemma follows directly from Lemmas 2.10, 2.11, and 2.12.

Lemma 2.13. *Let $t \in \mathbb{N}$ and let $A = x^aT_1^h$ or $A = (1+x)^aT_1^h$ be polynomials in $\mathbb{F}_2[x]$, where $a = 3.2^{t-1}-1$ and $h = 2^t-1$. Then $\sigma_2(A) = x^{2h}(1+x)^{2(a-1)}T_1^{h+1}$.*

Lemma 2.14. *If $a = 2^tu-1$ with u odd. Then,*

i- $\sigma_2(x^a) = (1+x)^{2^{t+1}-2}\left(\sigma\left(x^{u-1}\right)\right)^{2^{t+1}}$

ii- $\sigma_2(P^a) = (1+P)^{2^{t+1}-2}\left(\sigma\left(P^{u-1}\right)\right)^{2^{t+1}}$.

Lemma 2.15. *Let $t \in \mathbb{N}$ and let $A = x^a T_1^h$ or $A = (1+x)^a T_1^h \in \mathbb{F}_2[x]$. If A divides $\sigma_2(A)$, then $a = 3.2^{t-1} - 1$ and $h = 2^t - 1$.*

Definition 2.1. *Let $A \in \mathbb{F}_2[x]$ be a polynomial of degree m. Then,*

i. A inverts into itself if $A^* = A$.

ii. A is said to be $k-$complete if there exists $h \in \mathbb{N}^*$ such that $A = \sigma_k(x^h) = 1 + x^k + ... + x^{kh}$.

Lemma 2.16. *i. Any $k-$complete polynomial inverts to itself.*

ii. If $1 + x^k + ... + x^{km} = PQ$, then $P = P^*$ and $Q = Q^*$ or $P = Q^*$ and $Q = P^*$, where P and Q are irreducible polynomials in $\mathbb{F}_2[x]$.

Proof. i. Let A be a $k-$complete polynomial, then there exists $h \in \mathbb{N}$ such that

$$\begin{aligned} A &= \sigma_k(x^h) \\ &= 1 + x^k + ... + x^{kh} \\ A^* &= x^{kh} A\left(\frac{1}{x}\right) \\ &= x^{kh}\left(1 + \frac{1}{x^k} + ... + \frac{1}{x^{kh}}\right), A \text{ is } k-\text{complete} \\ &= A. \end{aligned}$$

Hence, A inverts to itself.

ii. If $1 + x^k + ... + x^{km} = PQ$, then PQ is $k-$complete. Using the above results, then PQ inverts to itself. Hence, $(PQ)^* = PQ = P^*Q^*$. Therefore, $P = P^*$ and $Q = Q^*$ or $P = Q^*$ and $Q = P^*$. □

3. Proof of Theorem 1.3

The following lemma is a direct consequence of Lemma 2.6.

Lemma 3.1. *The polynomial $A = P^\alpha$, $\alpha \geq 1$, is not a $k-$perfect polynomial over $\mathbb{F}_2$, for every $k \geq 1$.*

The preceding lemma shows that a $k-$perfect polynomial A over $\mathbb{F}_2$ has at least 2 prime factors.

Lemma 3.2. *Let $m \leq n$ be positive integers and let $A \in \mathbb{F}_2[x]$, then $\sigma_{2^m}(A)$ divides $\sigma_{2^n}(A)$.*

Proof.

$$\begin{aligned} \sigma_{2^n}(A) &= (\sigma(A))^{2^n} \\ &= (\sigma(A))^{2^m}(\sigma(A))^{2^{n-m}} \\ &= \sigma_{2^m}(A)(\sigma(A))^{2^{n-m}}. \end{aligned}$$

□

Notice that $\sigma_2(A)$ divides $\sigma_{2^n}(A)$ for any any $n \geq 1$. Hence, if A is a multi-perfect polynomial over $\mathbb{F}_2$, i.e. A divides $\sigma(A)$, then A is a $k-$multi-perfect polynomial over $\mathbb{F}_2$ when $k = 2^n$ for a positive integer n.

Lemma 3.3. *If $t \in \mathbb{N}$ and $A = x^a T_1^h$ or $A = (1+x)^a T_1^h$ be polynomials in $\mathbb{F}_2[x]$, where $a = 3.2^{t-1} - 1$ and $h = 2^t - 1$, then A divides $\sigma_{2^n}(A)$ for any $n \geq 1$.*

Proof. Since σ_2 divides σ_{2^n} and $\sigma_2(A) = x^{2h}(1+x)^{2(a-1)} T_1^{h+1}$ with $2h = a + 2^{t-1} - 1$. □

Lemma 3.4. *If $a = 2^t u - 1$ with u odd and $n \in \mathbb{Z}_{\geq 0}$). Then,*

i- $1 + x$ divides $\sigma_{2^n}(x^a)$

ii- $x(1+x)$ divides $\sigma_{2^n}(P^a)$

Proof. We have $\sigma_2(A)$ divides $\sigma_{2^n}(A)$ and $1 + x$ divides $\sigma_2(A)$ (Lemma 2.14). □

Lemma 3.5. *If A is $k-$perfect over $\mathbb{F}_2$, then $\overline{A}$ is also $k-$perfect over $\mathbb{F}_2$.*

Proof. Let $A(x) = \prod_{i=1}^{r} P_i^{\alpha_i}(x)$, where the primes $P_i(x) \in \mathbb{F}_2[x]$. Since A is $k-$perfect, then

(1) $$\sigma_k(A) = \prod_{i=1}^{r} \frac{P_i^{k(\alpha_i+1)} - 1}{P_i^k - 1} = A^k.$$

Let F_{2^t} be a splitting field for $A(x)$ over $\mathbb{F}_2$, then there exists $a_1, a_2, ..., a_k \in F_{2^t}$ such that for each i, $1 \leq i \leq k$, we have $P_i^{\alpha_i}(x) = \prod_{j=0}^{\beta_i - 1}(x - a_i^{2^j})^{\alpha_i}$, where $\deg(P_i(x)) = \beta_i$. Since $\gcd(P_i(x), P_j(x)) = 1$ over $\mathbb{F}_2$, for every $i \neq j$, then $\gcd(P_i(x), P_j(x)) = 1$ over F_{2^t}, for every $i \neq j$. Moreover,

$$P_i(x+1) = \prod_{j=0}^{\beta_i - 1}(x + 1 - a_i^{2^j}) = \prod_{j=0}^{\beta_i - 1}(x - (a_i - 1)^{2^j}).$$

Since $a_i - 1$ has degree β_i, it follows that each $Q_i(x) = P_i(x+1)$ is prime of degree β_i in $\mathbb{F}_2[x]$. We have $\gcd(Q_i(x), Q_j(x)) = 1$ in $\mathbb{F}_2[x]$, for every $i \neq j$, and hence the primes $Q_i(x)$ are distinct. Let $B(x) = \overline{A}(x) = \prod_{i=1}^{r} P_i^{\alpha_i}(x+1) = \prod_{i=1}^{r} Q_i^{\alpha_i}(x)$.

By substituting $B(x)$ in (1), we get

$$\begin{aligned}
\sigma_k(\overline{A}(x)) &= \sigma_k(B(x)) \\
&= \prod_{i=1}^{r} \frac{P_i^{k(\alpha_i+1)}(x+1) - 1}{P_i^k(x+1) - 1} \\
&= \prod_{i=1}^{r} \frac{Q_i^{k(\alpha_i+1)}(x) - 1}{Q_i^k(x) - 1} \\
&= B^k(x) \\
&= (\overline{A}(x))^k.
\end{aligned}$$

So, $B(x) = \overline{A}(x)$ is $k-$perfect over $\mathbb{F}_2$ □

Lemma 2.1 shows the relation between $\sigma_k(A)$ and $\sigma(A)$ when $k = 2^n$, and its important consequence, Theorem 3.1, completely characterizes all $k-$perfect polynomials over $\mathbb{F}_2$ when $k = 2^n$.

Theorem 3.1. *A is perfect over $\mathbb{F}_2$ if and only if A is 2^n-perfect over $\mathbb{F}_2$.*

Proof. Let $A = \prod_{i=1}^{r} P_i^{\alpha_i} \in \mathbb{F}_2[x]$ be a perfect polynomial over $\mathbb{F}_2$, where P_i is an irreducible polynomial, then

$$\sigma_{2^n}(A) = (\sigma(A))^{2^n} = A^{2^n}.$$

The converse is done by contrapositive. Assume that A is not perfect. Then,

$$\sigma_{2^n}(A) = (\sigma(A))^{2^n} \neq A^{2^n},$$

and we are done. □

Lemma 3.6. *Let $\omega(A) \geq 2$ and let A be a 2^n-perfect polynomial over $\mathbb{F}_2$, then $x(x+1)$ divides A.*

The proof of the following lemma can be done by a direct computation.

Lemma 3.7. *Let t be a positive integer, then the polynomial $x^{2^t-1}(x+1)^{2^t-1}$ is 2^n-perfect over $\mathbb{F}_2$.*

Lemma 3.8. *If $A = A_1A_2$ is 2^n-perfect over $\mathbb{F}_2$ and if $gcd(A_1, A_2) = 1$, then A_1 is 2^n-perfect if and only if A_2 is 2^n-perfect.*

The following lemma contains some interesting results from Canaday's paper (see [2], Lemma 6 and Theorem 8).

Lemma 3.9. *Let $A, B \in \mathbb{F}_2[x]$ and let $n, m \in \mathbb{N}$.*

(*i*) *If $\sigma(P^{2n}) = B^m A$, with $m > 1$ and $A \in \mathbb{F}_2[x]$ is nonconstant, then* $\deg(A)(P) > \deg(A)(B)$.

(*ii*) *If $\sigma\left(x^{2n}\right)$ has a Mersenne factor, then $n \in \{1, 2, 3\}$.*

Gallardo and Rahavandrainy [6] conjectured that $\sigma(T^{2m})$ is always divisible by a non-Mersenne prime, for any $m \in \mathbb{N}$, when $T = x^a(x+1)^b+1$ is a Mersenne prime with $a + b \neq 3$.

Lemma 3.10. *Let $A = x^a(x+1)^b \prod_i P_i^{h_i}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$ with each P_i is a Mersenne prime. Then $h_i = 2^{c_i} - 1$, for every i.*

Proof. Assume that h_i is even for every i. $A = x^a(x+1)^b \prod_i P_i^{h_i}$ be a 2^n-perfect then there exists a non Mersenne prime S such that S divides $\sigma(P_i^{h_i})$. So, S divides $\sigma_{2^n}(A) = A^{2^n}$. Therefore, $S = x$ or $S = x+1$ and this contradicts Lemmas 2.8 and 2.9 as h_i must be odd. Now, suppose that $h_i + 1 = 2^{c_i}u$, u is odd and $c_i \in \mathbb{N}$. But $\sigma(P_i^{h_i}) = (1+P_i)^{2^{c_i}-1}\left(\sigma(P_i^{u-1})\right)^{2^{c_i}}$. If $u-1 \geq 2$, again there exists a non Mersenne prime W such that W divides $\sigma(P_i^{u-1})$. So, W divides $\sigma_{2^n}(A) = A^{2^n}$. By Lemma 2.9, $W \neq x$ and $W \neq x+1$. But any prime divisor of A which is not a Mersenne prime is either x or $x+1$, a contradiction. Hence, $u = 1$ and the result follows. □

Lemma 3.11. *Let $c_i \in \mathbb{N}$, and let $A = x^a(x+1)^b \prod_i P_i^{2^{c_i}-1}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$ with each P_i is a Mersenne prime. Then, $P_i \in \{T_1, T_2, ..., T_5\}$, with $c_i = 1$ or 2.*

Proof. Since A is 2^n-perfect, then any irreducible factor Q of $\sigma(x^a)$ or $\sigma((1+x)^b)$ must divide A. So, $Q \in \{x, x+1, P_1, P_2, ...\}$. From Lemma 3.9(*ii.*), we have $Pi \in \{T_1, T_2, ..., T_5\}$. Now, we want to prove that $c_j \in \{1, 2\}$. Note that $\sigma(P_i^{2^{c_i}-1}) = (1+P_i)^{2^{c_i}-1}$ is not divisible by P_j, for any i, j. Moreover, if α_j are the exponents of P_j that are found in $\sigma(x^a)$ and in $\sigma((1+x)^b)$, then $\alpha_j \in \{0, 1, 2^r : r \in \mathbb{N}\}$ (Lemma 3.9(*ii.*)). Comparing exponents of P_j, we get $\alpha_j = 2^{c_j} - 1 \in \{0, 1, 2, 2^r, 2^r+1, 2^r+2^s : r, s \in \mathbb{N}\}$. Hence, $c_j = 1$ or 2. □

Lemma 3.12. *Let $c_i \in \mathbb{N}$, $P_i \in \{T_1, T_2, ..., T_5\}$, and $A = x^a(x+1)^b \prod_i P_i^{c_i}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$ with $c_i \in \{1, 3\}$. Then a or b must be even.*

Proof. For contradictional purpose, assume that a and b are both odd. By Lemma 3.13, we have $a = 2^r u - 1$ and $b = 2^s v - 1$ for some $t, s \in \mathbb{N}$, and u and v are odd positive integers less than or equal to 7. But,

$$\sigma(x^a) = (x+1)^{2^t-1}(1+x+...+x^{u-1})^{2^t}$$

and

$$\sigma((1+x)^b) = x^{2^s-1}\left(1+(1+x)+...+(1+x)^{v-1}\right)^{2^s}.$$

Also, P_i is not a factor of $\sigma(P_j^{c_j}) = (1+P_j)^{c_j}$ for any i, j. Suppose that P_i is a factor of $1+x+...+x^{u-1}$ but is not a factor of $1+(1+x)+...+(1+x)^{v-1}$ for some i, with $u \geq 3$. Hence, $2^t = c_i = 2^{h_i} - 1$, a contradiction.

Now, assume that P_i is a factor of both $1+x+...+x^{u-1}$ and $1+(1+x)+...+(1+x)^{v-1}$, then $2^t + 2^s = c_i = 2^{h_i} - 1$, also a contradiction. Therefore, $u = 1$ and in a similar manner we get $v = 1$. So, $\sigma(x^a) = \sigma(x^{2^t-1}) = (x+1)^a$ and $\sigma\left((x+1)^b\right) = \sigma\left((x+1)^{2^s-1}\right) = x^b$. Hence, $a = b$ and $x^a(x+1)^b$ is a 2^n-perfect (Lemma 3.7). By Lemma 3.8, the polynomial $\prod_{i=1}^r P_i^{h_i}$ is also 2^n-perfect. This contradicts Lemma 3.1. □

Lemma 3.13. *Let $c_i \in \mathbb{N}$, $u \geq 1$ and a be odd integers and let $A = x^a(x+1)^b \prod_i P_i^{2^{c_i}-1}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$, where each P_i is a Mersenne prime. Then, a is of the form $2^t u - 1$ with $u \leq 7$.*

Proof. Suppose that $a = 2^t u - 1$ with u is odd and $t \geq 1$. Since A is 2^n-perfect over $\mathbb{F}_2$, then

$$x^{2^n a}(x+1)^{2^n b} \prod_{i=1} P_i^{2^n(2^{c_i}-1)} = \left(\sigma(x^a)\sigma\left((x+1)^b\right) \prod_{i=1} \sigma\left(P_i^{2^{c_i}-1}\right)\right)^{2^n}.$$

But $\sigma(x^a) = 1 + x + ... + x^{2^t u - 1} = (1+x)^{2^t - 1} \sigma(x^{u-1})^{2^t}$. If $u > 2$, then as done in the proof of the preceding lemma we get $u - 1 \leq 6$ and hence the result. □

Lemma 3.14. *Let $a, b, c_i \in \mathbb{N}$ such that a is even and let $A = x^a(x+1)^b \prod_{i=1}^m P_i^{2^{c_i}-1}$ be a 2^n-perfect polynomial over $\mathbb{F}_2$, where each P_i is a Mersenne prime. Then, $a \leq 6$.*

Proof. Let $a = 2m$. Since A is 2^n-perfect over $\mathbb{F}_2$, then

$$\begin{aligned} x^{2^{n+1}m}(x+1)^{2^n b} \prod_{i=1} P_i^{2^n(2^{c_i}-1)} &= A^{2^n} \\ &= \sigma_{2^n}(A) \\ &= \left(\sigma(x^{2m})\sigma\left((x+1)^b\right) \prod_{i=1} \sigma\left(P_i^{2^{c_i}-1}\right)\right)^{2^n}. \end{aligned}$$

But x and $x+1$ do not divide $\sigma(x^{2m})$ and P_i does not divide $\sigma\left(P_i^{2^{c_i}-1}\right)$ so P_i divides $\sigma(x^{2m})$. We are done by Lemma 3.9 (*ii.*). □

3.1 Cases of the Proof

Let $A = x^a(x+1)^b \prod_{i=1}^r P_i^{h_i}$, where P_i, is a Mersenne prime be a 2^n-perfect over F_2. From Lemma 3.11, we have $h_i = 1$ or 3. By Lemma 3.12, we have a or b is even. To complete the proof of Theorem 1.3, we study the below cases:

Case 1. Both a and b are even:

In this case, we have

(2) $$1 + x + ... + x^a = P_{i_1} ... P_{i_s}.$$

Since the P_{i_j}'s are Mersenne primes, then $a, b \in \{2, 4, 6\}$. Since if A is a 2^n-perfect polynomial over F_2, then $\overline{A}$ is a 2^n-perfect polynomial over $\mathbb{F}_2$ so a and b can be chosen in the way $a \leq b$ and $a, b \in \{2, 4, 6\}$.

- If $a = b = 2$, then $1 + x + x^2 = 1 + (x+1) + (x+1)^2 = T_1$. Hence, $A = x^2(x+1)^2 T_1$ and $\sigma(A) = \sigma\left(x^2\right)\sigma\left((x+1)^2\right)\sigma(T_1) = (T_1)(T_1)(x(1+x)) = x(1+x)T_1^2 \neq A$. Therefore A is not perfect over $\mathbb{F}_2$ and hence A is not 2^n-perfect over $\mathbb{F}_2$ (Theorem 3.1).

- If $a = 2$ and $b = 4$, then $1 + x + x^2 = T_1$ and $1 + (x+1) + ... + (x+1)^4 = 1 + x^3(x+1) = T_5$. Hence, $A = x^2(x+1)^4 T_1 T_5$ and $\sigma(A) = \sigma\left(x^2\right)\sigma\left((x+1)^4\right)\sigma(T_1)\sigma(T_5) = (T_1)(T_5)(x(1+x))\left(x^3(1+x)\right) = x^4(1+x)^2 T_1 T_5 \neq A$. So, A is not 2^n-perfect over F_2 (Theorem 3.1).

- If $a = b = 4$, then $1 + x + ... + x^4 = T_4$ and $1 + (x + 1) + ... + (x + 1)^4 = 1 + x^3 + x^4 = T_5$. Hence, $A = x^4(x + 1)^4T_4T_5$ and $\sigma(A) = \sigma\left(x^4\right)\sigma\left((x + 1)^4\right)\sigma(T_4)\sigma(T_5) = (T_4)(T_5)\left(x(1 + x)^3\right)\left(x^3(1 + x)\right) = x^4(1+x)^4T_4T_5 = A$. So, A is 2^n-perfect over $\mathbb{F}_2$ (Theorem 3.1).

- If $a = 2$ and $b = 6$, then $1 + x + x^2 = T_1$ and $1 + (x + 1) + ... + (x + 1)^6 = (1 + x + x^3)(1 + x^2 + x^3) = T_2T_3$. Hence, $A = x^2(x + 1)^6T_1T_2T_3$ and

$$\begin{aligned}\sigma(A) &= \sigma\left(x^2\right)\sigma\left((x + 1)^6\right)\sigma(T_1)\sigma(T_2)\sigma(T_3)\\ &= (T_1)(T_2T_3)(x(1 + x))\left(x(1 + x)^2\right)\left(x^2(1 + x)\right)\\ &= x^4(1 + x)^4T_1T_2T_3\\ &\neq A.\end{aligned}$$

  Therefore, A is not 2^n-perfect over $\mathbb{F}_2$.

- If $a = 4$ and $b = 6$, then $1+x+...+x^4 = T_4$ and $1+(x+1)+...+(x+1)^6 = T_2T_3$. Hence, $A = x^4(x + 1)^6T_2T_3T_4$ and $\sigma(A) = A$. So, A is 2^n-perfect over $\mathbb{F}_2$.

- If $a = b = 6$, then $1 + x + ... + x^6 = (1 + x + x^3)(1 + x^2 + x^3) = T_2T_3 = 1 + (x + 1) + ... + (x + 1)^6$. Hence, $A = x^6(x + 1)^6T_2^2T_3^2$ and $\sigma(A) = \sigma\left(x^6\right)\sigma\left((x + 1)^6\right)\sigma\left(T_2^2\right)\sigma\left(T_3^2\right) = T_1^2T_2^2T_3^2T_4T_5 \neq A$. Therefore, A is not 2^n-perfect over $\mathbb{F}_2$.

Case 2. a is even and b is odd:

By Lemmas 3.13 and 3.14, we have $a \in \{2, 4, 6\}$ and $b = 2^tu - 1$ for some $t \in \mathbb{Z}_{\geq 1}$ and $u \in \{1, 3, 5, 7\}$.

- If $u = 1$ and $a = 2$, then $\sigma(x^2) = T_1, \sigma((x + 1)^{2^t-1}) = x^{2^t-1}$, and $\sigma(T_1) = x(x+1)$. Hence, $2^t-1+1 = b+1 \leq a = 2$. Thus, $t = 1$ and $A = x^2(x+1)T_1$.

- If $u = 1$ and $a = 4$, then $\sigma(x^4) = T_4, \sigma((x + 1)^{2^t-1}) = x^{2^t-1}$, and $\sigma(T_4) = x(x+1)^3$. Hence, $2^t-1+1 = b+1 \leq a = 4$. Thus, $t \leq 2$ and $3 \leq b = 2^t-1$, so $t = 2$ and $A = x^4(x + 1)^3T_4$.

- If $u = 1$ and $a = 6$, then $\sigma(x^6) = T_2T_3, \sigma((x + 1)^{2^t-1}) = x^{2^t-1}$, $\sigma(T_2) = x(x + 1)^2$ and $\sigma(T_3) = x^2(x + 1)$. Hence, $2^t - 1 + 2 + 1 = b + 3 \leq a = 6$. Thus, $t \leq 2$ and $3 \leq b = 2^t - 1$, so $t = 2$ and $A = x^6(x + 1)^3T_2T_3$.

- If $u = 3$ and $a = 2$, then $\sigma(x^2) = T_1, \sigma((x + 1)^{3.2^t-1}) = x^{2^t-1}T_1^{2^t}$. Hence, $T_1^{2^t+1}$ divides $\sigma(A) = A$ but $T_1^{2^t+2}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t + 1 \in \{1, 3\}$ and thus $t = 1$ and $A = x^2(1 + x)^5T_1$. But $\sigma(x^2(1 + x)^5T_1) \neq x^2(1 + x)^5T_1$ and hence A is not 2^n-perfect over F_2.

- If $u = 3$ and $a = 4$, then $\sigma(x^4) = T_4$. Since T_1 does not divide $\sigma(x^4)$, then $T_1^{2^t}$ divides $\sigma(A) = A$ but $T_1^{2^t+1}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t \in \{1, 3\}$, a contradiction.

- The case $u = 3$ and $a = 6$ is similar to the preceding one.
- If $u = 5$ and $a \in \{2, 6\}$, then $\sigma((x+1)^{5.2^t-1}) = x^{2^t-1}T_4^{2^t}$. Since T_4 does not divide $\sigma(x^a)$, then $T_4^{2^t}$ divides $\sigma(A) = A$ where $T_1^{2^t+1}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t \in \{1, 3\}$, a contradiction.
- If $u = 5$ and $a = 4$, then $\sigma(x^4) = T_4$. Since $T_4^{2^t+1}$ divides A and $T_1^{2^t+2}$ does not divide A. By Lemma 3.11, we have $2^t + 1 \in \{1, 3\}$. Thus $t = 1$ and $A = x^4(1+x)^9T_1^3$. But $\sigma(x^4(1+x)^9T_1^3) \neq x^4(1+x)^9T_1^3$. Hence, A is not 2^n-perfect over F_2.
- If $u = 7$ and $a \in \{2, 4\}$, then $\sigma((x+1)^{7.2^t-1}) = x^{2^t-1}T_2^{2^t}T_3^{2^t}$. Since T_2 and T_3 do not divide $\sigma(x^a)$, then $T_2^{2^t}$ divides A and $T_2^{2^t+1}$ does not divide $\sigma(A) = A$. By Lemma 3.11, we have $2^t \in \{1, 3\}$, a contradiction.
- If $u = 7$ and $a = 6$, then $\sigma\left(x^6\right) = T_2T_3$. So, $T_2^{2^t+1}$ (resp. $T_3^{2^t+1}$) divides A and $T_2^{2^t+1}$(resp. $T_3^{2^t+1}$) does not divide A. By Lemma 3.11, we have $2^t+1 \in \{1, 3\}$. Thus $t = 1$ and $A = x^6(1+x)^{13}T_2^3T_3^3$. But $\sigma\left(x^6(1+x)^{13}T_2^3T_3^3\right) \neq x^6(1+x)^{13}T_2^3T_3^3$. Hence, A is not 2^n-perfect over F_2.

The proof of Theorem 1.3 is now complete

4. Conclusion

We show the non existence of odd 2^n-perfect, $n \in \mathbb{N}$, polynomials over $\mathbb{F}_2$. A characterization of 2^n-perfect polynomials A over the prime field with two elements that are divisible by x, $x+1$, and Mersenne primes is given.

References

[1] T. Cai, D. Chen, Y. Zhang, *Perfect numbers and Fibonacci primes (I)*, Int. J. Number Theory, 11 (2015), 159-169.

[2] E. F. Canaday, *The sum of the divisors of a polynomial*, Duke Mathematical Journal, 8 (1941), 721-737.

[3] H. V. Chu, *Divisibility of divisor functions of even perfect numbers*, Journal of Integer Sequences, 24 (2021), p. 3.

[4] L. H. Gallardo, O. Rahavandrainy, *Odd perfect polynomials over* $\mathbb{F}_2$, J. Théor. Nombres Bordeaux, (2007), 165-174.

[5] L. H. Gallardo, O. Rahavandrainy, *There is no odd perfect polynomial over* $\mathbb{F}_2$ *with four prime factors*, Port. Math. (N.S.), 66 (2009), 131-145.

[6] L. H. Gallardo, O. Rahavandrainy, *On even (unitary) perfect polynomials over* $\mathbb{F}_2$, Finite Fields and Their Applications, 18 (2012), 920-932.

[7] L. H. Gallardo, O. Rahavandrainy, *All unitary perfect polynomials over* $\mathbb{F}_2$ *with at most four distinct irreducible factors*, Journal of Symbolic Computation, (2012), 429-502.

[8] L. H. Gallardo, O. Rahavandrainy, *Characterization of sporadic perfect polynomials over* $\mathbb{F}_2$, Functiones et Approximatio Commentarii Mathematici, 55 (2016), 7–21.

[9] X. Jiang, *On even perfect numbers*, Colloq. Math., 154 (2018), 131-135

[10] F. Luca, J. Ferdinands, *Sometimes n divides* $\sigma_k(n)$*:11090*, The American Mathematical Monthly, 113 (2006), 372-373.

[11] D. Suryanarayana, *Super perfect numbers*, Elem. Math., 24 (1969), 16-17.

Accepted: December 14, 2022

Prime-valent one-regular graphs of order $18p$

Qiao-Yu Chen
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang 471023
P.R. China
tj_chenqiaoyu@163.com

Song-Tao Guo*
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang 471023
P.R. China
gsongtao@gmail.com

Abstract. A graph is *one-regular* and *arc-transitive* if its full automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify connected one-regular graphs of prime valency and order $18p$ for each prime p. As a result there are two infinite families of such graphs, one is the cycle C_{18p} with valency two and the other is the normal Cayley graph on the generalized dihedral group $(\mathbb{Z}_{3p}\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2$ with valency three and $p \equiv 1 \pmod 6$.
Keywords: symmetric graph, arc-transitive graph, one-regular graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [21, 22] or [2, 3], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An s-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-

char 42 . Corresponding author

transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* or s-*transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n, that is, the cycle C_n, which is 1-regular with full automorphism group D_{2n}. Let p be a prime. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [6] and [7], respectively. Pan [20] characterized the prime-valent s-transitive graphs of square free order. Kutnar [17] classified cubic symmetric graphs of girth 6 and Oh [19] determined arc-transitive elementary abelian covers of the Pappus graph. The classification of pentavalent and heptavalent s-transitive graphs of order $18p$ was given in [1] and [13], respectively.

For 2-valent case, s-transitivity always means 1-regularity, and for cubic case, s-transitivity always means s-regularity by Miller [11]. However, for the other prime-valent case, this is not true, see for example [14] for pentavalent case and [15] for heptavalent case. Thus, characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graphs of order $18p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [18, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then, one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [7], we introduce the graphs

$G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p, q)$ the unique subgroup of Z_p^* of order q. Define the graph $G(2p, q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p, q primes. Then, X is isomorphic to K_{2p} with $q = 2p-1$, $K_{p,p}$ or $G(2p, q)$ with $q \,\big|\, (p-1)$. Furthermore, if $(p, q) \neq (11, 5)$ then $\mathrm{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p, q) = (11, 5)$ then $\mathrm{Aut}(G(2p, q)) = \mathrm{PGL}(2, 11)$.*

The following proposition is about the prime-valent symmetric graphs of order $6p$ with p a prime, which is deduced from [20, Theorem 1.2].

Proposition 2.3. *Let p and q be two primes. If $q > 7$, then there is no q-valent symmetric graph of order $6p$ admitting a solvable arc-transitive automorphism group.*

The following proposition is the famous "N/C-Theorem", see for example [16, Chapter I, Theorem 4.5]).

Proposition 2.4. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

From [10, p.12-14], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 2.5. *Let G be a non-abelian simple group. If the order $|G|$ has at most three different prime divisors, then G is called K_3-simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2, 3, p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	PSL(2, 17)	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	PSL(3, 3)	$2^4 \cdot 3^3 \cdot 13$
PSL(2, 7)	$2^3 \cdot 3 \cdot 7$	PSU(3, 3)	$2^5 \cdot 3^3 \cdot 7$
PSL(2, 8)	$2^3 \cdot 3^2 \cdot 7$	PSU(4, 2)	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $18p$ for each prime p. Let q be a prime. In what follows, we always denote by X a connected q-valent one-regular graph of order $18p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then, the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 18pq$.

Now, we first deal with the case $q \leq 7$. Clearly, any connected graph of order $18p$ and valency two is isomorphic to the cycle C_{18p}. Thus, for $q = 2$, $X \cong C_{18p}$ and $A \cong D_{36p}$. Let $q = 3$. Then, by [17, Theorem 1.2] and [19, Theorem 3.4], $X \cong CF_{18p}$ is a $\mathbb{Z}_p$-cover of the Pappus graph and also a normal Cayley graph of a generalized dihedral group $(\mathbb{Z}_{3p} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ with $p \equiv 1 (\mathrm{mod}\ 6)$. This implies that $A \cong ((\mathbb{Z}_{3p} \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3$. If $q = 5$ or 7, then by [1, Theorem 4.1] for $q = 5$ and [13, Theorem 3.1] for $q = 7$, there is no q-valent one-regular graph of order $18p$. Thus, in what follows we deal with the case $q > 7$. The next lemma is about the case $p = 2$.

Lemma 3.1. *Let X be a connected q-valent one-regular graph of order 36. Then, $X \cong C_{36}$.*

Proof. Since $|V(X)| = 36$, we have that $p = 2$. If $q \leq 7$, then by the above argument, the only possibility is $q = 2$ and X is isomorphic to the cycle C_{36}.

Let $q > 7$. Then, $|A| = 2^2 \cdot 3^2 \cdot q$. If A is non-solvable, then A has a composition factor isomorphic to a non-abelian simple group and hence this composition factor has order dividing $|A| = 2^2 \cdot 3^2 \cdot q$. This forces that this composition factor is a K_3-simple group. By Proposition 2.5, A has a composition factor isomorphic to A_5 and $q = 5$, contrary to our assumption. Thus, A is solvable. Let N be a minimal normal subgroup of A. Then, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_q$. Clearly, N is not transitive on $V(X)$. By Proposition 2.1, X_N is a q-valent symmetric graph of order $36/|N|$. Note that, $q > 7$ and there is no connected regular graph of odd order and odd valency. Thus, N is not isomorphic to $\mathbb{Z}_2^2$ or $\mathbb{Z}_q$.

Suppose that $N \cong \mathbb{Z}_2$. Then, X_N has order 18 and valency q. Since $q > 7$ is a prime, by [8], X_N is isomorphic to Pappus graph with $q = 3$ or the complete graph K_{18} with $q = 17$. For the former, X is a cubic symmetric graph of order 36. However, by [9], there is no cubic symmetric graph of order 36, a contradiction. For the latter, $A/N \lesssim \mathrm{Aut}(K_{18}) \cong \mathrm{S}_{18}$. Recall that $|A| = 2^2 \cdot 3^2 \cdot q$. We have $|A/N| = 18 \cdot 17$. However, by Magma [4], S_{18} has no subgroup of order $18 \cdot 17$, a contradiction.

Suppose that $N \cong \mathbb{Z}_3$. Then, X_N is a q-valent symmetric graph of order 12. By [8], $X_N \cong K_{12}$ with $q = 11$ because $q > 7$. It follows that $A/N \lesssim \mathrm{Aut}(K_{12}) \cong \mathrm{S}_{12}$. However, $|A/N| = 12 \cdot 11$ and by Magma [4], S_{12} has no subgroup of order $12 \cdot 11$, a contradiction.

Suppose that $N \cong \mathbb{Z}_3^2$. Then, X_N is a q-valent symmetric graph of order 4. Clearly, the only symmetric graphs of order 4 are C_4 with valency 2 and K_4 with valency 3. This is impossible because the valency $q > 7$. □

Finally, we treat with the case $p \geq 3$ and $q > 7$.

Lemma 3.2. *Let $p \geq 3$ and $q > 7$. Then, there is no new graph.*

Proof. Since $p \geq 3$ and $q > 7$, we have that $|A| = 18pq = 2 \cdot 3^2 \cdot p \cdot q$ is twice an odd integer. It follows that A has a normal subgroup of odd order and index 2. By Feit-Thompson's Theorem [12, Theorem], any group of odd order is solvable and so A is also solvable. Let N be a minimal normal subgroup of A. Then, N is also solvable and hence N is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$, $\mathbb{Z}_p$, $\mathbb{Z}_q$ or $\mathbb{Z}_p^2$ with $p = q$. By Proposition 2.1, X_N is a q-valent symmetric graph of order $9p$, $6p$, $2p$ or 18. Since there is no connected regular graph of odd order and odd valency, we have that $N \ncong \mathbb{Z}_2$. If $p \neq q$ and $N \cong \mathbb{Z}_q$, then X_N has order $18p/q$. This is impossible because q cannot divide $18p$. If $p = q$ and $N \cong \mathbb{Z}_p^2$, then $N_v \cong \mathbb{Z}_q = \mathbb{Z}_p$. However, by Proposition 2.1, X_N has order 18 and N is semiregular on $V(X)$. This forces that $N_v = 1$, a contradiction. Thus, $N \cong \mathbb{Z}_3$, $\mathbb{Z}_3^2$, $\mathbb{Z}_p$.

Let $N \cong \mathbb{Z}_3$. Then, X_N is a q-valent symmetric graph of order $6p$ and $A/N \lesssim \mathrm{Aut}(X_N)$. Recall that A is solvable. Thus, A/N is also solvable and acts arc-transitively on X_N. However, by Proposition 2.3, there is no q-valent symmetric graph admitting a solvable arc-transitive automorphism group with $q > 7$, a contradiction.

Let $N \cong \mathbb{Z}_p$. Then, X_N is a q-valent symmetric graph of order 18. By [8], there is only one q-valent symmetric graph of order 18 with $q > 7$, that is, the complete graph K_{18} and hence $q = 17$. It follows that $A/N \lesssim \mathrm{Aut}(K_{18}) \cong \mathrm{S}_{18}$ and $|A/N| = 2{\cdot}3^2{\cdot}17$. However, S_{18} has no subgroup of order $2{\cdot}3^2{\cdot}17$ by Magma [4], a contradiction.

Let $N \cong \mathbb{Z}_3^2$. Then, X_N is a q-valent symmetric graph of order $2p$. By Proposition 2.2, X_N is isomorphic to K_{2p} with $q = 2p - 1$ a prime, $K_{p,p}$ with $q = p$ or $G(2p, q)$ with $q \,\big|\, (p - 1)$.

Suppose that $X_N \cong K_{2p}$. Then, A/N has order $2{\cdot}p{\cdot}q$ and acts 2-transitively on $V(X_N)$. By Burnside's Theorem [5, p.192, Theorem IX], any 2-transitive permutation group is either almost simple or affine. Since A is solvable, A/N is also solvable. It forces that A/N is affine and hence A/N has a normal subgroup $M/N \cong \mathbb{Z}_p$. Note that, $N \cong \mathbb{Z}_3^2$. By Proposition 2.4, $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{Aut}(\mathbb{Z}_3^2) \cong \mathrm{GL}(2,3)$. Since $|\mathrm{GL}(2,3)| = 48$ and $q = 2p - 1 > 7$, we have that $C_M(N) = M$ and hence $M \cong \mathbb{Z}_3^2{\times}\mathbb{Z}_p$. It follows that M has a characteristic subgroup $K \cong \mathbb{Z}_p$. The normality of M in A implies that K is also normal in A. By Proposition 2.1, X_K is a q-valent symmetric graph of order 18 with $q > 7$, and by [8], $X_K \cong K_{18}$ with $q = 17$. Recall that $q = 2p - 1$. This forces that $p = 9$ is not a prime, a contradiction.

Suppose that $X_N \cong K_{p,p}$. Then, $p = q$ and $|A/N| = 2{\cdot}p^2$. Since $p > 7$, we have that A/N has a normal subgroup M/N of order p^2. Note that, $A/N \lesssim \mathrm{Aut}(K_{p,p}) \cong \mathrm{S}_p \,\mathrm{wr}\, \mathrm{S}_2$. Thus, a Sylow p-subgroup of A/N is isomorphic to $\mathbb{Z}_p^2$ and so $M/N \cong \mathbb{Z}_p^2$. By Proposition 2.4, $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{GL}(2,3)$. Since $|\mathrm{GL}(2,3)| = 48$ and $p > 7$, we have that $C_M(N) = M$. This forces that $M \cong \mathbb{Z}_p^2{\times}\mathbb{Z}_3^2$ has a characteristic subgroup $P \cong \mathbb{Z}_p^2$. By Proposition 2.1, X_P has order 18 and hence P is semiregular on $V(X)$. Clearly, this is impossible because $q = p$ and $P_v \cong \mathbb{Z}_p$.

Suppose that $X_N \cong G(2p, q)$. Then, $q \,\big|\, (p - 1)$ and $A/N \cong (\mathbb{Z}_p{\rtimes}\mathbb{Z}_q){\rtimes}\mathbb{Z}_2$. Similarly, by Proposition 2.4, we can easily deduce that A has a normal subgroup $P \cong \mathbb{Z}_p$. It follows that the quotient graph X_P has order 18 and is isomorphic to K_{18}. With a similar argument as the case "$N \cong \mathbb{Z}_p$", we have A/P has order $2{\cdot}3^2{\cdot}17$ and cannot be embedded in $\mathrm{Aut}(K_{18}) \cong \mathrm{S}_{18}$, a contradiction. $\square$

Combining the above arguments with the cases $q = 2, 3, 5, 7$, and Lemmas 3.1-3.2, we have the following result.

Theorem 3.1. *Let p, q be two primes and X a connected q-valent one-regular graph of order $18p$. Then, the only possibilities are $q = 2, 3$ and furthermore,*

(1) *for $q = 2$, $X \cong C_{18p}$ and $A \cong D_{36p}$;*
(2) *for $q = 3$, $X \cong CF_{18p}$ and $A \cong ((\mathbb{Z}_{3p}{\times}\mathbb{Z}_3){\rtimes}\mathbb{Z}_2) \rtimes \mathbb{Z}_3$ with $p \equiv 1(\mathrm{mod}\ 6)$.*

Acknowledgements

This work was supported by the National Natural Science Foundation of China (11301154).

References

[1] M. Alaeiyan, M. Akbarizadeh, *Classification of the pentavalent symmetric graphs of order 18p,* Indian J. Pur. Appl. Math., 50 (2019), 485-497.

[2] N. Biggs, *Algebraic graph theory,* Second ed., Cambridge University Press, Cambridge, 1993.

[3] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, Elsevier Science Ltd, New York, 1976.

[4] W. Bosma, C. Cannon, C. Playoust, *The MAGMA algebra system I: The user language,* J. Symbolic Comput., 24 (1997), 235-265.

[5] W. Burnside, *Theory of groups of finite order*, Cambridge University Press, Cambridge, 1897.

[6] C.Y. Chao, *On the classification of symmetric graphs with a prime number of vertices,* Trans. Amer. Math. Soc., 158 (1971), 247-256.

[7] Y. Cheng, J. Oxley, *On the weakly symmetric graphs of order twice a prime*, J. Combin. Theory B, 42 (1987), 196-211.

[8] M.D.E. Conder, *A complete list of all connected symmetric graphs of order 2 to 30,* `https://www.math.auckland.ac.nz/ conder/symmetricgraphs-orderupto30.txt`.

[9] M.D.E. Conder, P. Dobcsányi, *Trivalent symmetric graphs on up to 768 vertices,* J. Combin. Math. Combin. Comput., 40 (2002), 41-63.

[10] H.J. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A Wilson, *Atlas of finite group*, Clarendon Press, Oxford, 1985.

[11] D.Ž. Djoković, G.L. Miller, *Regular groups of automorphisms of cubic graphs,* J. Combin. Theory B, 29 (1980), 195-230.

[12] W. Feit, J.G. Thompson, *Solvability of groups of odd order,* Pac. J. Math., 13 (1963), 775-1029.

[13] S.T. Guo, *Heptavalent symmetric graphs of order* $18p$, Utilitas Math., 109 (2018), 3-15.

[14] S.T. Guo, Y.Q. Feng, *A note on pentavalent s-transitive graphs,* Discrete Math., 312 (2012), 2214-2216.

[15] S.T. Guo, Y.T. Li, X.H. Hua, (G, s)*-transitive graphs of valency* 7*,* Algebr. Colloq., 23 (2016), 493-500.

[16] B. Huppert, *Eudiche gruppen I*, Springer-Verlag, Berlin, 1967.

[17] K. Kutnar, D. Marušič, *A complete classification of cubic symmetric graphs of girth 6*, J. Combin. Theory Ser. B, 99 (2009), 162-184.

[18] P. Lorimer, *Vertex-transitive graphs: symmetric graphs of prime valency,* J. Graph Theory, 8 (1984), 55-68.

[19] J.M. Oh, *Arc-transitive elementary abelian covers of the Pappus graph,* Discrete Math., 309 (2009), 6590-6611.

[20] J. Pan, B. Ling, S. Ding, *On prime-valent symmetric graphs of square-free order,* Ars Math. Contemp., 15 (2018), 53-65.

[21] D.J. Robinson, *A Course in the theory of groups*, Springer-Verlag, New York, 1982.

[22] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964.

Accepted: February 14, 2023

The ω-continuity of group operation in the first (second) variable

Halgwrd M. Darwesh
Department of Mathematics
College of Science
University of Sulaimani
Sulaimani 46001, Kurdistan Region
Iraq
halgwrd.darwesh@univsul.edu.iq

Adil K. Jabbar
Department of Mathematics
College of Science
University of Sulaimani
Sulaimani 46001, Kurdistan Region
Iraq
adil.jabbar@univsul.edu.iq

Diyar M. Mohammed[*]
Department of Mathematics
College of Basic Education
University of Sulaimani
Sulaimani 46001, Kurdistan Region
Iraq
diyar.mohamed@univsul.edu.iq

Abstract. The present paper aims to introduce and study the ω-continuity of the group operation in the first (resp., second) variable and some basic properties and relationships concerning left and right translation functions are obtained. Also, we have shown that the group operation is ω-continuous at the first (resp., second) variable if and only if it is ω-irresolute at the first (resp., second) variable.
Keywords: ω-open, ω-closed, ω-continuous, ω-irresolute.

1. Introduction

Topology is a special type of geometry and includes several fields of study and it has many interesting applications in graph theory. Hdeib H. Z. [7] defined and studied ω-closed sets and ω-open sets. He used ω-closed sets to define a new type of mappings called ω-closed functions. He obtained many properties and relationships concerning these concepts. Also, he used ω-open sets to define $\omega - \textit{continuous}$ mappings [8] and he studied this new type of continuous

∗. Corresponding author

mapping and obtained certain properties and relationships concerning this type of continuous mappings.

The notion of a topological group goes back to the second half of the nineteenth century. Topological groups are objects that combine two separate algebraic structures with the topology structure and the requirement links them that multiplication and inversion are continuous functions.

In this article, we study the ω-continuity of a group operation at the first (resp., second) variable respectively and obtain some basic properties of this kind of ω-continuity of groups.

2. Preliminaries

Let A be a subset of a topological space (X, τ), the interior and closure of A are denoted by $Int(A)$ and $Cl(A)$, respectively. A point x of X is called a condensation point of A [9] if $G \cap A$ is uncountable for each open set G containing x. A is called ω-closed [7] if it contains all it is condensation points. The complement of an ω-closed set is called an ω-open set. The intersection of all ω-closed subsets of X which contain A is called ω-closure of A and is denoted by ωClA [4] and [7]. A point $x \in A$ is said to be an ω-interior point of A [8], if there exists an ω-open set U containing x such that $U \subseteq A$. The set of all ω-interior points of A is denoted by $\omega IntA$.

The discrete topology is denoted by τ_{dis}, and the family of all ω-open subsets of a space (X, τ), denoted by τ^{ω} from a topology on X finer than τ ([4]). A compact space is a topological space for which every covering of that space by a collection of open sets has a finite subcover.

Definition 2.1 ([5]). *A space (X, τ) is said to be $\omega - compact$ provided that every ω-open cover of X has a finite subcover.*

Definition 2.2 ([5]). *A space (X, τ) is said to be $\omega - lindelof$ provided that every ω-open cover of X has a countable subcover.*

Definition 2.3 ([4]). *A space (X, τ) is said to be locally countable if each point of X has a countable open neighbourhood.*

Theorem 2.1 ([3]). *Let (X, τ) be a topological space, then $\tau^{\omega} = \tau_{dis}$ if and only if the space (X, τ) is locally-countable.*

Theorem 2.2 ([4]). *For any topological space (X, τ) and any subset A of X, $(\tau_A)^{\omega} = \tau^{\omega}_A$.*

The proof of the following lemma can be found in [15]. Also, we can find a similar proof in [14], Lemma 3 and [16], Lemma 3.3].

Definition 2.4 ([8]). *Let $f : (X, \tau) \to (Y, \rho)$ be a mapping, f is said to be ω-continuous at a point $x \in X$, if for each open subset V in Y containing $f(x)$ there exists an ω-open subset U of X contains x such that $f(U) \subseteq V$, and f is called ω-continuous if it is ω-continuous at each point x of X.*

Definition 2.5 ([1]). *Let $f : (X, \tau) \to (Y, \rho)$ be a mapping. Then f is said to be ω-irresolute, if $f^{-1}(F)$ is an ω-closed in X for each ω-closed set F in Y.*

Definition 2.6 ([10]). *Let $f : (X, \tau) \to (Y, \rho)$ be a mapping, then f is an ω-homeomorphism if and only if f is bijective and f, f^{-1} are ω-irresolute.*

Definition 2.7 ([11]). *A space (X, τ) is said to be lindelof provided that every open cover of X has a countable subcover.*

Lemma 2.1 ([4]). *Let (X, τ) be a topological space. Then X is ω-lindelof if and only if it is lindelof.*

Definition 2.8 ([12]). *A topological space (X, τ) is called a normal space if given any disjoint closed sets E and F, there are neighbourhoods U of E and V of F with $U \cap V = \phi$.*

Definition 2.9 ([13]). *Let X be a nonempty set and $\mu : X \to X$ be a binary operation defined by $\mu(g_1, g_2) = g_1 * g_2$. The pair $(X, *)$ is a group if the following three properties hold:*

1. *For all $a, b, c \in X$ we have $(a * b) * c = a * (b * c)$ (associative law);*
2. *There exists an $e \in X$ such that for all $a \in X$ we have $a * e = e * a = a$ (existence of identity element);*
3. *For all $a \in X$ there exists $a^{-1} \in X$ such that $a * a^{-1} = a^{-1} * a = e$ (each element has inverse).*

Definition 2.10 ([13]). *Let $(X, *)$ be a group. If X has the property that $a * b = b * a$ for all $a, b \in X$, then we call X abelian.*

Definition 2.11 ([17]). *Let $(X, *)$ be a group and H be a subset of X. We call H a subgroup of X when the following hold:*

1. *$H \neq \phi$;*
2. *If $x, y \in H$, then $x * y \in H$;*
3. *If $x \in H$, then $x^{-1} \in H$.*

Definition 2.12 ([17]). *Let X be a group, H a subgroup of H and $g \in X$. The sets $gH = \{g * h, h \in H\}$ and $Hg = \{h * g, h \in H\}$ are called the left and right cosets of H in X, respectivly.*

3. The results

We introduce the following definition

Definition 3.1. *Let $(X, *)$ be a group and τ be a topology on X. The multiplication map $\mu : X * X \to X$ is said to be ω-continuous at the first (second) variable if, for any fixed point $a \in X$, any point $b \in X$ and any open set G in X which contains $\mu(b, a) = b * a$, $(\mu(a, b) = a * b)$, there exists an ω-open set V in X such that $b \in V$ and $V * a \subseteq G$, $(a * V \subseteq G)$.*

In our first result, we prove that for abelian groups, the ω-continuity of multiplication maps at the first and second variable are equivalent.

Theorem 3.1. *If $(X, *)$ is an abelian group and τ a topology on X. Then, the multiplication map μ is ω-continuous at the first variable if and only if it is ω-continuous at the second variable.*

Proof. Let μ be ω-continuous at the first variable. Suppose a is any fixed point of X and b is an arbitrary point of X. To show μ is ω-continuous at the second variable. Let O be any open subset of X which contains $a * b$. But, since $a * b = b * a$, so, $b * a \in O$, Since μ is ω-continuous at the first variable, then by Definition 3.1, there is an ω-open subset V of X which contains b and $V * a \subseteq O$. But, $V * a = a * V$, so $a * V \subseteq O$. Hence, μ is ω-continuous at the second variable. The converse part is followed similarly. □

Theorem 3.2. *If $(X, *)$ is any group and τ a topology on X such that (X, τ) is locally countable, then the multiplication map of X is $\omega -$ continuous at the first variable as well as at the second variable.*

Proof. Since (X, τ) is locally countable, so, by Theorem 2.1, $\tau^\omega = \tau_{dis}$. For any $a, b \in X$ and any open subset G of X such that $a * b \in G$, we have $\{a\}, \{b\} \in \tau^\omega$, $a * \{b\} = \{a * b\} \subseteq G$ and $\{a\} * b = \{a * b\} \subseteq G$. Thus, μ is ω-continuous at the first and second variables. □

Remark 3.1. The following example shows that the ω-continuity of the multiplication map in the first and second variable does not imply that the group is abelian and also does not imply that the group is semi-topological.

Example 3.1. Consider the symmetric group S_3 of the set $A = \{1, 2, 3\}$. The elements of this group are $f_1 = 1, f_2 = (1, 2), f_3 = (2, 3), f_4 = (1, 3), f_5 = (1, 2, 3), f_6 = (1, 3, 2)$, so, that S_3={1,(1,2),(1,3),(2,3), (1,2,3), (132)} with the usual composition of maps $(S_3, \circ)$ forms a non-commutative group, let $\tau = \{\phi, S_3, \{f_1\}, \{f_2, f_3, f_4\}, \{f_1, f_2, f_3, f_4\}, \{f_1, f_5, f_6\}\}$ be a topology on S_3 then, the multiplication map is not continuous neither in the first nor in the second variable because for $i = 2, 3, 4$ we have $f_i \circ f_i = f_1$ and $\{f_2, f_3, f_4\} \circ f_i \nsubseteq \{f_1\}$. Also, $f_1 \circ f_2, f_3, f_4 \subseteq f_1$ since $S_3 \times S_3$ is finite, so, by $(\tau \times \tau)_\omega = \tau_{dis}$, S_3 is finite $\tau^\omega = \tau_{dis}$, and $\tau^\omega \times \tau^\omega = \tau_{dis} \times \tau_{dis} = (\tau \times \tau)^\omega = \tau^\omega \times \tau^\omega$.

Theorem 3.3. *Let $(X, *)$ be a group and τ be a topology on X and the multiplication map μ is ω-continuous at the second (first) variable. For any $A, B \subseteq X$ and $a \in X$, the following statements are true:*

1. $a * \omega\ ClB \subseteq Cl(a * B)$*and* $((\omega ClB) * a \subseteq Cl(B * a))$.
2. $\omega Cl(a * B) \subseteq a * ClB$ *and* $(\omega Cl(B * a) \subseteq (ClB) * a)$.
3. $A * \omega ClB \subseteq Cl(A * B)$ *and* $((\omega ClB) * A \subseteq Cl(B * A))$.

Proof. 1. Let $y \in a * \omega ClB$, and let G be an open subset of X, such that $y \in G$. Then, there is $x \in \omega ClB$ such that y= $a * x$. Since μ is $\omega - continuous$ at the second variable, there exists an ω-open set V in X such that $x \in V$ and $a * V \subseteq G$. Since $x \in V$ and $x \in \omega ClB$, then $V \cap B \neq \phi$, so, there is, $s \in V \cap B$. Then, $a * s \in a * V$ and $a * s \in a * B$, so, $(a * V) \cap (a * B) \neq \phi$. Hence, $G \cap (a * B) \neq \phi$. This means that, $y \in Cl(a * B)$. Thus, $a * \omega ClB \subseteq Cl(a * B)$.

2. By (1) we have $a^{-1}(\omega(Cla*B)) \subseteq Cl(a^{-1}*(a*B)) = Cl(a^{-1}*a)*B = ClB$. Therefore, $a * (a^{-1} * (\omega Cl(a * B)) \subseteq a * ClB$. That is, $\omega Cl(a * B) \subseteq a * ClB$.

3. By (1) $A*\omega ClB = \bigcup_{a\in A}(a*\omega ClB) \subseteq \bigcup_{a\in A} Cl(a*B) \subseteq Cl \bigcup_{a\in A}(a*B) = Cl(A * B)$. □

Theorem 3.4. *Let* $(X, *)$ *be a group and* τ *be a topology on* X*, in which the multiplication map* μ *is* ω*-continuous at the second (first) variable. Then, for each* $A, B \subseteq X$ *and* $a \in X$*, the following statements hold:*

1. $Int(a * B) \subseteq a * \omega IntB$ *and* $(Int(B * a) \subseteq (\omega IntB) * a)$;
2. $a * IntB \subseteq \omega Int(a * B)$ *and* $((IntB) * a \subseteq \omega Int(B * a))$;
3. $A * IntB \subseteq \omega Int(A * B)$ *and* $((IntB) * A \subseteq \omega Int(B * A))$.

Proof. 1. Let $y \in Int(a * B)$. Then, there is an open set O in X such that $y \in O \subseteq a * B$, then there is $b \in B$ such that $y = a * b$. By ω-continuity of μ at the second variable, there exists an ω-open subset V of X such that $b \in V$ and $a * V \subseteq O$, that is, $a * V \subseteq a * B$, so, $a^{-1} * (a * V) \subseteq a^{-1} * (a * B)$, hence $V \subseteq B$. This means that, $b \in \omega IntB$, so, $y = a * b \in a * \omega IntB$. Hence, $Int(a * B) \subseteq a * \omega IntB$.

2. $a * IntB = a * Int(e * B) = a * Int(a^{-1} * (a * B)) \subseteq a * (a^{-1}\omega Int(a * B)) = (a * a^{-1}) * \omega Int(a * B) = e * \omega Int(a * B) = \omega Int(a * B)$.

3. $A * IntB = \bigcup_{a\in A}(a * IntB) \subseteq \bigcup_{a\in A} \omega Int(a * B) \subseteq \omega Int \bigcup_{a\in A}(a * B) = \omega Int(A * B)$. □

Theorem 3.5. *Let* $(X, *)$ *be a group and* τ *be a topology on* X*, then:*

1. *the multiplication map* μ *is* ω*-continuous at the second variable if and only if the left translation function* $\iota_a : X \to X$ *is* ω*-continuous, for each* $a \in X$*;*
2. *the multiplication map* μ *is* ω*-continuous at the first variable if and only if the right translation function* $r_a : X \to X$ *is* ω*-continuous, for each* $a \in X$*.*

Proof. We prove (1) and the proof of (2) is completely similar.

Let the multiplication map μ is $\omega-continuous$ at the second variable. To show that ι_a is $\omega-continuous$, for each $a \in X$.

Let $x \in X$ and O be any open subset of X such that $\iota_a(x) \in O$ (That is, $a * x \in O$.) So, there is an $\omega-open$ set V in X such that $x \in V$ and $a * V \subseteq O$, that is $\iota_a(V) \subseteq O$, this means that, ι_a is $\omega-continuous$ at x. But, since a and x are arbitrary points of X, therefore, ι_a is ω-continuous for each $a \in X$.

Suppose that ι_a is ω-continuous, for each $a \in X$. Now, let a be a fixed point of X, $x \in X$ and O be an arbitrary open subset of X such that $a * x \in O$. That is, $\iota_a \in O$. By ω-continuity of ι_a, there is an ω-open set V in X such that $x \in V$ and $\iota_a(V) \subseteq G$. Hence, $a * V \subseteq O$, so, that μ is $\omega-continuous$ at the second variable. □

Corollary 3.1. *Let τ be any topology on a group $(X, *)$, then:*

1. *the multiplication map μ is ω-continuous at the second variable if and only if the left translation function ι_a is ω-irresolute, for each $a \in X$;*

2. *the multiplication map μ is ω-continuous at the first variable if and only if the right translation function r_a is ω-irresolute, for each $a \in X$.*

Proof. 1. Since μ is $\omega-continuous$ at the second variable, so, by Theorem 3.5, the left translation function ι_a is $\omega-continuous$, for each $a \in X$. Since ι_a is bijective, ι_a is ω-irresolute, for each $a \in X$.

Conversely, let ι_a be $\omega-irresolute$ for each $a \in X$. Then, it is ω-continuous, for each $a \in X$. By Theorem 3.5, μ is ω-continuous at the second variable.

2. The proof is similar to the proof of (1). □

Proposition 3.1. *Let τ be a topology on a group $(X, *)$. Then, the left (right) translation function $\iota_a(\ r_a)$ is ω-continuous if and only if it is ω-homeomorphism, for each $a \in X$.*

Proof. Let ι_a (r_a) be an ω-continuous function, for each $a \in X$. Then, ι_a (r_a) respectively, is ω-irresolute for each $a \in X$. Since ι_a (r_a) is a bijective function with $(\iota a)^{-1}(V) = \iota a^{-1}(V) = V * a^{-1}$, and $a^{-1} \in X$, then $\iota_a^{-1}(\ r_a^{-1})resp.$, is an $\omega-irresolute$ function. Hence, ι_a (r_a) is ω-homeomorphism, for each $a \in X$. □

Proposition 3.2. *Let τ be a topology on a group $(X, *)$. Then:*

1. the multiplication map μ is ω-irresolute at the second variable if and only if the left translation function ι_a is ω-irresolute, for each $a \in X$.

2. the multiplication map μ is ω-irresolute at the first variable if and only if the right translation function r_a is ω-irresolute, for each $a \in X$.

Proof. The proof is completely similar to the proof of the Theorem 3.5. □

Proposition 3.3. *Let τ be a topology on a group $(X, *)$. The multiplication map μ is ω-irresolute at the second (resp., first) variable if and only if it is $\omega-continuous$ at the second (first) variable.*

Proof. Let μ is $\omega - irresolute$ at the second (first) variable if and only if $\iota_a(resp.,\ r_a)$ is $\omega - irresolute$, for each $a \in X$ by Proposition 3.3 if and only if μ is $\omega - continuous$ at the second (resp., second) variable by Corollary 3.1. □

Proposition 3.4. *Let τ be a topology on a group $(X, *)$. Then, the multiplication map μ is ω-irresolute at the second (first) variable if and only if it is ω-continuous at the second (first) variable.*

Proof. we can show that μ is $\omega - irresolute$ at the second (first) variable by the same way as we have proved Theorem 3.5 and Corollary 3.1, we will get the left translation ι_a (right translation r_a) function is $\omega - irresolute$, for each $a \in X$. If and only if μ is ω-continuous at the second (first) variable. □

Theorem 3.6. *If τ is a topology on a group $(X, *)$ such that the multiplication map μ is ω-continuous at the second variable, then for each $A, B \subseteq X$ and $a \in X$, we have:*

1. $a * \omega ClB = \omega Cl(a * B)$.
2. $a * \omega IntB = \omega Int(a * B)$.
3. *B is ω-open if and only if* $a * B\, is\, \omega - open$.
4. *B is ω-closed if and only if* $a * B\, is\, \omega - closed$.
5. $A * \omega ClB \subseteq \omega Cl(A * B)$.
6. $A * \omega IntB \subseteq \omega Int(A * B)$.
7. $\omega IntA * \omega IntB \subseteq \omega Int(A * B)$.
8. $\omega ClA * \omega ClB \subseteq \omega Cl(A * B)$.
9. *If B is $\omega - open$, then $A * B$ is $\omega - open$.*
10. *If B is ω-closed and A is finite, then $A * B$ is ω-closed.*

Proof. 1. Let $y \in a * \omega ClB$. Then, $y = a * b$ for some $b \in \omega ClB$. Let G be any $\omega - open$ subset of X such that $y = a * b \in G$. By Proposition 3.2 there exists an ω-open subset V of X such that $b \in V$ and $a * V \subseteq G$. Since $b \in \omega ClB$, so, $V \cap B \neq \phi$. Therefore, $a * V \cap a * B \neq \phi$. Since $a * V \subseteq G$, so, $G \cap (a * B) \neq \phi$. This means that, $y \in \omega Cl(a * B)$. That is, $a * \omega ClB \subseteq \omega Cl(a * B)$. Also, $a^{-1} * (\omega Cl(a * B) \subseteq \omega Cl(a^{-1} * (a * B)) = \omega Cl((a * a^{-1}) * B) = \omega Cl(e * B) = \omega ClB$. Then, $a * (a^{-1} * \omega Cl(a * B)) \subseteq a * \omega ClB$, so, that $\omega Cl(a * B) \subseteq a * \omega ClB$. Hence, $a * \omega ClB = \omega Cl(a * B)$.

2. Let $y \in \omega Int(a * B)$. Then, there exists $x \in B$ and an $\omega - open$ set V in X such that $y = a * x \in V \subseteq a * B$. By Proposition 3.2, there exists an $\omega - open$ set U in X such that $x \in U$ and $a * U \subseteq V$. Thus, $a * U \subseteq a * B$,

so, $U \subseteq B$. This means that, $x \in \omega IntB$. Then, $y = a * x \in a * \omega IntB$. So, $\omega Int\, a * B \subseteq a\,\omega IntB$. Now, Since $a^{-1} \in X$ and $a * B \subseteq X$, we get $\omega IntB = \omega Int(e * B) = \omega Int(a^{-1} * (a * B)) \subseteq a^{-1} * \omega Int(a * B)$. Therefore, $a*\omega IntB \subseteq (a*a^{-1}*\omega Int(a*B) = \omega Int(a*B)$. Hence, $a*\omega IntB = \omega Int(a*B)$.

3. Let B be $\omega - open$ in X. From Corollary 3.1, we have ι_a^{-1} is ω-irresolute, so, $(\iota_a^{-1})^{-1}(B)$ is ω-open in X. Since $(\iota_a^{-1})^{-1} = \iota_a$, so, $\iota_a(B)$ is ω-open in X. Thus, $a * B$ is ω-open in X.

Conversely, Let $a * B$ be $\omega - open$ in X. From Corollary 3.1, we have ι_a is ω-irresolute, then $\iota_a^{-1}(a * B)$ is ω-open in X. Since $(\iota_a)^{-1} = \iota_a^{-1}$, so, $\iota_a^{-1}(a * B)$ is ω-open in X. Since $\iota_a^{-1}(a * B) = a^{-1} * (a * B) = B$, so, B is $\omega - open$ in X.

4. Let B be $\omega - closed$ in X. Then, by (1), $a * B = a * \omega ClB = \omega Cl(a * B)$, so, $a * B$ is ω-closed.

Conversely, suppose that $a * B$ is an ω-closed subset of X, so, $a * B = \omega Cl(a*B)$. But, from (1), we have $\omega Cl(a*B) = a*\omega ClB$, so $a*B = a*\omega ClB$. This implies that $a^{-1} * (a * B) = a^{-1} * (a * \omega ClB)$. Hence, $B = \omega ClB$. Thus, B is $\omega - closed$ in X.

5. Let $y = a * b \in A * \omega ClB$, where $a \in A$ and $b \in \omega ClB$. To show $y \in \omega Cl(A * B)$. Let G be any ω-open subset of X such that $y = a * b \in G$. By Proposition 3.2, there exists an ω-open subset V of X such that $b \in V$ and $a * V \subseteq G$, since $b \in V$ and $b \in \omega ClB$, so, $V \cap B \neq \phi$, so, $(a * V) \cap (a * B) \neq \phi$. Since $a*V \subset G$, so, $G \cap (a*B) \neq \phi$ and since $a*B \subseteq A*B$, so, $G \cap (A*B) \neq \phi$. Hence, $y \in \omega Cl(A * B)$. Thus, $A * \omega ClB \subseteq \omega Cl(A * B)$.

6. By (2), we have $A * \omega IntB = \bigcup_{a \in A}(a * \omega IntB) = \bigcup_{a\in A}(\omega Int(a * B) \subseteq \omega Int(\bigcup_{a\in A}(a * B)) = \omega Int(A * B)$.

7. Since $\omega IntA \subseteq A$, so, $\omega IntA*\omega IntB \subseteq A*\omega IntB$ and since $A*\omega IntB \subseteq \omega Int(A * B)$. So, by (6) $\omega IntA * \omega IntB \subseteq \omega Int(A * B)$.

8. Let $y \in \omega ClA * \omega ClB$. Then, $y = a * b$, for some $a \in \omega ClA$, $b \in \omega ClB$. Let G be any $\omega - open$ subset of X such that $y = a * b \in G$. By Proposition 3.2, there is an $\omega - open\, subset\, V$ of X such that $b \in V$ and $a * V \subseteq G$. Since $b \in \omega ClB$, so, $V \cap B = \phi$. Since $a*(V \cap B) = (a*V) \cap (a*B)$, so, $G \cap (a*B) = \phi$. Since $a * B \subseteq A * B$, then $G \cap (A * B) = \phi$. Therefore, y$\in \omega Cl(A * B)$. Hence, $\omega ClA * \omega ClB \subseteq \omega Cl(A * B)$.

9. Let B be $\omega - open$ in X. Then by (3) $a * B$ is $\omega - open$, for each $a \in A$. Since, the union of any family of $\omega - open$ sets is ω-open, so, $\bigcup_{a\in A}(a * B)$ is $\omega - open$. But, since $A * B = \bigcup_{a\in A}(a * B)$, so, $A * B$ is $\omega - open$.

10. Let B be ω-closed and A be a finite subset of X. Then, by (4) $a * B$ is ω-closed, for each $a \in A$. Since $A * B = \bigcup_{a\in A}(a * B)$ and the finite union of $\omega - closed$ is ω-closed, so, $A * B$ is ω-closed. □

Theorem 3.7. *Let $(H, *)$ be a subgroup of a group $(X, *)$ and τ be any topology on X.*

1. *If $\mu : X * X \to X$ is ω-continuous at the second variable, then $\mu_H : H * H \to H$ is ω-continuous at the second variable.*

2. *If $\mu : X*X \to X$ is ω-continuous at the first variable, then $\mu_H : H*H \to H$ is ω-continuous at the first variable.*

Proof. We prove part (1) and the proof of the second part is almost similar.

Let a be a fixed point of H such that $\mu_H(a,b) = a*b \in G$. Then, there is an open set O in X such that $O = G \cap H$ and $\mu(a,b) = \mu_H(a,b) = a*b \in O$. Since μ is ω-continuous at the second variable, so, by Definition 3.1, there is an ω-open subset V of X such that $b \in V$ and $a*V \subseteq O$. Then, by Theorem 2.2, $V \cap H$ is ω-open in H and $a*(V \cap H) = a*V \cap a*H = a*V \cap H \subseteq O \cap H = G$. Hence, $\mu_H : H*H \to H$ is ω-continuous at the second variable. □

Theorem 3.8. *Let τ be a topology on a group $(X,*)$ such that the multiplication map μ is ω-continuous at the second (first) variable. If S is a semigroup subset of X for which $\omega IntS \neq \phi$, then $\omega IntS$ is also a semigroup.*

Proof. Without loss of generality, we assume that μ is ω-continuous at the second variable. It is given that, $\omega IntS \neq \phi$. Let $a, b \in \omega IntS$, then, there is an $\omega - open$ subset V of X such that $b \in V \subseteq S$. Since S is a semigroup, so, $a*b \in a*V \subseteq S$. But, from (3) of Theorem 3.6 we have $a*V$ is $\omega - open$ in X, so, $a*b \in \omega IntS$. Also, since $\omega IntS \subseteq S$ and μ is associative on S, so, μ is associative on $\omega IntS$. Hence, $\omega IntS$ is a semigroup. □

Theorem 3.9. *Let H be subgroup of a group X. Let τ be any topology on X such that the multiplication map μ is ω-continuous at the second (first) variable and $\omega Int(H) \neq \phi$. If the function $f : X \to X$ give by $f(x) = x^{-1}$ for each $x \in X$ is ω-continuous then $\omega Int(H)$ is a subgroup of X.*

Proof. Without loss of generality, we assuming that μ is ω-continuous at the second variable. By what we have done in the proof of Theorem 3.8 for any $a, b \in \omega Int(H)$ we obtain that $a*b \in \omega IntH$. Also, for any $a \in \omega IntH$, we have an ω-open subset G of X such that $a \in G \subseteq H$, $f : X \to X$ is a bijective function and it is ω-continuous function. Since V is ω-open in X, so, $f^{-1}(V) = \{x : f(x) \in V\} = \{x : x^{-1} \in V\} = V^{-1}$ so, V^{-1} is ω-open in X. Since $a \in V \subseteq H$, so, $a^{-1} \in V^{-1} \subseteq H^{-1} = H$. Hence, $a^{-1} \in \omega IntH$. Therefore, $a*b^{-1} \in \omega IntH$. Hence, $\omega IntH$ is subgroup of G. □

Theorem 3.10. *Let τ be a topology on a group $(X,*)$ such that the multiplication map μ is ω-continuous at the second (first) variable. If S is a semigroup subset of X, then ωClS is a semigroup subset of X.*

Proof. We prove this result for the case that μ is ω-continuous at the second variable, we left the other because it has a similar proof. Since $\phi \neq S \subseteq \omega ClS$. So, $\omega ClS \neq \phi$. Let a, b be any two points of ωClS and V is any ω-open subset of X which contains $a*b$. Since μ is ω-continuous at the second variable, so, by Corollary 3.1, the left translation function l_a is an ω-irresolute, for each $a \in X$. Now, for $a, b, c \in \omega ClS$, we have $a, (b*c), (a*b), c \in \omega ClS$. Therefore,

$a * (b * c), (a * b) * c \in \omega ClS$. Since $(X, *)$ is a group and $a, b, c \in X$, so, $a*(b*c) = (a*b)*c$. This means that $a*(b*c) = (a*b)*c$ in ωClS. Therefore, ωClS is a semigroup subset of X. □

Theorem 3.11. *Let H be a subgroup of a group $(X, *)$. Let τ be a topology on X such that the function $f : X \to X$ given by $f(x) = x^{-1}$ is ω-continuous. If the multiplication map μ is ω-continuous at either first or second variable, then ωClH is a subgroup of X.*

Proof. Since every subgroup of a group is a semigroup, so, by Theorem 3.10, ωClH is a semigroup subset of X. For all $a, b \in \omega ClH$, we get $a * b \in \omega ClH$. Since $f : X \to X$ given by $f(x) = x^{-1}$ is a bijective $\omega - continuous$ function, so, f is ω-irresolute, for each $a \in \omega ClH$, and any $\omega - open$ subset V of X such that $a^{-1} \in V$, we have $f(a) \in V$. So, by $\omega - irresolute$ of f, there exists an ω-open subset U of X such that $a \in U$ and $f(U) \subseteq V$. Since $a \in U$ and $a \in \omega ClH$, so, $U \cap H \neq \phi$. Thus, $(U \cap H)^{-1} = \phi$. Since $(U \cap H)^{-1} = U^{-1} \cap H^{-1} = U^{-1} \cap H$, so, $U^{-1} \cap H \neq \phi$, but $U^{-1} \subseteq V$, so, $V \cap H \neq \phi$. Hence, $a^{-1} \in \omega ClH$. Therefore, for each $a, b \in \omega ClH$, we have $a, b^{-1} \in \omega ClH$, and so, $a * b^{-1} \in \omega ClH$. This means that ωClH is a subgroup of X. □

Remark 3.2. It is easy to prove the same result for a topology on the group $(X, *)$ which makes the multiplication map μ as $\omega - continuous$ at the first variable, but, we need only to replace a with b, $a * V$ with $V * a$.

Theorem 3.12. *Let $(X, *)$ be a group and τ be any topology on X such that the multiplication map μ is ω-continuous at each variable. If S is a normal set of X such that $\omega IntS \neq \phi$, then both $\omega Int(S)$ and $\omega Cl(S)$ are normal.*

Proof. Let $x \in X$. Then, $x^{-1} \in X$. Since $\omega Int(S)$ is ω-open and μ is ω-continuous at each variable, then by (3) of Theorem 3.6 and as μ is ω-continuous at the first variable, we obtain that $x * \omega Int(S)x^{-1}$ is ω-open in X and $\omega Int(x * \omega Int(S) * x^{-1}) = x * \omega Int(S) * x^{-1}$. Since S is a normal set, so $x * \omega Int(S)x^{-1} \subseteq x * S * X^{-1} \subseteq S$, so $\omega Int(x * \omega Int(S)x^{-1}) \subseteq \omega Int(S)$. Therefore, $x * \omega Int(S)x^{-1} \subseteq \omega Int(S)$. Hence, $\omega Int(S)$ is a normal subset of X.

Now, we have to show $\omega Cl(S)$ is also a normal subset of X. To make this end, let $y \in x * \omega Cl(S) * x^{-1}$ and G be any ω-open subset of X such that $y \in G$. Then, there is $s \in \omega Cl(S)$ such that $y = x * s * x^{-1}$ by Proposition 3.2 there exists an ω-open subset V of X such that $s * x^{-1} \in V$ and $x * V \subseteq G$. Again by Proposition 3.2 there is an ω-open subset U in X such that $s \in U$ and $U*x^{-1} \subseteq V$. That is, $x*U*x^{-1} \subseteq x*V \subseteq G$. Now, since $s \in U$ and $s \in \omega Cl(S)$, then $U \cap S \neq \phi$, so, $(x * U * x^{-1}) \cap (x * S * x^{-1}) \neq \phi$. Since $(x * U * x^{-1}) \subseteq G$ and $x * S * x^{-1} \subseteq S$, so, $G \cap S \neq \phi$. This implies that $y \in \omega Cl(S)$. Thus, $x * \omega Cl(S) * x^{-1} \subseteq \omega Cl(S)$. Hence, $\omega Cl(S)$ is a normal subset of X. □

Corollary 3.2. *Let τ be a topology on a group $(X, *)$ such that the multiplication map is ω-continuous at the first (second) variable. If H is a normal subgroup* *of X and the function $f : X \to X$ given by $f(x) = x^{-1}$ for all $x \in X$, is ω-continuous, then $\omega IntH \neq \phi$ and ωClH both are normal subgroup of X.*

Proof. The proof follows from Theorem 3.9, Theorem 3.11 and Theorem 3.12. □

Theorem 3.13. *Let $(X, *)$ be a group and τ be any topology on X. If the multiplication map is ω-continuous in the second variable, then for any $A \subseteq X$, we have $A * B$ is $\omega - open$ for any open set $B \subseteq X$.*

Proof. If B is open, then $Int(B) = B$. Let $a \in A$. Then, $a * B = a * Int(B) \subseteq \omega Int(a * B)$ by (1) of Theorem 3.4. Hence, $A * B = A * Int(B) = \bigcup_{a\in A} a * Int(B) \subseteq \bigcup_{a\in A} \omega Int(a * B)$, so, $\bigcup_{a\in A} \omega Int(a * B) \subseteq \omega Int(\bigcup_{a\in A} a * B) = \omega Int(A * B)$, since $\omega Int(A) \cup \omega Int(B) \subseteq \omega Int(A * B)$, $A * B \subseteq \omega Int(A * B)$. Hence, $A * B = \omega Int(A * B)$. Thus, $A * B$ is $\omega - open$. □

Theorem 3.14. *Let the multiplication map μ of a group X with a topology τ on X is ω-continuous at the second (first) variable and $H \subseteq X$. Then:*

1. *If H is an ω-compact subset of X, then $a * H$ $(H * a)$ is a compact subset of X, for each $a \in X$.*
2. *If μ is ω-continuous at the second variable, then for each $a \in X$, where, H is ω-compact in X if and only if $a * H$ is ω-compact.*
3. *If μ is ω-continuous at the first variable, then for each $a \in X$, where, H is ω-compact in X if and only if $H * a$ is ω-compact.*

Proof. 1. Let H be an ω-compact subset of X and without loss of generality, we suppose that μ is ω-continuous at the second variable, so, by Theorem 3.5, ι_a is ω-continuous for each $a \in X$. Now, to show $a * H$ is compact. Let $\{\{V_\alpha\}_n; \alpha \in n\}$ be an open cover of $a * H$. Then, $(\iota_a)^{-1}(V_\alpha) = a^{-1} * V_\alpha$ is ω-open for each $\alpha \in \Lambda$. Since $H = (\iota_a)^{-1})(a * H) \subseteq (\iota_a)^{-1})(\bigcup_{\alpha\in\Lambda} V_\alpha) = \bigcup_{\alpha\in\Lambda}(\iota_a)^{-1})(V_\alpha) = \bigcup_{\alpha\in\Lambda} a^{-1} * V_\alpha$, so, $\{a^{-1} * (V_\alpha); \alpha \in \Lambda\}$ is an ω-open cover of H. So, by definition ω-compact, there exists a finite subset Λ_0 of Λ, such that $H \subseteq \bigcup_{\alpha\in\Lambda_0}(a^{-1} * V_\alpha)$. Hence, $a * H = \iota_a(H) \subseteq \iota_a(\bigcup_{\alpha\in\Lambda_0}(a^{-1} * V_\alpha) = a * (\bigcup_{\alpha\in\Lambda_0} a^{-1} * V_\alpha) = (a * a^{-1}) * (\bigcup_{\alpha\in\Lambda_0} V_\alpha = \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Thus, $a * H$ is a compact subset of X.

2. Let H be an ω-compact subset of X and $\{G_N : N \in \Lambda\}$ is an ω-open cover of $a*H$ where a is an arbitrary point of X. Since μ is ω-continuous at the second variable, so, by Corollary 3.1 ι_a is an ω-irresolute function. Therefor $(\iota_a)^{-1}(G)$ is ω-open in X, for each $N \in \Lambda$. Since $(\iota_a)^{-1}(G) = a^{-1} * G_N$. So, $a^{-1} * G_N$ is ω-open in X for each $N \in \Lambda$. Since $H = (\iota_a)^{-1}(a * H) \subseteq (\iota_a)^{-1}(\bigcup_{N\in\Lambda} G_N) = \bigcup_{N\in\Lambda}(\iota_a)^{-1}(G_N) = \bigcup_{N\in\Lambda}(a^{-1} * G_N)$. So, $\{a^{-1} * G_N : N \in \Lambda\}$ is an ω-open cover of H. Since H is ω-compact, so, there exists a finite subset Λ_0 of Λ such that $H = U_{N\in\Lambda_0}(a^{-1} * G_N)$. So, $aH = \iota_a(H) \subseteq \iota_a(\bigcup_{N\in\Lambda_0} a^{-1} * G_N) = \bigcup_{N\in\Lambda_0} \iota a(a^{-1} * G_N) = \bigcup_{N\in\Lambda} G_N$. Hence, $a * H$ is ω-compact.

Conversely, let $a * H$ be an ω-compact subset of X where $a \in X$. To show H is ω-compact.

Let $\{O_N, N \in \Lambda\}$ be any ω-open set in H. Then $a * H = \iota_a(H) = (\iota_a)^{-1})^{-1}(H) = (\iota_a^{-1})^{-1}(\bigcup_{N\in\Lambda} O_N) = (\bigcup_{N\in\Lambda}((\iota_a^{-1})^{-1}(O_N)) = (\bigcup_{N\in\Lambda} \iota_a(O_N) = \bigcup_{N\in\Lambda}(a * O_N)$.

Since O_N is $\omega - open$ for each $N \in \Lambda$ and $a \in X$, so, by part (3) of Theorem 3.6, we have $a * O_N$ is ω-open in X, for each $N \in \Lambda$. Since $a * H$ is ω-compact, so, there exists a finite subset Λ_0 of Λ such that $a * H \subseteq \bigcup_{N\in\Lambda}(a * O_N)$. So, $H = a^{-1} * (a * H) = \iota_a^{-1}(a * H) = (\iota_a)^{-1}(a * H) \subseteq (\iota_a)^{-1}(\bigcup_{N\in\Lambda_0}(a * O_N))$= $(\bigcup_{N\in\Lambda_0}(\iota_a)^{-1})(a * O_N)$=$\bigcup_{N\in\Lambda_0} a^{-1} * (a * O_N) = \bigcup_{N\in\Lambda_0} O_N$. Hence, H is ω-compact.

3. The proof is similar to part 2 with only replacing ι_a with r_a. □

Theorem 3.15. *Let τ be a topology on a group $(X, *)$ and $a \in X$, $H \subseteq X$.*

1. *If μ is $\omega - continuous$ at the second variable, then H is lindelof if and only if $a * H$ is lindelof.*
2. *If μ is $\omega - continuous$ at the first variable, then H is lindelof if and only if $H * a$ is lindelof.*

Proof. The Proof is almost similar to the proof of parts (2) and (3) of the Theorem 3.14 by using Lemma 2.1. □

References

[1] Al-T.A. Hawary, *ω-generalized, closed set*, Inter. J. Appl. Math., 16 (2004), 341-353.

[2] Alias B. Khalaf, Halgwrd M. Darwesh, K. Kannan, (2012), *Some types of separation axiom in topological spaces*, 28 (2012), 303-326.

[3] T.A. Al-Hawary, A. Al-Omari, *Between open and ω-open sets Q and A in general topology*, 24 (2006), 67-77.

[4] T.A. Al-Hawary, A. Al-Omari, *ω-continuous like mappings*, Al Manarah J., 13 (2007), 135-147.

[5] Al-Omari, M.S. Noorain, *Contra-ω-continuous and almost contra-ω-continous*, Int. J. of Math. and Math. Sci., 2007, Article ID 40469.13 pages.

[6] David Dummit, Richard Foote, *Abstract algebra,* Wiley, 2003.

[7] H.Z. Hdeib, *ω-closed mapping*, Revista Colobianade Mathematics, 16 (1982), 65-78.

[8] H.Z. Hdeib, *ω-continues functions*, Dirasta J., 16 (1989), 136-153.

[9] S. Willard, *General topology*, Addision Wesly London, 1970.

[10] Sh. M. Prabhavati, Sh. W. Revanasiddappa, *On $\alpha r\omega$-homeomorphism in topological space*, 21 (2018), 68-77.

[11] Lynn Arthur Steen, J. Arthur Jr. Seebach, *Countrer examples in topology*, (Dover reprint of 1978 ed.) Berlin, New York, Springer-Verlag, 1995.

[12] James R. Munkres, *Topology*, (2nd ed.), Prentice-Hall, 2000.

[13] David Dummit, Richard Foote, *Abstract algebra* (3rd Edition), Wiley, 2003.

[14] T. A. Al-Hawary, A. Al-Omari, *ω-decompositions of continuity*, Turk. J. Math., 30 (2006), 187-195.

[15] R. Engelking, *Dimension theory*, North Holland, Amsterdam, 1978.

[16] T. Noiri, A. Al-Omari, M.S. Noorani, *On ωb-open sets and be Lendelof spaces*, Eur. J. of Pure and Appl. Math., 1 (2008), 3-9.

[17] Dylan Spivak, *An introduction to topological group*, Lakehead University, Thunder Bay, Ontario, Canada. Math. 4301, 2015.

Accepted: June 9, 2022

The group of integer solutions of the Diophantine equation $x^2 + mxy + ny^2 = 1$

Ghader Ghasemi
Department of Mathematics
Faculty of Sciences
University of Mohaghegh Ardabili
56199-11367, Ardabil
Iran
ghasemi@uma.ac.ir

Abstract. Let m and n be two integers. It is shown that the set of all integer solutions of the Diophantine equation $x^2 + mxy + ny^2 = 1$ has an Abelian group structure. Furthermore, it is shown that this Abelian group is isomorphic to one of the groups $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}$.

Keywords: Abelian group, commutative ring, Diophantine equation, Pell's equation, torsion subgroup.

1. Introduction

In mathematics, a Diophantine equation is a polynomial equation, usually involving two or more unknowns, such that the only solutions of interest are the integer ones (an integer solution is such that all the unknowns take integer values).

Recall that an elliptic curve is a plane curve over a finite field (rather than the real numbers) which consists of the points satisfying the equation $y^2 = x^3 + ax + b$, along with a distinguished point at infinity. The coordinates here are to be chosen from a fixed finite field of characteristic not equal to 2 or 3. This set together with the group operation of elliptic curves is an Abelian group, with the point at infinity as an identity element.

Pursuing this point of view further, in this paper we focused on the set of the points satisfying the equation $x^2 + \eta xy + \xi y^2 = 1_R$, where the coordinates are to be chosen from a commutative ring R with the identity element 1_R. We prove that this set together with a suitable group operation is an Abelian group, with $e = (1_R, 0_R)$ as the identity element. Also, by using this result we study the set of all integer solutions of the Diophantine equation $x^2 + mxy + ny^2 = 1$, where $m, n \in \mathbb{Z}$. Recall that one special case of these equations is the *Pell's equation*, which has a historical background.

We prove that in general the Abelian group of all integer solutions of the Diophantine equation $x^2 + mxy + ny^2 = 1$ is isomorphic to one of the groups

$\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}$. Also, we show that the set of all integer solutions of the *Pell's equation* as an Abelian group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$.

Throughout this paper, for each element g of a given group $(G, *)$ we denote the order of g by $o(g)$. Also, for each subgroup H of G we denote the order of H by $|H|$. For any unexplained notation and terminology, we refer to [1] and [2].

2. The results

We start this section with the following theorem.

Theorem 2.1. *Let $(R, +, \cdot)$ be a commutative ring with the identity element 1_R and η, ξ be two arbitrary elements of R. Set*

$$G(R, \eta, \xi) := \left\{(a, b) \in R \times R \ : \ a^2 + \eta ab + \xi b^2 = 1_R\right\}.$$

Define the binary operation $*$ on $G(R, \eta, \xi)$ as $(a, b) * (c, d) := (ac - \xi bd, bc + ad + \eta bd)$, for each $(a, b), (c, d) \in G(R, \eta, \xi)$. Then, $(G(R, \eta, \xi), *)$ is an Abelian group with the identity element $e = (1_R, 0_R)$ such that $(a, b)^{-1} = (a + \eta b, -b)$, for each $(a, b) \in G(R, \eta, \xi)$.

Proof. For each $g = (a, b)$, $g' = (c, d) \in G(R, \eta, \xi)$, by the definition we have

$$a^2 + \eta ab + \xi b^2 = 1_R = c^2 + \eta cd + \xi d^2.$$

Therefore,

$$\begin{aligned} 1_R &= (1_R)(1_R) \\ &= (a^2 + \eta ab + \xi b^2)(c^2 + \eta cd + \xi d^2) \\ &= (ac - \xi bd)^2 + \eta(ac - \xi bd)(bc + ad + \eta bd) + \xi(bc + ad + \eta bd)^2, \end{aligned}$$

which shows that $g * g' = (ac - \xi bd, bc + ad + \eta bd) \in G(R, \eta, \xi)$.

We show that $*$ is associative. For each $g = (a, b)$, $g' = (c, d)$, $g'' = (u, v) \in G(R, \eta, \xi)$, one sees that

$$\begin{aligned} (g * g') * g'' &= (ac - \xi bd, bc + ad + \eta bd) * (u, v) \\ &= (r, s) \\ &= (a, b) * (cu - \xi dv, du + cv + \eta dv) \\ &= g * (g' * g''), \end{aligned}$$

where

$$\begin{aligned} r &= (ac - \xi bd)u - \xi(bc + ad + \eta bd)v \\ &= acu - \xi bdu - \xi bcv - \xi adv - \xi\eta bdv \\ &= acu - \xi adv - \xi bdu - \xi bcv - \xi\eta bdv \\ &= a(cu - \xi dv) - \xi b(du + cv + \eta dv), \end{aligned}$$

and

$$\begin{aligned} s &= (bc+ad+\eta bd)u+(ac-\xi bd)v+\eta(bc+ad+\eta bd)v \\ &= bcu+adu+\eta bdu+acv-\xi bdv+\eta bcv+\eta adv+\eta^2 bdv \\ &= bcu-\xi bdv+adu+acv+\eta adv+\eta bdu+\eta bcv+\eta^2 bdv \\ &= b(cu-\xi dv)+a(du+cv+\eta dv)+\eta b(du+cv+\eta dv). \end{aligned}$$

Moreover, for each $g=(a,b)$, $g'=(c,d)\in G(R,\eta,\xi)$, it is clear that

$$g*g'=(ac-\xi bd, bc+ad+\eta bd)=(ca-\xi db, cb+da+\eta db)=g'*g.$$

Hence, the binary operation $*$ is commutative.

Also, for each $g=(a,b)\in G(R,\eta,\xi)$, we see that

$$e*g=g=g*e,$$

where $e=(1_R,0_R)$. So e is the identity element of $G(R,\eta,\xi)$ with respect to the binary operation $*$.

Let $g=(a,b)\in G(R,\eta,\xi)$ and put $h=(c,d):=(a+\eta b,-b)$. By the definition from the assumption $g=(a,b)\in G(R,\eta,\xi)$ it follows that $a^2+\eta ab+\xi b^2=1_R$, and so

$$\begin{aligned} c^2+\eta cd+\xi d^2 &= (a+\eta b)^2+\eta(-b)(a+\eta b)+\xi(-b)^2 \\ &= a^2+2\eta ab+\eta^2b^2-\eta ab-\eta^2b^2+\xi b^2 \\ &= a^2+\eta ab+\xi b^2=1_R. \end{aligned}$$

Therefore, $h=(c,d)\in G(R,\eta,\xi)$. Also, we have

$$ac-\xi bd=a(a+\eta b)-\xi b(-b)=a^2+\eta ab+\xi b^2=1_R,$$

and

$$bc+ad+\eta bd=b(a+\eta b)+a(-b)+\eta b(-b)=ab+\eta b^2-ab-\eta b^2=0_R.$$

Thus, $h=(c,d)=(a+\eta b,-b)$ is an element of $G(R,\eta,\xi)$ such that

$$h*g=g*h=(ac-\xi bd, bc+ad+\eta bd)=(1_R,0_R)=e.$$

Hence, every element $g=(a,b)\in G(R,\eta,\xi)$ has an inverse in $G(R,\eta,\xi)$ and $g^{-1}=(a+\eta b,-b)$

Now, we are ready to deduce that $(G(R,\eta,\xi),*)$ is an Abelian group with the identity element $e=(1_R,0_R)$. □

The following lemma is needed in the proof of Lemma 2.3.

Lemma 2.1. *Let $(R,+,\cdot)$ be a commutative ring with an identity element and η, ξ be two arbitrary elements of R. Then, for each $g=(a,b)\in G(R,\eta,\xi)$ and each integer $k\geq 2$, there are elements $u_k, v_k\in R$ such that*

$$g^k=(a^k+u_kb^2, kba^{k-1}+v_kb^2).$$

Proof. We use induction on k. Since for $k = 2$ we have

$$g^2 = g * g = (a^2 - \xi b^2, 2ab + \eta b^2),$$

it is clear that the elements $u_2 = -\xi$ and $v_2 = \eta$ satisfy the desired condition. Now, let $k > 2$ and assume that the result has been proved for $k - 1$. Then, by inductive assumption there are elements $u_{k-1}, v_{k-1} \in R$ such that

$$g^{k-1} = (a^{k-1} + u_{k-1}b^2, (k-1)ba^{k-2} + v_{k-1}b^2).$$

Therefore,

$$\begin{aligned} g^k &= g * g^{k-1} \\ &= (a, b) * (a^{k-1} + u_{k-1}b^2, (k-1)ba^{k-2} + v_{k-1}b^2) \\ &= (a^k + u_k b^2, kba^{k-1} + v_k b^2), \end{aligned}$$

where

$$u_k = au_{k-1} - \xi bv_{k-1} - \xi(k-1)a^{k-2}, \text{ and } v_k = bu_{k-1} + (a+\eta b)v_{k-1} + \eta(k-1)a^{k-2}.$$

This completes the inductive step. □

In the sequel for each pair of integers m and n let $(\mathfrak{B}_{m,n}, *)$ denote the Abelian group $(G(\mathbb{Z}, m, n), *)$. The remainder of this section will be devoted to a discussion about the basic properties of the Abelian groups $(\mathfrak{B}_{m,n}, *)$, where $m, n \in \mathbb{Z}$.

Lemma 2.2. *Let m and n be two integers. Then, the following statements hold:*

i) *Suppose that $g = (a, b) \in \mathfrak{B}_{m,n}$. Then, $o(g) = 2$ if and only if $g = (-1, 0)$.*

ii) *Assume that $g = (a, b) \in \mathfrak{B}_{m,n}$ is an element of finite order k for some $k \geq 3$. Then, b divides k.*

iii) *Let p be a prime integer. If there is an element $g = (a, b) \in \mathfrak{B}_{m,n}$ of order p, then either $p = 2$ or $p = 3$.*

Proof. (i) If $o(g) = 2$, then it is clear that $(a, b) = g = g^{-1} = (a + mb, -b)$. Hence, $b = 0$ and $g = (a, 0)$. Since

$$(1, 0) = e = g^2 = (a, 0) * (a, 0) = (a^2, 0),$$

we see that $a = \pm 1$. Also, from the hypothesis $o(g) = 2$, we get $g \neq (1, 0)$, which implies that $a = -1$ and $g = (-1, 0)$. Conversely, if $g = (-1, 0) \in \mathfrak{B}_{m,n}$, then we see that $o(g) = 2$. Thus, $o(g) = 2$ if and only if $g = (-1, 0)$.

(ii) We claim that $b \neq 0$. Assume the opposite. Then, $g = (a, 0)$ and so

$$(1, 0) = e = g^k = (a^k, 0).$$

Hence, $a = \pm 1$ and so $g = e$ or $g = (-1, 0)$. Thus, $g^2 = e$ and so $k = o(g) \leq 2$, which is a contradiction. By the definition we have $g^k = e = (1, 0)$ and by Lemma 2.2 there are elements $u, v \in \mathbb{Z}$ such that

$$g^k = (a^k + ub^2, kba^{k-1} + vb^2).$$

Therefore, $kba^{k-1}+vb^2 = 0$. Since $a^2+mab+nb^2 = 1$, it is clear that the integers a and b are relatively prime and so the integers a^{k-1} and b are relatively prime as well. Also, from the assumption $b \neq 0$ and the relation $kba^{k-1} + vb^2 = 0$, we can deduce that $ka^{k-1} = -vb$. Therefore, b divides k.

(iii) Assume the opposite. Then, there is a prime integer $p > 3$ such that $o(g) = p$ for some element $g = (a, b) \in \mathfrak{B}_{m,n}$. We claim that $b = \pm 1$. Assume the opposite. Then, we have $b \neq \pm 1$. By (ii) we know that b divides p. Since p is a prime integer and $b \neq \pm 1$, it is concluded that $b = \pm p$. Furthermore, by Lemma 2.2 there are integers u', v' such that

$$(1, 0) = e = g^p = (a^p + u'b^2, pba^{p-1} + v'b^2) = (a^p + p^2u', \pm p^2a^{p-1} + p^2v').$$

From the relation $a^p + p^2u' = 1$ it follows that a^p is congruent to 1 (mod p). Also, by *Fermat's Theorem* we know that a^p is congruent to a (mod p). Thus, a is congruent to 1 (mod p) and hence $2a$ is congruent to 2 (mod p). Furthermore, since p is an odd prime it can be seen that the following element

$$g^2 = (a^2 - nb^2, 2ab + mb^2) = (a^2 - p^2n, \pm 2pa + p^2m),$$

is of order p as well. Now, if $\pm 2pa + p^2m \neq \pm 1$ then by the same argument it follows that

$$\pm 2pa + p^2m = \pm p.$$

Consequently, $\pm 2a + mp = \pm 1$. Therefore, $2a$ is congruent to ± 1 (mod p). Thus, 2 is congruent to ± 1 (mod p). Hence, we must have $p = 3$, which is a contradiction. Therefore, $\pm 2pa + p^2m = \pm 1$. So, we have $p(\pm 2a + pm) = \pm 1$. Hence, $p = \pm 1$, which is a contradiction. Therefore, $g = (a, b) = (a, \pm 1)$ and $b^2 = 1$. Moreover, since the element $g^2 = (a^2 - nb^2, 2ab + mb^2) = (a^2 - n, 2ab + m) \in \mathfrak{B}_{m,n}$ is of order p, by the same argument we see that $2ab + m = \pm 1$. Hence, $ab = \frac{-1-m}{2}$ or $ab = \frac{1-m}{2}$. By using the assumption $b = \pm 1$, from these relations we obtain

$$(a, b) \in \left\{ (\frac{-1-m}{2}, 1), (\frac{1+m}{2}, -1), (\frac{1-m}{2}, 1), (\frac{-1+m}{2}, -1) \right\}.$$

Hence, there are at most four elements $g = (a, b)$ of order p in $\mathfrak{B}_{m,n}$. Clearly, all of the $p - 1$ distinct elements $g, g^2, ..., g^{p-1}$ are of order p. This observation shows that the only possible case is $p = 5$. Also, in this situation the set $\left\{ (\frac{-1-m}{2}, 1), (\frac{1+m}{2}, -1), (\frac{1-m}{2}, 1), (\frac{-1+m}{2}, -1) \right\}$ is a subset of $\mathfrak{B}_{m,n}$ and all of its elements are of order p. Set $h = (u, v) := (\frac{-1-m}{2}, 1)$ and $t := (-1, 0)$. Then, we have $h, t \in \mathfrak{B}_{m,n}$ and $t * h = (\frac{1+m}{2}, -1)$. Therefore, $o(h) = o(t * h) = p$. Hence, $t^p = t^p * e = t^p * h^p = (t * h)^p = e$. Therefore, $o(t)$ divides p, which is a contradiction since $o(t) = 2$ and $p > 3$ is a prime integer. □

The following lemma and its corollary will be quite useful in this paper.

Lemma 2.3. *Let m and n be two integers. If H is a finite subgroup of $\mathfrak{B}_{m,n}$, then there are non-negative integers α and β such that $|H| = 2^\alpha \times 3^\beta$.*

Proof. Assume the opposite. Then, there is a prime integer $p > 3$ such that p divides $|H|$. So, in view of *Cauchy's Theorem*, (see [3]), the group H contains an element h of order p. But, by Lemma 2.3 this is a contradiction. □

Corollary 2.1. *Let m and n be two integers. If $g \in \mathfrak{B}_{m,n}$ is an element of finite order, then there are non-negative integers α and β such that $o(g) = 2^\alpha \times 3^\beta$.*

Proof. Let $H := \langle g \rangle$. Then, H is a subgroup of $\mathfrak{B}_{m,n}$ with $|H| = o(g) < \infty$. Now the assertion follows from Lemma 2.4. □

The following lemmas are of assistance in the proof of Theorem 2.14.

Lemma 2.4. *Let m and n be two integers. Then, each finite 2-subgroup of $\mathfrak{B}_{m,n}$ is cyclic.*

Proof. Assume the opposite. Then, there is a finite 2-subgroup H of $\mathfrak{B}_{m,n}$ such that H is not cyclic. Therefore, by the *Fundamental Theorem of Finite Abelian Groups* we have

$$H \simeq \prod_{i=1}^{k} \mathbb{Z}_{2^{\ell_i}},$$

for some positive integers $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_k$ with the property $|H| = 2^{\ell_1+\ell_2+\cdots+\ell_k}$ and $k \geq 2$. Furthermore, in this situation H has a subgroup K such that

$$K \simeq \prod_{i=1}^{k} \mathbb{Z}_2.$$

Thus, K contains exactly $2^k - 1$ distinct elements of order 2. Since $k \geq 2$ it follows that $\mathfrak{B}_{m,n}$ contains at least 3 distinct elements of order 2. But, by Lemma 2.3 there is precisely one element $g = (a, b) \in \mathfrak{B}_{m,n}$ with $o(g) = 2$, which is a contradiction. □

Lemma 2.5. *Let m and n be two integers. Then, each finite 3-subgroup of $\mathfrak{B}_{m,n}$ is cyclic.*

Proof. Assume the opposite. Then, there is a finite 3-subgroup H of $\mathfrak{B}_{m,n}$ such that H is not cyclic. Therefore, by the *Fundamental Theorem of Finite Abelian Groups* we have

$$H \simeq \prod_{i=1}^{k} \mathbb{Z}_{3^{\ell_i}},$$

for some positive integers $\ell_1 \le \ell_2 \le \cdots \le \ell_k$ with the property $|H| = 3^{\ell_1+\ell_2+\cdots+\ell_k}$ and $k \ge 2$. Furthermore, in this situation H has a subgroup K such that

$$K \simeq \prod_{i=1}^{k} \mathbb{Z}_3.$$

Thus, K contains exactly $3^k - 1$ distinct elements of order 3. Since $k \ge 2$ it follows that $\mathfrak{B}_{m,n}$ has at least 8 distinct elements of order 3. Assume that $g = (a,b) \in \mathfrak{B}_{m,n}$ is an element of order 3. Then, by Lemma 2.3 we know that b divides 3. Hence, $b \in \{\pm 1, \pm 3\}$. Moreover, from the relation $o(g) = 3$, we get $g^2 = g^{-1}$. Thus, $(a^2 - nb^2, 2ab + mb^2) = (a + mb, -b)$. So, $2ab + mb^2 = -b$, and by using the hypothesis $b \ne 0$, we obtain $a = \frac{-1-mb}{2}$. This observation shows that there are at most 4 distinct elements $g = (a,b) \in \mathfrak{B}_{m,n}$ with $o(g) = 3$, which is a contradiction. □

Lemma 2.6. *Let m and n be two integers. Then, each finite subgroup of $\mathfrak{B}_{m,n}$ is cyclic.*

Proof. Let H be a finite subgroup of $\mathfrak{B}_{m,n}$. Then, by Lemma 2.4 there are non-negative integers α and β such that $|H| = 2^\alpha \times 3^\beta$. Let P and Q denote the Sylow 2-subgroup and the Sylow 3-subgroup of H respectively. Then, by Lemmas 2.6 and 2.7, P and Q are cyclic groups. Therefore, from the relations $H = P \oplus Q$ and $(|P|, |Q|) = 1$, it is concluded that H is a cyclic group. □

Corollary 2.2. *Let $m, n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, $H_k := \{g \in \mathfrak{B}_{m,n} \ : \ g^k = e\}$ is a finite subgroup of $\mathfrak{B}_{m,n}$. In particular, $S_k := \{g \in \mathfrak{B}_{m,n} \ : \ o(g) = k\}$ is a finite set.*

Proof. Assume the opposite. Then, we can find a finite subgroup K of H_k with $|K| > k$. Therefore, by Lemma 2.8, K is a cyclic group. So, there exists an element $g \in K$ such that $K = \langle g \rangle$. By the hypothesis we have $g^k = e$ and hence $|K| = o(g) \le k$, which is a contradiction. Since $S_k \subseteq H_k$, we see that S_k is a finite set as well. □

Lemma 2.7. *Let $m, n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, for each $b \in \mathbb{Z}$ there is at most one integer a such that $(a,b) \in \mathfrak{B}_{m,n}$ and $o((a,b)) = k$.*

Proof. Assume that $g = (a,b) \in \mathfrak{B}_{m,n}$ and $o(g) = k$. Then, by the definition we have

$$(1,0) = g^k = (P(a,b,m,n), Q(a,b,m,n)),$$

for some polynomials $P(X_1,X_2,X_3,X_4), Q(X_1,X_2,X_3,X_4) \in \mathbb{Z}[X_1,X_2,X_3,X_4]$. Since $a^2 = 1 - mab - nb^2$, we can write $P(a,b,m,n) = H_1(b,m,n) + aH_2(b,m,n)$ and $Q(a,b,m,n) = H_3(b,m,n) + aH_4(b,m,n)$, for some $H_1(X_1,X_2,X_3)$, $H_2(X_1, X_2,X_3)$, $H_3(X_1,X_2,X_3)$, $H_4(X_1,X_2,X_3) \in \mathbb{Z}[X_1,X_2,X_3]$.

By Corollary 2.9, there are only a finite number of elements $g \in \mathfrak{B}_{m,n}$ with $o(g) = k$. Therefore, for each element $b \in \mathbb{Z}$, there are only a finite number of integers a such that $(a,b) \in \mathfrak{B}_{m,n}$ and $o((a,b)) = k$. This observation implies that for each $b \in \mathbb{Z}$, $H_2(b,m,n) \neq 0$ or $H_4(b,m,n) \neq 0$. So, we can find at most one integer a such that $H_1(b,m,n) + aH_2(b,m,n) = 1$ and $H_3(b,m,n) + aH_4(b,m,n) = 0$. Thus, for each $b \in \mathbb{Z}$, there is at most one integer a such that $(a,b) \in \mathfrak{B}_{m,n}$ and $o((a,b)) = k$. □

Let m and n be two integers. In the sequel, we will denote the torsion subgroup of $\mathfrak{B}_{m,n}$ by $\mathfrak{T}_{m,n}$. We recall that the torsion subgroup of $\mathfrak{B}_{m,n}$ is defined as:

$$\mathfrak{T}_{m,n} := \{g \in \mathfrak{B}_{m,n} \ : \ o(g) < \infty\}.$$

Lemma 2.8. *Let m and n be two integers. Then, the following statements hold:*

i) *Let $g_1 = (a_1, b_1) \in \mathfrak{T}_{m,n}$ be an element of order 4. Then, b_1 divides 2.*

ii) *Suppose that $g_2 = (a_2, b_2) \in \mathfrak{T}_{m,n}$ is an element of order 8. Then, b_2 divides 2.*

iii) *Assume that $g_3 = (a_3, b_3) \in \mathfrak{T}_{m,n}$ is an element of order 6. Then, b_3 divides 3.*

iv) *Let $g_4 = (a_4, b_4) \in \mathfrak{T}_{m,n}$ be an element of order 12. Then, b_4 divides 3.*

Proof. (i) By Lemma 2.3, b_1 divides 4 and so $b_1 \neq 0$. Since $o(g_1) = 4$, it is clear that $o(g_1^2) = 2$. Thus, by Lemma 2.3 we have $g_1^2 = (-1, 0)$. Hence, $(a_1^2 - nb_1^2, 2a_1b_1 + mb_1^2) = (-1, 0)$. So, from the relations $b_1 \neq 0$ and $b_1(2a_1 + mb_1) = 0$, it follows that $2a_1 + mb_1 = 0$. Since $a_1^2 + ma_1b_1 + nb_1^2 = 1$, it is clear that the integers a_1 and b_1 are relatively prime. Therefore, the relation $2a_1 = -mb_1$ shows that b_1 divides 2.

(ii) Since $o(g_2) = 8$, it is clear that $o(g_2^2) = 4$. Also, the relation $g_2^2 = (a_2^2 - nb_2^2, 2a_2b_2 + mb_2^2)$ together with (i) implies that $2a_2b_2 + mb_2^2$ divides 2. Since b_2 divides $2a_2b_2 + mb_2^2$, it follows that b_2 divides 2.

(iii) Since $o(g_3) = 6$, it follows that $o(g_3^3) = 2$. Thus, by Lemma 2.3 we have $g_3^3 = (-1, 0)$. Put $t := (-1, 0)$. Since $o(g_3) = 6$, it can be seen that

$$\begin{aligned}(nb_3^2 - a_3^2, -2a_3b_3 - mb_3^2) &= t * g_3^2 \\ &= g_3^3 * g_3^2 \\ &= g_3^5 \\ &= g_3^{-1} \\ &= (a_3 + mb_3, -b_3),\end{aligned}$$

which shows that $-2a_3b_3 - mb_3^2 = -b_3$. By Lemma 2.3, b_3 divides 6 and so $b_3 \neq 0$. Thus, $mb_3 = -2a_3 + 1$ and hence 2 doesn't divide b_3. Therefore, b_3 divides 3.

(iv) Since $o(g_4) = 12$, it is clear that $o(g_4^2) = 6$. Also, the relation $g_4^2 = (a_4^2 - nb_4^2, 2a_4b_4 + mb_4^2)$ together with (iii) implies that $2a_4b_4 + mb_4^2$ divides 3. Since b_4 divides $2a_4b_4 + mb_4^2$, it is concluded that b_4 divides 3. □

Lemma 2.9. *Let m and n be two integers. Then, the following statements hold:*

i) *Assume that $g \in \mathfrak{T}_{m,n}$ is an element of order 2^k for some $k \in \mathbb{N}_0$. Then, $k \leq 2$.*

ii) *Suppose that $h \in \mathfrak{T}_{m,n}$ is an element of order 3^k for some $k \in \mathbb{N}_0$. Then, $k \leq 2$.*

Proof. (i) Assume the opposite. Then, we have $o(g) = 2^k$ for some $k \geq 3$. Therefore, $o(g^{2^{k-3}}) = 8$. By replacing g with $g^{2^{k-3}}$, we may assume that $o(g) = 8$. Let $g_1 = (a, b) \in \mathfrak{T}_{m,n}$ be an element of order 8. Then, by Lemma 2.11 we see that b divides 2. Thus, $b \in \{\pm 1, \pm 2\}$. Since $o(g) = 8$, one sees that there are exactly 4 distinct elements of order 8 in the subgroup $\langle g \rangle$ of $\mathfrak{T}_{m,n}$. Thus, by Lemma 2.10 for each $b \in \{\pm 1, \pm 2\}$ there is a unique integer a such that $g_1 = (a, b) \in \langle g \rangle$ and $o(g_1) = 8$. Let $g_2 = (c, d)$ be an element of $\langle g \rangle$ with $o(g_2) = 4$. Then, by Lemma 2.11 we see that d divides 2. Hence, $d \in \{\pm 1, \pm 2\}$. So, there is a unique integer a such that $g_3 = (a, d) \in \langle g \rangle$ and $o(g_3) = 8$. From the relations $a^2 + mad + nd^2 = 1$ and $c^2 + mcd + nd^2 = 1$, we get $(a - c)(a + c + md) = 0$. Since $o(g_2) = 4$ and $o(g_3) = 8$, it follows that $g_2 \neq g_3$ and so $a \neq c$. Thus, from the relations $a - c \neq 0$ and $(a - c)(a + c + md) = 0$, it is concluded that $a + c + md = 0$. Therefore, $g_3^{-1} = (a + md, -d) = (-c, -d) = (-1, 0) * g_2$, which implies that $(g_3^{-1})^4 = (-1, 0)^4 * g_2^4 = e$. Hence, $8 = o(g_3) = o(g_3^{-1}) \leq 4$, which is a contradiction.

(ii) Assume the opposite. So, we have $o(h) = 3^k$ for some $k \geq 3$. Hence, $o(h^{3^{k-3}}) = 27$. By replacing h with $h^{3^{k-3}}$, we my assume that $o(h) = 27$. Let $h_1 = (r, s) \in \mathfrak{T}_{m,n}$ be an element of order 27. Then, by Lemma 2.3 we see that s divides 27. Hence, $s \in \{\pm 1, \pm 3, \pm 9, \pm 27\}$. Therefore, by Lemma 2.10 there are at most 8 elements $h_1 = (r, s) \in \mathfrak{T}_{m,n}$ with the property $o(h_1) = 27$. Since $o(h) = 27$, one sees that there are exactly 18 elements of order 27 in the subgroup $\langle h \rangle$ of $\mathfrak{T}_{m,n}$, which is a contradiction. □

Lemma 2.10. *Let m and n be two integers. Suppose that $h \in \mathfrak{T}_{m,n}$ is an element of order 3^k for some $k \in \mathbb{N}_0$. Then, $k \leq 1$.*

Proof. Assume the opposite. Since by Lemma 2.12 we have $k \leq 2$, it follows that $k = 2$. Let $h_1 = (a, b) \in \mathfrak{T}_{m,n}$ be an element of order 9. Then, by Lemma 2.3 we see that b divides 9. Hence, $b \in \{\pm 1, \pm 3, \pm 9\}$. Since $o(h) = 9$, one sees that there are exactly 6 elements of order 9 in the subgroup $\langle h \rangle$ of $\mathfrak{T}_{m,n}$. Thus, by Lemma 2.10 for each $b \in \{\pm 1, \pm 3, \pm 9\}$ there is a unique integer a such that $h_1 = (a, b) \in \langle h \rangle$ and $o(h_1) = 9$. Let $h_2 = (c, d)$ be an element of $\langle h \rangle$ with $o(h_2) = 3$. Then, by Lemma 2.3 we see that d divides 3. Hence, $d \in \{\pm 1, \pm 3\}$. So, there is a unique integer a such that $h_3 = (a, d) \in \langle h \rangle$ and $o(h_3) = 9$. From the relations $a^2 + mad + nd^2 = 1$ and $c^2 + mcd + nd^2 = 1$, we get $(a - c)(a + c + md) = 0$. Since $o(h_2) = 3$ and $o(h_3) = 9$, it follows that $h_2 \neq h_3$ and so $a \neq c$. Thus, from the relations $a - c \neq 0$ and $(a - c)(a + c + md) = 0$, it is concluded that $a + c + md = 0$. Therefore, $h_3^{-1} = (a + md, -d) = (-c, -d) = (-1, 0) * h_2$, which implies that $(h_3^{-1})^6 = (-1,0)^6 * h_2^6 = e$. Hence, $9 = o(h_3) = o(h_3^{-1}) \leq 6$, which is a contradiction. □

The following result plays a key role in the proof of our main theorem.

Theorem 2.2. *Let m and n be two integers. Then, the Abelian group $\mathfrak{T}_{m,n}$ is isomorphic to $\mathbb{Z}_k$ for some $k \in \{2,4,6\}$.*

Proof. We claim that $|\mathfrak{T}_{m,n}| \leq 12$. Assume the opposite. Then, we have $|\mathfrak{T}_{m,n}| > 12$. Therefore, we can find a finite subgroup H of $\mathfrak{T}_{m,n}$ with $|H| > 12$. By Lemma 2.8, H is a cyclic group. Thus, there is an element $g_1 \in H$ with $H = \langle g_1 \rangle$. In view of Corollary 2.5, there are integers $\alpha_1, \beta_1 \in \mathbb{N}_0$ such that $o(g_1) = 2^{\alpha_1} \times 3^{\beta_1}$. Since $2^{\alpha_1} \times 3^{\beta_1} = o(g_1) = |H| > 12 = 2^2 \times 3$, we can deduce that $\alpha_1 \geq 3$ or $\beta_1 \geq 2$. Moreover, it is clear that

$$o(g_1^{2^{\alpha_1}}) = 3^{\beta_1} \text{ and } o(g_1^{3^{\beta_1}}) = 2^{\alpha_1}.$$

Therefore, by Lemma 2.12 and Lemma 2.13 we get $\alpha_1 \leq 2$ and $\beta_1 \leq 1$, which is a contradiction. So, we have $|\mathfrak{T}_{m,n}| \leq 12$. Hence, by Lemma 2.8 it is concluded that $\mathfrak{T}_{m,n}$ is a cyclic subgroup of $\mathfrak{B}_{m,n}$. Therefore, there exists an element $g_2 \in \mathfrak{T}_{m,n}$ with $\mathfrak{T}_{m,n} = \langle g_2 \rangle$. In view of Corollary 2.5, there are integers $\alpha_2, \beta_2 \in \mathbb{N}_0$ such that $o(g_2) = |\mathfrak{T}_{m,n}| = 2^{\alpha_2} \times 3^{\beta_2}$. Since

$$o(g_2^{2^{\alpha_2}}) = 3^{\beta_2} \text{ and } o(g_2^{3^{\beta_2}}) = 2^{\alpha_2},$$

by Lemma 2.12 and Lemma 2.13 we can deduce that $\alpha_2 \leq 2$, $\beta_2 \leq 1$ and so $|\mathfrak{T}_{m,n}|$ divides 12. Since the element $t = (-1,0) \in \mathfrak{T}_{m,n}$ is of order 2, it follows that 2 divides $|\mathfrak{T}_{m,n}|$. Therefore, $|\mathfrak{T}_{m,n}| \in \{2,4,6,12\}$. We claim that $|\mathfrak{T}_{m,n}| \neq 12$. Assume the opposite. Then, there exists an element $h \in \mathfrak{T}_{m,n}$ such that $\mathfrak{T}_{m,n} = \langle h \rangle$ and $o(h) = 12$. Let $h_1 = (a,b) \in \langle h \rangle$ be an element of order 12. Then, by Lemma 2.11 we see that b divides 3. Hence, $b \in \{\pm 1, \pm 3\}$. Since $o(h) = 12$, one sees that there are exactly 4 distinct elements of order 12 in the group $\langle h \rangle = \mathfrak{T}_{m,n}$. Thus, by Lemma 2.10 for each $b \in \{\pm 1, \pm 3\}$ there is a unique integer a such that $h_1 = (a,b) \in \langle h \rangle$ and $o(h_1) = 12$. Let $h_2 = (c,d)$ be an element of $\langle h \rangle$ with $o(h_2) = 6$. Then, by Lemma 2.11 we see that d divides 3. Hence, $d \in \{\pm 1, \pm 3\}$. So, there is a unique integer a such that $h_3 = (a,d) \in \langle h \rangle$ and $o(h_3) = 12$. From the relations $a^2 + mad + nd^2 = 1$ and $c^2 + mcd + nd^2 = 1$, we get $(a-c)(a+c+md) = 0$. Since $o(h_2) = 6$ and $o(h_3) = 12$, it follows that $h_2 \neq h_3$ and so $a \neq c$. Thus, from the relations $a - c \neq 0$ and $(a-c)(a+c+md) = 0$, it is concluded that $a + c + md = 0$. Therefore, $h_3^{-1} = (a+md, -d) = (-c,-d) = (-1,0) * h_2$, which implies that $(h_3^{-1})^6 = (-1,0)^6 * h_2^6 = e$. Hence, $12 = o(h_3) = o(h_3^{-1}) \leq 6$, which is a contradiction. Therefore, $\mathfrak{T}_{m,n}$ is a finite cyclic group with $|\mathfrak{T}_{m,n}| \in \{2,4,6\}$. Consequently, $\mathfrak{T}_{m,n}$ is isomorphic to $\mathbb{Z}_k$ for some $k \in \{2,4,6\}$, as required. □

The following auxiliary lemmas are needed in the proof of Theorem 2.20.

Lemma 2.11. *Let m and n be two integers and set $\delta := m^2 - 4n$. If $\delta < 0$, then the Abelian group $\mathfrak{B}_{m,n}$ is isomorphic to $\mathbb{Z}_k$ for some $k \in \{2, 4, 6\}$.*

Proof. By Theorem 2.14 it is enough to prove that $\mathfrak{B}_{m,n} = \mathfrak{T}_{m,n}$. Also, in order to prove this assertion, it suffices for us to prove that $\mathfrak{B}_{m,n}$ is a finite group. Assume that $(a, b) \in \mathfrak{B}_{m,n}$. Then, by the definition we have $a^2 + mab + nb^2 = 1$. Therefore, $(2a + mb)^2 - \delta b^2 = 4(a^2 + mab + nb^2) = 4$. Hence,

$$0 \le (2a+mb)^2 \le (2a+mb)^2 - \delta b^2 = 4, \text{ and } 0 \le b^2 \le -\delta b^2 \le (2a+mb)^2 - \delta b^2 = 4.$$

Therefore, $\{2a + mb, b\} \subseteq \{0, \pm 1, \pm 2\}$. Thus, $\mathfrak{B}_{m,n}$ is a finite group, as required. □

Lemma 2.12. *Let m and n be two integers and set $\delta := m^2 - 4n$. If $\delta = 0$, then the Abelian group $\mathfrak{B}_{m,n}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$.*

Proof. Assume that $(a, b) \in \mathfrak{B}_{m,n}$. Then, by the definition we have $a^2 + mab + nb^2 = 1$. Therefore, $(2a + mb)^2 = (2a + mb)^2 - \delta b^2 = 4(a^2 + mab + nb^2) = 4$. Hence, $2a + mb = \pm 2$ and so (a, b) is a solution to one of the two-variable linear Diophantine equations $2x + my = 2$ or $2x + my = -2$. By solving these linear Diophantine equations we obtain, $(a, b) = (\pm 1 + \frac{mk}{\mu}, \frac{-2k}{\mu}) \in \mathfrak{B}_{m,n}$, where $k \in \mathbb{Z}$ and μ is the greatest common divisor of the integers 2 and m.

Set $t := (-1, 0)$ and $g := (1 + \frac{m}{\mu}, \frac{-2}{\mu})$. Then, by using induction on k and applying the relation $m^2 - 4n = 0$, it can be seen that

$$g^k = (1 + \frac{mk}{\mu}, \frac{-2k}{\mu}), \text{ and } g^{-k} = (1 - \frac{mk}{\mu}, \frac{2k}{\mu}),$$

for each $k \in \mathbb{N}$. Therefore, $g^k = (1 + \frac{mk}{\mu}, \frac{-2k}{\mu})$ and $t * g^{-k} = (-1 + \frac{mk}{\mu}, \frac{-2k}{\mu})$, for each $k \in \mathbb{Z}$. Hence,

$$\mathfrak{B}_{m,n} = \left\{(\pm 1 + \frac{mk}{\mu}, \frac{-2k}{\mu}) \;:\; k \in \mathbb{Z}\right\} = \left\{t^\ell * g^k \;:\; \ell, k \in \mathbb{Z}\right\} = \langle t \rangle * \langle g \rangle.$$

Furthermore, from the relations $o(g) = \infty$ and $o(t) = 2$, we can deduce that $\langle t \rangle \cap \langle g \rangle = \{e\}$. Therefore, $\mathfrak{B}_{m,n} = \langle t \rangle \oplus \langle g \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}$, as required. □

Lemma 2.13. *Let m and n be two integers and set $\delta := m^2 - 4n$. If δ is a positive perfect square integer, then the Abelian group $\mathfrak{B}_{m,n}$ is isomorphic to $\mathbb{Z}_2$.*

Proof. By assumption there is a positive integer λ such that $m^2 - 4n = \delta = \lambda^2$. Suppose that $(a, b) \in \mathfrak{B}_{m,n}$. Then, by the definition we have $a^2 + mab + nb^2 = 1$. Therefore,

$$(2a + mb + \lambda b)(2a + mb - \lambda b) = (2a + mb)^2 - \delta b^2 = 4(a^2 + mab + nb^2) = 4.$$

In fact, there are precisely six cases. In the following four cases:
Case 1. $2a + mb + \lambda b = 1$ and $2a + mb - \lambda b = 4$.

Case 2. $2a + mb + \lambda b = 4$ and $2a + mb - \lambda b = 1$.
Case 3. $2a + mb + \lambda b = -1$ and $2a + mb - \lambda b = -4$.
Case 4. $2a + mb + \lambda b = -4$ and $2a + mb - \lambda b = -1$, we see that $2a + mb = \pm\frac{5}{2}$, contradicting the fact that $2a + mb$ is an integer. Also, in the following two remainder cases,
Case 5. $2a + mb + \lambda b = 2$ and $2a + mb - \lambda b = 2$, Case 6. $2a + mb + \lambda b = -2$ and $2a + mb - \lambda b = -2$, we see that $(a, b) = (\pm 1, 0)$. Therefore, $\mathfrak{B}_{m,n} = \{(1, 0), (-1, 0)\} \simeq \mathbb{Z}_2$, as required. □

Recall that *Pell's equation*, also called the *Pell-Fermat equation*, is any Diophantine equation of the form $x^2 - ny^2 = 1$, where n is a given positive nonsquare integer. It is well-known that *Pell's equation* has a infinite solutions. Also, this equation has a solution (a_1, b_1) with $a_1 \geq 1$ and $b_1 \geq 1$ which has some special properties and is called *the fundamental solution.* Furthermore, once the fundamental solution is found, all remaining solutions may be calculated algebraically from

$$a_k + b_k\sqrt{n} = (a_1 + b_1\sqrt{n})^k,$$

expanding the right side, equating coefficients of $\sqrt{n}$ on both sides, and equating the other terms on both sides. This yields the recurrence relation

$$(a_{k+1}, b_{k+1}) = (a_1 a_k + n b_1 b_k, a_1 b_k + b_1 a_k).$$

In this situation, the set of all solutions of the equation $x^2 - ny^2 = 1$ is equal to

$$\{(\pm 1, 0)\} \cup \{(\pm a_k, \pm b_k) \ : \ k \in \mathbb{N}\}.$$

For more details see [1]. In order to establish our next lemma, we use a proof similar to the proof of [1, p. 180, Theorem 7].

Lemma 2.14. *Let m and n be two integers and set $\delta := m^2 - 4n$. If δ is a positive nonsquare integer, then the Abelian group $\mathfrak{B}_{m,n}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$.*

Proof. Let (u_1, v_1) denote the fundamental solution of the *Pell's equation* $x^2 - \delta y^2 = 1$. Set $(\alpha, \beta) := (u_1 - mv_1, 2v_1)$. Then, it is easy to see that $(\alpha, \beta) \in \mathfrak{B}_{m,n}$, $\alpha + \frac{m\beta}{2} = u_1 > 0$ and $\frac{\beta}{2} = v_1 > 0$. Set $M := \alpha + \frac{m\beta}{2} + \frac{\beta}{2}\sqrt{\delta}$. If $(\alpha', \beta') \in \mathfrak{B}_{m,n}$ is an element such that $\alpha' + \frac{m\beta'}{2} > 0$ and $\frac{\beta'}{2} > 0$, then the condition

$$\alpha' + \frac{m\beta'}{2} + \frac{\beta'}{2}\sqrt{\delta} \leq M,$$

implies that $\alpha' + \frac{m\beta'}{2} \leq M$ and $\frac{\beta'}{2} \leq M$. Therefore, $1 \leq \beta' \leq 2M$ and $-|m|M \leq \alpha' \leq (1 + |m|)M$. Thus, in particular, there are only finitely many choices for the integers α' and β'. Let us choose $g = (a_1, b_1) \in \mathfrak{B}_{m,n}$ for which $a_1 + \frac{mb_1}{2} > 0$, $\frac{b_1}{2} > 0$ and $a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}$ is least. This is possible since there are only finitely many elements $(\alpha', \beta') \in \mathfrak{B}_{m,n}$ such that $\alpha' + \frac{m\beta'}{2} > 0$ and $\frac{\beta'}{2} > 0$, and

$$\alpha' + \frac{m\beta'}{2} + \frac{\beta'}{2}\sqrt{\delta} \leq M.$$

For each positive integer k, define a_k and b_k by

$$a_k + \frac{mb_k}{2} + \frac{b_k}{2}\sqrt{\delta} = \left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)^k. \tag{2.1}$$

Indeed, since by the hypothesis δ is a positive nonsquare integer, we see that $\sqrt{\delta}$ is an irrational number. Therefore, for each positive integer k, the elements a_k and b_k can be calculated algebraically from (2.18.1), expanding the right side, equating coefficients of $\sqrt{\delta}$ on both sides, and equating the other terms on both sides.

By using induction on k, we prove that $a_k + \frac{mb_k}{2} > 0$, $\frac{b_k}{2} > 0$ and $(a_k, b_k) = g^k \in \mathfrak{B}_{m,n}$, for each $k \in \mathbb{N}$. For $k = 1$ the assertion holds by the hypothesis. Now, let $k > 1$ and assume that the result has been proved for $k - 1$. Then, by inductive assumption we know that $a_{k-1} + \frac{mb_{k-1}}{2} > 0$, $\frac{b_{k-1}}{2} > 0$ and $(a_{k-1}, b_{k-1}) = g^{k-1} \in \mathfrak{B}_{m,n}$. By using the fact that $\sqrt{\delta}$ is an irrational number, from the relations

$$\begin{aligned}
a_k + \frac{mb_k}{2} + \frac{b_k}{2}\sqrt{\delta} &= \left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)^k \\
&= \left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)\left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)^{k-1} \\
&= \left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)\left(a_{k-1} + \frac{mb_{k-1}}{2} + \frac{b_{k-1}}{2}\sqrt{\delta}\right),
\end{aligned}$$

we obtain, $b_k = a_1 b_{k-1} + b_1 a_{k-1} + m b_1 b_{k-1}$ and

$$\begin{aligned}
a_k + \frac{mb_k}{2} &= a_1 a_{k-1} + \frac{m(a_1 b_{k-1} + b_1 a_{k-1})}{2} + \frac{m^2 b_1 b_{k-1}}{4} + \frac{\delta b_1 b_{k-1}}{4} \\
&= a_1 a_{k-1} + \frac{m(a_1 b_{k-1} + b_1 a_{k-1})}{2} + \frac{m^2 b_1 b_{k-1}}{4} + \frac{(m^2 - 4n) b_1 b_{k-1}}{4} \\
&= a_1 a_{k-1} - n b_1 b_{k-1} + \frac{m(a_1 b_{k-1} + b_1 a_{k-1} + m b_1 b_{k-1})}{2} \\
&= a_1 a_{k-1} - n b_1 b_{k-1} + \frac{mb_k}{2},
\end{aligned}$$

which implies that $a_k = a_1 a_{k-1} - n b_1 b_{k-1}$. Thus,

$$(a_k, b_k) = (a_1 a_{k-1} - n b_1 b_{k-1}, a_1 b_{k-1} + b_1 a_{k-1} + m b_1 b_{k-1}) = g * g^{k-1} = g^k.$$

Also, the relations

$$\frac{b_k}{2} = \left(a_1 + \frac{mb_1}{2}\right)\left(\frac{b_{k-1}}{2}\right) + \left(a_{k-1} + \frac{mb_{k-1}}{2}\right)\left(\frac{b_1}{2}\right),$$

and

$$a_k + \frac{mb_k}{2} = \left(a_1 + \frac{mb_1}{2}\right)\left(a_{k-1} + \frac{mb_{k-1}}{2}\right) + \delta\left(\frac{b_1}{2}\right)\left(\frac{b_{k-1}}{2}\right),$$

together with the hypothesis $a_1 + \frac{mb_1}{2} > 0$, $a_{k-1} + \frac{mb_{k-1}}{2} > 0$, $\frac{b_1}{2} > 0$, $\frac{b_{k-1}}{2} > 0$, and $\delta > 0$, imply that $a_k + \frac{mb_k}{2} > 0$ and $\frac{b_k}{2} > 0$. This completes the inductive step.

We claim that $o(g) = \infty$. Assume the opposite and let $o(g) = j < \infty$. Then, we see that $g^j = (a_j, b_j) = (1, 0)$. Therefore, $b_j = 0$, which is impossible since $\frac{b_j}{2} > 0$. Thus, $o(g) = \infty$ and so the cyclic subgroup $\langle g \rangle$ of $\mathfrak{B}_{m,n}$ is isomorphic to $\mathbb{Z}$.

Let $h = (a', b') \in (\mathfrak{B}_{m,n} \setminus \mathfrak{T}_{m,n})$. Then, by the definition we have $o(h) = \infty$. Set $t = (-1, 0) \in \mathfrak{B}_{m,n}$. Since $o(t) = 2$, it is clear that $t \in \mathfrak{T}_{m,n}$. Also, from the assumption $o(h) = \infty$ it follows that $b' \neq 0$ and the elements $h, h^{-1}, t * h, t * h^{-1}$ are different. Therefore, the relation

$$\left\{h, h^{-1}, t * h, t * h^{-1}\right\} = \left\{(a', b'), (a' + mb', -b'), (-a', -b'), (-a' - mb', b')\right\},$$

implies that $2a' + mb' \neq 0$ and hence $a' + \frac{mb'}{2} \neq 0$. Since

$$\left\{(u + \frac{mv}{2}, \frac{v}{2}) \ : \ (u, v) \in \left\{h, h^{-1}, t * h, t * h^{-1}\right\}\right\} = \left\{(\pm(a' + \frac{mb'}{2}), \pm\frac{b'}{2})\right\},$$

we can find an element $(a, b) \in \{h, h^{-1}, t * h, t * h^{-1}\}$ such that $a + \frac{mb}{2} > 0$ and $\frac{b}{2} > 0$. We show that $(a, b) = g^\ell$ for some $\ell \in \mathbb{N}$.

Since (a_1, b_1) was chosen as the element of $\mathfrak{B}_{m,n}$ for which $a_1 + \frac{mb_1}{2} > 0$, $\frac{b_1}{2} > 0$ and $a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}$ is least, we see that

$$a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta} \leq a + \frac{mb}{2} + \frac{b}{2}\sqrt{\delta}. \tag{2.2}$$

We assert that there is a positive integer ℓ such that

$$\left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)^\ell \leq a + \frac{mb}{2} + \frac{b}{2}\sqrt{\delta} < \left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)^{\ell+1}. \tag{2.3}$$

Since by the hypothesis δ is a positive nonsquare integer, it follows that $\delta \geq 2$. Therefore, by using the hypothesis $a_1 + \frac{mb_1}{2} > 0$ and $\frac{b_1}{2} > 0$, one sees that

$$a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta} = \frac{2a_1 + mb_1}{2} + \frac{b_1}{2}\sqrt{\delta} \geq \frac{1}{2} + \frac{1}{2}\sqrt{\delta} \geq \frac{1}{2} + \frac{1}{2}\sqrt{2} > \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, $a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta} > 1$ and hence the powers of $a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}$, became arbitrary large. So, there is a largest value of ℓ for which

$$\left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)^\ell \leq a + \frac{mb}{2} + \frac{b}{2}\sqrt{\delta}.$$

Furthermore, by the relation (2.18.2) we know that this largest value of ℓ is at least 1. Moreover, it is clear that this largest value of ℓ forces (2.18.3) to hold. Let us multiply (2.18.3) by $(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta})^\ell$, which is positive since $a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta} > 0$ and

$$\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)\left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right) = a_1^2 + ma_1b_1 + nb_1^2 = 1.$$

Then, we see that

$$1 \le \left(a + \frac{mb}{2} + \frac{b}{2}\sqrt{\delta}\right)\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)^\ell < a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}. \tag{2.4}$$

Since $g = (a_1, b_1)$, $g^\ell = (a_\ell, b_\ell) \in \mathfrak{B}_{m,n}$, one sees that

$$\begin{aligned}&\left(a_\ell + \frac{mb_\ell}{2} + \frac{b_\ell}{2}\sqrt{\delta}\right)\left(a_\ell + \frac{mb_\ell}{2} - \frac{b_\ell}{2}\sqrt{\delta}\right)\\ &= \left(a_\ell + \frac{mb_\ell}{2}\right)^2 - \delta\left(\frac{b_\ell}{2}\right)^2 = a_\ell^2 + ma_\ell b_\ell + nb_\ell^2 = 1,\end{aligned}$$

and

$$\begin{aligned}&\left(a_\ell + \frac{mb_\ell}{2} + \frac{b_\ell}{2}\sqrt{\delta}\right)\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)^\ell\\ &= \left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)^\ell\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)^\ell\\ &= \left(\left(a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}\right)\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)\right)^\ell\\ &= \left(\left(a_1 + \frac{mb_1}{2}\right)^2 - \delta\left(\frac{b_1}{2}\right)^2\right)^\ell\\ &= \left(a_1^2 + ma_1b_1 + nb_1^2\right)^\ell = 1^\ell = 1.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(a_\ell + \frac{mb_\ell}{2} + \frac{b_\ell}{2}\sqrt{\delta}\right)\left(a_\ell + \frac{mb_\ell}{2} - \frac{b_\ell}{2}\sqrt{\delta}\right) = 1\\ &= \left(a_\ell + \frac{mb_\ell}{2} + \frac{b_\ell}{2}\sqrt{\delta}\right)\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)^\ell,\end{aligned}$$

and so

$$\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)^\ell = a_\ell + \frac{mb_\ell}{2} - \frac{b_\ell}{2}\sqrt{\delta}.$$

Set $c := aa_\ell + mab_\ell + nbb_\ell$, and $d := ba_\ell - ab_\ell$. Obviously, $c, d \in \mathbb{Z}$. Also, it is straightforward to see that

$$c + \frac{md}{2} = \left(a + \frac{mb}{2}\right)\left(a_\ell + \frac{mb_\ell}{2}\right) - \delta\left(\frac{b}{2}\right)\left(\frac{b_\ell}{2}\right)$$

and

$$\frac{d}{2} = \left(\frac{b}{2}\right)\left(a_\ell + \frac{mb_\ell}{2}\right) - \left(a + \frac{mb}{2}\right)\left(\frac{b_\ell}{2}\right).$$

So, we have

$$\begin{aligned}&\left(a + \frac{mb}{2} + \frac{b}{2}\sqrt{\delta}\right)\left(a_1 + \frac{mb_1}{2} - \frac{b_1}{2}\sqrt{\delta}\right)^\ell \\ &= \left(a + \frac{mb}{2} + \frac{b}{2}\sqrt{\delta}\right)\left(a_\ell + \frac{mb_\ell}{2} - \frac{b_\ell}{2}\sqrt{\delta}\right) = c + \frac{md}{2} + \frac{d}{2}\sqrt{\delta}.\end{aligned}$$

Moreover, it is easy to see that

$$\left(a + \frac{mb}{2} - \frac{b}{2}\sqrt{\delta}\right)\left(a_\ell + \frac{mb_\ell}{2} + \frac{b_\ell}{2}\sqrt{\delta}\right) = c + \frac{md}{2} - \frac{d}{2}\sqrt{\delta}.$$

By using these relations it can be seen that

$$\begin{aligned}c^2 + mcd + nd^2 &= \left(c + \frac{md}{2}\right)^2 - \delta\left(\frac{d}{2}\right)^2 \\ &= \left(\left(a + \frac{mb}{2}\right)^2 - \delta\left(\frac{b}{2}\right)^2\right)\left(\left(a_\ell + \frac{mb_\ell}{2}\right)^2 - \delta\left(\frac{b_\ell}{2}\right)^2\right) \\ &= \left(a^2 + mab + nb^2\right)\left(a_\ell^2 + ma_\ell b_\ell + nb_\ell^2\right) \\ &= (1)(1) = 1.\end{aligned}$$

Therefore, $(c,d) \in \mathfrak{B}_{m,n}$. Furthermore, (2.18.4) asserts that

$$1 \leq c + \frac{md}{2} + \frac{d}{2}\sqrt{\delta} < a_1 + \frac{mb_1}{2} + \frac{b_1}{2}\sqrt{\delta}. \tag{2.5}$$

We claim that $(c,d) = (1,0)$. Assume the opposite. If $d = 0$, then from the relation $c^2 + mcd + nd^2 = 1$ we get $c = \pm 1$ and so, by (2.18.5) it follows that $c = 1$. Thus, $(c,d) = (1,0)$ which is a contradiction. Also, if $c + \frac{md}{2} = 0$, then from the relation

$$\left(c + \frac{md}{2}\right)^2 - \delta\left(\frac{d}{2}\right)^2 = 1,$$

we can deduce $-\delta\left(\frac{d}{2}\right)^2 = 1$ and so $\delta < 0$, which is a contradiction. Hence, $d \neq 0$ and $c + \frac{md}{2} \neq 0$. In this situation we claim that $c + \frac{md}{2} + \frac{d}{2}\sqrt{\delta} \leq 1$. Assume the opposite. Then, we have $c + \frac{md}{2} + \frac{d}{2}\sqrt{\delta} > 1$. Let us consider the following three cases:

Case 1. $c + \frac{md}{2} < 0$ and $\frac{d}{2} < 0$. Then, $c + \frac{md}{2} + \frac{d}{2}\sqrt{\delta} < 0$, which contradicts the assumption that $c + \frac{md}{2} + \frac{d}{2}\sqrt{\delta} > 1$.

Case 2. $c + \frac{md}{2} < 0$ and $\frac{d}{2} > 0$. Then

$$-\left(c + \frac{md}{2}\right) + \frac{d}{2}\sqrt{\delta} > c + \frac{md}{2} + \frac{d}{2}\sqrt{\delta} > 1,$$

and so

$$
\begin{aligned}
-1 &= -(c^2+mcd+nd^2) = \delta\left(\frac{d}{2}\right)^2 - \left(c+\frac{md}{2}\right)^2 \\
&= \left(-\left(c+\frac{md}{2}\right)+\frac{d}{2}\sqrt{\delta}\right)\left(c+\frac{md}{2}+\frac{d}{2}\sqrt{\delta}\right) > 1,
\end{aligned}
$$

which is absurd.

Case 3. $c+\frac{md}{2}>0$ and $\frac{d}{2}<0$. Then

$$
c+\frac{md}{2}-\frac{d}{2}\sqrt{\delta} > c+\frac{md}{2}+\frac{d}{2}\sqrt{\delta} > 1,
$$

and so

$$
\begin{aligned}
1 = c^2+mcd+nd^2 &= \left(c+\frac{md}{2}\right)^2-\delta\left(\frac{d}{2}\right)^2 \\
&= \left(c+\frac{md}{2}-\frac{d}{2}\sqrt{\delta}\right)\left(c+\frac{md}{2}+\frac{d}{2}\sqrt{\delta}\right) > 1,
\end{aligned}
$$

which is also absurd. Thus, the only possible case is $c+\frac{md}{2}>0$ and $\frac{d}{2}>0$. However, if this is the case, then (2.18.5) contradicts the way in which (a_1,b_1) was chosen. Therefore, we must have

$$
c+\frac{md}{2}+\frac{d}{2}\sqrt{\delta} \le 1.
$$

Then, (2.18.5) implies that

$$
c+\frac{md}{2}+\frac{d}{2}\sqrt{\delta} = 1.
$$

So, by using the fact that $\sqrt{\delta}$ is an irrational number, we can deduce that $(c,d)=(1,0)$, which is a contradiction. Thus, we have $(c,d)=(1,0)$ and hence $c+\frac{md}{2}+\frac{d}{2}\sqrt{\delta}=1$. Therefore,

$$
\left(a+\frac{mb}{2}+\frac{b}{2}\sqrt{\delta}\right)\left(a_1+\frac{mb_1}{2}-\frac{b_1}{2}\sqrt{\delta}\right)^{\ell} = 1.
$$

Multiplying both sides of this equation by $\left(a_1+\frac{mb_1}{2}+\frac{b_1}{2}\sqrt{\delta}\right)^{\ell}$, we see that

$$
a+\frac{mb}{2}+\frac{b}{2}\sqrt{\delta} = \left(a_1+\frac{mb_1}{2}+\frac{b_1}{2}\sqrt{\delta}\right)^{\ell} = a_\ell+\frac{mb_\ell}{2}+\frac{b_\ell}{2}\sqrt{\delta}.
$$

Thus, by using the fact that $\sqrt{\delta}$ is an irrational number, we get $(a,b)=(a_\ell,b_\ell)=g^\ell$. Therefore, $g^\ell=(a,b)\in\{h,h^{-1},t{*}h,t{*}h^{-1}\}$ and so $h=t^r g^s$ for some integers r and s. Since $t\in\mathfrak{T}_{m,n}$, we see that $h\in\mathfrak{T}_{m,n}*\langle g\rangle$. So,

$$
(\mathfrak{B}_{m,n}\setminus\mathfrak{T}_{m,n}) \subseteq \mathfrak{T}_{m,n}*\langle g\rangle.
$$

Hence,

$$\mathfrak{B}_{m,n} = (\mathfrak{B}_{m,n} \setminus \mathfrak{T}_{m,n}) \cup \mathfrak{T}_{m,n} \subseteq \mathfrak{T}_{m,n} * \langle g \rangle \subseteq \mathfrak{B}_{m,n},$$

which means that $\mathfrak{B}_{m,n} = \mathfrak{T}_{m,n} * \langle g \rangle$. Also, by using the assumption $o(g) = \infty$, we can deduce that $\mathfrak{T}_{m,n} \cap \langle g \rangle = \{e\}$. Therefore, $\mathfrak{B}_{m,n} = \mathfrak{T}_{m,n} \oplus \langle g \rangle$. By Theorem 2.14 there exists an element $\theta \in \mathfrak{T}_{m,n}$ such that $\mathfrak{T}_{m,n} = \langle \theta \rangle$. Set $h' := \theta * g$. Since $h' \in (\mathfrak{B}_{m,n} \setminus \mathfrak{T}_{m,n})$, by the same argument we can find integers $r', s' \in \mathbb{Z}$ such that $\theta * g = h' = t^{r'} * g^{s'}$. Since $\mathfrak{B}_{m,n} = \mathfrak{T}_{m,n} \oplus \langle g \rangle$ and $\theta, t \in \mathfrak{T}_{m,n}$ it is concluded that $\theta = t^{r'} \in \langle t \rangle$. Therefore, $\mathfrak{T}_{m,n} = \langle \theta \rangle \subseteq \langle t \rangle \subseteq \mathfrak{T}_{m,n}$, which means that $\mathfrak{T}_{m,n} = \langle t \rangle = \{e, t\}$. Thus, $\mathfrak{B}_{m,n} = \mathfrak{T}_{m,n} \oplus \langle g \rangle = \langle t \rangle \oplus \langle g \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}$. □

Corollary 2.3. *Assume that n is a given positive nonsquare integer. Then, the Abelian group of all integer solutions of the Pell's equatuion $x^2 - ny^2 = 1$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$.*

Proof. The assertion follows from Lemma 2.18. □

We are now in a position to use the previous results to produce a proof of our main theorem.

Theorem 2.3. *Let m and n be two integers. Then, the Abelian group $\mathfrak{B}_{m,n}$ is isomorphic to one of the groups $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times \mathbb{Z}$.*

Proof. The assertion follows from Lemmas 2.15, 2.16, 2.17 and 2.18. □

Example 2.4. (i) Assume that $n > 0$ is a given perfect square integer. Then, by Lemma 2.17 we see that $\mathfrak{B}_{0,-n} = \mathfrak{T}_{0,-n} = \{(1,0), (-1,0)\} \simeq \mathbb{Z}_2$.

(ii) Assume that n is a given positive nonsquare integer. Then, by Corollary 2.19 we have $\mathfrak{B}_{0,-n} \simeq \mathbb{Z}_2 \times \mathbb{Z}$ and so $\mathfrak{T}_{0,-n} = \{(1,0), (-1,0)\} \simeq \mathbb{Z}_2$.

(iii) Let $g = (0,1) \in \mathfrak{B}_{0,1} = \{(1,0), (-1,0), (0,1), (0,-1)\}$. Then, one can see that $o(g) = 4$ and $\mathfrak{B}_{0,1} = \mathfrak{T}_{0,1} = \langle g \rangle \simeq \mathbb{Z}_4$.

(iv) Let $g = (0,-1) \in \mathfrak{B}_{-1,1}$. Then, it is easy to see that $o(g) = 6$ and so, by Theorem 2.14 and Lemma 2.15 we can deduce that $\mathfrak{B}_{-1,1} = \mathfrak{T}_{-1,1} = \langle g \rangle \simeq \mathbb{Z}_6$.

Remark 2.5. Let $(R, +, \cdot)$ be a commutative ring with the identity element and η, ξ, ζ be three arbitrary elements of R. Set

$$S(R, \eta, \xi, \zeta) := \{(u,v) \in R \times R \ : \ u^2 + \eta uv + \xi v^2 = \zeta\}.$$

Assume that $S(R, \eta, \xi, \zeta) \neq \emptyset$ and $(u,v) \in S(R, \eta, \xi, \zeta)$. Then, for each $(a,b) \in G(R, \eta, \xi)$ the element

$$(a,b) \cdot (u,v) := (au - \xi bv, bu + av + \eta bv),$$

belongs to $S(R, \eta, \xi, \zeta)$. In fact, by this definition the group $G(R, \eta, \xi)$ acts on the set $S(R, \eta, \xi, \zeta)$, provided that $S(R, \eta, \xi, \zeta) \neq \emptyset$.

References

[1] W. W. Adams and L. J. Goldstein, *Introduction to number theory*, Prentice-Hall, Inc, 1976.

[2] M. Hall, *The theory of groups*, New York: The Macmillan Company, 1959.

[3] J. McKay, *Another proof of Cauchy's group theorem*, American Math. Monthly, 66 (1959), 119.

Accepted: November 21, 2022

On open question of prominent interior GE-filters in GE-algebras

Yingmin Guo
College of Foreign Languages
Xi'an Shiyou University
Xi'an, 710065
China

Wei Wang*
College of Sciences
Xi'an Shiyou University
Xi'an, 710065
China
wwmath@xsyu.edu.cn

Hui Wu
Science and Information College
Qingdao Agricultural University
Qingdao 266109
China

Abstract. In an interior GE-algebra, the concept of prominent interior GE-filter of type 1 was introduced to serve as a generalization of prominent interior GE-filters. However, there are some work need to be done for this goal. For example, the extension property for prominent interior GE-filter of type 1 still remains unproved so there is an open question on the extension property of such GE-filters need to be proved that 'Let (X, f) be an interior GE-algebra. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter of type 1 in (X, f), then is G also a prominent interior GE-filter of type 1 in (X, f)?' In this paper, we propose the condition for an interior GE-filter to be a prominent interior GE-filter of type 1 , then we prove the extension property for prominent interior GE-filter of type 1 in an interior GE-algebra, and thus the open question is solved.
Keywords: GE-algebra, GE-filter, prominent interior GE-filter of type 1, extension property, open question.

1. Introduction

Henkin and Scolem introduced Hilbert algebra in the implication investigation inintuitionistic logics and other non classical logics [1, 2, 3, 4, 5, 6, 7, 8]. Bandaru et al. introduced GE-algebra as a generalization of Hilbert algebra, and studied its properties [9]. Later some scholars studied interior operators on different

∗. Corresponding author

algebraic structures, such as bounded residuated lattices, GMV-algebras and GE-algebras, and thus different kinds of interior GE-algebras were introduced [10, 11, 12, 13].

Filters theory plays a vital role not only in studying of algebraic structure, but also in non classical logic computer science and logical semantics From the aspect of logical point, filters correspond to various provable formulae sets [14, 15]. Song et al. introduced the notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter, and investigate their relations and properties [16]. They provided relations between belligerent interior GE-filter and interior GE-filter and conditions for an interior GE-filter to be a belligerent interior GE-filter is considered. Given a subset and an element, they established an interior GE-filter, and they considered conditions for a subset to be a belligerent interior GE-filter. They studied the extensibility of the belligerent interior GE-filter and established relationships between a weak interior GE-filter and a belligerent interior GE-filter of type 1, type 2 and type 3. Rezaei et al. [12]studied prominent GE-filters in GE algebras.

Afterwards, Song et al. introduced the concept of a prominent interior GE-filter (of type 1 and type 2), and investigated their properties. The relationship between a prominent GE-filter and a prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter are discussed. Also conditions for an interior GE-filter to be a prominent interior GE-filter are given and conditions under which an internal GE-filter larger than a given internal GE-filter can become a prominent internal GE-filter are considered. The relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1 is discussed [17].

After that, because of the lack of some properties for prominent interior GE-filters of type 1 and of type 2 to serve as a generalization of prominent interior GE-filters, [17] proposed an open question of prominent interior GE-filters of type 1 and of type 2 in GE-algebras that "Let (X, f) be an interior GE-algebra. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter of type 1 in (X, f), then is G also a prominent interior GE-filter of type 1 in (X, f)?"

The motivation of this paper is to further study the prominent interior GE-filter and solve the open question. We prove that in an interior GE-algebra, every prominent interior GE-filter of type 1 is a GE-filters as a complement of [17]. We propose the condition for an interior GE-filter to be a prominent interior GE-filter of type 1. Based on this, we prove the extension property for prominent interior GE-filter of type 1, and thus an open question on such GE-filters of type 1 is solved. As an application, the proof method of the extension property for prominent interior GE-filter of two types can improve the extension theory for other filters in GE-algebras and enrich the generalization theory for filter generation in other logic algebras.

2. Preliminaries

Definition 2.1 ([9]). *A GE-algebra is a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms for all $u, v, w \in X$:*
$(GE1)$ $u * u = 1$;
$(GE2)$ $1 * u = u$;
$(GE3)$ $u * (v * w) = u * (v * (u * w))$.

In a GE-algebra X, a binary relation $\leq$ is defined by $(\forall x, y \subseteq X)(x \leq y \Leftrightarrow x * y = 1)$.

Definition 2.2 ([9]). *A GE-algebra X is said to be transitive if it satisfies:*

$$(\forall x, y, z \in X)(x * y \leq (z * x) * (z * y)).$$

Proposition 2.1 ([9]). *Every GE-algebra X satisfies the following items for $\forall u, v, w \in X$:*
(1) $u * 1 = 1$;
(2) $u * (u * v) = u * v$;
(3) $u \leq v * u$;
(4) $u * (v * w) \leq v * (u * w)$;
(5) $1 \leq u \Rightarrow u = 1$;
(6) $u \leq (v * u) * u$;
(7) $u \leq (u * v) * v$;
(8) $u \leq v * w \Leftrightarrow v \leq u * w$.
If X is transitive, then:
(9) $u \leq v \Rightarrow w * u \leq w * v, v * w \leq u * w$;
(10) $u * v \leq (v * w) * (u * w)$.

Lemma 2.1 ([9]). *In a GE-algebra X, the following facts are equivalent for $\forall x, y, z \in X$:*
(1) $x * y \leq (z * x) * (z * y)$;
(2) $x * y \leq (y * z) * (x * z)$.

Definition 2.3 ([9]). *A subset F of a GE-algebra X is called a GE-filter of X if it satisfies for $\forall x, y \in X$:*
(1) $1 \in F$;
(2) $x * y \in F, x \in F \Rightarrow y \in F$.

Lemma 2.2 ([9]). *In a GE-algebra X , every non-empty subset F of X is a filter if and only if it satisfies:*
(1) $1 \in F$ *and* $(\forall x, y \in X)(x \leq y, x \in F \Rightarrow y \in F)$;
(2) $(\forall x, y \in F, z \in X)(x \leq y * z \Rightarrow z \in F)$.

Definition 2.4 ([10]). *A subset F of a GE-algebra X is called a prominent GE-filter of X if it satisfies* $1 \in F$ *and* $(\forall x, y, z \in X)(x * (y * z) \in F, x \in F \Rightarrow ((z * y) * y) * z \in F)$.

Note that every prominent GE-filter is a GE-filter in a GE-algebra (see [10]).

Definition 2.5 ([4]). *By an interior GE-algebra we mean a pair (X, f) in which X is a GE-algebra and $f : X \to X$ is a mapping such that for $\forall x, y \in X$:*

(1) $x \le f(x)$;
(2) $(f \circ f)(x) = f(x)$;
(3) $(x \le y \Rightarrow f(x) \le f(y))$.

Definition 2.6 ([11]). *Let (X, f) be an interior GE-algebra. A GE-filter F of X is said to be interior if it satisfies:*

$$(*) \qquad (\forall x \in X)(f(x) \in F \Rightarrow x \in F).$$

Definition 2.7 ([17]). *Let (X, f) be an interior GE-algebra. Then a subset F of X is called a prominent interior GE-filter in (X, f) if F is a prominent GE-filter of X which satisfies the condition $(*)$.*

Theorem 2.8 ([17]). *In an interior GE-algebra, every prominent interior GE-filter is an interior GE-filter.*

Theorem 2.9 ([17]). *Every interior GE-filter F in an interior GE-algebra (X, f) is a prominent interior GE-filter if and only if it satisfies:*
$(\forall x, y \in X)(x * y \in F \Rightarrow ((y * x) * x) * y \in F)$.

3. Prominent interior GE-filters of type 1

Definition 3.1 ([17]). *Let (X, f) be an interior GE-algebra and let F be a subset of X which satisfies $1 \in F$, then F is called a prominent interior GE-filter of type 1 in (X, f) if it satisfies:*

$$(\forall x, y, z \in X)(x * (y * f(z)) \in F, f(x) \in F \Rightarrow ((f(z) * y) * y) * f(z) \in F).$$

By example [17] shows that interior GE-filter and prominent interior GE-filter of type 1 are independent of each other.

Theorem 3.2 ([17]). *In an interior GE-algebra, every prominent interior GE-filter is of type 1, but the converse may not be true.*

Proposition 3.1. *In an interior GE-algebra, every prominent interior GE-filter of type 1 is a GE-filters.*

Proof. Let F be prominent interior GE-filter of type 1 of X and for any $x, y \in X$. Let $x * y \in F$, $x \in F$, if we let f be the identity mapping and $z = 1$, then we get $f(z) = 1$ and $f(x) \in F$, by definition , we have $(\forall x, y, z = 1 \in X) x * (z * f(y)) = x * (1 * y) = x * y \in F, f(x) = x \in F \Rightarrow (f(y) * z) * z) * f(y) = (y * 1) * 1) * y = y \in F$. It follows from F is a GE-filters. □

Theorem 3.3. *An interior GE-filter F of X is a prominent interior GE-filter of type 1 if and only if it satisfies:*

$$y * f(z) \in F \;\; \textit{implies} \;\; ((f(z) * y) * y) * f(z) \in F \;\; \textit{for all}\, y, z \in X.$$

Proof. Assume that F is a prominent interior GE-filter of type 1 of X and let $y, z \in X$ be such that $y * f(z) \in F$. Then $1 * (y * f(z)) = y * f(z) \in F$ and $f(1) = 1 \in F$. It follows from that $((f(z) * y) * y) * f(z) \in F$.

Conversely, let F be an interior GE-filter of X satisfying the above condition and let $x, y, z \in X$ be such that $x * (y * f(z)) \in F$ and $f(x) \in F$. Then $x \in F$, $y * f(z) \in F$ and hence $((f(z) * y) * y) * f(z) \in F$. Therefore, F is a prominent interior GE-filter of type 1 of X. □

[17] proposed an open question of prominent interior GE-filters of type 1 and of type 2 in GE-algebras: Let (X, f) be an interior GE-algebra. Let F and G be interior GE-filters in (X, f). If $F \subseteq G$ and F is a prominent interior GE-filter of type 1 in (X, f), then is G also a prominent interior GE-filter of type 1 in (X, f)?

For this open question for type 1, based on the previous work, we can solve it in the following theorem.

Theorem 3.4 (Extension property for prominent interior GE-filter of type 1.)**.** *Let F and G be prominent interior GE-filters of type 1 of X such that $F \subseteq G$. If F is a prominent interior GE-filter of type 1, then so is G.*

Proof. Let $y, z \in X$ be such that $y * f(z) \in G$. Then $y * ((y * f(z)) * f(z)) \leq (y * f(z)) * (y * f(z)) = 1 \in F$. Since F is prominent interior of type 1, it follows that $((((y * f(z)) * f(z)) * y) * y) * ((y * f(z)) * f(z)) \in F$ so, that $(y * f(z)) * (((((y * f(z)) * f(z)) * y) * y) * f(z)) \in F \subseteq G$.

Since $y * f(z) \in G$, therefore $((((y * f(z)) * f(z)) * y) * y) * f(z) \subseteq G$. But $1 = (y * f(z)) * 1 = (y * f(z)) * (f(z) * f(z)), \leq f(z) * ((y * f(z)) * f(z)), \leq (((y*f(z))*f(z))*y)*(f(z)*y), \leq ((f(z)*y)*y)*((((y*f(z))*f(z))*y)*y), \leq (((((y * f(z)) * f(z)) * y) * y) * f(z)) * (((f(z) * y) * y) * f(z))$. Using Lemma 2.6. (2), we get $(((((y * f(z)) * f(z)) * y) * y) * f(z)) * (((f(z) * y) * y) * f(z)) = 1$ and $((f(z) * y) * y) * f(z) \in G$.

Hence, by Theorem 3.4, G is a prominent interior GE-filter of type 1 of X. □

4. Conclusion

Filter theory is of great significance in the study of algebraic domain. Many scholars introduced the concepts and relationships among a varieties of filters from different aspects. Several open questions on this topic thus appeared. In GE-algebras, an open question of prominent interior GE-filters of type 1 and of type 2 is proposed.

The purpose of the study is to solve the open question. On the basis of previous work, in this paper, we prove that in an interior GE-algebra, every prominent interior GE-filter of type 1 is a GE-filters as a complement. We also propose the condition for an interior GE-filter to be a prominent interior GE-filter of type 1. Based on this, we prove the extension property for prominent interior GE-filter of type 1 is proved, and thus an open question on such GE-filters of type 1 is solved. We hope that will bring us enlightment in the study of this field.

For the future work, we will further study the prominent interior GE-filter of type 2 and solve the open question of it completely. If the extension property for prominent interior GE-filter of type 2 also holds, we will try to find a generalization of the two types in an interior GE-algebra.

Acknowledgment

This Research work is supported by the National Natural Science Foundation of P. R. China (Grant No. 11571281).

References

[1] R. A. Borzooei, J. Shohani, *On generalized Hilbert algebras*, Ital. J. Pure Appl. Math., 29 (2012), 71-86.

[2] I. Chajda, R. Halas, *Congruences and ideal as in Hilbert algebras*, Kyungpook Math. J., 39 (1999), 429-432.

[3] I. Chajda, R. Halas, Y. B. Jun, *Annihilators and deductive systems in commutative Hilbert algebras*, Commentationes Mathematicae Universitatis Carolinae, 43 (2002), 407-417.

[4] Y. B. Jun, *Commutative Hilbert algebras*, Soochow. J. Math., 22 (1996), 477-484.

[5] Y. B. Jun, K. H. Kim, *H-filters of Hilbert algebras*, Scientiae Mathematicae Japonicae, e-2005 (2005), 231-236.

[6] A. S. Nasab, A. B. Saeid, *Semi maximal filter in Hilbert algebra*, J. Intell. Fuzzy Syst., 30 (2016), 7-15.

[7] A. S. Nasab, A. B. Saeid, *Stonean Hilbert algebra*, J. Intell. Fuzzy Syst., 30 (2016), 485-492.

[8] A. S. Nasab, A. B. Saeid, *Study of Hilbert algebras, in point of filters*, Analele Ştiinţifice ale Universităţii Ovidius Constanţa, 24 (2016), 221-251.

[9] R. K. Bandaru, A. B. Saeid, Y. B. Jun, *On GE-algebras*, Bull. Sect. Logic, 50 (2021), 81-96.

[10] J. G. Lee, R. K. Bandaru, K. Hur, Y. B. Jun, *Interior GE-algebras*, J. Math., 2021 (2021), 1-10.

[11] J. Rachunek, Z. Svoboda, *Interior and closure operators on bounded residuated lattices*, Cent. Eur. J. Math., 12 (2014), 534-544.

[12] A. Rezaei, R. K. Bandaru, A. B. Saeid, Y. B. Jun, *Prominent GE-filters and GE-morphisms in GE-algebras*, Afr. Mat., 32 (2021), 1121-1136.

[13] F. Svrcek, *Operators on GMV-algebras*, Math. Bohem., 129 (2004), 337-347.

[14] W. Wang, Y. M. Guo, *On the intuitionistic fuzzy filter in Heyting algebra*, Journal of Xi'an Shiyou University, 23 (2008), 106-108.

[15] W. Wang, X. L. Xin, *On fuzzy filters of pseudo BL algebras,* Fuzzy Sets and Systems, 162 (2011), 27-38.

[16] S. Z. Song, R. K. Bandaru, D. A. Romano, Y. B. Jun, *Interior GE-filters of GE-algebras,* Discuss. Math. Gen. Algebra Appl., in press.

[17] S. Z. Song, R. K. Bandaru, Y. B. Jun, *Prominent interior GE-filters of GE-algebras*, AIMS Mathematics, 6 (2021), 13432-13447.

Accepted: April 2, 2022

On soft p_c-regular and soft p_c-normal spaces

Qumri H. Hamko
Department of Mathematics
College of Education
Salahaddin University
Kurdistan-Region
Iraq
qumri.hamko@su.edu.krd

Nehmat K. Ahmed
Department of Mathematics
College of Education
Salahaddin University
Kurdistan-Region
Iraq
nehmat.ahmed@su.edu.krd

Alias B. Khalaf*
Department of Mathematics
College of Science
University of Duhok
Kurdistan Region
Iraq
aliasbkhalaf@uod.ac

Abstract. The aim of this paper is to introduce two new types of soft separation axioms called soft $p_c$regular and soft p_c-normal spaces by using the concept of soft p_c-open sets in soft topological spaces. We explore several properties and relations of such spaces. Also, we investigate hereditary and soft invariance properties by considering certain soft mappings.
Keywords: soft p_c-open set, soft p_c-regular space, soft p_c-normal space.

1. Introduction

Molodtsov [18] initiated the concept of soft set theory in 1999 as a new mathematical tool to treat many complicated problems related to probability and fuzzy set theory. After that many researchers presented applications of soft set theory in many fields of mathematics such as operations researches, mathematical analysis and algebraic structures. Shabir and Naz [21] in 2011 applied the notion of soft sets to introduce the concept of soft topological spaces which are defined over an initial universe with a fixed set of parameters. They introduced

*. Corresponding author

almost all the essential classical notions in topology and defined the concept of soft open sets, soft closed sets, soft interior point, soft closure and soft separation axioms. Al-shami et al. [4] and [5], investigated several types of soft separation axioms and studied studied their images ang pre-images under soft mappings.

Husain and Ahmed [13] in 2015 studied the properties of soft interior, soft closure and soft boundary operators and they introduced separation axioms by using ordinary points in the universal set also Georgiou et. al [9] in 2013, studied some soft separation axioms, soft continuity in soft topological spaces using ordinary points of a topological space X. Bayramov et al. in [7], defined the notion of soft points and applied them to discuss the properties of soft interior, soft closure and soft boundary operators. They also defined and introduced soft neighborhoods and soft continuity in soft topological spaces using soft points.

It is noticed that a soft topological space gives a parametrized family of topologies on the initial universe but the converse is not true i.e. if some topologies are given for each parameter, we cannot construct a soft topological space from the given topologies. Consequently we can say that the soft topological spaces are more generalized than the classical topological spaces for more details we refer to [3] and [4].

Recently, Hamko and Ahmed [1] introduced the notion of soft p_c-open sets. They applied this notion to define and discuss the concept of soft p_c-interior, soft soft p_c-closure and soft p_c-boundary operators. Also, they introduced the concept of soft continuity and almost soft continuity by employing soft points and soft p_c-open sets in a soft topological space.

The aim of this paper, is to introduce and discuss a study of soft separation axioms which we call them soft p_c-regular and soft p_c-normal spaces which are defined over an initial universe with a fixed set of parameters. We indicate the relationships between them and present several of their properties.

Throughout the present paper, X will be a nonempty initial universal set and E will be a set of parameters. A pair (F, E) is called a soft set over X, where F is a mapping $F : E \to P(X)$. The collection of soft sets (F, E) over a universal set X with a parameter set E is denoted by $SP(X)_E$. Any logical operation (λ) on soft sets in soft topological spaces are denoted by usual set theoretical operations with symbol $(\widetilde{s}(\lambda))$.

2. Preliminaries

In this section we present the main definitions and results which will be used in the sequel. For some definitions or results which are not mentioned in this section, we refer to [2], [3], [7], [12], [17] and [22].

Definition 2.1 ([21])**.** *A soft set (F, E) over X is said to be null soft set denoted by $\widetilde{\phi}$ if, for all $e \in E$, $F(e) = \phi$ and (F, E) over X is said to be absolute soft set denoted by $\widetilde{X}$ if, for all $e \in E$, $F(e) = X$.*

Definition 2.2 ([21]). *The complement of a soft set (F,E) is denoted by $(F,E)^c$ or $\tilde{X}\setminus(F,E)$ and is defined by $(F,E)^c=(F^c,E)$ where $F^c: E\to P(X)$ is a mapping given by $F^c(e)=X\setminus F(e)$, for all $e\in E$.*

It is clear that $((F,E)^c)^c=(F,E)$, $\widetilde{\phi}^c=\widetilde{X}$ and $\widetilde{X}^c=\widetilde{\phi}$.

Definition 2.3 ([21]). *For two soft sets (F,E) and (G,B) over a common universe X, we say that (F,E) is a soft subset of (G,B), if*

1. *$E\subseteq B$;*
2. *for all $e\in E$, $F(e)\subseteq G(e)$.*

We write $(F,E)\sqsubseteq(G,B)$.

Definition 2.4 ([21]). *The union of two soft sets of (F,E) and (G,B) over the common universe X is the soft set $(H,C)=(F,E)\sqcup(G,B)$, where $C=E\cup B$ and for all $e\in C$,*

$$H(e)=\begin{cases}F(e), & if \quad e\in E-B,\\ G(e), & if \quad e\in B-E,\\ F(e)\cup G(e), & if \quad e\in E\cap B.\end{cases}$$

In particular, $(F,E)\sqcup(G,E)=F(e)\cup G(e)$, for all $e\in E$.

Definition 2.5 ([21]). *The intersection (H,C) of two soft sets (F,E) and (G,B) over a common universe X, denoted $(F,E)\sqcap(G,B)$, is defined as $C=E\cap B$, and $H(e)=F(e)\cap G(e)$, for all $e\in C$.*

In particular, $(F,E)\sqcap(G,E)=F(e)\cap G(e)$, for all $e\in E$.

Definition 2.6 ([7]). *Let $x\in X$, then (x,E) denotes the soft set over X for which $x(e)=\{x\}$, for all $e\in E$.*

Let (F,E) be a soft set over X and $x\in X$. We say that $x\widetilde{\in}(F,E)$ read as x belongs to the soft set (F,E) whenever $x\in F(e)$, for all $e\in E$.

Definition 2.7 ([7]). *The soft set (F,E) is called a soft point, denoted by (x_e,E) or x_e, if for the element $e\in E$, $F(e)=\{x\}$ and $F(e)=\phi$, for all $e\in E\setminus\{e\}$.*

We say that $x_e\widetilde{\in}(G,E)$ if $x\in G(e)$.

Two soft points x_e and $y_{e'}$ are distinct if either $x\neq y$ or $e\neq e'$.

Remark 2.1. From Definition 2.6 and Definition 2.7, it is clear that:

1. (x,E) is the smallest soft set containing x;
2. if $x\widetilde{\in}(F,E)$ then $x_e\widetilde{\in}(F,E)$;
3. $(F,E)=\sqcup\{(x_e,E): e\in E\}$.

Definition 2.8 ([21]). *Let $\widetilde{\tau}$ be a collection of soft sets over a universe X with a fixed set E of parameters. Then, $\widetilde{\tau} \subseteq SP(X)_E$ is called a soft topology if*

1. *$\tilde{\phi}$ and $\tilde{X}$ belongs to $\widetilde{\tau}$.*
2. *The union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.*
3. *The intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.*

The triplet $(X, \widetilde{\tau}, E)$ is called a soft topological space over X. The members of $\tilde{\widetilde{\tau}}$ are called soft open sets in $\tilde{X}$ and complements of them are called soft closed sets in $\tilde{X}$ and they are denoted by $SO(\tilde{X})$ and $SC(\tilde{X})$, respectively. Soft interior and soft closure are denoted by $\tilde{s}int$ and $\tilde{s}cl$, respectively.

Definition 2.9 ([21]). *Let $(X, \widetilde{\tau}, E)$ be a soft topological space and let (G, E) be a soft set. Then:*

1. *The soft closure of (G, E) is the soft set $\tilde{s}cl(G, E) = \sqcap\{(K, B)\widetilde{\in}SC(\tilde{X}) : (G, E) \sqsubseteq (K, B)\}$*
2. *The soft interior of (G, E) is the soft set $\tilde{s}int(G, E) = \sqcup\{(H, B)\widetilde{\in}SO(\tilde{X}) : (H, B) \sqsubseteq (G, E)\}$.*

Definition 2.10 ([12]). *Let $(X, \widetilde{\tau}, E)$ be a soft topological space, (G, E) be a soft set over $\tilde{X}$ and $x_e\widetilde{\in}\tilde{X}$. Then, (G, E) is said to be a soft neighborhood of x_e if there exists a soft open set (H, E) such that $x_e\widetilde{\in}(H, E) \sqsubseteq ((G, E)$.*

Proposition 2.1 ([21]). *Let $(Y, \widetilde{\tau}_Y, E)$ be a soft subspace of a soft topological space $(X, \widetilde{\tau}, E)$ and $(F, E)\widetilde{\in}SP(X)_E$. Then:*

1. *If (F, E) is a soft open set in $\widetilde{Y}$ and $\widetilde{Y}\widetilde{\in}\widetilde{\tau}$, then $(F, E)\widetilde{\in}\widetilde{\tau}$.*
2. *(F, E) is a soft open set in $\widetilde{Y}$ if and only if $(F, E) = \widetilde{Y}\sqcap(G, E)$ for some $(G, E)\widetilde{\in}\widetilde{\tau}$.*
3. *(F, E) is a soft closed set in $\widetilde{Y}$ if and only if $(F, E) = \widetilde{Y}\sqcap(H, E)$ for some soft closed (H, E) in $\widetilde{X}$.*

Definition 2.11 ([14]). *A soft subset (F, E) of a soft space $\widetilde{X}$ is said to be soft pre-open if $(F, E) \sqsubseteq \tilde{s}int(\tilde{s}cl(F, E))$. The complement of a soft pre-open set is said to be soft pre-closed. The family of soft pre-open (soft pre-closed) set is denoted by $\tilde{s}PO(X)$ ($\tilde{s}PC(X)$).*

Lemma 2.1 ([14]). *Arbitrary union of soft pre-open sets is a soft pre-open set.*

Lemma 2.2 ([2]). *A subset (F, E) of a soft topological spaces $(X, \widetilde{\tau}, E)$ is soft pre-open if and only if there exists a soft open set (G, E) such that*

$$(F, E)\sqsubseteq(G, E)\sqsubseteq\tilde{s}cl(F, E).$$

Lemma 2.3 ([2]). *Let $(F,E)\sqsubseteq\widetilde{Y}\sqsubseteq\widetilde{X}$, where $(X,\widetilde{\tau},E)$ is a soft topological space and $\widetilde{Y}$ is a soft pre-open subspace of $\widetilde{X}$. Then, $(F,E)\widetilde{\in}\ \widetilde{s}PO(X)$, if and only if $(F,E)\widetilde{\in}\ \widetilde{s}PO(Y)$.*

Theorem 2.1 ([19]). *If (U,E) is soft open and (F,E) is soft pre-open in $(X,\widetilde{\tau},E)$, then $(U,E)\sqcap(F,E)$ is soft pre-open.*

Lemma 2.4 ([1]). *Let $(F,E)\sqsubseteq\widetilde{Y}\sqsubseteq\widetilde{X}$, where $(X,\widetilde{\tau},E)$ is a soft topological space and $\widetilde{Y}$ is a soft subspace of $\widetilde{X}$. If $(F,E)\widetilde{\in}\tilde{s}PO(X)$, then $(F,E)\widetilde{\in}\tilde{s}PO(Y)$.*

Definition 2.12 ([1]). A soft pre-open set (F,E) in a soft topological space $(X,\widetilde{\tau},E)$ is called soft p_c-open if for each $x_e\widetilde{\in}(F,E)$, there exists a soft closed set (K,E) such that $x_e\widetilde{\in}(K,E)\sqsubseteq(F,E)$. The soft complement of each soft p_c-open set is called soft p_c-closed set.

The family of all soft p_c-open (resp., soft p_c-closed) sets in a soft topological space $(X,\widetilde{\tau},E)$ is denoted by $\widetilde{s}P_cO(X,\widetilde{\tau},E)$ (resp., $\widetilde{s}P_cC(X,\widetilde{\tau},E)$) or $\widetilde{s}P_cO(X)$ (resp., $\widetilde{s}P_cC(X)$).

Definition 2.13 ([2]). *Let $(X,\widetilde{\tau},E)$ be a soft topological space and let (G,E) be a soft set. Then:*

1. *The soft pre-closure of (G,E) is the soft set*

$$\tilde{s}pcl(G,E)=\sqcap\{(K,B)\widetilde{\in}\tilde{s}PC(\tilde{X}):(G,E)\sqsubseteq(K,B)\}.$$

2. *The soft pre-interior of (G,E) is the soft set*

$$\tilde{s}pint(G,E)=\sqcup\{(H,B)\widetilde{\in}\tilde{s}PO(\tilde{X}):(H,B)\sqsubseteq(G,E)\}.$$

Definition 2.14 ([11]). *Let $(X,\widetilde{\tau},E)$ be a soft topological space and let (G,E) be a soft set. Then:*

1. *A soft point $x_e\widetilde{\in}\widetilde{X}$ is said to be a soft p_c-limit soft point of a soft set (F,E) if for every soft p_c-open set (G,E) containing x_e, $(G,E)\sqcap[(F,E)\setminus\{x_e\}]\neq\widetilde{\phi}$.*

   The set of all soft p_c-limit soft points of (F,E) is called the soft p_c-derived set of (F,E) and is denoted by $\tilde{s}P_cD(F,E)$.

2. *The soft p_c-closure of (G,E) is the soft set*

$$\tilde{s}p_ccl(G,E)=\sqcap\{(K,B)\widetilde{\in}\tilde{s}P_cC(\tilde{X}):(G,E)\sqsubseteq(K,B)\}.$$

3. *The soft p_c-interior of (G,E) is the soft set*

$$\tilde{s}p_cint(G,E)=\sqcup\{(H,B)\widetilde{\in}\tilde{s}P_cO(\tilde{X}):(H,B)\sqsubseteq(G,E)\}.$$

Lemma 2.5 ([1]). *If $(F,E)\sqsubseteq\widetilde{Y}\sqsubseteq\widetilde{X}$ and $\widetilde{Y}$ is soft clopen. Then, $(F,E)\widetilde{\in}\tilde{s}P_cO(Y)$ if and only if $(F,E)\widetilde{\in}\tilde{s}P_cO(X)$.*

Lemma 2.6 ([1]). *Let $(F,E) \sqsubseteq \widetilde{Y} \sqsubseteq \widetilde{X}$ and $\widetilde{Y}$ be soft clopen. If $(F,E)\widetilde{\in}\tilde{s}P_cO(X)$, then $(F,E) \sqcap \widetilde{Y}\widetilde{\in}\tilde{s}P_cO(Y)$.*

Lemma 2.7 ([11]). *Let $(F,E) \sqsubseteq \widetilde{Y} \sqsubseteq \widetilde{X}$. If $\widetilde{Y}$ is soft clopen, then $\tilde{s}p_c cl_Y(F,E) = \tilde{s}p_c cl_X(F,E) \sqcap \widetilde{Y}$.*

Definition 2.15 ([10]). *A soft topological space $(X, \widetilde{\tau}, E)$ is said to be:*

1. *Soft T_0, if for each pair of distinct soft points x, $y\widetilde{\in}X$, there exist soft open sets (F,E) and (G,E) such that either $x\widetilde{\in}(F,E)$ and $y\widetilde{\notin}(F,E)$ or $y\widetilde{\in}(G,E)$ and $x\widetilde{\notin}(G,E)$.*

2. *Soft T_1, if for each pair of distinct soft points x, $y \in X$, there exist two soft open sets (F,E) and (G,E) such that $x\widetilde{\in}(F,E)$ but $y\widetilde{\notin}(F,E)$ and $y\widetilde{\in}(G,E)$ but $x\widetilde{\notin}(G,E)$.*

3. *Soft T_2, if for each pair of distinct soft points x, $y \in X$, there exist two disjoint soft open sets (F,E) and (G,E) containing x and y, respectively.*

In [7], S. Bayramov and C. G. Aras redefined soft T_i-spaces by replacing soft points x_e, $y_{e'}$ instead of the ordinary points x, y in Definition 2.15.

Proposition 2.2 ([7]). *1. Every soft T_2-space $\Rightarrow$ soft T_1-space $\Rightarrow$ soft T_0-space.*
2. A soft topological space $(X, \widetilde{\tau}, E)$ is soft T_1 if and only if each soft point is soft closed.

In [21], a soft regular space is defined by using ordinary points as:

Definition 2.16 ([21]). *If for every $x \in X$ and every soft closed set (F,E) not containing x, there exist two soft open sets (G,E) and (H,E) such that $x\widetilde{\in}(G,E)$, $(F,E) \sqsubseteq (H,E)$ and $(G,E)\sqcap(H,E) = \widetilde{\phi}$ then $\widetilde{X}$ is called soft regular.*

In [12] a soft regular space is defined by by replacing soft points x_e instead of the ordinary point x in Definition 2.16.

Definition 2.17 ([15]). *A soft topological space $(X, \widetilde{\tau}, E)$ is said to be*

1. *$\tilde{s}p_c$-T_0, if for each pair of distinct soft points x_e, $y_{e'}\widetilde{\in}SP(X)_E$, there exist soft p_c-open sets (F,E) and (G,E) such that $x_e\widetilde{\in}(F,E)$ and $y_{e'}\widetilde{\notin}(F,E)$ or $y_{e'}\widetilde{\in}(G,E)$ and $x_e\widetilde{\notin}(G,E)$.*

2. *$\tilde{s}p_c$-T_1, if for each pair of distinct soft points x_e, $y_{e'}\widetilde{\in}SP(X)_E$, there exist two soft p_c-open sets (F,E) and (G,E) such that $x_e\widetilde{\in}(F,E)$ but $y_{e'}\widetilde{\notin}(F,E)$ and $y_{e'}\widetilde{\in}(G,E)$ but $x_e\widetilde{\notin}(G,E)$.*

3. *$\tilde{s}p_cT_2$, if for each pair of distinct soft points x_e, $y_{e'}\widetilde{\in}SP(X)_E$, there exist two disjoint soft p_c-open sets (F,E) and (G,E) containing x_e and $y_{e'}$, respectively.*

Definition 2.18 ([15]). *A soft topological space $(X, \widetilde{\tau}, E)$ is said to be*

1. $\tilde{s}p_cT_0^*$, *if for each pair of distinct points $x, y \in X$, there exist soft p_c-open sets (F, E) and (G, E) such that $x \widetilde{\in} (F, E)$ and $y \widetilde{\notin} (F, E)$ or $y \widetilde{\in} (G, E)$ and $x \widetilde{\notin} (G, E)$.*

2. $\tilde{s}p_c$-T_1^*, *if for each pair of distinct points $x, y \in X$, there exist two soft p_c-open sets (F, E) and (G, E) such that $x \widetilde{\in} (F, E)$ but $y \widetilde{\notin} (F, E)$ and $y \widetilde{\in} (G, E)$ but $x \widetilde{\notin} (G, E)$.*

3. $\tilde{s}p_c$-T_2^*, *if for each pair of distinct soft points $x, y \in X$, there exist two disjoint soft p_c-open sets (F, E) and (G, E) containing x and y, respectively.*

Proposition 2.3 ([15]). *A space $(X, \widetilde{\tau}, E)$ is $\tilde{s}p_c - T_0$ if and only if every soft points $x_e \neq y_{e'}$ implies $\tilde{s}p_c cl\{x_e\} \neq \tilde{s}p_c cl\{y_{e'}\}$.*

Proposition 2.4 ([15]). *A space $(X, \widetilde{\tau}, E)$ is $\tilde{s}p_c - T_1$ if and only if every soft point of the space $(X, \widetilde{\tau}, E)$ is an soft p_c-closed set.*

Proposition 2.5 ([15]). *If a soft topological space $(X, \widetilde{\tau}, E)$ is $\tilde{s}p_c - T_1$, then it is soft $\tilde{s}p_c - T_1^*$.*

Definition 2.19 ([16]). *Let $SP(X)_E$ and $SP(Y)_B$ be families of soft sets. Let $u : X \to Y$ and $p : E \to B$ be mappings. Then, a mapping $f_{pu} : SP(X)_E \to SP(Y)_B$ is defined as:*

1. *Let (F, E) be a soft set in $SP(X)_E$. The image of (F, E) under f_{pu}, written as $f_{pu}(F, E) = (f_{pu}(F), p(E))$, is a soft set in $SP(Y)_B$ such that*

$$f_{pu}(F)(e') = \begin{cases} \bigcup\limits_{e \in p^{-1}(e') \cap E} u(F(e)), & if \quad p^{-1}(e') \cap E \neq \phi \\ \phi, & if \quad p^{-1}(e') \cap E = \phi, \end{cases}$$

for all $e' \in B$.

2. *Let (G, B) be a soft set in $SP(Y)_B$. Then, the inverse image of (G, B) under f_{pu}, written as $f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is a soft set in $SP(X)_E$ such that*

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}(G(p(e))), & if \quad p(e) \in B \\ \phi, & otherwise, \end{cases}$$

for all $e \in E$.

The soft function f_{pu} is called surjective if p and u are surjective and it is called injective if p and u are injective.

Definition 2.20 ([22]). *Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\mu}, B)$ be two soft topological spaces. A soft mapping $f_{pu}:(X, \widetilde{\tau}, E) \to (Y, \widetilde{\mu}, B)$ is called soft continuous if $f_{pu}^{-1}((G, B)) \widetilde{\in} \widetilde{\tau}$, for all $(G, B) \widetilde{\in} \widetilde{\mu}$.*

3. Soft p_c-regular spaces

In this section, we introduce some types of soft regular spaces by using soft p_c-open sets. Many characterizations of these spaces are found. Also, some hereditary properties and relations between these spaces are investigated.

Definition 3.1. *A soft space $\widetilde{X}$ is said to be $\tilde{s}p_c$-regular (resp., $\tilde{s}p_c^*$-regular), if for each $x_e \widetilde{\in} \widetilde{X}$ and each soft closed (resp., $\tilde{s}p_c$-closed) set (K,E) such that $x_e \widetilde{\notin} (K,E)$, there exist two disjoint soft p_c-open sets (F,E) and (G,E) such that $x_e \widetilde{\in} (F,E)$ and $(K,E) \sqsubseteq (G,E)$.*

Remark 3.1. In a finite soft space $SP(X)_E$, if (F,E) is any soft p_c-open set, then by definition it is soft pre-open and a union of soft closed sets and hence it is soft closed, so we obtain that (F,E) is both soft open and soft closed.

Equivalently, any soft p_c-closed set is both open and closed.

From the above remark, we get the following result

Proposition 3.1. *If $SP(X)_E$ is finite, then every $\tilde{s}p_c$-regular space is both $\tilde{s}p_c^*$-regular and soft regular.*

The following example shows that an $\tilde{s}p_c$-regular space is not necessary $\tilde{s}p_c-T_i$ for $i = 0, 1, 2$.

Example 3.1. Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and $\widetilde{X} = \{(e_1, X), (e_2, X)\}$ and let $\widetilde{\tau} = \{\widetilde{X}, \widetilde{\phi}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\}$, where $(F_1, E) = \{(e_1, \{x\}), (e_2, \phi)\}$, $(F_2, E) = \{(e_1, \{x\}), (e_2, X)\}$, $(F_3, E) = \{(e_1, \{y\}), (e_2, \phi)\}$, $(F_4, E) = \{(e_1, \{y\}), (e_2, X)\}$, $(F_5, E) = \{(e_1, X), (e_2, \phi)\}$, $(F_6, E) = \{(e_1, \phi), (e_2, X)\}$. Then, it can be checked that $\tilde{s}p_cO(X) = \widetilde{\tau}$. Since $x_{e_2} \neq y_{e_2}$ and each soft open set containing one of them contains the other, so it is not $\tilde{s}p_c - T_i$ for $i = 0, 1, 2$. This space is $\tilde{s}p_c - T_i^*$ for $i = 0, 1$ but it is not $\tilde{s}p_c - T_2^*$. By easy calculation it can be shown that this space is $\tilde{s}p_c$-regular and hence by Proposition 3.1 it is both $\tilde{s}p_c^*$-regular and soft regular.

Example 3.2. Let $X = \{x, y\}$, $E = \{e_1, e_2\}$ and $\widetilde{X} = \{(e_1, X), (e_2, X)\}$ and let $\widetilde{\tau} = \{\widetilde{X}, \widetilde{\phi}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$, where $(F_1, E) = \{(e_1, X), (e_2, \phi)\}$, $(F_2, E) = \{(e_1, \phi), (e_2, \{x, y\})\}$, $(F_3, E) = \{(e_1, \{y\}), (e_2, X)\}$ and $(F_4, E) = \{(e_1, \{y\}), (e_2, \phi)\}$. Since $y_{e_1} \widetilde{\notin} (F_3, E)^c$ but there are no disjoint soft p_c-open sets containing them. Hence, this space is not $\tilde{s}p_c$-regular and not soft regular but it can be checked that it is $\tilde{s}p_c^*$-regular.

Recall that a soft space $(X, \widetilde{\tau}, E)$ is called soft-Alexandroff space [20] if any arbitrary intersection of soft open sets is soft open. Equivalently, any arbitrary union of soft closed sets is soft closed.

Proposition 3.2. *Every soft-Alexandroff space is $\tilde{s}p_c^*$-regular.*

Proof. Similar to Remark 3.1, in a soft-Alexandroff space $(X, \widetilde{\tau}, E)$. If (F, E) is an $\tilde{s}p_c^*$-open set, then it is soft closed and hence (F, E) and its complement are both soft open and soft closed. Therefore, for each $x_e \widetilde{\notin} (F, E)$, we have (F, E) and $(F, E)^c$ are the required disjoint soft $\tilde{s}p_c^*$-open sets. □

If we take $X = \mathbb{R}$ with the usual topology and if E consists only one parameter, then $\mathbb{R}$ is both soft regular and $\tilde{s}p_c^*$-regular but it is not soft-Alexandroff.

Theorem 3.1. *The following statements about a space $\tilde{X}$ are equivalent:*

1. *$\tilde{X}$ is $\tilde{s}p_c^*$-regular (resp., $\tilde{s}p_c$-regular) space.*
2. *For each $x_e \widetilde{\in} \widetilde{X}$ and each soft p_c-open (resp., soft open) set (F, E) containing x_e, there exist soft p_c-open set (G, E) containing x_e such that $x_e \widetilde{\in} (G, E) \sqsubseteq \tilde{s}p_c cl(G, E) \sqsubseteq (F, E)$.*
3. *Each element of X has an $\tilde{s}p_c$- neighborhood (resp., soft neighborhood) base consisting of soft p_c-closed sets.*
4. *Every soft p_c-closed (resp., soft closed) set (K, E) is the intersection of all soft p_c-closed neighborhoods of (K, E).*
5. *For every non-empty soft subset (F, E) of $\tilde{X}$ and every soft p_c-open (resp., soft open) subset (G, E) of $\tilde{X}$ such that $(F, E) \sqcap (G, E) \neq \widetilde{\phi}$, there exist $\tilde{s}p_c$- open subset (W, E) of $\tilde{X}$ such that $(F, E) \sqcap (W, E) \neq \widetilde{\phi}$, and $\tilde{s}p_c cl(W, E) \sqsubseteq (G, E)$.*
6. *For every non-empty soft subset (F, E) of $\tilde{X}$ and every soft p_c-closed (resp., soft closed) subset (K, E) of $\tilde{X}$ such that $(F, E) \sqcap (K, E) = \widetilde{\phi}$, there exist two soft p_c-open subset (G, E) and (W, E) such that $(F, E) \sqcap (G, E) \neq \widetilde{\phi}$, $(W, E) \sqcap (G, E) = \widetilde{\phi}$ and $(K, E) \sqsubseteq (W, E)$.*

Proof. We only prove the $\tilde{s}p_c^*$-regular case. Since the other case can be proved similarly.

(1) → (2). Let (F, E) be soft p_c-open set and $x_e \widetilde{\in} (F, E)$. Then, $\widetilde{X} \setminus (F, E)$ is a soft p_c-closed set such that $x_e \widetilde{\notin} \widetilde{X} \setminus (F, E)$. By $\tilde{s}p_c^*$-regularity of X, there are soft p_c-open sets (G, E), (H, E) such that $x_e \widetilde{\in} (G, E)$, $\widetilde{X} \setminus (F, E) \sqsubseteq (H, E)$ and $(H, E) \sqcap (G, E) = \widetilde{\phi}$. Therefore, $x_e \widetilde{\in} (G, E) \sqsubseteq \widetilde{X} \setminus (H, E) \sqsubseteq (F, E)$, Hence, $x_e \widetilde{\in} (G, E) \sqsubseteq \tilde{s}p_c cl(G, E) \sqsubseteq \tilde{s}p_c cl(\widetilde{X} \setminus (H, E)) = \widetilde{X} \setminus (H, E) \sqsubseteq (F, E)$. This gives $\tilde{s}p_c cl(G, E) \sqsubseteq \widetilde{X} \setminus (H, E) \sqsubseteq (F, E)$. Consequently, $x_e \widetilde{\in} (G, E)$ and $\tilde{s}p_c cl(G, E) \sqsubseteq (F, E)$.

(2) → (3). Let $y_{e'} \widetilde{\in} \widetilde{X}$. Then, for every soft p_c-open set (G, E) such that $y_{e'} \widetilde{\in} (G, E)$, $\tilde{s}p_c cl(G, E) \sqsubseteq (F, E)$. Thus, for each $y_{e'} \widetilde{\in} \widetilde{X}$, the sets $\tilde{s}p_c cl(G, E)$ form an $\tilde{s}p_c$- neighborhood base consisting of soft p_c-closed sets of $\widetilde{X}$. This proves (3).

(3) → (1). Let (K, E) be soft p_c-closed set which does not contain x_e. Then, $\widetilde{X} \setminus (K, E)$ is soft p_c-open, so it is $\tilde{s}p_c$- neighborhood of x_e. By (3), there is

soft p_c-closed set (L,E) which contains x_e and it is an $\tilde{s}p_c$- neighborhood of x_e with $(L,E) \sqsubseteq \widetilde{X} \setminus (K,E)$. Consider the sets (L,E) and $\widetilde{X} \setminus (L,E)$. Then, $x_e \widetilde{\in} (L,E)$, $(K,E) \sqsubseteq \widetilde{X} \setminus (L,E) = (G,E)$ and $(K,E) \sqcap (L,E) = \widetilde{\phi}$. Therefore, $\widetilde{X}$ is $\tilde{s}p_c^*$-regular.

(2) $\to$ (4). Let (K,E) be soft p_c-closed and $x_e \widetilde{\notin} (K,E)$. Then, $x_e \widetilde{\in} \widetilde{X} \setminus (K,E)$ and $\widetilde{X} \setminus (K,E)$ is $\tilde{s}p_c$- open subset of $\widetilde{X}$. Using the hypothesis, there exists an soft p_c-open set (F,E) such that $x_e \widetilde{\in} (F,E) \sqsubseteq \tilde{s}p_c cl(F,E) \sqsubseteq \widetilde{X} \setminus (K,E)$. Hence, $(K,E) \sqsubseteq \widetilde{X} \setminus \tilde{s}p_c cl(F,E) \sqsubseteq \widetilde{X} \setminus (F,E)$. Consequently $\widetilde{X} \setminus (F,E)$ is soft p_c-closed neighborhood of (K,E) to which x_e does not belong. This proves (4).

(4) $\to$ (5). Let $\phi \neq (F,E) \sqsubseteq \tilde{X}$ and (G,E) be a soft p_c-open subset of $\widetilde{X}$ such that $(F,E) \sqcap (G,E) \neq \widetilde{\phi}$. Let $x_e \widetilde{\in} (F,\ E) \sqcap (G,\ E)$. Since $x_e \widetilde{\notin} \widetilde{X} \setminus (G,E)$ and $\widetilde{X} \setminus (G,E)$ is soft p_c-closed, so there exists an soft p_c-closed neighborhood of $\widetilde{X} \setminus (G,E)$ say (E,E), such that $x_e \widetilde{\notin} (E,E)$. Let $\widetilde{X} \setminus (G,E) \sqsubseteq (D,E) \sqsubseteq (E,E)$ where (D,E) is soft p_c-open set. Then, $(W,E) = \widetilde{X} \setminus (E,E)$ is soft p_c-open set, $x_e \widetilde{\in} (W,\ E)$ and $(F,E) \sqcap (W,E) \neq \widetilde{\phi}$. Also, $\widetilde{X} \setminus (D,E)$ being soft p_c-closed. $\tilde{s}p_c cl(W,E) = \tilde{s}p_c cl(\widetilde{X} \setminus (E,E)) \sqsubseteq \widetilde{X} \setminus (D,E) \sqsubseteq (G,E)$.

(5) $\to$ (6). Let $\phi \neq (F,E) \sqsubseteq \tilde{X}$ and (K,E) be soft p_c-closed subset of $\widetilde{X}$ such that $(K,E) \sqcap (F,E) = \widetilde{\phi}$, then $\widetilde{X} \setminus (K,E) \sqcap (F,E) \neq \widetilde{\phi}$, and $\widetilde{X} \setminus (K,E)$ is soft p_c-open. Using (5), there exists an soft p_c-open subset $G,E)$ of $\widetilde{X}$ such that $(G,E) \sqcap (F,E) \neq \widetilde{\phi}$ and $(G,E) \sqsubseteq \tilde{s}p_c cl(G,E) \sqsubseteq \widetilde{X} \setminus (K,E)$. Putting $(W,E) = \widetilde{X} \setminus \tilde{s}p_c cl(G,E)$, then $(K,E) \sqsubseteq (W,E) \sqsubseteq \widetilde{X} \setminus (G,E)$, and (W,E) is soft p_c-open. Hence the proof.

(6) $\to$ (1). Let $x_e \widetilde{\notin} (K,E)$, where (K,E) is soft p_c-closed, and let $(F,E) = \{x_e\} \neq \phi$, Then, $(K,E) \sqcap (F,E) = \widetilde{\phi}$ and hence, using (6) there exist two soft p_c-open sets (G,E), and (W,E) such that $(W,E) \sqcap (G,E) = \widetilde{\phi}$, $(G,E) \sqcap (F,E) \neq \widetilde{\phi}$ and $(K,E) \sqsubseteq (W,E)$, which implies that $\widetilde{X}$ is $\tilde{s}p_c^*$-regular. □

Theorem 3.2. *A topological space $(X,\widetilde{\tau},E)$ is $\tilde{s}p_c^*$-regular (resp., $\tilde{s}p_c$-regular) if and only if for each $x_e \widetilde{\in} \widetilde{X}$ and soft p_c-closed (resp., soft closed) set (K,E) such that $x_e \widetilde{\notin} (K,E)$, there exist soft p_c-open sets (G,E), (H,E) such that $x_e \widetilde{\in} (G,E)$, $(K,E) \sqsubseteq (H,E)$ and $\tilde{s}p_c cl(G,E) \sqcap \tilde{s}p_c cl(H,E) = \widetilde{\phi}$.*

Proof. We only prove the $\tilde{s}p_c^*$- regular case because the other case can be proved similarly.

Suppose that $\widetilde{X}$ is $\tilde{s}p_c^*$-regular, then for each $x_e \widetilde{\in} \widetilde{X}$ and soft p_c-closed set (K,E) such that $x_e \widetilde{\notin} (K,E)$, there exist two soft p_c-open sets (U,E) and (V,E) such that $x_e \widetilde{\in} (U,E)$, $(K,E) \sqsubseteq (V,E)$ and $(U,E) \sqcap (V,E) = \phi$. Which implies that $x_e \widetilde{\in} (U,E) \sqsubseteq \widetilde{X} \setminus (V,E) \sqsubseteq \widetilde{X} \setminus (K,E)$. That is $x_e \widetilde{\in} (U,E) \sqsubseteq \tilde{s}p_c cl(U,E) \sqsubseteq \widetilde{X} \setminus (V,E) \sqsubseteq \widetilde{X} \setminus (K,E)$. Using Theorem 3.1(2) and the fact that $x_e \widetilde{\in} (U,E)$, where (U,E) is soft p_c-open set there exist soft p_c-open (G,E) containing x_e such that $x_e \widetilde{\in} (G,E) \sqsubseteq \tilde{s}p_c cl(G,E) \sqsubseteq (U,E))$. Therefore, $(K,E) \sqsubseteq (V,E) \sqsubseteq \widetilde{X} \setminus \tilde{s}p_c cl(U,E) \sqsubseteq \widetilde{X} \setminus (U,E) \sqsubseteq \widetilde{X} \setminus \tilde{s}p_c cl(G,E)$ and $(K,E) \sqsubseteq (V,E) \sqsubseteq \tilde{s}p_c cl(V,E) \sqsubseteq \widetilde{X} \setminus (U,E)$, Now take $(H,E) = (V,E)$, we get $x_e \widetilde{\in} (G,E)$, $(K,E) \sqsubseteq (H,E)$ and

$\tilde{s}p_c cl(G,E) \sqcap \tilde{s}p_c cl(H,E) = \phi$. This proves the necessity part. The proof of sufficiency follows directly. □

Lemma 3.1. *Every soft clopen subspace of an $\tilde{s}p_c$-regular space $\widetilde{X}$ is $\tilde{s}p_c$-regular.*

Proof. Let $\widetilde{Y}$ be a soft clopen subspace of $\tilde{s}p_c$-regular space $\widetilde{X}$. Suppose that (H,E) is soft p_c-closed set in $\widetilde{Y}$ and $y_{e'}\widetilde{\in}\widetilde{Y}$ such that $y_{e'}\widetilde{\notin}(H,E)$. Then, $(H,E) = (G,E) \sqcap Y$, where (G,E) is soft p_c-closed in $\widetilde{X}$. Then, $y_{e'}\widetilde{\notin}(G,E)$. Since $\widetilde{X}$ is $\tilde{s}p_c$-regular, there exist disjoint soft p_c-open sets (U,E), (V,E) in $\widetilde{X}$ such that $y_{e'}\widetilde{\in}(U,E)$, $(H,E) \sqsubseteq (V,E)$. Then, $(U,E) \sqcap Y$ and $(V,E) \sqcap Y$ are disjoint soft p_c-open sets in $\widetilde{Y}$ containing $y_{e'}$ and (H,E), respectively. This completes the proof. □

Remark 3.2. If the soft space $\widetilde{X}$ is finite, then by Remark 3.1, every soft p_c-open set is both closed and open and hence we obtain that Lemma 3.1 is true foe every subspace. Lemma 3.1 is true because the intersection of an soft p_c-open set in $\widetilde{X}$ with a soft clopen subspace remains an soft p_c-open set in the subspace but still we ask the following question.

Every soft subspace of an $\tilde{s}p_c$-regular space $\widetilde{X}$ is $\tilde{s}p_c$-regular or not ?.

Theorem 3.3. *Every $\tilde{s}p_c$-regular and $\tilde{s}p_c - T_0$ space $\widetilde{X}$ is an $\tilde{s}p_c{-}T_2$ space.*

Proof. Let x_e, $y_{e'}\widetilde{\in}\widetilde{X}$ such that $x_e \neq y_{e'}$. Since $\widetilde{X}$ is $\tilde{s}p_c - T_0$, then there exists an soft p_c-open set (U,E) containing x_e but not $y_{e'}$. Using the hypothesis that $\widetilde{X}$ is $\tilde{s}p_c$-regular and since $x_e\widetilde{\in}(U,E)$, so there is an soft p_c-open set (V,E), such that $x_e\widetilde{\in}(V,E) \sqsubseteq \tilde{s}p_c cl(V,E) \sqsubseteq (U,E)$. But $y_{e'}\widetilde{\notin}(U,E)$ implies that $y_{e'}\widetilde{\notin}\tilde{s}p_c cl(V,E)$, then we get $y_{e'}\widetilde{\in}\widetilde{X} \setminus \tilde{s}p_c cl(V,E)$. Therefore, we have (U,E) and $\widetilde{X} \setminus \tilde{s}p_c cl(V,E)$ are soft p_c-open sets such that $x_e\widetilde{\in}(U,E)$, $y_{e'}\widetilde{\in}\widetilde{X} \setminus \tilde{s}p_c cl(V,E)$ and $\widetilde{X} \setminus \tilde{s}p_c cl(V,E) \sqcap (U,E) = \widetilde{\phi}$. Hence, the result follows. □

The proof of the following lemma is obvious.

Lemma 3.2. *Let $(X,\widetilde{\tau},E)$ be an $\tilde{s}p_c$-regular (resp., an $\tilde{s}p_c^*$-regular) space and let (H,E) be a soft closed (resp., soft p_c-closed) set such that $x_e\widetilde{\notin}(H,E)$, then there exists an soft p_c-open set (F,E) such that $x_e\widetilde{\in}(F,E)$ and $(F,E) \sqcap (H,E) = \widetilde{\phi}$.*

Proposition 3.3. *A soft topological space is $\tilde{s}p_c$-regular (resp., an $\tilde{s}p_c^*$-regular) if and only if for each soft point $x_e\widetilde{\in}SP(X)_E$ and for each soft open (resp., soft p_c-open) set (F,E) containing x_e, there exists an soft p_c-open set (U,E) of x_e such that $\tilde{s}p_c cl(U,E)\sqsubseteq(F,E)$.*

Proof. Let $(X,\widetilde{\tau},E)$ be $\tilde{s}p_c$-regular space. Let $x_e\widetilde{\in}\widetilde{X}$ and (F,E) is an soft p_c-open set containing x_e. Then, $X \setminus (F,E)$ is an soft p_c-closed set such that $x_e\widetilde{\notin}\widetilde{X} \setminus (F,E)$. Since $(X,\widetilde{\tau},E)$ is an $\tilde{s}p_c$-regular, so there exist soft p_c-open sets (V,E) and (U,E) such that $x_e\widetilde{\in}(U,E)$, $X\setminus(F,E)\sqsubseteq(V,E)$ and $(U,E)\sqcap(V,E) = \widetilde{\phi}$. Thus, $(U,E)\sqsubseteq X \setminus (V,E)$ and hence $\tilde{s}p_c cl(U,E)\sqsubseteq X \setminus (V,E)\sqsubseteq(F,E)$.

Conversely, let $x_e\widetilde{\in}\widetilde{X}$ and (H,E) be an soft p_c-closed set such that $x_e\widetilde{\notin}(H,E)$. Then, $X\setminus(H,E)$ is an soft p_c-open set containing x_e. So, by hypothesis there exist an soft p_c-open set (U,E) of x_e such that $\tilde{s}p_c cl(U,E)\sqsubseteq X\setminus(H,E)$. Thus, $(H,E)\sqsubseteq X\setminus\tilde{s}p_c cl(U,E)$ and $(U,E)\sqcap X\setminus\tilde{s}p_c cl(U,E)=\widetilde{\phi}$. Therefore, $(X,\widetilde{\tau},E)$ is $\tilde{s}p_c$-regular.

The proof when $(X,\widetilde{\tau},E)$ is $\tilde{s}p_c$-regular is analogues. □

Definition 3.2. *A soft topological space $(X,\widetilde{\tau},E)$ is said to be strongly $\tilde{s}p_c^*$-regular (resp., strongly $\tilde{s}p_c$-regular), if for every soft p_c-closed (resp., soft closed) set (H,E) and every point $x\widetilde{\notin}(H,E)$, there exists disjoint soft p_c-open sets (F,E) and (G,E) such that $x\widetilde{\in}(F,E)$ and $(H,E)\sqsubseteq(G,E)$.*

Example 3.3. Let $X=\{x,y\}$, $E=\{e_1,e_2\}$ and $\widetilde{X}=\{(e_1,X),(e_2,X)\}$ and let $\widetilde{\tau}=\{\widetilde{X},\widetilde{\phi},(F_1,E),(F_2,E)\}$, where $(F_1,E)=\{(e_1,\{x\}),(e_2,\{x\})\}$, $(F_2,E)=\{(e_1,\{y\}),(e_2,\{y\})\}$. Then, it is not difficult to check that $(X,\widetilde{\tau},E)$ is both strongly $\tilde{s}p_c^*$-regular and strongly $\tilde{s}p_c$-regular.

The following result is obvious.

Proposition 3.4. *Every strongly $\tilde{s}p_c^*$-regular (resp., strongly $\tilde{s}p_c$-regular) space is $\tilde{s}p_c^*$-regular (resp., $\tilde{s}p_c$-regular).*

The converse of Proposition 3.4 is not true in general. The space in Example 3.1, is $\tilde{s}p_c^*$-regular and $\tilde{s}p_c$-regular but it is neither strongly $\tilde{s}p_c^*$-regular nor strongly $\tilde{s}p_c$-regular.

We shall prove all the results related to strongly $\tilde{s}p_c^*$-regular spaces and the proof of the results related to strongly $\tilde{s}p_c$-regular can be done in a similar way.

Lemma 3.3. *If $(X,\widetilde{\tau},E)$ is strongly $\tilde{s}p_c^*$-regular (resp., strongly $\tilde{s}p_c$-regular) space and (H,E) is an soft p_c-closed (resp., soft closed) set such that $x\widetilde{\notin}(H,E)$, then there exists an soft p_c-open set (F,E) such that $x\widetilde{\in}(F,E)$ and $(F,E)\sqcap(H,E)=\widetilde{\phi}$.*

Proposition 3.5. *A soft topological space $(X,\widetilde{\tau},E)$ is strongly $\tilde{s}p_c^*$-regular (resp., strongly $\tilde{s}p_c$-regular) if and only if for each point $x\in X$ and for each soft p_c-open (resp., soft open) set (F,E) containing x, there exists an soft p_c-open set (U,E) containing x such that $\tilde{s}p_c cl(U,E)\sqsubseteq(F,E)$.*

Proof. Let $(X,\widetilde{\tau},E)$ be a strongly $\tilde{s}p_c^*$-regular space. Let $x\in X$ and (F,E) be an soft p_c-open set containing x. Then, $X\setminus(F,E)$ is an soft p_c-closed set such that $x\widetilde{\notin}\widetilde{X}\setminus(F,E)$. Since $(X,\widetilde{\tau},E)$ is $\tilde{s}p_c^*$-regular, then there exist soft p_c-open sets (V,E) and (U,E) such that $x\widetilde{\in}(U,E)$, $X\setminus(F,E)\sqsubseteq(V,E)$ and $(U,E)\sqcap(V,E)=\widetilde{\phi}$. Thus, $(U,E)\sqsubseteq X\setminus(V,E)$ and hence $\tilde{s}p_c cl(U,E)\sqsubseteq X\setminus(V,E)\sqsubseteq(F,E)$.

Conversely, let $x\in X$ and (H,E) be an soft p_c-closed set such that $x\widetilde{\notin}(H,E)$. Then, $X\setminus(H,E)$ is an soft p_c-open set containing x. So, by hypothesis there exists an soft p_c-open set (U,E) containing x such that $\tilde{s}p_c cl(U,E)\sqsubseteq X\setminus(H,E)$. Thus, $(H,E)\sqsubseteq X\setminus\tilde{s}p_c cl(U,E)$ and $(U,E)\sqcap X\setminus\tilde{s}p_c cl(U,E)=\widetilde{\phi}$. Therefore, $(X,\widetilde{\tau},E)$ is strongly $\tilde{s}p_c^*$-regular. □

Proposition 3.6. *Let $(X, \widetilde{\tau}, E)$ be a soft topological space and $x \in X$. If $(X, \widetilde{\tau}, E)$ is a strongly $\tilde{s}p_c^*$-regular (resp., strongly $\tilde{s}p_c$-regular) space, then the following statements are true:*

1. *$x \widetilde{\notin} (H, E)$ if and only if $(x, E) \sqcap (H, E) = \widetilde{\phi}$ for every soft p_c-closed (resp., soft closed) set (H, E).*

2. *$x \widetilde{\notin} (F, E)$ if and only if $(x, E) \sqcap (F, E) = \widetilde{\phi}$ for every soft p_c-open (resp., soft open) set (F, E).*

Proof. (1) Let $x \widetilde{\notin} (H, E)$, then by Lemma 3.3, there exists an $\tilde{s}p_c$- open set (F, E) such that $x \widetilde{\in} (F, E)$ and $(F, E) \sqcap (H, E) = \widetilde{\phi}$. Since $(x, E) \sqsubseteq (F, E)$, we have $(x, E) \sqcap (H, E) = \widetilde{\phi}$.

Conversely, straightforward.

(2) Let $x \widetilde{\notin} (F, E)$. Then, we have two cases:

(i) $x \notin F(\alpha)$, for all $e \in E$, it is obvious that $(x, E) \sqcap (F, E) = \widetilde{\phi}$.

(ii) $x \notin F(\alpha)$ and $x \in F(\beta)$ for some α, $\beta \in E$, then we have $x \in X \setminus F(\alpha)$ and $x \widetilde{\notin} \widetilde{X} \setminus F(\beta)$ for some α, $\beta \in E$ and so $\widetilde{X} \setminus (F, E)$ is an soft p_c-closed set such that $x \widetilde{\notin} \widetilde{X} \setminus (F, E)$, by (1), $(x, E) \sqcap \widetilde{X} \setminus (F, E) = \widetilde{\phi}$. So, $(x, E) \sqsubseteq (F, E)$ but this contradicts that $x \widetilde{\notin} F(\alpha)$ for some $\alpha \in E$. Consequently, we have $(x, E) \sqcap (F, E) = \widetilde{\phi}$.

The converse part is obvious. □

Proposition 3.7. *Let $(X, \widetilde{\tau}, E)$ be a soft topological space and $x \in X$. Then, the following statements are equivalent:*

1. *$(X, \widetilde{\tau}, E)$ is a strongly $\tilde{s}p_c^*$-regular (resp., strongly $\tilde{s}p_c$-regular) space,*

2. *For each soft p_c-closed (resp., soft closed) set (H, E) such that $(x, E) \sqcap (H, E) = \widetilde{\phi}$, there exist soft p_c-open sets (F, E) and (G, E) such that such that $(x, E) \sqsubseteq (F, E)$, $(H, E) \sqsubseteq (G, E)$ and $(F, E) \sqcap (G, E) = \widetilde{\phi}$.*

Proof. Follows from Proposition 3.6(1) and Lemma 3.3. □

Proposition 3.8. *Let $(X, \widetilde{\tau}, E)$ be a soft topological space and $x \in X$. If $(X, \widetilde{\tau}, E)$ is a strongly $\tilde{s}p_c^*$-regular (resp., strongly $\tilde{s}p_c$-regular), then the following statements are true:*

1. *For an soft p_c-open (resp., soft open) set (F, E), $x \widetilde{\in} (F, E)$ if and only if $x \in F(\alpha)$ for some $\alpha \in E$.*

2. *For an soft p_c-open (resp., soft open) set (F, E), $(F, E) = \sqcup \{(x, E) : x \in F(\alpha)$ for some $\alpha \in E\}$.*

Proof. (1). Suppose that $x \in F(\alpha)$ and $x \widetilde{\notin} (F, E)$ for some $\alpha \in E$. Then, by Proposition 3.7(2), $(x, E) \sqcap (F, E) = \widetilde{\phi}$. By our assumption, this is a contradiction and so $x \in (F, E)$. The Converse is obvious.

(2). Follows from part (1) and Remark 2.1. □

Proposition 3.9. *Let $(X, \widetilde{\tau}, E)$ be a soft topological space and $x \in X$. If $(X, \widetilde{\tau}, E)$ is strongly $\tilde{s}p_c^*$-regular, then the following statements are equivalent:*

1. *$(X, \widetilde{\tau}, E)$ is a $\tilde{s}p_c$-T_1^* space,*
2. *For $x, y \in X$ with $x \neq y$, there exist soft p_c-open sets (F, E) and (G, E) such that $(x, E) \sqsubseteq (F, E)$ and $(y, E) \sqcap (F, E) = \widetilde{\phi}$, $(y, E) \sqsubseteq (G, E)$ and $(x, E) \sqcap (G, E) = \widetilde{\phi}$.*

Proof. It is clear that $x \widetilde{\in} (F, E)$ if and only if $(x, E) \sqsubseteq (F, E)$, and by Proposition 3.8(2), $x \widetilde{\notin} (F, E)$ if and only if $(x, E) \sqcap (F, E) = \widetilde{\phi}$. Hence, statements (1) and (2) are equivalent. □

4. Soft p_c-normal spaces

In this section, we define $\tilde{s}p_c$-normal spaces and derive many of its properties. The relationship to other soft spaces and its image under $\tilde{s}p_c$-continuous functions are discussed.

Definition 4.1. *A soft space $\widetilde{X}$ is said to be $\tilde{s}p_c$-normal (resp., $\tilde{s}p_c^*$-normal) space, if for any disjoint soft closed (resp., $\tilde{s}p_c^*$-closed) sets (K, E) and (L, E) of $\widetilde{X}$, there exist soft p_c-open sets (U, E), (V, E) such that $(K, E) \sqsubseteq (U, E)$, $(L, E) \sqsubseteq (V, E)$ and $(V, E) \sqcap (U, E) = \widetilde{\phi}$.*

Example 4.1. Let $X = \{x_1, x_2, x_3\}$, $E = \{e_1, e_2\}$ and let $\widetilde{\tau} = \{\widetilde{X}, \widetilde{\phi}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$, where $(F_1, E) = \{(e_1, \{x_1, x_2\}), (e_2, \{x_3\})\}$, $(F_2, E) = \{(e_1, \{x_3\}), (e_2, \{x_1, x_2\})\}$, $(F_3, E) = \{(e_1, \{x_1\}), (e_2, \phi)\}$, $(F_4, E) = \{(e_1, \{x_1, x_3\}), (e_2, \{x_1, x_2\})\}$. Then, this space is both $\tilde{s}p_c$-normal and $\tilde{s}p_c^*$-normal but it is not $\tilde{s}p_c$-regular.

Theorem 4.1. *A space $\widetilde{X}$ is an $\tilde{s}p_c^*$-normal space, if for each pair of soft p_c-open sets (U, E) and (V, E) in $\widetilde{X}$ such that $\widetilde{X} = (U, E) \sqcup (V, E)$, there are soft p_c-closed sets (G, E) and (H, E) which are contained in (U, E) and (V, E), respectively and $\tilde{X} = (G, E) \sqcup (H, E)$.*

Proof. Straightforward. □

Theorem 4.2. *If $\widetilde{X}$ is any soft space, then the following statements are equivalent:*

1. *$\widetilde{X}$ is $\tilde{s}p_c^*$-normal,*
2. *For each $\tilde{s}p_c$- closed set (F_1, E) in $\widetilde{X}$ and soft p_c-open set (G, E) contains (F_1, E), there is an soft p_c-open set (U, E) such that $(F_1, E) \sqsubseteq (U, E) \sqsubseteq \tilde{s}p_c cl(U, E) \sqsubseteq (G, E)$,*
3. *For each $\tilde{s}p_c$- closed set (F_1, E) in $\widetilde{X}$ and soft p_c-open set (G, E) containing (F_1, E), there are soft p_c-open sets (U_n, E) for $n \in N$, such that $(F_1, E) \sqsubseteq \bigsqcup_{n \in N} (U_n, E)$ and $\tilde{s}p_c cl(U_n, E) \sqsubseteq (G, E)$, for each $n \in N$.*

Proof. (1) $\to$ (2). Since (G,E) is soft p_c-open set containing (F_1,E), then $\widetilde{X}\setminus(G,E)$ and (F_1,E) are disjoint soft p_c-closed sets in $\widetilde{X}$. Since $\widetilde{X}$ is $\tilde{s}p_c^*$-normal, so there exist soft p_c-open sets (U,E) and (V,E) such that $(F_1,E)\sqsubseteq(U,E)$, $\widetilde{X}\setminus(G,E)\sqsubseteq(V,E)$ and $(V,E)\sqcap(U,E)=\widetilde{\phi}$. Hence, $(F_1,E)\sqsubseteq(U,E)\sqsubseteq \tilde{s}p_c cl(U,E)\sqsubseteq \tilde{s}p_c cl(\widetilde{X}\setminus(V,E))=\widetilde{X}\setminus(V,E)\sqsubseteq(G,E)$, or $(F_1,E)\sqsubseteq(U,E)\sqsubseteq \tilde{s}p_c cl(U,E)\sqsubseteq(G,E)$.

(2) $\to$ (3). Let (F_1,E) be an soft p_c-closed set and (G,E) be an soft p_c-open set in an $\tilde{s}p_c^*$-normal space $\widetilde{X}$ such that $(F_1,E)\sqsubseteq(G,E)$. So, by hypothesis, there is an soft p_c-open set (U,E) such that $(F_1,E)\sqsubseteq(U,E)\sqsubseteq\tilde{s}p_c cl(U,E)\sqsubseteq(G,E)$. If we put $(U_n,E)=(U,E)$, for all $n\in N$, the proof follows.

(3) $\to$ (1). Let (F_1,E) and (F_2,E) be a pair of disjoint soft p_c-closed set in the space $\widetilde{X}$, then $\widetilde{X}\setminus(F_2,E)$ is an soft p_c-open set in $\widetilde{X}$ containing (F_1,E). So, by hypothesis, there are soft p_c-open sets (U_n,E) for $n\in N$ such that

$$(F_1,E)\sqsubseteq\bigsqcup_{n\in N}(U_n,E)$$

and $\tilde{s}p_c cl(U_n,E)\sqsubseteq\widetilde{X}\setminus(F_2,E)$ for each $n\in N$. Since $\widetilde{X}\setminus(F_1,E)$ is an soft p_c-open subset of $\widetilde{X}$ containing the soft p_c-closed set (F_2,E), then by applying the condition of the theorem again, we get soft p_c-open sets (V_n,E) for $n\in N$, such that

$$(F_2,E)\sqsubseteq\bigsqcup_{n\in N}(V_n,E)$$

and $\tilde{s}p_c cl(V_n,E)\sqsubseteq\widetilde{X}\setminus(F_1,E)$ for each $n\in N$. Thus, $\tilde{s}p_c cl(U_n,E)\sqcap(F_2,E)=\widetilde{\phi}$ and $\tilde{s}p_c cl(V_n,E)\sqcap(F_1,E)=\widetilde{\phi}$ for each $n\in N$. Setting

$$(G_n,E)=(U_n,E)\setminus\bigsqcup_{n\in N}\tilde{s}p_c cl(V_n,E)$$

and

$$(H_n,E)=(V_n,E)\setminus\bigsqcup_{n\in N}\tilde{s}p_c cl(U_n,E).$$

Then

$$(U,E)=\bigsqcup_{n\in N}(G_n,E)$$

and

$$(V,E)=\bigsqcup_{n\in N}(H_n,E)$$

are disjoint soft p_c-open sets in $\widetilde{X}$ containing (F_1,E) and (F_2,E), respectively. Hence, $\widetilde{X}$ is $\tilde{s}p_c^*$-normal. □

Theorem 4.3. *A soft topological space $\widetilde{X}$ is $\tilde{s}p_c$-normal if and only if for each soft closed set (F_1,E) in $\widetilde{X}$ and soft open set (G,E) contains (F_1,E), there is an soft p_c-open set (U,E) such that $(F_1,E)\sqsubseteq(U,E)\sqsubseteq\tilde{s}p_c cl(U,E)\sqsubseteq(G,E)$.*

Proof. Let (F_1, E) be any soft close subset in an $\tilde{s}p_c$-normal space $\widetilde{X}$ and (G, E) be any soft open subset of $\widetilde{X}$ containing (F_1, E). Then, $\widetilde{X} \setminus (G, E)$ is closed and $\widetilde{X} \setminus (G, E) \sqcap (F_1, E) = \widetilde{\phi}$. Hence, by hypothesis, there exist two disjoint soft p_c-open sets (U, E) and (V, E) such that $(F_1, E) \sqsubseteq (U, E)$, $\widetilde{X} \setminus (G, E) \sqsubseteq (V, E)$ and $(V, E) \sqcap (U, E) = \widetilde{\phi}$. Since $(V, E) \sqcap (U, E) = \widetilde{\phi}$, then $(U, E) \sqsubseteq \widetilde{X} \setminus (V, E)$. But $\widetilde{X} \setminus (G, E) \sqsubseteq (V, E)$, then $\widetilde{X} \setminus (V, E) \sqsubseteq (G, E)$ and so $(U, E) \sqsubseteq (G, E)$. And since (U, E) and (V, E) are soft p_c-open sets, then $\widetilde{X} \setminus (V, E)$ and $\widetilde{X} \setminus (U, E)$ are soft p_c-closed sets and so $\tilde{s}p_c cl(\widetilde{X} \setminus (V, E)) = \widetilde{X} \setminus (V, E)$ and $\tilde{s}p_c cl(\widetilde{X} \setminus (U, E)) = \widetilde{X} \setminus (U, E)$ and then $(F_1, E) \sqsubseteq (U, E) \sqsubseteq \tilde{s}p_c cl(U, E) \sqsubseteq \tilde{s}p_c cl(\widetilde{X} \setminus (V, E)) = \widetilde{X} \setminus (V, E) \sqsubseteq (G, E)$. Thus, $(F_1, E) \sqsubseteq (U, E) \sqsubseteq \tilde{s}p_c cl(U, E) \sqsubseteq (G, E)$.

Conversely, let the condition be satisfied and let (F_1, E), (F_2, E) be two disjoint soft closed subsets of $\widetilde{X}$. Then, $(F_1, E) \sqsubseteq \widetilde{X} \setminus (F_2, E)$ and since (F_2, E) is soft closed then $\widetilde{X} \setminus (F_2, E)$ is a soft open subset containing (F_1, E). So, by hypothesis, there exist soft p_c-open sets (U, E) such that $(F_1, E) \sqsubseteq (U, E) \sqsubseteq \tilde{s}p_c cl(U, E) \sqsubseteq \widetilde{X} \setminus (F_2, E)$. Putting $(V, E) = \widetilde{X} \setminus \tilde{s}p_c cl(U, E)$, then there exist two disjoint soft p_c-open sets (U, E) and (V, E) such that $(F_1, E) \sqsubseteq (U, E)$ and $(F_2, E) \sqsubseteq (V, E)$. Therefore, $\widetilde{X}$ is $\tilde{s}p_c$-normal. □

Theorem 4.4. *Every soft T_1, $\tilde{s}p_c$-normal space $\widetilde{X}$ is $\tilde{s}p_c$-regular.*

Proof. Let (F_1, E) be any soft closed subset in an $\tilde{s}p_c$-normal space $\widetilde{X}$ and $x_e \widetilde{\in} \widetilde{X}$ such that $x_e \widetilde{\notin} (F_1, E)$. Since $\widetilde{X}$ is soft T_1 space, then $\{x_e\}$ is soft closed subset in $\widetilde{X}$ with $\{x_e\} \sqcap (F_1, E) = \widetilde{\phi}$. By $\tilde{s}p_c$-normality of $\widetilde{X}$, there exist two disjoint soft p_c-open sets (U, E) and (V, E) of $\widetilde{X}$ such that $\{x_e\} \sqsubseteq (U, E)$, so $x_e \widetilde{\in} (U, E)$, $(F_1, E) \sqsubseteq (V, E)$ and $(U, E) \sqcap (V, E) = \widetilde{\phi}$. Thus, $\widetilde{X}$ is an $\tilde{s}p_c$-regular space. □

Theorem 4.5. *If $\widetilde{Y}$ is a soft clopen subspace of an $\tilde{s}p_c$-normal (resp., $\tilde{s}p_c^*$-normal) space $\widetilde{X}$, then $\widetilde{Y}$ is $\tilde{s}p_c$-normal (resp., $\tilde{s}p_c^*$-normal).*

Proof. Let $\widetilde{X}$ be an $\tilde{s}p_c^*$-normal space and $\widetilde{Y}$ be a soft clopen subspace of $\widetilde{X}$. Let (K_1, E) and (K_2, E) be two disjoint soft p_c-closed subsets of $\widetilde{Y}$, then By Lemma 2.7, (K_1, E) and (K_2, E) are two disjoint soft p_c-closed subsets of $\widetilde{X}$. By $\tilde{s}p_c^*$ -normality of $\widetilde{X}$, there exist two soft p_c-open sets (F_1, E) and (F_2, E) such that $(K_1, E) \sqsubseteq (F_1, E)$, $(K_2, E) \sqsubseteq (F_2, E)$ and $(F_1, E) \sqcap (F_2, E) = \phi$, then $(K_1, E) \sqsubseteq (F_1, E) \sqcap \widetilde{Y}$ and $(K_2, E) \sqsubseteq (F_2, E) \sqcap \widetilde{Y}$. It follows from, $(F_1, E) \sqcap (F_2, E) = \widetilde{\phi}$, that $((F_1, E) \sqcap \widetilde{Y}) \sqcap ((F_2, E) \sqcap \widetilde{Y}) = \widetilde{\phi}$ and By Lemma 2.5, we have $((F_1, E) \sqcap \widetilde{Y})$ and $((F_2, E) \sqcap \widetilde{Y})$ are soft p_c-open subsets of $\widetilde{Y}$. Hence, $\widetilde{Y}$ is $\tilde{s}p_c^*$ -normal. □

The following example shows that Theorem 4.5, is not true when $\widetilde{Y}$ is soft open or soft closed.

Example 4.2. Let $X = \{x, y, z\}$, $E = \{e_1, e_2\}$ and $\widetilde{X} = \{(e_1, X), (e_2, X)\}$, let $\widetilde{\tau} = \{\widetilde{X}, \widetilde{\phi}, (F_1, E), (F_2, E), (F_3, E), (F_4, E)\}$, where $(F_1, E) = \{(e_1, \{x\}), (e_2, X)\}$, $(F_2, E) = \{(e_1, \{y\}), (e_2, \{y\})\}$, $(F_3, E) = \{(e_1, \phi), (e_2, \{y\})\}$, $(F_4, E) = \{(e_1, \{x, y\}), (e_2, X)\}$. Then, $(X, \widetilde{\tau}, E)$ is both $\tilde{s}p_c^*$-normal and $\tilde{s}p_c$-normal space and the soft open set (F_4, E) is not $\tilde{s}p_c$-normal. Also, $(X, \widetilde{\tau}^c, E)$ is both $\tilde{s}p_c^*$-normal and $\tilde{s}p_c$-normal space and the soft closed set (F_4, E) is not $\tilde{s}p_c$-normal.

Theorem 4.6. *Every $\tilde{s}p_c^*$-normal $\tilde{s}p_c - T_2$ space $\widetilde{X}$ is $\tilde{s}p_c^*$-regular.*

Proof. Suppose that (F_1, E) is an soft p_c-closed set and $x_e \widetilde{\notin} (F_1, E)$ for each $x_e \widetilde{\in} \widetilde{X}$. Since $\widetilde{X}$ is an $\tilde{s}p_c - T_2$ space. Therefore, by Theorem 2.4, each $\{x_e\}$ is soft p_c-closed in $\widetilde{X}$. Since $\widetilde{X}$ is $\tilde{s}p_c^*$-normal, so there exist soft p_c-open sets (U, E), (V, E) such that $\{x_e\} \sqsubseteq (U, E)$, $(F_1, E) \sqsubseteq (V, E)$ and $(U, E) \sqcap (V, E) = \widetilde{\phi}$, this implies that $\widetilde{X}$ is $\tilde{s}p_c^*$-regular. □

Definition 4.2. *A soft mapping $f_{pu} : (X, \widetilde{\tau}, E) \to (Y, \widetilde{\mu}, B)$ is called an soft p_c-open mapping if and only if the image of every soft p_c-open set in $\widetilde{X}$ is an soft p_c-open in $\widetilde{Y}$.*

Proposition 4.1. *Let $(X, \widetilde{\tau}, E)$ and $(Y, \widetilde{\mu}, B)$ be soft topological spaces and $f_{pu} : SP(X)_E \to SP(Y)_B$ be a soft bijective and soft p_c-open mapping. If $(X, \widetilde{\tau}, E)$ is $\tilde{s}p_c$-T_i, then $(Y, \widetilde{\mu}, B)$ is $\tilde{s}p_c$-T_i spaces $(i = 0, 1, 2)$.*

Proof. We prove only the case for $\tilde{s}p_c$-T_0 space and the proof of the other are similar. Let $y_{\beta 1}$, $y_{\beta 2} \widetilde{\in} SP(Y)_B$ be two distinct soft points. Since f_{pu} is bijective, there exist distinct soft points x_{e1}, $x_{e2} \widetilde{\in} \widetilde{X}$ such that $f_{pu}(x_{e1}) = y_{\beta 1}$, $f_{pu}(x_{e2}) = y_{\beta 2}$. Since $(X, \widetilde{\tau}, E)$ is an $\tilde{s}p_c$-T_0 space, there exist soft p_c-open sets (F, E), (G, E) such that $x_{e1} \widetilde{\in} (F, E)$ and $x_{e2} \widetilde{\notin} (F, E)$ or $x_{e2} \widetilde{\in} (G, E)$ and $x_{e1} \widetilde{\notin} (G, E)$. As f_{pu} is an soft p_c-open mapping, then $f_{pu}(F, E)$, $f_{pu}(G, E)$ are soft p_c-open sets such that $y_{\beta 1} \widetilde{\in} f_{pu}(F, E)$ and $y_{\beta 2} \widetilde{\notin} f_{pu}(F, E)$ or $y_{\beta 2} \widetilde{\in} f_{pu}(G, E)$ and $y_{\beta 1} \widetilde{\notin} f_{pu}(G, E)$. This implies that, $(Y, \widetilde{\mu}, B)$ is $\tilde{s}p_c$-T_0. □

Definition 4.3. *A function $f_{pu} : (X, \widetilde{\tau}, E) \to (Y, \widetilde{\mu}, B)$ is injective soft point $\tilde{s}p_c$-closure if and only if for every $x_e, y_{e'} \widetilde{\in} \widetilde{X}$ such that $\tilde{s}p_c cl(\{x_e\}) \neq \tilde{s}p_c cl(\{y_{e'}\})$, then $\tilde{s}p_c cl(\{f(x_e)\}) \neq \tilde{s}p_c cl(\{f(y_{e'})\})$.*

It is clear that the identity function from any soft topological space onto itself is a function which satisfies Definition 4.3.

Theorem 4.7. *If a function $f_{pu} : (X, \widetilde{\tau}, E) \to (Y, \widetilde{\mu}, B)$ is injective soft point $\tilde{s}p_c$-closure and $\widetilde{X}$ is an $\tilde{s}p_c$-T_0 space, then f_{pu} is soft injective.*

Proof. Let $x_e, y_{e'} \widetilde{\in} \widetilde{X}$ with $x_e \neq y_{e'}$. Since $\widetilde{X}$ is $\tilde{s}p_c$-T_0, therefore by Proposition 2.3, $\tilde{s}p_c cl(\{x_e\}) \neq \tilde{s}p_c cl(\{y_{e'}\})$. But f_{pu} is $(1-1)$ soft point $\tilde{s}p_c$-closure, implies that $\tilde{s}p_c cl(\{f(x_e)\}) \neq \tilde{s}p_c cl(\{f(y_{e'})\})$. Hence, $f_{pu}(x_e)\}) \neq f_{pu}(y_{e'})$. Thus, f_{pu} is soft injective. □

5. Conclusion

Many topological notions are extended to the soft topology after introducing the concept of soft topological spaces. Several classes of soft sets are defined and applied to present many notions in soft topology. In this paper, we employ the notion of soft p_c-open set to introduce some types of soft regular and soft normal spaces and give many properties of these spaces. Also, we discuss relations between these spaces, hereditary properties and their images under soft p_c-continuous mappings.

References

[1] N. K. Ahmed, Q. H. Hamko, *Soft p_c-open sets and $\tilde{s}p_c$-continuity in soft topological spaces,* ZANCO Journal of Pure and Applied Sciences, 30 (2017) 72-84.

[2] M. Akdag, A. Ozkan, *On soft preopen sets and soft pre separation axioms*, Gazi University Journal of Science, 27 (2014), 1077-1083.

[3] T. M. Al-shami, L. D. R. Kocinac, *The equivalence between the enriched and extended soft topologies,* Appl. Comput. Math., 18 (2019), 149-162.

[4] T. M. Al-shami, M. E. El-Shafei, *Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone,* Soft Comput., 2019.

[5] T. M. Al-shami, M. E. El-Shafei, *Two new types of separation axioms on supra soft separation spaces*, Demonstratio Mathematica, 52 (2019), 147-165.

[6] T. M. Al-shami, M. E. El-Shafei, *On supra soft topological ordered spaces*, Arab Journal of Basic and Applied Sciences, 26 (2019), 433-445.

[7] S. Bayramov, C. G. Aras, *A new approach to separability and compactness in soft topological spaces*, (TWMS) Journal of Pure and Applied Mathematics, 9 (2018), 82-93.

[8] M. E. El-Shafei, M. Abo-Elhamayel, T. M. Al-Shami, *Partial soft separation axioms and soft compact spaces*, Filomat, 32 (2018), 4755-4771.

[9] D. N. Georgiou, A. C. Megaritis, V. I. Petropoulos, *On soft topological spaces*, Applied Mathematics and Information Sciences, 7 (2013), 1889-1901.

[10] O. Gocur, A. Kopuzlu, *On soft separation axioms*, Annals of Fuzzy Mathematics and Informatics, 9 (2015), 817-822.

[11] Q. H. Hamko, N. K. Ahmed, *Characterizations of soft p_c-open Sets and $\tilde{s}p_c$-almost continuous mapping in soft topological spaces,* Eurasian Journal of Science and Engineering, 4 (2018), 192-209.

[12] S. Hussain, B Ahmad, *Some properties of soft topological spaces*, Computers and Mathematics with Applications, 62 (2011), 4058-4067.

[13] S. Hussain, B. Ahmad, *Soft separation axioms in soft topological spaces*, Hacettepe Journal of Mathematics and Statistics, 44 (2015), 559-568.

[14] G. Ilango, M. Ravindran, *On soft pre-open sets in soft topological spaces*, International Journal of Mathematics Research, 5 (2013), 399-409.

[15] Alias B. Khalaf, Q. H. Hamko, N. K. Ahmed, *On soft p_c-separation axioms,* Demonstratio Mathematica, 53 (2020), 67-79.

[16] A. Kharal, B. Ahmad, *Mappings on soft classes*, New Mathematics and Natural Computation, 7 (2011), 471-481.

[17] P. K. Maji, R. Biswas, R. Roy, *Soft set theory*, Computers and Mathematics with Applications, 45 (2003), 555-562.

[18] D. Molodtsov, *Soft set theory-first results,* Computers and Mathematics with Applications, 37 (1999), 19-31.

[19] M. Ravindran, G. Ilango, *A note on soft pre-pen sets,* International Journal of Pure and Applied Mathematics, 106 (2016), 63-78.

[20] A. Selvi I. Arockiarani, *Soft Alexandroff spaces in soft ideal topological spaces*, Indian Journal of Applied Research, 6 (2016), 233-243.

[21] M. Shabir, M. Naz, *On soft topological spaces*, Computers and Mathematics with Applications, 61 (2011), 1786-1799.

[22] I. Zorlutuna, H. Cakır, *On continuity of soft mappings*, Applied Mathematics and Information Sciences, 9 (2015), 403-409.

Accepted: April 20, 2022

Schatten class weighted composition operators on weighted Hilbert Bergman spaces of bounded strongly pseudoconvex domain

Cheng-shi Huang
School of Mathematics and Statistics
Sichuan University of Science and Engineering
Zigong, 643000, Sichuan
P. R. China
dzhcsc6@163.com

Zhi-jie Jiang*
School of Mathematics and Statistics
South Sichuan Center for Applied Mathematics
Sichuan University of Science and Engineering
Zigong, 643000, Sichuan
P. R. China
matjzj@126.com

Abstract. Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$, $\delta(z) = d(z, \partial D)$ the Euclidean distance from the point z to the boundary ∂D and $H(D)$ the set of all holomorphic functions on D. For given $\beta \in \mathbb{R}$, the weighted Hilbert Bergman space on D, denoted by $A^2(D, \beta)$, consists of all $f \in H(D)$ such that

$$\|f\|_{2,\beta} = \Big[\int_D |f(z)|^2 \delta(z)^\beta dv(z)\Big]^{\frac{1}{2}} < +\infty,$$

where dv is the Lebesgue measure on D. The aim of the paper is to completely characterize the Schatten class of weighted composition operators on $A^2(D, \beta)$ when $\delta(z)$ satisfies certain integrable condition.
Keywords: weighted composition operator, strongly pseudoconvex domain, weighted Hilbert Bergman space, Schatten class.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and $H(\Omega)$ the set of all holomorphic functions on Ω. Let φ be a holomorphic self-map of Ω and $u \in H(\Omega)$. The well-known weighted composition operator on some subspaces of $H(\Omega)$ is defined by

$$W_{\varphi,u} f(z) = u(z) f(\varphi(z)), \;\; z \in \Omega.$$

When $u(z) \equiv 1$, it is reduced to the composition operator, usually denoted by C_φ. While $\varphi(z) = z$, it is reduced to the multiplication operator, usually

*. Corresponding author

denoted by M_u. Weighted composition operators have been widely studied (see, for example, [4, 5, 8, 9, 10, 15, 16, 17] and the related references therein).

Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, $\delta(z) = d(z, \partial D)$ the Euclidean distance from the point z to the boundary ∂D and dv the Lebesgue measure on D. The authors in [2] introduced the following weighted Bergman space by considering the distance function $\delta(z)$ as a weight on D. For given $\beta \in \mathbb{R}$ and $p \in [1, +\infty)$, the weighted Bergman space $A^p(D, \beta)$ consists of all $f \in H(D)$ such that

$$\|f\|_{p,\beta} = \Big[\int_D |f(z)|^p \delta(z)^\beta dv(z) \Big]^{\frac{1}{p}} < +\infty.$$

With the norm $\|\cdot\|_{p,\beta}$, $A^p(D, \beta)$ becomes a Banach space. If $\beta = 0$, then $A^p(D, \beta)$ is abbreviated to $A^p(D)$, usually called the Bergman space. In this paper, we consider the case of $p = 2$. For this case, it is a Hilbert space with the inner product

$$\langle f, g \rangle_\beta = \int_D f(z)\overline{g(z)}\delta(z)^\beta dv(z).$$

For a given separable Hilbert space H, the Schatten p-class of operators on H, $S_p(H)$, consists of those compact operators T on H with its sequence of singular numbers λ_n belonging to ℓ^p, the p-summable sequence space. When $p = 1$, it is usually called the trace class, and $p = 2$ is usually called the Hilbert-Schmidt class (see [22]). The theory of Schatten p-class of operators on the holomorphic function spaces has been widely studied (see, for example, [18, 7, 19, 14, 23, 12, 13, 6, 20] and the references therein). In particular, the authors in [20] characterized the Schatten p-class of weighted composition operators on $A^2(D)$.

Motivated by previous mentioned studies (in especial [20]), it is natural to consider how to characterize the Schatten p-class of weighted composition operators on $A^2(D, \beta)$. After a long time of careful consideration, we find that if the parameter β satisfies the condition

$$\int_D K(z, z)\delta(z)^\beta dv(z) = +\infty,$$

then it is a difficult problem. However, if β satisfies the condition

$$\int_D K(z, z)\delta(z)^\beta dv(z) < +\infty,$$

we can completely characterize the Schatten p-class of weighted composition operators on $A^2(D, \beta)$ by borrowing the methods obtained in [2] and [21]. We hope that this paper can attract people's more attention to such problems.

Let $K(z, w) : D \times D \to \mathbb{C}$ be the Bergman kernel of D. For every $w \in D$, the normalized Bergman kernel of D, denoted by $k_w(z)$, is defined by

$$k_w(z) = \frac{K(z, w)}{\sqrt{K(w, w)}} = \frac{K(z, w)}{\|K(\cdot, w)\|_{2,\beta}}.$$

For μ a finite complex Borel measure on D, the Berezin transform $\tilde{\mu}(z)$ is defined by

$$\tilde{\mu}(z)=\int_D |k_z(w)|^2 d\mu(w).$$

Let $\beta(z,w)$ be the Kobayashi distance function on D. For $z\in D$ and $r\in(0,1)$, let

$$B(z,r)=\{w\in D:\beta(z,w)<r\}$$

denote the Kobayashi ball with center z and radius $\frac{1}{2}\ln\frac{1+r}{1-r}$. We define $v_\beta(B(z,r))$ by

$$v_\beta(B(z,r))=\int_{B(z,r)}\delta(w)^\beta dv(w).$$

The function $\hat{\mu}^r(z)$ on D is defined by

$$\hat{\mu}^r(z)=\frac{\mu(B(z,r))}{v_\beta(B(z,r))}.$$

For φ the holomorphic self-map of D and $u\in H(D)$, we define $dv_{2,\beta}(z)=|u(z)|^2\delta(z)^\beta dv(z)$ and $\mu_{2,\beta}=v_{2,\beta}\circ\varphi^{-1}$, respectively. In this paper, we will use the Berezin transform $\tilde{\mu}_{2,\beta}$ and the function $\hat{\mu}^r_{2,\beta}$ to characterize the Schatten p-class of weighted composition operators on $A^2(D,\beta)$.

In this paper, the positive constants are denoted by C which may differ from one occurrence to the next.

2. Preliminary results

In this section, we present some results from [1] on the Kobayashi geometry of bounded strongly pseudoconvex domain.

Lemma 2.1. *Let $D\subseteq\mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $z_0\in D$ and $r\in(0,1)$, there exists a positive constant C independent of $z\in B(z_0,r)$ such that*

$$\frac{1-r}{C}\delta(z_0)\le\delta(z)\le\frac{C}{1-r}\delta(z_0).$$

Lemma 2.2. *Let $D\subseteq\mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $\beta\in\mathbb{R}$ and $r\in(0,1)$, there exist two positive constants C_1 and C_2 such that*

$$C_1\delta(\cdot)^{n+1+\beta}\le v_\beta(B(\cdot,r))\le C_2\delta(\cdot)^{n+1+\beta}.$$

By using Lemma 2.1 and Lemma 2.2, we have the following result.

Corollary 2.1. *For $r,s,R\in(0,1)$, there exists a positive constant C independent of z_1,z_2 with $\beta(z_1,z_2)\le R$ such that*

$$C^{-1}\le\frac{v_\beta(B(z_1,r))}{v_\beta(B(z_2,s))}\le C.$$

We also need the following result on the Bergman kernel obtained in [1] and [11].

Lemma 2.3. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $r \in (0,1)$, there exist positive constants C and δ such that, if $z_0 \in D$ satisfies $\delta(z_0) < \delta$, then*

$$\frac{C}{\delta(z_0)^{n+1}} \leq |K(z,z_0)| \leq \frac{1}{C\delta(z_0)^{n+1}}$$

and

$$\frac{C}{\delta(z_0)^{n+1}} \leq |k_{z_0}(z)|^2 \leq \frac{1}{C\delta(z_0)^{n+1}},$$

for all $z \in B(z_0, r)$.

From Lemmas 2.2 and 2.3, the following result follows.

Corollary 2.2. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $r \in (0,1)$, there exist positive constants C and δ such that, if $z_0 \in D$ satisfies $\delta(z_0) < \delta$, then*

$$\frac{C}{v_\beta(B(z_0,r))} \leq |K(z,z_0)| \leq \frac{1}{Cv_\beta(B(z_0,r))}$$

and

$$\frac{C}{v_\beta(B(z_0,r))} \leq |k_{z_0}(z)|^2 \leq \frac{1}{Cv_\beta(B(z_0,r))},$$

for all $z \in B(z_0, r)$.

We also need the following cover of D (see [1]).

Lemma 2.4. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then, for $r \in (0,1)$, there exist an $m \in \mathbb{N}$ and a sequence $\{z_i\} \subseteq D$ such that $D = \bigcup_{i=1}^{\infty} B(z_i, r)$ and any point in D belongs to at most m balls of the form $B(z_i, R)$ where $R = \frac{1}{2}(1+r)$.*

3. Main results and proofs

First, we have the following result.

Lemma 3.1. *If $T \in S_1(A^2(D,\beta))$, then*

$$tr(T) = \int_D \langle TK(\cdot,z), K(\cdot,z)\rangle_\beta \delta(z)^\beta dv(z).$$

Proof. Let $\{e_j(z)\}$ be an orthonormal basis for $A^2(D,\beta)$. We have

$$K(z,w)=\sum_{j=1}^{\infty} e_j(z)\overline{e_j(w)}.$$

Then, from this it follows that

$$\begin{aligned}
\operatorname{tr}(T) &= \sum_{j=1}^{\infty}\langle Te_j, e_j\rangle_\beta = \sum_{j=1}^{\infty}\int_D Te_j(z)\overline{e_j(z)}\delta(z)^\beta dv(z)\\
&= \sum_{j=1}^{\infty}\int_D \langle Te_j, K(\cdot,z)\rangle_\beta \overline{e_j(z)}\delta(z)^\beta dv(z)\\
&= \sum_{j=1}^{\infty}\int_D \langle e_j, T^*K(\cdot,z)\rangle_\beta \overline{e_j(z)}\delta(z)^\beta dv(z)\\
&= \int_D \delta(z)^\beta \int_D \Big(\sum_{j=1}^{\infty} e_j(w)\overline{e_j(z)}\Big)\overline{T^*K(\cdot,z)(w)}\delta(w)^\beta dv(w)dv(z)\\
&= \int_D \delta(z)^\beta \int_D K(w,z)\overline{T^*K(\cdot,z)(w)}\delta(w)^\beta dv(w)dv(z)\\
&= \int_D \langle K(\cdot,z), T^*K(\cdot,z)\rangle_\beta \delta(z)^\beta dv(z) = \int_D \langle TK(\cdot,z), K(\cdot,z)\rangle_\beta \delta(z)^\beta dv(z).
\end{aligned}$$

From this, the desired result follows. This completes the proof. □

In the following result, we give an estimate for the finite positive Borel measure on D.

Lemma 3.2. *Let $D\subseteq\mathbb{C}^n$ be a bounded strongly pseudoconvex domain, μ a finite positive Borel measure on D and $r\in(0,1)$. Then, there exists a positive constant C depending on r such that*

$$\mu(B(a,r))\le \frac{C}{v_\beta(B(a,r))}\int_{B(a,r)}\mu(B(z,r))\delta(z)^\beta dv(z).$$

Proof. For any $a\in D$, we have

$$\begin{aligned}
&\int_{B(a,r)}\mu(B(z,r))\delta(z)^\beta dv(z) = \int_{B(a,r)}\delta(z)^\beta dv(z)\int_{B(z,r)}d\mu(w)\\
&= \int_{B(a,r)}\delta(z)^\beta dv(z)\int_D \chi_{B(z,r)}(w)d\mu(w) = \int_D d\mu(w)\int_{B(a,r)}\chi_{B(z,r)}(w)\delta(z)^\beta dv(z).
\end{aligned}$$

Noting that $\chi_{B(w,r)}(z)=\chi_{B(z,r)}(w)$, for all w and z in D, we have

$$\begin{aligned}
\int_{B(a,r)}\mu(B(z,r))\delta(z)^\beta dv(z) &= \int_D d\mu(w)\int_{B(a,r)}\chi_{B(w,r)}(z)\delta(z)^\beta dv(z)\\
&= \int_D v_\beta(B(a,r)\cap B(w,r))d\mu(w) \ge \int_{B(a,r)} v_\beta(B(a,r)\cap B(w,r))d\mu(w),
\end{aligned}$$

where $\chi_{B(w,r)}(z)$ is the characteristic function of the set $B(w,r)$. Let $\alpha(t)$ $(0 \leq t < 1)$ be the geodesic (in the Bergman metric) from a to w and $m_{(a,w)} = \alpha(\frac{1}{2})$. By using Lemma 3 in [21], we obtain

$$\int_{B(a,r)} \mu(B(z,r))\delta(z)^\beta dv(z) \geq \int_{B(a,r)} v_\beta \left(B(m_{(a,w)}, \frac{r}{2})\right) d\mu(w).$$

From Corollary 2.1, it follows that there exists a positive constant C depending only on r such that

$$Cv_\beta \left(B \left(m_{(a,w)}, \frac{r}{2}\right)\right) \geq v_\beta(B(a,r)),$$

for all $w \in B(a,r)$. Therefore, we have

$$C \int_{B(a,r)} \mu(B(z,r))\delta(z)^\beta dv(z) \geq \int_{B(a,r)} v_\beta(B(a,r))d\mu(w),$$

that is,

$$\mu(B(a,r)) \leq \frac{C}{v_\beta(B(a,r))} \int_{B(a,r)} \mu(B(z,r))\delta(z)^\beta dv(z).$$

This completes the proof. □

Corollary 3.1. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, μ a finite positive Borel measure on D and $r \in (0,1)$. Then, there exists a positive constant C depending on r such that*

$$[\mu_{2,\beta}(B(z_j,r))]^{\frac{p}{2}} \leq \frac{C}{v_\beta(B(z_j,r))} \int_{B(z_j,r)} [\mu_{2,\beta}(B(z_j,r))]^{\frac{p}{2}} \delta(z)^\beta dv(z).$$

Corollary 3.2. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, μ a finite positive Borel measure on D and $r \in (0,1)$. Then, for every $r, R \in (0,1)$, there exists a positive constant C depending on r and R such that*

$$\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} \leq \frac{C}{v_\beta(B(z_j,r))} \int_{B(z_j,r)} \left[\frac{\mu_{2,\beta}(B(z,r))}{v_\beta(B(z,r))}\right]^{\frac{p}{2}} \delta(z)^\beta dv(z),$$

for all z_j, z with $\beta(z_j,z) \leq R$.

As an application of Corollary 3.2, we can introduce the following complex measure. For $p \in [2,+\infty)$, the complex measure $\mu_{2,\beta,\zeta}$ is defined by

$$\mu_{2,\beta,\zeta}(z) = \sum_{j=1}^{\infty} \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}\zeta-1} \chi_{B(z_j,r)}(z)\mu_{2,\beta}(z),$$

where ζ is a complex number with $0 \leq \mathrm{Re}\zeta \leq 1$ and $\chi_{B(z_j,r)}(z)$ is the characteristic function of the set $B(z_j,r)$.

Lemma 3.3. *Let $\zeta = \frac{2}{p}$. Then, it follows that*

$$T_{\mu_{2,\beta}} \leq T_{\mu_{2,\beta,\frac{2}{p}}} \leq mT_{\mu_{2,\beta}}.$$

Proof. Obviously, it follows that

$$\mu_{2,\beta,\frac{2}{p}}(z) = \sum_{j=1}^{\infty} \chi_{B(z_j,r)}(z)\mu_{2,\beta}(z) \geq \mu_{2,\beta}(z).$$

Then, we have

$$T_{\mu_{2,\beta,\frac{2}{p}}} f(z) = \int_D f(w)K(w,z)d\mu_{2,\beta,\frac{2}{p}}(w) \geq \int_D f(w)K(w,z)d\mu_{2,\beta}(w) = T_{\mu_{2,\beta}} f(z),$$

which shows $T_{\mu_{2,\beta,\frac{2}{p}}} \geq T_{\mu_{2,\beta}}$.

Conversely, it follows from Lemma 2.4 that $\mu_{2,\beta,\frac{2}{p}}(z) \leq m\mu_{2,\beta}(z)$. Similarly, we can get $T_{\mu_{2,\beta,\frac{2}{p}}} \leq mT_{\mu_{2,\beta}}$. This completes the proof. □

Lemma 3.4. *Let T_1, T_2 be two compact operators on Hilbert space H and $0 \leq T_1 \leq T_2$. Then*

$$\|T_1\|_{S_p(H)} \leq \|T_2\|_{S_p(H)}.$$

Proof. By Lemma 14 in [21], we have $s_j(T_1) \leq s_j(T_2)$ for $j \in \mathbb{N}$. Since

$$\|T\|_{S_p} = \left[\sum_{j=1}^{\infty}(s_j(T))^p\right]^{\frac{1}{p}},$$

we have

$$\|T_1\|_{S_p(H)} = \left[\sum_{j=1}^{\infty}(s_j(T_1))^p\right]^{\frac{1}{p}} \leq \left[\sum_{j=1}^{\infty}(s_j(T_2))^p\right]^{\frac{1}{p}} = \|T_2\|_{S_p(H)}.$$

This completes the proof. □

Now, we prove the main result of this paper. We assume that β satisfies the condition

$$\int_D K(z,z)\delta(z)^{\beta} dv(z) < +\infty. \tag{1}$$

Remark 3.1. We consider the condition (1) for the special case $D = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disk. For this case, we have (see, for example, [22])

$$K(z,w) = \frac{1}{(1 - z\overline{w})^2}.$$

For the case, it is easy to see that $\delta(z) = 1 - |z|^2$. Then, we have

$$\int_{\mathbb{D}} K(z,z)\delta(z)^{\beta} dv(z) = \int_{\mathbb{D}} (1 - |z|^2)^{\beta-2} dv(z) = 2\pi \int_0^1 (1 - r^2)^{\beta-2} r dr. \tag{2}$$

From a direct calculation, it follows that (2) is finite if and only if $\beta \in (1, +\infty)$. This shows that Theorem 3.1 excludes the result of the Bergman space (that is, corresponding to $\beta = 0$). Maybe it is caused by the different definitions of the weights. For example, in [21] the author defined the weighted Bergman space on bounded symmetric domains by the weight $K(z,z)^{\lambda}$.

Theorem 3.1. *Let $D \subseteq \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, $p \in [2, +\infty)$, φ a holomorphic self-map of D and $u \in H(D)$. Then, the following statements are equivalent:*

(*i*) $W_{\varphi,u} \in S_p(A^2(D,\beta))$;

(*ii*) $\tilde{\mu}_{2,\beta} \in L^{\frac{p}{2}}(D, K(z,z)\delta(z)^{\beta} dv(z))$;

(*iii*) $\hat{\mu}^r_{2,\beta} \in L^{\frac{p}{2}}(D, K(z,z)\delta(z)^{\beta} dv(z))$;

(*iv*) $\sum_{j=1}^{\infty} \left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}} < +\infty$, *where $\{z_j\}$ is the sequence in Lemma* 2.4.

Proof. For $f, g \in A^2(D,\beta)$, we have

$$\langle (W_{\varphi,u})^*(W_{\varphi,u})f, g\rangle_{\beta} = \langle (W_{\varphi,u})f, (W_{\varphi,u})g\rangle_{\beta} = \int_D |u(z)|^2 f(\varphi(z))\overline{g(\varphi(z))}\delta(z)^{\beta} dv(z)$$
$$= \int_D f(\varphi(z))\overline{g(\varphi(z))} dv_{2,\beta}(z) = \int_D f(w)\overline{g(w)} d\mu_{2,\beta}(w).$$

Considering the Toeplitz operator on $A^2(D,\beta)$

$$T_{\mu_{2,\beta}} f(z) = \int_D f(w) K(w,z) d\mu_{2,\beta}(w),$$

we have

$$\begin{aligned} \left\langle T_{\mu_{2,\beta}} f, g\right\rangle_{\beta} &= \int_D \int_D f(w) K(w,z) d\mu_{2,\beta}(w) \overline{g(z)} \delta(z)^{\beta} dv(z) \\ &= \int_D f(w) \overline{\int_D K(z,w) g(z) \delta(z)^{\beta} dv(z)} d\mu_{2,\beta}(w) \\ &= \int_D f(w)\overline{g(w)} d\mu_{2,\beta}(w), \end{aligned}$$

which shows that

$$T_{\mu_{2,\beta}} = (W_{\varphi,u})^*(W_{\varphi,u}).$$

This implies that $T_{\mu_{2,\beta}}$ is a positive operator on $A^2(D,\beta)$.

$(i) \Rightarrow (ii)$. From Theorem 1.4.6 in [22], we know that $W_{\varphi,u} \in S_p(A^2(D,\beta))$ if and only if $T_{\mu_{2,\beta}} \in S_{\frac{p}{2}}(A^2(D,\beta))$. Since $T_{\mu_{2,\beta}}$ is positive, by using Lemma 3.1, we have

$$\begin{aligned}\|T_{\mu_{2,\beta}}\|_{S_{\frac{p}{2}}}^{\frac{p}{2}} = \operatorname{tr}(T_{\mu_{2,\beta}}^{\frac{p}{2}}) &= \int_D \left\langle T_{\mu_{2,\beta}}^{\frac{p}{2}} K(\cdot,z), K(\cdot,z)\right\rangle_\beta \delta(z)^\beta dv(z)\\ &= \int_D K(z,z)\left\langle T_{\mu_{2,\beta}}^{\frac{p}{2}} k(\cdot,z), k(\cdot,z)\right\rangle_\beta \delta(z)^\beta dv(z).\end{aligned}$$

Since $\frac{p}{2} \geq 1$ and each k_z is a unit vector in $A^2(D,\beta)$, by Proposition 6.4 in [3] we get

$$\begin{aligned}\|T_{\mu_{2,\beta}}\|_{S_{\frac{p}{2}}(A^2(D,\underline{\ }))}^{\frac{p}{2}} &\geq \int_D K(z,z)\left[\left\langle T_{\mu_{2,\beta}} k(\cdot,z), k(\cdot,z)\right\rangle_\beta\right]^{\frac{p}{2}} \delta(z)^\beta dv(z)\\ &= \int_D K(z,z)(\tilde{\mu}_{2,\beta}(z))^{\frac{p}{2}} \delta(z)^\beta dv(z),\end{aligned}$$

which shows that $\tilde{\mu}_{2,\beta} \in L^{\frac{p}{2}}(D, K(z,z)\delta(z)^\beta dv(z))$.

$(ii) \Rightarrow (iii)$. Form Corollary 2.2, there exists a positive constant C such that

$$\begin{aligned}C\tilde{\mu}_{2,\beta}(z_0) = C\int_D |k_{z_0}(z)|^2 d\mu_{2,\beta}(z) &\geq C\int_{B(z_0,r)} |k_{z_0}(z)|^2 d\mu_{2,\beta}(z)\\ &\geq \frac{1}{v_\beta(B(z_0,r))}\int_{B(z_0,r)} d\mu_{2,\beta}(z) = \hat{\mu}_{2,\beta}^r(z_0).\end{aligned}$$

Thus

$$\int_D (\hat{\mu}_{2,\beta}^r(z))^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z) \leq C\int_D (\tilde{\mu}_{2,\beta}(z))^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z) < +\infty.$$

$(iii) \Rightarrow (iv)$. Let $\{z_j\}$ be the sequence in Lemma 2.4. By Corollary 3.2, we have

$$\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} \leq \frac{C}{v_\beta(B(z_j,r))}\int_{B(z_j,r)} \left[\frac{\mu_{2,\beta}(B(z,r))}{v_\beta(B(z,r))}\right]^{\frac{p}{2}} \delta(z)^\beta dv(z).$$

From Corollary 2.2, letting $z_0 = z$, there exists a positive constant C such that

$$\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} \leq C\int_{B(z_j,r)} \left[\frac{\mu_{2,\beta}(B(z,r))}{v_\beta(B(z,r))}\right]^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z).$$

By Lemma 2.4, there exists an $m \in \mathbb{N}$ such that

$$\sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} \leq Cm\int_D \left[\frac{\mu_{2,\beta}(B(z,r))}{v_\beta(B(z,r))}\right]^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z),$$

that is,

$$\sum_{j=1}^{\infty}\left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}} \leq Cm\int_D \left(\hat{\mu}^r_{2,\beta}(z)\right)^{\frac{p}{2}} K(z,z)\delta(z)^\beta dv(z).$$

$(iv) \Rightarrow (i)$. We use the complex interpolation method in [21] to prove this statement. We want to show that $T_{\mu_{2,\beta}} \in S_{\frac{p}{2}}(A^2(D,\beta))$ and

$$\|T_{\mu_{2,\beta}}\|^{\frac{p}{2}}_{S_{\frac{p}{2}}(A^2(D,\underline{)})} \leq C\sum_{j=1}^{\infty}\left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}}.$$

For $p=2$, by Corollary 2.2, there exists a positive constant C such that

$$\begin{aligned}
&\|T_{\mu_{2,\beta}}\|_{S_1(A^2(D,\beta))} = \int_D \left\langle T_{\mu_{2,\beta}}K(\cdot,z),K(\cdot,z)\right\rangle_\beta \delta(z)^\beta dv(z)\\
&= \int_D K(z,z)\left\langle T_{\mu_{2,\beta}}k_z(\cdot),k_z(\cdot)\right\rangle_\beta \delta(z)^\beta dv(z) = \int_D K(z,z)(\tilde{\mu}_{2,\beta}(z))\delta(z)^\beta dv(z)\\
&= \int_D K(z,z)\int_D |k_z(w)|^2 d\mu_{2,\beta}(w)\delta(z)^\beta dv(z) = \int_D\int_D |K(w,z)|^2 d\mu_{2,\beta}(w)\delta(z)^\beta dv(z)\\
&= \int_D\int_D |K(w,z)|^2\delta(z)^\beta dv(z)d\mu_{2,\beta}(w) = \int_D K(w,w)d\mu_{2,\beta}(w)\\
&= \int_D K(z,z)d\mu_{2,\beta}(z) \leq \sum_{j=1}^{\infty}\int_{B(z_j,r)}|K(z,z)|d\mu_{2,\beta}(z) \leq C\sum_{j=1}^{\infty}\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))},
\end{aligned}$$

for all $z_j \in B(z,r)$ and $j\in\mathbb{N}$. For $1<\frac{p}{2}<+\infty$, since $\sum_{j=1}^{\infty}\left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}} < +\infty$, we can assume that

$$\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))} < 1,$$

for all $j\in\mathbb{N}$. By Corollary 2.2 and Lemma 2.4, we have

$$\begin{aligned}
|\mu_{2,\beta,\zeta}|(D) &\leq \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}\mathrm{Re}\zeta-1}\mu_{2,\beta}(B(z_j,r))\\
&\leq \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{-1}\mu_{2,\beta}(B(z_j,r)) = \sum_{j=1}^{\infty}v_\beta(B(z_j,r))\\
&\leq C\sum_{j=1}^{\infty}\int_{B(z_j,r))}K(z,z)\delta(z)^\beta dv(z) \leq Cm\int_D K(z,z)\delta(z)^\beta dv(z) < +\infty.
\end{aligned}$$

For every ζ with $0\leq \mathrm{Re}\zeta\leq 1$, we consider the Toeplitz operator $T_{\mu_{2,\beta,\zeta}}$ on $A^2(D,\beta)$ defined by

$$T_{\mu_{2,\beta,\zeta}}f(z) = \int_D K(z,w)f(w)d\mu_{2,\beta,\zeta}(w).$$

By Lemma 3.3 and Lemma 3.4, we have

$$\|T_{\mu_{2,\beta}}\|_{S_p(A^2(D,\beta))} \le \|T_{\mu_{2,\beta,\frac{2}{p}}}\|_{S_p(A^2(D,\beta))} \le m\|T_{\mu_{2,\beta}}\|_{S_p(A^2(D,\beta))}.$$

Thus, $T_{\mu_{2,\beta}} \in S_{\frac{p}{2}}(A^2(D,\beta))$ is equivalent to $T_{\mu_{2,\beta,\frac{2}{p}}} \in S_{\frac{p}{2}}(A^2(D,\beta))$. By complex interpolation (see [21]), we have

$$\|T_{\mu_{2,\beta,\frac{2}{p}}}\|_{S_{\frac{p}{2}}(A^2(D,\beta))} \le M_0^{1-\frac{2}{p}} M_1^{\frac{2}{p}},$$

where

$$M_0 = \sup\left\{\|T_{\mu_{2,\beta,\zeta}}\| : \mathrm{Re}\zeta = 0\right\} \text{ and } M_1 = \sup\left\{\|T_{\mu_{2,\beta,\zeta}}\|_{S_1} : \mathrm{Re}\zeta = 1\right\}.$$

Now, we show that M_0 and M_1 are bounded. For $\mathrm{Re}\zeta = 0$,

$$\begin{aligned}|\mu_{2,\beta,\zeta}|(B(z_k,r)) &\le \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{-1}\int_{B(z_k,r)}\chi_{B(z_j,r)}(z)d\mu_{2,\beta}(z)\\ &= \sum_{j=1}^{\infty}\left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{-1}\mu_{2,\beta}(B(z_k,r)\cap B(z_j,r)).\end{aligned}$$

Since $B(z_k,r)\cap B(z_j,r) \ne 0$, by Lemma 2.4, for any fixed positive integer k, there exists $N_k \le N$ such that

$$\begin{aligned}|\mu_{2,\beta,\zeta}|(B(z_k,r)) &\le \sum_{i=1}^{N_k}\left[\frac{\mu_{2,\beta}(B(z_{j_i},r))}{v_\beta(B(z_{j_i},r))}\right]^{-1}\mu_{2,\beta}(B(z_k,r)\cap B(z_{j_i},r))\\ &\le \sum_{i=1}^{N_k}\left[\frac{\mu_{2,\beta}(B(z_{j_i},r))}{v_\beta(B(z_{j_i},r))}\right]^{-1}\mu_{2,\beta}(B(z_{j_i},r))\\ &= \sum_{i=1}^{N_k} v_\beta(B(z_{j_i},r)).\end{aligned}$$

Since $B(z_{j_i},r)\cap B(z_k,r) \ne 0$, by Corollary 2.1 there exists a positive constant C such that

$$v_\beta(B(z_{j_i},r)) \le Cv_\beta(B(z_k,r)).$$

Thus, for all $k \in \mathbb{N}$, we have

$$|\mu_{2,\beta,\zeta}|(B(z_k,r)) \le CN_k v_\beta(B(z_k,r)) \le CNv_\beta(B(z_k,r)).$$

From Theorem 3.4 in [1], we know that $|\mu_{2,\beta,\zeta}|$ is a Carleson measure of $A^2(D,\beta)$. By Corollary and Theorem 7 in [21], there exists a positive constant C such that

$$\int_D |f(z)|^2 d|\mu_{2,\beta,\zeta}|(z) \le C\int_D |f(z)|^2\delta(z)^\beta dv(z),$$

for all f in $A^2(D,\beta)$. Therefore,

$$\begin{aligned}\left|\langle T_{\mu_{2,\beta,\zeta}} f, g\rangle_\beta\right| &= \left|\int_D f(z)\overline{g(z)} d|\mu_{2,\beta,\zeta}|(z)\right| \\ &\leq \left[\int_D |f(z)|^2 d|\mu_{2,\beta,\zeta}|(z)\right]^{\frac{1}{2}} \left[\int_D |g(z)|^2 d|\mu_{2,\beta,\zeta}|(z)\right]^{\frac{1}{2}} \\ &\leq C\left[\int_D |f(z)|^2 \delta(z)^\beta dv(z)\right]^{\frac{1}{2}} \left[\int_D |g(z)|^2 \delta(z)^\beta dv(z)\right]^{\frac{1}{2}},\end{aligned}$$

which implies that $\|T_{\mu_{2,\beta,\zeta}}\| \leq C$, for all ζ with $\mathrm{Re}\zeta = 0$, that is, M_0 is bounded.

For $\mathrm{Re}\zeta = 1$, by Corollary 2.2, there exists a positive constant C such that

$$\begin{aligned}\int_D K(z,z) d|\mu_{2,\beta,\zeta}|(z) &\leq \sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}-1} \int_{B(z_j,r)} K(z,z) d\mu_{2,\beta}(z) \\ &\leq C\sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}-1} \frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))} \\ &= C\sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}}.\end{aligned}$$

For any orthonormal bases $\{f_j\}$ and $\{g_j\}$ of $A^2(D,\beta)$ and $\mathrm{Re}\zeta = 1$, we have

$$\begin{aligned}\sum_{j=1}^\infty \left|\langle T_{\mu_{2,\beta,\zeta}} f_j(z), g_j(z)\rangle_\beta\right| &\leq \int_D \sum_{j=1}^\infty |f_j(z)||g_j(z)| d|\mu_{2,\beta,\zeta}|(z) \\ &\leq \int_D \left[\sum_{j=1}^\infty |f_j(z)|^2\right]^{\frac{1}{2}} \left[\sum_{j=1}^\infty |g_j(z)|^2\right]^{\frac{1}{2}} d|\mu_{2,\beta,\zeta}|(z) \\ &= \int_D K(z,z) |d|\mu_{2,\beta,\zeta}|(z) \\ &\leq C\sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}}.\end{aligned}$$

Therefore, for all $\mathrm{Re}\zeta = 1$, we have

$$\|T_{\mu_{2,\beta,\zeta}}\|_{S_1(A^2(D,\beta))} \leq C\sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}},$$

that is,

$$M_1 \leq C\sum_{j=1}^\infty \left[\frac{\mu_{2,\beta}(B(z_j,r))}{v_\beta(B(z_j,r))}\right]^{\frac{p}{2}} = C\sum_{j=1}^\infty \left(\hat{\mu}^r_{2,\beta}(z_j)\right)^{\frac{p}{2}}.$$

Hence,

$$\|T_{\mu_{2,\beta}}\|_{S_p(A^2(D,\beta))} \leq M_0^{1-\frac{2}{p}} M_1^{\frac{2}{p}} \leq C \left(\sum_{j=1}^{\infty} \left(\hat{\mu}_{2,\beta}^r(z_j) \right)^{\frac{p}{2}} \right)^{\frac{2}{p}}.$$

This completes the proof. □

Acknowledgments

The authors would like to thank the referee and editor for providing valuable comments for the improvement of the paper. This study was supported by Sichuan Science and Technology Program (2022ZYD0010).

References

[1] M. Abate, A. Saracco, *Carleson measures and uniformly discrete sequences in strongly pseudoconvex domains,* J. London. Math. Soc., 83 (2011), 587-605.

[2] M. Abate, J. Raissy, A. Saracco, *Toeplitz operators and Carleson measures in strongly pseudoconvex domains,* J. Funct. Anal., 263 (2012), 3449-3491.

[3] J. Arazy, S. D. Fisher, J. Peetre, *Hankel operators on weighted Bergman spaces,* Amer. J. Math., 110 (1988), 989-1053.

[4] F. Colonna, S. Li, *Weighted composition operators from the minimal Möbius invariant space into the Bloch space,* Mediter. J. Math., 10 (2013), 395-409.

[5] K. Esmaeili, M. Lindström, *Weighted composition operators between Zygmund type spaces and their essential norms,* Integral Equ. Oper. Theory, 75 (2013), 473-490.

[6] E. A. Gallardo-Gutiérrez, R. Kumar, J. R. Partington, *Boundedness, compactness and Schatten-class membership of weighted composition operators*, Integr. Equ. Oper. Theory., 67 (2010), 467-479.

[7] J. Isralowitz, J. Virtanen, L. Wolf, *Schatten class Toeplitz operators on generalized Fock spaces,* J. Math. Anal. Appl., 421 (2015), 329-337.

[8] Z. J. Jiang, *Weighted composition operators from weighted Bergman spaces to some spaces of analytic functions on the upper half plane,* Util. Math., 93 (2014), 205-212.

[9] Z. J. Jiang, Z. A. Li, *Weighted composition operators on Bers-type spaces of Loo-keng Hua domains,* Bull. Korean Math. Soc., 57 (2020), 583-595.

[10] A. S. Kucik, *Weighted composition operators on spaces of analytic functions on the complex half-plane*, Complex Anal. Oper. Theory., 12 (2018), 1817-1833.

[11] H. Li, *BMO, VMO and Hankel operators on the Bergman space of strictly pseudoconvex domains,* J. Funct. Anal., 106 (1992), 375-408.

[12] S. Y. Li, B. Russo, *Schatten class composition operators on weighted Bergman spaces of bounded symmetric domains,* Ann. Mat. Pur Appl., 172 (1997), 379-394.

[13] T. Mengestie, *Schatten class weighted composition operators on weighted Fock spaces,* Arch. Math., 101 (2013), 349-360.

[14] B. F. Sehba, *Schatten class Toeplitz operators on weighted Bergman spaces of tube domains over symmetric cones,* Sehba Complex Anal. Synerg, 4 (2018), 3-27.

[15] S. Stević, *Weighted composition operators from Bergman-Privalov-type spaces to weighted-type spaces on the unit ball,* Appl. Math. Comput., 217 (2010), 1939-1943.

[16] S. Stević, Z. J. Jiang, *Differences of weighted composition operators on the unit polydisk*, Mat. Sb., 52 (2011), 358-371.

[17] S. Stević, R. Chen, Z. Zhou, *Weighted composition operators between Bloch type spaces in the polydisc,* Mat. Sb., 201 (2010), 289-319.

[18] X. F. Wang, J. Xia, G. F. Cao, *Schatten p-class Toeplitz operators with unbounded symbols on pluriharmonic Bergman space*, Acta Mathematica Sinica, English Series., 29 (2013), 2355-2366.

[19] L. H. Xiao, X. F. Wang, J. Xia, *Schatten p-class* $(0 < p \leq \infty)$ *Toeplitz operators on generalized Fock spaces,* Acta Mathematica Sinica, 31 (2015), 703-714.

[20] X. D. Yang, *Schatten class weighted composition operators on Bergman spaces of bounded strongly pseudoconvex domains,* Cent. Eur. J. Math., 11 (2013), 74-84.

[21] K. Zhu, *Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains,* J. Operator Theory., 20 (1988), 329-357.

[22] K. Zhu, *Operator theory in function spaces*, Marecl Dekker, New York, 1990.

[23] K. Zhu, *Schatten class composition operators on weighted Bergman spaces of the disk,* J. Operator Theory., 46 (2001), 173-181.

Accepted: February 13, 2023

Hermite-Hadamard inequality for preinvex functions

Akhlad Iqbal
Department of Mathematics
Aligarh Muslim University
Aligarh-202002
India
akhlad6star@gmail.com
akhlad.mm@amu.ac.in

Khairul Saleh
Department of Mathematics
King Fahd University of Petroleum and Minerals
Dhahran 31261
Saudi Arabia
khairul@kfupm.edu.sa

Izhar Ahmad*
Department of Mathematics
King Fahd University of Petroleum and Minerals
Dhahran 31261
Saudi Arabia
and
Center for Intelligent Secure Systems
King Fahd University of Petroleum and Minerals
Dhahran 31261
Saudi Arabia
drizhar@kfupm.edu.sa

Abstract. We derive integral inequalities of Hermite-Hadamard type for the functions that have preinvex absolute values of third order derivatives. Moreover, we also discuss applications to several special means.

Keywords: Hermite-Hadamard inequality, invex set, preinvex function, integral inequality.

1. Introduction

For convex functions, several inequalities have been studied by many authors, see [1], [2]-[9]. But the inequality obtained by Hadamard [8] is considered the most significant and rich in applications. Let $g : \mathcal{I} \subseteq R \to R$ be a convex

∗. Corresponding author

function on the interval $\mathcal{I}$. The inequality in [8] is given by

$$(1)\quad g\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(u)du \leq \frac{g(\alpha)+g(\beta)}{2}, \quad \alpha,\beta \in \mathcal{I} \text{ and } \alpha<\beta.$$

As mentioned in [8]: "inequality (1) is known as the Hermite-Hadamard (H-H) inequality for convex functions". The inequalities will be reversed for a concave function. Hadamard inequality refines the concept of convexity and various classical inequalities can be derived from it.

Recently, several extensions, refinements and generalizations have been discussed by the many authors, see [2, 7, 9, 16, 18]. Dragomir et. al. [5] proved the following lemma for the class of convex functions.

Lemma 1.1 ([5]). *Suppose that $g : \mathcal{I}^o \subseteq R \to R$ be a differentiable mapping on $\mathcal{I}^o$, $\alpha, \beta \in \mathcal{I}^o$, such that $\alpha < \beta$. If $g' \in L[\alpha, \beta]$, then*

$$(2)\quad \frac{g(\alpha)+g(\beta)}{2} - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(u)du = \frac{\beta-\alpha}{2}\int_{0}^{1}(1-2t)g'(t\alpha+(1-t)\beta)dt.$$

Hanson [10] introduced the concept of invexity which is a significant generalization of covexity. The concept of preinvex functions was introduced by Weir and Mond [17], later Jeyakumar et. al. [13] investigated some properties of these functions. They [13] also studied the role of preinvex functions in optimization and mathematical programming. Yuan et. al. [18] investigated some new characterizations of preinvex and prequasi-invex function under some assumptions. Noor [14] derived H-H inequality for preinvex and log-preinvex functions, later Iqbal et. al. [11] investigated some refined integral inequalities and discussed its applications to special means.

The objective of this work is to formulate some new refined inequalities of H-H type for the functions that have preinvex absolute values of third derivatives. We have considered various special means to show its applications. Our findings extend the previously known results.

2. Preliminaries

The following definitions and known result will be used in the sequel.

Definition 2.1 ([10]). *A set $X \subseteq R^n$ is said to be invex with respect to $\eta : X \times X \to R^n$ if*

$$(3)\quad v + t\eta(u,v) \in X, \forall\ u,v \in X\ \&\ t \in [0,1].$$

As discussed in [10], "the definition says that there is a path starting from v which is contained in X. It is not necessary that u should be one of the end points of the path. However, if we require that u be an end point of the path for every pair $u, v \in X$, then $\eta(u,v) = u - v$, reduces to convexity."

Define
$$P_{uy} := \{w : w = u + t\eta(v,u) : t \in [0,1]\}.$$
It represents the η-path joining the points u and $y := u + \eta(v,u)$ for every $u, v \in X$.

Definition 2.2 ([17]). *Let $X \subseteq R^n$ be an invex set with respect to $\eta : X \times X \to R^n$. Then, the function $g : X \to R$ is called preinvex with respect to η, if*

$$g(v + t\eta(u,v)) \leq tg(u) + (1-t)g(v), \ \forall\ u, v \in X \ \&\ t \in [0,1]. \tag{4}$$

Preinvex function is the generalized class of convex functions. The function $f(u) = - \mid u \mid$ is preinvex with respect to η, where

$$\eta(u,v) := \begin{cases} u - v, & \text{if } u \leq 0, v \leq 0 \text{ and } u \geq 0, v \geq 0, \\ v - u, & \text{otherwise.} \end{cases}$$

But it is not convex. Recently, Barani et. al. [1] extended the Lemma 1.1 for invex sets as follows:

Lemma 2.1 ([1]). *Suppose that $A \subseteq R$ be an open invex subset with respect to $\eta : A \times A \to R$ and $\alpha, \beta \in A$ with $\eta(\alpha,\beta) \neq 0$ and that $g : A \to R$ be differentiable function. If g' is integrable on the η path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$, then*

$$\begin{aligned} &-\frac{g(\beta) + g(\beta + \eta(\alpha,\beta))}{2} + \frac{1}{\eta(\alpha,\beta)} \int_{\beta}^{\beta+\eta(\alpha,\beta)} g(u)du \\ &= \frac{\eta(\alpha,\beta)}{2} \int_0^1 (1-2t) g'(\beta + t\eta(\alpha,\beta))dt. \end{aligned}$$

Using Lemma 2.1, Barani et. al. [1] established H-H type inequalities for preinvex functions.

3. Main results

We now extend the previous known results for the functions whose third derivatives absolute values are preinvex. Consider the function $\eta : A \times A \to R$ with $\eta(\alpha,\beta) \neq 0$, for $\alpha, \beta \in A$. Henceforth, we assume that $A \subseteq R$ is an open invex set with respect to η.

Lemma 3.1. *Let $g : A \to R$ be three times differentiable function and g''' is integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$, then*

$$\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta)) + g(\beta)] + \frac{1}{\eta(\alpha,\beta)} \int_{\beta}^{\beta+\eta(\alpha,\beta)} g(u)du$$

$$= \frac{\eta(\alpha,\beta)^3}{12} \int_0^1 t(1-t)(2t-1) g'''(\beta + t\eta(\alpha,\beta))dt. \tag{5}$$

Proof. Let $\alpha, \beta \in A$. Since A is an invex set with respect to η, $\beta + t\eta(\alpha,\beta) \in A$ for every $t \in [0,1]$. Integrating by parts, we get

$$\int_0^1 t(1-t)(2t-1)g'''(b+t\eta(\alpha,\beta))dt$$
$$= \left[\frac{t(1-t)(2t-1)g''(\beta+t\eta(\alpha,\beta))}{\eta(\alpha,\beta)}\right]_0^1$$
$$- \frac{1}{\eta(\alpha,\beta)}\int_0^1 (-6t^2+6t-1)g''(\beta+t\eta(\alpha,\beta))dt$$
$$= \frac{1}{\eta(\alpha,\beta)}\left[\frac{(6t^2-6t+1)g'(\beta+t\eta(\alpha,\beta))}{\eta(\alpha,\beta)}\right]_0^1$$
$$- \frac{1}{\eta(\alpha,\beta)^2}\int_0^1 (12t-6)g'(\beta+t\eta(\alpha,\beta))dt$$
$$= \frac{1}{\eta(\alpha,\beta)^2}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{6}{\eta(\alpha,\beta)^3}[g(\beta+\eta(\alpha,\beta)) + g(\beta)]$$
$$+ \frac{12}{(\eta(\alpha,\beta)^4}\int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du. \qquad \square$$

Using above lemma, we prove some interesting results for the preinvex functions.

Theorem 3.1. *Let $g : A \to R$ be three times differentiable function and g''' is integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$. If $|g'''|$ is preinvex on A, then*

$$\left|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)]\right.$$
$$\left. -\frac{1}{2}[g(\beta+\eta(\alpha,\beta)) + g(\beta)] + \frac{1}{\eta(\alpha,\beta)}\int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du\right|$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{384}\left[\frac{25}{2}|g'''(\beta)| - |g'''(\alpha)|\right]$$

Proof. Applying Lemma 3.1 and using the preinvexity of $|g'''|$, we get

$$\left|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta)) + g(\beta)]\right.$$
$$\left. +\frac{1}{\eta(\alpha,\beta)}\int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du\right|$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{12}\int_0^1 t(1-t)|(2t-1)||g'''(\beta+t\eta(\alpha,\beta))|dt$$
$$\leq \frac{|\eta(\alpha,\beta)|^3}{12}\,[\int_0^1 t(1-t)|(2t-1)|(t|g'''(\alpha)| + (1-t)|g'''(\beta)|)dt$$

$$= \frac{|\eta(\alpha,\beta)|^3}{12} \, [|g'''(\alpha)| \int_0^1 t^2(1-t)|(2t-1)|dt + |g'''(\beta)| \int_0^1 t(1-t)^2|(2t-1)|dt$$

$$= \frac{|\eta(\alpha,\beta)|^3}{384} \left[\frac{25}{2}|g''(\beta)| - |g''(\alpha)|\right]. \qquad \square$$

Theorem 3.2. *Let $g : A \to R$ be three times differentiable function and g''' be integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$. If $|g'''|^{p/p-1}$ is preinvex on A for $p > 1$, then*

$$\Bigg| \frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta)) + g(\beta)] + \frac{1}{\eta(\alpha,\beta)} \int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du \Bigg| \leq \frac{|\eta(\alpha,\beta)|^3}{96} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(|g'''(\alpha)|^q + |g'''(\beta)|^q\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 3.1, preinvexity of $|g'''|^{p/p-1}$ and Holder's integral inequality, we obtain

$$\begin{aligned}
&\Bigg| \frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[f(\beta+\eta(\alpha,\beta)) + g(\beta)] + \frac{1}{\eta(\alpha,\beta)} \int_\beta^{\beta+\eta(\alpha,\beta)} g(u)du \Bigg| \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12} \int_0^1 t(1-t)|(2t-1)||g'''(\beta + t\eta(\alpha,\beta))|dt \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12} \left(\int_0^1 t^p(1-t)^p|(2t-1)|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 |g'''(\beta + t\eta(\alpha,\beta))|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{12} \left(\frac{1}{2^{2p+1}(p+1)}\right)^{\frac{1}{p}} \left(\int_0^1 t|g'''(\alpha)|^q + (1-t)|g'''(\beta)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\alpha,\beta)|^3}{96} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(|g'''(\alpha)|^q + |g'''(\beta)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. $\square$

Theorem 3.3. *Let $g : A \to R$ be three times differentiable function and g''' be integrable on the η-path $P_{\beta\gamma}$, $\gamma = \beta + \eta(\alpha,\beta)$. If $|g'''|^q$ is preinvex on A for $q > 1$, then*

$$\Bigg| \frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta)) - g'(\beta)] - \frac{1}{2}[g(\beta+\eta(\alpha,\beta))$$

$$+g(\beta)]+\frac{1}{\eta(\alpha,\beta)}\int_{\beta}^{\beta+\eta(\alpha,\beta)}g(u)du\Bigg|$$
$$\leq\frac{|\eta(\alpha,\beta)|^3}{192}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{25}{2}|g'''(\beta)|^q-|g'''(\alpha)|^q\right)^{\frac{1}{q}}.$$

Proof. Since $|g'''|^q$ is preinvex, using Lemma 3.1 and power-mean inequality, we obtain

$$\Bigg|\frac{\eta(\alpha,\beta)}{12}[g'(\beta+\eta(\alpha,\beta))-g'(\beta)]-\frac{1}{2}[f(\beta+\eta(\alpha,\beta))$$
$$+g(\beta)]+\frac{1}{\eta(\alpha,\beta)}\int_{\beta}^{\beta+\eta(\alpha,\beta)}g(u)du\Bigg|$$
$$\leq\frac{|\eta(\alpha,\beta)|^3}{12}\int_0^1 t(1-t)|(2t-1)||g'''(\beta+t\eta(\alpha,\beta))|dt$$
$$\leq\frac{|\eta(\alpha,\beta)|^3}{12}\left(\int_0^1 t(1-t)|(2t-1)|dt\right)^{1-\frac{1}{q}}$$
$$\cdot\left(\int_0^1 t(1-t)|(2t-1)||g'''(\beta+t\eta(\alpha,\beta))|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{|\eta(\alpha,\beta)|^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)|(2t-1)|[t|g'''(\alpha)|^q+(1-t)|g'''(\beta)|^q]dt\right)^{\frac{1}{q}}$$
$$\leq\frac{|\eta(\alpha,\beta)|^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left(|g''(\alpha)|^q\int_0^1 t^2(1-t)|(2t-1)|dt\right.$$
$$\left.+|g''(\beta)|^q\int_0^1 t(1-t)^2|(2t-1)|dt\right)^{\frac{1}{q}}$$
$$\leq\frac{|\eta(\alpha,\beta)|^3}{12}\left(\frac{1}{16}\right)^{1-\frac{1}{q}}\left[|g'''(\alpha)|^q(-\frac{1}{32})+|g'''(\beta)|^q(\frac{25}{64})\right]^{\frac{1}{q}}$$
$$=\frac{|\eta(\alpha,\beta)|^3}{192}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{25}{2}|g'''(\beta)|^q-|g'''(\alpha)|^q\right)^{\frac{1}{q}}.\qquad\square$$

4. Some applications

For distinct positive real numbers a_1 and a_2, we have:

Arithmetic mean: $A(a_1,a_2)=\frac{a_1+a_2}{2}$,

Logarithmic mean: $L_p(a_1,a_2)=\frac{a_1-a_2}{\ln a_1-\ln a_2}$, and

generalized logarithmic mean: $L_p(a_1,a_2)=\left[\frac{a_1^{p+1}-a_2^{p+1}}{(p+1)(a_1-a_2)}\right]^{1/p}, p\neq-1,0.$

Let us suppose that

$$g(u)=\frac{u^{n+3}}{(n+1)(n+2)(n+3)}$$

be a function and $a_3=a_2+\eta(a_1,a_2)$, then

$$\frac{g(a_3)+g(a_2)}{2}=\frac{1}{(n+1)(n+2)(n+3)}A(a_3^{n+3},a_2^{n+3}),$$

$$\frac{1}{\eta(a_1,a_2)}\int_{a_2}^{a_3}g(u)du=\frac{1}{\eta(a_1,a_2)}\frac{1}{(n+1)(n+2)(n+3)}\left[\frac{a_3^{n+4}-a_2^{n+4}}{n+4}\right].$$

For $\eta(a_1,a_2)=a_1-a_2$, it becomes

$$\frac{1}{a_1-a_2}\int_{a_2}^{a_1}g(u)du=\frac{1}{(n+1)(n+2)(n+3)}L_{n+3}^{n+3}(a_1,a_2),$$

$$g'(a_2+\eta(a_1,a_2)-g'(a_2)=\frac{(a_2+\eta(a_1,a_2))^{n+2}-a_2^{n+2}}{(n+1)(n+2)}.$$

For $\eta(a_1,a_2)=a_1-a_2$, it becomes

$$g'(a_1)-g'(a_2)=\frac{(a_1-a_2)}{(n+1)}L_{n+1}^{n+1}(a_1^{n+2},a_2^{n+2}).$$

Now, using the results of section 3, we discuss some applications to special means of real numbers.

Proposition 4.1. *For positive numbers a_1 and a_2 such that $a_1>a_2$ and $0<n\le 1$, we have*

$$\begin{aligned}&\big|(a_1-a_2)^2(n+2)(n+3)L_{n+1}^{n+1}(a_1^{n+2},a_2^{n+2})\\&-12A(a_1^{n+3},a_2^{n+3})+12L_{n+3}^{n+3}(a_1^{n+4},a_2^{n+4})\big|\\&\le\frac{|(a_1-a_2)|^3}{32}(n+1)(n+2)(n+3)\left[\frac{25}{2}|a_2^n|-|a_1^n|\right].\end{aligned}$$

Proof. The statement takes after from Theorem 3.3 connected to the function

$$g(u)=\frac{u^{n+3}}{(n+1)(n+2)(n+3)},$$

for $\eta(a_1,a_2)=a_1-a_2$. □

Proposition 4.2. *For positive numbers a_1 and a_2 such that $a_1>a_2$ and $0<n\le 1$, we have*

$$\begin{aligned}&\big|(a_1-a_2)^2(n+2)(n+3)L_{n+1}^{n+1}(a_1^{n+2},a_2^{n+2})-12A(a_1^{n+3},a_2^{n+3})+12L_{n+3}^{n+3}(a_1^{n+4},a_2^{n+4})\big|\\&\le\frac{|(a_1-a_2)|^3}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(n+1)(n+2)(n+3)(|a_1^n|^q+|a_2^n|^q)^{\frac{1}{q}}.\end{aligned}$$

Proof. The statement takes after from Theorem 3.3 connected to the function

$$g(u) = \frac{u^{n+3}}{(n+1)(n+2)(n+3)},$$

for $\eta(a_1, a_2) = a_1 - a_2$. □

Proposition 4.3. *For positive numbers a_1 and a_2 such that $a_1 > a_2$, $0 < n \leq 1$ and $q > 1$, we have*

$$\left|(a_1 - a_2)^2(n+2)(n+3)L_{n+1}^{n+1}(a_1^{n+2}, a_2^{n+2}) - 12A(a_1^{n+3}, a_2^{n+3}) + 12L_{n+3}^{n+3}(a_1^{n+4}, a_2^{n+4})\right|$$
$$\leq \frac{|(a_1 - a_2)|^3}{16}\left(\frac{1}{2}\right)^q (n+1)(n+2)(n+3)\left[\frac{25}{2}|a_2^n|^q - |a_1^n|^q\right]^{\frac{1}{q}}.$$

Proof. The statement takes after from Theorem 3.3 connected to the function

$$g(u) = \frac{u^{n+3}}{(n+1)(n+2)(n+3)},$$

for $\eta(a_1, a_2) = a_1 - a_2$. □

5. Conclusion

In this paper, we have extended the estimates of right hand side of Hermite-Hadamard type inequality for the functions having pre-invex third derivative absolute values. To show its application, we have considered several special means for arbitrary real numbers. In the future, the results can be generalized for higher order derivatives. Moreover, it can be studied in the context of q-calculus, and various applications can be explored.

References

[1] A. Barani, A.G. Ghazanfari, S.S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex,* J. Inequal. Appl., 247 (2012).

[2] M. Bombardelli, S. Varosanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejer inequalities,* Comput. Math. Appl., 58 (2010), 1869-1877.

[3] S.S. Dragomir, *Two mappings in connection to Hadamard's inequalities,* J. Math. Anal. Appl., 167 (1992), 49-56.

[4] S.S. Dragomir, *On Hadamards inequalities for convex functions,* Math. Balk., 6 (1992), 215-222.

[5] S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,* Appl. Math. Lett., 11 (1998), 91-95.

[6] S.S. Dragomir, C.E.M. Pearce, *Quasi-convex functions and Hadamard's inequality,* Bull. Austr. Math. Soc., 57 (1998), 377-385.

[7] S.S. Dragomir, Y.J. Cho, S.S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications,* J. Math. Anal. Appl., 245 (2000), 489-501.

[8] J. Hadamard, *Tude sur les proprietes des fonctions entieres en particulier dune fonction considre par Riemann,* J. Math. Pures Appl., 58 (1893), 171-215.

[9] N. Hadjisavvas, *Hadamard-type inequalities for quasi-convex functions,* J. Ineq. Pure Appl. Math., 4 (2003), article 13.

[10] M.A. Hanson, *On sufficiency of the Kuhn-Tucker conditions,* J. Math. Anal. Appl., 80 (1981), 545-550.

[11] A. Iqbal, V. Samhita, *Some integral inequalities for log-preinvex functions. Applied analysis in biological and physical sciences,* Springer Proceedings in Mathematics & Statistics, Springer, 2016, 373-384.

[12] D.A. Ion, *Some estimates on the Hermite-Hadamard inequality through quasi-convex functions,* Annals of University of Craiova. Math. Comp. Sci. Ser., 34 (2007), 19-22.

[13] V. Jeyakumar, *Strong and weak invexity in mathematical programming,* Meth. Oper. Res., 55 (1985), 109-125.

[14] M.A. Noor, *Hermite-Hadamard integral inequality for log-preinvex functions,* J. Math. Anal. Approx. Theory., 2 (2007), 126-131.

[15] P.M. Pardalos, P.G. Georgiev, H.M. Srivastava, (Eds), *Nonlinear analysis: stabiliy, approximation and inequalities,* Springer, 2012.

[16] S. Qaisar, S. Hussain, H. Chuanjiang, *On new inequalities of Hermite-Hadamard type for functions whose third derivatives absolute values are quasi-convex with applications,* J. Egyptian Math. Soc., 22 (2014), 137-146.

[17] T. Weir, B. Mond, *Preinvex functions in multiobjective optimization,* J. Math. Anal. Appl., 136 (1988), 29-38.

[18] G.S. Yang, D.Y. Hwang, K.L. Tseng, *Some inequalities for differentiable convex and concave mappings,* Comput. Math. Appl., 47 (2004), 207-216.

[19] D.H. Yuan, X. Liu, G. Lai, *Note on generalized invex functions,* Optim Lett., 7 (2013), 617-623.

Accepted: September 6, 2022

Constructions of indecomposable representations of algebras via reflection functors

Wanwan Jia
Department of Mathematics
Zhejiang University Hangzhou
Zhejiang 310027
China
and
School of Science
Xihua University Chengdu
Sichuan 610039
China
jiawanwan@zju.edu.cn

Fang Li*
Department of Mathematics
Zhejiang University
Hangzhou, Zhejiang 310027
China
fangli@zju.edu.cn

Abstract. The main aim of this paper is to reform the Bernstein-Gelfand-Ponomarev theory in order to characterize representations of some (non-basic) artinian algebras. All non-isomorphic indecomposable projective and injective representations are constructed via Coxeter functors for a generalized path algebra of acyclic quiver and then for an artinian hereditary algebra of Gabriel-type with an admissible ideal. The methods given via natural quivers and reformed modulations are helpful for one to study some properties which are *not Morita-invariant* in representation theory.
Keywords: modulation, natural quiver, reflection functor, artinian algebra, generalized path algebra, indecomposable representation.

1. Introduction

Reflection functors were introduced into the representation theory of quivers by Bernstein, Gelfand and Ponomarev in their work on the 4-subspace problem [13] and on Gabriel's Theorem, e.g. [5, 2, 3]. Due to the latter result, one obtains the classifications of finite type and tame type of basic hereditary artinian algebras, that is, acyclic quiver algebras, over an algebraically closed field. Furthermore, there have been several generalizations, see [6, 11, 10, 4, 1, 9]. In [11, 10, 9], Bernstein-Gelfand-Ponomarev theory was generalized to hereditary tensor algebras of quivers over division rings. In [6], the authors gave an extension of

*. Corresponding author

the concept of reflection functors and some applications to quivers with relations (equivalently say, to some special basic non-hereditary artinian algebras). A special case of this theory has been developed by Marmaridis [25] and applied to certain quivers with relations. In [1], a theory of partial Coxeter functors was developed for a basic artin algebra with a simple projective noninjective module.

The fact that each finite dimensional basic algebra over an algebraically closed field is some quotient of path algebra plays an important role in algebraical representation theory, since it characterizes the structures of basic algebras and provides a method to give various examples of basic algebras using quivers. More importantly, it can be used to characterize finitely generated modules over an algebra. However, there are limitations to this approach. Firstly, the ground field has to be an algebraically closed field. Secondly, the characterization of representations of a finite dimensional algebra must be based on its corresponding basic algebra. But, some information of representations of the original algebra will be lost via its basic algebra. To solve this problem, Coelho and Liu[8] first introduced the concept of generalized path algebras, so as to have a more direct and new understanding for the structures and representations of algebras.

It is noted that artinian algebras having be studied in all former papers are basic. Although the module category of an artinian algebra and that of its corresponding basic algebra are equivalent which means the representation types of these two algebras are coherent, in usual it is difficult to consider the relation between the dimensions of their modules. It is the motivation for us to use the method of reflection functors to study non-basic artinian algebras and some data of their representations which are not Morita-invariant.

The main aim of this paper is to reform the Bernstein-Gelfand-Ponomarev theory to characterize the representation categories of some (non-basic) artinian algebras and to give a method for constructing indecomposable projective and injective representations via reflection functors and Coxeter functors. This makes it possible to compute the dimensions of indecomposable representations of a (non-basic) artinian algebra. The tool we use is the natural quiver of an artinian algebra.

In the classical setting, mathematicians dealt with the module theory of the path algebras of quivers. In this paper, we use the natural quivers of (non-basic) hereditary algebras and the reformed modulations via generalized path algebras which are isomorphic to hereditary algebras, see [15, 21, 8, 7], to solve the corresponding problems in modules over the generalized path algebras.

The natural quiver will have fewer arrows than the Ext-quiver when the algebra A is not basic. Natural quivers are not invariant under the Morita equivalence and much closer to reflect the structure of the algebra, rather than just its module category. There are numerous cases even in the representation theory that one needs the structure of the algebras, for example, the character values of finite groups in a block cannot be preserved through Morita equivalence.

We think natural quivers and generalized path algebras are valid to study some properties which are not Mortia invariant in representation theory.

When an artinian algebra A is of Gabriel-type [18], that is, A is isomorphic to some quotient of the generalized path algebra of its natural quiver Δ_A, then any representations of A can be induced directly from some representations of the generalized path algebra of Δ_A. From [18], we know that any artinian algebra splitting over its radical must be of Gabriel-type. It is more straightforward through representations of the generalized path algebra of Δ_A to set up an approach to representations of an artinian algebra.

Associated with any representation of a quiver is a dimension vector, and the dimension vectors of indecomposable modules are the positive roots of the quadratic form associated to the quiver (see e.g. [5, 11, 14]). Similar results seem to hold for certain quivers with relations. Some applications of reflection functors involve the study of the transformations of dimension vectors they induce. It turns out in [6] that there are applications of our functors which make use of the analogous transformations which is considered as a change of basis for a fixed root-system - a tilting of the axes relative to the roots which results in a different subset of roots lying in the positive cone.

For our need, for an artinian algebra, the dimension vectors of modules and the Cartan matrix are introduced in Section 2. First, some properties of dimension vectors are given, which are generalizations of the corresponding properties for a basic algebra. When the global dimension of an artinian algebra is finite, its Cartan matrix is invertible and can be computed through an integer matrix and two diagonal matrices. The Euler characteristic and the Euler quadratic form of an artinian algebra is defined from the Cartan matrix. On the other hand, the Euler form and quadratic form of a pre-modulation is defined. It was shown in [2] that the quadratic form and the Euler quadratic form coincide for a path algebra through the homological interpretation of the Euler characteristic. However, for a generalized path algebra, it is difficult to get the similar relation between its Euler quadratic form and the quadratic form from its corresponding pre-modulation in the reason that in the general case the homological interpretation of the Euler characteristic can not be computed via the inverse matrix of its Cartan matrix. So, in this paper, the homological interpretation of the Euler form, as well as the quadratic form, is characterized directly.

As analogue of the dimension vectors of indecomposable modules of quivers, it is interesting for one to discuss the relationship between the dimension vectors of indecomposable representations of artinian algebras and the positive roots of the quadratic forms associated to pre-modulations. Since the dimension vector and Cartan matrix of an artinian algebra are not invariant under the Morita equivalence, the mentioned relation above has only been a conjecture. This will be our further expectation for researching with this new method given via natural quivers and generalized modulations.

In Section 3, first, the reflection functors are given for the representation category $rep(\mathcal{M}, \Omega)$ of a pre-modulation $\mathcal{M}$ with acyclic connected valued quiver and using of them as a pair of mutual invertible functors Δ_i^- and Δ_i^+, the categorical equivalence is obtained between the full subcategories $rep^{(i)}(\mathcal{M}, \Omega)$ and $rep_{(i)}(\mathcal{M}, \Omega)$ for $i = 1, n$.

Moreover, we get the construction of all non-isomorphic indecomposable projective and injective representations of a generalized path algebra with acyclic quiver and then of an artinian hereditary algebra of Gabriel-type with admissible ideal.

At last, in Section 4, as application, we discuss the relationship between representation-type of a generalized path algebra and its natural quiver.

2. Dimension vectors of representations

2.1 Dimension vectors of modules over an artinian algebra

One attaches to each module of a basic algebra a vector with integral coordinates, called its dimension vector. This allows one to use methods of linear algebra when studying modules over a basic algebra. For example, an important application is in the famous Kac theorem which means the relation between dimension vectors of indecomposable modules and the so-called positive root system of a basic (hereditary) algebra. However, as we have known, the natural quiver is a tool to characterize an artinian (non-basic) algebra. In this paper, we try to give directly, but not through the theory of basic algebras, the description of the relationship between indecomposable modules of artinian (non-hereditary) algebras and the generalization of dimension vectors via natural quivers. Note that the dimension as a linear space and dimension vector defined below of a module are not Morita-invariant. This explains the validity of our discussion here.

Throughout this paper, we will always use k to be an algebraically closed field.

An artinian algebra A over k with Jacobson radical $r = r(A)$ is called *splitting over radical* if the natural homomorphism $A \to A/r$ is a splitting algebra homomorphism. In this case, A/r can be embedded into A as a subalgebra.

For two rings A and B, a finitely generated A-B-bimodule M, define $rk_{A,B}(M)$ to be the minimal number of generators of M as an A-B-bimodule among all genarating sets. Then we call $rk_{A,B}(M)$ the *rank* of M as A-B-bimodule.

The concept of generalized path algebra was introduced early in [8]. Here we review the different but equivalent definition which is given in [18].

Let $Q = (Q_0, Q_1)$ be a quiver. Given a collection of k-algebras $\mathcal{A} = \{A_i \mid i \in Q_0\}$ with the identity $e_i \in A_i$. Let $A_0 = \prod_{i \in Q_0} A_i$ be the direct product k-algebra. Clearly, each e_i is an orthogonal central idempotent of A_0. For $i, j \in Q_0$, let $\Omega(i, j)$ be the subset of arrows in Q_1 from i to j. Write

$${}_iM_j \overset{def}{=} A_i \Omega(i, j) A_j$$

be the free A_i-A_j-bimodule with basis $\Omega(i,j)$. This is the free $A_i \otimes_k A_j^{op}$-module over the set $\Omega(i,j)$. Thus,

$$M = \bigoplus_{(i,j)\in Q_0\times Q_0} A_i\Omega(i,j)A_j \tag{1}$$

is an A_0-A_0-bimodule. The *generalized path algebra*[8, 15, 18] is defined to the tensor algebra

$$T(A_0, M) = \bigoplus_{n=0}^{\infty} M^{\otimes_{A_0} n}.$$

Here $M^{\otimes_{A_0} n} = M \otimes_{A_0} M \otimes_{A_0} \ldots \otimes_{A_0} M$ and $M^{\otimes_{A_0} 0} = A_0$. We denote by $k(Q, \mathcal{A})$ the generalized path algebra. $k(Q, \mathcal{A})$ is called *(semi-)normal* if all A_i are (semi-)simple k-algebras.

Suppose that A is a left artinian k-algebra and $r = r(A)$ is its Jacobson radical. Write $A/r = A_1 \oplus \ldots \oplus A_s$, where A_i are two-sided simple ideals of A/r. Such a decomposition of A/r is also called *a block decomposition of the algebra A/r*. Then, r/r^2 is an A/r-bimodule. Let ${}_iM_j = A_i \cdot r/r^2 \cdot A_j$, which is finitely generated as an A_i-A_j-bimodule for each pair (i,j).

Now we introduce the concept of natural quiver and corresponding generalized path algebra of A.

Definition 2.1 ([18])**.** *Suppose that A is a left artinian k-algebra and $r = r(A)$ is its Jacobson radical. Write $A/r = A_1 \oplus \ldots \oplus A_s$, where A_i are two-sided simple ideals of A/r.*

(i) The natural quiver of A is defined by $\Delta_A = (\Delta_0, \Delta_1)$ with the vertex set Δ_0 to be the index set $\{1, 2, \ldots, s\}$ of the isomorphism classes of simple A-modules corresponding to the set of blocks of A/r; with the arrow set Δ_1 consisting of $t_{i,j}$ arrows from i to j for $i, j \in \Delta_0$ where $t_{i,j} = rk_{A_j,A_i}({}_jM_i)$. Obviously, there is no arrow from i to j if ${}_jM_i = 0$.

(ii) Denote $\mathcal{A} = \{A_i \mid i \in Q_0\}$. The generalized path algebra $k(\Delta_A, \mathcal{A})$ is called the corresponding generalized path algebra of A.

By Definition 2.1, the natural quiver of artinian algebra A is always finite.

In [18], we have known the following characterization of an artinian algebra A splitting over radical via its generalized path algebra.

Theorem 2.1 ([18])**.** *An artinian k-algebra A is splitting over radical if and only if there is an ideal I of the corresponding generalized path algebra $k(\Delta_A, \mathcal{A})$ of A and a positive integer s such that $A \cong k(\Delta_A, \mathcal{A})/I$ with $J^s \subset I \subset J$ where J is the ideal of $k(\Delta_A, \mathcal{A})$ generated by all $\mathcal{A}$-paths of length 1.*

This means that an artinian k-algebra splitting over radical is of Gabriel-type.

Definition 2.2. *Suppose that A is an artinian algebra splitting over radical r with ideal I satisfying $A \cong k(\Delta_A, \mathcal{A})/I$ due to Theorem 2.1. Write $(\Delta_A)_0 = \{1, 2, \ldots, s\}$. Let $A/r = A_1 \oplus \ldots \oplus A_s$ where A_i are simple ideals of A/r. For a right A-module M, the dimension vector of M is defined to be the vector*

$$\mathbf{dim}M = \begin{pmatrix} \frac{dim_k MA_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k MA_s}{dim_k A_s} \end{pmatrix}$$

in $\mathbb{Q}^s$ for the field of rational numbers $\mathbb{Q}$, where A_i acts on M as subalgebras of A.

The notion of dimension vectors of modules of a basic algebra in [2] is in the special case of this definition. Clearly, dimension vector is not Morita-invariant.

Lemma 2.1. *Let A be an artinian k-algebra splitting over radical r such $A/r = A_1 \oplus \ldots \oplus A_s$ where A_i are simple ideals of A/r, and M be a right A-module. Embedding A_i into A, consider A_iA and A_iAA_i through the multiplication of A. Then, for any $i = 1, \ldots, s$,*

(i) the k-linear map

$$\theta_M^{(i)}: \ Hom_A(A_iA, M) \to MA_i \tag{2}$$

defined by the formula $\varphi \mapsto \varphi(1_{A_i}) = \varphi(1_{A_i})1_{A_i}$ for $\varphi \in Hom_A(A_iA, M)$, is an isomorphism of right A_iAA_i-modules, and it is functorial in M;

(ii) the isomorphism $\theta_{A_iA}^{(i)}: \ End(A_iA) \overset{\cong}{\to} A_iAA_i$ of right A_iAA_i-modules induces an isomorphism of k-algebras.

Proof. (i) For any $\bar{a}_i x \bar{b}_i \in A_iAA_i$,

$\theta_M^{(i)}(\varphi \bar{a}_i x \bar{b}_i) = (\varphi \bar{a}_i x \bar{b}_i)(1_{A_i}) = \varphi(\bar{a}_i x \bar{b}_i) = \varphi(1_{A_i})\bar{a}_i x \bar{b}_i = (\theta_M^{(i)}(\varphi))\bar{a}_i x \bar{b}_i$.

Then, $\theta_M^{(i)}$ is a homomorphism of right A_iAA_i-modules. And, $\theta_M^{(i)}$ is functorial in M from the following commutative diagram:

$$\begin{array}{ccc} Hom_A(A_iA, M) & \xrightarrow{\theta_M^{(i)}} & MA_i \\ {\scriptstyle Hom_A(A_iA,\ f)}\downarrow & & \downarrow{\scriptstyle f_{A_i}} \\ Hom_A(A_iA, N) & \xrightarrow{\theta_N^{(i)}} & NA_i \end{array}$$

where $f: M \to N$ is an A-homomorphism and f_{A_i} is the restriction of f on MA_i.

In order to prove $\theta_M^{(i)}$ is invertible, define a map $\zeta_M^{(i)}: \ MA_i \to Hom_A(A_iA, M)$ by the formula $\zeta_M^{(i)}(m\bar{a}_i)(\bar{b}_i x) = m\bar{a}_i\bar{b}_i x$ for $\bar{a}_i, \bar{b}_i \in A_i, x \in A$. It is easy to check that $\zeta_M^{(i)}(m\bar{a}_i): A_iA \to M$ is well-defined and is an A-homomorphism.

For any $m\bar{a}_i \in MA_i, \bar{b}_i x\bar{c}_i \in A_iAA_i, \bar{d}_i b \in A_iA$,

$$\zeta_M^{(i)}(m\bar{a}_i\bar{b}_ix\bar{c}_i)(\bar{d}_ib) = m\bar{a}_i\bar{b}_ix\bar{c}_i\bar{d}_ib = \zeta_M^{(i)}(m\bar{a}_i)(\bar{b}_ix\bar{c}_i\bar{d}_ib) = (\zeta_M^{(i)}(m\bar{a}_i)\bar{b}_ix\bar{c}_i)(\bar{d}_ib),$$

then $\zeta_M^{(i)}(m\bar{a}_i\bar{b}_ix\bar{c}_i) = \zeta_M^{(i)}(m\bar{a}_i)\bar{b}_ix\bar{c}_i$, which means $\zeta_M^{(i)}$ is a homomorphism of A_iAA_i-modules.

Moreover, for $f \in Hom_A(A_iA, M)$, $\bar{d}_ib \in A_iA$,

$$(\zeta_M^{(i)}\theta_M^{(i)})(f)(\bar{d}_ib)=(\zeta_M^{(i)}(\theta_M^{(i)}(f)))(\bar{d}_ib)=\theta_M^{(i)}(f)\bar{d}_ib=\theta_M^{(i)}(f\bar{d}_ib)=(f\bar{d}_ib)(1_{A_i})=f(\bar{d}_ib)$$

then $\zeta_M^{(i)}\theta_M^{(i)} = id_{Hom_A(A_iA,M)}$. Similarly, $\theta_M^{(i)}\zeta_M^{(i)} = id_{MA_i}$. Hence, $\theta_M^{(i)}$ is an isomorphism.

(ii) This follows from (i) for $M = A_iA$. □

Lemma 2.2. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2. For each right A-module and $i \in \Delta_0$, the k-linear map (2) induces functorial isomorphisms of k-vector spaces*

$$Hom_A(P(i), M) \overset{\cong}{\to} MA_i \overset{\cong}{\to} DHom_A(M, I(i)).$$

where D is the standard duality $Hom_k(-,k)$, $P(i)=A_iA$ and $I(i)=Hom_k(AA_i,k)$.

Proof. The first isomorphism follows directly from Lemma 2.1 (i). The second isomorphism is the composition

$$\begin{aligned} DHom_A(M, I(i)) &= DHom_A(M, D(AA_i)) \cong DHom_A(D(D(M)), D(AA_i)) \\ &\cong DHom_{A^{op}}(AA_i, D(M)) \cong D(A_iD(M)) \text{ (by Lemma 2.1)} \\ &\cong Hom_k(A_iD(M), k) \cong Hom_k(D(M), k)A_i = D(D(M))A_i \\ &\cong MA_i. \quad \square \end{aligned}$$

This lemma yields $\mathbf{dim}M = \begin{pmatrix} \frac{dim_k Hom_A(P(1),M)}{dim_k A_1} \\ \vdots \\ \frac{dim_k Hom_A(P(s),M)}{dim_k A_s} \end{pmatrix} = \begin{pmatrix} \frac{dim_k Hom_A(M,I(1))}{dim_k A_1} \\ \vdots \\ \frac{dim_k Hom_A(M,I(s))}{dim_k A_s} \end{pmatrix}.$

When A is an artinian k-algebra splitting over radical r, i.e. $A = r + A/r$, we have $1_A = r_0 + 1_{A/r}$ for some $r_0 \in r$. Then, $1_{A/r} = 1_A 1_{A/r} = r_0 1_{A/r} + 1_{A/r}$, thus, $r_0 1_{A/r} = 0$. Similarly, $1_{A/r} r_0 = 0$. Then, $1_A = 1_A^2 = (r_0 + 1_{A/r})^2 = r_0^2 + 1_{A/r}$. Moreover, we can get $1_A = r_0^t + 1_{A/r}$ for any natural number t. But, r is nilpotent, so there is t such that $r_0^t = 0$. Hence,

$$1_A = 1_{A/r}.$$

For $A/r = A_1 + \ldots + A_s$, we have $A \supseteq A_1A + \ldots + A_sA \supseteq (A_1 + \ldots + A_s)A \supseteq 1_{A/r}A = 1_AA = A$. Therefore,

$$A = A_1A + \ldots + A_sA$$

which means that for all $i = 1, \ldots, s$, $P(i) = A_iA$ are projective right A-modules. It follows that $Hom_A(P(i), -)$ are exact functors for $i = 1, \ldots, s$.

Proposition 2.1. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2 and $0 \to L \to M \to N \to 0$ be a short exact sequence of right A-modules. Then,* $\mathbf{dim}M = \mathbf{dim}L + \mathbf{dimN}$.

Proof. Using of the exact functor $Hom_A(P(i), -)$ to the short exact sequence $0 \to L \to M \to N \to 0$, we get the exact sequence of k-vector spaces:

$$0 \to Hom_A(P(i), L) \to Hom_A(P(i), M) \to Hom_A(P(i), N) \to 0.$$

By Lemma 2.2, this short exact sequence becomes to the following:

$$0 \to LA_i \to MA_i \to NA_i \to 0.$$

Hence, for each $i \in (\Delta_A)_0$,

$$dim_k MA_i = dim_k LA_i + dim_k NA_i.$$

The statement follows from the definition of dimension vectors. □

Since A_i is isomorphic to the matrix algebra of order n_i over a division k-algebra D_i, in the sequel of this section we always *let n_i denote this notation of the order of the matrix algebra*. We know that there are primitive idempotents $e_{i1}, e_{i2}, \ldots, e_{in_i}$ of A_i such that $P(i) = A_iA = e_{i1}A \oplus e_{i2}A \oplus \ldots \oplus e_{in_i}A$ but $e_{i1}A \cong e_{i2}A \cong \ldots \cong e_{in_i}A$ as right A-modules. So, we can write $P(i) \cong \oplus n_i e_{i1}A$. Here, for $i = 1, \ldots, s$, $P_i = e_{i1}A$ are all indecomposable projective right A-modules. Moreover, $S_i = P_i/P_ir$, $i = 1, \ldots, s$, are all simple A-modules.

It is easy to see that $dim_k S_i = n_i dim_k D_i$ and $dim_k A_i = n_i^2 dim_k D_i$.

Since $A_iA_j = 0$ for $i \neq j$, we have $S_iA_j = \begin{cases} S_i, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$ Therefore, for $i = 1, \ldots, s$,

$$\mathbf{dim}S_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{n_i dim_k D_i}{n_i^2 dim_k D_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{n_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3}$$

which we denote as X_i. For an artinian k-algebra A, denote by $K_0(A)$ the Grothendieck group of A, $[M]$ the corresponding element in $K_0(A)$ for an A-module M.

Proposition 2.2. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2 and let $S_1, \ldots, S_s$ be a complete set of the isomorphism classes of simple right A-modules. Then,* *the Grothendieck group $K_0(A)$ of A is a free abelian group having as a basis the set $\{[S_1], \ldots, [S_s]\}$. Define $\mathbf{dim}[M] = \mathbf{dim}M$ as the dimension vector of $[M]$ for each A-module M and moreover $\mathbf{dim}(-[M]) = -\mathbf{dim}M$, then $\mathbf{dim}$ is a group homomorphism from $K_0(A)$ to $\mathbb{Q}^s$ and the set of dimension vectors is, i.e. the image of $\mathbf{dim}$,*

$$\mathbf{dim}K_0(A) = \{u_1X_1 + \ldots + u_sX_s : \; u_1, \ldots, u_s \in \mathbb{Z}\}.$$

Proof. Let M be a module in mod$A$ and let $0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$ be a composition series for M. By the definition of $K_0(A)$, we have

$$\begin{aligned} [M] &= [M_t/M_{t-1}] + [M_{t-1}] = [M_t/M_{t-1}] + [M_{t-1}/M_{t-2}] + [M_{t-2}] = \ldots \\ &= \sum_{j=1}^{t}[M_j/M_{j-1}] = \sum_{i=1}^{s} c_i(M)[S_i] \end{aligned}$$

where $c_i(M)$ is the number of composition factors M_j/M_{j-1} of M that are isomorphic to S_i. Hence, $\{[S_1], \ldots, [S_s]\}$ generates the free abelian group $K_0(A)$.

Thus, by the definition of $\mathbf{dim}$ on $K_0(A)$ and Proposition 2.1, we know $\mathbf{dim}$ is a group homomorphism. □

Since $K_0(A)$ of A is a free abelian group with rank s having as a basis the set $\{[S_1], \ldots, [S_s]\}$, it is also isomorphic to $\mathbb{Z}^s$ as groups, but not through $\mathbf{dim}$.

As a consequence, we show the relation between the dimension vector of a module M and the number of simple composition factors of M that are isomorphic to each simple modules S_i.

Corollary 2.1. *Let $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.2 and let $S_1, \ldots, S_s$ be a complete set of the isomorphism classes of simple right A-modules. For any module M in mod$A$, let $c_i(M)$ be the number of composition factors M_j/M_{j-1} of M that are isomorphic to S_i and let $l(M)$ be the composition length of M. Then,*

$$c_i(M) = (dim_k MA_i)/(n_i dim_k D_i)$$

and thus $l(M) = \sum_{i=1}^{s}(dim_k MA_i)/(n_i dim_k D_i)$, where D_i is the division k-algebra such that A_i is isomorphic to the matrix algebra of order n_i over D_i for $A/r = A_1 \oplus \ldots \oplus A_s$ where A_i are simple ideals of A/r.

Proof. In the proof of Proposition 2.2, we have $[M] = \sum_{i=1}^{s} c_i(M)[S_i]$. Then, $\mathbf{dim}M = \mathbf{dim}[M] = \sum_{i=1}^{s} c_i(M)\mathbf{dim}[S_i] = \sum_{i=1}^{s} c_i(M)\mathbf{dim}S_i$. By (3), we get

$$dim_k MA_i = c_i(M) n_i dim_k D_i.$$

Thus, $l(M) = \sum_{i=1}^{s} c_i(M) = \sum_{i=1}^{s}(dim_k MA_i)/(n_i dim_k D_i)$. □

Definition 2.3. *Let A be an artinian k-algebra splitting over radical r with $A/r = A_1 \oplus \ldots \oplus A_s$. The Cartan matrix of A is the $s \times s$ matrix*

$$C_A = \begin{pmatrix} c_{11} & \ldots & c_{1s} \\ \vdots & \ddots & \vdots \\ c_{s1} & \ldots & c_{ss} \end{pmatrix},$$

where $c_{ji} = dim_k A_i A A_j$ for $i, j = 1, \ldots, s$.

Let $e_1, \ldots, e_s$ be the complete set of primitive orthogonal idempotents. Then $A_i A \cong n_i e_i A$ as right A-modules where $n_i e_i A$ means the direct sum of n_i copies of $e_i A$, that is, $P(i) = n_i P_i$ for the indecomposable projective A-modules $P_i = e_i A$ $(i = 1, \ldots, s)$.

By Lemma 2.2,

$$\begin{aligned} A_i A A_j &\cong Hom_A(P(j), A_i A) \cong Hom_A(A_j A, A_i A) \\ &\cong Hom_A(n_j e_j A, n_i e_i A) \cong n_j n_i Hom_A(e_j A, e_i A). \end{aligned}$$

Thus, $dim_k Hom_A(P_j, P_i) = c_{ji}/(n_j n_i)$ for $i = 1, \ldots, s$.

On the other hand, by Lemma 2.2,

$$I(i) = Hom_k(AA_i, k) \cong Hom_k(n_i A e_i, k) \cong n_i Hom_k(Ae_i, k) = n_i I_i,$$

where $I_i = Ae_i$ $(i = 1, \ldots, s)$ are the indecomposable injective A-modules. Moreover, by Lemma 2.2,

$$\begin{aligned} Hom_A(P(j), P(i)) &\cong DHom_A(P(i), I(j)) \cong DDHom_A(I(j), I(i)) \\ &\cong Hom_A(I(j), I(i)) \cong n_i n_j Hom_A(I_j, I_i). \end{aligned}$$

Thus,

$$A_i A A_j \cong Hom_A(I(j), I(i)) \tag{4}$$

and $A_i A A_j \cong n_i n_j Hom_A(I_j, I_i)$. Hence, $dim_k Hom_A(I_j, I_i) = c_{ji}/(n_j n_i)$.

Therefore, through modulo $n_i n_j$ for each c_{ij}, the Cartan matrix of A records the numbers of linearly independent homomorphisms between the indecomposable projective A-modules and the numbers of linearly independent homomorphisms between the indecomposable injective A-modules.

Below we discuss some elementary facts on the Cartan matrix.

Proposition 2.3. *Let C_A be the Cartan matrix of an artinian algebra $A \cong k(\Delta_A, \mathcal{A})/I$ as in Definition 2.3. Then,*

(i) *The i-th column of C_A is* $\begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix} \mathbf{dim} P(i)$ *and*

$$\begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix} \mathbf{dim} P(i) = n_i C_A \mathbf{dim} S_i;$$

(ii) The i-th row of C_A is $(\mathbf{dim}I(i))^t \begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix}$ *and*

$$\begin{pmatrix} n_1^2 dim_k D_1 & & \\ & \ddots & \\ & & n_s^2 dim_k D_s \end{pmatrix} \mathbf{dim}I(i) = n_i C_A^t \mathbf{dim}S_i.$$

Proof. (ii) $(\mathbf{dim}I(i))^t = (\frac{dim_k I(i)A_1}{dim_k A_1}, \dots, \frac{dim_k I(i)A_s}{dim_k A_s})$. By Lemma 2.2, we have $I(i)A_j \cong DHom_A(I(i), I(j))$. By (4), $Hom_A(I(i), I(j)) \cong A_j A A_i$. But,

$$dim_k DHom_A(I(i), I(j)) = dim_k Hom_A(I(i), I(j)).$$

Thus, $\frac{dim_k I(i)A_j}{dim_k A_j} = \frac{dim_k A_j A A_i}{dim_k A_j}$ for $j = 1, \dots, s$, which means the first result. From this and (3), the second result follows.

(i) Its proof is similar, since it is easy to be obtained from the definition of $\mathbf{dim}P(i)$ and (3). □

Proposition 2.4. *Let A be an artinian algebra as in Definition 2.2 with $A \cong k(\Delta_A, \mathcal{A})/I$. Suppose the global dimension of A is finite. Then, the Cartan matrix C_A is invertible and there exists $B \in \mathcal{M}_s(\mathbb{Z})$ such that*

$$C_A^{-1} = \begin{pmatrix} \frac{1}{n_1^3 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^3 dim_k D_s} \end{pmatrix} B \begin{pmatrix} n_1 & & \\ & \ddots & \\ & & n_s \end{pmatrix}$$

where $\mathcal{M}_s(\mathbf{Z})$ denotes the $s \times s$ full matrix ring over the integer ring $\mathbb{Z}$.

Proof. Here $s = | \Delta_0 |$. Since A is of finite global dimension, for any $i \in \{1, \dots, s\}$ and the corresponding simple A-module S_i there is a projective resolution

$$0 \to Q_{m_i} \to \dots \to Q_1 \to Q_0 \to S_i \to 0$$

in mod$A$ for a positive integer m_i.

From Proposition 2.1, it follows that $\mathbf{dim}S_i = \sum_{l=1}^{m_i} (-1)^l \mathbf{dim}Q_l$. Because $P_1, \dots, P_s$ are the complete set of non-isomorphic indecomposable projective A-modules, each Q_l is a direct sum of finitely many copies of $P_1, \dots, P_s$. Thus, for each i, $\mathbf{dim}S_i$ is a linear combination of the vectors $\mathbf{dim}P_1, \dots, \mathbf{dim}P_s$ with integral coefficients. Thus, there exists $B \in \mathcal{M}_s(\mathbb{Z})$ such that

$$\begin{pmatrix} n_1^{-1} & & \\ & \ddots & \\ & & n_s^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{dim}S_1 & \dots & \mathbf{dim}S_s \end{pmatrix} = \begin{pmatrix} \mathbf{dim}P_1 & \dots & \mathbf{dim}P_s \end{pmatrix} B.$$

But, $P(i) = n_iP_i$, so $\mathbf{dim}P(i) = n_i\mathbf{dim}P_i$ for $i = 1, \dots, s$. Hence,

$$\begin{pmatrix} n_1^{-1} & & \\ & \ddots & \\ & & n_s^{-1} \end{pmatrix} = \begin{pmatrix} n_1^{-1}\mathbf{dim}P(1) & \dots & n_s^{-1}\mathbf{dim}P(s) \end{pmatrix} B$$

$$= \begin{pmatrix} C_A\mathbf{dim}S_1 & \dots & C_A\mathbf{dim}S_s \end{pmatrix} \begin{pmatrix} \frac{1}{n_1^2 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^2 dim_k D_s} \end{pmatrix} B$$
$$= C_A \begin{pmatrix} \frac{1}{n_1^3 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^3 dim_k D_s} \end{pmatrix} B.$$

Thus,

$$C_A^{-1} = \begin{pmatrix} \frac{1}{n_1^3 dim_k D_1} & & \\ & \ddots & \\ & & \frac{1}{n_s^3 dim_k D_s} \end{pmatrix} B \begin{pmatrix} n_1 & & \\ & \ddots & \\ & & n_s \end{pmatrix}. \quad \square$$

Note that, when $n_i = 1$ and $dim_kD_i = 1$ for all i, A is a basic algebra and $C_A^{-1} = B$ is an integer matrix.

We use the Cartan matrix C_A to define a nonsymmetric $\mathbb{Z}$-bilinear form on the $\mathbb{Z}^s$.

Definition 2.4. *Let A be an artinian algebra with radical r of finite global dimension such that $A/r = A_1 \oplus \dots \oplus A_s$ where each A_i is simple ideals of A/r which is isomorphic to the matrix algebra of order n_i over a division k-algebra D_i. Let C_A be the Cartan matrix of A.*

(i) The Euler characteristic of A is the $\mathbb{Z}$-bilinear form $\langle -, - \rangle_A : \mathbb{Z}^s \times \mathbb{Z}^s \to \mathbb{Z}$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \mathbf{x}^t \begin{pmatrix} n_1^{-1} & & \\ & \ddots & \\ & & n_s^{-1} \end{pmatrix} (C_A^{-1})^t \begin{pmatrix} n_1^3 dim_k D_1 & & \\ & \ddots & \\ & & n_s^3 dim_k D_s \end{pmatrix} \mathbf{y}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^s$;

(ii) The Euler quadratic form of A is the quadratic form $q_A : \mathbb{Z}^s \to \mathbb{Z}$ defined by $q_A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle_A$ for $\mathbf{x} \in \mathbb{Z}^s$.

This definition makes sense due to Proposition 2.4.

2.2 Dimension vectors of representations of a pre-modulation

Given a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ and a vertex $i \in \mathcal{G}$, define an operation, denoted by δ_i, on the orientation Ω to get the orientation $\delta_i\Omega$ as follows: we reverse all arrows along edges containing i and leave all others unchanged in Ω.

With respect to the orientation Ω, call *admissible sequence of sinks* an ordering

$$(k_1, k_2, \dots, k_n)$$

of all the vertices of $\mathcal{G}$ such that k_1 is a sink with respect to Ω, k_2 a sink with respect to $\delta_{k_1}\Omega$, and so on, that is, k_t is a sink with respect to $\delta_{k_{t-1}} \dots \delta_{k_1}\Omega$ for $2 \le t \le n$. Similarly, *admissible sequence of sources* can be defined. We shall call an orientation admitting an admissible sequence of sinks *admissible.* As known in [10], the orientation Ω is admissible if and only if the valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ has no oriented cycle. In general, there are many different admissible sequences with respect to a given orientation.

Suppose that $\mathcal{M} = (A_i, {}_iM_j)$ is a k-pre-modulation of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ whose orientation Ω is admissible. Let $k(Q, \mathcal{A}) = T(M, A_0)$ be the constructed corresponding normal generalized path algebra in [16], where $M = \bigoplus_{i,j} A_i\Omega(i,j)A_j$ for $A_i\Omega(i,j)A_j \cong {}_iM_j$ and $A_0 = \oplus_{i\in Q_0}A_i$. Then, $Q_0 = \mathcal{G} = \{1, 2, \dots, s\}$ and the arrow set $Q_1 = \bigcup_{i,j} \Omega(i,j)$ is decided by the number t_{ij} of generators in the A_i-A_j-basis of ${}_iM_j$ as free A_i-A_j-bimodule.

Denote $A = k(Q, \mathcal{A})$. Q is a finite acyclic quiver since the orientation Ω is admissible. Then, A is artinian. Due to [15], $k(Q, \mathcal{A})$ is just the corresponding generalized path algebra $k(\Delta_A, \mathcal{A})$, that is, the ideal I is zero in Theorem 2.1.

Let $\mathcal{V} = (V_i, {}_j\varphi_i)$ be a representation of $\mathcal{M}$. Then, $V = \oplus_{i\in Q_0}V_i$ is a right module over $A = k(Q, \mathcal{A})$ with right A_i-module V_i such that $VA_i = V_i$ but $V_iA_j = 0$ for $i, j \in Q_0, i \neq j$. However, $A/r \cong A_0 = \oplus_{i\in Q_0}A_i$ for the radical of A. So, let $Q_0 = \mathcal{G} = \{1, 2, \dots, s\}$, the dimension vector of V

$$\mathbf{dim}V = \begin{pmatrix} \frac{dim_k VA_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k VA_s}{dim_k A_s} \end{pmatrix} = \begin{pmatrix} \frac{dim_k V_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k V_s}{dim_k A_s} \end{pmatrix}$$

in $\mathbb{Q}^s$. We call $\mathbf{dim}V$ the dimension vector of the representation $\mathcal{V} = (V_i, {}_j\varphi_i)$ of $\mathcal{M}$, denoted as $\mathbf{dim}\mathcal{V} = \begin{pmatrix} \frac{dim_k V_1}{dim_k A_1} \\ \vdots \\ \frac{dim_k V_s}{dim_k A_s} \end{pmatrix}$.

For a k-pre-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$, we define the bilinear forms $B(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}, \mathbf{y})$ by

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i\in\mathcal{G}} x_iy_i dim_k A_i - \sum_{i\to j} d_{ij}x_iy_j dim_k A_j,$$
$$(\mathbf{x}, \mathbf{y}) = B(\mathbf{x}, \mathbf{y}) + B(\mathbf{y}, \mathbf{x}),$$

where $\mathbf{x} = (x_i)_{i\in\mathcal{G}}$ and $\mathbf{y} = (y_i)_{i\in\mathcal{G}}$ in $\mathbb{Q}^s$. We call $B(-,-)$ the *Euler form* and $(-,-)$ the *symmetric Euler form* respectively. Moreover, we can define the quadratic form $q_{\mathcal{M}} : \mathbb{Q}^s \to \mathbb{Q}^s$ by $q_{\mathcal{M}}(x) = B(x,x)$ for $x \in \mathbb{Q}^s$, which is called *the quadratic form of the pre-modulation* $\mathcal{M}$.

In the trivial case that $A_i = k$ for all $i \in \mathcal{G}$, we can get a quiver Q with $Q_0 = \mathcal{G}$ and Q_1 consisting of t_{ij} arrows from i to j by $t_{ij} = d_{ij}/\varepsilon_i = d_{ji}/\varepsilon_j$. Then, the quadratic form $q_{\mathcal{M}}$ is just that of the quiver Q defined in [2]. In this trivial case, it was shown in Lemma VII4.1 of [2] that this quadratic form $q_{\mathcal{M}}$ and the Euler quadratic form q_A coincide for $A = kQ$. The proof of this result in [2] was dependent on the homological interpretation of the Euler characteristic.

However, for a general $A = k(Q,\mathcal{A})$, it is difficult for us to try to get the similar relation between the quadratic form $q_{\mathcal{M}}$ and the Euler quadratic form q_A in the reason that the inverse matrix of the Cartan matrix C_A is so complicated for computing that we cannot give the homological interpretation of the Euler characteristic $\langle -,-\rangle_A$. Hence, on the other hand, we will give the homological interpretation of the Euler form $B(-,-)$ as follows.

Theorem 2.2. *Assume that $\mathcal{M} = (A_i,\ {}_iM_j)$ is a pre-modulation over a field k of a valued quiver $(\mathcal{G},\mathcal{D},\Omega)$. For two representations $\mathcal{X} = (X_i,\ {}_i\varphi_j)$ and $\mathcal{Y} = (Y_i,\ {}_i\psi_j)$ in $rep(\mathcal{M})$,*

$$B(\mathbf{dim}\mathcal{X},\mathbf{dim}\mathcal{Y}) = dim_k Hom(\mathcal{X},\mathcal{Y}) - dim_k Ext^1(\mathcal{X},\mathcal{Y}).$$

Proof. Firstly, define a map:

$$\Delta_{\mathcal{X},\mathcal{Y}} : \bigoplus_{i\in\mathcal{G}} Hom_{A_i}(X_i,\ Y_i) \longrightarrow \bigoplus_{j\to i} Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, Y_i)$$

with $\Delta_{\mathcal{X},\mathcal{Y}}((\alpha_i)_{i\in\mathcal{G}}) = (\,{}_i\psi_j(\alpha_j\otimes 1) - \alpha_i\,{}_i\varphi_j)_{j\to i}$, for any

$$(\alpha_i)_{i\in\mathcal{G}} \in \bigoplus_{i\in\mathcal{G}} Hom_{A_i}(X_i,\ Y_i).$$

Due to the definition of morphisms between representations, it is easy to see that $Ker\Delta_{\mathcal{X},\mathcal{Y}} = Hom(\mathcal{X},\mathcal{Y})$.

Secondly, we can show that $Coker\Delta_{\mathcal{X},\mathcal{Y}} = Ext^1(\mathcal{X},\mathcal{Y})$ as follows.

Let $\Sigma = ({}_i\sigma_j)$ belong to $\bigoplus_{j\to i} Hom_{A_i}(X_j \otimes {}_jM_i, Y_i)$. Then we can get an extension $E(\Sigma) = (Y_j \oplus X_j, \begin{pmatrix} {}_i\psi_j & {}_i\sigma_j \\ 0 & {}_i\varphi_j \end{pmatrix})$ of representations $\mathcal{X}$ and $\mathcal{Y}$. Conversely, any extension of $\mathcal{X}$ and $\mathcal{Y}$ can be denoted as this form. So, there exists the one-one correspondence between all elements of $\bigoplus_{j\to i} Hom_{A_i}(X_j \otimes {}_jM_i, Y_i)$ and all of extensions of representations $\mathcal{X}$ and $\mathcal{Y}$.

Let $\Sigma' = ({}_i\sigma'_j)$ be another element in $\bigoplus_{j\to i} Hom_{A_i}(X_j \otimes {}_jM_i, Y_i)$ with its corresponding extension $E(\Sigma') = (Y_j \oplus X_j, \begin{pmatrix} {}_i\psi_j & {}_i\sigma'_j \\ 0 & {}_i\varphi_j \end{pmatrix})$.

Then $E(\Sigma)$ and $E(\Sigma')$ are equivalent if and only if there exists an invertible morphism τ of $rep(\mathcal{M})$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{Y} & \xrightarrow{i} & E(\Sigma) & \xrightarrow{p} & \mathcal{X} & \longrightarrow & 0 \\ & & \downarrow id & & \downarrow \tau & & \downarrow id & & \\ 0 & \longrightarrow & \mathcal{Y} & \xrightarrow{i'} & E(\Sigma') & \xrightarrow{p'} & \mathcal{X} & \longrightarrow & 0 \end{array}$$

commutes where i and i' are both embedding maps, p and p' are both projectors. It can be easily checked that τ must be the form of $\tau = \{\begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix} : i \in \mathcal{G}\}$ where τ_i is an A_i-homomorphism from X_i to Y_i, $i \in \mathcal{G}$. And, obviously, for any such τ_i, the given τ always makes this diagram to be commutative. Hence, $E(\Sigma)$ and $E(\Sigma')$ are equivalent if and only if there exists a morphism $\tau = \{\begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix} : i \in \mathcal{G}\}$ of $rep(\mathcal{M})$ for an A_i-homomorphism τ_i from X_i to Y_i, $i \in \mathcal{G}$.

Since the τ is admitted to be a morphism in $rep(\mathcal{M})$, the following square commutes:

$$\begin{array}{ccc} (Y_j \oplus X_j) \otimes {}_jM_i & \xrightarrow{\begin{pmatrix} {}_i\psi_j & {}_i\sigma_j \\ 0 & {}_i\varphi_j \end{pmatrix}} & Y_i \oplus X_i \\ \begin{pmatrix} 1 & \tau_j \\ 0 & 1 \end{pmatrix} \otimes 1 \downarrow & & \downarrow \begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix} \\ (Y_j \oplus X_j) \otimes {}_jM_i & \xrightarrow[\begin{pmatrix} {}_i\psi_j & {}_i\sigma'_j \\ 0 & {}_i\varphi_j \end{pmatrix}]{} & Y_i \oplus X_i \end{array}$$

Then

$$\begin{pmatrix} {}_i\psi_j & {}_i\sigma'_j \\ 0 & {}_i\varphi_j \end{pmatrix} (\begin{pmatrix} 1 & \tau_j \\ 0 & 1 \end{pmatrix} \otimes 1) = \begin{pmatrix} 1 & \tau_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}_i\psi_j & {}_i\sigma_j \\ 0 & {}_i\varphi_j \end{pmatrix}.$$

It follows that ${}_i\sigma_j + \tau_i \, {}_i\varphi_j = {}_i\psi_j(\tau_j \otimes 1) + {}_i\sigma'_j$, hence

$${}_i\sigma_j - {}_i\sigma'_j = {}_i\psi_j(\tau_j \otimes 1) - \tau_i \, {}_i\varphi_j.$$

It means that $\Sigma - \Sigma' \in Im(\Delta_{\mathcal{X},\mathcal{Y}})$ due to the definition of $\Delta_{\mathcal{X},\mathcal{Y}}$.

Hence, we get that $E(\Sigma)$ and $E(\Sigma')$ are equivalent if and only if $\Sigma - \Sigma' \in Im(\Delta_{\mathcal{X},\mathcal{Y}})$, which implies that $Cok(\Delta_{\mathcal{X},\mathcal{Y}}) \cong Ext^1(\mathcal{X},\mathcal{Y})$.

Next, we need the following lemma:

Lemma 2.3. *Suppose A and B are simple algebras over a field k and X, Y are both right A-modules, Z is a right B-module and M is a free B-A-bimodule. Then,*

(5) $\quad dim_kHom_A(X,Y) = (dim_kX \; dim_kY)/dim_kA,$

(6) $\quad dim_kHom_A(Z \otimes_B M, Y) = (rank_AM dim_kZ dim_kY)/dim_kB.$

Proof. Since A is a simple algebra, we have $A \cong M_n(D)$ for some positive integer n and D a divisible k-algebra.

It is easy to see that for any simple A-modules X and Y, we have $X \cong Y$, then $Hom_A(X,Y) \cong D$ and $dim_k Hom_A(X,Y) = dim_k D$; simultaneously, $dim_k X = ndim_k D$, $dim_k Y = ndim_k D$ and $dim_k A = n^2 dim_k D$. Therefore,

$$dim_k Hom_A(X,Y) = (dim_k X\ dim_k Y)/dim_k A = dim_k D.$$

In general, let A-modules X and Y be any A-modules which are not necessarily simple. Since A is simple, X and Y are semisimple A-modules. Let $X = X_1 \oplus \cdots \oplus X_s$ and $Y = Y_1 \oplus \cdots \oplus Y_t$.

Then, $dim_k Hom_A(X,Y) = dim_k Hom_A(X_1 \oplus \cdots \oplus X_s, Y_1 \oplus \cdots \oplus Y_t) = dim_k \oplus_{i,j} Hom_A(X_i, Y_j) = \oplus_{i,j} dim_k Hom_A(X_i, Y_j) = (st) dim_k D$.

On the other hand,

$$\begin{aligned}(dim_k X\ dim_k Y)/dim_k A &= (dim_k(X_1 \oplus \cdots \oplus X_s)\ dim_k(Y_1 \oplus \cdots \oplus Y_t))/dim_k A \\ &= (\oplus_{i=1}^{s} dim_k X_i)(\oplus_{i=1}^{t} dim_k Y_i)/dim_k A \\ &= ((sn) dim_k D (tn) dim_k D)/(n^2 dim_k D) = (st) dim_k D.\end{aligned}$$

Therefore, we get $dim_k Hom_A(X,Y) = (dim_k X\ dim_k Y)/dim_k A$.

According to the adjoint-isomorphism theorem,

$$Hom_A(Z \otimes_B M, Y) \cong Hom_B(Z, Hom_A(M,Y)).$$

Hence, due to (5), we have

$$\begin{aligned}dim_k Hom_A(Z \otimes_B M, Y) &= dim_k Hom_B(Z, Hom_A(M,Y)) \\ &= dim_k Z dim_k Hom_A(M,Y)/dim_k B \\ &= dim_k Z(dim_k M dim_k Y/dim_k A)/dim_k B \\ &= (rank_A M dim_k Z dim_k Y)/dim_k B.\end{aligned}$$

□

Now, return to the proof of the proposition:

By the definition of B, we have

$$\begin{aligned}B(\mathbf{dim}\mathcal{X}, \mathbf{dim}\mathcal{Y}) &= \sum_{i\in\mathcal{G}} dim_k A_i \frac{dim_k X_i}{dim_k A_i} \frac{dim_k Y_i}{dim_k A_i} \\ &- \sum_{j\to i} d_{ji} dim_k A_i \frac{dim_k X_j}{dim_k A_j} \frac{dim_k Y_i}{dim_k A_i} \\ &= \sum_{i\in\mathcal{G}} (dim_k X_i dim_k Y_i)/dim_k A_i \\ &- \sum_{j\to i} (rank_{A_i}({}_iM_j) dim_k X_j dim_k Y_i)/dim_k A_j \\ &= \sum_{i\in\mathcal{G}} dim_k Hom_{A_i}(X_i, Y_i) - \sum_{j\to i} dim_k Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, Y_i)\end{aligned}$$

$$= dim_k \bigoplus_{i \in \mathcal{G}} Hom_{A_i}(X_i, Y_i) - dim_k \bigoplus_{j \to i} Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, Y_i)$$
$$= dim_k Ker \Delta_{\mathcal{X},\mathcal{Y}} - dim_k Coker \Delta_{\mathcal{X},\mathcal{Y}}$$
$$= dim_k Hom(\mathcal{X}, \mathcal{Y}) - dim_k Ext^1(\mathcal{X}, \mathcal{Y}). \qquad \square$$

According to the discussion above before this theorem, we leave as a question as follows.

Problem 2.1. *Characterize the relationship between the Euler characteristic and the Euler form and that between their corresponding quadratic forms.*

3. Berstein-Gelfand-Ponomarev theory in category of pre-modulations

3.1 Reflection functors of a pre-modulation

A k-pre-modulation $\mathcal{M} = (A_i, {}_iM_j)$ of a valued graph $(\mathcal{G}, \mathcal{D})$ is defined in [16] as a set of artinian k-algebras $\{A_i\}_{i \in \mathcal{G}}$, together with a set $\{{}_iM_j\}_{(i,j) \in \mathcal{G} \times \mathcal{G}}$ of finitely generated free unital A_i-A_j-bimodules ${}_iM_j$ such that $rank({}_iM_j)_{A_j} = d_{ij}$ and $rank_{A_i}({}_iM_j) = d_{ji}$.

Assume that $\mathcal{M} = (A_i, {}_iM_j)$ is a k-pre-modulation over a connected valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ with the admissible sequence of sinks $\{1, 2, \ldots, n\}$, that is, $(\mathcal{G}, \mathcal{D}, \Omega)$ has no oriented cycles. Let $dim_k A_i = f_i$ which is finite by the definition for any $i \in \mathcal{G}$, and let $rank({}_iM_j)_{A_j} = d_{ij}$ and $rank_{A_i}({}_iM_j) = d_{ji}$. Then $d_{ji} f_i = dim_k {}_iM_j = d_{ij} f_j$.

Denote by $\underline{A_l}$ the representation of $rep(\mathcal{M})$ corresponding to the vertex $l \in \mathcal{G}$ defined by $\underline{A_l} = (X_i, {}_i\varphi_j)$ where $X_i = \begin{cases} A_l, & \text{if } i = l \\ 0, & \text{if } i \neq l \end{cases}$ and ${}_i\varphi_j = 0$ for all $i \to j$. All $\underline{A_l}$ $(l \in \mathcal{G})$ are called the *elementary representations* of $rep(\mathcal{M})$.

Since A_l $(l \in \mathcal{G})$ is a simple algebra, let $dim_k A_l = s_l^2$ for a positive integer s_l. As A_l-module, A_l can be decomposed into a direct sum of s_l simple A_l-modules which are isomorphic each other, that is, every A_l has a unique simple A_l-submodule under isomorphism. Equivalently, every $\underline{A_l}$ can be decomposed into a direct sum of some simple representations which are isomorphic each other, that is, we have:

Fact 3.1. *For any vertex $l \in \mathcal{G}$, $\underline{A_l}$ in the category $rep(\mathcal{M})$ has a unique simple direct summand under isomorphism.*

Lemma 3.1. *$\underline{A_1}$ is projective and $\underline{A_n}$ is injective in $rep(\mathcal{M})$.*

Proof. Since $\underline{A_1}$ is non-zero only in the first coordinate, suppose there is the diagram:
where $\beta_i = 0$ for any $i \neq 1$ and the row sequence is exact. Thus, it follows that:

$$\begin{array}{ccccc} & & \underline{A}_1 & & \\ & & \downarrow{\scriptstyle \underline{\beta} = (\beta_i)} & & \\ \mathcal{X} & \xrightarrow{\pi} & \mathcal{Y} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} & & A_1 & & \\ & & \downarrow{\scriptstyle \beta_1} & & \\ X_1 & \xrightarrow{\pi_1} & Y_1 & \longrightarrow & 0 \end{array}$$

But, since A_1 is a simple algebra, A_1 is projective as A_1-module. So, there is γ_1 such that the following diagram commutes:

$$\begin{array}{ccccc} & & A_1 & & \\ & \swarrow{\scriptstyle \gamma_1} & \downarrow{\scriptstyle \beta_1} & & \\ X_1 & \xrightarrow{\pi_1} & Y_1 & \longrightarrow & 0 \end{array}$$

Hence, the first diagram can be completed by $\underline{\gamma} = (\gamma_i)$ with $\gamma_i = 0$ for $i \neq 1$, that is, the following diagram commutes:

Moreover, it is necessary to explain that $\underline{\gamma}$ is a morphism in $rep(\mathcal{M})$. Indeed, since 1 is a sink, there exists no arrow $1 \to i$ for any i. If there is an arrow $j \to 1$ for some j, the following diagram is always commutative:

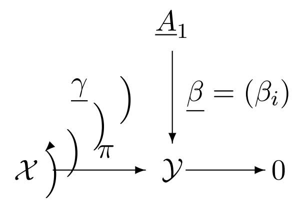
$$\begin{array}{ccc} 0 \otimes_{A_j} {}_jM_1 & \xrightarrow{0} & A_1 \\ \downarrow{\scriptstyle 0 \otimes id} & & \downarrow{\scriptstyle \gamma_1} \\ X_j \otimes_{A_j} {}_jM_1 & \xrightarrow{{}_j\psi_1} & X_1 \end{array}$$

From this diagram and $A_j = 0$, $\gamma_j = 0$ for any $j \neq 1$, it follows that $\underline{\gamma}$ is a morphism in $rep(\mathcal{M})$.

Therefore $\underline{A}_1$ is projective in $rep(\mathcal{M})$.

Dually, it can be proved similarly that $\underline{A}_n$ is injective in $rep(\mathcal{M})$ since A_n is injective as A_n-module. □

Corollary 3.1. *In the category $rep(\mathcal{M})$, the unique simple direct summand S_1 under isomorphism of $\underline{A}_1$ is projective and that of $\underline{A}_n$ is injective.*



Corollary 3.2. *For* $i,\, j \in \mathcal{G}$, $dim_k Ext^1(\underline{A}_i,\, \underline{A}_j) = d_{ij} f_j$, $dim_k Ext^1(\underline{A}_i,\, \underline{A}_j) = d_{ji} f_i$.

Proof. If $i \to j$, by the definition, we have

$$B(\mathbf{dim}\, \underline{A}_i,\, \mathbf{dim}\, \underline{A}_j) = -d_{ij} f_j.$$

Since $Hom(\underline{A}_i,\, \underline{A}_j) = 0$, by the Theorem 2.2 we deduce that

$$dim_k Ext^1(\underline{A}_i,\, \underline{A}_j) = -B(\mathbf{dim}\, \underline{A}_i,\, \mathbf{dim}\, \underline{A}_j) = d_{ij} f_j,$$

and hence the first equality follows. On the other hand, if there is no arrow $i \to j$, the first equality are trivial as $0 = 0$.

The second equality is an immediate consequence of the fact that $d_{ij} f_j = d_{ji} f_i$. □

Given any vertex $k \in \mathcal{G}$ of a valued quiver $(\mathcal{G},\, \mathcal{D},\, \Omega)$, we define a reflection $\delta_k : \mathbb{Q}^{\mathcal{G}} \to \mathbb{Q}^{\mathcal{G}}$ satisfying that if $\underline{x} = (x_i)_{i \in \mathcal{G}}$, then $\delta_k \underline{x} = \underline{y} = (y_i)_{i \in \mathcal{G}}$ is given by:

$$\begin{aligned} y_i &= x_i, \forall i \neq k, \\ y_k &= -x_k + \sum_{i \in \mathcal{G}} d_{ik} x_i. \end{aligned}$$

Corollary 3.3. (i) *Let $\mathcal{X}$ be a representation with no direct summand isomorphic to the unique simple direct summand of $\underline{A}_1$, then*

$$(\delta_1(\mathbf{dim}\mathcal{X}))_1 = \frac{dim_k Ext^1(\mathcal{X},\, \underline{A}_1)}{dim_k A_1}.$$

(ii) *Let $\mathcal{X}$ be a representation with no direct summand isomorphic to $\underline{A}_n$, then*

$$(\delta_n(\mathbf{dim}\mathcal{X}))_n = \frac{dim_k Ext^1(\underline{A}_n,\, \mathcal{X})}{dim_k A_n}.$$

Proof. (i) If $\mathcal{X}$ has no direct summand isomorphic to the unique simple direct summand of $\underline{A}_1$, then $Hom(\mathcal{X},\, \underline{A}_1) = 0$. Hence

$$B(\mathbf{dim}\mathcal{X},\, \mathbf{dim}\, \underline{A}_1) = -dim_k Ext^1(\mathcal{X},\, \underline{A}_1).$$

On the other hand,

$$\begin{aligned} B(\mathbf{dim}\mathcal{X}, \mathbf{dim}\ \underline{A}_1) &= f_1\frac{dim_k X_1}{f_1} - \sum_{i\to 1} d_{i1} f_1 \frac{dim_k X_i}{f_i} \\ &= -f_1(-\frac{dim_k X_1}{f_1} + \sum_{i\in\mathcal{G}} d_{i1}\frac{dim_k X_i}{f_i}) \\ &= -f_1(\delta_1(\mathbf{dim}\mathcal{X}))_1, \end{aligned}$$

where the second equality uses the fact that the vertex 1 is a sink.

(ii) can be proved dually. □

Now, to any sink (respectively, source) k of the graph $\mathcal{G}$, we shall associate a functor Δ_k^+ (respectively, Δ_k^-) of $rep(\mathcal{M}, \Omega)$ into $rep(\mathcal{M}, \delta_k\Omega)$, which are called *the reflection functors of the pre-modulation* $\mathcal{M} = (A_i, {}_iM_j)$.

In accordance with our convention, 1 is a sink, and n a source of Ω, thus we shall content ourselves with defining Δ_1^+ and Δ_n^-.

Let $\mathcal{X} = (X_i, {}_i\varphi_j)$ be an object of $rep(\mathcal{M}, \Omega)$, we recall that ${}_i\varphi_j : X_j \otimes_{A_j} {}_jM_i \to X_i$ is an A_i-map. We can attach to it an A_j-map $\overline{{}_i\varphi_j} : X_j \to X_i \otimes_{A_i} {}_iM_j$ in the following way.

By the adjoint isomorphism theorem, we have

$$Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i, X_i) \cong Hom_{A_j}(X_j, Hom_{A_i}({}_jM_i, X_i)).$$

Lemma 3.2. *Let A be a semisimple algebra and B another finite-dimensional algebra over a field k, X be right an A-module and M a left-right free B-A-bimodule with basis of a finite number of generators. Then, as right B-modules,*

$$Hom_A(M, X) \cong X \otimes_A Hom_A(M, A).$$

Proof. Define $\pi : X \otimes_A Hom_A(M, A) \to Hom_A(M, X)$ satisfying

$$\pi(\sum_i x_i \otimes f_i)(m) = x_i f_i(m),$$

for all $x_i \in X, f_i \in Hom_A(M, A)$ and $m \in M$. Then, π is a right B-module homomorphism. In fact, $\pi((\sum_i x_i \otimes f_i)b)(m) = \pi(\sum_i x_i \otimes f_i b)(m) = \sum_i x_i(f_i b)(m) = \sum_i x_i f_i(bm) = \pi(\sum_i x_i \otimes f_i)(bm) = (\pi(\sum_i x_i \otimes f_i)b)(m)$,
it follows that $\pi((\sum_i x_i \otimes f_i)b) = \pi(\sum_i x_i \otimes f_i)b$.

Let $\{\varepsilon_1, \ldots, \varepsilon_s\}$ be the basis of M as right A-module. Define f_i be from M to A satisfying $f_i(\varepsilon_j) = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}$. Then f_i can be expended into a right A-homomorphism and $\{f_1, \ldots, f_s\}$ is the basis of $Hom_A(M, A)$ as left free A-module. For any $g \in Hom_A(M, X)$, let $\chi = \sum_{i=1}^s g(\varepsilon_i) \otimes f_i$, then $\chi \in X \otimes_A Hom_A(M, A)$ satisfying $\pi(\chi) = g$. Therefore, π is surjective.

Write $M_A \cong \oplus_\lambda A_A$, thus, we get the following right A-isomorphisms:

$Hom_A(M, X) \cong Hom_A(\oplus_\lambda A_A, X) \cong \oplus_\lambda Hom_A(A_A, X) \cong \oplus_\lambda X_A \cong \oplus_\lambda X_A \otimes A \cong X_A \otimes (\oplus_\lambda A_A) \cong X_A \otimes Hom(\oplus_\lambda A_A, A) \cong X_A \otimes Hom(M, A)$.
Then, $dim_k(Hom_A(M, X)) = dim_k(X \otimes_A Hom_A(M, A))$.

Hence from the fact that the surjective right B-module homomorphism π is also a surjective k-linear map of spaces, we know that π is an isomorphism. $\square$

Dealing with finite-dimensional modules, by Lemma 3.2, we get that

$$Hom_{A_i}({}_jM_i,\, X_i) \cong X_i \otimes_{A_i} Hom_{A_i}({}_jM_i,\, A_i) \cong X_i \otimes_{A_i} {}_iM_j,$$

giving a canonical isomorphism

$$Hom_{A_i}(X_j \otimes_{A_j} {}_jM_i,\, X_i) \cong Hom_{A_j}(X_j,\, X_i \otimes_{A_i} {}_iM_j). \tag{7}$$

Thus to ${}_i\varphi_j$ there corresponds $\overline{{}_i\varphi_j} : X_j \to X_i \otimes_{A_i} {}_iM_j$ which will be referred to as the *adjoint* of ${}_i\varphi_j$. Now we can define $\Delta_1^+\mathcal{X} = \mathcal{Y} = (Y_i,\, {}_i\psi_j)$ as follows:

If $j \neq 1$, take $Y_j = X_j$, and ${}_i\psi_j = {}_i\varphi_j$.

If $j = 1$,for every $i \in \mathcal{G}$ such $\exists\, i \to 1$, we have a mapping ${}_1\varphi_i : X_i \otimes_{A_i} {}_iM_1 \to X_1$. Let $\varphi_1 = \bigoplus_{j\to 1} {}_1\varphi_j : \bigoplus_{j\to 1} X_j \otimes_{A_j} {}_jM_1 \to X_1$. Let $Y_1 = Ker\varphi_1$, κ_1 the embedding map from $Ker\varphi_1$ to $\bigoplus_{j\to 1} X_j \otimes_{A_j} {}_jM_1$ and ${}_i\kappa_1 = \pi_i\kappa_1 : Y_1 \to X_i \otimes_{A_i} {}_iM_1 = Y_i \otimes_{A_i} {}_iM_1$ (where π_i is the canonical projection if there exists an arrow $i \to 1$):

$$\begin{array}{ccccccc} & & & & X_i \otimes_{A_i} {}_iM_1 & & \\ & & & {}_i\kappa_1 \nearrow & \uparrow \pi_i & \searrow {}_1\varphi_i & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{\kappa_1} & \bigoplus_{j\to 1}(X_j \otimes_{A_j} {}_jM_1) & \xrightarrow{\varphi_1} & X_1 \end{array}$$

According to (7), we put ${}_i\psi_1 = \overline{{}_i\kappa_1} : Y_1 \otimes_{A_1} {}_1M_i \to X_i = Y_i$. Thus we have defined $\Delta_1^+\mathcal{X} = \mathcal{Y}$ in $rep(\mathcal{M},\, \delta_1\Omega)$.

If $\alpha : \mathcal{X} \to \mathcal{X}'$ is a morphism of $rep(\mathcal{M},\, \Omega)$, $\beta = \Delta_1^+\alpha$ is defined as follows: if $j \neq 1$, take $\beta_j = \alpha_j$ and $\beta_1 : Y_1 \to Y_1'$ is the restriction to Y_1 of the mapping

$$\bigoplus_{i\to 1}(\alpha_i \otimes 1) : \bigoplus_{i\to 1} X_i \otimes_{A_i} {}_iM_1 \to \bigoplus_{i\to 1} X_i' \otimes_{A_i} {}_iM_1.$$

If $\exists$ arrow $i \to 1$ in Ω, then

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y_1 & \xrightarrow{\kappa_1} & \bigoplus_{j\to 1} X_j \otimes_{A_j} {}_jM_1 & \xrightarrow{\varphi_1} & X_1 \\ & & \downarrow \beta_1 & & \downarrow \bigoplus_{j\to 1}(\alpha_j \otimes 1) & & \downarrow \alpha_1 \\ 0 & \longrightarrow & Y_1' & \xrightarrow{\kappa_1'} & \bigoplus_{j\to 1} X_j' \otimes_{A_j} {}_jM_1 & \xrightarrow{\varphi_1'} & X_1' \end{array}$$

Thus,

$$\begin{array}{ccc} Y_1 & \xrightarrow{{}_i\kappa_1} & Y_i \otimes_{A_i} {}_iM_1 \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_i\otimes 1} \\ Y_1' & \xrightarrow{{}_i\kappa_1'} & Y_i' \otimes_{A_i} {}_iM_1 \end{array}$$

It follows that

$$\begin{array}{ccc} Y_1 \otimes_{A_1} {}_1M_i & \xrightarrow{{}_i\psi_1} & Y_i \\ \downarrow{\scriptstyle \beta_1\otimes 1} & & \downarrow{\scriptstyle \beta_i} \\ Y_1' \otimes_{A_1} {}_1M_i & \xrightarrow{{}_i\psi_1'} & Y_i' \end{array}$$

And, if $i \neq 1$, $\beta_i = \alpha_i$ which are morphisms in $rep(\mathcal{M}, \Omega)$.

Hence, all β_i are morphisms in $rep(\mathcal{M}, \delta_1\Omega)$. Thus, β is a morphism of $rep(\mathcal{M}, \delta_1\Omega)$.

In summary, Δ_1^+ is a functor from $rep(\mathcal{M}, \Omega)$ to $rep(\mathcal{M}, \delta_1\Omega)$.

Dually, $\Delta_n^-\mathcal{X} = \mathcal{Y} = (Y_i, {}_i\psi_j)$ is the object of $rep(\mathcal{M}, \delta_n\Omega)$ defined as follows:

(i) If $i \neq n$, take $Y_i = X_i$, and ${}_i\psi_j = {}_i\varphi_j$; (ii) If $i = n$, let Y_n be the cokernel in the diagram:

$$\begin{array}{ccccccc} & & & X_j \otimes_{A_j} {}_jM_n & & & \\ & \nearrow{\scriptstyle {}_j\bar{\varphi}_n} & & \uparrow{\scriptstyle \pi_j}\ \downarrow{\scriptstyle \iota_j} & \searrow{\scriptstyle {}_n\eta_i} & & \\ X_n & \xrightarrow[({}_j\bar{\varphi}_n)]{} & & \oplus_{n\to j}(X_j \otimes_{A_j} {}_jM_n) & \xrightarrow[\eta_n]{} & Y_n & \longrightarrow 0 \end{array}$$

and ${}_n\psi_j = {}_n\eta_j$.

For a morphism $\alpha : \mathcal{X} \to \mathcal{X}'$, we define $\beta = \Delta_n^-\alpha$ by letting $\beta_i = \alpha_i$ for $i \neq n$, while $\beta_n : Y_n \to Y_n'$ is the mapping induced on the cokernels by

$$\bigoplus_{n\to j}(\alpha_j \otimes 1) : \ \bigoplus_{n\to j} X_j \otimes_{A_j} {}_jM_n \to \bigoplus_{n\to j} X_j' \otimes_{A_j} {}_jM_n.$$

In summary, Δ_n^- is a functor from $rep(\mathcal{M}, \Omega)$ to $rep(\mathcal{M}, \delta_n\Omega)$.

As a direct consequence of the definition, Δ_1^+ preserves monomorphisms, while Δ_n^- preserves epimorphisms, and both preserve finite direct sums.

3.2 Construction of indecomposable projectives/injective representations

In this part, we use reflection functors to construct indecomposable projective/injective representations of a hereditary algebra.

Lemma 3.3. *Let $(\mathcal{G}, \mathcal{D}, \Omega)$ be a connected valued quiver with admissible orientation Ω and $\mathcal{M}$ be a k-pre-modulation. Then for every representation $\mathcal{X}$ of $\mathcal{M}$:*

(i) $\mathcal{X} \cong \Delta_1^- \Delta_1^+ \mathcal{X} \oplus \mathcal{P}$, where $\mathcal{P} = (P_i, {}_i\pi_j)$ with $P_i = 0$ if $i \neq 1$ and P_1 is a (semisimple) A_1-module. Thus if $\mathcal{X}$ is indecomposable, either (a) $\mathcal{X} \cong \mathcal{P}$ (equivalently, $\Delta_1^+ \mathcal{X} = 0$) in which case $\mathcal{P}$ is the unique simple direct summand of $\underline{A}_1$ under isomorphism or (b) $\mathcal{X} \cong \Delta_1^- \Delta_1^+ \mathcal{X}$ (equivalently, $\Delta_1^+ \mathcal{X} \neq 0$) in which case End$(\Delta_1^+ \mathcal{X}) \cong$End$(\mathcal{X})$ and thus $\Delta_1^+ \mathcal{X}$ is indecomposable and $\mathbf{dim}(\Delta_1^+ \mathcal{X}) = \delta_1(\mathbf{dim}\mathcal{X})$;

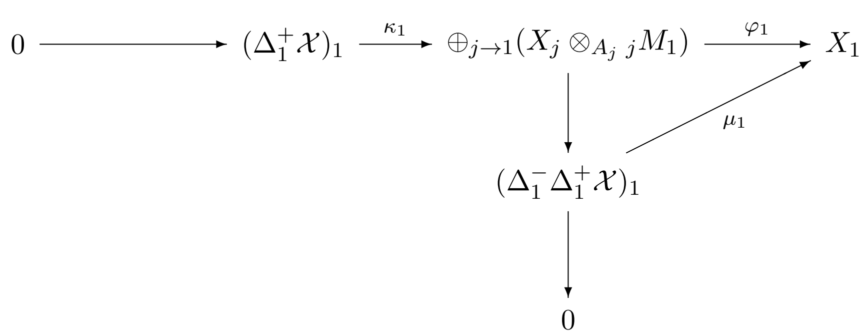
(ii) $\mathcal{X} \cong \Delta_n^+ \Delta_n^- \mathcal{X} \oplus \mathcal{I}$, where $\mathcal{I} = (I_i, {}_i\tau_j)$ with $I_i = 0$, if $i \neq n$ and I_n is a (semisimple) A_n-module. Thus, if $\mathcal{X}$ is indecomposable, either (a) $\mathcal{X} \cong \mathcal{I}$ (equivalently, $\Delta_n^- \mathcal{X} = 0$) in which case $\mathcal{I}$ is the unique simple direct summand of $\underline{A}_n$ under isomorphism or (b) $\mathcal{X} \cong \Delta_n^+ \Delta_n^- \mathcal{X}$ (equivalently, $\Delta_n^- \mathcal{X} \neq 0$) in which case End$(\Delta_n^- \mathcal{X}) \cong$End$(\mathcal{X})$ and thus $\Delta_n^- \mathcal{X}$ is indecomposable and $\mathbf{dim}(\Delta_n^- \mathcal{X}) = \delta_n(\mathbf{dim}\mathcal{X})$.

Proof. Firstly, We give the prove of (i).

Since $\mathcal{X} \in rep(\mathcal{M}, \Omega)$ and 1 is a sink in Ω, $\mathcal{Y} = \Delta_1^+ \mathcal{X} \in rep(\mathcal{M}, \delta_1\Omega)$ and 1 is a source in $\delta_1\Omega$. Then, by the definition of Δ_1^-, we have

$$(\Delta_1^- \Delta_1^+ \mathcal{X})_1 = cokY_1 = cok(ker\varphi_1) = Im\varphi_1 \overset{\mu_1}{\hookrightarrow} X_1.$$

Thus, we obtain the following diagram in the first coordinate from the construction of $\Delta_1^- \Delta_1^+ \mathcal{X}$:



Due to the above mention, $(\Delta_1^- \Delta_1^+ \mathcal{X})_1$ can be seen as an A_1-submodule of X_1. But, since A_1 is a simple algebra, all its modules are projective and then $(\Delta_1^- \Delta_1^+ \mathcal{X})_1$ is a direct summand of X_1 as an A_1-module. Let $X_1 \cong (\Delta_1^- \Delta_1^+ \mathcal{X})_1 \oplus P_1$ where P_1 is a semisimple A_1-module. Thus, by the definition of $\Delta_1^- \Delta_1^+ \mathcal{X}$, $\mathcal{X} \cong \Delta_1^- \Delta_1^+ \mathcal{X} \oplus \mathcal{P}$, where $\mathcal{P} = (P_i, {}_i\pi_j)$ with $P_i = 0$ if $i \neq 1$.

Hence, if $\mathcal{X}$ is indecomposable, we have either (a) $\mathcal{X} \cong \mathcal{P}$, equivalently, $\Delta_1^- \Delta_1^+ \mathcal{X} = 0$ or (b) $\mathcal{X} \cong \Delta_1^- \Delta_1^+ \mathcal{X}$, equivalently, $\mathcal{P} = 0$.

In the case (a), if $\Delta_1^+ \mathcal{X} = 0$, clearly $\Delta_1^- \Delta_1^+ \mathcal{X} = 0$; conversely, if $\Delta_1^- \Delta_1^+ \mathcal{X} = 0$, then in the above diagram all $X_j = 0$ $(j \neq 1)$ which means $(\Delta_1^+ \mathcal{X})_1 = 0$ and it follows that $\Delta_1^+ \mathcal{X} = 0$. Therefore, $\mathcal{X} \cong \mathcal{P}$ is equivalent to $\Delta_1^+ \mathcal{X} = 0$.

Moreover, in the case (b), $\mathcal{X} \cong \Delta_1^- \Delta_1^+ \mathcal{X}$ is equivalent to $\Delta_1^+ \mathcal{X} \neq 0$. Then, $\Delta_1^+ \Delta_1^- \Delta_1^+ \mathcal{X} \cong \Delta_1^+ \mathcal{X}$ and φ_1 is surjective.

From X_1 to get $(\Delta_1^+ \mathcal{X})_1$, we have the following:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\Delta_1^+\mathcal{X})_1 & \xrightarrow{\kappa_1} & \bigoplus_{j\to 1} X_j \otimes_{A_j} {}_jM_1 & \xrightarrow{\varphi_1} & X_1 \\
 & & \downarrow{\scriptstyle \widetilde{f}_1} & & \downarrow{\scriptstyle \bigoplus_{j\to 1}(f_j\otimes 1)} & & \downarrow{\scriptstyle f_1} \\
0 & \longrightarrow & (\Delta_1^+\mathcal{X})_1 & \xrightarrow{\kappa_1} & \bigoplus_{j\to 1} X_j \otimes_{A_j} {}_jM_1 & \xrightarrow{\varphi_1} & X_1
\end{array}$$

From $(\Delta_1^+ \mathcal{X})_1$ to get $(\Delta_1^- \Delta_1^+ \mathcal{X})_1$, we have the following:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\Delta_1^+\mathcal{X})_1 & \xrightarrow{\kappa_1} & \bigoplus_{j\to 1} X_j \otimes_{A_j} {}_jM_1 & \xrightarrow{\varphi_1} & (\Delta_1^-\Delta_1^+\mathcal{X})_1 \\
 & & \downarrow{\scriptstyle \widetilde{f}_1} & & \downarrow{\scriptstyle \bigoplus_{j\to 1}(f_j\otimes 1)} & & \downarrow{\scriptstyle \widetilde{\widetilde{f}}_1} \\
0 & \longrightarrow & (\Delta_1^+\mathcal{X})_1 & \xrightarrow{\kappa_1} & \bigoplus_{j\to 1} X_j \otimes_{A_j} {}_jM_1 & \xrightarrow{\varphi_1} & (\Delta_1^-\Delta_1^+\mathcal{X})_1
\end{array}$$

where $(\Delta_1^- \Delta_1^+ \mathcal{X})_1 = Im\varphi_1$ is embedded into X_1 by μ_1. But, φ_1 is surjective in the case (b), $(\Delta_1^- \Delta_1^+ \mathcal{X})_1 = Im\varphi_1$ is isomorphic to X_1. So, f_1 and $\widetilde{\widetilde{f}}_1$ are one-one correspondence via $\Delta_1^- \Delta_1^+$. Therefore, in the series of maps:

$$End(\mathcal{X}) \xrightarrow{\Delta_1^+} End(\Delta_1^+ \mathcal{X}) \xrightarrow{\Delta_1^-} End(\Delta_1^- \Delta_1^+ \mathcal{X}) \xrightarrow{\Delta_1^+} End(\Delta_1^+ \Delta_1^- \Delta_1^+ \mathcal{X}),$$

we get $End(\mathcal{X}) \overset{\Delta_1^- \Delta_1^+}{\cong} End(\Delta_1^- \Delta_1^+ \mathcal{X})$ and similarly,

$$End(\Delta_1^+ \mathcal{X}) \overset{\Delta_1^+ \Delta_1^-}{\cong} End(\Delta_1^+ \Delta_1^- \Delta_1^+ \mathcal{X}).$$

From them, it follows that $End(\Delta_1^+ \mathcal{X}) \overset{\Delta_1^-}{\cong} End(\Delta_1^- \Delta_1^+ \mathcal{X})$, and then $End(\mathcal{X}) \overset{\Delta_1^+}{\cong} End(\Delta_1^+ \mathcal{X})$. Naturally, the above isomorphisms still hold under the meaning of the endomorphism algebras of these representations.

Now the indecomposability of $\mathcal{X}$ implies that $End(\mathcal{X})$ is local, hence so is $\mathrm{End}(\Delta_1^+ \mathcal{X})$ through the isomorphism and then $\Delta_1^+ \mathcal{X}$ is indecomposable.

Lastly, we verify that $\mathbf{dim}(\Delta_1^+ \mathcal{X}) = \delta_1(\mathbf{dim}\mathcal{X})$ in the case (b). By the definitions of $\Delta_1^+ \mathcal{X}$ and δ_1, it is enough to show that $\frac{dim_k (\Delta_1^+ \mathcal{X})_1}{dim_k A_1} = (\delta_1(\mathbf{dim}\mathcal{X}))_1$.

On the one hand,

$$(\delta_1(\mathbf{dim}\mathcal{X}))_1 = -\frac{dim_k X_1}{dim_k A_1} + \sum_{i\to 1} d_{i1}\frac{dim_k X_i}{dim_k A_i}.$$

On the other hand, in this case, φ_1 is surjective, then we have the short exact sequence

$$0 \longrightarrow (\Delta_1^+\mathcal{X})_1 \longrightarrow \bigoplus_{i\to 1}(X_i \otimes_{A_i} {}_iM_1) \longrightarrow X_1 \longrightarrow 0$$

which gives $dim_k(\Delta_1^+\mathcal{X})_1 = \sum_{i\to 1} dim_k(X_i \otimes_{A_i} {}_iM_1) - dim_k X_1$. Thus,

$$\frac{dim_k(\Delta_1^+\mathcal{X})_1}{dim_k A_1} = -\frac{dim_k X_1}{dim_k A_1} + \sum_{i\to 1}\frac{dim_k(X_i \otimes_{A_i} {}_iM_1)}{dim_k A_1}.$$

Hence, it is enough for us to prove that for any arrow $i \to 1$, $\frac{dim_k(X_i\otimes {}_iM_1)}{dim_k A_1} = d_{i1}\frac{dim_k X_i}{dim_k A_i}$.

In fact, $dim_k({}_iM_1) = d_{i1} dim_k A_1$, so $d_{i1}\frac{dim_k X_i}{dim_k A_i} = \frac{dim_k({}_iM_1)}{dim_k A_1}\frac{dim_k X_i}{dim_k A_i}$. Since A_i is a simple algebra over k and X_i is its right module, there is $dim_k A_i = s_i^2$ for some positive integer s_i, $X_i = W_1 \oplus \ldots \oplus W_t$ for some right A_i-simple submodules $W_1, \ldots, W_t$ and $dim_k W_i = s_i$ for all i. And, ${}_iM_1$ is a left free A_i-module with d_{1i} the rank of a basis which we write $d_{1i} = rank_{A_i}({}_iM_1)$. Then, ${}_iM_1 = \oplus\, d_{1i}A_i$ and

$$X_i \otimes_{A_i} {}_iM_1 = (\oplus_{j=1}^t W_j)\otimes_{A_i}(\oplus\, d_{1i}A_i) = \oplus\, d_{1i}(\oplus_{j=1}^t W_j \otimes_{A_i} A_i) = \oplus\, d_{1i}(\oplus_{j=1}^t W_j).$$

Thus, $dim_k(X_i \otimes_{A_i} {}_iM_1) = d_{1i} t s_i$.

On the other hand, $(dim_k({}_iM_1) dim_k X_i)/dim_k A_i = rank_{A_i}({}_iM_1) dim_k X_i = d_{1i} t s_i$.

Hence, $\frac{dim_k(X_i\otimes {}_iM_1)}{dim_k A_1} = d_{i1}\frac{dim_k X_i}{dim_k A_i}$, then $\frac{dim_k(\Delta_1^+\mathcal{X})_1}{dim_k A_1} = (\delta_1(\mathbf{dim}\mathcal{X}))_1$. It means that $\mathbf{dim}(\Delta_1^+\mathcal{X}) = \delta_1(\mathbf{dim}\mathcal{X})$.

The proof of (ii) can be given dually by considering the following diagram:

$$\begin{array}{ccccccc}
X_n & \xrightarrow{({}_i\bar{\varphi}_n)} & \oplus_{n\to i}(X_i \otimes_{A_i} {}_iM_1) & \xrightarrow{\eta_n} & (\Delta_n^-\mathcal{X})_n & \longrightarrow & 0 \\
 & & \uparrow & & & & \\
 & & (\Delta_n^+\Delta_n^-\mathcal{X})_n = ker({}_i\bar{\varphi}_n) & & & & \\
 & & \uparrow & & & & \\
 & & 0 & & & &
\end{array}$$

The direct sum $\mathcal{X} \cong \Delta_n^+\Delta_n^-\mathcal{X} \oplus \mathcal{I}$ is from the fact A_n is a simple algebra and then X_n is projective as A_n-module. The further discussion is similar in dual. □

Theorem 3.2. (*i*) *The full subcategory* $rep^{(1)}(\mathcal{M}, \Omega)$ *of all representations in* $rep(\mathcal{M}, \Omega)$ *with no direct summand isomorphic to the unique projective simple direct summand* (*under isomorphism*) *of* $\underline{A}_1$ *is equivalent to the full subcategory* $rep_{(1)}(\mathcal{M}, \Omega)$ *of all representations in* $rep(\mathcal{M}, \delta_1\Omega)$ *with no direct summand isomorphic to the unique injective simple direct summand* (*under isomorphism*) *of* $\underline{A}_1$.

(*ii*) *The full subcategory* $rep_{(n)}(\mathcal{M}, \Omega)$ *of all representations in* $rep(\mathcal{M}, \Omega)$ *with no direct summand isomorphic to the unique injective simple direct summand* (*under isomorphism*) *of* $\underline{A}_n$ *is equivalent to the full subcategory* $rep^{(n)}(\mathcal{M}, \Omega)$ *of all representations in* $rep(\mathcal{M}, \delta_n\Omega)$ *with no direct summand isomorphic to the unique projective simple direct summand (under isomorphism) of* $\underline{A}_n$.

Proof. By Lemma 3.1, $\underline{A}_1$ is projective and $\underline{A}_n$ is injective in $rep(\mathcal{M}, \Omega)$. Then by the definitions of δ_1 and δ_n, in $rep(\mathcal{M}, \delta_1\Omega)$ and $rep(\mathcal{M}, \delta_n\Omega)$ respectively, $\underline{A}_1$ is injective and $\underline{A}_n$ is projective. Then, so are their direct summands respectively.

Any $\mathcal{X} \in rep(M, \Omega)$ can be written as $\mathcal{X} = \mathcal{P}^{(1)} + \ldots + \mathcal{P}^{(s)} + \mathcal{X}^{(1)} + \ldots + \mathcal{X}^{(t)}$ where all $\mathcal{P}^{(i)}$are indecomposable and $\Delta_1^+\mathcal{P}^{(i)} = 0$, all $\mathcal{X}^{(j)}$are indecomposable and $\Delta_1^+\mathcal{X}^{(j)} \neq 0$. Then by Lemma 3.3, $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(s)}$ are all the (possible) direct summands of X isomorphic to the unique simple direct summand of $\underline{A}_1$, and $\Delta_1^-\Delta_1^+\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}^{(1)} + \ldots + \Delta_1^-\Delta_1^+\mathcal{X}^{(t)} = \mathcal{X}^{(1)} + \ldots + \mathcal{X}^{(t)}$. Therefore, $\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}$ if and only if X has no direct summands isomorphic to the unique simple direct summand of $\underline{A}_1$. It means $\mathcal{X} \in rep^{(1)}(M, \Omega)$ if and only if $\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}$. Moreover, through the functors Δ_1^-, Δ_1^+ in $rep^{(1)}(M, \Omega)$, for any morphism $\alpha : \mathcal{X} \to \mathcal{X}'$, we get also $\alpha = \Delta_1^-\Delta_1^+\alpha$.

Similarly, $\mathcal{Y} = \Delta_1^-\Delta_1^+\mathcal{Y}$ for any object $\mathcal{Y}$ in $rep_{(1)}(\mathcal{M}, \Omega)$ and $\beta = \Delta_1^+\Delta_1^-\beta$ for a morphism β in $rep_{(1)}(\mathcal{M}, \Omega)$. Thus, $\mathcal{X} \in rep^{(1)}(M, \Omega)$ means $\Delta_1^+\mathcal{X} = \Delta_1^+\Delta_1^-(\Delta_1^+\mathcal{X})$. So, $\Delta_1^+\mathcal{X}$ is in $rep_{(1)}(M, \Omega)$. Similarly, for any morphism α in $rep^{(1)}(M, \Omega)$, $\Delta_1^+\alpha$ is in $rep_{(1)}(M, \Omega)$. That is, Δ_1^+ is a functor from $rep^{(1)}(M, \Omega)$ to $rep_{(1)}(M, \Omega)$.

Similarly, Δ_1^- is a functor from $rep_{(1)}(M, \Omega)$ to $rep^{(1)}(M, \Omega)$.

Trivially, Δ_1^- and Δ_1^+ are mutual invertible. Hence, Δ_1^+ and Δ_1^- implement the desired equivalence.

The part (ii) can be discussed similarly. □

The following corollary can be got easily from the relations $\mathcal{X} = \Delta_1^-\Delta_1^+\mathcal{X}$ and $\alpha = \Delta_1^-\Delta_1^+\alpha$:

Corollary 3.4. (*i*) *For two objects* $\mathcal{X}$, $\mathcal{X}'$ *in* $rep^{(1)}(\mathcal{M},\Omega)$,

$$Ext^1(\mathcal{X},\mathcal{X}') \cong Ext^1(\Delta_1^+\mathcal{X},\Delta_1^+\mathcal{X}');$$

(*ii*) *For two objects* $\mathcal{Y}$, $\mathcal{Y}'$ *in* $Rep_{(1)}(\mathcal{M},\Omega)$,

$$Ext^1(\mathcal{Y},\mathcal{Y}') \cong Ext^1(\Delta_1^-\mathcal{Y},\Delta_1^-\mathcal{Y}').$$

Now, define the functors:

$$\Delta^+ = \Delta_n^+ \Delta_{n-1}^+ \dots \Delta_2^+ \Delta_1^+ : \ rep(\mathcal{M}, \Omega) \to rep(\mathcal{M}, \Omega)$$

and

$$\Delta^- = \Delta_1^- \Delta_2^- \dots \Delta_{n-1}^- \Delta_n^- : \ rep(\mathcal{M}, \Omega) \to rep(\mathcal{M}, \Omega).$$

These endofunctors are called the *Coxter functors*. For each $u \in \mathcal{G}$, define the representations $\underline{P}_u = \Delta_1^- \Delta_2^- \dots \Delta_{u-1}^- \underline{A}_u$ with $\underline{A}_u \in rep(\mathcal{M}, \delta_u \delta_{u+1} \dots \delta_n \Omega)$, $\underline{Q}_u = \Delta_n^+ \Delta_{n-1}^+ \dots \Delta_{u+1}^+ \underline{A}_u$ with $\underline{A}_u \in rep(\mathcal{M}, \delta_u \delta_{u-1} \dots \delta_1 \Omega)$.

Since A_u is a simple algebra over k, let $dim A_u = s_u^2$ for a positive integer s_u, then $A_u = W_u^{(1)} + \dots + W_u^{(s_u)}$ with the mutual-isomorphic simple A_u-modules $W_u^{(1)}, \dots, W_u^{(s_u)}$, and $\underline{A}_u = \underline{W}_u^{(1)} + \dots + \underline{W}_u^{(s_u)}$ where all mutual-isomorphic simple representations $\underline{W}_u^{(i)}$ are defined by $\underline{W}_u^{(i)} = (X_j, \ {}_j\varphi_l)$ for $X_j = \begin{cases} W_u^{(i)}, & \text{if } j = u \\ 0, & \text{if } j \neq u \end{cases}$ and ${}_j\varphi_l = 0$ for all $j \to l$.

It is clear to understand that the set $\{\underline{W}_u^{(1)}\}_{1 \le u \le n}$ consists of the set of all mutual non-isomorphic simple representations in $rep(\mathcal{M}, \Omega)$. Then, $\underline{P}_u = \mathcal{P}_u^{(1)} \oplus \dots \oplus \mathcal{P}_u^{(s_u)}$ and $\underline{Q}_u = \mathcal{Q}_u^{(1)} \oplus \dots \oplus \mathcal{Q}_u^{(s_u)}$ with mutual-isomorphic indecomposable representations $\mathcal{P}_u^{(i)} = \Delta_1^- \Delta_2^- \dots \Delta_{u-1}^- \underline{W}_u^{(i)}$ for $i = 1, \dots, s_u$ and $\mathcal{Q}_u^{(i)} = \Delta_n^+ \Delta_{n-1}^+ \dots \Delta_{u+1}^+ \underline{W}_u^{(i)}$ for $i = 1, \dots, s_u$ by Lemma 3.3.

For any distinct u, v, $\mathcal{P}_u^{(i)}$ and $\mathcal{P}_v^{(j)}$ are non-isomorphic each other for all i, j, since $\underline{W}_u^{(i)}$ and $\underline{W}_v^{(j)}$ are so. Now, we can obtain:

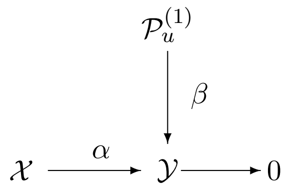
Theorem 3.3. *The set $\{\mathcal{P}_u^{(1)}\}_{1 \le u \le n}$ (respectively, $\{\mathcal{Q}_u^{(1)}\}_{1 \le u \le n}$) consists of the set of all non-isomorphic indecomposable projective (respectively, injective) representations in $rep(\mathcal{M}, \Omega)$ for a connected valued quiver $(\mathcal{G}, \mathcal{D}, \Omega)$ with the admissible orientation Ω and the admissible sequence of sinks $\{1, 2, \dots, n\}$.*

Proof. According to the one-one correspondence between simple representations and indecomposable projective representations via modulo the latter radical in $rep(\mathcal{M}, \Omega)$ and the above fact all $\mathcal{P}_u^{(i)}$ are indecomposable representations, it suffices to prove all $\mathcal{P}_u^{(1)}$ are projective, for this implies these indecomposable representations are, indeed, all non-isomorphic indecomposable projective ones.

We use induction u. First, for $u = 1$, $\mathcal{P}_1^{(1)}$ is just the unique simple direct summand under isomorphism of $\underline{A}_1$ which is projective by Corollary 3.1. Next, assume that for all $l < u$, $\mathcal{P}_l^{(1)}$ is projective for its corresponding admissible orientation of the graph. Then, in particular, $\widetilde{\mathcal{P}}_u^{(1)} = \Delta_2^- \dots \Delta_{u-1}^- \underline{W}_u^{(1)}$ is projective. We have $\mathcal{P}_u^{(1)} = \Delta_1^- \widetilde{\mathcal{P}}_u^{(1)}$.

Firstly, since $\widetilde{\mathcal{P}}_u^{(1)}$ is indecomposable, we have

$$\Delta_1^- \Delta_1^+ \mathcal{P}_u^{(1)} = \Delta_1^- (\Delta_1^+ \Delta_1^- \widetilde{\mathcal{P}}_u^{(1)}) = \Delta_1^- \widetilde{\mathcal{P}}_u^{(1)} = \mathcal{P}_u^{(1)} \tag{8}$$



by Lemma 3.3, which means $\mathcal{P}_u^{(1)}$ is indecomposable. In order to prove the projectivity of $\mathcal{P}_u^{(1)}$, consider the diagram
whose row is exact. We show that it may be assumed that such a diagram is in the category $rep^{(1)}(\mathcal{M}, \Omega)$ in Theorem 3.2.

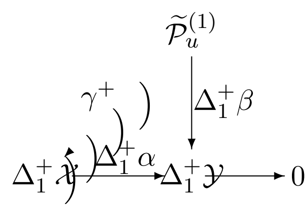
Indeed, by Lemma 3.3, we have $\mathcal{X} \cong \Delta_1^- \Delta_1^+ \mathcal{X} \oplus \mathcal{P}$, where $\mathcal{P} = (P_i, {}_i\pi_j)$ with $P_i = 0$ if $i \neq 1$ and P_1 is a (semisimple) A_1-module. We claim that

$$\alpha(\Delta_1^- \Delta_1^+ \mathcal{X}) = \Delta_1^- \Delta_1^+ \mathcal{Y}.$$

In fact, clearly $\Delta_1^- \Delta_1^+ \mathcal{X} \in rep^{(1)}(\mathcal{M}, \Omega)$, then $\alpha(\Delta_1^- \Delta_1^+ \mathcal{X}) \subseteq \Delta_1^- \Delta_1^+ \mathcal{Y}$. If this inclusion is proper, the fact that α is an epimorphism implies that some copy of the unique simple direct summand S_1 of $\underline{A}_1$ lies in $\Delta_1^- \Delta_1^+ \mathcal{Y}$. It is a contradiction.

Also, $\beta(\mathcal{P}_u^{(1)}) \subseteq \Delta_1^- \Delta_1^+ \mathcal{Y}$. Otherwise, there would exist a non-zero map $\mathcal{P}_u^{(1)} \to S_1$. This map must be an epimorphism since S_1 is simple and thus S_1 is a direct summand of $\mathcal{P}_u^{(1)}$ since S_1 is projective by Corollary 3.1. But, due to (8), $\mathcal{P}_u^{(1)}$ is indecomposable and non-isomorphic to S_1. This is a contradiction.

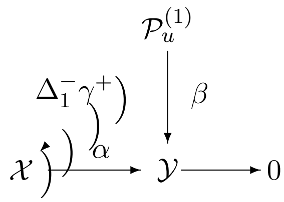
Thus, without loss of generality, assume that the above diagram lies in the category $rep^{(1)}(\mathcal{M}, \Omega)$ in Theorem 3.2. Then, applying Δ_1^+, we have $\Delta_1^+ \mathcal{P}_u^{(1)} = \Delta_1^+ \Delta_1^- \widetilde{\mathcal{P}}_u^{(1)} \cong \widetilde{\mathcal{P}}_u^{(1)}$ and get the following diagram:



where γ^+ exists by the projectivity of $\widetilde{\mathcal{P}}_u^{(1)}$ which makes this diagram to be commutative.

By Theorem 3.2, Δ_1^- and Δ_1^+ are mutual invertible between $rep_{(1)}(M, \Omega)$ and $rep^{(1)}(M, \Omega)$. So, $\Delta_1^- \Delta_1^+ \mathcal{X} \cong \mathcal{X}$, $\Delta_1^- \Delta_1^+ \mathcal{Y} \cong \mathcal{Y}$, $\Delta_1^- \Delta_1^+ \alpha \cong \alpha$, $\Delta_1^- \Delta_1^+ \beta \cong \beta$. But, $\mathcal{P}_u^{(1)} = \Delta_1^- \widetilde{\mathcal{P}}_u^{(1)}$. Thus, we get the following commutative diagram:
which means the projectivity of $\mathcal{P}_u^{(1)}$.

The statement on $\{\mathcal{Q}_u^{(i)}\}_{1 \leq i \leq s_u;\, 1 \leq u \leq n}$ can be shown in dual, according to the one-one correspondence between simple representations and indecomposable



injective representations via the frontal, as the socles, are embedded into the latter in $rep(\mathcal{M}, \Omega)$. □

According to Theorem 3.3 and the mutual constructions between a normal generalized path algebra and the corresponding pre-modulation in Section 2, we can give all indecomposable projective or injective representations of a normal generalized path algebra as follows:

Corollary 3.5. *Let $k(Q, \mathcal{A})$ be a normal $\mathcal{A}$-path algebra over a field k with connected acyclic quiver Q and the corresponding k-pre-modulation $\mathcal{M} = (A_i,\ {}_iM_j)$. Denote by $\{1, 2, \ldots, n\}$ the admissible sequence of sinks in Q and $\{\mathcal{P}_u^{(1)}\}_{1\leq u\leq n}$ (respectively, $\{\mathcal{Q}_u^{(1)}\}_{1\leq u\leq n}$) the set of all mutual non-isomorphic indecomposable projective (respectively, injective) representations in $rep(\mathcal{M}, \Omega)$ as in Theorem 3.3. Write $\mathcal{P}_u^{(1)} = (X_j^{(u)},\ {}_j\varphi_i)_{i,j\in Q_0}$ and $\mathcal{Q}_u^{(1)} = (Y_j^{(u)},\ {}_j\psi_i)_{i,j\in Q_0}$, let $\mathbf{P}_u = \sum_{j\in Q_0} X_j^{(u)}$ and $\mathbf{Q}_u = \sum_{j\in Q_0} Y_j^{(u)}$ for $u = 1, \ldots, n$. Then, in the category $modk(Q, \mathcal{A})$, under isomorphism, $\{\mathbf{P}_u\}_{1\leq u\leq n}$ (respectively, $\{\mathbf{Q}_u\}_{1\leq u\leq n}$) is the set of all indecomposable projective (respectively, injective) modules.*

We have known in [18] that if an artinian algebra A of Gabriel-type with admissible ideal is hereditary, then A is isomorphic to its related generalized path algebra $k(\Delta_A, \mathcal{A})$. Therefore, we can construct all indecomposable projective and injective modules over this kind of artinian hereditary algebras using of the method given in Corollary 3.5.

Remark 3.4. In [11], V.Dlab and C.M.Ringel generalize the Bernstein-Gelfand-Ponomarev theory in two directions. On one hand, they use valued graphs instesd of graphs, and show the relationship between the dimension vectors of indecomposable representations of elementary artinian algebras over skew-fields and the positive roots of the quadratic forms which is a bijection. On the other hand, they discuss the extended Dynkin diagrams and describe all there indecomposable representations. Note when the skew-fields are fields then the elementary artinian algebras are basic.

In our work, we use the natural quiver of a (non-basic) hereditary artinian algebra and the reformed modulations via generalized path algebras isomorphic to the hereditary algebras to construct all non-isomorphic indecomposable projective and injective representations of the generalized path algebras with acyclic quivers.

4. Representation-type of a generalized path algebra and its natural quiver

As one knows, according to Gabriel theory, representation type of a classical path algebra over an algebraically closed field or the modulation of a valued quiver is decided by the type of the quiver. Naturally, it is motivated to consider representation type of a generalized path algebra, equivalently, of a generalized modulation through the type of the corresponding natural quiver. First let us review the discussion given in [18].

We say a quiver to be of *almost Dynkin-affine type* provide that when one looks upon all arrows with same direction between an ordered pair of vertices as an arrow then the quiver becomes a quiver of either Dynkin or affine type; moreover, if it is of neither Dynkin nor affine type, we call this *proper almost Dynkin-affine type*. Respectively, we can give the definitions of (*proper*) *almost Dynkin type* and (*proper*) *almost affine type*.

By the classical Gabriel theory, if A is a hereditary k-spitting artinian algebra, A is of finite type if and only if Γ_A is of Dynkin type, A is of tame type if and only if Γ_A is of affine type. About the natural quiver Δ_A, it firstly was given that:

Proposition 4.1 ([18]). *For a hereditary k-splitting artinian algebra A, let m_{ij} be the number of arrows from a vertex i to another vertex j in the Ext-quiver Γ_A of A. Then, the natural quiver $\Delta_A = \Gamma_A$ if $m_{ij} \leq 1$ for any $i, j \in \Gamma_A$. Moreover, if A is of either finite type or tame type, then its natural quiver Δ_A is of either Dynkin type or affine type respectively.*

By Drozd's tame-and-wild Theorem, a finite-dimensional algebra A over an algebraically closed field k, which is not of finite type, is of either tame type or wild type. Then, the following holds:

Corollary 4.1 ([18]). *A finite-dimensional hereditary algebra A over an algebraically closed field k is of wild type if its natural quiver Δ_A is of neither Dynkin type nor affine type.*

The converse result is not true, that is, when A is of wild type, Δ_A is also possible to be of either Dynkin type or affine type.

Motivated by this discussion, it is asked how to characterize the kind of finite-dimensional (more generally, artinian) hereditary algebras of wild type whose natural quivers are of either Dynkin type or affine type?

As a part of this question, a class of wild algebras whose natural quivers are of either Dynkin type or affine type was constructed as in the following:

Proposition 4.2 ([18]). *For a normal generalized path algebra $k(Q, \mathcal{A})$ over an algebraically closed field k with Q a finite acyclic quiver, let $\mathcal{A} = \{A_i : i \in Q_0\}$ and $n_i = \sqrt{dim_k A_i}$ for any $i \in Q_0$.*

(*i*) *If there is an arrow from i to j in Q with $n_i n_j > 1$, then $k(Q, \mathcal{A})$ is of wild type;*

(ii) If the quiver Q is of either Dynkin or affine type and there is an arrow from i to j in Q with $n_in_j > 1$, then the Ext-quiver of $k(Q, \mathcal{A})$ is of proper almost Dynkin-affine type.

Theorem 4.1. *For a normal generalized path algebra $k(Q, \mathcal{A})$ over an algebraically closed field k with Q a finite connected acyclic quiver, let $\mathcal{A} = \{A_i : i \in Q_0\}$ and $n_i = \sqrt{dim_k A_i}$ for any $i \in Q_0$. If Q is of Dynkin type (resp. affine type), then*

(i) $k(Q, \mathcal{A})$ is of finite type (resp. tame type) if and only if $A_i \cong k$ for each vertex $i \in Q_0$, or equivalently say, $k(Q, \mathcal{A}) \cong kQ$;

(ii) in the otherwise case, $k(Q, \mathcal{A})$ is of wild type.

Proof. (i) "if": It is trivial according to the classical Gabriel theory.

"only if": As we have known in [17, 18], Q is just the natural quiver of $k(Q, \mathcal{A})$. Let Γ denote the Ext-quiver of $k(Q, \mathcal{A})$. Then, the relation is given in [21, 18] that $g_{ij} = n_i n_j t_{ij}$ for the numbers g_{ij} and t_{ij} arrows from i to j in Γ and Q respectively.

Suppose there is one $p \in Q_0$ such that $A_p \ncong k$, that is, $n_p > 1$. Since Q is connected, p is either a head or a tail of some arrow in Q. No loss of generality, let p be the head of an arrow $\alpha : p \to q$ in Q. Then, $g_{pq} = n_p n_q t_{pq} > 1$ due to $n_p > 1$. Thus, Γ is neither of Dynkin type nor of affine type. By Gabriel theory, $k(Q, \mathcal{A})$ is neither of finite type nor of tame type.

(ii): It follows from the proof of "only if" above and Drozd's tame-and-wild Theorem. □

In the case of basic hereditary algebras, Gabriel's theorem tell us the hereditary algebra KQ is representation-finite if and only if the underlying graph of Q is one of the Dynkin diagrams. Theorem 4.1 discusses the representation type of normal generalized path algebra $k(Q, A)$, where Q is Dynkin quiver. It shows that a normal generalized path algebra $k(Q, A)$ to be representation-finite type in the case the quiver is of Dynkin type if and only if all algebras at the vertices are isomorphic to fields. As analogue for affine type, we also discuss the condition for a generalized path algebra to be of tame type in the case the quiver is of affine type.

It is easy to see that in the case of Theorem 4.1 (ii), the Ext-quiver of $k(Q, \mathcal{A})$ is certainly of proper almost Dynkin-affine type.

Acknowledgements

Project supported by the National Natural Science Foundation of China (No. 12131015, No. 12071422).

References

[1] M. Auslander, M.I. Platzeck, I. Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc., 250 (1979), 1-46.

[2] I. Assem, D. Simson, A. Skowronski, *Elements of the representation theory of associative algebras Vol I: Techniques of representation theory*, Cambridge University Press, 2006.

[3] M. Auslander, I. Reiten, S.O. Smalø, *Representation theory of artin algebras*, Cambridge University Press, 1995.

[4] D.J. Benson, *Representations and cohomology (I): Basic representation theory of finite groups and associative algebras*, Cambridge University Press, 1995.

[5] I.N. Bernstein, I.M. Gelfand, V.A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Russian Math. Survey, 28 (1973), 17-32.

[6] S. Brenner, M.C.R. Butler, *Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors*, Representation Theory II, Lecture Notes in Mathematics, 832 (1980), 103-169.

[7] L.L. Chen, *Quivers and Hopf algebras*, Doctoral Dissertation, Zhejiang University, China, 2008.

[8] F.U. Coelho, S.X. Liu, *Generalized path algebras*, Lecture Notes in Pure and Appl. Math, (2000), 53-66.

[9] B. Deng, J. Du, B. Parshall, J. Wang, *Finite dimensional algebras and quantum groups*, Mathematical Surveys and Monographs, Amer. Math. Soc., 150 (2008).

[10] V. Dlab, *Representations of valued graph*, Seminaire de mathematiques superieures, Les presses de luniversite de montreal, 1980.

[11] V. Dlab, C.M. Ringel, *Indecomposable representations of graphs and algebras*, Memoirs of the Amer. Math. Soc., 6 (1976), 1-57.

[12] Y.A. Drozd, V.V. Kirichenko, *Finite dimensional algebras,* Springer-Verlag, 1994.

[13] I.M. Gelfand, V.A. Pononarev, *Problems of linear algebra and the classification of quadruples of subspaces in a finite-dimensional vector space,* Coll. Math. Soc. Bolyai 5, Tihany, (1970), 163-237.

[14] V.G. Kac, *Infinite root systems, representations of graphs, and invariant theory,* Inventiones Math., 56 (1980), 57-92.

[15] F. Li, *Characterization of left Artinian algebras through pseudo path algebras,* J. Australia Math. Soc., 83(2007), 385-416.

[16] F. Li, *Modulation and natural valued quiver of an algebra,* Pacific J. of Mathematics, 256(2012),105-128.

[17] F. Li, L.L. Chen, *The natural quiver of an artinian algebra*, Algebras and Representation Theory, 13 (2010), 623-636.

[18] F. Li, Z.Z. Lin, *Approach to artinian algebras via natural quivers*, Trans. Amer. Math. Soc., 364 (2012), 1395-1411.

[19] F. Li, G.X. Liu, *Generalized path coalgebra and generalized dual Gabriel theorem,* Acta Mathematica Sinica (Chinese Series), 51 (2008),853-862.

[20] F. Li, D.W. Wen, *Ext-quiver, AR-quiver and natural quiver of an algebra,* in Geometry, Analysis and Topology of Discrete Groups, Advanced Lectures in Mathematics 6, Editors: Lizhen Ji, Kefeng Liu, Lo Yang, Shing-Tung Yau, Higher Education Press and International Press, Beijing, 2008.

[21] G.X. Liu, *Classification of finite dimensional basic Hopf algebras and related topics,* Doctoral Dissertation, Zhejiang University, China, 2005.

[22] S.X. Liu, *Isomorphism problem for tensor algebras over valued graphs*, Sci. China Ser. (A), 34 (1991), 267-272.

[23] S.X. Liu, J.Y. Guo, B. Zhu, Y. Han, *Rings and algebras*, Science Press, 2009.

[24] S.X. Liu, J. Xiao, Y.L. Luo, *Isomorphism of path algebras,* Journal of Beijing Normal University, 31 (1986), 483-487.

[25] N. Marmaridis, *Reflection functors*, Proc. ICRA II, Ottawa 1979, Lecture Notes in Math., 832 (1980), 382-395.

[26] R.S. Pierce, *Associative algebras*, Springer-Verlag, 1982.

Accepted: September 17, 2022

Semigroup of transformations with restricted partial range: regularity, abundance and some combinatorial results

Jiulin Jin
School of Science
Guiyang University
Guiyang, Guizhou, 550005
China
j.l.jin@hotmail.com

Taijie You*
School of Mathematical Sciences
Guizhou Normal University
Guiyang, Guizhou, 550025
China
youtaijie1111@163.com

Abstract. Suppose that X be a nonempty set. Denote by $\mathcal{T}(X)$ the full transformation semigroup on X. For $\emptyset \neq Z \subseteq Y \subseteq X$, let $\mathcal{T}(X,Y,Z) = \{\alpha \in \mathcal{T}(X) : Y\alpha \subseteq Z\}$. Then, $\mathcal{T}(X,Y,Z)$ is a subsemigroup of $\mathcal{T}(X)$. In this paper, we characterize the regular elements of the semigroup $\mathcal{T}(X,Y,Z)$, and present a necessary and sufficient condition under which $\mathcal{T}(X,Y,Z)$ is regular. Furthermore, we investigate the abundance of the semigroup $\mathcal{T}(X,Y,Z)$ for the case $Z \subsetneq Y \subsetneq X$. In addition, we compute the cardinalities of $\mathcal{T}(X,Y,Z)$, $\mathrm{Reg}(\mathcal{T}(X,Y,Z))$ and $\mathrm{E}(\mathcal{T}(X,Y,Z))$ when X is finite, respectively.
Keywords: transformation semigroup, restricted partial range, regular element, $\mathcal{L}^*$-relation, $\mathcal{R}^*$-relation.

1. Introduction

An element x of semigroup S is called a *regular element* of S if $x = xyx$ for some $y \in S$, and S is said to be a *regular semigroup* if every element of S is regular. The set of all regular elements of a semigroup S is denoted by $\mathrm{Reg}(S)$. An element x of semigroup S is called an *idempotent* of S if $x^2 = x$. The set of all idempotents of a semigroup S is denoted by $\mathrm{E}(S)$. Regular element (resp., idempotent) is one of the most studied topics in semigroup theory due to its nice algebraic properties and wide applications. There have been many research works studying regularity of semigroups (see, [1, 11, 15, 16, 18, 19, 20, 21, 29]).

Let S be a semigroup and $a, b \in S$. We say that a and b are *$\mathcal{L}$-related* (*$\mathcal{R}$-related*) in S if $S^1a = S^1b$ ($aS^1 = bS^1$) where S^1 denotes the monoid obtained from S by adding an identity if S has no identity, otherwise, $S^1 = S$. If a and b are $\mathcal{L}$-related ($\mathcal{R}$-related), we can write $(a,b) \in \mathcal{L}$ ($(a,b) \in \mathcal{R}$). Again, if $(a,b) \in$

∗. Corresponding author

$\mathcal{L}$ in some oversemigroup of S, then a and b are called *$\mathcal{L}^*$-related* and write $(a, b) \in \mathcal{L}^*$. The relation $\mathcal{R}^*$ can be defined dually. Clearly, $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$, and $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations on S. Fountain [7] pointed out that a semigroup S is said to be *left abundant* (*right abundant*) if every $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) contains an idempotent. Moreover, a semigroup S is called *abundant* if it is both left abundant and right abundant. It is obvious that regular semigroups are abundant, but the converse is not true. For example, Umar [27] shown that the semigroup of order-decreasing finite full transformations is abundant but not regular. Many papers have been written describing the abundance of various semigroups.

For a nonempty set X, let $\mathcal{T}(X)$ be the full transformation semigroup on X that is, the semigroup under composition of all maps from X into itself. It is well known that $\mathcal{T}(X)$ is a regular semigroup (see [9]). Transformation semigroups are ubiquitous in semigroup theory because of Cayley's Theorem which states that every semigroup S embeds in some transformation semigroup $\mathcal{T}(X)$ (see [9, Theorem 1.1.2]).

Given a nonempty subset Y of X, let

$$\overline{\mathcal{T}}(X, Y) = \{\alpha \in \mathcal{T}(X) : Y\alpha \subseteq Y\}, \quad \mathcal{T}(X, Y) = \{\alpha \in \mathcal{T}(X) : X\alpha \subseteq Y\}.$$

Then, $\overline{\mathcal{T}}(X, Y)$ is a subsemigroup of $\mathcal{T}(X)$ and $\mathcal{T}(X, Y)$ is a subsemigroup of $\overline{\mathcal{T}}(X, Y)$. In 1966, Magill [17] introduced and studied the semigroup $\overline{\mathcal{T}}(X, Y)$. In 1975, Symons [25] introduced the semigroup $\mathcal{T}(X, Y)$, and also described all automorphisms of $\mathcal{T}(X, Y)$. Recently, $\overline{\mathcal{T}}(X, Y)$ and $\mathcal{T}(X, Y)$ have been studied in a variety of contexts (see [10, 12, 13, 18, 20, 21, 22, 23, 24, 28]).

The study of the related combinatorial properties of subsemigroups of finite full transformation semigroup has always been one of the most important topics in the semigroup theory. Many scholars have obtained results (see [2, 3, 6, 8]). Although they have studied semigroups $\overline{\mathcal{T}}(X, Y)$ and $\mathcal{T}(X, Y)$ from different perspectives, very little research has been found to deal with other literatures have studied other related combinatorial properties of semigroups $\overline{\mathcal{T}}(X, Y)$ and $\mathcal{T}(X, Y)$ except that Nenthein, Youngkhong and Kemprasit [18] determined the number of all regular elements in $\overline{\mathcal{T}}(X, Y)$ and $\mathcal{T}(X, Y)$.

For X, Y and Z are all nonempty sets with $Z \subseteq Y \subseteq X$, the first author [14] defined

$$\mathcal{T}(X, Y, Z) = \{\alpha \in \mathcal{T}(X) : Y\alpha \subseteq Z\}.$$

Clearly, for each $\alpha, \beta \in \mathcal{T}(X, Y, Z)$, $Y(\alpha\beta) = (Y\alpha)\beta \subseteq Z\beta \subseteq Y\beta \subseteq Z$ and so $\alpha\beta \in \mathcal{T}(X, Y, Z)$. Therefore, we have $\mathcal{T}(X, Y, Z)$ is a subsemigroup of $\mathcal{T}(X)$, and we call it *the semigroup of transformations with restricted partial range* on X. The semigroup $\mathcal{T}(X, Y, Z)$ is a generalization of semigroups $\mathcal{T}(X)$, $\overline{\mathcal{T}}(X, Y)$ and $\mathcal{T}(X, Z)$, that is,

- if $Z = Y$, then $\mathcal{T}(X, Y, Z) = \overline{\mathcal{T}}(X, Y)$;
- if $Y = X$, then $\mathcal{T}(X, Y, Z) = \mathcal{T}(X, Z)$;
- if $Z = Y = X$, then $\mathcal{T}(X, Y, Z) = \mathcal{T}(X)$.

For the case $Z = Y = X$, it is well known that $\mathcal{T}(X,Y,Z) = \mathcal{T}(X)$ is a regular semigroup and so $\mathcal{T}(X,Y,Z)$ is abundant.

For the case $Z = Y \subsetneq X$, Sun [22] shown the following result.

Lemma 1.1. ([22, Theorem 4.2]) *The semigroup $\overline{\mathcal{T}}(X,Y)$ is abundant.*

For the case $Z \subsetneq Y = X$. Then, $\mathcal{T}(X,Y,Z) = \mathcal{T}(X,Z)$ contains exactly one element if $|Z| = 1$. And if $|Z| \geq 2$, Sun [23] presented the following result.

Lemma 1.2. ([23, Theorem 1]) *The semigroup $\mathcal{T}(X,Z)$ is left abundant but not right abundant.*

The paper is organized as follows. In Section 2, we characterize the regular elements of the semigroup $\mathcal{T}(X,Y,Z)$, and present a necessary and sufficient condition under which $\mathcal{T}(X,Y,Z)$ is regular. In Section 3, we investigate the abundance of the semigroup $\mathcal{T}(X,Y,Z)$ for the case $Z \subsetneq Y \subsetneq X$. In Section 4, we compute the cardinalities of $\mathcal{T}(X,Y,Z)$, $\mathrm{Reg}(\mathcal{T}(X,Y,Z))$ and $\mathrm{E}(\mathcal{T}(X,Y,Z))$ when X is finite, respectively. All combinatorial formulas in $\mathcal{T}(X,Y,Z)$ also apply to the semigroup $\overline{\mathcal{T}}(X,Y)$ (resp. $\mathcal{T}(X,Z)$ or $\mathcal{T}(X)$).

Throughout this paper, we always write functions on the right; in particular, this means that for a composition $\alpha\beta$, α is applied first. For any sets A and B, we denote by $|A|$ the cardinality of A, and write $A \setminus B = \{a \in A : a \notin B\}$. For each $\alpha \in \mathcal{T}(X,Y,Z)$, we denote by $X\alpha$ the range of α. And if A is a nonempty subset of X then the restriction of α to the set A is denoted by $\alpha|_A$. Moreover, for the general background of Semigroup Theory and standard notation, we refer the readers to Howie's book [9].

2. Regularity

In this section, we characterize the regularity of the semigroup $\mathcal{T}(X,Y,Z)$. First, we describe the regular elements of the semigroup $\mathcal{T}(X,Y,Z)$.

Theorem 2.1. *Let $\alpha \in \mathcal{T}(X,Y,Z)$. Then, the following conditions are equivalent:*

(i) $\alpha \in \mathrm{Reg}(\mathcal{T}(X,Y,Z))$;

(ii) $X\alpha \cap Y \subseteq Z\alpha$;

(iii) $X\alpha \cap Y = Z\alpha$.

Proof. (i) $\Rightarrow$(ii). Let $\alpha \in \mathrm{Reg}(\mathcal{T}(X,Y,Z))$. Then, exists $\beta \in \mathcal{T}(X,Y,Z)$ such that $\alpha = \alpha\beta\alpha$. For each $x \in X\alpha \cap Y$, we have $x \in Y$ and $x = a\alpha$ for some $a \in X$. Consequently, $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha \in Y\beta\alpha \subseteq Z\alpha$ and so, (ii) holds.

(ii)$\Rightarrow$(iii). It is obvious that $Z\alpha \subseteq Y\alpha \subseteq X\alpha \cap Z \subseteq X\alpha \cap Y$, together with condition (ii), we get (iii).

(iii)$\Rightarrow$(i). Suppose that $X\alpha \cap Y = Z\alpha$, and let

$$X\alpha \cap Y = \{\overline{y_1}, \overline{y_2}, \ldots, \overline{y_s}\}.$$

Then, exist $z_i \in Z$ $(i = 1, 2, \ldots, s)$ such that $z_i\alpha = \overline{y_i}$. We consider two cases. If $X\alpha \setminus Y = \emptyset$, we define a mapping $\beta : X \to X$ by

$$x\beta = \begin{cases} z_i, & \text{if } x = \overline{y_i} \text{ for some } i = 1, 2, \ldots, s, \\ z_1, & \text{otherwise.} \end{cases}$$

If $X\alpha \setminus Y \neq \emptyset$. Then, for each $x \in X\alpha \setminus Y$, choose and fix $t_x \in \{k \in X : k\alpha = x\}$, and define a mapping $\beta : X \to X$ by

$$x\beta = \begin{cases} z_i, & \text{if } x = \overline{y_i} \text{ for some } i = 1, 2, \ldots, s, \\ t_x, & \text{if } x \in X\alpha \setminus Y \\ z_1, & \text{otherwise.} \end{cases}$$

For both cases, it is easy to verify that $\alpha = \alpha\beta\alpha$ and $\beta \in \mathcal{T}(X, Y, Z)$. Hence, (i) holds. □

In particular, we take $Z = Y$ (resp., $Y = X$) in Theorem 2.1. Then, we get the following Corollary 2.1 (resp., Corollary 2.2) which are proved by Nenthein, Youngkhong and Kemprasit [18, Theorem 2.1] (resp., [18, Theorem 2.3]).

Corollary 2.1. *Let* $\alpha \in \overline{\mathcal{T}}(X, Y)$. *Then, the following conditions are equivalent:*

(i) $\alpha \in \mathrm{Reg}(\overline{\mathcal{T}}(X, Y))$.

(ii) $X\alpha \cap Y \subseteq Y\alpha$.

(iii) $X\alpha \cap Y = Y\alpha$.

Corollary 2.2. *Let* $\alpha \in \mathcal{T}(X, Z)$. *Then, the following conditions are equivalent:*

(i) $\alpha \in \mathrm{Reg}(\mathcal{T}(X, Z))$.

(ii) $X\alpha \subseteq Z\alpha$.

(iii) $X\alpha = Z\alpha$.

Nenthein, Youngkhong and Kemprasit presented a necessary and sufficient condition under which $\mathcal{T}(X, Z)$ (resp., $\overline{\mathcal{T}}(X, Y)$) is regular in [18] that is,

Lemma 2.1 ([18], Corollary 2.2)**.** $\mathcal{T}(X, Z)$ *is a regular semigroup if and only if* $|Z| = 1$ *or* $X = Z$.

Lemma 2.2 ([18], Corollary 2.4)**.** $\overline{\mathcal{T}}(X, Y)$ *is a regular semigroup if and only if* $|Y| = 1$ *or* $X = Y$.

Next, a necessary and sufficient condition for $\mathcal{T}(X, Y, Z)$ to be a regular semigroup can be given as follows:

Theorem 2.2. *$\mathcal{T}(X,Y,Z)$ is a regular semigroup if and only if one of the following statements holds:*

(i) $|Y| = 1$.

(ii) $X = Y$ *and* $|Z| = 1$.

(iii) $Z = Y = X$.

Proof. For $|Y| = 1$. It is note that Z be a nonempty subset of Y, then $Y = Z$ and so $\mathcal{T}(X,Y,Z) = \overline{\mathcal{T}}(X,Y)$. According to Lemma 2.2, we have $\mathcal{T}(X,Y,Z)$ is regular. For $X = Y$ and $|Z| = 1$. It is easy to see that $\mathcal{T}(X,Y,Z) = \mathcal{T}(X,Z)$ and so from Lemma 2.1 it follows that $\mathcal{T}(X,Y,Z)$ is regular. For $Z = Y = X$, we have $\mathcal{T}(X,Y,Z) = \mathcal{T}(X)$ which is regular.

Conversely, suppose that $\mathcal{T}(X,Y,Z)$ is a regular semigroup, and let (i), (ii) and (iii) be not established. Note that X, Y and Z are all nonempty sets with $Z \subseteq Y \subseteq X$. To do this, we distinguish three cases:

Case 1. $Z \subsetneq Y \subsetneq X$. Let z be an element of Z, and choose $y \in Y$ such that $y \neq z$. Since $X \setminus Y \neq \emptyset$, we define a mapping $\alpha : X \to X$ by

$$x\alpha = \begin{cases} z, & \text{if } x \in Y, \\ y, & \text{if } x \in X \setminus Y. \end{cases}$$

It is easy to verify that $\alpha \in \mathcal{T}(X,Y,Z)$. However, $X\alpha \cap Y = \{z,y\} \supsetneq \{z\} = Z\alpha$. By Theorem 2.1, we immediately deduce that α is not a regular element of $\mathcal{T}(X,Y,Z)$, which contradicts the fact that $\mathcal{T}(X,Y,Z)$ is regular.

Case 2. $|Z| > 1$ and $Z = Y \subsetneq X$. Then, $\mathcal{T}(X,Y,Z) = \overline{\mathcal{T}}(X,Y)$ with $|Y| \neq 1$ and $X \neq Y$. Also, we have $\mathcal{T}(X,Y,Z)$ is not regular by Lemma 2.2. This is a contradiction.

Case 3. $|Z| > 1$ and $Z \subsetneq Y = X$. Then, $\mathcal{T}(X,Y,Z) = \mathcal{T}(X,Z)$ with $|Z| \neq 1$ and $X \neq Z$. Similar to the above, we have $\mathcal{T}(X,Y,Z)$ is not regular by Lemma 2.1. This is a contradiction. □

3. Abundance

In this section, we investigate the abundance of the semigroup $\mathcal{T}(X,Y,Z)$ for the case $Z \subsetneq Y \subsetneq X$. The following two lemmas give characterizations of $\mathcal{L}^*$ and $\mathcal{R}^*$ that can be found, for instance, in [7].

Lemma 3.1 ([7], Lemma 1.1)**.** *Let S be a semigroup and $a, b \in S$. Then, the following statements are equivalent:*

(i) $(a,b) \in \mathcal{L}^*$.

(ii) *For all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.*

Dually, we have:

Lemma 3.2. *Let S be a semigroup and $a, b \in S$. Then, the following statements are equivalent:*

(i) $(a, b) \in \mathcal{R}^*$.

(ii) For all $x, y \in S^1$, $xa = ya$ *if and only if* $xb = yb$.

To facilitate the description of the following lemma, we introduce a binary relation Λ on $\mathcal{T}(X, Y, Z)$ as follows: For each $\alpha, \beta \in \mathcal{T}(X, Y, Z)$, $(\alpha, \beta) \in \Lambda$ if and only if one of the following statements holds:

(i) $(X \setminus Y)\alpha \cap (Y \setminus Z) = \emptyset$ and $(X \setminus Y)\beta \cap (Y \setminus Z) = \emptyset$.

(ii) $(X \setminus Y)\alpha \cap (Y \setminus Z) \neq \emptyset$ and $(X \setminus Y)\beta \cap (Y \setminus Z) \neq \emptyset$.

Clearly, Λ is an equivalence relation on $\mathcal{T}(X, Y, Z)$.

Lemma 3.3. *Let $Z \subsetneq Y \subsetneq X$ and $\alpha, \beta \in \mathcal{T}(X, Y, Z)$. Then, the following statements hold:*

(i) for $|Z| = 1$, $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $(\alpha, \beta) \in \Lambda$ *and* $X\alpha \cap (X \setminus Y) = X\beta \cap (X \setminus Y)$.

(ii) for $|Z| \geq 2$, $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $X\alpha = X\beta$.

Proof. (i) Suppose that $(\alpha, \beta) \in \Lambda$ and $X\alpha \cap (X \setminus Y) = X\beta \cap (X \setminus Y)$. By $|Z| = 1$, we say that $Z = \{z_0\}$. From $(\alpha, \beta) \in \Lambda$, we distinguish two cases:
Case 1. $(X \setminus Y)\alpha \cap (Y \setminus Z) = \emptyset$ and $(X \setminus Y)\beta \cap (Y \setminus Z) = \emptyset$. Clearly,

$$\begin{aligned}
X\alpha =& Y\alpha \cup (X \setminus Y)\alpha \\
=& \{z_0\} \cup \{(X \setminus Y)\alpha \cap [Z \cup (X \setminus Y)]\} \\
=& \{z_0\} \cup \{[(X \setminus Y)\alpha \cap Z] \cup [(X \setminus Y)\alpha \cap (X \setminus Y)]\} \\
=& \{z_0\} \cup [(X \setminus Y)\alpha \cap (X \setminus Y)] \\
=& \{z_0\} \cup [(X \setminus Y)\alpha \cap (X \setminus Y)] \cup [Y\alpha \cap (X \setminus Y)] \text{ (By } Y\alpha \cap (X \setminus Y) \subseteq \\
& Z \cap (X \setminus Y) = \emptyset) \\
=& \{z_0\} \cup \{[(X \setminus Y)\alpha \cup Y\alpha] \cap (X \setminus Y)\} \\
=& \{z_0\} \cup [X\alpha \cap (X \setminus Y)].
\end{aligned}$$

Similarly, we have $X\beta = \{z_0\} \cup [X\beta \cap (X \setminus Y)]$. Since $X\alpha \cap (X \setminus Y) = X\beta \cap (X \setminus Y)$, we have $X\alpha = X\beta$. This implies that α and β are $\mathcal{L}$-related in the full transformation semigroup $\mathcal{T}(X)$ (see [9, page 63]). Hence, $(\alpha, \beta) \in \mathcal{L}^*$.
Case 2. $(X \setminus Y)\alpha \cap (Y \setminus Z) \neq \emptyset$ and $(X \setminus Y)\beta \cap (Y \setminus Z) \neq \emptyset$. For each $\eta, \theta \in \mathcal{T}^1(X, Y, Z)$, we consider the following three subcases:
Case 2.1. $\eta = 1$ and $\theta = 1$. Clearly, $(\alpha, \beta) \in \mathcal{L}^*$.
Case 2.2. $\eta = 1$ and $\theta \neq 1$. Then, $\theta \in \mathcal{T}(X, Y, Z)$ and so $Y\theta \subseteq Z = \{z_0\}$. Let $\gamma\eta = \gamma\theta$ $(\gamma \in \{\alpha, \beta\})$. Then, $\gamma = \gamma\theta$ and so $x\theta = x$, for all $x \in X\gamma$. This means that $(X \setminus Y)\gamma \cap (Y \setminus Z) = \emptyset$ (If not, there exist $b_\gamma \in X \setminus Y$ and $y_\gamma \in Y \setminus Z$ such

that $y_\gamma = b_\gamma\gamma \in X\gamma$. Then, $y_\gamma = y_\gamma\theta \in Z$, this contradicts the condition that $y_\gamma \in Y \setminus Z$). This is a contradiction.

Case 2.3. $\eta \neq 1$ and $\theta \neq 1$. That is, $\eta, \theta \in \mathcal{T}(X, Y, Z)$. Then, $Y\eta = \{z_0\} = Y\theta$ and so $\eta|_Y = \theta|_Y$. Therefore,

$$\begin{aligned}\alpha\eta = \alpha\theta &\Leftrightarrow \eta|_{X\alpha} = \theta|_{X\alpha}\\ &\Leftrightarrow \eta|_{X\alpha\cap Y} = \theta|_{X\alpha\cap Y} \text{ and } \eta|_{X\alpha\cap(X\setminus Y)} = \theta|_{X\alpha\cap(X\setminus Y)}\\ &\Leftrightarrow \eta|_{X\beta\cap Y} = \theta|_{X\beta\cap Y} \text{ and } \eta|_{X\beta\cap(X\setminus Y)} = \theta|_{X\beta\cap(X\setminus Y)}\\ &\Leftrightarrow \eta|_{X\beta} = \theta|_{X\beta}\\ &\Leftrightarrow \beta\eta = \beta\theta.\end{aligned}$$

By Lemma 3.1 we conclude that $(\alpha, \beta) \in \mathcal{L}^*$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}^*$ such that $(\alpha, \beta) \notin \Lambda$ or $X\alpha\cap(X\setminus Y) \neq X\beta \cap (X \setminus Y)$. We distinguish two cases:

Case 1. $(\alpha, \beta) \notin \Lambda$. Then, we have $((X \setminus Y)\alpha \cap (Y \setminus Z) = \emptyset$ and $(X \setminus Y)\beta \cap (Y \setminus Z) \neq \emptyset)$ or $((X \setminus Y)\alpha \cap (Y \setminus Z) \neq \emptyset$ and $(X \setminus Y)\beta \cap (Y \setminus Z) = \emptyset)$. By symmetry, let $(X \setminus Y)\alpha \cap (Y \setminus Z) = \emptyset$ and $(X \setminus Y)\beta \cap (Y \setminus Z) \neq \emptyset$. Define two mappings $\eta : X \to X$ and $\theta : X \to X$ by $\eta = 1$ and

$$x\theta = \begin{cases} x, & \text{if } x \in X\alpha \\ z_0, & \text{if } x \notin X\alpha. \end{cases}$$

Clearly, $\theta \in \mathcal{T}(X, Y, Z)$ and $\alpha\eta = \alpha\theta$. Howerver, $\beta\eta \neq \beta\theta$. This contradicts the fact that $(\alpha, \beta) \in \mathcal{L}^*$.

Case 2. $X\alpha \cap (X \setminus Y) \neq X\beta \cap (X \setminus Y)$. Then, exists $a \in X\beta \cap (X \setminus Y)$ such that $a \notin X\alpha \cap (X \setminus Y)$ and so $a_0\beta = a$ for some $a_0 \in X$ and $x\alpha \neq a$, for all $x \in X$. In fact, $a_0 \in X \setminus Y$ (If not, $a = a_0\beta \in Z$, this contradicts the fact that $a \in X \setminus Y$). We consider two cases. If $|X \setminus Y| = 1$. It is clear that $X \setminus Y = \{a\}$. Define two mappings $\eta : X \to X$ and $\theta : X \to X$ by $X\eta = z_0$ and

$$x\theta = \begin{cases} z_0, & \text{if } x \in Y \\ a_0, & \text{if } x \in X \setminus Y. \end{cases}$$

If $|X \setminus Y| \geq 2$. Define two mappings $\eta : X \to X$ and $\theta : X \to X$ by

$$x\eta = \begin{cases} z_0, & \text{if } x \in Y \cup \{a\} \\ a_0, & \text{if } x \in X \setminus (Y \cup \{a\}) \end{cases} \quad \text{and} \quad x\theta = \begin{cases} z_0, & \text{if } x \in Y \\ a_0, & \text{if } x \in X \setminus Y. \end{cases}$$

For both cases, we have $\eta, \theta \in \mathcal{T}(X, Y, Z)$ and $\alpha\eta = \alpha\theta$. However,

$$a_0\beta\eta = a\eta = z_0 \neq a_0 = a\theta = a_0\beta\theta$$

and so $\beta\eta \neq \beta\theta$. This contradicts the fact that $(\alpha, \beta) \in \mathcal{L}^*$.

Hence, $(\alpha, \beta) \in \Lambda$ and $X\alpha \cap (X \setminus Y) = X\beta \cap (X \setminus Y)$.

(ii) Let $X\alpha = X\beta$. This implies that α, β are $\mathcal{L}$-related in the full transformation semigroup $\mathcal{T}(X)$. Hence, $(\alpha, \beta) \in \mathcal{L}^*$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}^*$ and $X\alpha \neq X\beta$. Then, exists $a \in X\beta$ such that $a \notin X\alpha$ and so $a_0\beta = a$ for some $a_0 \in X$ and $x\alpha \neq a$, for all $x \in X$. Note that $|Z| \geq 2$ and $|X| \geq 4$. Then, we can take distinct $z_1, z_2 \in Z$, and choose nonempty subsets X_1, X_2 of X with $|X_i| \geq 2$ $(i = 1, 2)$ such that X is a disjoint union of X_1 and X_2. Define two mappings $\eta : X \to X$ and $\theta : X \to X$ by

$$x\eta = \begin{cases} z_1, & \text{if } x \in X_1 \cup \{a\} \\ z_2, & \text{if } x \in X_2 \setminus \{a\} \end{cases} \quad \text{and} \quad x\theta = \begin{cases} z_1, & \text{if } x \in X_1 \setminus \{a\} \\ z_2, & \text{if } x \in X_2 \cup \{a\}. \end{cases}$$

Clearly, $\eta, \theta \in \mathcal{T}(X, Y, Z)$ and $\alpha\eta = \alpha\theta$. However,

$$a_0\beta\eta = a\eta = z_1 \neq z_2 = a\theta = a_0\beta\theta$$

and so $\beta\eta \neq \beta\theta$. This contradicts the fact that $(\alpha, \beta) \in \mathcal{L}^*$. Hence, $X\alpha = X\beta$. □

A necessary and sufficient condition for $\alpha \in \mathcal{T}(X, Y, Z)$ to be an idempotent can be given as follows:

Lemma 3.4. *Let $\alpha \in \mathcal{T}(X, Y, Z)$. Then, α is an idempotent if and only if the following statements hold:*

(i) $X\alpha \subseteq Z \cup (X \setminus Y)$.

(ii) $t\alpha = t$, for all $t \in X\alpha$.

Proof. Suppose that $X\alpha \subseteq Z \cup (X \setminus Y)$ and $t\alpha = t$, for all $t \in X\alpha$. For each $x \in X$, there exists $t \in X\alpha$ such that $x\alpha = t$. Then, $x\alpha^2 = (x\alpha)\alpha = t\alpha = t = x\alpha$. Hence, α is an idempotent.

Conversely, suppose that α is an idempotent, and let (i) or (ii) do not hold. To do this, we distinguish two cases:
Case 1. (i) not holds and (ii) holds. Then, there exists $y \in X\alpha$ such that $y \in Y \setminus Z$ and so $y\alpha = y \notin Z$. This is a contradiction.
Case 2. (ii) not holds. There exists $t_0 \in X\alpha$ such that $t_0\alpha \neq t_0$. Note that $x_0\alpha = t_0$ for some $x_0 \in X$. Then, $x_0\alpha^2 = (x_0\alpha)\alpha = t_0\alpha \neq t_0 = x_0\alpha$, which contradicts the fact that α is an idempotent. □

Lemma 3.5. *Let $Z \subsetneq Y \subsetneq X$. Then, not each $\mathcal{L}^*$-class of $\mathcal{T}(X, Y, Z)$ contains an idempotent.*

Proof. Let $f \in \mathcal{T}(X, Y, Z)$ such that $(X \setminus Y)f \cap (Y \setminus Z) \neq \emptyset$. Next, we prove that the $\mathcal{L}^*$-class $\mathcal{L}^*_f$ containing f has no idempotents. Assume that $(f, e) \in \mathcal{L}^*$ for some idempotent $e \in \mathcal{T}(X, Y, Z)$, then two cases are considered as follows:
Case 1. $|Z| = 1$. Since Lemma 3.3 (1) it follows that $(X \setminus Y)e \cap (Y \setminus Z) \neq \emptyset$ and so $Xe \cap (Y \setminus Z) \neq \emptyset$.

Case 2. $|Z| \geq 2$. From Lemma 3.3 (2) it follows that $Xe = Xf$ and so $Xe \cap (Y \setminus Z) \neq \emptyset$.

However, we have $Xe \subseteq Z \cup (X \setminus Y)$ since Lemma 3.4 (1). Note that $Z \subsetneq Y \subsetneq X$, then $Xe \cap (Y \setminus Z) = \emptyset$. This is a contradiction. □

After that, we consider the $\mathcal{R}^*$-relation. Let π_α be the partition of X induced by $\alpha \in \mathcal{T}(X,Y,Z)$, namely,

$$\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}.$$

Lemma 3.6. *Let $Z \subsetneq Y \subsetneq X$ and $\alpha, \beta \in \mathcal{T}(X,Y,Z)$. Then, $(\alpha,\beta) \in \mathcal{R}^*$ if and only if $\pi_\alpha = \pi_\beta$.*

Proof. Let $\pi_\alpha = \pi_\beta$. This implies that α, β are $\mathcal{R}$-related in the full transformation semigroup $\mathcal{T}(X)$ (see [9, page 63]). Hence, $(\alpha,\beta) \in \mathcal{R}^*$.

Conversely, suppose that $(\alpha,\beta) \in \mathcal{R}^*$ and $x_1\alpha = x_2\alpha$ for some distinct $x_1, x_2 \in X$. We show that $x_1\beta = x_2\beta$. There are three cases to be considered.
Case 1. $x_1, x_2 \in Z$. Define two mappings $\eta : X \to X$ and $\theta : X \to X$ by

$$x\eta = \begin{cases} x_1, & \text{if } x \in Y \\ x, & \text{if } x \in X \setminus Y \end{cases} \quad \text{and} \quad x\theta = \begin{cases} x_2, & \text{if } x \in Y \\ x, & \text{if } x \in X \setminus Y. \end{cases}$$

Clearly, $\eta, \theta \in \mathcal{T}(X,Y,Z)$ and $\eta\alpha = \theta\alpha$. Then, $\eta\beta = \theta\beta$ and so $x_1\beta = Y\eta\beta = Y\theta\beta = x_2\beta$.
Case 2. $x_1, x_2 \in X \backslash Z$. Choose and fix $z_0 \in Z$. Define two mappings $\eta : X \to X$ and $\theta : X \to X$ by

$$x\eta = \begin{cases} z_0, & \text{if } x \in Y \\ x_1, & \text{if } x \in X \setminus Y \end{cases} \quad \text{and} \quad x\theta = \begin{cases} z_0, & \text{if } x \in Y \\ x_2, & \text{if } x \in X \setminus Y. \end{cases}$$

Clearly, $\eta, \theta \in \mathcal{T}(X,Y,Z)$ and $\eta\alpha = \theta\alpha$. Then, $\eta\beta = \theta\beta$ and so $x_1\beta = (X \setminus Y)\eta\beta = (X \setminus Y)\theta\beta = x_2\beta$.
Case 3. $x_1 \in Z$ and $x_2 \in X \backslash Z$. Define two mappings $\eta : X \to X$ and $\theta : X \to X$ by $X\eta = x_1$ and

$$x\theta = \begin{cases} x_1, & \text{if } x \in Y \\ x_2, & \text{if } x \in X \setminus Y. \end{cases}$$

Clearly, $\eta, \theta \in \mathcal{T}(X,Y,Z)$ and $\eta\alpha = \theta\alpha$. Then, $\eta\beta = \theta\beta$ and so $x_1\beta = (X \setminus Y)\eta\beta = (X \setminus Y)\theta\beta = x_2\beta$.

For both cases, we have $\pi_\alpha \subseteq \pi_\beta$. Dually, we may show that $\pi_\beta \subseteq \pi_\alpha$. Consequently, $\pi_\alpha = \pi_\beta$. □

Lemma 3.7. *Let $Z \subsetneq Y \subsetneq X$. Then, the following statements hold:*

(i) for $|Z| = 1$, each $\mathcal{R}^*$-class of $\mathcal{T}(X,Y,Z)$ contains an idempotent;

(ii) for $|Z| \geq 2$, not each $\mathcal{R}^*$-class of $\mathcal{T}(X,Y,Z)$ contains an idempotent.

Proof. (i) Let $\alpha \in \mathcal{T}(X,Y,Z)$. Then, exists an index set I such that $\pi_\alpha = \{A_i : i \in I\}$. Note that $Y\alpha \subseteq Z$ and $|Z| = 1$, there exists $i \in I$ such that $Y \subseteq A_i$. Take $z_0 \in Z$ and $a_j \in A_j$, for all $j \in I \setminus \{i\}$. Define a mapping $e : X \to X$ by

$$xe = \begin{cases} z_0, & \text{if } x \in A_i \\ a_j, & \text{if } x \in A_j, \text{ for all } j \in I \setminus \{i\}. \end{cases}$$

Clearly, $e \in \mathcal{T}(X,Y,Z)$ is an idempotent and $\pi_\alpha = \pi_e$. By Lemma 3.6, we have $(\alpha, e) \in \mathcal{R}^*$. Hence, each $\mathcal{R}^*$-class of $\mathcal{T}(X,Y,Z)$ contains an idempotent.

(ii) By $|Z| \geq 2$, we can take distinct $z_1, z_2 \in Z$. Define $f \in \mathcal{T}(X,Y,Z)$ such that $Zf = z_1$ and $(Y \setminus Z)f = z_2$. Then, $Z \subseteq A_i$ and $(Y \setminus Z) \subseteq A_j$ for some distinct $A_i, A_j \in \pi_f = \{A_j : j \in J\}$ where J be some index set. We assert that the $\mathcal{R}^*$-class $\mathcal{R}^*_f$ containing f has no idempotents. Indeed, if $(f, e) \in \mathcal{R}^*$ for some idempotent $e \in \mathcal{T}(X,Y,Z)$. Then, by Lemma 3.6 it follows that $\pi_e = \pi_f$. According to Lemma 3.4, $|A_j e \cap A_j| = 1$ and so $(Y \setminus Z)e = A_j e \in A_j$. Note that $Z \subseteq A_i$ and $A_i \cap A_j = \emptyset$. Then, $(Y \setminus Z)e \cap Z = \emptyset$. This contradicts the fact that $(Y \setminus Z)e \subsetneq Ye \subseteq Z$. □

By Lemmas 3.5 and 3.7, we obtain the main result in this section.

Theorem 3.1. *Let $Z \subsetneq Y \subsetneq X$. Then, the following statements hold:*

(i) for $|Z| = 1$, the semigroup $\mathcal{T}(X,Y,Z)$ is right abundant;

(ii) for $|Z| \geq 2$, the semigroup $\mathcal{T}(X,Y,Z)$ is neither left abundant nor right abundant.

As a consequence of Lemma 1.1, Lemma 1.2 and Theorem 3.1, we have the following conclusion.

Corollary 3.1. (I) *for $Z = Y = X$, the semigroup $\mathcal{T}(X,Y,Z) = \mathcal{T}(X)$ is abundant.*

(II) *for $Z \subsetneq Y = X$,*

(i) $|Z| = 1$, the semigroup $\mathcal{T}(X,Y,Z) = \mathcal{T}(X,Z)$ is abundant;

(ii) $|Z| \geq 2$, the semigroup $\mathcal{T}(X,Y,Z) = \mathcal{T}(X,Z)$ is left abundant but not right abundant.

(III) *for $Z = Y \subsetneq X$, the semigroup $\mathcal{T}(X,Y,Z) = \overline{\mathcal{T}}(X,Y)$ is abundant;*

(IV) *for $Z \subsetneq Y \subsetneq X$,*

(i) $|Z| = 1$, the semigroup $\mathcal{T}(X,Y,Z)$ is right abundant;

(ii) $|Z| \geq 2$, the semigroup $\mathcal{T}(X,Y,Z)$ is neither left abundant nor right abundant.

4. Some combinatorial results

The Stirling number of the second kind $S(n, r)$ counts the number of partitions of a set of n elements into r indistinguishable boxes in which no box is empty.

Recall that the number of ways that r objects can be chosen from n distinct objects written $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

It is shown in [4, Theorem 8.26] that

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n.$$

for integers n and r with $0 \le r \le n$. In particular, $S(p,0) = 0$ $(p \ge 1)$ and $S(0,0) = 1$. Bóna [5] also presented a formula related Stirling number, that is,

Lemma 4.1. ([5, page 32]) *Let $m, k \in \mathbb{N}$ such that $1 \le k \le m$. Then:*

$$\sum_{r=1}^{k}\binom{k}{r}r!S(m,r) = k^m.$$

Lemma 4.2. *Let $|X| = n$, $|Y| = m$ and $|Z| = k$. Then, for each $r \in \mathbb{N}$ with $1 \le r \le k$,*

$$|\{\alpha \in \mathcal{T}(X,Y,Z) : |Y\alpha| = r\}| = \binom{k}{r}r!S(m,r)n^{n-m}. \tag{1}$$

Proof. Let Z' be a nonempty subset of Z with $|Z'| = r$, we have $1 \le r \le k$ since $|Z| = k$. It is easy to see that the number of mappings $\alpha : X \to X$ such that $Y\alpha = Z'$ and $(X \setminus Y)\alpha \subseteq X$ is $r!S(m,r)n^{n-m}$, that is,

$$|\{\alpha \in \mathcal{T}(X,Y,Z) : Y\alpha = Z'\}| = r!S(m,r)n^{n-m}.$$

Consequently, Equation (1) holds for each $r \in \mathbb{N}$ with $1 \le r \le k$. □

Theorem 4.1. *Let $|X| = n$, $|Y| = m$ and $|Z| = k$. Then:*

$$|\mathcal{T}(X,Y,Z)| = \sum_{r=1}^{k}\binom{k}{r}r!S(m,r)n^{n-m} = k^m n^{n-m}. \tag{2}$$

Proof. According to Lemma 4.2, we have

$$|\{\alpha \in \mathcal{T}(X,Y,Z) : |Y\alpha| = r\}| = \binom{k}{r}r!S(m,r)n^{n-m},$$

for each $r \in \mathbb{N}$ with $1 \le r \le k$. Then, $|\mathcal{T}(X,Y,Z)| = \sum_{r=1}^{k}\binom{k}{r}r!S(m,r)n^{n-m}$ by the summing up over all r. Moreover, from Lemma 4.1 it follows that $\sum_{r=1}^{k}\binom{k}{r}r!S(m,r)n^{n-m} = k^m n^{n-m}$. Hence, Equation (2) as required. □

Since Theorem 4.1, we obtain the following corollary which appears in [18, page 311].

Corollary 4.1. *Let* $|X| = n$, $|Y| = m$ *and* $|Z| = k$. *Then:*

(*i*) $|\overline{\mathcal{T}}(X,Y)| = \sum_{r=1}^{m} \binom{m}{r} r! S(m,r) n^{n-m} = m^m n^{n-m}$;

(*ii*) $|\mathcal{T}(X,Z)| = \sum_{r=1}^{k} \binom{k}{r} r! S(n,r) = k^n$;

(*iii*) $|\mathcal{T}(X)| = \sum_{r=1}^{n} \binom{n}{r} r! S(n,r) = n^n$.

Next, we determine the number of all regular elements in the semigroup $\mathcal{T}(X,Y,Z)$ when X is finite.

Theorem 4.2. *Let* $|X| = n$, $|Y| = m$ *and* $|Z| = k$. *Then:*

$$|\mathrm{Reg}(\mathcal{T}(X,Y,Z))| = \sum_{r=1}^{k} \binom{k}{r} r! S(k,r) r^{m-k} (n-m+r)^{n-m}. \tag{3}$$

Proof. For each $\alpha \in \mathrm{Reg}(\mathcal{T}(X,Y,Z))$, we have $X\alpha \cap Y = Z\alpha \subseteq Y\alpha \subseteq Z$ by Theorem 2.1. Then, exists a nonempty subset Z' of Z with $|Z'| = r$ such that $Z\alpha = X\alpha \cap Y = Z'$. Clearly, $(Y \setminus Z)\alpha \subseteq Y\alpha \subseteq X\alpha \cap Z \subseteq X\alpha \cap Y = Z'$ and so

$$(Y \setminus Z)\alpha \subseteq Z'. \tag{4}$$

We can also assert

$$(X \setminus Y)\alpha \subseteq Z' \cup (X \setminus Y) \tag{5}$$

(If not, there exists some $y \in X \setminus Y$ such that $y\alpha \in Y \setminus Z'$, then $y\alpha \in X\alpha \cap Y = Z'$. This is a contradiction). Conversely, if a mapping $\alpha \in \mathcal{T}(X,Y,Z)$ satisfies $Z\alpha = Z'$, formulas (4) and (5), it is easy to see that

$$X\alpha \cap Y = [Z \cup (Y \setminus Z) \cup (X \setminus Y)]\alpha \cap Y \subseteq [Z' \cup (X \setminus Y)] \cap Y = Z' = Z\alpha$$

since $Z' \subseteq Z \subseteq Y \subseteq X$. Then, by Theorem 2.1, we have $\alpha \in \mathrm{Reg}(\mathcal{T}(X,Y,Z))$ and $Z\alpha = Z'$. Hence, for each nonempty set $Z' \subseteq Z$, we have

$$\begin{aligned} &\{\alpha \in \mathrm{Reg}(\mathcal{T}(X,Y,Z)) : Z\alpha = Z'\} \\ =&\{\alpha \in \mathcal{T}(X,Y,Z) : \alpha \text{ satisfies } Z\alpha = Z',\ \text{formulas (4) and (5)}\}. \end{aligned}$$

It follows that the number of maps $\alpha \in \mathcal{T}(X,Y,Z)$ satisfying $Z\alpha = Z'$, formulas (4) and (5) is $r!S(k,r)r^{m-k}(n-m+r)^{n-m}$ since $|Z' \cup (X \setminus Y)| = |X \setminus Y| + |Z'| = n - m + r$, that is,

$$|\{\alpha \in \mathrm{Reg}(\mathcal{T}(X,Y,Z)) : Z\alpha = Z'\}| = r!S(k,r)r^{m-k}(n-m+r)^{n-m}.$$

Consequently, for each $r \in \mathbb{N}$ with $1 \leq r \leq k$,

$$|\{\alpha \in \mathrm{Reg}(\mathcal{T}(X,Y,Z)) : |Z\alpha| = r\}| = \binom{k}{r} r! S(k,r) r^{m-k} (n-m+r)^{n-m}$$

and so Equation (3) holds by the summing up over all r. □

Since Theorem 4.2, we obtain the following corollary which appears in [18, Theorem 2.6 and Theorem 2.7].

Corollary 4.2. *Let* $|X| = n$, $|Y| = m$ *and* $|Z| = k$. *Then,*

(*i*) $|\mathrm{Reg}(\overline{\mathcal{T}}(X,Y))| = \sum_{r=1}^{m} \binom{m}{r} r! S(m,r)(n-m+r)^{n-m}$.

(*ii*) $|\mathrm{Reg}(\mathcal{T}(X,Z))| = \sum_{r=1}^{k} \binom{k}{r} r! S(k,r) r^{n-k}$.

Moreover, we compute the cardinality of $\mathrm{E}(\mathcal{T}(X,Y,Z))$.

Theorem 4.3. *Let* $|X| = n$, $|Y| = m$ *and* $|Z| = k$. *Then:*

$$|\mathrm{E}(\mathcal{T}(X,Y,Z))| = \sum_{r=1}^{n-m+k} \sum_{i=\max\{1,m-n+r\}}^{\min\{k,r\}} \binom{k}{i} \binom{n-m}{r-i} i^{m-i} r^{n-m-r+i}. \tag{6}$$

Proof. Define an idempotent α with $|X\alpha| = r$, we have to choose a r-element set $X\alpha$, then exists $i \in \mathbb{N}$ such that $|X\alpha \cap Z| = i$ and $|X\alpha \cap (X \setminus Y)| = r - i$ by Lemma 3.4 (There are $\binom{k}{i}\binom{n-m}{r-i}$ different ways). Also, we have to define a mapping $\varphi : X \setminus X\alpha \to X\alpha$ such that $\varphi(Y \setminus X\alpha) \subseteq Z$ and $\varphi((X \setminus Y) \setminus X\alpha) \subseteq X\alpha$ in an arbitrary way (This can be done in $i^{m-i} r^{n-m-r+i}$ different ways). Note that i meets $1 \leq i \leq k$ and $0 \leq r - i \leq n - m$. Then, $\max\{1, m-n+r\} \leq i \leq \min\{k, r\}$. Hence,

$$|\{\alpha \in \mathrm{E}(\mathcal{T}(X,Y,Z)) : |X\alpha| = r\}| = \sum_{i=\max\{1,m-n+r\}}^{\min\{k,r\}} \binom{k}{i} \binom{n-m}{r-i} i^{m-i} r^{n-m-r+i}$$

by summing up over all i. Note that

$$1 \leq r = |X\alpha| \leq |Y\alpha| + |(X \setminus Y)\alpha| \leq |Z| + |X \setminus Y| = n - m + k.$$

Therefore Equation (6) is now obtained by summing up over all r. □

Since Theorem 4.3, we obtain the following corollary.

Corollary 4.3. *Let* $|X| = n$, $|Y| = m$ *and* $|Z| = k$. *Then:*

(*i*) $|\mathrm{E}(\overline{\mathcal{T}}(X,Y))| = \sum_{r=1}^{n} \sum_{i=\max\{1,m-n+r\}}^{\min\{m,r\}} \binom{m}{i} \binom{n-m}{r-i} i^{m-i} r^{n-m-r+i}$.

(*ii*) $|\mathrm{E}(\mathcal{T}(X,Z))| = \sum_{r=1}^{k} \binom{k}{r} r^{n-r}$.

(*iii*) $|\mathrm{E}(\mathcal{T}(X))| = \sum_{r=1}^{n} \binom{n}{r} r^{n-r}$.

Acknowledgements

This work was supported by the doctoral research start-up fund of Guiyang University (GYU-KY-2023).

References

[1] J. Araujo, J. Konieczny, *Semigroups of transformations preserving an equivalence relation and a cross-section*, Communications in Algebra, 32 (2004), 1917-1935.

[2] G. Ayık, H. Ayık, M. Koç, *Combinatorial results for order-preserving and order-decreasing transformations*, Turkish Journal of Mathematics, 35 (2011), 1-9.

[3] L. Bugay, M. Yağcı, H. Ayık, *Combinatorial results for semigroups of order-preserving and A-decreasing finite transformations*, Bulletin of the Malaysian Mathematical Sciences Society, 42 (2019), 921-932.

[4] R. Brualdi, *Introductory combinatorics*, Prentice Hall Press, Englewood Cliffs, 2009.

[5] M. Bóna, *Combinatorics of Permutations*, CRC Press, Boca Raton, 2004.

[6] V. H. Fernandes, G. M. S. Gomes, M. M. Jesus, *The cardinal and the idempotent number of various monoids of transformations on a finite chain*, Bulletin of the Malaysian Mathematical Sciences Society, 34 (2011), 79-85.

[7] J. B. Fountain, *Abundant semigroups*, Proceedings of the London Mathematical Society, 44 (1982) 1, 103-129.

[8] O. Ganyushkin, V. Mazorchuk, *Classical finite transformation semigroups*, Springer Press, London, 2009.

[9] J. Howie, *Fundamentals of semigroup theory*, Oxford University Press, New York, 1995.

[10] P. Honyam, J. Sanwong, *Semigroups of transformations with invariant set*, Journal of the Korean Mathematical Society, 48 (2011) 2, 289-300.

[11] S. Jitman, R. Srithus, C. Worawannotai, *Regularity of semigroups of transformations with restricted range preserving an alternating orientation order*, Turkish Journal of Mathematics, 42 (2018), 1913-1926.

[12] J. L. Jin, F. Y. Zhu, T. J. You, Y. Y. Qu, *Rank of the finite transformation semigroup of weak Y-stabilizer*, Journal of Northeast Normal University (Natural Science Edition), 52 (2020) 3, 50-55.

[13] J. L. Jin, W. Teng, F. Y. Zhu, T. J. You, Y. Y. Qu, *Maximal subsemigroups of the finite transformation semigroup of weak Y-stabilizer*, Journal of Shandong University (Natural Science), 55 (2020) 10, 55-62.

[14] J. L. Jin, *On the rank of semigroup of transformations with restricted partial range*, Filomat, 35 (2021) 14, 4925-4936.

[15] V. I. Kim, I. B. Kozhukhov, *Regularity conditions for semigroups of isotone transformations of countable chains*, Journal of Mathematical Sciences, 152 (2008), 203-208.

[16] M. Y. Ma, T. J. You, S. S. Luo, Y. X. Yang, L. C. Wang, *Regularity and Green's relation for finite E-order-preserving transformations semigroups*, Semigroup Forum, 80 (2010), 164-173.

[17] K. D. Magill, *Subsemigroups of* $\mathcal{S}(X)$, Math. Japan, 11 (1966), 109-115.

[18] S. Nenthein, P. Youngkhong, Y. Kemprasit, *Regular elements of some transformation semigroups*, Pure mathematics and applications, 16 (2005) 3, 307-314.

[19] P. Purisang, J. Rakbud, *Regularity of transformation semigroups defined by a partion*, Communications of The Korean Mathematical Society, 31 (2016) 2, 217-227.

[20] J. Sanwong, W. Sommanee, *Regularity and Green's relations on a semigroup of transformations with restricted range*, International Journal of Mathematics and Mathematical Sciences, (2008) 794013.

[21] J. Sanwong, *The regular part of a semigroup of transformations with restricted range*, Semigroup Forum, 83 (2011), 134-146.

[22] L. Sun, L. M. Wang, *Natural partial order in semigroups of transformations with invariant set*, Bulletin of the Australian Mathematical Society, 87 (2013) 1, 94-107.

[23] L. Sun, *A note on abundance of certain semigroups of transformations with restricted range*, Semigroup Forum, 87 (2013), 681-684.

[24] L. Sun, J. L. Sun, *A natural partial order on certain semigroups of transformations with restricted range*, Semigroup Forum, 92 (2016) 1, 135-141.

[25] J. S. V. Symons, *Some results concerning a transformation semigroup*, Journal of the Australian Mathematical Society, 19 (1975) 4, 413-425.

[26] K. Toker, H. Ayik, *On the rank of transformation semigroup* $\mathcal{T}_{(n,m)}$, Turkish Journal of Mathematics, 42 (2018), 1970-1977.

[27] A. Umar, *On the semigroups of order-decreasing finite full transformations*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 120 (1992), 129-142.

[28] Q. F. Yan, S. F. Wang, *Some results on semigroups of transformations with restricted range*, Open Mathematics, 19 (2021), 69-76.

[29] W. Yonthanthum, *Regular elements of the variant semigroups of transformations preserving double direction equivalences*, Thai Journal of Mathematics, 16 (2018) 1, 165-171.

Accepted: February 7, 2023

On some properties of Nörlund ideal convergence of sequence in neutrosophic normed spaces

Vakeel A. Khan*
Department of Mathematics
Aligarh Muslim University
Aligarh–202002
India
vakhanmaths@gmail.com

Mohammad Arshad
Department of Mathematics
Aligarh Muslim University
Aligarh- 202002
India
mohammadarshad3828@gmail.com

Abstract. The purpose of this paper is to introduce the Nörlund ideal convergent sequence spaces with respect to these spaces $\mathscr{N}^f_{I_0(\mathcal{S})}$, $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_\infty(\mathcal{S})}$. Also, we studied the Nörlund ideal Cauchy criterion in neutrosophic normed space and its properties. Also, we define an open ball $B(x,\epsilon,\gamma)$ and closed ball $B[x,\epsilon,\gamma]$ in neutrosophic norm space. Furthermore, we also look at some of these convergent sequence spaces' topological and algebraic properties.

Keywords: ideal convergent, ideal Cauchy, Nörland mean, Nörlund matrix, sequence space, Nörlund ideal convergent, Nörlund ideal Cauchy sequence and neutrosophic normed space.

1. Introduction

The fuzzy set was first developed in 1965 by Zadeh [27], and they have since been used in a variety of domains, including artificial intelligence, robotics, and control theory. According to him, a fuzzy set assigns a membership value from $[0,1]$ to each element of a given crisp universe set.

Atanassov K.T. in [14], [13] introduced the intuitionistic fuzzy set (IFS) on a universe X as an extension of the fuzzy set. Coker [15] used this concept to develop intuitionistic fuzzy topological spaces. Saadati and Park [20] investigated these spaces and their extension, resulting in the idea of intuitionistic fuzzy normed space.

In 1998, Samarandache [3] presented the first philosophical point for neutrosophic set. The concept of classic set theory has been extended in the form of the neutrosophic set by adding an intermediate membership function. Examples

∗. Corresponding author

of other generalizations are the Fuzzy set [27], and intuitionistic fuzzy set [14]. The actual definition of neutrosophic sets was given based on the independence of membership, non-membership, and hesitation function.

In 2006, F. Samarandache and W.B. Vasantha Kanasamy in [26] introduced the concept of neutrosophic algebraic structures.

Bera and Mahapatra [21] first introduced the neutrosophic soft linear space. Neutrosophic soft norm linear space, convexity, metric [34], and Cauchy sequence were examined by Bera and Mahapatra [22]. The purpose of the current paper is to change the intuitionistic fuzzy normed space of the structure into neutrosophic normed space. The Cauchy sequence has been studied on neutrosophic normed space in an attempt to investigate some beautiful results in this structure.

H. Fast [5] and I. J. Schoenberg [6] introduce the idea of statistical convergence, whereas J. C̃erveñanský [28] and J.S. Connor [29, 30] develop it. R.C. Buck [31, 32] and D.S. Mitrinović [33] include some examples of statistical convergence in mathematical analysis and number theory. The idea of statistical convergence with regard to the intuitionistic fuzzy norm was introduced by Mursaleen [16]. In neutrosophic normed space, statistical convergence was first investigated by Kirisci and Simsek [7]. The concept of "ideal convergence" is an extension of the notion of "statistical convergence", and it is dependent on the idea of the ideal of subsets of the set $\mathbb{N}$. Śalát et al. [23], [24], Filipów and Tryba [19], Khan and Nazreen [12], Khan et al. [11], Khan and Nazreen [12] and several more writers further investigated the concept of I-convergent from the perspective of sequence space and related it with the summability theory. To better understand the I-convergence in neutrosophic normed space, we have been inspired by this.

The purpose of this study is to define new neutrosophic sequence spaces using the Nörlund matrix and the neutrosophic norm. Also, we will study Nörlund I-convergent and Nörlund I-Cauchy in neutrosophic normed spaces, and by using the Nörlund matrix $\mathscr{N}^f$ and the notion of Nörlund I-convergent of sequence in neutrosophic normed space, we introduce some new spaces of Nörlund I-convergent sequence with regard to the neutrosophic norm $(\mathcal{U}, \mathcal{V}, \mathcal{W})$. We also investigate at some of these convergent sequence spaces' topological and algebraic properties, as well as some interesting connections between these spaces $\mathscr{N}^f_{I_0(\mathcal{S})}$, $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_\infty(\mathcal{S})}$.

2. Preliminaries

Definition 2.1 ([9])**.** Let I be the power set of any set Z, where Z is the set. Then, I is called ideal, if:

(1) $\emptyset \in I$;

(2) $\vartheta_1, \vartheta_2 \in I \Rightarrow \vartheta_1 \cup \vartheta_2 \in I$, additive;

(3) $\vartheta_1 \in I, \vartheta_2 \subseteq \vartheta_1 \Rightarrow \vartheta_2 \in I$, hereditary.

If $I \neq 2^Z$ then $I \subseteq 2^Z$ is called nontrivial. If I contain every singleton subset of X. then nontrivial ideal $I \subseteq 2^Z$ is called admissible. If there are no non-trivial ideal $K \neq I$ then nontrivial ideal I is maximal such that $I \subset K$.

Definition 2.2 ([9]). Let $\mathscr{F}$ be the power set of any set Z, where Z is the set. Then, $\mathscr{F}$ is said to be filter. If: (1) $\emptyset \notin \mathscr{F}$;

(2) For $\vartheta_1, \vartheta_2 \in \mathscr{F}$; $\vartheta_1 \cap \vartheta_2 \in \mathscr{F}$;

(3) If $\vartheta_1 \in \mathscr{F}$ and $\vartheta_2 \supset \vartheta_1$ imply $\vartheta_2 \in \mathscr{F}$.

$\mathscr{F}(I)$ is the filter associated with each ideal$I$ of Z such that $\mathscr{F}(I) = \big\{A \subset Z : A^c \in I\big\}$ is true for each ideal of Z. Then, using the article, we present I as an admissible ideal.

Note. Class $\mathscr{F}(I) = \{\vartheta_1 \subset Z : \vartheta_1 = Z/\vartheta_2, \text{ for some } \ \vartheta_2 \in I\}$ is a filter on Z, where $I \subset P(Z)$ is a non-trivial ideal. $\mathscr{F}(I)$ is described as the filter associated with the ideal I.

Definition 2.3 ([8]). In any set Z, let I be a non trivial ideal subset of a power set $(P(Z))$. So, it is said that a sequence $x = (x_k)$ is ideally convergent to α, iff the set $\{k \in Z : |x_k - \alpha| \geq \epsilon\} \in I$ and we write it as $I - \lim x = \alpha$, for every $\epsilon > 0$.

Definition 2.4 ([8]). In any set Z, let I be a non trivial ideal subset of a power set $(P(Z))$. So, it is said that a number sequence $x = (x_k)$ is ideally Cauchy. If, for any $\epsilon > 0, \exists\ L = L(\epsilon)$, the set $\{k \in Z : |x_k - x_L| \geq \epsilon\} \in I$.

The Nörlund matrix $\mathscr{N}^f$ was initially used in the theory of sequence space by Wang [25]. Remember that $t = (t_k)$ is a non negative sequence of real numbers and $A_n = \sum_{k=0}^{n} t_k, \forall\ n \in \mathbb{N}$ with $t_0 > 0$. Then, with regard to the sequence $t = (t_k)$, the Norlund matrix $\mathscr{N}^f = (a^t_{nm})$ is defined as follows:

$$a^t_{nm} = \begin{cases} \dfrac{t_{n-m}}{A_n}, & \text{if } 0 \leq m \leq n \\ 0, & \text{if } m > n, \end{cases} \tag{1}$$

for all $n, m \in \mathbb{N}$. It is known that the Nörlund matrix $\mathscr{N}^f$ is regular iff $t_n/T_n \to 0$ as $n \to \infty$.

Let $t_0 = D_0 = 1$ and define L_n for $n \in \{1, 2, 3, ...\}$ by

$$D_n = \begin{bmatrix} t_1 & 1 & 0 & 0 & ...0 \\ t_2 & t_1 & 1 & 0 & ...0 \\ t_3 & t_2 & t_1 & 1 & ...0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & ...1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & ...t_1 \end{bmatrix} \tag{2}$$

Then, the inverse matrix $L^t = (l^t_{nm})$ of Nörland matrix $\mathscr{N}^f = (a^t_{nm})$ was define by Mears in [4], for all $n \in \mathbb{N}$, as follows

$$l_{nm} = \begin{cases} (-1)^{n-m} D_{n-m} T_k, & \text{if } (0 \leq m \leq n), \\ 0, & \text{otherwise}, \end{cases}$$

for all $n, m \in \mathbb{N}$.

One can refer to [4, 2, 1] for more background about Norland space.

In this paper, the natural and real number sets, respectively, are denoted by the letters $\mathbb{N}$ and $\mathbb{R}$. ω also represents for the linear space having all real sequences. The sequence spaces c_0, c and l_∞ represent the spaces of all null, convergent, and bounded sequences, respectively. We now define the Nörlund sequence space established by Wang in [25] as follows

$$\mathscr{N}^f = \left\{ x = (x_k) \in \omega = \sum_{n=0}^{\infty} |\frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k|^p < \infty, 1 \leq p < \infty \right\},$$

where $A_n = \sum_{k=0}^{n} a_k$. All sequences whose Norlund transformations are in the spacel_∞ and l_p with $1 \leq p < \infty$ are contained in the spaces $l_\infty(\mathscr{N}^f)$ and $l_p(\mathscr{N}^f)$.

Motivated by [17], Khan [8] recently presented the sequence spaces $c^I_0(\mathscr{N}^f)$, $c^I(\mathscr{N}^f)$, and $l^I_\infty(\mathscr{N}^f)$ as the sets of all sequences whose $\mathscr{N}^f$ transformations are in spaces c_0, c, and l_∞, respectively. Khan did this by using the concept of Nörlund $I-$convergence, Nörlund $I-$ null and Nörlund $I-$ bounded sequence space,where I is an admissible ideal of subset of $\mathbb{N}$. For more details on these spaces, we refer to [18, 8]. Define

$$c^I_0(\mathscr{N}^f) := \left\{ y = (y_k) \in \omega : \{n \in \mathbb{N} : |\mathscr{N}^f_n(y)| \geq \epsilon\} \in I \right\},$$
$$c^I(\mathscr{N}^f) := \{y{=}(y_k){\in}\omega : \{n{\in}\mathbb{N} : |\mathscr{N}^f_n(y) - K|{\geq}\epsilon \text{ for some } K \in \mathbb{R}\}{\in}I\},$$
$$l^I_\infty(\mathscr{N}^f) := \left\{ y = (y_k) \in \omega : \exists M > 0 \ \text{ s.t } \{n \in \mathbb{N} : |\mathscr{N}^f_n(y)| \geq M\} \in I \right\},$$

where

$$\mathscr{N}^f_n(y) := \frac{1}{T_n} \sum_{k=0}^{n} t_{n-k} y_k, \qquad \text{for all } n \in \mathbb{N}. \tag{3}$$

Definition 2.5 ([10, 7])**.** Given an binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if:

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $\vartheta * 1 = \vartheta \ \forall \ \vartheta \in [0,1]$;

(d) $\vartheta_1 * \vartheta_2 \leq \vartheta_3 * \vartheta_4$ whenever $\vartheta_1 \leq \vartheta_3$ and $\vartheta_2 \leq \vartheta_4$ for each $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in [0,1]$.

Example 2.1. For $\vartheta_1, \vartheta_2 \in [0,1]$, define $\vartheta_1 * \vartheta_2 = \vartheta_1 \vartheta_2$ or $\vartheta_1 * \vartheta_2 = \min\{\vartheta_1, \vartheta_2\}$, then $*$ is continuous t-norm.

Definition 2.6 ([10, 7])**.** Given an binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if:

(a) $\diamond$ is commutative and associative;
(b) $\diamond$ is continuous;
(c) $\vartheta \diamond 0 = \vartheta \; \forall \; \sigma \in [0,1]$;
(d) $\vartheta_1 \diamond \vartheta_2 \leq \vartheta_3 \diamond \vartheta_4$ whenever $\vartheta_1 \leq \vartheta_3$ and $\vartheta_2 \leq \vartheta_4$ for each $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in [0,1]$.

Example 2.2. Let $\vartheta_1, \vartheta_2 \in [0,1]$. Define $\vartheta_1 \diamond \vartheta_2 = \min\{\vartheta_1 + \vartheta_2, 1\}$ or $\vartheta_1 \diamond \vartheta_2 = \max\{\vartheta_1, \vartheta_2\}$, then $\diamond$ is continuous t-conorm.

Definition 2.7 ([20])**.** Take Z as a linear space and $\mathcal{S} = \{< x, \mathcal{U}(x), \mathcal{V}(x), \mathcal{W}(x) >: x \in Z\}$ be a normed space such that $\mathcal{S} : Z \times (0, \infty) \longrightarrow [0,1]$. Assume $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm respectively. The four-tuple $V = (Z, \mathcal{S}, *, \diamond)$ is said to be neutrosophic normed space (NNS) if the subsequent conditions are hold, for all $x, y, \in Z$ and $\gamma, \delta > 0$:

(1) $0 \leq \mathcal{U}(x,\gamma) \leq 1, 0 \leq \mathcal{V}(x,\gamma) \leq 1, 0 \leq \mathcal{W}(x,\gamma) \leq 1, \gamma \in \mathbb{R}^+$;
(2) $\mathcal{U}(x,\gamma) + \mathcal{V}(x,\gamma) + \mathcal{W}(x,\gamma) \leq 3$, for $\gamma \in \mathbb{R}^+$;
(3) $\mathcal{U}(x,\gamma) = 1$ for $\gamma > 0$ iff $x = 0$;
(4) $\mathcal{U}(\alpha x, \gamma) = \mathcal{U}\ (x, \frac{\gamma}{|\alpha|})$;
(5) $\mathcal{U}(x,\gamma) * \mathcal{U}(y,\delta) \leq \mathcal{U}(x+y, \gamma+\delta)$;
(6) $\mathcal{U}(x,*)$ is continuous nondecreasing function;
(7) $\lim_{\gamma\to\infty} \mathcal{U}(x,\gamma) = 1$;
(8) $\mathcal{V}(x,\gamma) = 0$ for $\gamma > 0$ iff $x = 0$;
(9) $\mathcal{V}(\alpha x, \gamma) = \mathcal{V}(x, \frac{\gamma}{|\alpha|})$;
(10) $\mathcal{V}(x,\gamma) \diamond \mathcal{V}(y,\delta) \geq \mathcal{V}(x+y, \gamma+\delta)$;
(11) $\mathcal{V}(x,\diamond)$ is continuous nonincreasing function;
(12) $\lim_{\gamma\to\infty} \mathcal{V}(x,\gamma) = 0$;
(13) $\mathcal{W}(x,\gamma) = 0$ for $\gamma > 0$ iff $x = 0$;
(14) $\mathcal{W}(\alpha x, \gamma) = \mathcal{W}(x, \frac{\gamma}{|\alpha|})$;
(15) $\mathcal{W}(x,\gamma) \diamond \mathcal{W}(y,\delta) \geq \mathcal{W}(x+y, \gamma+\delta)$;
(16) $\mathcal{W}(x,\diamond)$ is continuous nonincreasing function;
(17) $\lim_{\gamma\to\infty} \mathcal{W}(x,\gamma) = 0$;
(18) if $\gamma \leq 0$, then $\mathcal{U}(x,\gamma) = 0, \mathcal{V}(x,\gamma) = 1, \mathcal{W}(x,\gamma) = 1$.

In such case, $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ is said to be neutrosophoic norm (NN).

Example 2.3 ([10])**.** Suppose $(Z, \|.\|)$ be a normed space. Using the $*$ and $\diamond$ operations, as t-norm $x * y = x.y$ and t-conorm $x \diamond y = x + y - xy$, for $\gamma > ||x||$ and $\gamma > 0$

$$\mathcal{U}(x,\gamma) = \frac{\gamma}{\gamma + \|\mu\|}\ ,\ \mathcal{V}(x,\gamma) = \frac{\|x\|}{\gamma + \|x\|} \text{ and } \mathcal{W}(x,\gamma) = \frac{\|x\|}{\gamma},$$

for all $x, y \in Z$. If we take $\gamma \leq \|x\|$, then $\mathcal{U}(x,\gamma)$=0, $\mathcal{V}(x,\gamma) = 1$ and $W(x,\gamma)$=1. Then, $(Z, \mathcal{S}, *, \diamond)$ is NNS in such a way that $\mathcal{S} : Z \times \mathbb{R}^+ \to [0,1]$.

Example 2.4. Suppose $(Z=\mathbb{R},\|.\|)$ be a normed space, where $\|a\|=|a|,\forall\, a\in\mathbb{R}$. Using the $*$ and $\diamond$ operations, as t-norm $x*y=\min\{x,y\}$ and t-conorm $x\diamond y=\max\{x,y\}$, $\forall x,y\in[0,1]$ and define

$$\mathcal{U}(x,\gamma)=\frac{\gamma}{\gamma+r\|x\|}\ ,\ \mathcal{V}(x,\gamma)=\frac{r\|x\|}{\gamma+\|x\|}\ \text{ and }\ \mathcal{W}(x,\gamma)=\frac{r\|x\|}{\gamma},$$

where $r>0$ Then, $\mathcal{S}=\{(x,\gamma),\mathcal{U}(x,\gamma),\mathcal{V}(x,\gamma),\mathcal{W}(x,\gamma):(x,\gamma)\in Z\times\mathbb{R}^+\}$ is a NN on Z.

Definition 2.8 ([7]). Let V be an NNS. A sequence $x=\{x_k\}$ is said to be convergent to α with respect to $\mathcal{S}=(\mathcal{U},\mathcal{V},\mathcal{W})$, if for every $0<\epsilon<1$ and $\gamma>0$, there exists $k\in\mathbb{N}$, such that $\mathcal{U}(x_k-\alpha,\gamma)>1-\epsilon$, $\mathcal{V}(x_k-\alpha,\gamma)<\epsilon$ and $\mathcal{W}(x_k-\alpha,\gamma)<\epsilon$. That is, for all $\gamma>0$, we have

$$\lim_{k\to\infty}\mathcal{U}(x_k-\alpha,\gamma)=1,\ \lim_{k\to\infty}\mathcal{V}(x_k-\alpha,\gamma)=0 \text{ and } \lim_{k\to\infty}\mathcal{W}(x_k-\alpha,\gamma)=0.$$

The convergent in NNS $V=(Z,\mathcal{S},*,\diamond)$ is denoted by $\mathcal{S}-\lim x_k=\alpha$.

Definition 2.9 ([7]). Let V be an NNS. A sequence $x=\{x_k\}$ is Cauchy sequence with respect to $\mathcal{S}=(\mathcal{U},\mathcal{V},\mathcal{W})$, if for every $0<\epsilon<1$ and $\gamma>0$, there exists $K\in\mathbb{N}$, such that $\mathcal{U}(x_n-x_k,\gamma)>1-\epsilon$, $\mathcal{V}(x_n-x_k,\gamma)<\epsilon$ and $\mathcal{W}(x_n-x_k,\gamma)<\epsilon$, for all $n,k\in K$.

Definition 2.10 ([7]). Let V be an NNS. Then, open ball with center x and radius ϵ is defined as, for $0<\epsilon<1$, $x\in Z$ and $\gamma>0$,

$$B(x,\epsilon,\gamma)=\{y\in Z:\mathcal{U}(x-y,\gamma)\ >1-\epsilon,\ \ \mathcal{V}(x-y,\gamma)<\epsilon,\ \ \mathcal{W}(x-y,\gamma)<\epsilon\}.$$

Definition 2.11 ([7]). Let V be an NNS and $Y\subseteq Z$. Then, Y is said to be open if for each $y\in Y$, there exist $\gamma>0$, $0<\epsilon<1$ such that $B(y,\epsilon,\gamma)\subseteq Y$.

3. Main results (on the Nörlund sequence)

Throughout the article, we assume that the sequences $x=\{x_k\}\in\omega$ and $\mathscr{N}_n^f(x)$ are connected as shown in (3) and I is an admissible ideal of a subset of $\mathbb{N}$. In this section, by using a domain of the Nörlund matrix which is used in [8] and $I-$convergence w.r.t. neutrosophic norm $\mathcal{S}=(\mathcal{U},\mathcal{V},\mathcal{W})$. As shown below, we define new Norlund sequence spaces:

$$\begin{aligned}\mathscr{N}^f_{I_0(\mathcal{S})} := \big\{x=\{x_n\}\in\omega:\{n\in\mathbb{N}:\mathcal{U}\big(\mathscr{N}_n^f(x),\gamma\big)\le 1-\epsilon \\ \text{or } \mathcal{V}\big(\mathscr{N}_n^f(x),\gamma\big)\ge\epsilon,\mathcal{W}\big(\mathscr{N}_n^f(x),\gamma\big)\ge\epsilon\}\in I\big\}\end{aligned} \tag{4}$$

$$\begin{aligned}\mathscr{N}^f_{I(\mathcal{S})} := \big\{x{=}\{x_n\}{\in}\omega:\{n{\in}\mathbb{N}:\text{for some }\gamma\in\mathbb{R}\ ,\ \mathcal{U}\big(\mathscr{N}_n^f(x)-\alpha,\gamma\big){\le}1-\epsilon \\ \text{or } \mathcal{V}\big(\mathscr{N}_n^f(x)-\alpha,\gamma\big)\ge\epsilon,\mathcal{W}\big(\mathscr{N}_n^f(x)-\alpha,\gamma\big)\ge\epsilon\}\in I\big\}\end{aligned} \tag{5}$$

$$\begin{aligned}\mathscr{N}^f_{I^\infty(\mathcal{S})} := \big\{x{=}\{x_n\}{\in}\omega : \{n{\in}\mathbb{N}, \exists\epsilon{\in}(0,1) \text{ s.t } \mathcal{U}\big(\mathscr{N}^f_n(x),\gamma\big){\leq}1-\epsilon \\ \text{or } \mathcal{V}\big(\mathscr{N}^f_n(x),\gamma\big) \geq \epsilon, \mathcal{W}\big(\mathscr{N}^f_n(x),\gamma\big) \geq \epsilon\} \in I\big\}.\end{aligned} \tag{6}$$

We describe an open ball and a closed ball with a center at x and a radius $\gamma > 0$ with regard to the neutrosophic $\epsilon \in (0,1)$ parameter, indicated by $\mathscr{B}(x,\epsilon,\gamma)$ and $\mathscr{B}[x,\epsilon,\gamma]$, as follows:

$$\begin{aligned}\mathscr{B}(x,\epsilon,\gamma){=}\big\{z{=}\{z_k\}{\in}\omega : \{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big){\leq}1{-}\epsilon \\ \text{or } \mathcal{V}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) \geq \epsilon, \mathcal{W}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) \geq \epsilon\} \in I\big\}\end{aligned} \tag{7}$$

and

$$\begin{aligned}\mathscr{B}[x,\epsilon,\gamma]{=}\big\{z{=}\{z_k\}{\in}\omega : \{n{\in}\mathbb{N} : \mathcal{U}\big(\mathscr{N}^f_n(x){-}\mathscr{N}^f_n(z),\gamma\big){<}1-\epsilon \\ \text{or } \mathcal{V}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) > \epsilon, \mathcal{W}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z),\gamma\big) > \epsilon\} \in I\big\}.\end{aligned} \tag{8}$$

In this case, we write $I_{(\mathcal{S})}{-}\lim(x) = \alpha$ since $\{x_n\}$ converges to some $\alpha \in \mathbb{C}$ represented by $x_n \xrightarrow{I_{(\mathcal{S})}} \alpha$ if $\{x_n\} \in \mathscr{N}^t_{I(\mathcal{S})}$.

Theorem 3.1. The inclusion relation $\mathscr{N}^f_{I_0(\mathcal{S})} \subset \mathscr{N}^f_{I(\mathcal{S})} \subset \mathscr{N}^f_{I^\infty(\mathcal{S})}$ holds.

Proof. We know that $\mathscr{N}^f_{I_0(\mathcal{S})} \subset \mathscr{N}^f_{I(\mathcal{S})}$. Then, we only show that $\mathscr{N}^f_{I(\mathcal{S})} \subset \mathscr{N}^f_{I^\infty(\mathcal{S})}$. Consider $x = \{x_n\} \in \mathscr{N}^f_{I(\mathcal{S})}$. Then, there exists $\alpha \in \mathbb{C}$, such that $I_{(\mathcal{S})}{-}\lim(x_k) = \alpha$. Thus, for any $0 < \epsilon < 1$ and $\gamma > 0$ the set

$$\begin{aligned}P = \Big\{n \in \mathbb{N} : \mathcal{U}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) > 1 - \epsilon \text{ and } \mathcal{V}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) < \epsilon, \\ \mathcal{W}\left(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\right) < \epsilon\Big\} \in \mathscr{F}(I).\end{aligned}$$

Suppose $\mathcal{U}\big(\alpha, \frac{\gamma}{2}\big) = u$, $\mathcal{V}\big(\alpha, \frac{\gamma}{2}\big) = v$ and $\mathcal{W}\big(\alpha, \frac{\gamma}{2}\big) = w$, for all $\gamma > 0$. Since $u, v, w \in (0,1)$ and $0 < \epsilon < 1$, there exists $r_1, r_2, r_3 \in (0,1)$, such that $(1{-}\epsilon){*}u > 1 - r_1$, $\epsilon \diamond v < r_2$ and $\epsilon \diamond w < r_3$, we have

$$\begin{aligned}\mathcal{U}\big(\mathscr{N}^f_n(x),\gamma\big) &= \mathcal{U}\big(\mathscr{N}^f_n(x) - \alpha + \alpha,\gamma\big) \\ &\geq \mathcal{U}\Big(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\Big) * \mathcal{U}\left(\alpha, \frac{\gamma}{2}\right) \\ &> (1-\epsilon) * u \\ &> 1 - r_1,\end{aligned}$$

$$\begin{aligned}\mathcal{V}\big(\mathscr{N}^f_n(x),\gamma\big) &= \mathcal{V}\big(\mathscr{N}^f_n(x) - \alpha + \alpha,\gamma\big) \\ &\leq \mathcal{V}\Big(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}\Big) \diamond \mathcal{V}\left(\alpha, \frac{\gamma}{2}\right) \\ &< \epsilon \diamond v \\ &< r_2\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}\big(\mathscr{N}_n^f(x),\gamma\big) =&\mathcal{W}\big(\mathscr{N}_n^f(x)-\alpha+\alpha,\gamma\big)\\ \leq&\mathcal{W}\Big(\mathscr{N}_n^f(x)-\alpha,\frac{\gamma}{2}\Big)\diamond\mathcal{W}\Big(\alpha,\frac{\gamma}{2}\Big)\\ <&\epsilon\diamond w\\ <&r_3.\end{aligned}$$

Taking $r=\max\{r_1,r_2,r_3\}$, then $\big\{n\in\mathbb{N},\exists\ r\in(0,1):\mathcal{U}\big(\mathscr{N}_n^f(x),\gamma\big)>1-r$ and $\mathcal{V}\big(\mathscr{N}_n^f(x),\gamma\big)<r,\mathcal{W}\big(\mathscr{N}_n^f(x),\gamma\big)<r\big\}\in\mathscr{F}(I)\implies x=\{x_k\}\in\mathscr{N}^f_{I^\infty(\mathcal{S})}$ implies $\mathscr{N}^f_{I(\mathcal{S})}\subset\mathscr{N}^f_{I^\infty(\mathcal{S})}$. □

The contrary of an inclusion relation does not hold. To defend our claim, consider the following examples.

Example 3.1. Suppose $(\mathbb{R},\|.\|)$ be a normed space such that $\|x\|=\sup_k|x_k|$, and $\vartheta_1*\vartheta_2=\min\{\vartheta_1,\vartheta_2\}$ and $\vartheta_1\diamond\vartheta_2=\max\{\vartheta_1,\vartheta_2\}$, $\forall\vartheta_1,\vartheta_2\in(0,1)$. Now, define norms $\mathcal{S}=(\mathcal{U},\mathcal{V},\mathcal{W})$ on $\mathbb{R}^2\times(0,\infty)$ as follows

$$\mathcal{U}(x,\gamma)=\frac{\gamma}{\gamma+\|x\|}\quad,\ \mathcal{V}(x,\gamma)=\frac{\|x\|}{\gamma+\|x\|}\text{ and }\mathcal{W}(x,\gamma)=\frac{\|x\|}{\gamma}.$$

Then, $(\mathbb{R},\mathcal{S},*,\diamond)$ is a NNS. Consider the sequence (x_k)=$\{1\}$. It can be easily seen that $(x_k)\in\mathscr{N}^f_{I(\mathcal{S})}$ and $x_k\xrightarrow{I(\mathcal{S})}1$, but $x_k\notin\mathscr{N}^f_{I_0(\mathcal{S})}$.

Theorem 3.2. The spaces $\mathscr{N}^f_{I_0(\mathcal{S})}$ and $\mathscr{N}^f_{I(\mathcal{S})}$ are linear spaces.

Proof. We know that $\mathscr{N}^f_{I_0(\mathcal{S})}\subset\mathscr{N}^f_{I(\mathcal{S})}$. Then, we'll illustrate the outcome for $\mathscr{N}^f_{I(\mathcal{S})}$ The proof of linearity of the space $\mathscr{N}^f_{I_0(\mathcal{S})}$ follows similarly. Suppose sequences $x=\{x_k\},y=\{y_k\}\in\mathscr{N}^f_{I(\mathcal{S})}$. Then, there exist $\alpha_1,\alpha_2\in\mathbb{C}$, such that $\{x_k\}$ and $\{y_k\}$ I–converge to α_1 and α_2 respectively.

We will show that the sequence $\mu x_k+\nu y_k$ I–converges to $\mu\alpha_1+\nu\alpha_2$ for any scalars μ and ν. Consider the following sets for c and d

$$\mathscr{P}_1=\Bigg\{n\in\mathbb{N}:\mathcal{U}\Bigg(\mathscr{N}_n^f(x)-\alpha_1,\frac{\gamma}{2|\mu|}\Bigg)\leq1-\epsilon\text{ or }\mathcal{V}\Bigg(\mathscr{N}_n^f(x)-\alpha_1,\frac{\gamma}{2|\mu|}\Bigg)\geq\epsilon,\\ \mathcal{W}\Bigg(\mathscr{N}_n^f(x)-\alpha_1,\frac{\gamma}{2|\mu|}\Bigg)\geq\epsilon\Bigg\}\in I,$$

$$\mathscr{P}_2=\Bigg\{n\in\mathbb{N}:\mathcal{U}\Bigg(\mathscr{N}_n^f(y)-\alpha_2,\frac{\gamma}{2|\nu|}\Bigg)\leq1-\epsilon\text{ or }\mathcal{V}\Bigg(\mathscr{N}_n^f(y)-\alpha_2,\frac{\gamma}{2|\nu|}\Bigg)\geq\epsilon,\\ \mathcal{W}\Bigg(\mathscr{N}_n^f(y)-\alpha_2,\frac{\gamma}{2|\nu|}\Bigg)\geq\epsilon\Bigg\}\in I.$$

Now, we take the complement of $\mathscr{P}_1$ and $\mathscr{P}_2$

$$\mathscr{P}_1^c = \Big\{n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon\Big\} \in F(I),$$

$$\mathscr{P}_2^c = \Big\{n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_n^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_n^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon\Big\} \in F(I).$$

Consequently, set $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ produces $\mathscr{P} \in I$. Thus, $\mathscr{P}^c$ is a set that is not empty in $\mathcal{F}(I)$. We'll illustrate this for each $\{x_k\}, \{y_k\} \in \mathscr{N}_{I(\mathcal{S})}^f$

$$\mathscr{P}^c \subset \Big\{n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) > 1-\epsilon \\ \text{and } \mathcal{V}\Big(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) < \epsilon\Big\}.$$

Let $i \in \mathscr{P}^c$. In this case,

$$\mathcal{U}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) < \epsilon, \\ \mathcal{U}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) > 1-\epsilon \text{ and } \mathcal{V}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon, \\ \mathcal{W}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) < \epsilon.$$

Consider

$$\begin{aligned} &\mathcal{U}\Big(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) \\ &\geq \mathcal{U}\Big(\mu\mathscr{N}_i^f(x) - \mu\alpha_1, \frac{\gamma}{2}\Big) * \mathcal{U}\Big(\nu\mathscr{N}_i^f(y) - \nu\alpha_2, \frac{\gamma}{2}\Big) \\ &= \mathcal{U}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) * \mathcal{U}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big) \\ &> (1-\epsilon) * (1-\epsilon) \\ &> 1-\epsilon. \end{aligned}$$

$$\implies \mathcal{U}\Big(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) > 1 - \epsilon$$

$$\begin{aligned}\mathcal{V}\Big(\mathscr{N}_i^f(\mu x+\nu y)-(\mu\alpha_1+\nu\alpha_2), \gamma\Big) &\leq \mathcal{V}\Big(\mu\mathscr{N}_i^f(x)-\mu\alpha_1, \frac{\gamma}{2}\Big) \diamond \mathcal{V}\Big(\nu\mathscr{N}_i(y)-\nu\alpha_2, \frac{\gamma}{2}\Big)\\ &= \mathcal{V}\Big(\mathscr{N}_i^f(x)-\alpha_1, \frac{\gamma}{2|\mu|}\Big) \diamond \mathscr{V}\Big(\mathscr{N}_i^f(y)-\alpha_2, \frac{\gamma}{2|\nu|}\Big)\\ &< \epsilon \diamond \epsilon\\ &< \epsilon.\end{aligned}$$

$$\implies \mathcal{V}\Big(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big) < \epsilon \text{ and}$$

$$\begin{aligned}&\mathcal{W}\Big(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma\Big)\\ &\leq \mathcal{W}\Big(\mu\mathscr{N}_i^f(x) - \mu\alpha_1, \frac{\gamma}{2}\Big) \diamond \mathcal{W}\Big(\mu\mathscr{N}_i(y) - \nu\alpha_2, \frac{\gamma}{2}\Big)\\ &= \mathcal{W}\Big(\mathscr{N}_i^f(x) - \alpha_1, \frac{\gamma}{2|\mu|}\Big) \diamond \mathcal{W}\Big(\mathscr{N}_i^f(y) - \alpha_2, \frac{\gamma}{2|\nu|}\Big)\\ &< \epsilon \diamond \epsilon\\ &< \epsilon.\end{aligned}$$

$\implies \mathcal{W}(\mathscr{N}_i^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon$. Thus, $\mathscr{P}^c \subset \{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(\mu x+\nu y)-(\mu\alpha_1+\nu\alpha_2), \gamma) > 1-\epsilon$ and $\mathcal{V}(\mathscr{N}_n^f(\mu x+\nu y)-(\mu\alpha_1+\nu\alpha_2), \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon\}$. Since $\mathscr{P}^c \in \mathscr{F}(I)$.

By the properties of $\mathscr{F}(I)$, we have $\{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) > 1 - \epsilon$ and $\mathcal{V}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(\mu x + \nu y) - (\mu\alpha_1 + \nu\alpha_2), \gamma) < \epsilon\} \in \mathscr{F}(I)$. It indicates that the sequence $(\mu x_k + \nu y_k)$ I–converge to $\mu\alpha_1 + \nu\alpha_2$. Therefore, $(\mu x_k + \nu y_k) \in \mathscr{N}_{I(\mathcal{S})}^f$. Hence, $\mathscr{N}_{I(\mathcal{S})}^f$ is linear space. □

Theorem 3.3. Each open ball in neutrosophic $0 < \epsilon < 1$ with centre at x and radius $0 < \jmath < 1$, i.e., $\mathscr{B}(x, \gamma, \epsilon)$ is an open set in $\mathscr{N}_{I(\mathcal{S})}^f$, where $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ is a neutrosophic norm.

Proof. Suppose that $\mathscr{B}(x, \gamma, \epsilon)$ is an open ball with a radius of $\gamma > 0$ and a neutrosophic $0 < \epsilon < 1$ parameter, with its centre at $x = (x_k) \in \mathscr{N}_{I(\mathcal{S})}^f$

$$\begin{aligned}\mathscr{B}(x, \gamma, \epsilon) = \big\{y = (y_k) \in \omega : \{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) \leq 1 - \epsilon\\ \text{or } \mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) \geq \epsilon, \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) \geq \epsilon\} \in I\big\}.\end{aligned}$$

Then

$$\begin{aligned}&\mathscr{B}^c(x, \gamma, \epsilon) = \big\{y = (y_k) \in \omega : \{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) > 1 - \epsilon \text{ and}\\ &\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) < \epsilon, \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y), \gamma\big) < \epsilon\} \in F(I)\big\}.\end{aligned}$$

Suppose $y = (y_k) \in \mathscr{B}^c(x,\gamma,\epsilon)$. Then, for $\mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma\big) > 1-\epsilon$, $\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma\big) < \epsilon$ and $\mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma\big) < \epsilon$ so, there exists $\gamma_0 \in (0,\gamma)$ such that $\mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma_0\big) > 1-\epsilon$, $\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma_0\big) < \epsilon$ and $\mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma_0\big) < \epsilon$.

Putting $\epsilon_0 = \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma_0\big)$, we have $\epsilon_0 > 1-\epsilon$. Then, $\exists\, p \in (0,1)$ such that $\epsilon_0 > 1-p > 1-\epsilon$. For $\epsilon_0 > 1-p$, we can have $\epsilon_1,\epsilon_2,\epsilon_3 \in (0,1)$, such that $\epsilon_0 * \epsilon_1 > 1-p$, $(1-\epsilon_0)\diamond(1-\epsilon_2) < p.$ and $(1-\epsilon_0)\diamond(1-\epsilon_3) < p$. Let $\epsilon_4 = \max\{\epsilon_1,\epsilon_2,\epsilon_3\}$.

Now, consider the open ball $\mathscr{B}^c(y,\gamma-\gamma_0,1-\epsilon_4)$. We shall show that $\mathscr{B}^c(y,\gamma-\gamma_0,1-\epsilon_4) \subset \mathscr{B}^c(x,\gamma,\epsilon)$.

Let $z = \{z_k\} \in \mathscr{B}^c(y,\gamma-\gamma_0,1-\epsilon_4)$, then $\mathcal{U}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z),\gamma-\gamma_0\big) > \epsilon_4$ and $\mathcal{V}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z),\gamma-\gamma_0\big) < 1-\epsilon_4$, $\mathcal{W}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z),\gamma-\gamma_0\big) < 1-\epsilon_4$. Therefore,

$$\begin{aligned}\mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z),\gamma\big) &\geq \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma_0\big) * \mathcal{U}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z),\gamma-\gamma_0\big)\\ &\geq \epsilon_0 * \epsilon_4 \geq \epsilon_0 * \epsilon_1\\ &> (1-p)\\ &> (1-\epsilon)\end{aligned}$$

$$\begin{aligned}\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z),\gamma\big) &\leq \mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma_0\big) \diamond \mathcal{V}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z),\gamma-\gamma_0\big)\\ &\leq (1-\epsilon_0)\diamond(1-\epsilon_4) \leq \epsilon_0 \diamond \epsilon_2\\ &< p\\ &< \epsilon\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z),\gamma\big) &\leq \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(y),\gamma_0\big) \diamond \mathcal{W}\big(\mathscr{N}_n^f(y) - \mathscr{N}_n^f(z),\gamma-\gamma_0\big)\\ &\leq \epsilon_0 \diamond \epsilon_4 \leq \epsilon_0 \diamond \epsilon_3\\ &< p\\ &< \epsilon\end{aligned}$$

Therefore, the set $\big\{n \in \mathbb{N} : \mathcal{U}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z),\gamma\big) > 1-\epsilon$ and $\mathcal{V}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z),\gamma\big) < \epsilon$, $\mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z),\gamma\big) < \epsilon\big\} \in \mathscr{F}(I)$.

$\implies z = (z_k) \in \mathscr{B}^c(x,\gamma,\epsilon)$,

$\implies \mathscr{B}^c\big(y,\gamma-\gamma_0,1-\epsilon_4\big) \subset \mathscr{B}^c\big(x,\gamma,\epsilon\big)$. □

Remark 3.1. The spaces $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_0(\mathcal{S})}$ are Nörland I-convergent and Nörland I-null in NNS with respect to neutrosophic norms $\mathcal{S} = (\mathcal{U},\mathcal{V},\mathcal{W})$.

Now, define a collection $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ of a subset of $\mathscr{N}^f_{I(\mathcal{S})}$ as follows: $\tau^{\mathscr{N}^f}_{I(\mathcal{S})} = \{P \subset \mathscr{N}^f_{I(\mathcal{S})}$: for every $x = (x_k) \in P\ \exists\ \gamma > 0$ and $\epsilon \in (0,1)$ s.t $\mathscr{B}(x,\gamma,\epsilon) \subset P\}$. Then, $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ constructs a topology on sequence space $\mathscr{N}^f_{I(\mathcal{S})}$. The collection described by $\mathcal{B} = \{\mathscr{B}(x,\gamma,\epsilon) : b \in \mathscr{N}^f_{I(\mathcal{S})}, r > 0 \text{ and } \epsilon \in (0,1)\}$ is the topology's base $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ on the space $\mathscr{N}^f_{I(\mathcal{S})}$.

Theorem 3.4. The topology $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ on the space $\mathscr{N}^f_{I_0(\mathcal{S})}$ is first countable.

Proof. For every $x = \{x_k\} \in \mathscr{N}^f_{I(\mathcal{S})}$, consider the set $\mathcal{B} = \{\mathscr{B}(x, \frac{1}{n}, \frac{1}{n})\} : n = 1,2,3,4,...\}$, which is a local countable basis at $x = (x_k)$. As a result, the topology $\tau^{\mathscr{N}^f}_{I(\mathcal{S})}$ on the space $\mathscr{N}^f_{I_0(\mathcal{S})}$ is first countable. □

Theorem 3.5. The spaces $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_0(\mathcal{S})}$ are Hausdorff spaces.

Proof. We know that $\mathscr{N}^f_{I_0(\mathcal{S})} \subset \mathscr{N}^f_{I(\mathcal{S})}$.

We will only show the solution for $\mathscr{N}^f_{I(\mathcal{S})}$. Suppose $x = (x_k), y = (y_k) \in \mathscr{N}^f_{I(\mathcal{S})}$ as well as $x \neq y$. Then, for any $n \in \mathbb{N}$ and $\gamma > 0$, implies $0 < \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) < 1$, $0 < \mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) < 1$ and $0 < \mathcal{W}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma) < 1$.

Putting $\epsilon_1 = \mathcal{U}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma\big)$, $\epsilon_2 = \mathcal{V}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma\big)$, $\epsilon_3 = \mathcal{W}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma\big)$ and $\epsilon = \max\{\epsilon_1, 1-\epsilon_2, 1-\epsilon_3\}$. Then, for each $\epsilon_0 \in (\epsilon, 1)$ there exist $\epsilon_4, \epsilon_5, \epsilon_6 \in (0,1)$, such that $\epsilon_4 * \epsilon_4 \geq \epsilon_0$, $(1-\epsilon_5)\diamond(1-\epsilon_5) \leq (1-\epsilon_0)$ and $(1-\epsilon_6)\diamond(1-\epsilon_6) \leq (1-\epsilon_0)$. Once again putting $\epsilon_7 = \max\{\epsilon_4, 1-\epsilon_5, 1-\epsilon_6, \}$, think about the open balls. $\mathscr{B}(x, 1-\epsilon_7, \frac{\gamma}{2})$ and $\mathscr{B}(y, 1-\epsilon_7, \frac{\gamma}{2})$ respectively centred at x and y. Then, it is obvious that $\mathscr{B}^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathscr{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2}) = \phi$.

If possible let $x = \{x_k\} \in \mathscr{B}^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathscr{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2})$. Then, we have

$$\begin{aligned}
\epsilon_1 &= \mathcal{U}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma\big) \\
&\geq \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z), \frac{\gamma}{2}) * \mathcal{U}(\mathscr{N}^f_n(z) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) \\
&> \epsilon_7 * \epsilon_7 \\
&\geq \epsilon_4 * \epsilon_4 \\
&\geq \epsilon_0 \\
&> \epsilon_1,
\end{aligned} \tag{9}$$

$$\begin{aligned}
\epsilon_2 &= \mathcal{V}\big(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \gamma\big) \\
&\leq \mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(z), \frac{\gamma}{2}) \diamond \mathcal{V}(\mathscr{N}^f_n(z) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) \\
&< (1-\epsilon_7) \diamond (1-\epsilon_7) \\
&\leq (1-\epsilon_5) \diamond (1-\epsilon_5) \\
&\leq (1-\epsilon_0) \\
&< \epsilon_2
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\epsilon_3 &= \mathcal{W}\big(\mathscr{N}_n^f(x) - \mathscr{N}^f(y), \gamma\big) \\
&\leq \mathcal{W}(\mathscr{N}_n^f(x) - \mathscr{N}_n^f(z), \frac{\gamma}{2}) \diamond \mathcal{W}(\mathscr{N}_n^f(z) - \mathscr{N}_n^f(y), \frac{\gamma}{2}) \\
&< (1-\epsilon_7) \diamond (1-\epsilon_7) \\
&\leq (1-\epsilon_6) \diamond (1-\epsilon_6) \\
&\leq (1-\epsilon_0) \\
&< \epsilon_3.
\end{aligned} \tag{11}$$

We have a contradiction from equations (9), (10) and (11). Therefore, $\mathscr{B}^c(x, 1-\epsilon_7, \frac{\gamma}{2}) \cap \mathscr{B}^c(y, 1-\epsilon_7, \frac{\gamma}{2}) = \phi$. Hence, the space $\mathscr{N}^f_{I(\mathcal{S})}$ is a Hausdorff space. □

Theorem 3.6. Suppose $\tau^{\mathscr{N}f}_{I(\mathcal{S})}$ be a topology on a neutrosophic norm spaces $\mathscr{N}^f_{I(\mathcal{S})}$, then a sequence $x = \{x_k\} \in \mathscr{N}^f_{I(\mathcal{S})}$, such that $(x_k) \longrightarrow \alpha$, iff $\mathcal{U}(\mathscr{N}_n^f(x) - \alpha) \longrightarrow 1, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha) \longrightarrow 0$ and $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Consider a sequence $\{x_k\} \to \alpha$, and Fix $\gamma_0 > 0$, then for $\gamma \in (0,1)$, $\exists$ $n_0 \in \mathbb{N}$ s.t. $\{x_k\} \in \mathscr{B}(x, \gamma, \epsilon)$, $\forall\ k \geq n_0$, then for a $\gamma > 0$, $\mathscr{B}(x, \gamma, \epsilon) = \{x = (x_k) \in \omega : \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) \leq 1-\epsilon$ or $\mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) \geq \epsilon$, $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) \geq \epsilon\} \in I$, such that $\mathscr{B}^c(x, \gamma, \epsilon) \in \mathscr{F}(I)$ then

$$1 - \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathscr{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon.$$

Hence, $\mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 1, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0,$ and $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0$ as $n \to \infty$. Conversely, if $\forall\ \gamma > 0, \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 1, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0,$ and $\mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) \to 0$ as $n \to \infty$. Then, for each $\epsilon \in (0,1)$, $\exists\ n_0 \in \mathbb{N}$ s.t. $1 - \mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon\ \forall\ n \geq n_0$. Hence, we have

$$\mathcal{U}(\mathscr{N}_n^f(x) - \alpha, \gamma) > 1-\epsilon, \mathcal{V}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \mathcal{W}(\mathscr{N}_n^f(x) - \alpha, \gamma) < \epsilon, \forall\ n \geq n_0.$$

Thus, $\{x_k\} \in \mathscr{B}^c(x, \gamma, \epsilon)$, $\forall\ k \geq n_0$ and hence $\{x_k\} \to \alpha$. □

Now, we establish results about the relationship between Nörlund I-convergent and Nörlund I-Cauchy sequence in NNS.

Definition 3.1. In an NNS V. A sequence $x = \{x_n\} \in V$ is said to be Nörlund I-convergent to $\alpha \in \mathbb{C}$ with regard to neutrosophic norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, denoted by $x_n \to \alpha$, if for every $\epsilon \in (0,1)$ and $\gamma > 0$, where

$$\begin{aligned}
N_1 = \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha, \gamma\Big) \leq 1-\epsilon \\
\text{or } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha, \gamma\Big) \geq \epsilon, \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha, \gamma\Big) \geq \epsilon \Big\} \in I
\end{aligned}$$

and we write $I_{\mathcal{S}}\text{–}\lim(x_n) = \alpha$.

Definition 3.2. A sequence $x = \{x_n\} \in V$ is said to Nörlund I-Cauchy with respect to neutrosophic norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, if for every $\epsilon \in (0,1)$ and $\gamma > 0$, $\exists\ k \in \mathbb{N}$, such that

$$N_2 = \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_k^f(x), \gamma\Big) \leq 1 - \epsilon \\ \text{or } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_k^f(x), \gamma\Big) \geq \epsilon, \mathcal{W}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_k^f(x), \gamma\Big) \geq \epsilon \Big\} \in I.$$

Theorem 3.7. Let $\mathscr{N}_{I(\mathcal{S})}^f$ be an NNS. If a sequence $x = \{x_k\} \in$ is Nörlund I–convergent w.r.t NN $\mathcal{S}$, then the $I_{(\mathcal{S})}$–$lim(x)$ is unique.

Proof. Let $x = \{x_k\}$ is Nörlund I–convergent in NNS. Let on contrary that α_1 and α_2 are two distinct elements, thus $I_{(\mathcal{S})}-\lim(x_k) = \alpha_1$ and $I_{(\mathcal{S})}-\lim(x_k) = \alpha_2$. For a given $\epsilon > 0$, choose $p > 0$ such that $(1-p) * (1-p) > 1 - \epsilon$, $p \diamond p < \epsilon$ and $p \diamond p < \epsilon$, for $\gamma > 0$.

We show that $\alpha_1 = \alpha_2$. We define $P_1 = \{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(x) - \alpha_1, \gamma) \leq 1 - \epsilon\}$, $P_2 = \{n \in \mathbb{N} : \mathcal{V}(\mathscr{N}_n^f(x) - \alpha_1, \gamma) \geq \epsilon\}$, $P_3 = \{n \in \mathbb{N} : \mathcal{W}(\mathscr{N}_n^f(x) - \alpha_1, \gamma) \geq \epsilon\}$, $Q_1 = \{n \in \mathbb{N} : \mathcal{U}(\mathscr{N}_n^f(x) - \alpha_2, \gamma) \leq 1 - \epsilon\}$, $Q_2 = \{n \in \mathbb{N} : \mathcal{V}(\mathscr{N}_n^f(x) - \alpha_2, \gamma) \geq \epsilon\}$, $Q_3 = \{n \in \mathbb{N} : \mathcal{W}(\mathscr{N}_n^f(x) - \alpha_2, \gamma) \geq \epsilon\}$, where $A = (P_1 \cup Q_1) \cap (P_2 \cup Q_2) \cap (P_3 \cup Q_3)$ sets $P_1, P_2, P_3, Q_1, Q_2, Q_3$ and A must be belongs to I, since $\{x_k\}$ has two distinct I-limits with regard to neutrosophic norm $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, *i.e.* α_1, α_2. As a result, $A^c \in \mathscr{F}(I)$ implies that A^c is not empty. Let us write some $n_0 \in A^c$ then either $n_0 \in P_1^c \cap Q_1^c$ or $n_0 \in P_2^c \cap Q_2^c$ or $n_0 \in P_3^c \cap Q_3^c$.

If $n_0 \in P_1^c \cap Q_1^c$, it follows that

$$\mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_1, \frac{\gamma}{2}\Big) > 1 - p \text{ and } \mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_2, \frac{\gamma}{2}\Big) > 1 - p.$$

Hence,

$$\begin{aligned} \mathcal{U}\Big(\alpha_1 - \alpha_2, \gamma\Big) &\geq \mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_1, \frac{\gamma}{2}\Big) * \mathcal{U}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_2, \frac{\gamma}{2}\Big) \\ &> (1-p) * (1-p) \\ &> (1 - \epsilon). \end{aligned}$$

Because $\epsilon > 0$ was arbitrary, $\mathcal{U}(\alpha_1 - \alpha_2, \gamma) = 1$ was given to all $\gamma > 0$. Thus, we have $\alpha_1 = \alpha_2$, which is a contradiction.

If $n_0 \in P_2^c \cap Q_2^c$, it follows that

$$\mathcal{V}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_1, \frac{\gamma}{2}\Big) < p \text{ and } \mathcal{V}\Big(\mathscr{N}_{n_0}^f(x) - \alpha_2, \frac{\gamma}{2}\Big) < p.$$

Hence,

$$\begin{aligned}\mathcal{V}\Big(\alpha_1-\alpha_2,\gamma\Big) &\le \mathcal{V}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_1,\frac{\gamma}{2}\Big)\diamond\mathcal{V}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_2,\frac{\gamma}{2}\Big)\\ &< p\diamond p\\ &< \epsilon.\end{aligned}$$

Because $\epsilon > 0$ was arbitrary, $\mathcal{V}(\alpha_1-\alpha_2,\gamma)=0$ was given to all $\gamma>0$. Thus, we have $\alpha_1=\alpha_2$, which is a contradiction.

If $n_0\in P_3^c\cap Q_3^c$, it follows that

$$\mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_1,\frac{\gamma}{2}\Big)<p \text{ and } \mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_2,\frac{\gamma}{2}\Big)<p.$$

Hence,

$$\begin{aligned}\mathcal{W}\Big(\alpha_1-\alpha_2,\gamma\Big) &\le \mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_1,\frac{\gamma}{2}\Big)\diamond\mathcal{W}\Big(\mathscr{N}^f_{n_0}(x)-\alpha_2,\frac{\gamma}{2}\Big)\\ &< p\diamond p\\ &< \epsilon.\end{aligned}$$

Because $\epsilon > 0$ was arbitrary, $\mathcal{W}(\alpha_1-\alpha_2,\gamma)=0$ was given to all $\gamma>0$. Thus, we have $\alpha_1=\alpha_2$, which is a contradiction.

As an outcome, in all cases, α_1=α_2, implying that the $I_{(\mathcal{S})}$–limit is unique. □

Now, we establish results about the relationship between Nörlund I-convergent and Nörlund I-Cauchy sequence in NNS.

Theorem 3.8. A sequence $x=\{x_k\}\in\mathscr{N}^f_{I(\mathcal{S})}$ is I–convergent with regard to neutrosophic norms $\mathcal{S}=(\mathcal{U},\mathcal{V},\mathcal{W})$ if and olny if it is I–Cauchy with respect to the same norms.

Proof. Let $x=(x_k)$ is Nörlund I–convergent with regard to neutrosophic norms $(\mathcal{S})$ such that $I_{(\mathcal{S})}$–$\lim(x_k)=\alpha$. For given $\epsilon\in(0,1)$ there exists $p_1\in(0,1)$, such that $(1-p_1)*(1-p_1)>1-\epsilon$ and $p_1\diamond p_1<\epsilon$. Since $I_{(\mathcal{S})}$–$\lim(x_k)=\alpha$ therefore, for all $\gamma>0$, $A_1=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)\le 1-p_1$ or $\mathcal{V}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)\ge p_1,\mathcal{W}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)\ge p_1\big\}\in I$, that implies $A_1^c=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)>1-p_1$ and $\mathcal{V}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)<p_1,\mathcal{W}\big(\mathscr{N}^f_n(x)-\alpha,\gamma\big)<p_1\big\}\in\mathscr{F}(I)$. Let a natural number $J\in A_1^c$, we have $\mathcal{U}\big(\mathscr{N}^f_J(x)-\alpha,\gamma\big)>1-p_1$ and $\mathcal{V}\big(\mathscr{N}^f_J(x)-\alpha,\gamma\big)<p_1,\mathcal{W}\big(\mathscr{N}^f_J(x)-\alpha,\gamma\big)<p_1$.

Now, we show that for $x\in\mathscr{N}^f_{I(\mathcal{S})}$ $\exists$ a natural number $J=J(x,\epsilon,\gamma)$ s.t. the set $A_2=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathscr{N}^f_n(x)-\mathscr{N}^f_J(x),\gamma\big)\le 1-\epsilon$ or $\mathcal{V}\big(\mathscr{N}^f_n(x)-\mathscr{N}^f_J(x),\gamma\big)\ge\epsilon,\mathcal{W}\big(\mathscr{N}^f_n(x)-\mathscr{N}^f_J(\vartheta),\gamma\big)\ge\epsilon\big\}\in I$. For this, we need prove that $A_2\subset A_1$., Let on contrary that $A_2 \nsubseteq A_1$. Then, $\exists\, l \in A_2$, but not in A_1 we have $\mathcal{U}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \leq 1-\epsilon$. Then, $\mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) > 1 - p_1$.

In particular, $\mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) > 1 - p_1$. Then

$$\begin{aligned} 1-\epsilon &\geq \mathcal{U}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \\ &\geq \mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) * \mathcal{U}\big(\mathcal{N}_J^f(x) - \alpha, \frac{\gamma}{2}\big) \\ &> (1-p_1) * (1-p_1) \\ &> (1-\epsilon) \end{aligned}$$

which is a contradiction.

$\implies \mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \leq 1 - p_1$.
Similarly, consider $\mathcal{V}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \geq \epsilon$. Then, $\mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$. In particular, $\mathcal{V}\big(\mathcal{N}_n^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$. Then

$$\begin{aligned} \epsilon &\leq \mathcal{V}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \\ &\leq \mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \diamond \mathcal{V}\big(\mathcal{N}_J^f(x) - \alpha, \frac{\gamma}{2}\big) \\ &< p_1 \diamond p_1 \\ &< \epsilon \end{aligned}$$

which is a contradiction.

$\implies \mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \geq p_1$ and similarly consider $\mathcal{W}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \geq \epsilon$. Then, $\mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$.

In particular $\mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) < p_1$. Then

$$\begin{aligned} \epsilon &\leq \mathcal{W}\big(\mathcal{N}_l^f(x) - \mathcal{N}_J^f(x), \gamma\big) \\ &\leq \mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \diamond \mathcal{W}\big(\mathcal{N}_J^f(x) - \alpha, \frac{\gamma}{2}\big) \\ &< p_1 \diamond p_1 \\ &< \epsilon \end{aligned}$$

which is again a contradiction.

$\implies \mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \frac{\gamma}{2}\big) \geq p_1$.

Therefore, for $l \in A_2$, we have $\mathcal{U}\big(\mathcal{N}_l^f(x) - \alpha, \gamma\big) \leq 1 - p_1$ or $\mathcal{V}\big(\mathcal{N}_l^f(x) - \alpha, \gamma\big) \geq p_1, \mathcal{W}\big(\mathcal{N}_l^f(x) - \alpha, \gamma\big) \geq p_1$.

$\implies l \in A_1$. Hence, $A_2 \subset A_1$. Since $A_1 \in I$, so $A_2 \in I$. Consequently, the sequence $x = \{x_k\}$ is Nörlund I–Cauchy with regard to norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$.

Conversely, suppose the sequence $x = \{x_k\}$ is Nörlund I–Cauchy with regard to the norms $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{W})$. Then, $\exists\, j \in \mathbb{N}$ such that $B_1 = \big\{n \in \mathbb{N} : \mathcal{U}\big(\mathcal{N}_n^f(x) - \mathcal{N}_j^f(x), \gamma\big) \leq 1-\epsilon$ or $\mathcal{V}\big(\mathcal{N}_n^f(x) - \mathcal{N}_j^f(x), \gamma\big) \geq \epsilon, \mathcal{W}\big(\mathcal{N}_n^f(x) - \mathcal{N}_j^f(x), \gamma\big) \geq \epsilon\big\} \in I$. But, on the other hand, the sequence $x = (x_k)$ is not Nörlund I–convergent denoted by B_2,

$$B_2 = \Big\{ n \in \mathbb{N} : \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) > 1 - p_1 \text{ or } \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) < p_1, \\ \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) < p_1 \Big\} \in I,$$

$\Longrightarrow$

$$\begin{aligned} 1 - \epsilon &\geq \mathcal{U}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_j^f(x), \gamma\Big) \\ &\geq \mathcal{U}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) * \mathcal{U}\Big(\mathscr{N}_j^f(x) - \alpha, \frac{\gamma}{2}\Big) \\ &> (1 - p_1) * (1 - p_1) \\ &> 1 - \epsilon \end{aligned}$$

which is a contradiction. Now,

$$\begin{aligned} \epsilon &\leq \mathcal{V}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_j^f(x), \gamma\Big) \\ &\leq \mathcal{V}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) \diamond \mathcal{V}\Big(\mathscr{N}_j^f(x) - \alpha, \frac{\gamma}{2}\Big) \\ &< p_1 \diamond p_1 \\ &< \epsilon \end{aligned}$$

which is again a contradiction and

$$\begin{aligned} \epsilon &\leq \mathcal{W}\Big(\mathscr{N}_n^f(x) - \mathscr{N}_j^f(x), \gamma\Big) \\ &\leq \mathcal{W}\Big(\mathscr{N}_n^f(x) - \alpha, \frac{\gamma}{2}\Big) \diamond \mathcal{W}\Big(\mathscr{N}_j^f(x) - \alpha, \frac{\gamma}{2}\Big) \\ &< p_1 \diamond p_1 \\ &< \epsilon. \end{aligned}$$

This again contradicts it. Therefore, $B_2 \in \mathscr{F}(I)$, and hence $x = \{x_k\}$ is Nörlund I–convergent. □

The following theorems are easy to prove.

Theorem 3.9. In NNS V, a sequence $x = \{x_k\} \in V$ is Nörlund Cauchy with regard to NN $\mathcal{S}$. and $\mathscr{N}_{I(\mathcal{S})}^f$ cluster to α in Z then $\{x_k\}$ is Nörlund I-convergent to α with regard to same NN S.

Theorem 3.10. In NNS V, a sequence $x = \{x_k\} \in V$ is Nörlund Cauchy with regard to NN $\mathcal{S}$ then it is Nórlund I-Cauchy with regard to NN $\mathcal{S}$.

Now, follows the notations:

The space of all sequences whose N^f- transform is neutrosophic bounded sequence is denoted as $l^{\infty}_{(\mathcal{S})}(\mathscr{N}^f)$.

$\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$ indicates the space containing all sequences with neutrosophic bounded N^f- transforms and neutrosophic Norland ideal convergent sequences.

Theorem 3.11. Space $\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$ is closed linear space of $l^{\infty}_{(\mathcal{S})}(\mathscr{N}^f)$.

Proof. The given space is a subspace of $l^{\infty}_{(\mathcal{S})}(\mathscr{N}^f)$, as we are aware. Now, that $\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$ must be proved to be closed, we demonstrate that $\overline{\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}} = \mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$. (where $\overline{\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}}$ denoted the closure of $\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$).

It is clear that $\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}} \subset \overline{\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}}$.

Conversely, we show that $\overline{\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}} \subset \mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$.

Let $x \in \overline{\mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}}$ then , $\mathscr{B}(x,\gamma,\epsilon) \cap \mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}} \neq \phi$, for evey open ball $\mathscr{B}(x,\gamma,\epsilon)$ of any radius $\gamma > 0$ and $\epsilon > 0$ centred at x. So, let $x \in \mathscr{B}(x,\gamma,\epsilon) \cap \mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$ and $0 < p < 1$ and $\gamma > 0$, choose $\epsilon \in (0,1)$ s.t. $(1-p)*(1-p) > 1-\epsilon$ and $p \diamond p < \epsilon$.

Since $y \in \mathscr{B}(x,\gamma,\epsilon) \cap \mathscr{N}^f_{I^{\infty}_{(\mathcal{S})}}$ so, there exits a subset A of $\mathbb{N}$ s.t $A \in \mathscr{F}(I)$ and $\forall n \in A$, we have $\mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) > 1-p$ and $\mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) < p$, $\mathcal{W}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) < p$ and $\mathcal{U}(\mathscr{N}^f_n(y) - \alpha, \frac{\gamma}{2}) > 1-p$ and $\mathcal{V}(\mathscr{N}^f_n(y) - \alpha, \frac{\gamma}{2}) < p$, $\mathcal{W}(\mathscr{N}^f_n(y) - \alpha, \frac{\gamma}{2}) < p$.

Hence, $\forall n \in A$, we obtain

$$\begin{aligned}
\mathcal{U}(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}) &= \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y) + \mathscr{N}^f_n(y) - \alpha, \gamma) \\
&\geq \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) * \mathcal{U}(\mathscr{N}^f_n(y) - \alpha, \frac{\gamma}{2}) \\
&> (1-p)*(1-p) \\
&> 1-\epsilon,
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}(\mathscr{N}^f_n(x) - \alpha, \frac{\gamma}{2}) &= \mathcal{V}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y) + \mathscr{N}^f_n(y) - \alpha, \gamma) \\
&\leq \mathcal{U}(\mathscr{N}^f_n(x) - \mathscr{N}^f_n(y), \frac{\gamma}{2}) \diamond \mathcal{V}(\mathscr{N}^f_n(y) - \alpha, \frac{\gamma}{2}) \\
&< p \diamond p \\
&< \epsilon
\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}(\mathcal{N}_n^f(x)-\alpha,\frac{\gamma}{2}) &= \mathcal{W}(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y)+\mathcal{N}_n^f(y)-\alpha,\gamma)\\ &\leq \mathcal{W}(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2})\diamond\mathcal{W}(\mathcal{N}_n^f(y)-\alpha,\frac{\gamma}{2})\\ &< p\diamond p\\ &< \epsilon.\end{aligned}$$

Thus, $A\subset\{n\in\mathbb{N}:\mathcal{U}(\mathcal{N}_n^f(x)-\alpha,\gamma)>1-\epsilon$ and $\mathcal{V}(\mathcal{N}_n^f(x)-\alpha,\gamma)<\epsilon\ ,\ \mathcal{W}(\mathcal{N}_n^f(x)-\alpha,\gamma)<\epsilon\}$.

As $A\in\mathscr{F}(I)$, which implies that $\{n\in\mathbb{N}:\mathcal{U}(\mathcal{N}_n^f(x)-\alpha,\gamma)>1-\epsilon$ and $\mathcal{V}(\mathcal{N}_n^f(x)-\alpha,\gamma)<\epsilon\ ,\ \mathcal{W}(\mathcal{N}_n^f(x)-\alpha,\gamma)<\epsilon\}\in\mathscr{F}(I)$. Therefore, $x\in\mathcal{N}^f_{I^\infty_{(\mathcal{S})}}$. Hence, $\overline{\mathcal{N}^f_{I^\infty_{(\mathcal{S})}}}\subset\mathcal{N}^f_{I^\infty_{(\mathcal{S})}}$. □

Theorem 3.12. Let $x=\{x_k\}\in\omega$ be a sequence. If $\exists$ a sequence $y=\{y_k\}\in\mathcal{N}^f_{I(\mathcal{S})}$, such that $\mathcal{N}_n^f(x)=\mathcal{N}_n^f(y)$ for almost all n relative to neutrosophic I, then $x\in\mathcal{N}^f_{I(\mathcal{S})}$.

Proof. Consider $\mathcal{N}_n^f(x)=\mathcal{N}_n^f(y)$ for almost all n relative to I. Then $\{n\in\mathbb{N}:\mathcal{N}_n^f(x)\neq\mathcal{N}_n^f(y)\}\in I$. This implies $\{n\in\mathbb{N}:\mathcal{N}_n^f(x)=\mathcal{N}_n^f(y)\}\in\mathscr{F}(I)$. Therefore, for $n\in\mathscr{F}(I)\ \forall\ \gamma>0$, $\mathcal{U}\big(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2}\big)=1$, $\mathcal{V}\big(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2}\big)=0$ and $\mathcal{W}\big(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2}\big)=0$. Since $\{y_k\}\in\mathcal{N}^f_{I(\mathcal{S})}$, let $I_{(\mathcal{S})}\text{–}\lim(y_k)=\alpha$. Then, for any $\epsilon\in(0,1)$ and $\gamma>0$,

$$\begin{aligned}A_1=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathcal{N}_n^f(y)-\alpha,\frac{\gamma}{2}\big)>1-\epsilon \text{ and } \mathcal{V}\Big(\mathcal{N}_n^f(y)-\alpha,\frac{\gamma}{2}\Big)<\epsilon,\\ \mathcal{W}\big(\mathcal{N}_n^f(y)-\alpha,\frac{\gamma}{2}\big)<\epsilon\big\}\in\mathscr{F}(I).\end{aligned}$$

Consider the set $A_2=\big\{n\in\mathbb{N}:\mathcal{U}\big(\mathcal{N}_n^f(x)-\alpha,\gamma\big)>1-\epsilon$ and $\mathcal{V}\big(\mathcal{N}_n^f(x)-\alpha,\gamma\big)<\epsilon,\mathcal{W}\big(\mathcal{N}_n^f(x)-\alpha,\gamma\big)<\epsilon\big\}$.

We show that $A_1\subset A_2$. So, for $n\in A_1$ we have

$$\begin{aligned}\mathcal{U}\big(\mathcal{N}_n^f(x)-\alpha,\gamma\big) &\geq \mathcal{U}\big(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2}\big)*\mathcal{U}\big(\mathcal{N}_n^f(y)-\alpha,\frac{\gamma}{2}\big)\\ &> 1*(1-\epsilon)\\ &= 1-\epsilon,\end{aligned}$$

$$\begin{aligned}\mathcal{V}\big(\mathcal{N}_n^f(x)-\alpha,\gamma\big) &\leq \mathcal{V}\big(\mathcal{N}_n^f(x)-\mathcal{N}_n^f(y),\frac{\gamma}{2}\big)\diamond\mathcal{V}\big(\mathcal{N}_n^f(y)-\alpha,\frac{\gamma}{2}\big)\\ &< 0\diamond\epsilon\\ &= \epsilon\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}\big(\mathscr{N}_n^f(x)-\alpha,\gamma\big) &\leq \mathcal{W}\big(\mathscr{N}_n^f(x)-\mathscr{N}_n^f(y),\frac{\gamma}{2}\big)\diamond\mathcal{W}\big(\mathscr{N}_n^f(y)-\alpha,\frac{\gamma}{2}\big)\\ &< 0\diamond\epsilon\\ &=\epsilon.\end{aligned}$$

This implies that $n\in A_2$ and hence $A_1\subset A_2$. Since $A_1\in\mathscr{F}(I)$, therefore $A_2\in\mathscr{F}(I)$. Hence, $x=\{x_k\}\in\mathscr{N}^f_{I(\mathcal{S})}$. □

Conclusion

In this research, we investigated the ideal convergence of extended Nörlund sequences in NNS and defined a new type of sequence space $\mathscr{N}^f_{I_0(\mathcal{S})}$, $\mathscr{N}^f_{I(\mathcal{S})}$ and $\mathscr{N}^f_{I_\infty(\mathcal{S})}$ utilising the previously studied Nörlund matrix $\mathscr{N}^f$. In NNS, the concepts of Nörlund ideal convergence and Nörlund ideal Cauchy sequence are examined, and significant findings are established. We may also investigate the topological properties of these spaces, which will give a better technique for dealing with ambiguity and inexactness in numerous fields of science, engineering, and economics.

Acknowledgement

The author appreciates the anonymous referees careful reading and suggestions, which improved the article more understandable.

References

[1] B. C. Tripathy, P. J. Dowari, *Nórlund, and Riesz mean of a sequence of complex uncertain variables*, Filomat, 32 (2018), 2875-2881.

[2] B. C. Tripathy, A. Baruah, *Nórlund and Riesz mean of sequences of fuzzy real numbers*, Applied Mathematics Letters, 23 (2010), 651-655.

[3] F. Samarandache, *Neutrosophic set - A generalization of the intuitionistic fuzzy set*, Inter. J. Pure Appl. Math, 24 (2005), 287-297.

[4] F. M. Mears, *The inverse nórlund mean,* Annals of Mathematics, (1943), 401-410.

[5] H. Fast, *Sur la convergence statistique*, Colloq. Math., 2 (1951), 241-244.

[6] I. J. Schoenberg, *The integrability of certain function abd related summability methods*, Amer. Math. Monthly, 66 (1959), 361-375.

[7] M. Kirisci, N. Simsek, *Neutrosophic normed spaces and statistical convergence*, The Journal of Analysis, 28 (2020), 1059-1073.

[8] V.A. Khan, S.A.A. Abdullah, K.M.A.S. Alshlool, *A study of Nórlund ideal convergent sequence spaces,* Yugoslav Journal of Operations Research, DOI: https://doi.org/10.2298/YJOR200716044K

[9] V.A. Khan, I.A. Khan, *Spaces of intuitionistic fuzzy Nörlund $I-$convergent sequences,* Afrika Matematika, 33 (2022).

[10] V. A. Khan, M. Arshad, M.D. Khan, *Some results of neutrosophic normed space VIA Tribonacci convergent sequence spaces*, J. Inequal. Appl, 42 (2022), 1-27.

[11] V.A Khan. R.K.A Rababah, K.M.A. Alshlool, S.A.A Abdullah, A. Ahmad, *On ideal convergence Fibonacci difference sequence spaces*, Advances in Difference Equations, 2018(1):199, 2018.

[12] V.A. Khan, N. Khan, Y. Khan, *On Zweier paranorm I-convergent double sequence spaces*, Cogent Mathematics, 3 (2016).

[13] K.T. Atanassov, *More on intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 33 (1989), 37-45.

[14] K.T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87-96.

[15] S. Özçağ, Ç. Doğan, *On connectedness in intuitionistic fuzzy special topological spaces*, Internat. J. Math. and Math. Sci., 21 (1998), 33-40.

[16] M. Mursaleen, S.A. Mohiuddine, *Statistical convergence of double sequence in intuitionistic fuzzy normed space*, Chaos, Solitons & Fractals, 41 (2009), 2414-2421.

[17] O. Tug, F. Basar, *On the spaces of Norlund null and Norlund convergent sequences,* This Journal of Pure and Applied Mathematics, 7 (2016), 76-87.

[18] O. Tug, *On some generalized Nörlund ideal convergent sequence spaces,* ZANCO Journal of Pure and Applied Sciences, 28 (2016), 97-103.

[19] R. Filipów, J. Tryba, *Ideal convergence versus matrix summability*, Studia Math., 245 (2019), 101-127.

[20] R. Saadati, J.H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos, Solitons & Fractals, 27 (2006), 331-334.

[21] T. Bera, N.K. Mahapatra, *On neutrosophic soft linear space*, Fuzzy Information and Engineering, 9 (2017), 299-324.

[22] T. Bera, N.K. Mahapatra, *Neutrosophic soft normed linear spaces*, Neutrosophic Sets and Systems, 23 (2018), 1-6.

[23] T. Śalát, B.C. Tripathy, Miloś, *On some properties of I-convergence*, Tatra Mt. Math. Publ, 28 (2004), 274-286.

[24] T. Śalát, B.C. Tripathy, Miloś, *On the I-convergence field*, Ital. J. Pure Appl. Math., 17 (2005), 1-8.

[25] C.S. Wang, *On Nörlund sequence space*, Tamkang J. Math., 9 (1978), 269-274.

[26] W.B.V. Kandasamy, F. Samarandache, *Neutrosophic rings*, Published by Hexis, Phoenix, Arizona (USA) in 2006 (2006).

[27] L.A. Zadeh, *Fuzzy sets*, Inf. Control, 8 (1965), 338-353.

[28] J. Cervenansky, *Statistical convergence and statistical continuity*, Zbornik Vedeckych Prac M_tFSTU, 6 (1998), 207-212.

[29] J.S. Connor, *The statistical and strong p-Cesaro convergence of sequence*, Analysis, 8 (1988), 47-63.

[30] J.S. Connor, *Two valued measures and summability*, Analysis, 10 (1990), 373-385.

[31] R.C. Buck, *The measure theoretic approach to density*, Amer. J. Math, 68 (1946), 560-580.

[32] R.C. Buck, *Generalized asymptotic density*, Amer. J. Math, 75 (1953), 335-346.

[33] D.S. Mitrinović, J. Sandor, B. Crstici, *Handbook of number Theory*, Kluwer Acad. Publ., Dordrecht-Boston-London, 1996.

[34] M. Kirisci, N. Simsek, *Neutrosophic metric spaces*, Mathematical Sciences, Math. Sci., 14 (2020), 241-248.

Accepted: November 9, 2022

Some aspects of the vertex-order graph

Kiruthika G.[*]
Department of Mathematics
Bharathiar University PG Extension and Research Centre
Erode
India
aisswaryakiruthika@gmail.com

Kalamani D.
Department of Mathematics
Bharathiar University PG Extension and Research Centre
Erode
India
kalamanikec@gmail.com

Abstract. The vertex-order graph of the finite cyclic group G is based on its components C_d of the vertex-order graph $\Im(G)$, whose vertices are of order $'d'$ as the divisors of the order of the group G. The important properties of the vertex-order graph and its complements namely girth, radius, diameter, clique number, independence number and rank are derived. Further, the complement $\overline{\Im(G)}$ of the vertex-order graph is proved as a complete t-partite graph and shown with an example. Later, we compute the first, second and third Zagreb indices of the graph $\Im(G)$, $\overline{\Im(Z_p)}$ and $\overline{\Im(Z_{pq})}$.
Keywords: vertex-order graph, complete t-partite, Zagreb index.

1. Introduction

Group theoretical facts with disconnected graph will yield the finest application in the real world problems like protein-protein interaction, genetically disorders, existence of new virus with pandemic potential, handling drug discovery situation etc., in the medical science field. Over the past four decades, researchers developed enormous amount of applications in the area of algebraic graph theory, especially algebraic facts with connected graphs [2, 3, 13].

A graph H is said to be connected if there exists a path between every pair of vertices; Otherwise, the graph is disconnected. A disconnected graph consists of two or more connected subgraphs of H. Each of these connected subgraphs are called component of H.

A clique C of H is a subgraph of a graph H such that all vertices in the subgraph are completely connected with each other. The clique number of the graph H, denoted by $Clique(H)$, is the number of vertices in the maximal clique of H. An independent set or stable set in a graph H is a set of pairwise non

∗. Corresponding author

adjacent vertices of H. The independence number of a graph H, denoted by $\alpha(H)$, is the maximum size of an independent set of vertices. The girth of a graph H with a cycle is the length of the shortest cycle. The eccentricity of a vertex u, denoted by $e(u)$, is the greatest distance from u to all other vertices in the graph H. That is,

$$e(u) = \max_{x \in V(G)} d(u, x).$$

The radius of the graph H, denoted by $rad(H)$, is the value of smallest eccentricity. The diameter of the graph H, denoted by $diam(H)$, is the value of greatest eccentricity.

The eigenvalues of a graph H are defined to be the eigenvalues of its adjacency matrix. The rank of the graph H, denoted by $\rho(H)$, is defined as the number of non zero eigenvalues of its adjacency matrix.

Kiruthika and Kalamani [15] found generalization of the vertex partition and edge partition of the power graph of the finite abelian group of an order pq if $p < q$, where p and q are distinct primes. They found some types of topological indices of graphs related to the groups. Jahandideh, Sarmin and Omer computed the various types of indices like Szeged index, edge Wiener index, first Zagreb index and the second Zagreb index for the non commuting graph [5]. Ramanae, Gundloor and Jummannaver [18] investigated the third Zagreb index, forgotten index and coindices of cluster graphs. Veylaki, Nikmehr and Tavalla [19] explained some basic mathematical properties for the third and hyper Zagreb coindices of graph operations.

Topological descriptors are based on the graph impression of the molecule and can also encode chemical information concerning atom type and bond multiplicity. It plays a vital role in the area of Quantitative Structure Activity Relation(QSAR), Quantitative Structure Property Relation(QSPR) and Fuzzy Lattice Neural Network(FLNN) [14]. One of the classical topological index is the familiar Zagreb index that was first introduced in [11] where Gutman and Trinajstic[10] examined the dependence of total-electron energy on molecular structure and this was elaborated in [8, 9].

The first and second Zagreb indices of a graph H, are defined as

$$M_1(H) = \sum_{uv \in E(H)} [d(u) + d(v)],$$
$$M_2(H) = \sum_{uv \in E(H)} [d(u)d(v)],$$

respectively.

Another Zagreb index called the third Zagreb index of a graph H, denoted by $M_3(H)$, is defined by Fath-Tabar [4] as

$$M_3(H) = \sum_{uv \in E(H)} |d(u) - d(v)|.$$

The Zagreb indices [8, 12, 16, 17] play a very important key role in the past, present and future research developments.

In this research, we newly defined the vertex-order graph $\Im(G)$ of the finite cyclic group G. The study of certain properties of the vertex-order graph of the finite cyclic group G is the main outcome and is presented in this research. The complement of the vertex-order graph is also defined with simple proofs.

Throughout this paper, we follow the terminologies and notations of [6] for groups and [20, 7] for graphs.

2. Some theoretical properties of the vertex-order graph

In this section, some simple characteristics of the graph $\Im(G)$ are studied with theorems and examples.

Definition 2.1. *A **vertex-order** graph of a finite cyclic group G is a simple graph whose vertices are elements of the group G and there is an edge between any two distinct vertices iff its orders are equal and is denoted by $\Im(G)$.*

Example 2.1. The vertex-order graph $\Im(Z_9)$ of the finite cyclic group G is shown in Figure 1.

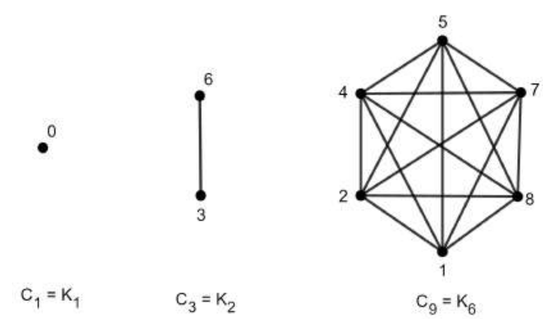


Figure 1: The vertex-order graph $\Im(Z_9)$.

Theorem 2.1. *The girth $gr(\Im(G))$ of the vertex-order graph $\Im(G)$ is given by*

$$gr(\Im(G)) = \begin{cases} 3, & \text{if } \phi(n) \geq 3 \\ \infty, & \text{otherwise.} \end{cases}$$

where ϕ is the euler totient function.

Proof. Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G of order n.

It is noted that the length of the shortest cycle of $\Im(G)$ is the minimum length of the cycles of all the components C_d where d is the divisor of n. The graph

$\Im(G)$ is disconnected and the components C_d are all complete. The complete subgraph is denoted by K_m where $m = \phi(d)$ and each K_m is $(m-1)$ regular.

If $\phi(n) \geq 3$ then the vertex-order graph contains the complete graph K_m and $m \geq 3$. In this case, the length of the shortest cycle is 3.

In all other cases, the graph does not contain any cycle.

Hence,

$$gr(\Im(G)) = \begin{cases} 3, & \text{if } \phi(n) \geq 3 \\ \infty, & \text{otherwise.} \end{cases} \quad \square$$

Theorem 2.2. *For any vertex-order graph $\Im(G)$, the eccentricity of the vertex v is $e(v) = \infty$*

Proof. Let $e(v)$ be the eccentricity of the vertex v of the vertex-order graph $\Im(G)$.

The distance between the any two vertices v_i, v_j is ∞, if v_i, v_j are the vertices of two different components C_d of the vertex order graph. Since each component C_d is complete, the distance between v_i and v_j is 1 if v_i, v_j are vertices of the same component. Thus, $e(v) = \max_j d(v,\ v_j) = \infty$. Therefore, the eccentricity of the vertex v of the vertex-order graph is ∞. □

Lemma 2.1. *Let G be the finite cyclic group of order n. Then, the following holds:*

(*i*) *diam*$(\Im(G)) = \infty$.

(*ii*) *rad*$(\Im(G)) = \infty$.

Theorem 2.3. *The independence number of the vertex-order graph denoted by $\alpha(\Im(G))$ is always t where t is the number of components of the graph.*

Proof. Let $\alpha(\Im(G))$ be the independence number of the vertex-order graph. It is clear to see that the independence number of a complete graph is 1. Since each component C_d is a clique, the independence number of complete graph is one i.e., $\alpha(C_d) = 1$ and is denoted by I_d. Also the the vertex-order graph is the disjoint union of its components C_d. Thus

$$\begin{aligned} \alpha(\Im(G)) &= \sum_d \alpha(C_d) \\ &= Number\ of\ components\ of\ the\ vertex\ order\ graph = t. \end{aligned}$$

$\therefore$ Independence number of the vertex-order graph is t. □

Corollary 2.1. *The independence number $\alpha(\Im(Z_n))$ of the vertex-order graph is four if $n = pq$, where p and q are any two distinct primes.*

Example 2.2. The independence number of the vertex-order graph $\Im(Z_{15})$ is four, $\alpha[\Im(Z_{15})] = 4$ which is shown in Figure 3.

Theorem 2.4. *Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G. Then*

$$\rho(\Im(G)) = \begin{cases} n-1, & \textit{if } n \textit{ is odd,} \\ n-2, & \textit{if } n \textit{ is even,} \end{cases}$$

where $\rho(\Im(G))$ is the rank of the vertex-order graph.

Proof. Let λ_1, λ_2, λ_3,...λ_n be the eigenvalues of the vertex-order graph of the finite cyclic group G order n.

If λ_i, $i = 1, 2, ...m$ are the eigenvalues of the complete graph K_m where $m = \phi(d)$ and d is the divisor of n, then for $m \neq 1$,

$$\lambda_i = \begin{cases} -1, & \text{if } i = 1, 2, 3....m-1, \\ m-1, & \text{if } i = m. \end{cases}$$

If $m = 1$, the eigenvalue of K_m is zero.

The set of eigenvalues of the vertex-order graph $\Im(G)$ is the union of all the eigenvalues of the complete graph K_m for all m.

So, the number of non-zero eigenvalues value of the vertex-order graph is $n-1$ if n is odd and $n-2$ if n is even, since the number of isolated vertices is 1 if n is odd and 2 if n is even.

Hence, the rank of the vertex-order graph

$$\rho(\Im(G)) = \begin{cases} n-1, & \text{if } n \text{ is} \quad \text{odd}, \\ n-2, & \text{if } n \text{ is} \quad \text{even}. \end{cases}$$ □

3. Properties of the complement of the vertex-order graph

Let $\overline{\Im(G)}$ be the complement of the vertex-order graph of the finite cyclic group G. In this section, some important properties of the complement of the vertex-order graph are discussed.

Theorem 3.1. *The complement $\overline{\Im(G)}$ of the vertex-order graph is a complete t-partite graph.*

Proof. Let t be the number of connected component of the vertex-order graph $\Im(G)$ of the finite cyclic group G of order n.

Each component C_d of the vertex-order graph is a complete subgraph K_m and the vertex-order graph $\Im(G)$ is the disjoint union of the complete subgraphs K_m where $m = \phi(d)$ and d is the divisors of n.

It is clear that no two vertices in the same component C_d of the vertices are adjacent in their complements. This implies that each component C_d of $\Im(G)$ is independent in their complement $\overline{\Im(G)}$. Hence, the complement graph $\overline{\Im(G)}$ is the complete t-partite graph where t is the number of components of the vertex-order graph. □

Example 3.1. Let $\overline{\Im(Z_{555})}$ be the complement of the the vertex-order graph of the group Z_{555}. Let C_1, C_3, C_5, C_{15}, C_{37}, C_{111}, C_{185}, C_{555} are the 8 components of the graph $\Im(Z_{555})$ where 1, 3, 5, 15, 37, 111, 185, 555 are the divisors of 555. Then, the number of vertices in the each component C_d of the vertex-order graph $\overline{\Im(Z_{555})}$ is given below:

$$\begin{aligned}
|C_1| &= \phi(1) = 1,\\
|C_3| &= \phi(3) = 2,\\
|C_5| &= \phi(5) = 4,\\
|C_{15}| &= \phi(15) = 8,\\
|C_{37}| &= \phi(37) = 36,\\
|C_{111}| &= \phi(111) = 72,\\
|C_{185}| &= \phi(185) = 144,\\
|C_{555}| &= \phi(555) = 288.
\end{aligned}$$

Hence, K_1, K_2, K_4, K_8, K_{36}, K_{72}, K_{144}, K_{288} are the complete subgraphs of $\Im(Z_{555})$. The set of vertices in K_m are independent in their complement $\overline{\Im(Z_{555})}$ for any m. Thus, $\overline{\Im(Z_{555})}$ is the complete 8-partite graph and is denoted by $K_{1,2,4,8,36,72,144,288}$ which is shown in Figure 2.

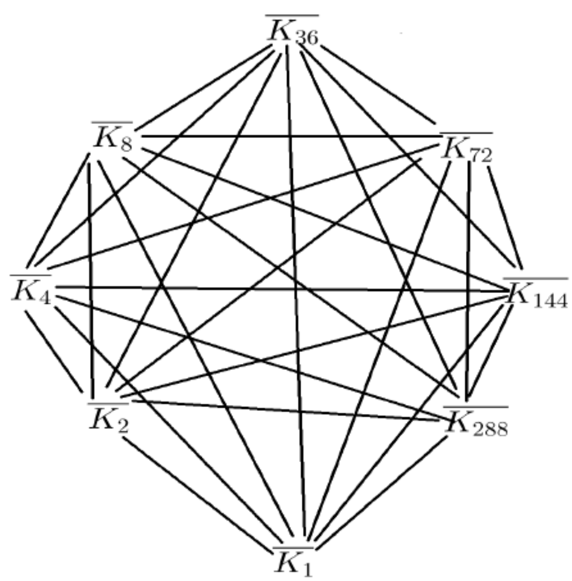


Figure 2: The edge adjacency of the complement of the Vertex-order graph $\overline{\Im(Z_{555})}$.

Lemma 3.1. *The complement $\overline{\Im(G|e)}$ of the vertex-order graph is complete bi-partite if n is the square of the prime number.*

Proof. Let C_d be the component of the vertex-order graph. If $n = p^2$, then $\Im(G)$ has exactly three distinct components namely C_1, C_p, C_{p^2} where p is a prime. Each of these are complete which is shown in Figure 1. By omitting the identity element, there are only two components C_p and C_{p^2} which are independent in their complement. Hence, $\overline{\Im(G|e)}$ is a complete bi-partite graph. □

Theorem 3.2. *The complement $\overline{\Im(G)}$ of the vertex-order graph has a clique as the number of independent set.*

Proof. Let $\overline{\Im(G)}$ be the complement of the vertex-order graph of the finite cyclic group G of order n.

Let $Clique(\overline{\Im(G)})$ be the clique number of the complement of the vertex-order graph. From Theorem 3.1, the complement of the vertex-order graph $\overline{\Im(G)}$ is the complete t-partite graph. From this it is clear that the largest complete subgraph $\overline{\Im(G)}$ contains t vertices

$$\therefore\ Clique(\overline{\Im(G)}) = t. \square$$

Corollary 3.1. *The independence number of the complement of the vertex-order graph $\overline{\Im(G)}$ is the number of the generators of the finite cyclic group G, i.e., $\alpha(\overline{\Im(G)}) = \phi(n)$.*

Theorem 3.3. *The girth of the complement of the vertex order graph $\Im(G)$ is given by*

$$gr\overline{(\Im(G))} = \begin{cases} \infty, & \text{if } n = p, \\ 3, & \text{if } n \neq p. \end{cases}$$

Proof. Let $\Im(G)$ be the vertex-order graph of order n. Let $gr\overline{(\Im(G))}$ be the girth of the complement of the vertex-order graph $\Im(G)$.

If $n = p$, a prime, then the complement graph does not contain cycle since $\overline{\Im(G)}$ is a star graph.

In this case, the girth of the complement of the vertex-order graph is ∞.

If $n \neq p$ where p is a prime, then $\overline{\Im(G)}$ is the complete t-partite graph for every $t \geq 3$ and the complement graph contains the cycle of length 3.

In this case, the girth of the complement of the graph $\Im(G)$ is 3. Thus,

$$gr\overline{(\Im(G))} = \begin{cases} \infty, & \text{if } n = p, \\ 3, & \text{if } n \neq p. \end{cases} \quad \square$$

Theorem 3.4. *Let $\overline{\Im(G)}$ be the complement of the vertex-order graph of the finite cyclic group G. Then, the following holds:*

(*i*) $rad(\overline{\Im(G)}) = 1$;

(*ii*) $diam(\overline{\Im(G)}) = 2$.

Proof. Let $\overline{\Im(G)}$ be the vertex-order graph associated with finite cyclic group G of order n. It is noticed that $\overline{\Im(G)}$ is connected, since $\Im(G)$ is disconnected. The minimum and maximum eccentricities are 1 and 2 respectively.

Henceforth, the proof follows $diam(\overline{\Im(G)}) = 2$ and $rad(\overline{\Im(G)}) = 1$. □

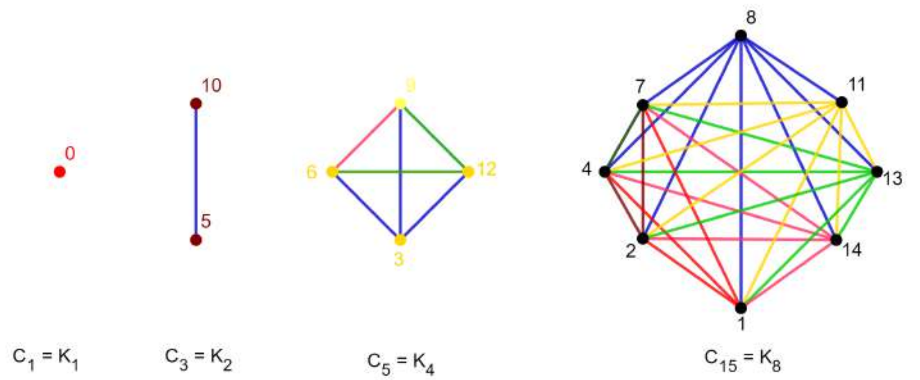


Figure 3: The vertex-order graph $\Im(Z_{15})$ with its four components.

Corollary 3.2. *The edge set of the complement of the vertex-order graph* $\Im(G)$ *is partitioned into* tC_2 *edge sets which is equal to the number of independent set of the graph* $\Im(G)$.

Example 3.2. The number of independent set of the graph $\Im(Z_{15})$ is 4 which is shown in Figure 3. From Corollary 3.2, the number of edge sets of the complement of the vertex-order graph $\Im(G)$ is 6 which is shown in Figure 4.

Then, the number of edges

$$\begin{aligned}|E(\overline{\Im(Z_{15})})| &= |E_1| + |E_2| + |E_3| + |E_4| + |E_5| + |E_6| \\ &= 4 + 2 + 8 + 8 + 32 + 16 \\ &= 70.\end{aligned}$$

Lemma 3.2. *The complement* $\overline{\Im(G|e)}$ *of the vertex-order graph is a null graph iff* n *is prime.*

Proof. If $n = p$, a prime, then $\overline{\Im(G)}$ is the star graph in which identity element e is an universal vertex. By omitting the identity element e, the star graph $K_{1,n-1}$ becomes null graph. Hence, $\overline{\Im(G|e)}$ is the null graph. □

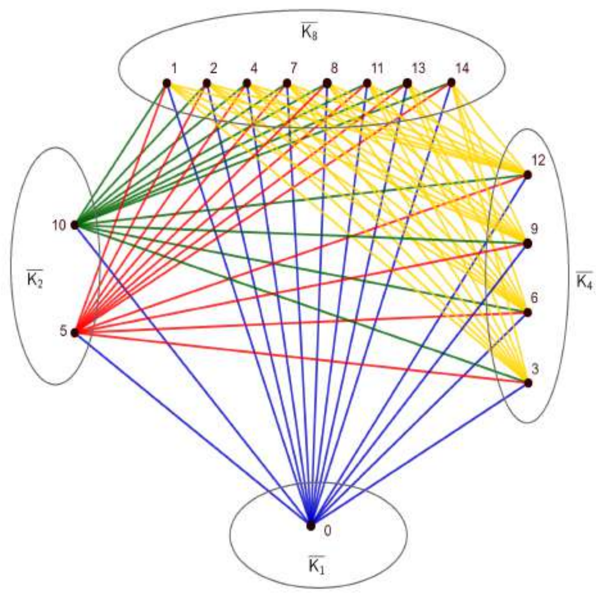


Figure 4: The complement of the Vertex-order graph $\Im(Z_{15})$ with its four independent sets.

Corollary 3.3. *The complement $\overline{\Im(G|i_v)}$ of the vertex-order graph is uni-cyclic if n is 6 where i_v is the isolated vertices of the graph $\Im(G)$.*

Theorem 3.5. *For any vertex-order graph $\Im(G)$, the rank of the complement of the vertex-order graph $\Im(G)$ is $\rho(\overline{\Im(G)}) = t$.*

Proof. Let $\rho(\overline{\Im(G)})$ be the rank of the complement of the vertex-order graph. From [1] the vertex-order graph $\Im(G)$ of rank t has clique number at most t; equality holds if and only if $\Im(G)$ is a complete t-partite graph. Thus, it is found that the rank of the complement of the vertex-order graph $\overline{\Im(G)}$ is the maximum clique of the graph $\overline{\Im(G)}$, since the every component C_d of $\Im(G)$ contain clique which is complete.

Rank of the complement of the vertex-order graph is the clique number of the complement of the vertex-order graph. From Theorem 3.2, the rank of the complement $\overline{\Im(G)}$ of the vertex-order graph is the number of connected components t $\rho(\overline{\Im(G)}) = Clique(\overline{\Im(G)}) = t$. □

Example 3.3. The rank of the complement of the vertex-order graph $\Im(Z_8)$ is four which is shown in Figure 5. $\rho(\overline{\Im(Z_8)}) = 4$.

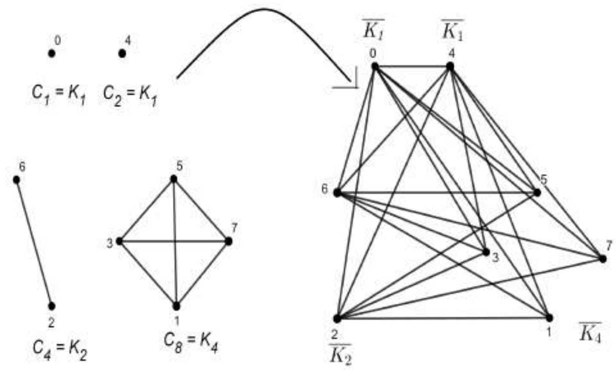


Figure 5: The vertex-order graph $\Im(Z_8)$ with its four components and transformation of its complement.

4. Computation of Zagreb indices of the vertex-order graph

In this section, we derive some Zagreb indices of the vertex-order graph.

Theorem 4.1. *The first Zagreb index of the vertex-order graph $M_1(\Im(G))$ is $\sum_m m(m-1)^2$, where $m = \phi(d)$.*

Proof. Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G. Since $\Im(G)$ is a disconnected graph, there are finite number of connected components C_d, each of which is a complete graph K_m where $m = \phi(d)$ and d is the divisor of the order of the group G. Then, the number of edges and vertices in K_m for $m \neq 1$ are mC_2 and m respectively. If $m = 1$, there is no edge. Thus, the first Zagreb index

$$\begin{aligned} M_1(\Im(G)) &= \sum_{uv \in E(G)} [d(u) + d(v)] \\ &= \sum_m mC_2[(m-1) + (m-1)] \\ &= \sum_m m(m-1)^2. \end{aligned}$$ □

Example 4.1. The first Zagreb index of the vertex-order graph $M_1(\Im(Z_{15}))$ is 430.

By the definition of vertex-order graph $\Im(Z_{15})$, the connected components are given by K_1, K_2, K_4, K_8 which is shown in Figure 3. Then, the first Zagreb index

$$M_1(\Im(Z_{15}) = \sum_m m(m-1)^2 = 430.$$

Theorem 4.2. *The second Zagreb index of the vertex-order graph* $M_2(\Im(G))$ *is* $\sum_m \frac{m(m-1)^3}{2}$ *where* $m = \phi(d)$.

Proof. Let $\Im(G)$ be the vertex-order graph of the finite cyclic group G. Since $\Im(G)$ is a disconnected graph, there are finite number of connected components C_d, each of which is a complete graph K_m where $m = \phi(d)$ and d is the divisor of the order of the group G. Then, the number of edges and vertices in K_m for $m \neq 1$ are mC_2 and m respectively. If $m = 1$ there is no edge. Thus, the second Zagreb index

$$\begin{aligned} M_2(\Im(G)) &= \sum_{uv \in E(G)} d(u)d(v) \\ &= \sum_m mC_2[(m-1)(m-1)] \\ &= \sum_m \frac{m(m-1)^3}{2}. \end{aligned}$$

□

Example 4.2. The second Zagreb index of the vertex-order graph $M_2(\Im(Z_{18}))$ is 752.

By the definition of vertex-order graph $\Im(Z_{18})$, the connected components are given by K_1,K_1, K_2, K_2, K_6, K_6. Then, the second Zagreb index $M_2(\Im(Z_{18}))$ is given by

$$M_2(\Im(Z_{18})) = \sum_m \frac{m(m-1)^3}{2} = 752.$$

Lemma 4.1. *Let* $\Im(G)$ *be the vertex-order graph of the finite cyclic group* G*. Then, the third Zagreb index of the vertex-order graph* $M_3(\Im(G))$ *is zero, since the order of the vertices of all the components* C_d *are equal.*

5. Computation of Zagreb indices of the complement of the vertex-order graph

In this section, some Zagreb indices of the complement of the vertex-order graph are derived with its generalizations.

Theorem 5.1. *Let* $\overline{\Im(G)}$ *be the complement of the vertex-order graph of the finite cyclic group* G *of order* pq *where* p *and* q *are any distinct primes. Then, its Zagreb indices are given by*

(1) $M_1(\overline{\Im(G)}) = p^2q^2(p+q+1) - pq(p^2+q^2+p+q-1)$;

(2) $M_3(\overline{\Im(G)}) = p^2q^2(p+q-5) - pq[3(p^2+q^2+5) - 9p - 11q] + p + q - 1) + 2(p^3+q^3) - 4p(p-1) - 6q(q-1)$.

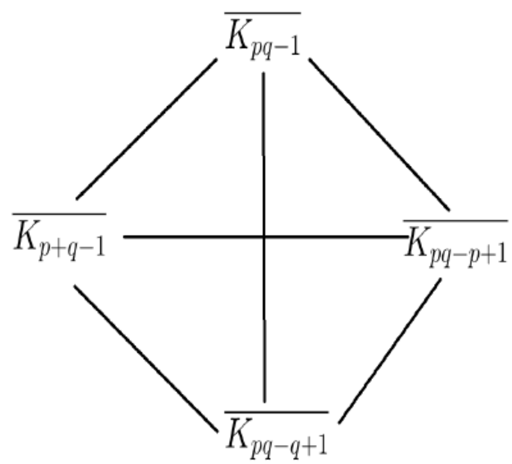


Figure 6: The edge adjacency of the complement of the vertex-order graph $\overline{\Im(Z_{pq})}$.

Proof. Consider the complement of the vertex-order graph $\Im(G)$ of order n where $n = pq$, p and q are any two distinct primes. The total number of vertices and edges of $\overline{\Im(G)}$ is given by pq and $p^2q + pq^2 - p^2 - q^2 - 2pq + 2p + 2q - 2$ respectively. Then, the vertex set can be divided into 1 vertex of degree $pq - 1$, $p - 1$ vertices of degree $pq - p + 1$, $q - 1$ vertices of degree $pq - q + 1$ and $pq - p - q + 1$ vertices of degree $p + q - 1$. Let $d_{\overline{\Im(Z_{pq})}}(u)$ and $d_{\overline{\Im(Z_{pq})}}(v)$ be the degrees of the end vertices u and v respectively.

The edge set $E(\overline{\Im(G)})$ can be divided into two edge partitions based on the degrees of end vertices. These can easily done by using the four independent sets which is shown in Figure 6.

The first partition of edges $E_1(\overline{\Im(Z_{pq})})$ contains $p - 1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq - 1$, $d_{\overline{\Im(Z_{pq})}}(v) = pq - p + 1$, the second partition of edges $E_2(\overline{\Im(Z_{pq})})$ contains $q - 1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq - 1$, $d_{\overline{\Im(Z_{pq})}}(v) = pq - q + 1$, the third partition of edges $E_3(\overline{\Im(Z_{pq})})$ contains $pq - p - q + 1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq - 1$, $d_{\overline{\Im(Z_{pq})}}(v) = p + q - 1$.

The fourth partition of edges $E_4(\overline{\Im(Z_{pq})})$ contains $pq - p - q + 1$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq - p + 1$, $d_{\overline{\Im(Z_{pq})}}(v) = pq - q + 1$.

The fifth partition of the edges $E_5(\overline{\Im(Z_{pq})})$ contains $(p - 1)(pq - p - q + 1)$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq - p + 1$, $d_{\overline{\Im(Z_{pq})}}(v) = p + q - 1$.

The sixth partition of the edges $E_6(\overline{\Im(Z_{pq})})$ contains $(q - 1)(pq - p - q + 1)$ edges uv, where $d_{\overline{\Im(Z_{pq})}}(u) = pq - q + 1$, $d_{\overline{\Im(Z_{pq})}}(v) = p + q - 1$. Then, the required results for the graph $\overline{\Im(Z_{pq})}$ by using the number of its edge partition as follows:

(1) The first Zagreb index

$$\begin{aligned} M_1(\overline{\Im(Z_{pq})}) &= \sum_{uv\in E(\overline{\Im(Z_{pq})})} [d(u)+d(v)] \\ &= (p-1)[pq-1+pq-p+1]+(q-1)[pq-1+pq-q+1] \\ &+ (pq-p-q+1)[pq-1+p+q-1] \\ &+ (pq-p-q+1)[pq-p+1+pq-q+1] \\ &+ (p-1)(pq-p-q+1)[pq-p+1+p+q-1] \\ &+ (q-1)(pq-p-q+1)[pq-p+1+p+q-1] \\ &= p^2q^2(p+q+1)-pq(p^2+q^2+p+q-1). \end{aligned}$$

(2) The third Zagreb index

$$\begin{aligned} M_3(\overline{\Im(Z_{pq})}) &= \sum_{uv\in E(\overline{\Im(Z_{pq})})} |d(u)-d(v)| \\ &= (p-1)[pq-1-pq+p-1]+(q-1)[pq-1-pq+q-1] \\ &+ (pq-p-q+1)[pq-1-p-q+1] \\ &+ (pq-p-q+1)[pq-p+1-pq+q-1] \\ &+ (p-1)(pq-p-q+1)[pq-p+1-p-q+1] \\ &+ (q-1)(pq-p-q+1)[pq-p+1-p-q+1] \\ &= p^2q^2(p+q-5)-pq[3(p^2+q^2+5)-9p-11q] \\ &+ 2(p^3+q^3)-4p(p-1)-6q(q-1). \end{aligned}$$

Similarly, we can generalize the second Zagreb index of the graph $\overline{\Im(Z_{pq})}$. □

Example 5.1. Let $\overline{\Im(Z_{15})}$ be the complement of the vertex-order graph of the finite cyclic group Z_{15}. Then, its Zagreb indices are given by

(1) $M_1(\overline{\Im(Z_{15})}) = 1410$;

(2) $M_3(\overline{\Im(Z_{15})}) = 310$.

Using Theorem 5.1 the results obtained for $\Im(Z_{15})$ are as follows:

(1) The first Zagreb index

$$\begin{aligned} M_1(\overline{\Im(Z_{15})}) &= p^2q^2(p+q+1)-pq(p^2+q^2+p+q-1) \\ &= 3^25^2(3+5+1)-3.5(3^2+5^2+3+5-1) \\ &= 225(9)-15(41) \\ &= 1410. \end{aligned}$$

(2) The third Zagreb index

$$\begin{aligned}M_3(\overline{\Im(Z_{15})}) &= p^2q^2(p+q-5) - pq[3(p^2+q^2+5) - 9p - 11q] \\ &+ 2(p^3+q^3) - 4p(p-1) - 6q(q-1) \\ &= 310.\end{aligned}$$

Similarly, we can enumerate the second Zagreb index of the graph $\overline{\Im(Z_{15})}$.

Theorem 5.2. *Let $\overline{\Im(G)}$ be the complement of the vertex-order graph of the finite cyclic group G of prime order p. Then, its Zagreb indices are given by*

(1) $M_1(\overline{\Im(G)}) = p(p-1)$;

(2) $M_2(\overline{\Im(G)}) = (p-1)^2$;

(3) $M_3(\overline{\Im(G)}) = (p-1)(p-2)$.

Proof. Let $\overline{\Im(Z_p)}$ be the complement of the vertex-order graph of the finite cyclic group G. Since $\Im(G)$ is a disconnected graph of complete graphs K_1, K_{p-1}, the total number of vertices and edges of $\overline{\Im(Z_p)}$ (or) $K_{1,p-1}$ are given by p and $p-1$ respectively. Then, the vertex set can be partitioned into 1 vertices of degree $p-1$, and $p-1$ vertices of degree 1. Thus, the only one edge set which is given by $E_{1,p-1} = p-1$. (1) The first Zagreb index

$$M_1(\overline{\Im(Z_p)}) = \sum_{uv\in E(\overline{\Im(Z_p)})} [d(u)+d(v)] = p(p-1).$$

(2) The second Zagreb index

$$\begin{aligned}M_2(\overline{\Im(Z_p)}) &= \sum_{uv\in E(\overline{\Im(Z_p)})} d(u)d(v) \\ &= (p-1)[(p-1)(1)] \\ &= (p-1)^2.\end{aligned}$$

(3) The third Zagreb index

$$M_3(\overline{\Im(Z_p)}) = \sum_{uv\in E(\overline{\Im(Z_p)})} |d(u)-d(v)| = (p-1)(p-2). \qquad \square$$

Example 5.2. The first, second and third Zagreb indices of the complement of the vertex-order graph $\Im(Z_{19})$ of the finite cyclic group Z_{19} are 342, 324, 306 respectively.

6. Conclusion

In this paper, the graph theoretical properties of the vertex-order graph and its complements are interpreted with their proofs. Also the some Zagreb indices of the vertex-order graph $\Im(G)$ and the complement of the vertex-order graph $\Im(Z_{pq})$, $\Im(Z_p)$ are derived with their examples.

References

[1] S. Akbari, P. J. Cameron, G. B. Khosrovshahi, *Ranks and signatures of adjacency matrices*, manuscript, available online at http://www.maths.qmu.ac.uk/lsoicher/designtheory.org/library/preprints /ranks.pdf. 2004.

[2] P. Balakrishnan, M. Sattanathan, R. Kala, *The center graph of a group*, South Asian J. of Math., 1 (2011), 21-28.

[3] P. J. Cameron, *The power graph of a finite group-II*, J. Group Theory, 13 (2010), 779-783.

[4] H. Fath-Tabar, *Old and new Zagreb indices of graphs*, Match. Commun. Math. Comput. Chem., 65 (2011), 79-84.

[5] M. Jahandideh, N.H. Sarmin, S.M.S. Omer, *The topological indices of non commuting graph of a finite group*, Int. J. of Pure & Appl. Math. 105 (2015), 27-38.

[6] J. Gallian, *Contemperary abstract algebra*, Cengage Learning, (2017), 1-557.

[7] F. Harary, *Graph theory*, Narosa Publishing House, New Delhi, 1988.

[8] I. Gutman, B. Furtula, Z. K. Vukicevic, G. Popvoda, *On Zagreb indices and co-indices*, Match. Commun. Math. and Comput. Chem., 74 (2015), 5-16.

[9] I. Gutman, K. C. Das, *The first Zagreb index 30 years after*, Match. Commun. Math. Comput. Chem., 50 (2004), 83-92.

[10] I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals total-electron energy of alternant hydrocarbons*, Chem. Phys. Lett., 17 (1972), 535-538.

[11] I. Gutman, N. Trinajstic, C.F. Wilcox, *Graph theory and molecular orbitals. XII. Acyclic polyenes*, J. Chem. Phys., 62 (1975), 3399-3405.

[12] D. Kalamani, G. Kiruthika, *Subdivision vertex corona and subdivision vertex neighbourhood corona of cyclic graphs*, Advan. in Math. Scient. J., 9 (2020), 607-617.

[13] D. Kalamani, G. Ramya, *Product maximal graph of a finite commutative ring*, Bull. of Cal. Math. Soci., 113 (2021), 127-134.

[14] D. Kalamani, P. Balasubramanie, *Age classification using fuzzy lattice neural network*, Inter. Conf. on Intelli. Syst. Desi. and Appli., IEEE, (2006).

[15] G. Kiruthika, D. Kalamani, *Degree based partition of the power graphs of a finite abelian group*, Malaya J. of Matematik, 1 (2020), 66-71.

[16] G. Kiruthika, D. Kalamani, *Computation of Zagreb indices on K-gamma graphs*, Inter. J. of Research in Advent Tech., 7 (2019), 413-417.

[17] K. Pattabiraman *The third Zagreb indices and its coindices of two classes of graphs*, Bulletin of the International Mathematical Virtual Institute, 8 (2018), 213-219.

[18] H.S. Ramanae, M.M. Gundloor, R.B. Jummannaver, *Third Zagreb, forgotten index and coindices of some cluster graph*, Asian J. Mathe. and Comp. Research., 21 (2017), 210-216.

[19] M. Veylaki, M.J. Nikmehr, H.A. Tavallaee, *The third and hyper-Zagreb coindices of some graph operations*, J. Appl. Math. Comput., 50 (2016), 315-325.

[20] D. B. West, *Introduction to graph theory*, Prentice Hall, 2008-third impression.

Accepted: April 19, 2022

Lebesgue's theorem and Egoroff's theorem for complex uncertain sequences

Ömer Kişi[*]
Department of Mathematics
Bartin University
Bartın
Turkey
okisi@bartin.edu.tr

Mehmet Gürdal
Department of Mathematics
Suleyman Demirel University
Isparta
Turkey
gurdalmehmet@sdu.edu.tr

Abstract. In this paper, within framework uncertain theory, we investigate Lebesgue's theorem, Egoroff's theorem and Riesz's theorem for complex uncertain sequences.
Keywords: uncertain theory, strongly order continuous, Lebesgue type theorems, Riesz type theorem.

1. Introduction

Uncertainty theory was initiated by Liu [2] in 2007 and advanced by Liu [3] in 2011 which based on an uncertain measure which supplies normality, duality, subadditivity, and product axioms. Recently, uncertainty theory has effectively been applied to uncertain programming (see, e,g., Liu [4], Liu and Chen [5]), uncertain risk analysis (see, e.g., Liu [6]), uncertain calculus (see, e.g., Liu [7]) and uncertain statistics (see, e.g., Tripathy and Nath [8]), etc.

Peng [9] proposed the notions of complex uncertain variables that are measurable functions from uncertainty spaces to the set of complex numbers. As convergence of sequences plays an essential role in the basic theory of mathematics, there are many mathematicians who have worked these in the field of uncertain measure. Liu [2] presented convergence in measure, convergence in mean, convergence almost surely (a.s.) and convergence in distribution in 2007. You [12] gave a kind type of convergence called convergence uniformly almost surely (u.a.s.) and proved the relationships among the convergence notions. Based on these concepts, the convergence of complex uncertain sequences was first worked by Chen, Ning and Wang [13]. Tripathy and Nath [8] investigated the statistical convergence concepts of complex uncertain sequences.

*. Corresponding author

Several kinds of convergence were investigated for sequence of measurable functions on a measure space, and fundamental relations between these types were examined [14]. Fuzzy measure theory is a generalisation of classical measure theory. This generalisation is acquired by exchanging the additivity axiom of classical measures with weak axioms of monotonicity and continuity [15]. As detailed in [16, 17, 18], several generalizations of Lebesgue's theorem, Egoroff's theorem and Riesz's theorem for sequence of measurable functions on classical measure spaces hold for fuzzy measures with the autocontinuity and finiteness.

This paper is devoted to presenting classical theorems such as Lebesgue's theorem, Egoroff's theorem and Riesz's theorem for complex uncertain sequences in uncertain theory.

2. Preliminaries

First, some basic notions and theorems in uncertainty theory are given, which are utilized in this paper.

Definition 2.1. *Assume that $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ is named an uncertain measure if it supplies the subsequent axioms:*

(*i*) $\mathcal{M}\{\Gamma\} = 1$;

(*ii*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any* $\Lambda \in \Gamma$

(*iii*) *For all countable sequence of* $\{\Lambda_p\} \subset \mathcal{L}$, *we obtain*

$$\mathcal{M}\left\{\bigcup_{p=1}^{\infty}\Lambda_p\right\} \leq \sum_{p=1}^{\infty}\mathcal{M}\{\Lambda_p\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ *is named an uncertainty space, and every element* Λ *in* $\mathcal{L}$ *is known as an event.*

Definition 2.2. *A complex uncertain variable is a measurable function from the space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of complex numbers, namely, for any Borel set of* T *of complex numbers, the set*

$$\{\zeta \in T\} = \{\gamma \in \Gamma : \zeta(\gamma) \in T\}$$

is an event.

Definition 2.3. *The sequence* $\{\zeta_w\}$ *is named to be convergent a.s. to* ζ *provided that there is an event* Λ *with* $\mathcal{M}\{\Lambda\} = 1$ *such that*

$$\lim_{w\to\infty} \|\zeta_w(\gamma) - \zeta(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$.

Definition 2.4. *The sequence* $\{\zeta_w\}$ *is named to be convergent u.a.s. to* ζ *provided that there is a* $\{R_k\}$, $\mathcal{M}\{R_k\} \to 0$ *such that* $\{\zeta_w\}$ *converges uniformly to* ζ *in* $R_k^c = \Gamma - R_k$, *for any fixed* $k \in \mathbb{N}$.

Let T be an abstract space. $\mathcal{F}$ a σ-algebra of subsets of T, X a real normed space with the origin 0, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X; $\mathcal{P}_f(X)$ the family of closed, nonvoid sets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$ given by:

$$h(M;N) = \max\{e(M,N); e(N,M)\}, \text{ for every } M,N \in \mathcal{P}_f(X),$$

where $e(M,N) = \sup_{x\in X} d(x,N)$ is the excess of M over N.

Definition 2.5 ([1, 10, 11])**.** *A set multifunction* $\mu : \mathcal{F} \to \mathcal{P}_f(X)$ *is said to be:*

- (i) *continuous from below if* $\lim_{n\to\infty} h(\mu(A_n), A) = 0$, *for each increasing sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \nearrow A$.
- (ii) *continuous from above if* $\lim_{n\to\infty} h(\mu(A_n), A) = 0$, *for each decreasing sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \searrow A$.
- (iii) *order continuous if* $\lim_{n\to\infty} |\mu(A_n)| = 0$, *for every sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \searrow \emptyset$.
- (iv) *strongly order continuous if* $\lim_{n\to\infty} |\mu(A_n)| = 0$, *for every sequence of sets* $(A_n)_n \subset \mathcal{F}$, *with* $A_n \searrow A$ *and* $\mu(A_n) = \{0\}$.

3. Main results

The aim of this study is to examine Lebesgue's theorem, Egoroff's theorem and Riesz's theorem in uncertain measure theory. Throughout the study, assume $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, Λ_w and Λ are both events in $\mathcal{L}$. Now, we give two notions of uncertain measure $\mathcal{M}$.

Definition 3.1. $\mathcal{M}$ *is named strongly order continuous, if it supplies that* $\lim_{w\to\infty} \mathcal{M}(\Lambda_w) = 0$ *whenever* $\Lambda_w \searrow \Lambda$ *and* $\mathcal{M}(\Lambda) = 0$.

Definition 3.2. $\mathcal{M}$ *is named strongly continuous at* Γ, *if it supplies that*

$$\lim_{w\to\infty} \mathcal{M}(\Lambda_w) = 1$$

whenever $\Lambda_w \nearrow \Lambda$ *and* $\mathcal{M}(\Lambda) = 1$.

Theorem 3.1 (Lebesgue's theorem)**.** *Assume that* $\{\zeta_w\}$ *be a complex uncertain sequence and* ζ *be a complex uncertain variable in* $(\Gamma, \mathcal{L}, \mathcal{M})$, *which supply the subsequent condition that* $\{\zeta_w\}$ *converges almost surely (a.s.) to* ζ. *Then,* $\{\zeta_w\}$ *converges in measure to* ζ *iff* $\mathcal{M}$ *is strongly order continuous.*

Proof. Presume that the sequence $\{\zeta_w\}$ converges to ζ a.s., and take H as the set of these points $\gamma \in \Gamma$ at which $\zeta_w(\gamma)$ does not convergence to $\zeta(\gamma)$, hen

$$H = \bigcup_{p=1}^{\infty} \bigcap_{w=1}^{\infty} \bigcup_{r=w}^{\infty} \left\{ \gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p} \right\}$$

and $\mathcal{M}(H) = 0$. In addition, we get

$$\mathcal{M}\left(\bigcap_{w=1}^{\infty} \bigcup_{r=w}^{\infty} \left\{ \gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p} \right\} \right) = 0$$

for any $p \geq 1$. If we accept

$$\Lambda_w^{(p)} = \bigcup_{r=w}^{\infty} \left\{ \gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p} \right\}$$

and

$$\Lambda^{(p)} = \bigcap_{w=1}^{\infty} \bigcup_{r=w}^{\infty} \left\{ \gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p} \right\}$$

for any $p \geq 1$, then

$$\bigcup_{r \geq w}^{\infty} \left\{ \gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p} \right\} \searrow \bigcap_{w=1}^{\infty} \bigcup_{r=w}^{\infty} \left\{ \gamma : \|\zeta_r(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p} \right\}, \ (w \to \infty)$$

and $\mathcal{M}(\Lambda^{(p)}) = 0$. According to strongly order continuity of $\mathcal{M}$, we can acquire $\lim_{w \to \infty} \mathcal{M}(\Lambda_w^{(p)}) = 0$ for any $p \geq 1$ and, so

$$\lim_{w \to \infty} \mathcal{M}\left(\left\{ \gamma : \|\zeta_w(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p} \right\} \right) \leq \lim_{w \to \infty} \mathcal{M}\left(\Lambda_w^{(p)} \right) = 0, \forall p \geq 1.$$

This demonstrates that $\{\zeta_r\}$ converges in measure to ζ. For any sequence $\{\Lambda_w\}_w$ of events with $\Lambda_w \searrow \Lambda$ and $\mathcal{M}(\Lambda) = 0$, we determine a complex uncertain sequence $\{\zeta_w\}$ by

$$\zeta_w(\gamma) = \begin{cases} 0, & \text{if } \gamma \in \Gamma - \Lambda_w, \\ 1, & \text{if } \gamma \in \Lambda_w \end{cases}$$

for any $w \geq 1$. It is easy to understand that $\{\zeta_w\}$ converges to 0 a.s. If $\{\zeta_w\}$ converges to 0 in measure, then we can acquire

$$\lim_{w \to \infty} \mathcal{M}(\Lambda_w) \leq \lim_{n \to \infty} \mathcal{M}\left(\left\{ \gamma : \zeta_w(\gamma) \geq \frac{1}{2} \right\} \right) = 0.$$

As a result, $\mathcal{M}$ is strongly order continuous. □

Now, we generalize Egoroff's theorem in classical measure theory to uncertain measure theory.

Definition 3.3. *$\mathcal{M}$ is called to have feature (S), if for any sequence $\{\Lambda_w\}_w$ of events with $\lim_{w\to\infty}\mathcal{M}(\Lambda_w)=0$, there is a subsequence $\{\Lambda_{w_i}\}_i$ of $\{\Lambda_w\}_w$ such that $\mathcal{M}(\limsup \Lambda_{w_i})=0$.*

Theorem 3.2 (Egoroff's theorem)**.** *Assume that $\{\zeta_w\}$ be a complex uncertain sequence and ζ be a complex uncertain variable in $(\Gamma,\mathcal{L},\mathcal{M})$. If $\mathcal{M}$ is strongly order continuous and has feature (S), then*

$$\zeta_w \to \zeta(a.s.) \Rightarrow \zeta_w \to \zeta(u.a.s.)$$

Proof. Presume that $\mathcal{M}$ is strongly order continuous and has feature (S). Take H as the set of points $\gamma\in\Gamma$ whenever $\{\zeta_w\}$ does not convergence to ζ. Then, $\mathcal{M}(H)=0$ and $\{\zeta_w\}$ converges a.s. to ζ on $\Gamma-H$. If we indicate

$$H_w^{(r)}=\bigcap_{i=w}^{\infty}\left\{\gamma\in\Gamma:\|\zeta_i(\gamma)-\zeta(\gamma)\|<\frac{1}{r}\right\}$$

for any $r\geq 1$, then $H_w^{(r)}$ is increasing in w for all fixed r, and we obtain

$$\Gamma-H=\bigcap_{r=1}^{\infty}\bigcup_{w=1}^{\infty}H_w^{(r)}.$$

As for any fixed $r\geq 1$, $\Gamma-H\subseteq\bigcup_{w=1}^{\infty}H_w^{(r)}$, we get

$$\Gamma-H_w^{(r)}\searrow\bigcap_{w=1}^{\infty}\left(\Gamma-H_w^{(r)}\right).$$

Noting that $\bigcap_{w=1}^{\infty}(\Gamma-H_w^{(r)})\subset H$ for any fixed $r\geq 1$, so $\mathcal{M}(\bigcap_{w=1}^{\infty}(\Gamma-H_w^{(r)}))=0$ $(r=1,2,...)$. By utilizing the strong order continuity of $\mathcal{M}$, we get

$$\lim_{w\to\infty}\mathcal{M}\left(\Gamma-H_w^{(r)}\right)=0,\ \forall r\geq 1.$$

So, there is a subsequence $\{\Gamma-H_{w(r)}^{(r)}\}_r$ of $\{\Gamma-H_w^{(r)}:w,r\geq 1\}$ supplying

$$\mathcal{M}\left(\Gamma-H_{w(r)}^{(r)}\right)\leq\frac{1}{r},\ \forall r\geq 1$$

and so

$$\lim_{w\to\infty}\mathcal{M}\left(\Gamma-H_{w(r)}^{(r)}\right)=0.$$

By applying the feature (S) of $\mathcal{M}$ to the sequence $\{\Gamma-H_{w(r)}^{(r)}\}_r$, then there is a subsequence of $\{\Gamma-H_{w(r_i)}^{(r_i)}\}_i$ of $\{\Gamma-H_{w(r)}^{(r)}\}_r$ such that

$$\mathcal{M}\left(\overline{\lim_{i\to\infty}}\left(\Gamma-H_{w(r_i)}^{(r_i)}\right)\right)=0$$

and $r_1 < r_2 < ....$
At the same time, since

$$\left(\bigcup_{i=t}^{\infty} \left(\Gamma - H_{w(r_i)}^{(r_i)} \right) \right) \searrow \overline{\lim_{i \to \infty}} \left(\Gamma - H_{w(r_i)}^{(r_i)} \right)$$

so, by utilizing the strong order continuity of $\mathcal{M}$, we get

$$\lim_{t \to \infty} \mathcal{M} \left(\bigcup_{i=t}^{\infty} \left(\Gamma - H_{w(r_i)}^{(r_i)} \right) \right) = 0.$$

For any $\rho > 0$, we take t_0 such that $\mathcal{M}(\bigcup_{i=t_0}^{\infty} (\Gamma - H_{w(r_i)}^{(r_i)})) < \rho$, namely, $\mathcal{M}(\Gamma - \bigcap_{i=t_0}^{\infty} H_{w(r_i)}^{(r_i)}) < \rho$.

Take $H_\rho = \bigcap_{i=t_0}^{\infty} H_{w(r_i)}^{(r_i)}$, then $\mathcal{M}(\Gamma - H_\rho) < \rho$. Now, we need to demonstrate that $\{\zeta_w\}$ converges to ζ on H_ρ uniformly a.s. Since

$$H_\rho = \bigcap_{i=t_0}^{\infty} \bigcap_{j=w(r_i)}^{\infty} \left\{ \gamma \in \Gamma : \|\zeta_i(\gamma) - \zeta(\gamma)\| < \frac{1}{r_i} \right\},$$

therefore, for any fixed $i \geq k_0$,

$$H_\rho \subset \bigcap_{j=w(r_i)}^{\infty} \left\{ \gamma \in \Gamma : \|\zeta_j(\gamma) - \zeta(\gamma)\| < \frac{1}{r_i} \right\}.$$

For any given $\sigma > 0$, we take $i_0 \, (\geq t_0)$ such that $\frac{1}{r_{i_0}} < \sigma$. Thus, as $j > w(r_{i_0})$, for any $\gamma \in H_\rho$, $\|\zeta_j(\gamma) - \zeta(\gamma)\| < \frac{1}{r_{i_0}} < \sigma$. This denotes that $\{\zeta_w\}$ converges to ζ on Γ_ρ uniformly a.s. The proof of the theorem is finalized. □

Definition 3.4. *$\mathcal{M}$ is named order continuous if it supplies that* $\lim_{w \to \infty} \mathcal{M}(\Lambda_w) = 0$ *whenever* $\Lambda_w \searrow \emptyset$.

Theorem 3.3. *Let $\mathcal{M}$ be an uncertain measure, assume that $\{\zeta_w\}$ be a complex uncertain sequence and ζ be a complex uncertain variable in $(\Gamma, \mathcal{L}, \mathcal{M})$. $\zeta_w \to \zeta(a.s.)$ implies $\zeta_w \to \zeta(u.a.s.)$, then $\mathcal{M}$ is strongly order continuous and hence order continuous.*

Proof. For any decreasing sequence $\{\Lambda_w\}_w$ of events with $\Lambda_w \searrow \Lambda$ and $\mathcal{M}(\Lambda) = 0$, we consider a complex uncertain sequence $\{\zeta_w\}$ as

$$\zeta_w(\gamma) = \begin{cases} 0, & \text{if } \gamma \in \Gamma - \Lambda_w, \\ 1, & \text{if } \gamma \in \Lambda_w \end{cases}$$

for any $w \geq 1$. It is easy to obtain that $\zeta_w \to 0$ $(a.s.)$. If $\zeta_w \to 0$ $(u.a.s.)$, then we can acquire for any $\sigma > 0$,

$$\lim_{w \to \infty} \mathcal{M}\{\gamma : \|\zeta_w(\gamma)\| \geq \sigma\} = 0.$$

As a result
$$\lim_{w\to\infty}\mathcal{M}\left(\Lambda_w\right)=\lim_{w\to\infty}\mathcal{M}\left\{\gamma:\zeta_w\left(\gamma\right)\geq\frac{1}{2}\right\}=0.$$
This gives $\mathcal{M}$ is strongly order continuous and hence order continuous. □

Theorem 3.4 (Riesz's theorem)**.** *Assume that $\mathcal{M}$ be an uncertain measure with the feature (S). If $\{\zeta_w\}$ converges to ζ in measure, then there is a subsequence $\{\zeta_{w_r}\}_r$ of $\{\zeta_w\}_w$ such that $\zeta_{w_r}\to\zeta(a.s.)$.*

Proof. Let $\{\zeta_w\}$ converges to ζ in measure. Then
$$\lim_{w\to\infty}\mathcal{M}\left\{\gamma:\left\|\zeta_w\left(\gamma\right)-\zeta\left(\gamma\right)\right\|\geq\frac{1}{r}\right\}=0,\ \forall r\geq 1.$$
If we take $\Lambda_w^{(r)}=\left\{\gamma:\left\|\zeta_w\left(\gamma\right)-\zeta\left(\gamma\right)\right\|\geq\frac{1}{r}\right\}$, then there is a subsequence $\{w_r\}_r$ such that $\mathcal{M}(\Lambda_{w_r}^{(r)})\leq\frac{1}{r}$ for any $r\geq 1$. Since $\mathcal{M}$ has the feature (S), there is a subsequence $\{\Lambda_{w_{r_i}}^{(r_i)}\}$ of $\{\Lambda_{w_r}^{(r)}\}$ such that $\mathcal{M}(\overline{\lim}_{i\to\infty}\Lambda_{w_{r_i}}^{(r_i)})=0$. This gives that $\zeta_{w_{r_i}}\to\zeta(a.s.)$. □

References

[1] A. Gavrilut, *Non-atomicity and the Darboux property for fuzzy and non-fuzzy Borel/Baire multivalued set functions*, Fuzzy Sets and Systems, 160 (2009), 1308-1317.

[2] B. Liu, *Uncertainty theory*, Second edition, Springer-Verlag, Berlin, 2007.

[3] B. Liu, *Uncertainty theory: a branch of mathematics for modelling human uncertainty*, Springer-Verlag, Berlin, 2011.

[4] B. Liu, *Theory and practice of uncertain programming*, Second edition, Springer-Verlag, Berlin, 2009.

[5] B. Liu, X. Chen, *Uncertain multiobjective programming and uncertain goal programming*, Journal of Uncertainty Analysis and Applications, 3 (2015), Article 10.

[6] B. Liu, *Uncertain risk analysis and uncertain reliability analysis*, Journal of Uncertain Systems, 4 (2010), 163-170.

[7] B. Liu, *Some research problems in uncertainty theory*, Journal of Uncertain Systems, 3 (2009), 3-10.

[8] B.C. Tripathy, P.K. Nath, *Statistical convergence of complex uncertain sequences*, New Mathematics and Natural Computation, 13 (2017), 359-374.

[9] Z. Peng, *Complex uncertain variable*, Doctoral Dissertation, Tsinghua University, 2012.

[10] A. Precupanu, A. Gavrilut, *A set-valued Egoroff type theorem*, Fuzzy Sets and Systems, 175 (2011), 87-95.

[11] A. Precupanu, A. Gavrilut, *Set-valued Lusin type theorem for null-null-additive set multifunctions*, Fuzzy Sets and Systems, 204 (2012), 106-116.

[12] C. You, *On the convergence of uncertain sequences*, Mathematical and Computer Modelling, 49 (2009), 482-487.

[13] X. Chen, Y. Ning, X. Wang, *Convergence of complex uncertain sequences*, Journal of Intelligent & Fuzzy Systems, 30 (2016), 3357-3366.

[14] P.R. Halmos, *Measure theory*, Springer-New York, 1968.

[15] Z. Wang, G.J. Klir, *Fuzzy measure theory*, Plenum-New York, 1992.

[16] Q. Sun, *Property (S) of fuzzy measure and Riesz's theorem*, Fuzzy Sets and Systems, 62 (1994), 117-119.

[17] Q. Sun, Z. Wang, *On the autocontinuity of fuzzy measures*, Cybernetics and Systems, 88 (1988), 717-721.

[18] Z. Wang, *The autocontinuity of set function and the fuzzy integral*, Journal of Mathematical Analysis and Applications, 99 (1984), 195-218.

Accepted: June 23, 2022

Novel concepts in fuzzy graphs

Kishore Kumar P.K
Department of Information Technology (Mathematics Section)
University of Technology and Applied Sciences
Al Musannah
Sulatanate of Oman
kishorePK@act.edu.om

Alimohammad Fallah Andevari*
Department of Mathematics
Farhangian University
Mazandaran
Iran
alimohamad.fallah@yahoo.com

Abstract. Today, fuzzy graphs have a variety of applications in other fields of study, including medicine, engineering, and psychology, and for this reason many researchers around the world are trying to identify their properties and use them in computer science as well as finding the shortest problem in a network. So, in this paper, some new fuzzy graphs are introduced and some properties of them are investigated. As a consequence of our results, some well-known assertions in the graph theory are obtained.
Keywords: fuzzy set, fuzzy graph, fuzzy line graph, fuzzy common neighborhood graph.

1. Introduction

The concept of graph theory was first introduced by Euler. In 1965, L. A. Zadeh discussed the fuzzy set [37]. Graphs are basically the bonding of objects. To emphasis on a real life problem, the objects are being bonded by some relations, such as friendship is the bonding of people. But when the ambiguousness or uncertainty in bonding exists, then the corresponding graph can be modeled as fuzzy graph model.

The first definition of a fuzzy graph was given by Kaufmann, which was based on Zadeh's fuzzy relations in 1973. A fuzzy graph has good capabilities in dealing with problems that cannot be explained by weight graphs. They have been able to have wide applications even in fields such as psychology and identifying people based on cancerous behaviors. One of the advantages of fuzzy graph is its flexibility in reducing time and costs in economic issues, which has been welcomed by all managers of institutions and companies. Fuzzy graph models are advantageous mathematical tools for dealing with combinatorial problems

∗. Corresponding author

of various domains including operations research, optimization, social science, algebra, computer science, and topology. They are obviously better than graphical models due to natural existence of vagueness and ambiguity. Mordeson studied fuzzy line graphs and developed its basic properties, in 1993 [14]. The theory of fuzzy graph is growing rapidly, with numerous applications in many domains, including networking, communication, data mining, clustering, image capturing, image segmentation, planning, and scheduling. Rashmanlou et al. [21, 22, 25, 26, 27, 28] defined bipolar fuzzy graphs with categorical properties, product vague graphs, and shortest path problem in vague graphs. Akram et al. [1, 2] introduced certain types of vague graphs and strong intuitionistic fuzzy graphs. Borzooei et al. [4, 5, 6, 7, 8, 9, 10] investigated new concepts on vague graphs. Parvathi et al. [16, 17] introduced intuitionistic fuzzy graphs and domination in intuitionistic fuzzy graphs. Kou et al. [11] given novel description on vague graph with application in transportation systems. Samanta et al. [18, 19, 20, 23, 24]presented new definitions on fuzzy graphs. Kosari et al. [12] introduced vague graph structure with application in medical diagnosis. Talebi et al. [34, 35, 36] studied interval-valued intuitionistic fuzzy competition graph, and new concept of an intuitionistic fuzzy graph with applications. Rao et al. [29, 30, 31, 32] defined domination and equitable domination in vague graphs. Zeng et al. [38] investigated certain properties of single-valued neutrosophic graphs. In this paper, we introduce many basic notions concerning a fuzzy graph and investigate a few related properties.

First we go through some basic definitions from [14, 15]

Definition 1.1. *A fuzzy subset of a non-empty set S is a map $\sigma : S \to [0,1]$ which assigns to each element x in S a degree of membership $\sigma(x)$ in* $[0,1]$ *such that* $0 \leq \sigma(x) \leq 1$.

If S represents a set, a fuzzy relation μ on S is a fuzzy subset of $S \times S$. In symbols, $\mu : S \times S \to [0,1]$ such that $0 \leq \mu(x,y) \leq 1$ for all $(x,y) \in S \times S$.

Definition 1.2. *Let σ be a fuzzy subset of a set S and μ a fuzzy relation on S. Then μ is called a fuzzy relation on σ if $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in S$ where $\wedge$ denote minimum.*

Let V be a nonempty set. Define the relation $\sim$ on $V \times V$ by for all $(x,y),(u,v) \in V \times V$, $(x,y) \sim (u,v)$ if and only if $x = u$ and $y = v$ or $x = v$ and $y = u$. Then it is easily shown that $\sim$ is an equivalence relation on $V \times V$. For all $x, y \in V$, let $[(x,y)]$ denote the equivalence class of (x,y) with respect to $\sim$. Then $[(x,y)] = \{(x,y),(y,x)\}$. Let $\mathcal{E}_V = \{[(x,y)] | x, y \in V, x \neq y\}$. For simplicity, we often write $\mathcal{E}$ for $\mathcal{E}_V$ when V is understood. Let $E \subseteq \mathcal{E}$. A graph is a pair (V,E). The elements of V are thought of as vertices of the graph and the elements of E as the edges. For $x, y \in V$, we let xy denote $[(x,y)]$. Then clearly $xy = yx$. We note that graph (V,E) has no loops or parallel edges.

Definition 1.3. *A fuzzy graph* $G = (V, \sigma_G, \mu_G)$ *is a triple consisting of a nonempty set* V *together with a pair of functions* $\sigma := \sigma_G : V \to [0,1]$ *and* $\mu := \mu_G : \mathcal{E} \to [0,1]$ *such that for all* $x, y \in V$, $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$.

The fuzzy set σ is called the fuzzy vertex set of G and μ the fuzzy edge set of G. Clearly μ is a fuzzy relation on σ.

Definition 1.4. *A path* P *in a fuzzy graph* $G = (V, \sigma, \mu)$ *is a sequence of distinct vertices* $x_0, x_1, \cdots, x_n$ *(except possibly* x_0 *and* x_n*) such that* $\mu(x_{i-1}x_i) > 0$ *for* $i = 1, \cdots, n$. *Here* n *is called the length of the path. We call* P *a cycle if* $x_0 = x_n$ *and* $n \geq 3$. *Two vertices that are joined by a path are called connected.*

Definition 1.5. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph. The degree* $x \in V$ *is denoted by* $d_G(x)$ *and defined as* $d_G(x) = \sum\limits_{y \in V} \mu(xy)$.

2. Introducing some new fuzzy graphs

In this section after introducing some new fuzzy graphs, we study some properties of them. These new fuzzy graphs and their properties are important not only as fuzzy graphs, but also for the crisp graph in the special case.

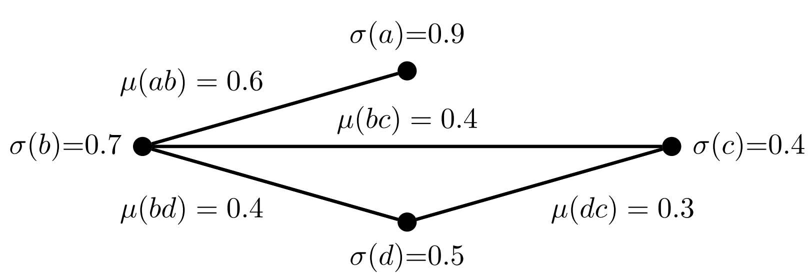
Definition 2.1. *Let* $G = (V, \sigma, \mu)$ *be a fuzzy graph. We define the complement of* G *by* $\overline{G} = (\overline{V}, \overline{\sigma}, \overline{\mu})$ *such that*

a) $\overline{V} = V$ *and* $\overline{\sigma}(v) = \sigma(v)$ *for all* $v \in V$;

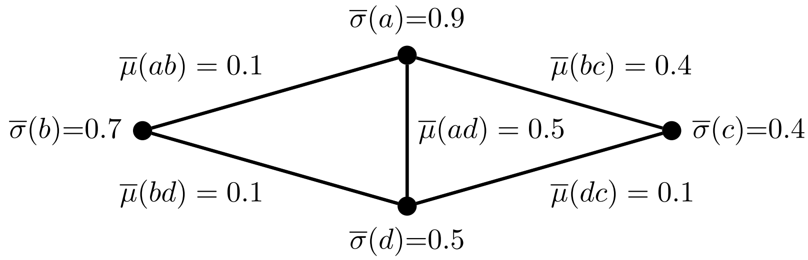
b) $\overline{\mu}(uv) = \sigma(u) \wedge \sigma(v) - \mu(uv)$, *for all* $u, v \in V$.

It is easy to show that $\overline{G}$ is a fuzzy graph on V.

Example 2.1. Let $V = \{a, b, c, d\}$ and $\sigma : V \to [0,1]$ be a map such that $\sigma(a) = 0.9$, $\sigma(b) = 0.7$, $\sigma(c) = 0.4$ and $\sigma(d) = 0.5$. Also, let $\mu : V \times V \to [0,1]$ be a map such that $\mu(ab) = 0.6$, $\mu(bc) = 0.4$, $\mu(bd) = 0.4$ and $\mu(dc) = 0.3$. We have the following diagram for the fuzzy graph $G = (V, \sigma, \mu)$.



Also, the diagram of the fuzzy graph $\overline{G} = (\overline{V}, \overline{\sigma}, \overline{\mu})$ is as follows:



Lemma 2.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then $\overline{\overline{G}} = G$.*

Proof. Suppose that $\overline{G} = (\overline{V}, \overline{\sigma}, \overline{\mu})$ and $\overline{\overline{G}} = (\overline{\overline{V}}, \overline{\overline{\sigma}}, \overline{\overline{\mu}})$. By the definition of the complement fuzzy graph, we have $\overline{\overline{V}} = \overline{V} = V$ and $\overline{\overline{\sigma}} = \overline{\sigma} = \sigma$. It suffices to prove that $\overline{\overline{\mu}}(uv) = \mu(uv)$ for all $u, v \in V$. We have

$$\begin{aligned}&\overline{\overline{\mu}}(uv) = \overline{\sigma}(u) \wedge \overline{\sigma}(v) - \overline{\mu}(uv)\\&= \sigma(u) \wedge \sigma(v) - \overline{\mu}(uv) = \sigma(u) \wedge \sigma(v) - (\sigma(u) \wedge \sigma(v) - \mu(uv)) = \mu(uv).\end{aligned}$$ □

Definition 2.2. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \varnothing$. Union of two fuzzy graphs G_1 and G_2 is denoted by $G_1 \cup G_2 = (V, \sigma, \mu)$ such that $V = V_1 \cup V_2$,*

$$\sigma(v) = \begin{cases} \sigma_1(v), & v \in V_1 \\ \sigma_2(v), & v \in V_2 \end{cases} \quad \textit{and} \quad \mu(uv) = \begin{cases} \mu_1(uv), & u, v \in V_1 \\ \mu_2(uv), & u, v \in V_2 \\ 0, & o.w \end{cases}$$

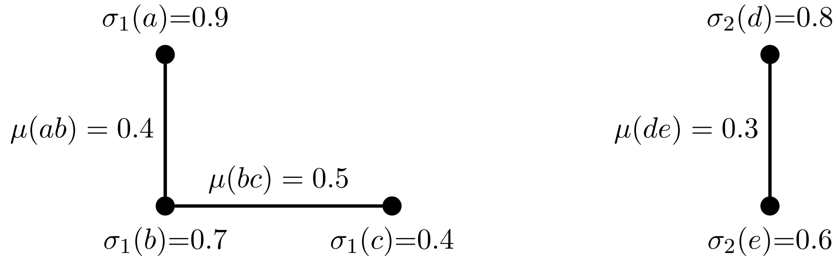
It is easy to see $G_1 \cup G_2$ is a fuzzy graph.

Definition 2.3. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \phi$. Sum of two fuzzy graphs G_1 and G_2 is denoted by $G_1 + G_2 = (V, \sigma, \mu)$ such that $V = V_1 \cup V_2$,*

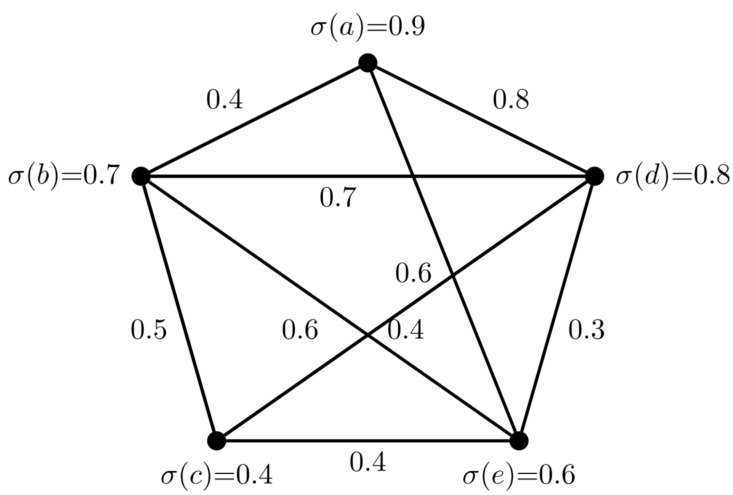
$$\sigma(v) = \begin{cases} \sigma_1(v), & v \in V_1 \\ \sigma_2(v), & v \in V_2 \end{cases} \quad \textit{and} \quad \mu(uv) = \begin{cases} \mu_1(uv), & u, v \in V_1 \\ \mu_2(uv), & u, v \in V_2 \\ \sigma_1(u) \wedge \sigma_2(v), & u \in V_1,\ v \in V_2 \end{cases}.$$

Example 2.2. Let $G_1 = (V_1, \sigma_1, \mu_1)$ be a fuzzy graph such that $V_1 = \{a, b, c\}$ and $\sigma_1 : V_1 \to [0, 1]$ and $\mu_1 : V_1 \times V_1 \to [0, 1]$ be maps such that $\sigma_1(a) = 0.9$, $\sigma_1(b) = 0.7$, $\sigma_1(c) = 0.4$, $\mu_1(ab) = 0.4$ and $\mu_1(bc) = 0.5$. Also, let $G_2 = (V_2, \sigma_2, \mu_2)$ be a fuzzy graph such that $V_2 = \{d, e\}$ and $\sigma_2 : V_2 \to [0, 1]$ and $\mu_2 : V_2 \times V_2 \to [0, 1]$ be maps such that $\sigma_2(d) = 0.8$, $\sigma_2(e) = 0.6$, $\mu_2(de) = 0.3$.

The fuzzy graphs G_1 and G_2 are drawn as follows, respectively:



By the definition of the sum of two graphs, $\mu(ab) = 0.4$, $\mu(ad) = 0.8$, $\mu(ae) = 0.6$, $\mu(de) = 0.3$, $\mu(be) = 0.6$, $\mu(bc) = 0.5$, $\mu(bd) = 0.7$, $\mu(ce) = 0.4$, $\mu(dc) = 0.4$ and the diagram of the fuzzy graph $G_1 + G_2$ is as follows:



Lemma 2.2. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. Then*

a) $\overline{G_1 \cup G_2} = \overline{G_1} + \overline{G_2}$;

b) $\overline{G_1 + G_2} = \overline{G_1} \cup \overline{G_2}$.

Proof. Suppose that $G_1 \cup G_2 = (V, \sigma, \mu)$, $\overline{G_1 \cup G_2} = (\overline{V}, \overline{\sigma}, \overline{\mu})$, $\overline{G_1} = (\overline{V_1}, \overline{\sigma_1}, \overline{\mu_1})$, $\overline{G_2} = (\overline{V_2}, \overline{\sigma_2}, \overline{\mu_2})$ and $\overline{G_1} + \overline{G_2} = (V', \sigma', \mu')$. By the definition of the union and sum of two graphs, we have $V = \overline{V} = V_1 \cup V_2 = \overline{V_1} \cup \overline{V_2} = V'$ and $\overline{\sigma(v)} = \sigma(v)$ for all $v \in V$. It suffices to prove that $\mu_{\overline{G_1 \cup G_2}}(uv) = \mu'_{\overline{G_1} + \overline{G_2}}(uv)$ for all $uv \in \mathcal{E}$. We have

$$\mu_{\overline{G_1 \cup G_2}}(uv) = \sigma(u) \wedge \sigma(v) - \mu(uv) = \sigma(u) \wedge \sigma(v) - \begin{cases} \mu_1(uv), & u, v \in V_1 \\ \mu_2(uv), & u, v \in V_2 \\ 0, & o.w \end{cases}$$

$$= \begin{cases} \sigma_1(u) \wedge \sigma_1(v) - \mu_1(uv), & u, v \in V_1 \\ \sigma_2(u) \wedge \sigma_2(v) - \mu_2(uv), & u, v \in V_2 \\ \sigma_1(u) \wedge \sigma_2(v), & u \in V_1,\ v \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu_1}(uv), & u, v \in V_1 \\ \overline{\mu_2}(uv), & u, v \in V_2 \\ \sigma_1(u) \wedge \sigma_2(v), & u \in V_1,\ v \in V_2 \end{cases} = \mu'_{\overline{G_1}+\overline{G_2}}(uv).$$

The other conclusion is proved similarly. □

Definition 2.4. *Let $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ be two fuzzy graphs. The Cartesian product of graphs G_1 and G_2 is denoted by $G_1 \times G_2 = (V, \sigma, \mu)$ is a fuzzy graph such that $V = V_1 \times V_2$,*

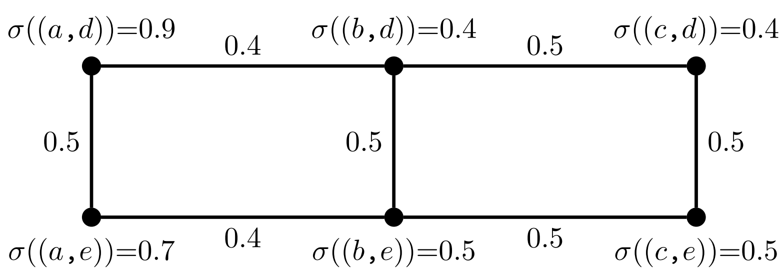
$$\sigma((u, v)) = \sigma_1(u) \vee \sigma_2(v),$$

where $\vee$ is denoted maximum and

$$\mu((u, v)(u', v')) = \begin{cases} \mu_2(vv'), & \text{if } u = u' \\ \mu_1(uu'), & \text{if } v = v' \\ 0, & o.w \end{cases}.$$

It is easy to show that $d_{G_1 \times G_2}((u, v)) = d_{G_1}(u) + d_{G_2}(v)$.

Example 2.3. Let G_1 and G_2 be the fuzzy graphs of Example 2.2. We have the following diagram for the fuzzy graph $G_1 \times G_2$.



Let $G = (V, \sigma, \mu)$ be a fuzzy graph the neighbor of vertex v is denoted by $N_G(v)$ and is defined as follows:

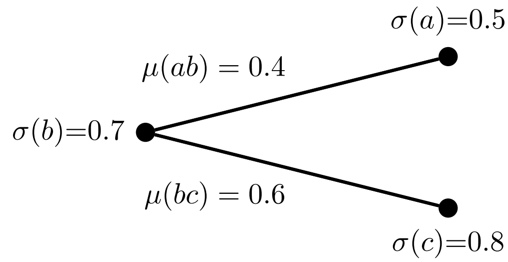
$$N_G(v) = \{u \in V \mid \mu(uv) > 0\}.$$

Definition 2.5. *The fuzzy common neighborhood graph or briefly fuzzy congraph of $G = (V, \sigma, \mu)$ is a fuzzy graph as $con(G) = (V, \omega, \lambda)$ such that $\omega(x) = \sigma(x)$ and*

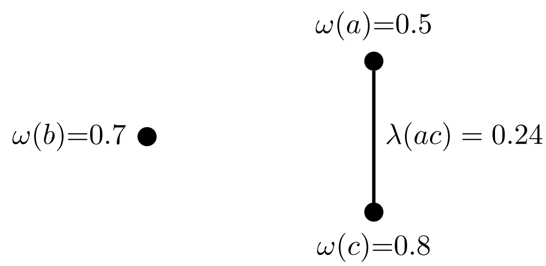
$$\lambda(uv) = \min_{x \in H}\{\mu(ux).\mu(vx)\},$$

where $H = N_G(u) \cap N_G(v)$.

Example 2.4. Let $V = \{a, b, c\}$ and $\sigma : V \to [0, 1]$ be a map such that $\sigma(a) = 0.5$, $\sigma(b) = 0.7$, $\sigma(c) = 0.8$. Also, let $\mu : V \times V \to [0, 1]$ be a map such that $\mu(ab) = 0.4$ and $\mu(bc) = 0.6$. We have the following diagram for the fuzzy graph $G = (V, \sigma, \mu)$.

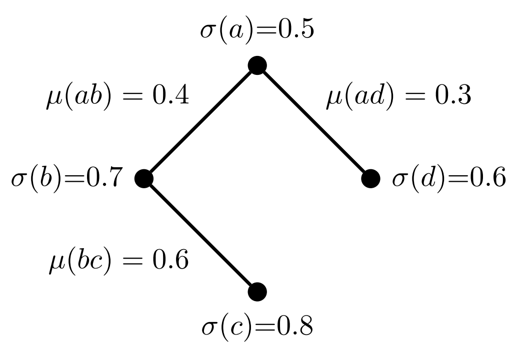


By using the definition of the fuzzy congraph, we have $\lambda(ab) = 0$, $\lambda(bc) = 0$, $\lambda(ac) = 0.24$ and $con(G) = (V, \omega, \lambda)$ is as follows:



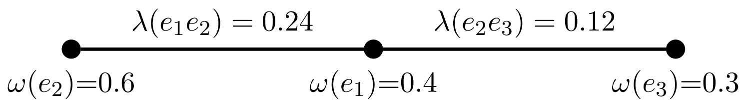
Definition 2.6. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The fuzzy line graph of G is a fuzzy graph as $L(G) = (\mathcal{E}, \omega, \lambda)$ such that $\omega(e) = \mu(uv)$ for all $e = uv \in \mathcal{E}$ and $\lambda(e_1e_2) = \omega(e_1).\omega(e_2)$ for all $e_1 = uv_1$, $e_2 = uv_2$ in $\mathcal{E}$.*

Example 2.5. Let $V = \{a, b, c, d\}$ and $\sigma : V \to [0, 1]$ be a map such that $\sigma(a) = 0.5$, $\sigma(b) = 0.7$, $\sigma(c) = 0.8$ and $\sigma(d) = 0.6$. Also, let $\mu : V \times V \to [0, 1]$ be a map such that $\mu(e_1) = \mu(ab) = 0.4$, $\mu(e_2) = \mu(bc) = 0.6$ and $\mu(e_3) = \mu(ad) = 0.3$. We have the following diagram for the fuzzy graph $G = (V, \sigma, \mu)$.



By using the definition of the fuzzy line graph, we have $\omega(e_1) = 0.4$, $\omega(e_2) = 0.6$, $\omega(e_3) = 0.3$, $\lambda(e_1e_2) = 0.24$, $\lambda(e_2e_3) = 0.12$ and the diagram of $L(G) = (\mathcal{E}, \omega, \lambda)$

is as follows:



Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $V = \{v_1, v_2, ..., v_p\}$, $\mathcal{E} = \{e_1, e_2, ..., e_q\}$ the vertex set and the edge set of G, respectively.

The *adjacency matrix* of fuzzy graph G is the $p \times p$ matrix $A_F = A_F(G)$ whose (i, j) entry denoted by a_{ij}, is defined by $a_{ij} = \mu(v_iv_j)$.

The (vertex-edge) *incidence matrix* of fuzzy graph G is the $p \times q$ matrix M_F, with rows indexed by the vertices and columns indexed by the edges, whose (i, j) entry denoted by m_{ij}, is defined as follows:

$$m_{ij} = \begin{cases} \mu(e_j), & \text{if } v_i \text{ is an endpoint of edge } e_j \\ 0, & \text{o. w} \end{cases}$$

The *fuzzy degree matrix* of G is the $p \times p$ matrix D_F whose (i, j) entry denoted by d_{ij}, is defined as follows:

$$d_{ij} = \begin{cases} \sum\limits_{v_k \in V} \mu^2(v_iv_k), & \text{if } i = j \\ 0, & \text{o. w} \end{cases}$$

The *edge matrix* of fuzzy graph G is the $q \times q$ matrix E_F whose (i, j) entry denoted by e_{ij}, is defined as follows:

$$e_{ij} = \begin{cases} \mu(e_i), & \text{if } i = j \\ 0, & \text{o. w} \end{cases}$$

Definition 2.7. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrix of size $m \times n$. Then we define $C = A \odot B$ is the $m \times n$ matrix whose (i, j) entry denoted by $a_{ij} \times b_{ij}$.*

Theorem 2.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph such that A_F, M_F and D_F are the adjacency, incidence and fuzzy degree matrices of G, respectively. Then*

$$M_F \times M_F^T = A_F \odot A_F + D_F.$$

Proof. Let $A_F = [a_{ij}]_{p \times p}$, $M_F = [m_{ij}]_{p \times q}$, $D_F = [d_{ij}]_{p \times p}$, $A_F \odot A_F = [t_{ij}]_{p \times p}$ and $M_F \times M_F^T = [b_{ij}]_{p \times p}$. First, let $i \neq j$. Then we get

$$\begin{aligned} b_{ij} &= \sum_{k=1}^{q} m_{ik}.m^T_{kj} = \sum_{k=1}^{q} m_{ik}.m_{jk} \\ &= \begin{cases} m_{ik'}.m_{jk'}, & \text{if } v_i \text{ and } v_j \text{ are endpoints of edge } e_{k'} \\ 0, & \text{o. w} \end{cases}, \end{aligned}$$

for some $1 \leq k' \leq q$. It follows that

$$b_{ij} = \begin{cases} \mu^2(e_{k'}), & \text{if } v_i \text{ and } v_j \text{ are endpoints of edge } e_{k'} \\ 0, & \text{o. w} \end{cases}$$
$$= a_{ij}.a_{ij} = t_{ij} = t_{ij} + 0 = t_{ij} + d_{ij},$$

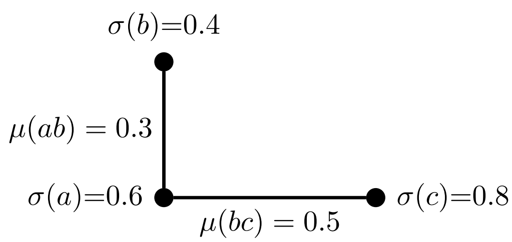
which proves our assertion.

If $i = j$, then

$$b_{ii} = \sum_{k=1}^{q} m_{ik}.m_{ki}^T = \sum_{k=1}^{q} m_{ik}.m_{ik} = \sum_{e_k = v_i v_t \in \mathcal{E}} \mu^2(e_k) = \sum_{v_t \in V} \mu^2(v_i v_t) = d_{ii}.$$

Then $b_{ii} = 0 + d_{ii} = a_{ii}.a_{ii} + d_{ii}$, which completes the proof. □

Example 2.6. Let $G = (V, \sigma, \mu)$ be the following fuzzy graph:



By the definitions of the adjacency, incidence, and fuzzy degree matrix in the fuzzy graph, we have:

$$A_F = \begin{bmatrix} 0 & 0.3 & 0.5 \\ 0.3 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}, M_F = \begin{bmatrix} 0.3 & 0.5 \\ 0.3 & 0 \\ 0 & 0.5 \end{bmatrix}, D_F = \begin{bmatrix} 0.34 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}.$$

It is easy to see that $M_F \times M_F^T = A_F \odot A_F + D_F$.

In the fuzzy graph $G = (V, \sigma, \mu)$, if for every $v \in V$ set $\sigma(v) = 1$ and for every edge e set $\mu(e) = 1$, then we can assume that every crisp graph is a fuzzy graph. Therefore, we can obtain similar results for crisp graphs. So, we have the following result which is well-known in the graph theory [3].

Corollary 2.1. *Let G be a graph and A, M and D be the adjacency, incidence and degree matrix of G, respectively. Then*

$$M \times M^T = A + D.$$

The next theorem characterized the degree of every vertex in the fuzzy con-graph.

Theorem 2.2. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $con(G) = (V, \omega, \lambda)$ the fuzzy congraph of G. If G has no cycles of size 4, then*

$$d_{con(G)}(v) = \sum_{u \in V} \mu(uv) \times d_G(u) - \sum_{u \in V} \mu^2(uv), \qquad v \in V.$$

Proof.

$$d_{con(G)}(v) = \sum_{u \in V} \lambda(uv) = \sum_{u \in V} \min_{x \in H}\{\mu(vx) \times \mu(xu)\},$$

where $H = N_G(u) \cap N_G(v)$. Since G has no cycle of size 4, it follows that $H \subseteq \{w\}$ for some $w \in V$. Therefore,

$$\begin{aligned} d_{con(G)}(v) &= \sum_{vw, wu \in \mathcal{E}(G)} \mu(vw) \times \mu(wu) \\ &= \sum_{vw \in \mathcal{E}(G)} \mu(vw) \times \sum_{u \in V} \mu(uw) - \sum_{w \in V} \mu^2(vw) \\ &= \sum_{w \in V} \mu(vw) \times d_G(w) - \sum_{w \in V} \mu^2(vw) \\ &= \sum_{u \in V} \mu(vu) \times d_G(u) - \sum_{u \in V} \mu^2(vu). \end{aligned}$$ □

From the above theorem, one can immediately deduce the following corollary, which has proved in [13].

Corollary 2.2. *Let $G = (V, E)$ be a graph. If G has no cycles of size 4, then*

$$d_{con(G)}(v) = \sum_{u \in N_G(v)} d_G(u) - d_G(v), \qquad v \in V.$$

The next theorem characterize the degree of every vertex in the fuzzy line graph.

Theorem 2.3. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $L(G) = (\mathcal{E}, \omega, \lambda)$ its fuzzy line graph. Then*

$$d_{L(G)}(e) = \mu(v_i v_j)(d_G(v_i) + d_G(v_j) - 2\mu(v_i v_j)), \qquad e = v_i v_j \in \mathcal{E}(G).$$

Proof. For an arbitrary edge $e = v_i v_j \in \mathcal{E}(G)$, set $e' = v_s v_t$, where $s \neq i$ and $t \neq j$. We have

$$d_{L(G)}(e) = \sum_{e \neq e'} \lambda(ee') = \sum_{e \neq e' = v_i v_t \in \mathcal{E}(G)} \lambda(ee') + \sum_{e \neq e' = v_j v_s \in \mathcal{E}(G)} \lambda(ee')$$

$$
\begin{aligned}
&= \sum_{v_t \neq v_i, t\neq j} \mu(v_iv_j)\mu(v_iv_t) + \sum_{v_s \neq v_j, s\neq i} \mu(v_iv_j)\mu(v_jv_s) \\
&= \mu(v_iv_j) \sum_{v_t \neq v_i, t\neq j} \mu(v_iv_t) + \mu(v_iv_j) \sum_{v_s \neq v_j, s\neq i} \mu(v_jv_s) \\
&= \mu(v_iv_j)(\sum_{v_t\in V} \mu(v_iv_t) - \mu(v_iv_j)) + \mu(v_iv_j)(\sum_{v_s\in V} \mu(v_jv_s) - \mu(v_iv_j)) \\
&= \mu(v_iv_j)(d_G(v_i) + d_G(v_j) - 2\mu(v_iv_j)). \qquad \square
\end{aligned}
$$

From the above theorem, we can conclude the following result, which is trivial in the line graph.

Corollary 2.3. *Let $G = (V, E)$ be a graph and $L(G) = (E, W)$ the line graph of G. Then*

$$d_{L(G)}(e) = d_G(u) + d_G(v) - 2, \qquad e \in E.$$

Theorem 2.4. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with the incidence and edge matrix M_F and E_F, respectively. Suppose that $L(G) = (\mathcal{E}, \omega, \lambda)$ is the fuzzy line graph of G with the adjacency matrix L_F. Then*

$$M_F^T \times M_F = L_F + 2E_F \odot E_F.$$

Proof. Let $M_F = [m_{ij}]_{p\times q}$, $L_F = [l_{ij}]_{q\times q}$, $E_F = [e_{ij}]_{q\times q}$ and $M_F^T \times M_F = [b_{ij}]_{q\times q}$. For $i \neq j$, we get

$$
\begin{aligned}
b_{ij} &= \sum_{k=1}^{p} m_{ik}^T.m_{kj} = \sum_{k=1}^{p} m_{ki}.m_{kj} \\
&= \begin{cases} m_{k'i}.m_{k'j}, & \text{if } v_{k'} \text{ is an endpoint of edges } e_i \text{ and } e_j \\ 0, & \text{otherwise} \end{cases},
\end{aligned}
$$

for some $1 \leq k' \leq p$. Hence

$$b_{ij} = \begin{cases} \mu(e_i).\mu(e_j), & \text{if } v_{k'} \text{ is an endpoint of edges } e_i \text{ and } e_j \\ 0, & \text{otherwise;} \end{cases}$$

thus

$$
\begin{aligned}
b_{ij} &= \begin{cases} \lambda(e_ie_j), & \text{if } v_{k'} \text{ is an endpoint of edges } e_i \text{ and } e_j \\ 0, & \text{otherwise} \end{cases} \\
&= l_{ij} = l_{ij} + 0 = l_{ij} + 2e_{ij}.e_{ij}.
\end{aligned}
$$

which proves our assertion.

Now, suppose that $i = j$ and $e_i = v_tv_s$. We have

$$b_{ii} = \sum_{k=1}^{p} m_{ik}^T.m_{kj} = \sum_{k=1}^{p} m_{ki}.m_{ki} = m_{ti}^2 + m_{si}^2 = 2\mu^2(e_i) = 2e_{ii}.e_{ii} = l_{ii} + 2e_{ii}.e_{ii}.$$

Therefore, the proof is complete. $\square$

Example 2.7. Let G be the fuzzy graph of example 2.5. By the definitions of the incidence, edge matrix, and the adjacency matrix of the line graph of G, we have the following matrices:

$$M_F = \begin{bmatrix} 0.4 & 0 & 0.3 \\ 0.4 & 0.6 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, L_F = \begin{bmatrix} 0 & 0.24 & 0.12 \\ 0.24 & 0 & 0 \\ 0.12 & 0 & 0 \end{bmatrix}, E_F = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

It is easy to check that $M_F^T \times M_F = L_F + 2E_F \odot E_F$.

From the above theorem, we deduce the following result, which has proved in [33].

Corollary 2.4. *Let G be a graph with the incidence matrix M and the adjacency matrix line graph L. Then*

$$M^T \times M = L + 2I_{q\times q}.$$

Theorem 2.5. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with the adjacency and fuzzy degree matrix A_F and D_F, respectively. Suppose that $con(G) = (V, \omega, \lambda)$ is the fuzzy congraph of G with the adjacency matrix B_F. If G has no cycles of size 4, then*

$$A_F^2 = B_F + D_F.$$

Proof. Let $A_F = [a_{ij}]_{p\times p}$, $B_F = [b_{ij}]_{p\times p}$, $D_F = [d_{ij}]_{p\times p}$ and $A_F^2 = [c_{ij}]_{p\times p}$. For $i \neq j$, we have

$$c_{ij} = \sum_{k=1}^{p} a_{ik}.a_{kj} = \begin{cases} a_{it}.a_{tj}, & \text{if the vertices } v_i \text{ and } v_j \text{ are connected to } v_t \\ 0, & \text{o. w} \end{cases}$$

for some $1 \leq t \neq i, j \leq p$. It follows that

$$c_{ij} = \begin{cases} \mu(v_iv_t).\mu(v_tv_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are connected to } v_t \\ 0, & \text{o. w.} \end{cases}$$

Since G has no cycles of size 4, then

$$c_{ij} = \begin{cases} \lambda(v_iv_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are connected to } v_t \\ 0, & \text{o. w.} \end{cases}$$

Thus, $c_{ij} = b_{ij} + 0 = b_{ij} + d_{ij}$, which proves our assertion in this case.

If $i = j$, then

$$c_{ii} = \sum_{k=1}^{p} a_{ik}.a_{ki} = \sum_{k=1}^{p} a_{ik}^2 = \sum_{v_k\in V} \mu^2(v_iv_k) = d_{ii} = b_{ii} + d_{ii}.$$

This completes the proof of the theorem. □

Example 2.8. Let G be the fuzzy graph of Example 2.4. By the definitions of the adjacency matrix of G, $con(G)$ and the fuzzy degree matrix of G, we have the following matrices:

$$A_F = \begin{bmatrix} 0 & 0.4 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.6 & 0 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0 & 0 & 0.24 \\ 0 & 0 & 0 \\ 0.24 & 0. & 0 \end{bmatrix}, \quad D_F = \begin{bmatrix} 0.16 & 0 & 0.24 \\ 0 & 0.52 & 0 \\ 0 & 0. & 0.36 \end{bmatrix}.$$

It is easy to see that $A_F^2 = B_F + D_F$.

Corollary 2.5. *Let G be a graph such that A and B are the adjacency matrices of G and $con(G)$, respectively. If G has no cycles of size 4, then $A^2 = B + D$, where D is the degree matrix of G.*

3. Conclusion

It is well known that fuzzy graphs are among the most ubiquitous models of both natural and humman-made structures. They can be used to model many types of relations and process dynamics in computer science, biological, social systems and physical. Theoretical concepts of fuzzy graphs are highly utilized by computer science applications. Especially in research areas of computer science such as data mining, image segmentation, clustering, image capturing and networking. So, in this paper, some new fuzzy graphs are presented and some properties of them are studied. As a consequence of our results, some well-known assertions in the graph theory are given. in our future work, we will introduce cubic vague fuzzy graphs and define new operations such as strong product, direct product, lexicographic product, union, and composition on it.

References

[1] M. Akram, F. Feng, S. Sarwar, Y.B. Jun, *Certain types of vague graphs*, U.P.B. Sci. Bull., Series A, 76 (2014), 141-154.

[2] M. Akram, B. Davvaz, *Strong intuitionistic fuzzy graphs*, Filomat, 26 (2012), 177-195.

[3] J.A. Bondy, U.S.R. Murty, *Graph theory,* Springer, 2008.

[4] R.A. Borzooei, H. Rashmanlou, S. Samanta, M. Pal, *A study on fuzzy labeling graphs*, Journal of Intelligent and Fuzzy Systems, 30 (2016), 3349-3355.

[5] R.A. Borzooei, H. Rashmanlou, M. Pal, *Regularity of vague graphs*, Journal of Intelligent and Fuzzy Systems, 30 (2016), 3681-3689.

[6] R.A. Borzooei, H. Rashmanlou, *Ring sum in product intuitionistic fuzzy graphs,* Journal of Advanced Research in Pure Mathematics, 7 (2015), 16-31.

[7] R.A. Borzooei, H. Rashmanlou, *Domination in vague graphs and its applications,* Journal of Intelligent and Fuzzy Systems, 29 (2015), 1933-1940.

[8] R.A. Borzooei, H. Rashmanlou, *Degree of vertices in vague graphs,* Journal of Applied Mathematics and Informatics, 33 (2015), 545-557.

[9] R.A. Borzooei, H. Rashmanlou, *New concepts of vague graphs*, International Journal of Machine Learning and Ceybernetic, 8 (2016), 1081-1092.

[10] R.A. Borzooei, H. Rashmanlou, *Semi global domination sets in vague graphs with application,* J. Intell. Fuzzy Syst., 30 (2016), 3645-3652.

[11] Z. Kou, S. Kosari, M. Akhoundi, *A novel description on vague graph with application in transportation systems,* Journal of Mathematics, Volume 2021, Article ID 4800499, 11 pages.

[12] S. Kosari, Y. Rao, H. Jiang, X. Liu, P. Wu, Z. Shao, *Vague graph structure with application in medical diagnosis,* Symmetry, 12 (2020), 1582.

[13] D.W. Lee, S. Sedghi, N. Shobe, *Zagreb indices of a graph and its common neighborhood graph*, Malaya J. Mat., 4 (2016), 468-475.

[14] J.N. Mordeson, *Fuzzy line graphs,* Pattern Recognit. Lett., 14 (1993), 381-384.

[15] S. Mathew, J.N. Mordeson, D. S. Malik, *Fuzzy graph theory,* Springer, 2018.

[16] R. Parvathi, M.G. Karunambigai, *Intuitionistic fuzzy graphs, computational intelligence, theory and applications,* International Conference in Germany, 2006, 18–20.

[17] R. Parvathi, G. Thamizhendhi, *Domination in intuitionistic fuzzy graphs,* Fourteenth Int.Conf. on IFSs, Sofia, 16 (2010), 39-49.

[18] H. Rashmanlou, S. Samanta, M. Pal, R.A. Borzooei, *A study on bipolar fuzzy graphs,* Journal of Intelligent and Fuzzy Systems, 28 (2015), 571-580.

[19] H. Rashmanlou, S. Samanta, M. Pal, R.A. Borzooei, *Bipolar fuzzy graphs with categorical properties*, International Journal of Computational Intelligent Systems, 8 (2015), 808-818.

[20] H. Rashmanlou, S. Samanta, M. Pal, R.A. Borzooei, *Product of bipolar fuzzy graphs and their degree,* International Journal of General Systems, doi.org/10.1080/03081079.2015.1072521.

[21] H. Rashmanlou, R.A. Borzooei, *Product vague graphs and its applications,* Journal of Intelligent and Fuzzy Systems, 30 (2016), 371-382.

[22] H. Rashmanlou, R.A. Borzooei, Sankar Sahoo, M. Pal, *On searching vague shortest path in a network,* Journal of Multiple-Valued Logic and Soft Computing, 29 (2017), 355-372.

[23] H.Rashmanlou, S. Samanta, M. Pal, R.A. Borzooei, *Properties of interval valued intuitionistic (S-T) fuzzy graphs, Pacific Science review*, doi.org/10.1016/j.psra.1016.06.003.

[24] H. Rashmanlou, S. Samanta, M. Pal, R.A. Borzooei, *Intuitionistic fuzzy graphs with categorical properties,* Fuzzy Information and Engineering, 7 (2015), 317-334.

[25] H. Rashmanlou, M. Pal, *Some properties of highly irregular interval-valued fuzzy graphs*, World Applied Sciences Journal, 27 (2013), 1756-1773.

[26] H. Rashmanlou, M. Pal, *Balanced interval-valued fuzzy graph,* Journal of Physical Sciences, 17 (2013), 43-57.

[27] H. Rashmanlou, Y.B. Jun, R.A. Borzooei, *More results on highly irregular bipolar fuzzy graphs*, Annals of Fuzzy Mathematics and Informatics, 8 (2014), 149-168.

[28] H. Rashmanlou, G. Muhiuddin, S.K. Amanathulla, F. Mofidnakhaei, M. Pal, *A study on cubic graphs with novel application*, Journal of Intelligent & Fuzzy Systems, 40 (2021), 89-101.

[29] Y. Rao, S. Kosari, Z. Shao, *Certain properties of vague graphs with a novel application*, Mathematics, 8 (2020), 1647.

[30] Y. Rao, S. Kosari, Z. Shao, R. Cai, L. Xinyue, *A study on domination in vague incidence graph and its application in medical sciences,* Symmetry, 12 (2020), 1885.

[31] Y. Rao, S. Kosari, Z. Shao, X. Qiang, M. Akhoundi, X. Zhang, *Equitable domination in vague graphs with application in medical sciences*, Frontiers in Physics, 37 (2021).

[32] X. Shi, S. Kosari, *Certain properties of domination in product vague graphs with novel application in medicine,* Frontiers in Physics, 9 (2021), 3-85.

[33] S. Skiena, *Implementing discrete mathematics*, Addison-Wesley, 1990.

[34] A.A. Talebi, Janusz Kacprzyk, H. Rashmanlou, S.H. Sadati, *A new concept of an intuitionistic fuzzy graph with applications*, J. of Mult.-Valued Logic & Soft Computing, 35 (2020), 431-454.

[35] A.A. Talebi, H. Rashmanlou, S.H. Sadati, *Interval-valued intuitionistic fuzzy competition graph,* J. of Mult.-Valued Logic & Soft Computing, 34 (2020), 335-364.

[36] A.A. Talebi, H. Rashmanlou, S.H. Sadati, *New concepts on m-polar interval-valued intuitionistic fuzzy graph,* TWMS J. App. and Eng. Math,, 10 (2020), 806-818

[37] L.A. Zadeh, *Fuzzy sets,* Inf. Control, 8, 1965, 338-353.

[38] S. Zeng, M. Shoaib, S. Ali, F. Smarandache, H. Rashmanlou, F. Mofidnakhaei, *Certain properties of single-valued neutrosophic graph with application in food and agriculture organization,* International Journal of Computational Intelligence Systems, 14 (2021), 1516-1540.

Accepted: January 24, 2023

On the sub-η-n-polynomial convexity and its applications

Lei Xu
Three Gorges Mathematical Research Center
China Three Gorges University
Yichang 443002
P. R. China
leixu9903@163.com

Tingsong Du*
Three Gorges Mathematical Research Center
China Three Gorges University
Yichang 443002
P. R. China
and
Department of Mathematics
College of Science
China Three Gorges University
Yichang 443002
P. R. China
tingsongdu@ctgu.edu.cn

Abstract. This study addresses a new family of functions, to be named as the sub-η-n-polynomial convex functions, which is defined as a general form of the n-polynomial convex functions and the sub-η-convex functions, and some of their significant properties are presented as well. In addition, by means of the sub-η-n-polynomial convexity, certain Hermite–Hadamard-type inequalities are established here. The sufficient conditions regarding optimality for sub-η-n-polynomial convex programming are discussed as applications.
Keywords: n-polynomial convex functions, sub-η-n-polynomial convex programming, optimality conditions.

1. Introduction

Convexity, as well as generalized convexity, provide forceful principles and approaches in both mathematics and certain areas of engineering, in particular, in optimization theory, see [13, 29, 15, 31, 33] and the references therein cited in them. With regard to generalizations and extensions of classical convexity, a variety of interesting articles have been published by plenty of mathematicians. For example, Bector and Singh [5] considered a type of B-vex functions. Long and Peng [24] discussed a family of functions, which is a general form of the B-

*. Corresponding author

vex mappings, called semi-B-preinvex mappings. Chao et al. [8] investigated a group of extented sub-b-convex mappings, as well as demonstrated the sufficient optimality criteria regarding sub-b-convex programming within unconstrained and inequality constrained conditions. Ahmad et al. [2] proposed the concept of geodesic sub-b-s-convex mappings, as well as gave certain properties on Riemannian manifolds. Liao and Du considered two groups of mappings in [21] and [22], named as the sub-b-s-convex mappings and sub-(b, m)-convex mappings, respectively, from which certain significant properties were studied, and optimality conditions for the introduced families of generalized convex programming were reported.

On the other hand, convexity acts on a crucial role in the area of inequalities by its significance of mathematics definition. Recently, a large number of researchers, including mathematicians, engineers and scientists, have tried to conduct an in-depth research regarding properties and inequalities in association with convexity from distinct directions. For instance, Toplu et al. [32] found a class of non-negative mappings, called n-polynomial convex mappings, as well as several related Hermite–Hadamard-type inequalities have been discussed. Deng et al. [10] constructed an integral identity, as well as received certain error bounds involving integral inequalities with regard to a family of strongly convex mappings, which is named as strongly n-polynomial preinvex mappings. By virtue of n-polynomial s-type preinvexity, Butt et al. [7] studied certain refinements of Hermite–Hadamard-type integral inequalities. For more significant findings in connection with n-polynomial convex mappings, we recommend the minded readers to consult [6, 27] and the bibliographies quoted in them.

Trying to get the further discussion, let us consider to the subsequent extraordinary Hermite–Hadamard's inequality in association with convexity.

Suppose that $\psi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval Ω, for each $\zeta_1, \zeta_2 \in \Omega$ together with $\zeta_1 \neq \zeta_2$. The subsequent inequalities, to be named as Hermite–Hadamard's inequalities, are frequently put into use in engineering mathematical and applied analysis

$$\psi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma \leq \frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}. \tag{1}$$

The distinguished integral inequalities, which have given rise to considerable attention from plenty of authors, provide error bounds for the mean value regarding a continuous convex mapping $\psi : [\zeta_1, \zeta_2] \to \mathbb{R}$. There have been a large amount of studies, with regard to the Hermite–Hadamard-type inequalities involving other diverse types of convex mappings, such as N-quasiconvex mappings [1], s-convex mappings [20], (α, m)-convex mappings [30], strongly exponentially generalized preinvex mappings [17], h-convex mappings [9], γ-preinvex mappings [4] and so on. For more vital outcomes pertaining to the Hermite–Hadamard-type inequalities, the reader may refer to [3, 11, 16, 23, 28, 25, 34] and the bibliographies quoted in them.

Enlightened by the above-mentioned research works, in particular, those created in [18, 8, 32], we study a new group of generalized convex sets, as well as generalized convex functions, to be called as sub-η-n-polynomial convex sets and sub-η-n-polynomial convex functions, respectively. And we explore certain fascinating properties of such group of sets and functions. Moreover, we investigate quite a few Hermite–Hadamard's type inequalities in relation to the sub-η-n-polynomial convex functions. As applications, we pursue the sufficient optimality conditions for unconstrained, as well as inequality constrained programming, which are under the sub-η-n-polynomial convexity.

Through out the paper, let us suppose that Λ is a nonempty convex set in $\mathbb{R}^n$. To this end, this section retrospects certain conceptions regarding generalized convexity, and related momentous results.

Definition 1.1 ([8]). *The real function $\psi\colon \Lambda \to \mathbb{R}$ is named as a sub-η-convex mapping defined on the interval Λ with regard to the mapping $\eta : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$, if the successive inequality*

$$\psi\big(\nu\gamma + (1-\nu)\varrho\big) \le \nu\psi(\gamma) + (1-\nu)\psi(\varrho) + \eta(\gamma, \varrho, \nu)$$

holds true for all $\gamma, \varrho \in \Lambda$ and $\nu \in [0, 1]$.

Definition 1.2 ([32]). *Assume that $n \in \mathbb{N}$, the nonnegative mapping $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is named as an n-polynomial convex mapping if the subsequent inequality*

$$\psi\big(\nu\gamma + (1-\nu)\varrho\big) \le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\varrho)$$

holds true for all $\gamma, \varrho \in \Omega$ and $\nu \in [0, 1]$.

In the published article [14], the author proposed a refinement version with regard to the extraordinary Hölder's integral inequality, called as Hölder–İşcan's integral inequality as below.

Theorem 1.1 ([14]). *Suppose that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ψ and ρ are two real mappings defined on the interval $[\zeta_1, \zeta_2]$, as well as if $|\psi|^p$, $|\rho|^q$ are both integrable mappings on the interval $[\zeta_1, \zeta_2]$, then we have the coming inequality*

$$\begin{aligned}\int_{\zeta_1}^{\zeta_2} |\psi(x)\rho(x)|\mathrm{d}\gamma \le \frac{1}{\zeta_2-\zeta_1}\Bigg[&\left(\int_{\zeta_1}^{\zeta_2}(\zeta_2-\gamma)|\psi(\gamma)|^p\mathrm{d}\gamma\right)^{\frac{1}{p}}\left(\int_{\zeta_1}^{\zeta_2}(\zeta_2-\gamma)|\rho(\gamma)|^q\mathrm{d}\gamma\right)^{\frac{1}{q}}\\ &+\left(\int_{\zeta_1}^{\zeta_2}(\gamma-\zeta_1)|\psi(\gamma)|^p\mathrm{d}\gamma\right)^{\frac{1}{p}}\left(\int_{\zeta_1}^{\zeta_2}(\gamma-\zeta_1)|\rho(\gamma)|^q\mathrm{d}\gamma\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

2. Sub-η-n-polynomial convex functions and their properties

The fact that the convexity, n-polynomial convexity, and sub-η-convexity have almost the analogous structures impels us to generalize these distinct families of convex functions. Now, let us consider to introduce the conception of the sub-η-n-polynomial convex functions and sub-η-n-polynomial convex sets as below. Then certain basic characterization theorems are proposed, as well as preservation of the sub-η-n-polynomial convexity with regard to some functional operations such as composition, sum and maximum are studied. In particular, two property theorems with regard to differentiable sub-η-n-polynomial convex functions are investigated in this section.

Definition 2.1. *Assume that $n \in \mathbb{N}$, the non-negative function $\psi\colon \Lambda \to \mathbb{R}$ is named as sub-η-n-polynomial convex defined on the interval Λ with regard to the mapping $\eta : \Lambda \times \Lambda \times [0,1] \to \mathbb{R}$, if the subsequent inequality*

$$\psi\big(\nu\gamma+(1-\nu)\varrho\big) \leq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\varrho)+\eta(\gamma,\varrho,\nu) \tag{2}$$

holds true for each $\gamma, \varrho \in \Lambda$ and $\nu \in [0,1]$. On the other hand, if the successive inequality

$$\psi\big(\nu\gamma+(1-\nu)\varrho\big) \geq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\varrho)+\eta(\gamma,\varrho,\nu) \tag{3}$$

holds true for each $\gamma, \varrho \in \Lambda$ and $\nu \in [0,1]$, then the function ψ is named as sub-η-n-polynomial concave. If the inequality notations in the above-mentioned inequalities are strict, then the function ψ is named as strictly sub-η-n-polynomial convex, as well as strictly sub-η-n-polynomial concave, respectively.

Remark 2.1. If we consider to take $n = 1$, then the sub-η-n-polynomial convex function reduces to the sub-η convex functions. Moreover, when we attempt to put $n = 1$ and claim $\eta(\gamma, \varrho, \nu) \leq 0$, the sub-$\eta$-$n$-polynomial convex function transforms to convex functions.

Remark 2.2. In accordance with Remark 3 in Ref. [32], we know that each nonnegative convex function is an n-polynomial convex function. When the mapping $\eta(\gamma, \varrho, \nu) \geq 0$, each nonnegative convex function is also a sub-η-n-polynomial convex function. In the same way, when we claim $\eta(\gamma, \varrho, \nu) \geq 0$, it is obvious that each n-polynomial convex function is also a sub-η-n-polynomial convex function.

Now, we try to study certain operations that preserve the sub-η-n-polynomial convexity with regard to positive linear combination and securing pointwise maximum. Because the proofs of these properties are simplified, they are omitted.

Proposition 2.1. *If the functions ψ, ρ: $\Lambda \to \mathbb{R}$ are both sub-η-n-polynomial convex with regard to the same mapping η, then $\psi + \rho$ is sub-η-n-polynomial convex with regard to the mapping 2η, and $\alpha\psi (\alpha > 0)$ is sub-η-n-polynomial convex with regard to the mapping $\alpha\eta$.*

Corollary 2.1. *If ψ_κ: $\Lambda \to \mathbb{R}$ $(\kappa = 1, 2, \ldots, \delta)$ are a series of sub-η-n-polynomial convex functions regarding the mappings $\eta_\kappa : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$ $(\kappa = 1, 2, \ldots, \delta)$, correspondingly, then the function*

$$\psi = \sum_{\kappa=1}^{\delta} a_\kappa \psi_\kappa, a_\kappa \geq 0, (\kappa = 1, 2, \ldots, \delta) \tag{4}$$

is sub-η-n-polynomial convex with regard to $\eta = \sum_{\kappa=1}^{\delta} a_\kappa \eta_\kappa$.

Proposition 2.2. *If ψ_κ: $\Lambda \to \mathbb{R}$ $(\kappa = 1, 2, \ldots, \delta)$ are a series of sub-η-n-polynomial convex functions with respect to the mappings $\eta_\kappa : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$ $(\kappa=1, 2, \ldots, \delta)$, correspondingly, then the function $\psi=\max\{\psi_\kappa, i=1, 2, \ldots, \delta\}$ is a sub-η-n-polynomial convex function with regard to the mapping $\eta = \max\{\eta_\kappa, \kappa = 1, 2, \ldots, \delta\}$.*

Theorem 2.1. *Assume that ψ: $\Lambda \to \mathbb{R}$ is a sub-η-n-polynomial convex function with regard to the mapping $\eta : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$, as well as ρ: $\mathbb{R} \to \mathbb{R}$ is an increasing function. If ρ meets the coming conditions:*

(5) (i) $\rho(\alpha\gamma) = \alpha\rho(\gamma), \forall\gamma \in \mathbb{R}, \alpha > 0,$

(6) (ii) $\rho(\gamma + \varrho) = \rho(\gamma) + \rho(\varrho), \forall\gamma, \varrho \in \mathbb{R},$

then the function $\psi^\triangle = \rho \circ \psi$ is sub-η-n-polynomial convex with regard to $\eta^\triangle = \rho \circ \eta$.

Proof. Since the function ψ is sub-η-n-polynomial convex regarding the mapping η and the function ρ is increasing, it follows that

$$\begin{aligned}
&(\rho \circ \psi)\big(\nu\gamma + (1-\nu)\varrho\big) \\
&= \rho\big(\psi\big(\nu\gamma + (1-\nu)\varrho\big)\big) \\
&\leq \rho\left(\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]\psi(\gamma) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^\kappa\big]\psi(\varrho) + \eta(\gamma, \varrho, \nu)\right).
\end{aligned}$$

By virtue of the provided conditions in (5) and (6), it readily yields that

$$\begin{aligned}
&(\rho \circ \psi)\big(\nu\gamma + (1-\nu)\varrho\big) \\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]\rho\big(\psi(\gamma)\big) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^\kappa\big]\rho\big(\psi(\varrho)\big) + \rho\big(\eta(\gamma, \varrho, \nu)\big) \\
&= \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big](\rho \circ \psi)(\gamma) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^\kappa\big](\rho \circ \psi)(\varrho) + (\rho \circ \eta)(\gamma, \varrho, \nu).
\end{aligned}$$

That is, the function $\psi^{\triangle} = \rho \circ \psi$ is sub-η-n-polynomial convex with regard to $\eta^{\triangle} = \rho \circ \eta$. This ends the proof. $\square$

Theorem 2.2. *Assume that $\eta_1 : \Lambda\times\Lambda\times[0,1] \to \mathbb{R}$ and $\eta_2 : [0,1]\times[0,1]\times[0,1] \to \mathbb{R}$ are two mappings along with $\eta_1(\gamma, \varrho, \nu) \leq \eta_2(\zeta_1, \zeta_2, \nu)$. If $\psi : \Lambda \subset \mathbb{R}^n \to \mathbb{R}$ is a sub-η_1-n-polynomial convex function on Λ with regard to η_1, then for all $\gamma, \varrho \in \Lambda$, the function $\Phi : [0,1] \to \mathbb{R}$, $\Phi(t) = \psi(\nu\gamma + (1-\nu)\varrho)$ is sub-η_2-n-polynomial convex on $[0,1]$ with regard to the mapping η_2.*

Proof. Assume that ψ is a sub-η_1-n-polynomial convex function on Λ regarding the mapping η_1. Let $\gamma, \varrho \in \Lambda$, $\nu \in [0,1]$ and $\zeta_1, \zeta_2 \in [0,1]$. Then, we know that

$$0 \leq \nu\zeta_1 + (1-\nu)\zeta_2 \leq 1,$$

and

$$\begin{aligned}
&\Phi(\nu\zeta_1 + (1-\nu)\zeta_2)\\
&= \psi\big[\big(\nu\zeta_1 + (1-\nu)\zeta_2\big)\gamma + \big(1 - \nu\zeta_1 - (1-\nu)\zeta_2\big)\varrho\big]\\
&= \psi\big[\nu(\zeta_1\gamma + (1-\zeta_1)\varrho) + (1-\nu)(\zeta_2\gamma + (1-\zeta_2)\varrho)\big]\\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\zeta_1\gamma + (1-\zeta_1)\varrho) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\zeta_2\gamma + (1-\zeta_2)\varrho)\\
&\quad + \eta_1(\zeta_1\gamma + (1-\zeta_1)\varrho, \zeta_2\gamma + (1-\zeta_2)\varrho, \nu)\\
&= \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\Phi(\zeta_1) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\Phi(\zeta_2)\\
&\quad + \eta_1(\zeta_1\gamma + (1-\zeta_1)\varrho, \zeta_2\gamma + (1-\zeta_2)\varrho, \nu)\\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\Phi(\zeta_1) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\Phi(\zeta_2) + \eta_2(\zeta_1, \zeta_2, \nu).
\end{aligned}$$

Hence, the function Φ is sub-η_2-n-polynomial convex on $[0,1]$ with regard to η_2. The proof of Theorem 2.2 is completed. $\square$

In what following, let us consider a novel concept regarding sub-η-n-polynomial convex set.

Definition 2.2. *Assume that the set $X \subseteq \mathbb{R}^{n+1}$ is a nonempty set. A set X is named as a sub-η-n-polynomial convex set with regard to the mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}$, if the subsequent inclusion relation*

$$\left(\nu\gamma + (1-\nu)\varrho,\ \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\beta + \eta(\gamma, \varrho, \nu)\right) \in X \tag{7}$$

holds true for $\forall\ (\gamma, \alpha), (\varrho, \beta) \in X, \gamma, \varrho \in \mathbb{R}^n$ and $\nu \in [0,1]$.

Here, let us take into account a characterization of sub-η-n-polynomial convex function ψ: $\Lambda \to \mathbb{R}$, by means of its epigraph $E(\psi)$, which is described by

$$E(\psi) = \{(\gamma, \alpha) | \gamma \in \Lambda, \alpha \in \mathbb{R}; \psi(\gamma) \leq \alpha\}. \tag{8}$$

Theorem 2.3. *A function $\psi\colon \Lambda \to \mathbb{R}$ is a sub-η-n-polynomial convex function regarding the mapping $\eta : \Lambda \times \Lambda \times [0,1] \to \mathbb{R}$ when, and only when $E(\psi)$ is a sub-η-n-polynomial convex set regarding the same mapping η.*

Proof. Suppose that the function ψ is sub-η-n-polynomial convex regarding the mapping η. Let $(\gamma_1, \alpha_1), (\gamma_2, \alpha_2) \in E(\psi)$. Then $\psi(\gamma_1) \le \alpha_1, \psi(\gamma_2) \le \alpha_2$, we know that

$$\begin{aligned}&\psi\big(\nu\gamma_1 + (1-\nu)\gamma_2\big)\\ &\le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma_1) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\gamma_2) + \eta(\gamma_1,\gamma_2,\nu)\\ &\le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha_1 + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\alpha_2 + \eta(\gamma_1,\gamma_2,\nu)\end{aligned}$$

holds true for $\forall \gamma_1, \gamma_2 \in \Lambda, \nu \in [0,1]$.

Hence, it is not difficult to check that

$$\Big(\nu\gamma_1 + (1-\nu)\gamma_2,\ \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha_1 + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\alpha_2 + \eta(\gamma_1,\gamma_2,\nu)\Big) \in E(\psi).$$

Therefore, the set $E(\psi)$ is a sub-η-n-polynomial convex set regarding the mapping η.

In turn, let us assume that $E(\psi)$ is a sub-η-n-polynomial convex set regarding the mapping η. Let $\gamma_1, \gamma_2 \in \Lambda$, we have $(\gamma_1, \alpha_1), (\gamma_2, \alpha_2) \in E(\psi)$. Thus, for $\nu \in [0,1]$, we find that

$$\left(\nu\gamma_1 + (1-\nu)\gamma_2,\ \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\alpha_1 + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\alpha_2 + \eta(\gamma_1,\gamma_2,\nu\right) \in E(\psi).$$

It suffices to show that

$$\psi\big(\nu\gamma_1 + (1-\nu)\gamma_2\big) \le \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\gamma_1) + \frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\gamma_2) + \eta(\gamma_1,\gamma_2,\nu).$$

That is, the function ψ is a sub-η-n-polynomial convex regarding the mapping η. This finishes the proof. □

We have the succedent propositions without proof.

Proposition 2.3. *If $X_\kappa (\kappa \in \Omega)$ is a series of sub-η-n-polynomial convex sets regarding the same mapping $\eta(\gamma, \varrho, \nu)$, then $\bigcap_{\kappa\in\Omega} X_\kappa$ is a sub-η-n-polynomial convex set with regard to the same mapping $\eta(\gamma, \varrho, \nu)$.*

Proposition 2.4. *If $\{\psi_\kappa | \kappa \in \Omega\}$ is a group of numerical functions, as well as any ψ_κ is a sub-η-n-polynomial convex function regarding the same mapping $\eta(\gamma, \varrho, \nu)$, then the numerical function $\psi = \sup_{\kappa\in\Omega} \psi_\kappa(\gamma)$ is a sub-η-n-polynomial convex function regarding the same mapping $\eta(\gamma, \varrho, \nu)$.*

To explore the optimal conditions regarding sub-η-n-polynomial convex programming, we next discuss certain properties in relation to a family of the differentiable sub-η-n-polynomial convex functions. Further, we assume that the limit $\lim_{\nu\to 0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu}$ exists for certain fixed $\gamma,\varrho\in\Lambda$.

Theorem 2.4. *Suppose that the function $\psi\colon \Lambda\to\mathbb{R}$ is differentiable and sub-η-n-polynomial convex regarding the mapping η. Then we have*

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)\le \frac{n+1}{2}\psi(\gamma)-\frac{1}{n}\psi(\gamma^*)+\lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu}. \tag{9}$$

Proof. By virtue of Taylor expansion and the sub-η-n-polynomial convexity of ψ defined on Λ, we find that

$$\begin{aligned}&\psi\big(\nu\gamma+(1-\nu)\gamma^*\big)\\ &=\psi(\gamma^*)+\nu\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)+o(\nu)\\ &\le\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]\psi(\gamma)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^\kappa\big]\psi(\gamma^*)+\eta(\gamma,\gamma^*,\nu).\end{aligned}$$

This implies that

$$\begin{aligned}&\nu\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)+o(\nu)\\ &\le\frac{1}{n}\Big[\sum_{\kappa=1}^{n}[1-(1-\nu)^\kappa]\psi(\gamma)-\sum_{\kappa=1}^{n}\nu^\kappa\psi(\gamma^*)\Big]+\eta(\gamma,\gamma^*,\nu).\end{aligned} \tag{10}$$

Dividing the above inequality (10) by ν and taking $\nu\to 0^+$, it yields that

$$\begin{aligned}\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)\le{}&\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]}{\nu}\psi(\gamma)\\ &-\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\nu^\kappa}{\nu}\psi(\gamma^*)+\lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu}.\end{aligned} \tag{11}$$

Employing the L'Hospital's rule, we can figure out that

$$\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]}{\nu}=\frac{n+1}{2},$$

and

$$\lim_{\nu\to 0^+}\frac{\frac{1}{n}\sum_{\kappa=1}^{n}\nu^\kappa}{\nu}=\frac{1}{n}.$$

Making use of the inequality (11), we deduce that

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*)\le \frac{n+1}{2}\psi(\gamma)-\frac{1}{n}\psi(\gamma^*)+\lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu},$$

which proves the required inequality in (9). This concludes the proof. □

Remark 2.3. If one attempts to pick up $n = 1$, in Theorem 2.4, then one receives Theorem 1.3 proven by Chao et al. in [8].

Theorem 2.5. *With the same hypotheses considered in Theorem 2.4, we have*

$$\begin{aligned}&\big(\nabla\psi(\varrho) - \nabla\psi(\gamma)\big)^T(\gamma - \varrho)\\ &\le \frac{(n-1)(n+2)}{2n}\big[\psi(\varrho) + \psi(\gamma)\big] + \lim_{\nu\to 0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu} + \lim_{\nu\to 0^+}\frac{\eta(\varrho,\gamma,\nu)}{\nu}.\end{aligned} \tag{12}$$

Proof. In accordance with Theorem 2.4, it follows that

$$\nabla\psi(\varrho)^T(\gamma - \varrho) \le \frac{n+1}{2}\psi(\gamma) - \frac{1}{n}\psi(\varrho) + \lim_{\nu\to 0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu}, \tag{13}$$

and

$$\nabla\psi(\gamma)^T(\varrho - \gamma) \le \frac{n+1}{2}\psi(\varrho) - \frac{1}{n}\psi(\gamma) + \lim_{\nu\to 0^+}\frac{\eta(\varrho,\gamma,\nu)}{\nu}. \tag{14}$$

Adding the above two inequalities, we obtain that

$$\begin{aligned}&\big(\nabla\psi(\varrho) - \nabla\psi(\gamma)\big)^T(\gamma - \varrho)\\ &\le \frac{(n-1)(n+2)}{2n}\big[\psi(\varrho) + \psi(\gamma)\big] + \lim_{\nu\to 0^+}\frac{\eta(\gamma,\varrho,\nu)}{\nu} + \lim_{\nu\to 0^+}\frac{\eta(\varrho,\gamma,\nu)}{\nu}.\end{aligned}$$

This ends the proof. □

Remark 2.4. If one attempts to pick up $n = 1$, in Theorem 2.5, then one captures Theorem 1.4 presented by Chao et al. in [8].

3. Inequalities in connection with sub-η-n-polynomial convexity

In this part, we construct the successive Hermite–Hadamard-type inequalities under sub-η-n-polynomial convexity.

Theorem 3.1. *Assume that the function ψ: $[\zeta_1, \zeta_2] \to \mathbb{R}$ is sub-η-n-polynomial convex with $\zeta_1 < \zeta_2$, and the mapping η: $[\zeta_1, \zeta_2] \times [\zeta_1, \zeta_2] \times [0, 1] \to \mathbb{R}$ is continuous. If the function $\psi \in L([\zeta_1, \zeta_2])$, then the subsequent Hermite–Hadamard-type inequalities*

$$\begin{aligned}&\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right) - \eta\left(\xi_0\zeta_1+(1-\xi_0)\zeta_2, (1-\xi_0)\zeta_1+\xi_0\zeta_2, \frac{1}{2}\right)\right]\\ &\le \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma \le \left(\frac{\psi(\zeta_1)+\psi(\zeta_2)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1} + \eta(\zeta_1, \zeta_2, \xi_0)\end{aligned} \tag{15}$$

hold true for certain fixed $\xi_0 \in (0, 1)$.

Proof. On account of the sub-η-n-polynomial convexity of ψ defined over the interval $[\zeta_1, \zeta_2]$, we can figure out that

$$\begin{aligned}
&\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
&=\psi\left(\frac{[\nu\zeta_1+(1-\nu)\zeta_2]+[(1-\nu)\zeta_1+\nu\zeta_2]}{2}\right)\\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(1-\frac{1}{2}\right)^{\kappa}\right]\psi\big(\nu\zeta_1+(1-\nu)\zeta_2\big)\\
&\quad+\frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\psi\big((1-\nu)\zeta_1+\nu\zeta_2\big)\\
&\quad+\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\\
&=\frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\Big[\psi\big(\nu\zeta_1+(1-\nu)\zeta_2\big)+\psi\big((1-\nu)\zeta_1+\nu\zeta_2\big)\Big]\\
&\quad+\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right).
\end{aligned}$$

Integrating the resulting inequality above regarding the variate ν over $[0,1]$, it follows that

$$\begin{aligned}
&\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
&\leq \frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\left[\int_0^1 f\big(\nu\zeta_1+(1-\nu)\zeta_2\big)\mathrm{d}\nu+\int_0^1\psi\big((1-\nu)\zeta_1+\nu\zeta_2\big)\mathrm{d}\nu\right]\\
&\quad+\int_0^1\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\mathrm{d}\nu\\
&=\frac{2}{\zeta_2-\zeta_1}\left(\frac{n+2^{-n}-1}{n}\right)\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\\
&\quad+\int_0^1\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\mathrm{d}\nu.
\end{aligned}$$

According to the mean value theorem of integrals, it yields that

$$\begin{aligned}
&\int_0^1\eta\left(\nu\zeta_1+(1-\nu)\zeta_2,(1-\nu)\zeta_1+\nu\zeta_2,\frac{1}{2}\right)\mathrm{d}\nu\\
&=\eta\left(\xi_0\zeta_1+(1-\xi_0)\zeta_2,(1-\xi_0)\zeta_1+\xi_0\zeta_2,\frac{1}{2}\right),\ \xi_0\in(0,1).
\end{aligned}$$

This finishes the proof of the first inequality in (15).

In the same way, by taking advantage of the sub-η-n-polynomial convexity of ψ on the interval $[\zeta_1, \zeta_2]$, as well as the mean value theorem of integrals, if the variable is changed as $\gamma = \nu\zeta_1 + (1-\nu)\zeta_2$, then we know that

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma \\
&= \int_0^1 \psi\big(\nu\zeta_1+(1-\nu)\zeta_2\big)\mathrm{d}\nu \\
&\le \int_0^1 \left[\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\psi(\zeta_1)+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\psi(\zeta_2)+\eta(\zeta_1,\zeta_2,\nu)\right]\mathrm{d}\nu \\
&= \frac{\psi(\zeta_1)}{n}\int_0^1\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu+\frac{\psi(\zeta_2)}{n}\int_0^1\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\mathrm{d}\nu+\int_0^1\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu \\
&= \frac{\psi(\zeta_1)}{n}\sum_{\kappa=1}^{n}\int_0^1\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu+\frac{\psi(\zeta_2)}{n}\sum_{\kappa=1}^{n}\int_0^1\big[1-\nu^{\kappa}\big]\mathrm{d}\nu \\
&\quad +\eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).
\end{aligned}$$

Also, we observe that

$$\int_0^1\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu=\int_0^1\big[1-\nu^{\kappa}\big]\mathrm{d}\nu=\frac{\kappa}{\kappa+1}.$$

This finishes the proof. □

Remark 3.1. If one attempts to pick up the mapping $\eta = 0$ in Theorem 3.1, then one receives Theorem 4 deduced by Toplu et al. in [32]. In particular, if one considers to pick up $\eta = 0$ and $n = 1$, then the inequalities (15) coincides with the extraordinary Hermite–Hadamard's inequalities (1).

Theorem 3.2. *Suppose that $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on the interval Ω° with $\zeta_1 < \zeta_2$. If ψ is a sub-η-n-polynomial convex function regarding continuous mapping $\eta\colon \Omega\times\Omega\times[0,1]\to\mathbb{R}$, then the successive inequalities*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)-\eta\left(\xi_0\zeta_1+(1-\xi_0)\zeta_2,(1-\xi_0)\zeta_1+\xi_0\zeta_2,\frac{1}{2}\right)\right] \\
&\le \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)d\gamma \\
&\le \left(\frac{n+2^{-n}-1}{n}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\left(\frac{\psi\left(\frac{3\zeta_1-\zeta_2}{2}\right)+\psi\left(\frac{3\zeta_2-\zeta_1}{2}\right)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}\right. \\
&\left.\quad+\eta\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_0\right)\right]+\eta\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right),
\end{aligned} \tag{16}$$

and

$$\left|\left[\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)d\gamma-\eta\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right)\right]-\left(\frac{n+2^{-n}-1}{n}\right)\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right| \tag{17}$$
$$\leq\left|\left(\frac{n+2^{-n}-1}{n}\right)\left[\left(\frac{\psi\left(\frac{3\zeta_1-\zeta_2}{2}\right)+\psi\left(\frac{3\zeta_2-\zeta_1}{2}\right)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}+\eta\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_0\right)\right]\right|$$

hold true for certain fixed $\xi_0\in(0,1)$ *and* $\xi_1\in\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}\right)$.

Proof. Applying the mean value theorem of integrals, as well as by substituting the variables $\gamma=\frac{3}{4}\nu+\frac{\zeta_1+\zeta_2}{4}$, $\nu\in[\frac{3\zeta_1-\zeta_2}{3},\frac{3\zeta_2-\zeta_1}{3}]$, we deduce that

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\\
&=\frac{3}{4(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{3}}^{\frac{3\zeta_2-\zeta_1}{3}}\psi\left(\frac{3}{4}\nu+\frac{\zeta_1+\zeta_2}{4}\right)\mathrm{d}\nu\\
&=\frac{3}{4(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{3}}^{\frac{3\zeta_2-\zeta_1}{3}}\psi\left(\frac{1}{2}\left(\frac{3}{2}\nu\right)+\frac{1}{2}\left(\frac{\zeta_1+\zeta_2}{2}\right)\right)\mathrm{d}\nu\\
&\leq\frac{3}{4(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{3}}^{\frac{3\zeta_2-\zeta_1}{3}}\left(\frac{1}{n}\sum_{\kappa=1}^{n}\left[1-\left(\frac{1}{2}\right)^{\kappa}\right]\left[\psi\left(\frac{3}{2}\nu\right)+\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right]\right.\\
&\quad\left.+\eta\left(\frac{3}{2}\nu,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right)\right)\mathrm{d}\nu \qquad (18)\\
&=\left(\frac{n+2^{-n}-1}{n}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}\psi(\nu)\mathrm{d}\nu\right]\\
&+\eta\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right).
\end{aligned}$$

According to the right hand side of outcome (15), we find that

$$\begin{aligned}
&\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}f(\nu)\mathrm{d}\nu\\
&\leq\left(\frac{\psi\left(\frac{3\zeta_1-\zeta_2}{2}\right)+\psi\left(\frac{3\zeta_2-\zeta_1}{2}\right)}{n}\right)\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}+\eta\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_0\right).
\end{aligned} \tag{19}$$

Combining the above-mentioned inequalities (18) and (19), one achieves the findings (16) and (17). This ends the proof. □

Remark 3.2. Under the assumptions mentioned in Theorem 3.2 with $\eta = 0$ and $n = 1$, we receive Lemma 3 presented by Mehrez in [25].

For mappings whose derivatives in absolute value are sub-η-n-polynomial convex, we will try to develop a series of Hermite–Hadamard-type integral inequalities. To achieve this object, we need the successive lemmas.

Lemma 3.1 ([12]). *Assume that the mapping $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable defined over the interval Ω°, $\zeta_1, \zeta_2 \in \Omega^\circ$ with $\zeta_1 < \zeta_2$. If the mapping $\psi' \in L([\zeta_1, \zeta_2])$, then we have the subsequent identity*

$$\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma = \frac{\zeta_2-\zeta_1}{2}\int_0^1 (1-2\nu)\psi'(\nu\zeta_1+(1-\nu)\zeta_2)\mathrm{d}\nu.$$

Lemma 3.2 ([19]). *Assume that $f\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping defined over the interval Ω°, $\zeta_1, \zeta_2 \in \Omega^\circ$ with $\zeta_1 < \zeta_2$. If the mapping $\psi' \in L([\zeta_1, \zeta_2])$, then we have the coming identity*

$$\begin{aligned}&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma - \psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\\ &= (\zeta_2-\zeta_1)\left[\int_0^{\frac{1}{2}} \nu\psi'\big(\zeta_2+(\zeta_1-\zeta_2)\nu\big)\mathrm{d}\nu + \int_{\frac{1}{2}}^1 (\nu-1)\psi'\big(\zeta_2+(\zeta_1-\zeta_2)\nu\big)\mathrm{d}\nu\right].\end{aligned}$$

Theorem 3.3. *Assume that $\psi\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable function defined on the interval Ω°, $\zeta_1, \zeta_2 \in \Omega^\circ$ with $\zeta_1 < \zeta_2$, and let the function $\psi' \in L([\zeta_1, \zeta_2])$. If the function $|\psi'|$ is sub-η-n-polynomial convex defined over the interval $[\zeta_1, \zeta_2]$ and the mapping $\eta\colon \Omega\times\Omega\times[0,1] \to \mathbb{R}^+$ is continuous, then the coming inequality*

$$\begin{aligned}&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right|\\ &\le \frac{\zeta_2-\zeta_1}{2n}\left(\sum_{\kappa=1}^{n}\left[\frac{(\kappa^2+\kappa+2)2^\kappa-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}\right]\big(|\psi'(\zeta_1)|+|\psi'(\zeta_2)|\big)+\frac{n}{2}\eta(\zeta_1,\zeta_2,\xi_0)\right)\end{aligned} \tag{20}$$

holds true for some fixed $\xi_0 \in (0,1)$.

Proof. Taking advantage of Lemma 3.1, as well as the sub-η-n-polynomial convexity of $|\psi'|$ defined on the interval $[\zeta_1, \zeta_2]$, it yields that

$$\begin{aligned}&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right|\\ &\le \frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu||\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|\mathrm{d}\nu\\ &\le \frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu|\left(\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-(1-\nu)^\kappa\big]|\psi'(\zeta_1)|\right.\\ &\quad\left.+\frac{1}{n}\sum_{\kappa=1}^{n}\big[1-\nu^\kappa\big]|\psi'(\zeta_2)|+\eta(\zeta_1,\zeta_2,\nu)\right)\mathrm{d}\nu\end{aligned}$$

$$= \frac{\zeta_2-\zeta_1}{2n}\Bigg(|\psi'(\zeta_1)|\int_0^1 |1-2\nu|\sum_{\kappa=1}^{n}\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu$$
$$+|\psi'(\zeta_2)|\int_0^1 |1-2\nu|\sum_{\kappa=1}^{n}\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\Bigg)+\frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu|\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu$$
$$= \frac{\zeta_2-\zeta_1}{2n}\Bigg(|\psi'(\zeta_1)|\sum_{\kappa=1}^{n}\int_0^1 |1-2\nu|\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu$$
$$+|\psi'(\zeta_2)|\sum_{\kappa=1}^{n}\int_0^1 |1-2\nu|\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\Bigg)+\frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu|\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu.$$

According to the mean value theorem of generalized integrals, we derive that

$$\int_0^1 |1-2\nu|\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu=\frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).$$

Also, we observe that

$$\int_0^1 |1-2\nu|\mathrm{d}\nu=\frac{1}{2},$$

and

$$\int_0^1 |1-2\nu|\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu=\int_0^1 |1-2\nu|\big[1-\nu^{\kappa}\big]\mathrm{d}\nu=\frac{(\kappa^2+\kappa+2)2^{\kappa}-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}.$$

Therefore, the proof of Theorem 3.3 is completed. □

Remark 3.3. If one considers to pick up $\eta=0$ in Theorem 3.3, then one receives Theorem 5 established by Toplu et al. in [32]. In particular, if we attempt to take $\eta=0$ and $n=1$, then we gain Theorem 2.2 provided by Dragomir et al. in [12].

Theorem 3.4. *Assume that $\psi\colon\Omega\subseteq\mathbb{R}\to\mathbb{R}$ is a differentiable function on Ω°, $\zeta_1,\zeta_2\in\Omega^{\circ}$ with $\zeta_1<\zeta_2$, and $\psi'\in L([\zeta_1,\zeta_2])$. If the function $|\psi'|^q$ is sub-η-n-polynomial convex on the interval $[\zeta_1,\zeta_2]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and the mapping $\eta\colon\Omega\times\Omega\times[0,1]\to\mathbb{R}^{+}$ is continuous, then the succedent inequality*

$$\begin{aligned}&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right|\\ &\le\frac{\zeta_2-\zeta_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{n}\sum_{\kappa=1}^{n}\frac{\kappa}{\kappa+1}\big(|\psi'(\zeta_1)|^q+|\psi'(\zeta_2)|^q\big)+\eta(\zeta_1,\zeta_2,\xi_0)\right]^{\frac{1}{q}}\end{aligned}\tag{21}$$

holds true for some fixed $\xi_0\in(0,1)$.

Proof. By means of Lemma 3.1, Hölder's integral inequality, and the sub-η-n-polynomial convexity of $|\psi'|^q$ defined on the interval $[\zeta_1, \zeta_2]$, it follows that

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right| \\
&\le \frac{\zeta_2-\zeta_1}{2}\int_0^1 |1-2\nu||\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|\mathrm{d}\nu \\
&\le \frac{\zeta_2-\zeta_1}{2}\left(\int_0^1 |1-2\nu|^p\mathrm{d}\nu\right)^{\frac{1}{p}}\left(\int_0^1 |\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\right)^{\frac{1}{q}} \\
&\le \frac{\zeta_2-\zeta_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1[1-(1-\nu)^\kappa]\mathrm{d}\nu\right. \\
&\quad\left. +\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1[1-\nu^\kappa]\mathrm{d}\nu + \int_0^1\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu\right)^{\frac{1}{q}}.
\end{aligned}$$

According to the mean value theorem of integrals, we obtain that

$$\int_0^1 \eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu = \eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).$$

Direct computation yields that,

$$\int_0^1 |1-2\nu|^p\mathrm{d}\nu = \frac{1}{p+1},$$

and

$$\int_0^1[1-(1-\nu)^\kappa]\mathrm{d}\nu = \int_0^1[1-\nu^\kappa]\mathrm{d}\nu = \frac{\kappa}{\kappa+1}.$$

This finishes the proof. □

Remark 3.4. If one attempts to pick up the mapping $\eta = 0$ in Theorem 3.4, then one acquires Theorem 6 derived by Toplu et al. in [32]. In particular, if we consider to take $\eta = 0$ and $n = 1$, we capture Theorem 2.3 provided by Dragomir et al. in [12].

Theorem 3.5. *Assume that $\psi\colon \Omega\subseteq\mathbb{R}\to\mathbb{R}$ is a differentiable function on the interval Ω°, $\zeta_1,\zeta_2\in\Omega^\circ$ with $\zeta_1<\zeta_2$, and $\psi'\in L([\zeta_1,\zeta_2])$. If $|\psi'|^q$ is a sub-η-n-polynomial convex function on $[\zeta_1,\zeta_2]$ for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and the mapping $\eta\colon \Omega\times\Omega\times[0,1]\to\mathbb{R}^+$ is continuous, then the succeding inequality*

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right| \\
&\le \frac{\zeta_2-\zeta_1}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \qquad (22)\\
&\times\left[\left(\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa}{2(\kappa+2)}+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa(\kappa+3)}{2(\kappa+1)(\kappa+2)}+\frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0)\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa(\kappa+3)}{2(\kappa+1)(\kappa+2)}+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\frac{\kappa}{2(\kappa+2)}+\frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0)\right)^{\frac{1}{q}}\Bigg]$$

holds true for certain fixed $\xi_0\in(0,1)$.

Proof. By taking advantage of Lemma 3.1, as well as the Hölder–İşcan's integral inequality, it yields that

$$\begin{aligned}
&\left|\frac{\psi(\zeta_1)+\psi(\zeta_2)}{2}-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right|\\
&\le\frac{\zeta_2-\zeta_1}{2}\int_0^1|1-2\nu||\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|\mathrm{d}\nu\\
&\le\frac{\zeta_2-\zeta_1}{2}\left[\left(\int_0^1(1-\nu)|1-2\nu|^p\mathrm{d}\nu\right)^{\frac{1}{p}}\left(\int_0^1(1-\nu)|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\nu|1-2\nu|^p\mathrm{d}\nu\right)^{\frac{1}{p}}\left(\int_0^1\nu|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Making use of the sub-η-n-polynomial convexity of $|\psi'|^q$, it follows that

$$\begin{aligned}
&\int_0^1(1-\nu)|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\\
&\le\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1(1-\nu)[1-(1-\nu)^\kappa]\mathrm{d}\nu+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1(1-\nu)[1-\nu^\kappa]\mathrm{d}\nu\\
&+\int_0^1(1-\nu)\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\nu|\psi'(\nu\zeta_1+(1-\nu)\zeta_2)|^q\mathrm{d}\nu\\
&\le\frac{|\psi'(\zeta_1)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1\nu[1-(1-\nu)^\kappa]\mathrm{d}\nu+\frac{|\psi'(\zeta_2)|^q}{n}\sum_{\kappa=1}^{n}\int_0^1\nu[1-\nu^\kappa]\mathrm{d}\nu\\
&+\int_0^1\nu\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu.
\end{aligned}$$

According to the mean value theorem of generalized integrals, we know that

$$\int_0^1(1-\nu)\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu=\int_0^1\nu\eta(\zeta_1,\zeta_2,\nu)\mathrm{d}\nu=\frac{1}{2}\eta(\zeta_1,\zeta_2,\xi_0),\ \ \xi_0\in(0,1).$$

Direct computation yields that

$$\int_0^1(1-\nu)|1-2\nu|^p\mathrm{d}\nu=\int_0^1\nu|1-2\nu|^p\mathrm{d}\nu=\frac{1}{2(p+1)},$$

$$\int_0^1 (1-\nu)[1-(1-\nu)^\kappa]\mathrm{d}\nu = \int_0^1 \nu[1-\nu^\kappa]\mathrm{d}\nu = \frac{\kappa}{2(\kappa+2)},$$

and

$$\int_0^1 \nu[1-(1-\nu)^\kappa]\mathrm{d}\nu = \int_0^1 (1-\nu)[1-\nu^\kappa]\mathrm{d}\nu = \frac{\kappa(\kappa+3)}{2(\kappa+1)(\kappa+2)}.$$

Thus, this concludes the proof. □

Remark 3.5. If one attempts to pick up the mapping $\eta = 0$, in Theorem 3.5, then one receives Theorem 8 constructed by Toplu et al. in [32]. In particular, if we consider to take $\eta = 0$ and $n = 1$, we capture Theorem 8 presented by İşcan in [14].

Theorem 3.6. *Suppose that the mapping $\eta_1\colon \Lambda \times \Lambda \times [0,1] \to \mathbb{R}^+$ and the mapping $\eta_2 : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^+$ are two continuous mappings together with $\eta_1(\gamma,\varrho,\nu) \le \eta_2(\zeta_1,\zeta_2,\nu)$, and the function $\psi\colon \Lambda \subseteq \mathbb{R}^n \to \mathbb{R}^+$ is sub-η_1-n-polynomial convex on Λ with regard to η_1. Then for any $\gamma,\varrho \in \Lambda$ and $\zeta_1,\zeta_2 \in [0,1]$ with $\zeta_1 < \zeta_2$, the subsequent inequality*

$$\begin{aligned}
&\left|\frac{1}{2}\int_0^{\zeta_1} \psi(s\gamma+(1-s)\varrho)\mathrm{d}s + \frac{1}{2}\int_0^{\zeta_2} \psi(s\gamma+(1-s)\varrho)\mathrm{d}s \right.\\
&\left. -\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\left(\int_0^\theta \psi(s\gamma+(1-s)\varrho)\mathrm{d}s\right)\mathrm{d}\theta\right|\\
&\le \frac{\zeta_2-\zeta_1}{2n}\left[\sum_{\kappa=1}^n\left(\frac{(\kappa^2+\kappa+2)2^\kappa-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}\right)\big(\psi(\zeta_1\gamma+(1-\zeta_1)\varrho)\right.\\
&\left. +\psi(\zeta_2\gamma+(1-\zeta_2)\varrho)\big) + \frac{n}{2}\eta_2(\zeta_1,\zeta_2,\xi_0)\right]
\end{aligned} \tag{23}$$

holds true for certain fixed $\xi_0 \in (0,1)$.

Proof. Assume that $\gamma,\varrho \in \Lambda$ and $\zeta_1,\zeta_2 \in [0,1]$ with $\zeta_1 < \zeta_2$. Since ψ is a sub-η_1-n-polynomial convex function, by Theorem 2.2, it yields that the function

$$\Phi : [0,1] \to \mathbb{R},\ \Phi(\nu) = \psi(\nu\gamma+(1-\nu)\varrho)$$

is a sub-η_2-n-polynomial convex function on $[0,1]$ with regard to η_2.

Define $\Psi\colon [0,1] \to \mathbb{R}$

$$\Psi(\nu) = \int_0^\nu \Phi(s)\mathrm{d}s = \int_0^\nu \psi(s\gamma+(1-s)\varrho)\mathrm{d}s.$$

Evidently, $\Psi'(\nu) = \Phi(\nu)$ for $\forall\ \nu \in (0,1)$.

Owing to $\psi(\Lambda) \subseteq \mathbb{R}^+$, it shows that $\Phi \ge 0$ on $[0,1]$. Thus, $\Psi' \ge 0$ on $[0,1]$. If one employs Theorem 3.3 to the function Ψ, then one knows that

$$\begin{aligned}
&\left|\frac{\Psi(\zeta_1)+\Psi(\zeta_2)}{2} - \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\Psi(\theta)\mathrm{d}\theta\right|\\
&\le \frac{\zeta_2-\zeta_1}{2n}\left(\sum_{\kappa=1}^n\left[\frac{(\kappa^2+\kappa+2)2^\kappa-2}{(\kappa+1)(\kappa+2)2^{\kappa+1}}\right]\big(|\Psi'(\zeta_1)|+|\Psi'(\zeta_2)|\big) + \frac{n}{2}\eta_2(\zeta_1,\zeta_2,\xi_0)\right),
\end{aligned}$$

and we conclude that the desired outcome (23) holds true. □

Theorem 3.7. *Suppose that the function $\psi\colon\ \Omega\subseteq\mathbb{R}\to\mathbb{R}$ has differentiable sub-η_1-n-polynomial convexity on Ω° regarding continuous mapping $\eta_1\colon \Omega\times\Omega\times[0,1]\to\mathbb{R}^+$, $\zeta_1,\zeta_2\in\Omega^\circ$ with $\zeta_1<\zeta_2$, and its derivative $\psi'\colon [\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}]\to\mathbb{R}$ is a continuous function on $[\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}]$. For $q\geq 1$, if the function $|\psi'|^q$ is sub-η_2-n-polynomial convex on $[\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}]$ regarding continuous mapping η_2: $[\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}]\times[\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}]\times[0,1]\to\mathbb{R}^+$, then the successive inequality*

$$\begin{aligned}
&\left|\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right.\right.\\
&\left.\left.-\eta_1\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right)\right]-\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right|\leq(\zeta_2-\zeta_1)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\frac{|\psi'(\frac{3\zeta_1-\zeta_2}{2})|^q}{n}K_1+\frac{|\psi'(\frac{3\zeta_2-\zeta_1}{2})|^q}{n}K_2+\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_2\right)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{|\psi'(\frac{3\zeta_1-\zeta_2}{2})|^q}{n}K_2+\frac{|\psi'(\frac{3\zeta_2-\zeta_1}{2})|^q}{n}K_1+\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_3\right)\right)^{\frac{1}{q}}\right]
\end{aligned}\tag{24}$$

holds for certain fixed $\xi_1\in(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2})$, $\xi_2\in(0,\frac{1}{2})$, and $\xi_3\in(\frac{1}{2},1)$, where

$$K_1=\sum_{\kappa=1}^{n}\left[\frac{1}{8}+\frac{\kappa+3-2^{\kappa+2}}{(\kappa+1)(\kappa+2)2^{\kappa+2}}\right],$$

and

$$K_2=\sum_{\kappa=1}^{n}\left[\frac{1}{8}-\frac{1}{(\kappa+2)2^{\kappa+2}}\right].$$

Proof. Making use of inequality (18), we know that

$$\begin{aligned}
&\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\\
&\leq\left(\frac{n+2^{-n}-1}{n}\right)\left[\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}\psi(\nu)\mathrm{d}\nu\right]\\
&+\eta_1\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right).
\end{aligned}\tag{25}$$

Taking advantage of Lemma 3.2, we derive that

$$
\begin{aligned}
&\frac{1}{2(\zeta_2-\zeta_1)}\int_{\frac{3\zeta_1-\zeta_2}{2}}^{\frac{3\zeta_2-\zeta_1}{2}}\psi(\nu)\mathrm{d}\nu \\
&=\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)+2(\zeta_2-\zeta_1)\left[\int_0^{\frac{1}{2}}\nu\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\mathrm{d}\nu\right. \\
&\left.+\int_{\frac{1}{2}}^{1}(\nu-1)\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\mathrm{d}\nu\right].
\end{aligned}
\tag{26}
$$

By putting (26) into (25), and by virtue of the properties of modulus, it yields that

$$
\begin{aligned}
&\left|\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\left[\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\psi(\gamma)\mathrm{d}\gamma\right.\right. \\
&\left.\left.-\eta_1\left(\xi_1,\frac{\zeta_1+\zeta_2}{2},\frac{1}{2}\right)\right]-\psi\left(\frac{\zeta_1+\zeta_2}{2}\right)\right| \\
&\leq(\zeta_2-\zeta_1)\left[\int_0^{\frac{1}{2}}\nu\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\right. \\
&\left.+\int_{\frac{1}{2}}^{1}(1-\nu)\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\right].
\end{aligned}
\tag{27}
$$

Let us take into account the coming two cases. Suppose that $q=1$. We observe that

$$
\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)=\psi'\left(\nu\left(\frac{3\zeta_1-\zeta_2}{2}\right)+(1-\nu)\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right).
$$

Since the function $|\psi'|$ is a sub-η_2-n-polynomial convex on $[\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2}]$, we know that for any $\nu\in[0,1]$

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\nu\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu \\
&\leq\frac{\left|\psi'\left(\frac{3\zeta_1-\zeta_2}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_0^{\frac{1}{2}}\nu\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu \\
&+\frac{\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_0^{\frac{1}{2}}\nu\big[1-\nu^{\kappa}\big]\mathrm{d}\nu \\
&+\int_0^{\frac{1}{2}}\nu\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu.
\end{aligned}
\tag{28}
$$

Similarly, it follows that

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}(1-\nu)\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\\
&\leq \frac{\left|\psi'\left(\frac{3\zeta_1-\zeta_2}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)\big[1-(1-\nu)^{\kappa}\big]\mathrm{d}\nu\\
&+\frac{\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right|}{n}\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)\big[1-\nu^{\kappa}\big]\mathrm{d}\nu\\
&+\int_{\frac{1}{2}}^{1}(1-\nu)\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu.
\end{aligned}\tag{29}$$

According to the mean value theorem of generalized integrals, we derive that

$$\int_0^{\frac{1}{2}}\nu\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu=\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_2\right),\xi_2\in\left(0,\frac{1}{2}\right),$$

and

$$\int_{\frac{1}{2}}^{1}(1-\nu)\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\nu\right)\mathrm{d}\nu=\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_3\right),\xi_3\in\left(\frac{1}{2},1\right).$$

Direct computation yields that

$$\sum_{\kappa=1}^{n}\int_0^{\frac{1}{2}}\nu[1-(1-\nu)^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)[1-\nu^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\left[\frac{1}{8}+\frac{\kappa+3-2^{\kappa+2}}{(\kappa+1)(\kappa+2)2^{\kappa+2}}\right],$$

and

$$\sum_{\kappa=1}^{n}\int_0^{\frac{1}{2}}\nu[1-\nu^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\int_{\frac{1}{2}}^{1}(1-\nu)[1-(1-\nu)^{\kappa}]\mathrm{d}\nu=\sum_{\kappa=1}^{n}\left[\frac{1}{8}-\frac{1}{(\kappa+2)2^{\kappa+2}}\right].$$

Consequently, this concludes the proof for this case.

Assume that $q>1$. On account of the power–mean inequality, as well as the sub-η_2-n-polynomial convexity of $|\psi'|^q$, we deduce that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\nu\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right|\mathrm{d}\nu\\
&=\int_0^{\frac{1}{2}}\nu\left|\psi'\left(\nu\left(\frac{3\zeta_1-\zeta_2}{2}\right)+(1-\nu)\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right)\right|\mathrm{d}\nu\\
&\leq\left(\int_0^{\frac{1}{2}}\nu\mathrm{d}\nu\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\nu\left|\psi'\left(\nu\left(\frac{3\zeta_1-\zeta_2}{2}\right)+(1-\nu)\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right)\right|^q\mathrm{d}\nu\right)^{\frac{1}{q}}\\
&\leq\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\frac{|\psi'(\frac{3\zeta_1-\zeta_2}{2})|^q}{n}K_1+\frac{|\psi'(\frac{3\zeta_2-\zeta_1}{2})|^q}{n}K_2\right.\\
&\left.+\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_2\right)\right)^{\frac{1}{q}}.
\end{aligned}\tag{30}$$

In the same way, it yields that

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} \nu \left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}+2(\zeta_1-\zeta_2)\nu\right)\right| \mathrm{d}\nu \\
&\leq \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left(\frac{\left|\psi'\left(\frac{3\zeta_1-\zeta_2}{2}\right)\right|^q}{n}K_2+\frac{\left|\psi'\left(\frac{3\zeta_2-\zeta_1}{2}\right)\right|^q}{n}K_1 \right. \\
&\quad \left. +\frac{1}{8}\eta_2\left(\frac{3\zeta_1-\zeta_2}{2},\frac{3\zeta_2-\zeta_1}{2},\xi_3\right)\right)^{\frac{1}{q}}.
\end{aligned} \tag{31}$$

Employing (30) and (31) in (27), one achieves the desired outcome (24), which concludes the proof. □

Remark 3.6. Under the same assumptions considered in Theorem 3.7 with $\eta_1 = \eta_2 = 0$ and $n = 1$, we successfully gain Theorem 1 presented by Mehrez in [25].

4. Applications

In order to identify the applications of the outcomes derived in the study, the unconstraint nonlinear programming is considered as below:

$$(P) \qquad \min\left\{\psi(\gamma) | \gamma \in \Lambda \subset \mathbb{R}^n\right\}, \tag{32}$$

where $\psi : \Lambda \to \mathbb{R}$ is a differentiable sub-η-n-polynomial convex function on Λ.

Theorem 4.1. *Assume that the function ψ: $\Lambda \to \mathbb{R}$ has differentiable sub-η-n-polynomial convexity with regard to the mapping $\eta : \Lambda \times \Lambda \times [0, 1] \to \mathbb{R}$. If $\gamma^* \in \Lambda$ and the successive condition*

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \geq \frac{n-1}{n}\psi(\gamma^*)+\frac{n-1}{2}\psi(\gamma), \tag{33}$$

holds true for each $\gamma \in \Lambda$, $\nu \in [0, 1]$, then γ^* is the optimal solution of ψ on Λ.

Proof. For any $\gamma \in \Lambda$, by Theorem 2.4, we find that

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \leq \frac{n+1}{2}\psi(\gamma)-\frac{1}{n}\psi(\gamma^*).$$

In combination with the condition (33), it readily yields that

$$\frac{n+1}{2}\psi(\gamma)-\frac{1}{n}\psi(\gamma^*) \geq \frac{n-1}{n}\psi(\gamma^*)+\frac{n-1}{2}\psi(\gamma),$$

i.e., $\psi(\gamma) - \psi(\gamma^*) \geq 0$. Therefore, γ^* is an optimal solution of ψ on Λ. This concludes the proof. □

Remark 4.1. If one considers to pick up $n = 1$, in Theorem 4.1, then one successfully receives Theorem 2.1 deduced by Chao et al. in [8].

Now, let us apply the outcomes investigated in this study to the nonlinear programming along with the subsequent inequality constraints:

$$(P_g) \quad \begin{array}{ll} \min & \psi(\gamma) \\ s.t. & \omega_i(\gamma) \leq 0, \ i \in U = \{1, 2, \ldots, m\}, \\ & \gamma \in \mathbb{R}^n, \end{array} \tag{34}$$

where ψ and ω_i are all differentiable defined on the set $D = \{\gamma \in \mathbb{R}^n | \omega_i(\gamma) \leq 0, \ i \in U\}$, which is assumed to be a nonempty feasible set of (P_g). In addition, for $\gamma^* \in D$, we define $U^* = \{\gamma \in \mathbb{R}^n | \omega_i(\gamma^*) = 0, \ i \in U\}$, $\lambda_i = (\lambda_1, \ldots, \lambda_m)^T$.

The successive theorem displays the Karush–Kuhn–Tucker (KKT) sufficient conditions.

Theorem 4.2. (KKT sufficient conditions) *Assume that $\psi(\gamma) : \mathbb{R}^n \to \mathbb{R}$ is a differentiable and sub-η-n-polynomial convex function with regard to the mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}$, and the functions $\omega_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in U)$ are a series of differentiable sub-η-n-polynomial convex with regard to the mappings $\eta_i : \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \to \mathbb{R}$ $(i \in U)$. Assume that $\gamma^* \in D$ is a KKT point regarding (P_g), that is, there exist multipliers $\lambda_i \geq 0$ $(i \in U)$ satisfying that*

$$\begin{aligned} &\nabla\psi(\gamma^*) + \sum_{i \in U} \lambda_i \nabla\omega_i(\gamma^*) = 0, \\ &\lambda_i \omega_i(\gamma^*) = 0. \end{aligned} \tag{35}$$

If the subsequent condition

$$\begin{aligned} &\frac{n-1}{n}\psi(\gamma^*) + \frac{n-1}{2}\psi(\gamma) + \lim_{\nu \to 0^+} \frac{\eta(\gamma, \gamma^*, \nu)}{\nu} \\ &\quad \leq -\sum_{i \in U} \lambda_i \lim_{\nu \to 0^+} \frac{\eta_i(\gamma, \gamma^*, \nu)}{\nu}, \forall \ \gamma \in D, \end{aligned} \tag{36}$$

also holds true, then γ^* is an optimal solution regarding the problem (P_g).

Proof. For each $\gamma \in D$, one observes that

$$\omega_i(\gamma) \leq 0 = \omega_i(\gamma^*), \ i \in U^* = \{i \in U | \omega_i(\gamma^*) = 0\}.$$

Making use of the sub-η-n-polynomial convexity of ω_i and Theorem 2.4, for $i \in U^*$, we find that

$$\nabla\omega_i(\gamma^*)^T(\gamma - \gamma^*) - \lim_{\nu \to 0^+} \frac{\eta_i(\gamma, \gamma^*, \nu)}{\nu} \leq \frac{n+1}{2}\omega_i(\gamma) - \frac{1}{n}\omega_i(\gamma^*) \leq 0. \tag{37}$$

According to the conditions (35), we know that

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) = -\sum_{i\in U}\lambda_i\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*) = -\sum_{i\in U^*}\lambda_i\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*). \tag{38}$$

By virtue of the inequality (36), we can figure out that

$$\begin{aligned}&\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \frac{n-1}{n}\psi(\gamma^*) - \frac{n-1}{2}\psi(\gamma) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu}\\ &\geq -\sum_{i\in U^*}\lambda_i\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*) + \sum_{i\in U^*}\lambda_i\lim_{\nu\to 0^+}\frac{\eta_i(\gamma,\gamma^*,\nu)}{\nu}\\ &\geq -\sum_{i\in U^*}\lambda_i\left[\nabla\omega_i(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+}\frac{\eta_i(\gamma,\gamma^*,\nu)}{\nu}\right].\end{aligned} \tag{39}$$

Here, we use (37) and (39) to derive the coming inequality

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \frac{n-1}{n}\psi(\gamma^*) - \frac{n-1}{2}\psi(\gamma) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \geq 0,$$

that is,

$$\nabla\psi(\gamma^*)^T(\gamma-\gamma^*) - \lim_{\nu\to 0^+}\frac{\eta(\gamma,\gamma^*,\nu)}{\nu} \geq \frac{n-1}{n}\psi(\gamma^*) + \frac{n-1}{2}\psi(\gamma),$$

and in accordance with Theorem 4.1 , it yields that

$$\psi(\gamma) \geq \psi(\gamma^*), \forall\ \gamma \in D.$$

Therefore, γ^* is an optimal solution regarding the problem (P_g). This concludes the proof. □

5. Conclusions

Sub-η-n-polynomial convexity, as well as sub-η-n-polynomial convex sets, are introduced in the present paper. Because of their significance, a series of interesting properties for newly defined functions and sets are discussed, respectively. Certain Hermite–Hadamard-type integral inequalities, in connection with sub-η-n-polynomial convex functions, are also presented. We conclude the article by showing that the derived inequalities also hold for convex functions and n-polynomial convex functions. As applications, under the sub-η-n-polynomial convexity, the KKT sufficient optimality conditions, under the sub-η-n-polynomial convex programming with unconstrained and constrained inequalities, are deduced in the present paper, respectively. We have reason to confirm that it is an interesting and innovative problem, for forthcoming researchers who will enable them to establish analogous integral inequalities for other diverse types of sub-η-convexity, and corresponding KKT optimality conditions for the generalized sub-η-convex programming in their future work.

Acknowledgements

The authors would like to thank the reviewer for his/her valuable comments and suggestions.

References

[1] S. Abramovich, L. E. Persson, *Fejér and Hermite-Hadamard type inequalities for N-quasiconvex functions,* Math. Notes, 102 (2017), 599-609.

[2] I. Ahmad, A. Jayswal, B. Kumari, *Characterizations of geodesic sub-b-s-convex functions on Riemannian manifolds,* J. Nonlinear Sci. Appl., 11 (2018), 189-197.

[3] B. Ahmad, A. Alsaedi, M. Kirane, B. T. Torebek, *Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals,* J. Comput. Appl. Math., 353 (2019), 120-129.

[4] M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, *On γ-preinvex functions,* Filomat, 34 (2020), 4137-4159.

[5] C. R. Bector, C. Singh, *B-vex functions,* J. Optim. Theory Appl., 71 (1991), 237-253.

[6] S. I. Butt, S. Rashid, M. Tariq, M. K. Wang, *Novel refinements via n-polynomial harmonically s-type convex functions and application in special functions,* J. Funct. Spaces, 2021 (2021) Article ID 6615948, 17 pages.

[7] S. I. Butt, H. Budak, M. Tariq, M. Nadeem, *Integral inequalities for n-polynomial s-type preinvex functions with applications,* Math. Methods Appl. Sci., 44 (2021), 11006-11021.

[8] M. T. Chao, J. B. Jian, D. Y. Liang, *Sub-b-convex functions and sub-b-convex programming,* J. Operations Research Transactions (China), 16 (2012), 1-8.

[9] M. R. Delavar, M. De La Sen, *A mapping associated to h-convex version of the Hermite-Hadamard inequality with applications,* J. Math. Inequal., 14 (2020), 329-335.

[10] Y. P. Deng, M. U. Awan, S. Talib, M. A. Noor, K. I. Noor, P. O. Mohammed, S. H. Wu, *Inequalities for estimations of integrals related to higher-order strongly n-polynomial preinvex functions,* Journal of Mathematics, 2020 (2020), Article ID 8841356, 12 pages.

[11] T. S. Du, H. Wang, M. A. Khan, Y. Zhang, *Certain integral inequalities considering generalized m-convexity on fractal sets and their applications,* Fractals, 27 (2019), Article ID 1950117, 17 pages.

[12] S. S. Dragomir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,* Appl. Math. Lett., 11 (1998), 91-95.

[13] S. Dragomir, M. Jleli, B. Samet, *Generalized convexity and integral inequalities,* Math. Methods Appl. Sci., 2020, 1-15, Online: https://doi.org/10.1002/mma.6293.

[14] İ. İşcan, *New refinements for integral and sum forms of Hölder inequality,* J. Inequal. Appl., 2019 (2019) Article ID 304, 11 pages.

[15] B. C. Joshi, *Generalized convexity and mathematical programs,* Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 82 (2020), 151-160.

[16] M. Kadakal, İ. İşcan, P. Agarwal, M. Jleli, *Exponential trigonometric convex functions and Hermite-Hadamard type inequalities,* Math. Slovaca, 71 (2021), 43-56.

[17] A. Kashuri, T. M. Rassias, *Fractional trapezium-type inequalities for strongly exponentially generalized preinvex functions with applications,* Appl. Anal. Discrete Math., 14 (2020), 560-578.

[18] A. Kiliçman, W. Saleh, *Generalized preinvex functions and their applications,* Symmetry, 2018 (2018), Article ID 493, 13 pages.

[19] U. S. Kirmaci, M. E. Özdemir, *Some inequalities for mappings whose derivatives are bounded and applications to special means of real numbers,* Appl. Math. Lett., 17 (2004), 641-645.

[20] M. A. Latif, *On some new inequalities of Hermite-Hadamard type for functions whose derivatives are s-convex in the second sense in the absolute value,* Ukrainian Math. J., 67 (2016), 1552-1571.

[21] J. G. Liao, T. S. Du, *On some characterizations of sub-b-s-convex functions,* Filomat, 30 (2016), 3885-3895.

[22] J. G. Liao, T. S. Du, *Optimality conditions in sub-(b, m)-convex programming,* Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 79 (2017), 95-106.

[23] J. G. Liao, S. H. Wu, T. S. Du, *The Sugeno integral with respect to α-preinvex functions,* Fuzzy Sets and Systems, 379 (2020), 102-114.

[24] X. J. Long, J. W. Peng, *Semi-B-preinvex functions,* J. Optim. Theory Appl., 131 (2006), 301-305.

[25] K. Mehrez, P. Agarwal, *New Hermite-Hadamard type integral inequalities for convex functions and their applications,* J. Comput. Appl. Math., 350 (2019), 274-285.

[26] P. O. Mohammed, *Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function,* Math. Meth. Appl. Sci., 44 (2021), 2314-2324.

[27] E. R. Nwaeze, M. A. Khan, A. Ahmadian, M. N. Ahmad, A. K. Mahmood, *Fractional inequalities of the Hermite-Hadamard type for m-polynomial convex and harmonically convex functions,* AIMS Mathematics, 6 (2020), 1889-1904.

[28] S. Sezer, *The Hermite-Hadamard inequality for s-convex functions in the third sense,* AIMS Mathematics, 6 (2021), 7719-7732.

[29] S. Sezer, Z. Eken, G. Tınaztepe, G. Adilov, *p-convex functions and some of their properties*, Numer. Funct. Anal. Optim., 42 (2021), 443-459.

[30] W. B. Sun, Q. Lin, *New Hermite-Hadamard type inequalities for* (α, m)-*convex functions and applications to special means,* J. Math. Inequal., 11 (2017), 383-397.

[31] S. Suzuki, *Optimality conditions and constraint qualifications for quasiconvex programming,* J. Optim. Theory Appl., 183 (2019), 963-976.

[32] T. Toplu, M. Kadakal, İ. İşcan, *On n-polynomial convexity and some related inequlities,* AIMS Mathematics, 5 (2020), 1304-1318.

[33] B. Yu, J. G. Liao, T. S. Du, *Optimality and duality with respect to b-(ε, m)-convex programming,* Symmetry, 2018 (2018), Article ID 774, 16 pages.

[34] S. H. Wu, M. U. Awan, *Estimates of upper bound for a function associated with Riemann-Liouville fractional integral via h-convex functions,* J. Funct. Spaces, 2019 (2019), Article ID 9861790, 7 pages.

Accepted: June 27, 2022

A weighted power distribution mechanism under transferable-utility systems: axiomatic results and dynamic processes

Yu-Hsien Liao
Department of Applied Mathematics
National Pingtung University
Taiwan
twincos@ms25.hinet.net

Abstract. By applying the notion of the efficient Banzhaf value, any additional fixed utility should be distributed equally among the players who are concerned. However, in several applications, this notion seems unrealistic for the situation being modeled. Therefore, we adopt weights to introduce a modification of the efficient Banzhaf value, which we name the weighted Banzhaf value. To present the rationality, we adopt some reasonable properties to characterize this weighted value. Based on different viewpoints, we further define excess functions to propose alternative formulations and related dynamic processes for this weighted value.

Keywords: the weighted Banzhaf value, excess function, dynamic process.

1. Introduction

In the framework of transferable-utility (TU) games, the power indices have been defined to measure the political power of each member of a voting system. A member in a voting system can be a party in a parliament or a country in a confederation. Each member will have a certain number of votes, and so their power indices will differ. The power index results may be found in Algaba et al. [1], Alonso et al. [2], Alonso and Fiestras [3], van den Brink and van der Laan [5], Dubey and Shapley [7], Haller [8], Lehrer [12], Ruiz [18], etc. Banzhaf [4] defined a power index in the framework of voting games that was essentially identical to that given by Coleman [6], and later extended it to arbitrary games by Owen [15, 16], who introduced two formulas. The Banzhaf value defined by Banzhaf [4] does not necessarily distribute the entire utility over all players in a grand coalition. Therefore, the *efficient Banzhaf value* and related results were proposed by Hwang and Liao [11] and Liao et al. [13], respectively.

In real-world situations, players might represent constituencies of different sizes or have different bargaining abilities. In addition, a lack of symmetry may arise when different bargaining abilities for different players are modeled. In various applications of TU games, it seems to be natural to assume that the players are given some *a priori measures of importance*, called *weights*. The study of weighted Banzhaf values was introduced by Radzik et al. [17]. Considering that there are exogenously given some positive weights between players, Radzik et al. [17] proposed an axiomatization of weighted Banzhaf values for a given vector of positive weights of players. Further, the family of all possible weighted Banzhaf values is described axiomatically. However, these weighted Banzhaf values introduced by Radzik et al. [17] are not efficient.

Based on the notion of the efficient Banzhaf value due to Hwang and Liao [11], all players first receive their marginal contributions from all coalitions in which they have participated; the remaining utilities are allocated equally. That is, any additional fixed utility (e.g., the cost of a common facility) is distributed equally among the players who are concerned. However, in several applications, the efficient Banzhaf value seems unrealistic for the situation being modeled. Therefore, we desire that any additional fixed utility could be distributed among players in proportion to their weights.

To modify relative discrimination among players under various situations, we adopt weights to propose different results as follows.

1. In Section 2, we adopt weights to propose the *weighted Banzhaf value*. Further, we present an alternative formulation of the weighted Banzhaf value in terms of *excess functions*. The excess of a coalition could be treated as the *variation* between the productivity and total payoff of the coalition.

2. In Section 3, we adopt the *efficiency-sum-reduced game* to characterize the weighted Banzhaf value. In Section 4, we propose dynamic processes to illustrate that the weighted Banzhaf value can be approached by players who start from an arbitrary efficient payoff vector. In Section 5, more discussions and interpretations are presented in detail.

2. The weighted Banzhaf value

A coalitional game with transferable-utility (TU game) is a pair (N, v) where N is the grand coalition and v is a mapping such that $v : 2^N \longrightarrow I\!R$ and $v(\emptyset) = 0$. Denote the class of all TU games by G. A solution on G is a function ψ which associates with each game $(N, v) \in G$ an element $\psi(N, v)$ of $I\!R^N$.

Definition 2.1. *The efficient Banzhaf value* (*Hwang and Liao* [*10*]), $\overline{\eta}$, *is the solution on* G *which associates with* $(N, v) \in G$ *and each player* $i \in N$ *the value*

$$\overline{\eta_i}(N, v) = \eta_i(N, v) + \frac{1}{|N|} \cdot \big[v(N) - \sum_{k \in N} \eta_k(N, v)\big], \tag{1}$$

where $\eta_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \big[v(S) - v(S \setminus \{i\})\big]$ *is the Banzhaf value* (*Owen* [*15, 16*]) *of* i. *It is known that the Banzhaf value violates EFF, and the efficient Banzhaf value satisfies EFF.*

Let $(N, v) \in G$. A function $w : N \to \mathbb{R}^+$ is called a weight function if w is a non-negative function. In different situations, players in N could be assigned different weights by weight functions. These weights could be interpreted as *a-priori measures of importance*; they are taken to reflect considerations not captured by the characteristic function. For example, we may be dealing with a problem of cost allocation among investment projects. Then the weights could be associated to the profitability of the different projects. In a problem of allocating travel costs among various institutions visited (cf. Shapley [20]), the weights may be the number of days spent at each one.

Given $(N, v) \in G$ and a weight function w, we define $|S|_w = \sum_{i\in S} w(i)$, for all $S \subseteq N$. The weighted Banzhaf value is defined as follows.

Definition 2.2. *Let w be a weight function. The weighted Banzhaf value $\overline{\eta^w}$, is the solution on G which associates with $(N, v) \in G$ and all players $i \in N$ the value*

$$\overline{\eta^w_i}(N, v) = \eta_i(N, v) + \frac{w(i)}{|N|_w} \cdot \big[v(N) - \sum_{k\in N} \eta_k(N, v)\big]. \tag{2}$$

By the definition of $\overline{\eta^w}$, all players firstly receive their marginal contributions from all coalitions, and further allocate the remaining utilities proportionally by applying weights.

Here, we provide a brief application of TU games and the weighted Banzhaf value in the setting of "utility distribution for management systems," such as Microsoft and NBA. In an organization, each department may consider management operation strategies. Besides competing in merchandising, all departments, such as the research department, purchasing department, and logistics department, should develop to increase the utility of the entire organization. Such a utility distribution problem could be formulated as follows. Let $N = \{1, 2, \ldots, n\}$ be a collection of all departments of an organization that could be provided jointly by some coalitions $S \subseteq N$ and let $v(S)$ be the profit of providing the cooperative coalition $S \subseteq N$ jointly. Each coalition $S \subseteq N$ could be formed by considering a specific operational aim. The function v could be treated as a utility function that assigns to each cooperative coalition $S \subseteq N$ the worth that the coalition S can obtain. Modeled in this notion, the utility distribution management system of an organization could be considered a cooperative TU game, with v being its characteristic function. However, as mentioned in the Introduction, it may be inappropriate in many situations if any additional fixed utility should be distributed equally among the players who are concerned. Thus, it is reasonable that weights are assigned to players and any fixed utility should be divided according to these weights. In the following sections, some more results will be proposed to show that the weighted Banzhaf value could be applied in the setting of utility distribution.

A solution ψ satisfies efficiency (EFF) if $\sum_{i\in N}\psi_i(N,v) = v(N)$, for all $(N,v) \in G$. Property EFF asserts that all players distribute all the utility completely.

Lemma 2.1. *The weighted Banzhaf value $\overline{\eta^w}$ satisfies EFF.*

Proof of Lemma 2.1. Let $(N,v) \in G$. By Definition 2.2,

$$\begin{aligned}\sum_{i\in N}\overline{\eta_i^w}(N,v) &= \sum_{i\in N}\eta_i(N,v) + \sum_{i\in N}\frac{w(i)}{|N|_w}\cdot\big[v(N) - \sum_{k\in N}\eta_k(N,v)\big]\\ &= \sum_{i\in N}\eta_i(N,v) + \frac{|N|_w}{|N|_w}\cdot\big[v(N) - \sum_{k\in N}\eta_k(N,v)\big]\\ &= v(N).\end{aligned}$$

Hence, the weighted Banzhaf value $\overline{\eta^w}$ satisfies EFF.

Next, we present an alternative formulation for the weighted Banzhaf value in terms of *excess functions*. If $x \in \mathbb{R}^N$ and $S \subseteq N$, write x_S for the restriction of x to S and write $x(S) = \sum_{i\in S} x_i$. Denote that $X(N,v) = \{x \in I\!R^N : \sum_{i\in N} x_i = v(N)\}$, for all $(N,v) \in G$. The excess of a coalition $S \subseteq N$ at x is the real number $e(S,v,x) = v(S) - x(S)$.

Lemma 2.2. *Let $(N,v) \in G$, $x \in X(N,v)$ and w be a weight function. Then*

$$\begin{aligned}&w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\\ &= w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]\ \forall i,j\in N\\ &\Longleftrightarrow x = \overline{\eta^w}(N,v).\end{aligned}$$

Proof of Lemma 2.2. Let $(N,v) \in G$, $x \in X(N,v)$ and w be a weight function. For all $i,j \in N$,

$$\begin{aligned}&w(j)\sum_{S\subseteq N\setminus\{i\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\\ &= w(i)\sum_{S\subseteq N\setminus\{j\}}\big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big]\\ &\Longleftrightarrow w(j)\sum_{S\subseteq N\setminus\{i\}}\big[v(S) - \frac{x}{2^{|N|-1}}(S) + \frac{x}{2^{|N|-1}}(S\cup\{i\}) - v(S\cup\{i\})\big]\\ &= w(i)\sum_{S\subseteq N\setminus\{j\}}\big[v(S) - \frac{x}{2^{|N|-1}}(S) + \frac{x}{2^{|N|-1}}(S\cup\{j\}) - v(S\cup\{j\})\big]\end{aligned}$$

$$\Longleftrightarrow w(j)\sum_{S\subseteq N\setminus\{i\}}\big[\frac{x_i}{2^{|N|-1}} - v(S\cup\{i\}) + v(S)\big] \tag{3}$$

$$= w(i) \sum_{S \subseteq N \setminus \{j\}} \Big[\frac{x_j}{2^{|N|-1}} - v(S \cup \{j\}) + v(S)\Big]$$
$$\Longleftrightarrow w(j)\big[x_i - \sum_{S \subseteq N \setminus \{i\}} [v(S \cup \{i\}) - v(S)]\big]$$
$$= w(i)\big[x_j - \sum_{S \subseteq N \setminus \{j\}} [v(S \cup \{j\}) - v(S)]\big]$$
$$\Longleftrightarrow w(j) \cdot \big[x_i - \eta_i(N,v)\big] = w(i) \cdot \big[x_j - \eta_j(N,v)\big].$$

By Definition 2.2,

$$w(j) \cdot \big[\overline{\eta_i^w}(N,v) - \eta_i(N,v)\big] = w(i) \cdot \big[\overline{\eta_j^w}(N,v) - \eta_j(N,v)\big]. \tag{4}$$

By equations (3) and (4),

$$\big[x_i - \overline{\eta_i^w}(N,v)\big] \sum_{j \in N} w(j) = w(i) \sum_{j \in N} \big[x_j - \overline{\eta_j^w}(N,v)\big].$$

Since $x \in X(N,v)$ and $\overline{\eta^w}$ satisfies EFF,

$$\big[x_i - \overline{\eta_i^w}(N,v)\big] \cdot |N|_w = w(i) \cdot \big[v(N) - v(N)\big] = 0.$$

Therefore, $x_i = \overline{\eta_i^w}(N,v)$, for all $i \in N$.

3. Axiomatic results

In this section, we adopt reductions and excess functions to introduce some axiomatic results and dynamic processes of the weighted Banzhaf value.

Subsequently, we adopt the efficiency-average-reduced game to characterize the weighted Banzhaf value.

Definition 3.1 (Liao et al. [13]). *Let $(N,v) \in G$, $S \subseteq N$ and ψ be a solution. The efficiency-sum-reduced game $(S, v_{S,\psi})$ with respect to ψ and S is defined by*

$$v_{S,\psi}(T) = \begin{cases} 0, & T = \emptyset, \\ v(N) - \sum_{i \in N \setminus S} \psi_i(N,v), & T = S, \\ \sum_{Q \subseteq N \setminus S} \big[v(T \cup Q) - \sum_{i \in Q} \psi_i(N,v)\big], & T \subsetneq S. \end{cases}$$

The efficiency-sum-reduction asserts that given a proposed payoff vector $\psi(N,v)$, the worth of a coalition T in $(S, v_{S,\psi})$ is computed under the assumption that T can secure the cooperation of any subgroup Q of $N \setminus S$, provided each member of Q receives his component of $\psi(N,v)$. After these payments are made, what remains for T is the value $v(T \cup Q) - \sum_{i \in Q} \psi_i(N,v)$. Summing behavior on the part of T involves finding the sum of the values $v(T \cup Q) - \sum_{i \in Q} \psi_i(N,v)$, for all $Q \subseteq N \setminus S$. A solution ψ satisfies bilateral efficiency-sum-consistency (BESCON) if $\psi_i(S, v_{S,\psi}) = \psi_i(N,v)$, for all $(N,v) \in G$ with $|N| \geq 2$, for all $S \subseteq N$ with $|S| = 2$ and, for all $i \in S$.

Lemma 3.1. *The weighted Banzhaf value* $\overline{\eta^w}$ *satisfies BESCON.*

Proof of Lemma 3.1. Let $(N, v) \in G$, $S \subseteq N$ with $|S| = 2$ and w be a weight function. Let $x = \overline{\eta^w}(N, v)$. Suppose $S = \{i, j\}$ then

$$\begin{aligned}
&\sum_{T\subseteq S\setminus\{i\}} \big[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T\cup\{i\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\big] \\
&= \big[e(\{j\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2}) - e(S, v_{S,\overline{\eta^w}}, \frac{x_S}{2})\big] + \big[e(\emptyset, v_{S,\overline{\eta^w}}, \frac{x_S}{2}) - e(\{i\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2})\big] \\
&= \big(v_{S,\overline{\eta^w}}(\{j\}) - \frac{x_j}{2}\big) - \big(v_{S,\overline{\eta^w}}(S) - \frac{x_S}{2}(S)\big) + 0 - \big(v_{S,\overline{\eta^w}}(\{i\}) - \frac{x_i}{2}\big) \\
&= \big(v_{S,\overline{\eta^w}}(\{j\}) - \frac{x_j}{2}\big) - 0 + 0 - \big(v_{S,\overline{\eta^w}}(\{i\}) - \frac{x_i}{2}\big) \\
&= \Big(\Big[\sum_{Q\subseteq N\setminus S} \big[v(\{j\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2}\big]\Big] - \frac{x_j}{2}\Big) \\
&- \Big(\Big[\sum_{Q\subseteq N\setminus S} \big[v(\{i\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2}\big]\Big] - \frac{x_i}{2}\Big) \qquad (5) \\
&= \sum_{Q\subseteq N\setminus S} \Big(\Big[v(\{j\}\cup Q) - \frac{x_j}{2^{|N|-1}}\Big] - \Big[v(\{i\}\cup Q) - \frac{x_i}{2^{|N|-1}}\Big]\Big) \\
&= \sum_{Q\subseteq N\setminus S} \Big(\Big[v(\{j\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2^{|N|-1}} - \frac{x_j}{2^{|N|-1}}\Big] \\
&- \Big[v(\{i\}\cup Q) - \sum_{k\in Q}\frac{x_k}{2^{|N|-1}} - \frac{x_i}{2^{|N|-1}}\Big]\Big) \\
&= \sum_{Q\subseteq N\setminus S} \Big(\Big[v(\{j\}\cup Q) - \sum_{k\in\{j\}\cup Q}\frac{x_k}{2^{|N|-1}}\Big] - \Big[v(\{i\}\cup Q) - \sum_{k\in\{j\}\cup Q}\frac{x_k}{2^{|N|-1}}\Big]\Big) \\
&= \sum_{Q\subseteq N\setminus S} \Big[(e(\{j\}\cup Q, v, \frac{x}{2^{|N|-1}}) - (e(\{i\}\cup Q, v, \frac{x}{2^{|N|-1}})\Big] \\
&= \sum_{Q\subseteq N\setminus\{i,j\}} \Big[(e(\{j\}\cup Q, v, \frac{x}{2^{|N|-1}}) - (e(\{i\}\cup Q, v, \frac{x}{2^{|N|-1}})\Big] \\
&= \sum_{Q\subseteq N\setminus\{i\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q\cup\{i\}, v, \frac{x}{2^{|N|-1}})\Big].
\end{aligned}$$

Similar to equation (5),

$$\begin{aligned}
&\sum_{T\subseteq S\setminus\{j\}} \big[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T\cup\{j\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\big] \\
&= \sum_{Q\subseteq N\setminus\{j\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q\cup\{j\}, v, \frac{x}{2^{|N|-1}})\Big].
\end{aligned}$$

By EFF of $\overline{\eta^w}$ and the definition of efficiency-sum-reduced game, $x_S \in X(S, v_{S,\overline{\eta^w}})$. Therefore, by Lemma 2.2,

$$w(j) \cdot \sum_{T \subseteq S \setminus \{i\}} \left[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T \cup \{i\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\right]$$
$$= w(j) \cdot \sum_{Q \subseteq N \setminus \{i\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q \cup \{i\}, v, \frac{x}{2^{|N|-1}})\Big]$$
(by equation (5))
$$= w(i) \cdot \sum_{Q \subseteq N \setminus \{j\}} \Big[(e(Q, v, \frac{x}{2^{|N|-1}}) - (e(Q \cup \{j\}, v, \frac{x}{2^{|N|-1}})\Big]$$
(by Lemma 2.2)
$$= w(i) \cdot \sum_{T \subseteq S \setminus \{j\}} \left[e(T, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}}) - e(T \cup \{j\}, v_{S,\overline{\eta^w}}, \frac{x_S}{2^{|S|-1}})\right]$$
(similar to equation (5)).

By Lemma 2 and $x_S \in X(S, v_{S,\overline{\eta^w}})$, we have that $x_S = \overline{\eta^w}(S, v_{S,\overline{\eta^w}})$. Hence, $\overline{\eta^w}$ satisfies BESCON.

Inspired by Hart and Mas-Colell [9], we provide an axiomatic result of the weighted Banzhaf value as follows. A solution ψ satisfies weighted Banzhaf standard for games (WBSFG) if $\psi(N, v) = \overline{\eta^w}(N, v)$, for all $(N, v) \in G$ with $|N| \leq 2$. Property WBSFG is a generalization of the two-person standardness axiom of Hart and Mas-Colell [9].

Lemma 3.2. *If a solution ψ satisfies WBSFG and BESCON, then it satisfies EFF.*

Proof of Lemma 3.2. Suppose ψ satisfies WBSFG and BESCON. Let $(N, v) \in G$. If $|N| \leq 2$, then ψ satisfies EFF by BESCON of ψ. Suppose $|N| > 2$, $i, j \in N$ and $S = \{i, j\}$. Since ψ satisfies EFF in two-person games,

$$\psi_i(S, v_{S,\psi}) + \psi_j(S, v_{S,\psi}) = v_{S,\psi}(S) = v(N) - \sum_{k \neq i,j} \psi_k(N, v). \tag{6}$$

By BESCON of ψ,

$$\psi_t(S, v_{S,\psi}) = \psi_t(N, v), \quad \text{for all } t \in S. \tag{7}$$

By equations (6) and (7), $v(N) = \sum_{k \in N} \psi_k(N, v)$, i.e., ψ satisfies EFF.

Theorem 3.1. *A solution ψ satisfies WBSFG and BESCON if and only if $\psi = \overline{\eta^w}$.*

Proof of Theorem 3.1. By Lemma 3.1, $\overline{\eta^w}$ satisfies BESCON. Clearly, $\overline{\eta^w}$ satisfies WBSFG.

To prove uniqueness, suppose ψ satisfies WBSFG and BESCON. By Lemma 3.2, ψ satisfies EFF. Let $(N,v) \in G$. If $|N| \leq 2$, it is trivial that $\psi(N,v) = \overline{\eta^w}(N,v)$ by SFG. Assume that $|N| > 2$. Let $i \in N$ and $S = \{i,j\}$ for some $j \in N \setminus \{i\}$. Then

$$\begin{aligned}
&\psi_i(N,v) - \overline{\eta^w_i}(N,v) \\
&= \psi_i(S, v_{S,\psi}) - \overline{\eta^w_i}(S, v_{S,\overline{\eta^w}}) \qquad \text{(by BESCON of } \psi, \overline{\eta^w}) \\
&= \overline{\eta^w_i}(S, v_{S,\psi}) - \overline{\eta^w_i}(S, v_{S,\overline{\eta^w}}) \qquad \text{(by WBSFG of } \psi, \overline{\eta^w}) \\
&= \eta^w_i(S, v_{S,\psi}) + \frac{w(i)}{|S|_w} \cdot \Big[v_{S,\psi}(S) - \big[\eta^w_i(S, v_{S,\psi}) + \eta^w_j(S, v_{S,\psi})\big]\Big] \qquad (8) \\
&- \eta^w_i(S, v_{S,\overline{\eta^w}}) - \frac{w(i)}{|S|_w} \cdot \Big[v_{S,\overline{\eta^w}}(S) - \big[\eta^w_i(S, v_{S,\overline{\eta^w}}) + \eta^w_j(S, v_{S,\overline{\eta^w}})\big]\Big] \\
&= \Big[v_{S,\psi}(S) + v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\}) \Big] + \frac{w(i)}{|S|_w} \cdot \Big[- v_{S,\psi}(S) \Big] \\
&- \Big[v_{S,\overline{\eta^w}}(S) + v_{S,\overline{\eta^w}}(\{i\}) - v_{S,\overline{\eta^w}}(\{j\}) \Big] - \frac{w(i)}{|S|_w} \cdot \Big[- v_{S,\overline{\eta^w}}(S) \Big].
\end{aligned}$$

By definitions of $v_{S,\psi}$ and $v_{S,\overline{\eta^w}}$,

$$\begin{aligned}
v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\}) &= \sum_{Q \subseteq N \setminus S} \Big[v(\{i\} \cup Q) - v(\{j\} \cup Q) \Big] \\
&= v_{S,\overline{\eta^w}}(\{i\}) - v_{S,\overline{\eta^w}}(\{j\}). \qquad (9)
\end{aligned}$$

By equations (8) and (9),

$$\begin{aligned}
\psi_i(N,v) - \overline{\eta^w_i}(N,v) &= \Big[1 - \frac{w(i)}{|S|_w}\Big] \cdot \Big[v_{S,\psi}(S) - v_{S,\overline{\eta^w}}(S) \Big] \\
&= \frac{w(j)}{|S|_w} \cdot \Big[\psi_i(N,v) + \psi_j(N,v) - \overline{\eta^w_i}(N,v) - \overline{\eta^w_j}(N,v) \Big].
\end{aligned}$$

That is,

$$w(i) \cdot \Big[\psi_i(N,v) - \overline{\eta^w_i}(N,v) \Big] = w(j) \cdot \Big[\psi_j(N,v) - \overline{\eta^w_j}(N,v) \Big].$$

By EFF of ψ and $\overline{\eta^w}$,

$$\begin{aligned}
0 &= v(N) - v(N) \\
&= \sum_{j \in N} \Big[\psi_j(N,v) - \overline{\eta^w_j}(N,v) \Big] \\
&= w(i) \cdot \Big[\psi_i(N,v) - \overline{\eta^w_i}(N,v) \Big] \sum_{j \in N} \frac{1}{w(j)}.
\end{aligned}$$

Hence, $\psi_i(N,v) = \overline{\eta^w_i}(N,v)$, for all $i \in N$.

The following examples are to show that each of the axioms used in Theorem 3.1 is logically independent of the remaining axioms.

Example 3.1. Define a solution ψ by, for all $(N, v) \in G$ and, for all $i \in N$,

$$\psi_i(N, v) = \frac{v(N)}{|N|}.$$

Clearly, ψ satisfies BESCON, but it violates WBSFG.

Example 3.2. Define a solution ψ by for all $(N, v) \in G$ and, for all $i \in N$,

$$\psi_i(N, v) = \begin{cases} \overline{\eta_i^w}(N, v), & \text{if } |N| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, ψ satisfies WBSFG, but it violates BESCON.

4. Dynamic results

In this section, we introduce two dynamic processes of the weighted Banzhaf value by applying excess functions and reductions.

In the following, we adopt excess functions to propose a correction function and related dynamic process for the weighted Banzhaf value.

Definition 4.1. *Let $(N, v) \in G$, $i \in N$ and w be a weight function. The e-correction function $f_i^{\overline{\eta^w}} : X(N, v) \to \mathbb{R}$ is defined by*

$$f_i^{\overline{\eta^w}}(x) = x_i + t \sum_{j \in N \setminus \{i\}} \Big(w(i) \sum_{S \subseteq N \setminus \{j\}} \big[e(S, v, \frac{x}{2^{|N|-1}}) - e(S \cup \{j\}, v, \frac{x}{2^{|N|-1}}) \big]$$
$$- w(j) \sum_{S \subseteq N \setminus \{i\}} \big[e(S, v, \frac{x}{2^{|N|-1}}) - e(S \cup \{i\}, v, \frac{x}{2^{|N|-1}}) \big] \Big),$$

where $t \in (0, \infty)$, which reflects the assumption that player i does not ask for full correction (when $t = 1$) but only (usually) a fraction of it.

When a player withdraws from the coalitions he/she/it joined, some of the other players may complain. The e-correction function is based on the idea that, each agent shortens the weighted excess relating to his own and others' non-participation in all coalitions, and adopts these regulations to correct the original payoff.

The following lemma shows that the e-correction function is well-defined, i.e., the efficiency is preserved under the e-correction function.

Lemma 4.1. *Let $(N, v) \in G$, w be a weight function and $f^{\overline{\eta^w}} = (f_i^{\overline{\eta^w}})_{i \in N}$. If $x \in X(N, v)$, then $f^{\overline{\eta^w}}(x) \in X(N, v)$.*

Proof of Lemma 4.1. Let $(N,v) \in G$, $i,j \in N$, $x \in X(N,v)$ and w be a weight function. Similar to the equation (3),

$$\begin{aligned} & w(i) \sum_{S \subseteq N\setminus\{j\}} \big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big] \\ & - w(j) \sum_{S \subseteq N\setminus\{i\}} \big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big] \\ & = w(i)\big[x_j - \overline{\eta^w_j}(N,v)\big] - w(j)\big[x_i - \overline{\eta^w_i}(N,v)\big]. \end{aligned} \tag{10}$$

By equation (10),

$$\begin{aligned} & \sum_{j \in N\setminus\{i\}} \Big(w(i) \sum_{S \subseteq N\setminus\{j\}} \big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big] \\ & - w(j) \sum_{S \subseteq N\setminus\{i\}} \big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\Big) \\ & = w(i) \sum_{j \in N\setminus\{i\}} \big[x_j - \overline{\eta^w_j}(N,v)\big] - \big[x_i - \overline{\eta^w_i}(N,v)\big] \sum_{j \in N\setminus\{i\}} w(j) \\ & = w(i)\cdot\big[v(N) - v(N)\big] - \big[x_i - \overline{\eta^w_i}(N,v)\big]\cdot|N|_w \\ & \text{(by EFF of } \overline{\eta^w},\ x \in X(N,v)) \\ & = |N|_w \cdot \Big(\overline{\eta^w_i}(N,v) - x_i\Big). \end{aligned} \tag{11}$$

Moreover

$$\begin{aligned} & \sum_{i \in N} |N|_w \cdot \Big(\overline{\eta^w_i}(N,v) - x_i\Big) \\ & = |N|_w \cdot \big(v(N) - v(N)\big) \qquad \text{(by EFF of } \overline{\eta^w},\ x \in X(N,v)) \\ & = \ 0. \end{aligned} \tag{12}$$

So, we have that

$$\begin{aligned} & \sum_{i \in N} f_i^{\overline{\eta^w}}(x) \\ & = \sum_{i \in N} \Big[x_i + t \sum_{j \in N\setminus\{i\}} \Big(w(i) \sum_{S \subseteq N\setminus\{j\}} \big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{j\},v,\frac{x}{2^{|N|-1}})\big] \\ & - w(j) \sum_{S \subseteq N\setminus\{i\}} \big[e(S,v,\frac{x}{2^{|N|-1}}) - e(S\cup\{i\},v,\frac{x}{2^{|N|-1}})\big]\Big)\Big] \\ & = v(N) + t\cdot 0 \quad \big(\text{by equations (11), (12) and } x \in X(N,v)\big) \\ & = v(N). \end{aligned}$$

Hence, $f^{\overline{\eta^w}}(x) \in X(N,v)$ if $x \in X(N,v)$.

Based on Lemma 4.1, we can define $x^0 = x$, $x^1 = f^{\overline{\eta^w}}(x^0), \ldots, x^q = f^{\overline{\eta^w}}(x^{q-1})$, for all $(N,v) \in G$, for all $x \in X(N,v)$ and, for all $q \in \mathbb{N}$. Next, we adopt the correction function to propose a dynamic process.

Theorem 4.1. *Let $(N,v) \in G$ and w be a weight function. If $0 < t < \frac{2}{|N|_w}$, then $\{x^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\eta^w}(N,v)$, for all $x \in X(N,v)$.*

Proof of Theorem 4.1. Let $(N,v) \in G$, $i \in N$, $x \in X(N,v)$ and w be a weight function. By equation (11) and definition of $f^{\overline{\eta^w}}$,

$$\begin{aligned} f_i^{\overline{\eta^w}}(x) - x_i = t \sum_{j \in N \setminus \{i\}} \Big(& w(i) \sum_{S \subseteq N \setminus \{j\}} \big[e(S, v, \frac{x}{2^{|N|-1}}) - e(S \cup \{j\}, v, \frac{x}{2^{|N|-1}}) \big] \\ & - w(j) \sum_{S \subseteq N \setminus \{i\}} \big[e(S, v, \frac{x}{2^{|N|-1}}) - e(S \cup \{i\}, v, \frac{x}{2^{|N|-1}}) \big] \Big) \\ = t \cdot |N|_w & \Big(\overline{\eta_i^w}(N,v) - x_i \Big). \end{aligned}$$

Hence,

$$\begin{aligned} \overline{\eta_i^w}(N,v) - f_i^{\overline{\eta^w}}(x) &= \overline{\eta_i^w}(N,v) - x_i + x_i - f_i^{\overline{\eta^w}}(x) \\ &= \overline{\eta_i^w}(N,v) - x_i - t \cdot |N|_w \cdot (\overline{\eta_i^w}(N,v) - x_i) \\ &= \Big(1 - t \cdot |N|_w\Big) \Big[\overline{\eta_i^w}(N,v) - x_i\Big]. \end{aligned}$$

For all $q \in \mathbb{N}$,

$$\overline{\eta^w}(N,v) - x^q = \Big(1 - t \cdot |N|_w\Big)^q \Big[\overline{\eta^w}(N,v) - x\Big].$$

If $0 < t < \frac{2}{|N|_w}$, then $-1 < (1 - t \cdot |N|_w) < 1$ and $\{x^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\eta^w}(N,v)$.

By applying a specific reduction, Maschler and Owen [14] defined a correction function to introduce a dynamic process for the Shapley value [19]. In the following, we propose a dynamic process by applying the notion due to Maschler and Owen [14].

Definition 4.2. *Let ψ be a solution, $(N,v) \in G$, $S \subseteq N$ and $x \in X(N,v)$. The (x,ψ)-reduced game*[1] *$(S, v^r_{\psi,S,x})$ is defined by for all $T \subseteq S$,*

$$v^r_{\psi,S,x}(T) = \begin{cases} v(N) - \sum_{i \in N \setminus S} x_i, & T = S, \\ v_{S,\psi}(T), & \textit{otherwise}. \end{cases}$$

1. For the discussion of x-dependent reduction, please see Maschler and Owen [14].

Inspired by Maschler and Owen [14], we define a correction function as follow. Let $(N, v) \in G$ and w be a weight function. The R-correction function to be $g = (g_i)_{i\in N}$ and $g_i : X(N, v) \to \mathbb{R}$ is define by

$$g_i(x) = x_i + t \sum_{k\in N\setminus\{i\}} \Big(\overline{\eta_i^w}\big(\{i,k\}, v^r_{\overline{\eta^w},\{i,k\},x}\big) - x_i\Big),$$

where $t \in (0, \infty)$, which reflects the assumption that player i does not ask for full correction (when $t = 1$) but only (usually) a fraction of it. Define $x^0 = x,\ x^1 = g(x^0), \dots,\ x^q = g(x^{q-1})$, for all $q \in \mathbb{N}$.

Lemma 4.2. *$g(x) \in X(N, v)$, for all $(N, v) \in G$ and, for all $x \in X(N, v)$.*

Proof of Lemma 4.2. Let $(N, v) \in G$, w be a weight function, $i, k \in N$ and $x \in X(N, v)$. Let $S = \{i, k\}$, by EFF of $\overline{\eta^w}$ and Definition 5,

$$\overline{\eta_i^w}(S, v^r_{\overline{\eta^w},S,x}) + \overline{\eta_k^w}(S, v^r_{\overline{\eta^w},S,x}) = x_i + x_k.$$

By Definition 4.2 and BESCON and WBSFG of $\overline{\beta}$,

$$\begin{aligned}\overline{\eta_i^w}(S, v^r_{\overline{\eta^w},S,x}) - \overline{\eta_k^w}(S, v^r_{\overline{\eta^w},S,x}) &= \overline{\eta_i^w}(S, v_{S,\overline{\eta^w}}) - \overline{\eta_k^w}(S, v_{S,\overline{\eta^w}})\\ &= \overline{\eta_i^w}(N, v) - \overline{\eta_k^w}(N, v).\end{aligned}$$

Therefore,

$$2 \cdot \Big[\overline{\eta_i^w}(S, v^r_{\overline{\eta^w},S,x}) - x_i\Big] = \overline{\eta_i^w}(N, v) - \overline{\eta_k^w}(N, v) - x_i + x_k. \tag{13}$$

By definition of g and equation (13),

$$\begin{aligned} g_i(x) &= x_i + \frac{t}{2} \cdot \Big[\sum_{k\in N\setminus\{i\}} \overline{\eta_i^w}(N, v) - \sum_{k\in N\setminus\{i\}} x_i\\ &\quad - \sum_{k\in N\setminus\{i\}} \overline{\eta_k^w}(N, v) + \sum_{k\in N\setminus\{i\}} x_k\Big]\\ &= x_i + \frac{w}{2} \cdot \Big[\big(|N| - 1\big)\overline{\eta_i^w}(N, v) - \big(|N| - 1\big)x_i \qquad (14)\\ &\quad - \big(v(N) - \overline{\eta_i^w}(N, v)\big) + \big(v(N) - x_i\big)\Big]\\ &= x_i + \frac{|N| \cdot t}{2} \cdot \Big[\overline{\eta_i^w}(N, v) - x_i\Big]. \end{aligned}$$

So, we have that

$$\begin{aligned}\sum_{i\in N} g_i(x) &= \sum_{i\in N} \Big[x_i + \frac{|N| \cdot t}{2} \cdot \big[\overline{\eta_i^w}(N, v) - x_i\big]\Big]\\ &= \sum_{i\in N} x_i + \frac{|N| \cdot t}{2} \cdot \big[\sum_{i\in N} \overline{\eta_i^w}(N, v) - \sum_{i\in N} x_i\big]\\ &= v(N) + \frac{|N| \cdot t}{2} \cdot \big[v(N) - v(N)\big]\\ &= v(N).\end{aligned}$$

Thus, $g\big(x\big) \in X(N, v)$, for all $x \in X(N, v)$.

Theorem 4.2. *Let $(N,v) \in G$ and w be a weight function. If $0 < \alpha < \frac{4}{|N|}$, then $\{x^q\}_{q=1}^{\infty}$ converges to $\overline{\eta^w}(N,v)$ for each $x \in X(N,v)$.*

Proof of Theorem 4.2. Let $(N,v) \in G$, w be a weight function and $x \in X(N,v)$. By equation (14), $g_i(x) = x_i + \frac{|N|\cdot t}{2} \cdot [\overline{\eta_i^w}(N,v) - x_i]$, for all $i \in N$. Therefore,

$$\big(1 - \frac{|N| \cdot t}{2}\big) \cdot \big[\overline{\eta_i^w}(N,v) - x_i\big] = \big[\overline{\eta_i^w}(N,v) - g_i(x)\big]$$

So, for all $q \in \mathbb{N}$,

$$\overline{\eta^w}(N,v) - x^q = \Big(1 - \frac{|N| \cdot t}{2}\Big)^q \Big[\overline{\eta^w}(N,v) - x\Big].$$

If $0 < t < \frac{4}{|N|}$, then $-1 < (1 - \frac{|N|\cdot t}{2}) < 1$ and $\{x^q\}_{q=1}^{\infty}$ converges to $\overline{\eta^w}(N,v)$, for all $(N,v) \in G$, for all weight function w and for all $i \in N$.

5. Conclusions

Weights come up naturally in the framework of utility allocation. For example, we may face the problem of utility allocation among investment projects. Then, the weights could be associated with the profitability of the different projects. Weights are also included in contracts signed by the owners of a condominium and used to divide the cost of building or maintaining common facilities. Another example is data or patent pooling among firms where the firms' sizes, measured for instance by their market shares, are natural weights. Therefore, we adopt weight functions to propose the weighted Banzhaf value. To present the rationality of the weighted Banzhaf value, we employ the efficiency-sum-reduction characterization. Based on excess functions, an alternative formulation is proposed to provide an alternative viewpoint for the weighted Banzhaf value. By applying excess functions and reductions, we also define correction functions to propose dynamic processes for the weighted Banzhaf value. Below are the comparisons of our results with related pre-existing results.

- The weighted Banzhaf value and related results are introduced initially in the framework of standard TU games.
- Inspired by Maschler and Owen [14], we propose dynamic processes for the weighted Banzhaf value. The major difference is that our e-correction function (Definition 4.1) is based on "excess functions," and Maschler and Owen's [14] correction function is based on "reductions".

Our results proposed raise two issues.

- Whether there exist weighted modifications and related results for some more solutions.

- Whether there exist different formulations and related results for some more solutions.

These issues are left to the readers.

References

[1] E. Algaba, J.M. Bilbao, R. van den Brink, A. Jiménez-Losada, *An axiomatization of the Banzhaf value for cooperative games on antimatroids*, Mathematical Methods of Operations Research, 59 (2004), 147-166.

[2] J.M. Alonso-Meijide, C. Bowles, M.J. Holler, S. Napel, *Monotonicity of power in games with a priori unions*, Theory and Decision, 66 (2009), 17–37.

[3] J.M. Alonso-Meijide, M.G. Fiestras-Janeiro, *Modification of the Banzhaf value for games with a coalition structure*, Ann. Oper. Res., 109 (2002), 213–227.

[4] J.F. Banzhaf, *Weighted voting doesn't work: a mathematical analysis*, Rutgers Law Rev, 19 (1965), 317-343.

[5] R. van den Brink, G. van der Laan, *Axiomatizations of the normalized banzhaf value and the Shapley value*, Social Choice and Welfare, 15 (1998), 567-582.

[6] J.S. Coleman, *Control of collectivities and the power of a collectivity to act*, In B.Lieberman, Ed., Social Choice, Gordon and Breach, London, U.K. (1971), 269-300.

[7] P. Dubey, L.S. Shapley, *Mathematical properties of the Banzhaf power index*, Mathematics of Operations Research, 4 (1979), 99-131.

[8] H. Haller, *Collusion properties of values*, International Journal of Game Theory, 23 (1994), 261-281.

[9] S. Hart, A. Mas-Colell, *Potential, value, and consistency*, Econometrica, 57 (1989), 589-614.

[10] Y.A. Hwang, Y.H. Liao, *Consistency and dynamic approach of indexes*, Social Choice and Welfare, 34 (2010), 679-694.

[11] Y.A. Hwang, Y.H. Liao, *Alternative formulation and dynamic process for the efficient Banzhaf–Owen index*, Operations Research Letters, 45 (2017), 60-62.

[12] E. Lehrer, *An axiomatization of the Banzhaf value*, International Journal of Game Theory, 17 (1988), 89-99.

[13] Y.H. Liao, P.T. Liu, L.Y. Chung, *The normalizations and related dynamic processes for two power indexes*, Journal of Control and Decision, 4 (2017), 179-193.

[14] M. Maschler, G. Owen, *The consistent Shapley value for hyperplane games*, International Journal of Game Theory, 18 (1989), 389-407.

[15] G. Owen, *Multilinear extensions and the Banzhaf value*, Naval Research Logistics Quart, 22 (1975), 741-750.

[16] G. Owen, *Characterization of the Banzhaf-Coleman index*, SIAM Journal on Applied Mathematics, 35 (1978), 315-327.

[17] S. Radzik, A. Nowak, T. Driessen, *Weighted Banzhaf values*, Mathematical Methods of Operations Research, 45 (1997), 109-118.

[18] L.M. Ruiz, *On the consistency of the Banzhaf semivalue*, TOP, 7 (1999), 163-168.

[19] L.S. Shapley, *A value for n-person game*, In Contributions to the Theory of Games II, Annals of Mathematics Studies No. 28, Kuhn HW, Tucker AW eds., Princeton University Press, Princeton, New Jersey (1953), 307-317.

[20] L.S. Shapley, *Discussant's comment*, In: Moriarity S (ed) Joint cost allocation, University of Oklahoma Press, Tulsa, 1982.

Accepted: February 1, 2023

Recognition of decomposable posets by using the poset matrix

Salah Uddin Mohammad*
Department of Mathematics
Shahjalal University of Science and Technology
Sylhet-3114
Bangladesh
salahuddin-mat@sust.edu

Md. Rashed Talukder
Department of Mathematics
Shahjalal University of Science and Technology
Sylhet-3114
Bangladesh
r.talukder-mat@sust.edu

Shamsun Naher Begum
Department of Mathematics
Shahjalal University of Science and Technology
Sylhet-3114
Bangladesh
snbegum-mat@sust.edu

Abstract. We introduce the notion of a composition of square matrices. We recall the notion of poset matrix, a square $(0,1)$-matrix, to represent posets. We show that this composition of poset matrices gives generalizations of the ordinal product as well as the direct sum and ordinal sum of poset matrices. We give an interpretation of the composition of poset matrices in posets. We show that the composition of poset matrices is also a poset matrix, and it represents a decomposable poset. This result gives, consequently, a matrix recognition of the decomposable posets.

Keywords: decomposable poset, composite poset, matrix recognition, poset matrix, composition, ordinal product.

1. Introduction

To maximize efficiency, methods for solving many optimization problems on the structure theory begin with some decomposition techniques. These techniques are used to reduce a bigger structure into smaller ones of the same kind, like posets into autonomous sets [3], graphs into clumps [1], comparability graphs into stable sets [10], schedules into job-modules [4], and networks into simplifiable subnetworks [9]. As a result, due to the computational tractability property of the decomposable posets, various methods for the recognition of this type of posets are considered by numerous authors. Khamis [3] recalled the notion of

*. Corresponding author

composition of posets and described an algorithmic method for the recognition of prime (indecomposable) posets. In this article, we give a matrix recognition of the decomposable posets by using the poset matrix, an incidence matrix introduced by Mohammad and Talukder [6] to represent posets.

Since the incidence matrices have many computational aspects, these are chosen repeatedly in recognizing different classes of posets [6, 11] and graphs [5, 12]. As a result, special operations on incidence matrices, due to the classical applications in the adjacent fields, are considered in the literature [5, 7, 8]. In this paper, we introduce the notion of a composition of square matrices and give an interpretation of this composition of poset matrices in posets. Tucker [12] recognized the circular-arc graphs and proper circular-arc graphs by using the properties of perfect 0s, circular 1s, and circularly compatible 1s defined on an augmented adjacency matrix. These results give us the idea of defining the property of transitive blocks of 1s on a block poset matrix and giving a matrix recognition of the decomposable posets.

In Section 2, we recall some basic terminologies related to the ordinal product and composition of posets. We also recall the common operations in the poset matrices and their interpretations in posets. In Section 3, we define the aforesaid composition of square matrices. Here, we mainly show that the composition of poset matrices is also a poset matrix, and it represents a decomposable poset. We also show that this composition of poset matrices generalizes the ordinal product of poset matrices, and every composite poset is decomposable. In Section 4, we define the property of transitive blocks of 1s in a block poset matrix and give a matrix recognition of the decomposable posets.

2. Preliminaries

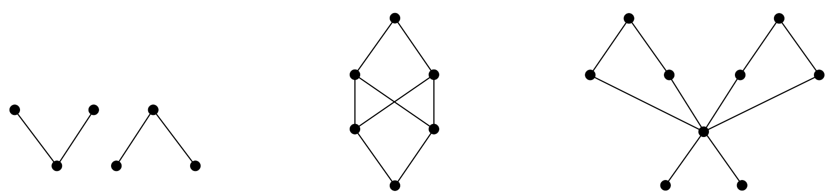
A *poset* (*partially ordered set*) is a structure $\mathbf{A} = \langle A, \leqslant \rangle$ consisting of the nonempty set A with the order relation $\leqslant$ on A, that is, the relation $\leqslant$ is reflexive, antisymmetric, and transitive on A. A poset $\mathbf{A}$ is called *finite* if the underlying set A is finite. Here, we assume that every poset is finite. Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ and $\mathbf{B} = \langle B, \leqslant_B \rangle$ be two posets. A bijective map $\phi : A \to B$ is called an *order isomorphism* if for all $x, y \in A$, we have $x \leqslant_A y$ if and only if $\phi(x) \leqslant_B \phi(y)$. We write $\mathbf{A} \cong \mathbf{B}$ whenever $\mathbf{A}$ and $\mathbf{B}$ are order isomorphic. For further essentials of posets, readers are referred to the classical book by Davey and Priestley [2].

We use the notation $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n (n \geq 1)$ for the n-element chain posets, $\mathbf{I}_n (n \geq 1)$ for the n-element antichain posets, $\mathbf{D}_n (n \geq 4)$ for the n-element diamond posets, $\mathbf{Z}_n (n \geq 4)$ for the n-element zigzag posets, and $\mathbf{B}_{m,n} (m \geq 1, n \geq 1)$ for the complete bipartite posets with m minimal elements and n maximal elements.

We also use the notation $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} \oplus \mathbf{B}$ to denote the direct sum and ordinal sum, respectively, of the posets $\mathbf{A}$ and $\mathbf{B}$. For any poset $\mathbf{A}$, we write shortly $n\mathbf{A}$ for $\mathbf{A} + \mathbf{A} + \cdots + \mathbf{A}$ and $\oplus^n \mathbf{A}$ for $\mathbf{A} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}$. In general, for any posets $\mathbf{B}_i, 1 \leq i \leq n$, we write shortly $\sum_{i=1}^{n} \mathbf{B}_i$ for $\mathbf{B}_1 + \mathbf{B}_2 + \cdots + \mathbf{B}_n$ and $\bigoplus_{i=1}^{n} \mathbf{B}_i$ for $\mathbf{B}_1 \oplus \mathbf{B}_2 \oplus \cdots \oplus \mathbf{B}_n$.

A poset $\mathbf{G}$ is called a *P-graph* if there exist the singleton or antichain posets $\mathbf{A}_i, 1 \leq i \leq n$ such that $\mathbf{G} \cong \bigoplus_{i=1}^{n} \mathbf{A}_i$. A poset $\mathbf{S}$ is called a *P-series* if there exist the P-graphs $\mathbf{G}_i, 1 \leq i \leq n$ such that $\mathbf{S} \cong \sum_{i=1}^{n} \mathbf{G}_i$. Every P-graph is trivially a P-series. A poset is called *series-parallel* if it can be expressed as the sum of the singleton posets using only the direct sum and ordinal sum. Every P-series, as well as every P-graph, is trivially series-parallel.

The *ordinal product* of the posets $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as the poset $\langle A \times B, \leqslant_{\otimes} \rangle$ such that for all $(x, y), (x', y') \in A \times B$, we have $(x, y) \leqslant_{\otimes} (x', y')$ if either (i) $x \leqslant_A x'$ or (ii) $x = x'$ and $y \leqslant_B y'$. Here, the posets $\mathbf{A}$ and $\mathbf{B}$ are called the *ordinal factors* of $\mathbf{A} \otimes \mathbf{B}$. In Figure 1, the ordinal product $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$ along with the direct sum $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$ and the ordinal sum $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$ are shown by using the Hasse diagrams. In general, $\mathbf{A} \otimes \mathbf{B} \ncong \mathbf{B} \otimes \mathbf{A}$.



$\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$ $\quad$ $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$ $\quad$ $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$

Figure 1: Hasse diagrams of $\mathbf{B}_{1,2} + \mathbf{B}_{2,1}$, $\mathbf{B}_{1,2} \oplus \mathbf{B}_{2,1}$, and $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$.

A poset $\mathbf{C}$ is said to be *composite* if and only if their exist nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$. For example, the poset $\mathbf{B}_{1,2} \otimes \mathbf{B}_{2,1}$ (Figure 1) is composite. Also, for any poset $\mathbf{B}$, the poset $n\mathbf{B}$ and $\oplus^n \mathbf{B}$ are composite, because $n\mathbf{B} \cong \mathbf{I}_n \otimes \mathbf{B}$ and $\oplus^n \mathbf{B} \cong \mathbf{C}_n \otimes \mathbf{B}$. A proof by using the poset matrix of the result relating the ordinal sum was given by Mohammad and Talukder [7].

We now recall the definition of the composition of posets. Let $\mathbf{A} = \langle A, \leqslant_A \rangle$ with $A = \{x_1, x_2, ..., x_m\}$ and $\mathbf{B}_r = \langle B_r, \leqslant_{B_r} \rangle$, $1 \leq r \leq m$ with $B_r = \{y_{t+i} : 1 \leq i \leq n_r\}$ where $t = \sum_{k=1}^{r-1} n_k$, be posets on the disjoint sets A and B_r, $1 \leq r \leq m$. Then the *composition* of the posets $\mathbf{A}$ and $\mathbf{B}_r$, $1 \leq r \leq m$, denoted by $\mathbf{A}\ [\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m]$, is defined as the poset $\langle \bigcup_{k=1}^{m} B_k, \leqslant_c \rangle$ such that for all $y_i, y_j \in \bigcup_{r=1}^{m} B_r$, we have $y_i \leqslant_c y_j$ if and only if one of the following conditions is satisfied.

1. $y_{t+i'}, y_{l+j'} \in B_r$ for some r (when $t = l = \sum_{k=1}^{r-1} n_k$, $i' = i - t$ and $j' = j - l$) and $y_{t+i'} \leqslant_{B_r} y_{l+j'}$,

2. $y_{t+i'} \in B_r$ and $y_{l+j'} \in B_s$ for some $r < s$ (when $\sum_{k=1}^{r-1} n_k = t < l = \sum_{k=1}^{s-1} n_k$, $i' = i - t$ and $j' = j - l$) and $x_r \leqslant_A x_s$.

Here, $\mathbf{A}$ is called the *outer* poset or *quotient* poset, and $\mathbf{B}_r, 1 \leq r \leq m$ are called the *inner* posets and their ground sets are called *autonomous* sets. An example of the composition of posets is shown in Figure 2 by using the Hasse diagrams. Obviously, for any posets $\mathbf{B}_i, 1 \leq i \leq n$, we have $\sum_{i=1}^{n} \mathbf{B}_i \cong \mathbf{I}_n[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n]$ and $\bigoplus_{i=1}^{n} \mathbf{B}_i \cong \mathbf{C}_n[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n]$. In particular, for any poset $\mathbf{A}$ with $|A| = n$, we have $\mathbf{A} \cong \mathbf{A}[\underbrace{\mathbf{1}, \mathbf{1}, ..., \mathbf{1}}_{n \text{ times}}]$.

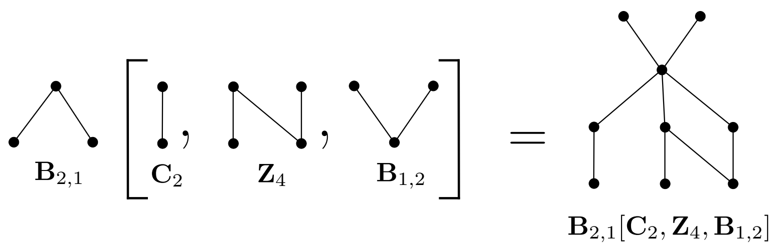


Figure 2: Hasse diagrams giving the composition $\mathbf{B}_{2,1}[\mathbf{C}_2, \mathbf{Z}_4, \mathbf{B}_{1,2}]$.

A poset $\mathbf{D}$ is called *decomposable* if and only if $\mathbf{D}$ is isomorphic to some posets obtained as the composition of two or more inner posets where at least one inner poset is nonsingleton. Thus, a poset $\mathbf{D}$ is decomposable if and only if there exist the poset $\mathbf{A}$ and the posets $\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n, n \geq 2$, where at least one $\mathbf{B}_i$ is nonsingleton, such that $\mathbf{D} \cong \mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_n]$. For example, the posets $\mathbf{D}_4$ and $\mathbf{Z}_4 \oplus \mathbf{1}$ are decomposable because $\mathbf{D}_4 \cong \mathbf{C}_2[\mathbf{1}, \mathbf{B}_{2,1}] \cong \mathbf{C}_2[\mathbf{B}_{1,2}, \mathbf{1}] \cong \mathbf{C}_3[\mathbf{1}, \mathbf{I}_2, \mathbf{1}]$ and $\mathbf{Z}_4 \oplus \mathbf{1} \cong \mathbf{C}_2[\mathbf{Z}_4, \mathbf{1}]$. Here, we see that the posets $\mathbf{1}$, $\mathbf{I}_2$, and $\mathbf{C}_2$ are not decomposable. We assume that these posets are trivially decomposable. On the other hand, a poset is called *prime* or *indecomposable* if and only if it is not decomposable. For example, the poset $\mathbf{Z}_4$ is a prime poset with the least number of elements.

Note that for any nontrivial P-graph $\mathbf{G}$, we have $\mathbf{G} \cong \mathbf{C}_n[\mathbf{I}_{m_1}, \mathbf{I}_{m_2}, ..., \mathbf{I}_{m_n}]$ for some $m_i, 1 \leq i \leq n$. Also, for any nontrivial P-series $\mathbf{S}$, we have $\mathbf{S} \cong \mathbf{I}_n[\mathbf{G}_1, \mathbf{G}_2, ..., \mathbf{G}_n]$ for some P-graphs $\mathbf{G}_i, 1 \leq i \leq n$. These show that every P-series as well as every P-graph is decomposable. Similarly, we can show that every series-parallel poset is decomposable. Note also that, since $\mathbf{Z}_4$ is not a P-graph, $\mathbf{Z}_4 \oplus \mathbf{1}$ is not series-parallel. Thus, a decomposable poset may not be series-parallel. However, we will show by using the poset matrix that every composite poset is decomposable (Corollary 3.2).

Mohammad and Talukder [6] introduced the notion of poset matrix, where they gave matrix recognitions of some subclasses of series-parallel posets. A square $(0,1)$-matrix $M = [a_{ij}], 1 \leq i, j \leq m$ is called a *poset matrix* if and only if the following conditions hold.

1. $a_{ii} = 1$ for all $1 \leq i \leq m$ i.e. M is reflexive,

2. $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$ i.e. M is antisymmetric,

3. $a_{ij} = 1$ and $a_{jk} = 1$ imply $a_{ik} = 1$ i.e. M is transitive.

Both the matrices M and M' in the following example are poset matrices

Example 2.1.

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad M' = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Throughout this paper, we use the notation $M_{m,n}$ for an m-by-n matrix and M_m for a square matrix of order m. In particular, we use the notation I_n, O_n, and Z_n for the n-th order identity matrix, the matrix with all entries 1s, and the matrix with all entries 0s, respectively. We also use the notation C_n for the matrix $[c_{ij}], 1 \leq i, j \leq n$ defined as $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$ otherwise. For every $n \geq 1$, both the matrices I_n and C_n are trivially poset matrices.

To each poset matrix $M_m = [a_{ij}]$, $1 \leq i, j \leq m$, a poset $\mathbf{A} = \langle A, \leqslant \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and x_i corresponds the i-th row (or column) of M_m, is associated by defining the order relation $\leqslant$ on A such that for all $1 \leq i, j \leq m$, we have $x_i \leqslant x_j$ if and only if $a_{ij} = 1$. Then it is said that the poset matrix M_m *represents* the poset $\mathbf{A}$ and vice versa. For example, the poset matrix I_n represents the poset $\mathbf{I}_n$ and the poset matrix C_n represents the poset $\mathbf{C}_n$. Also, the poset matrices M and M', as given in Example 2.1, represent the posets $\mathbf{D}_4$ and $\mathbf{Z}_4$, respectively.

Let M_m be a poset matrix. Then for some $1 \leq i, j \leq m$, interchanges of i-th and j-th rows along with the interchanges of i-th and j-th columns in M_m is called the *(i,j)-relabeling* of M_m. The following results are obtained by Mohammad and Talukder [6] where the authors gave some interpretations of the relabeling of poset matrices in posets.

Theorem 2.1. *Any relabeling of a poset matrix is a poset matrix, and it represents the same poset up to isomorphism.*

Theorem 2.2. *Every poset matrix can be relabeled to an upper (or lower) triangular matrix with 1s in the main diagonal by a finite number of relabeling.*

From now on, by a poset matrix we mean a poset matrix in the upper triangular form.

3. Composition of poset matrices

In this section, we give the construction of the composition of square matrices. We show that the composition of poset matrices generalizes the ordinal product of poset matrices. We also show that the composition of poset matrices represents a decomposable poset.

Definition 3.1. *The* composition *of the square matrices* $M_m = [a_{ij}], 1 \leq i, j \leq m$ *and* $N_{n_r}, 1 \leq r \leq m$, *denoted by* $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}]$, *is a block matrix defined as follows:*

$$M_m[N_{n_1}, ..., N_{n_m}] = \begin{bmatrix} a_{11}N_{n_1} & a_{12}O_{n_1,n_2} & \cdots & a_{1m}O_{n_1,n_m} \\ a_{21}O_{n_2,n_1} & a_{22}N_{n_2} & \cdots & a_{2m}O_{n_2,n_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}O_{n_m,n_1} & a_{m1}O_{n_m,n_2} & \cdots & a_{mm}N_{n_m} \end{bmatrix}.$$

Let $M_m = [a_{ij}], 1 \leq i, j \leq m$ and $N_{n_r}, 1 \leq r \leq m$ be poset matrices. Since M_m is a (0,1)-matrix, the (i,j)-th block Q_{ij} of the block matrix $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}] = [Q_{ij}], 1 \leq i, j \leq m$ can be expressed as follows:

$$Q_{ij} = \begin{cases} N_{n_i}, & \text{if } i = j, \\ O_{n_i,n_j}, & \text{if } i < j \text{ and } a_{ij} = 1, \\ Z_{n_i,n_j}, & \text{if } i < j \text{ and } a_{ij} = 0, \\ O_{n_j,n_i}, & \text{if } i > j \text{ and } a_{ij} = 1, \\ Z_{n_j,n_i}, & \text{if } i > j \text{ and } a_{ij} = 0. \end{cases} \tag{1}$$

Example 3.1.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

$$= \left[\begin{array}{cc|cccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

In the above example, we give the composition $B[C_2, M', B']$ of the poset matrices B, C_2, M' (Example 2.1), and B', where the matrices B and B' represent the posets $\mathbf{B}_{2,1}$ and $\mathbf{B}_{1,2}$, respectively.

Mohammad and Talukder [7] introduced the notion of the ordinal product of matrices. The ordinal product $M_m \boxtimes N_n$ of the poset matrices $M_m = [a_{ij}]$, $1 \leq i, j \leq m$ and N_n is a block matrix where the (i,j)-th block P_{ij} of the matrix

$M_m \boxtimes N_n = [P_{ij}]$, $1 \leq i, j \leq m$ is expressed as follows:

$$P_{ij} = \begin{cases} N_n, & \text{if } i = j, \\ O_n, & \text{if } i \neq j \text{ and } a_{ij} = 1, \\ Z_n, & \text{otherwise.} \end{cases} \tag{2}$$

The authors [7] then gave an interpretation of the ordinal product of poset matrices in posets as follows:

Theorem 3.1. *Let M_m represent the poset* $\mathbf{A}$ *and N_n represent the poset* $\mathbf{B}$. *Then the matrix $M_m \boxtimes N_n$ is a poset matrix and it represents the poset* $\mathbf{A} \otimes \mathbf{B}$.

Corollary 3.1. *Let* $\mathbf{B}$ *be any poset. Then* $\mathbf{C}_n \otimes \mathbf{B} \cong \oplus^n \mathbf{B}$.

The result in Corollary 3.1 was proved by using the fact that the ordinal product of poset matrices gives a generalization of the ordinal sum of poset matrices. Below, we show that the composition of poset matrices generalizes the ordinal product of poset matrices.

Lemma 3.1. *Let M_m and N_n be poset matrices. Then*

$$M_m[\underbrace{N_n, N_n, ..., N_n}_{m \text{ times}}] = M_m \boxtimes N_n. \tag{3}$$

Proof. Substitute $n_i = n, 1 \leq i \leq m$ in the expression for Q_{ij} in equation (1). Then (i,j)-th block Q_{ij} of $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}] = [Q_{ij}], 1 \leq i, j \leq m$ takes the following form.

$$Q_{ij} = \begin{cases} N_n, & \text{if } i = j, \\ O_{n,n}, & \text{if } i < j \text{ and } a_{ij} = 1, \\ Z_{n,n}, & \text{if } i < j \text{ and } a_{ij} = 0, \\ O_{n,n}, & \text{if } i > j \text{ and } a_{ij} = 1, \\ Z_{n,n}, & \text{if } i > j \text{ and } a_{ij} = 0. \end{cases}$$

This implies

$$Q_{ij} = \begin{cases} N_n, & \text{if } i = j, \\ O_n, & \text{if } i \neq j \text{ and } a_{ij} = 1, \\ Z_n, & \text{otherwise} \end{cases}$$

which equals the expression for P_{ij} in equation (2). Thus, for all $1 \leq i, j \leq m$, the (i,j)-th block of the poset matrix $M_m[\underbrace{N_n, N_n, ..., N_n}_{m \text{ times}}]$ equals the (i,j)-th block of the poset matrix $M_m \boxtimes N_n$. Hence the equality in equation (3) holds. □

The following result gives an interpretation of the composition of poset matrices in posets.

Theorem 3.2. *Let M_m represent the poset* $\mathbf{A}$ *and N_{n_i} represent the poset* $\mathbf{B}_i$, $1 \leq i \leq m$. *Then the matrix $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}]$ is a poset matrix and it represents the poset* $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, ..., \mathbf{B}_m]$.

Proof. Let $M_m = [a_{ij}], 1 \leq i, j \leq m$, $N_{n_r} = [b_{ij}], 1 \leq i, j \leq n_r$ and $1 \leq r \leq m$. Also let $M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}] = Q_T = [q_{ij}], 1 \leq i, j \leq T$, where $T = \sum_{r=1}^{m} n_r$, with block representation $[Q_{ij}], 1 \leq i, j \leq m$. Since M_m and $N_{n_r}, 1 \leq r \leq m$ are all upper triangular matrices with 1s in the main diagonal, $Q_{ij} = Z_{n_i,n_j}$ for all $i > j$. Thus Q_T is upper triangular with elements 1s in the main diagonal and hence Q_T is reflexive and antisymmetric. For transitivity of Q_T, let $q_{ij} = q_{jk} = 1$ for some $1 \leq i \leq j \leq k \leq T$. Then, we have the following cases:

1. $q_{ij}, q_{jk} \in Q_{rr} = N_{n_r}$ for some $1 \leq r \leq m$. Then there exist $b_{i'j'}$, $b_{j'k'}$, $b_{i'k'} \in N_{n_r}$ such that $b_{i'j'} = q_{ij} = 1$, $b_{j'k'} = q_{jk} = 1$ and $b_{i'k'} = q_{ik}$. Since N_{n_r} is transitive, $q_{ik} = b_{i'k'} = 1$.

2. $q_{ij} \in Q_{rs} = O_{n_r,n_s}$ and $q_{jk} \in Q_{ss} = N_{n_s}$ for some $1 \leq r < s \leq m$. Then $q_{ik} \in Q_{rs} = O_{n_r,n_s}$ and clearly, $q_{ik} = 1$.

3. $q_{ij} \in Q_{rs} = O_{n_r,n_s}$ and $q_{jk} \in Q_{st} = O_{n_s,n_t}$ for some $1 \leq r < s < t \leq m$. Then $q_{ik} \in Q_{rt}$. Then by the definition of composition of poset matrices, $a_{rs}, a_{st} \in M_m$; and $a_{rs} = a_{st} = 1$. Since M_m is transitive, $a_{rt} = 1$. Therefore, $Q_{rt} = O_{n_r,n_t}$ and clearly, $q_{ik} = 1$.

Thus, Q_T is transitive and hence a poset matrix.

Now, we show that Q_T represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. Let $A = \{x_1, x_2, ..., x_m\}$ and $B_r = \{y_{t+i} : 1 \leq i \leq n_r\}$ where $t = \sum_{k=1}^{r-1} n_k$. Let $q_{ij} = 1$ in Q_T for some $1 \leq i \leq j \leq T$. Then $q_{ij} \in Q_{rs}$ for some $1 \leq r \leq s \leq m$, and we have the following two cases.

1. $r = s$. Then $Q_{rs} = N_{n_r}$ and $b_{i'j'} = q_{ij} \in Q_{kl} = N_{n_r}$ for $t = \sum_{k=1}^{r-1} n_k$, $i' = i - t$ and $j' = j - t$. Since $b_{i'j'} = 1$ and N_{n_r} represents $\mathbf{B}_r$, we have $y_{t+i'} \leqslant_{B_r} y_{t+j'}$. Then, by the definition of composition of posets, $y_i \leqslant_c y_j$.

2. $r < s$. Then $Q_{rs} = O_{n_r,n_s}$ for $\sum_{k=1}^{r-1} n_k = t < l = \sum_{k=1}^{s-1} n_k$. Then $y_{t+i'} \in B_r$ and $y_{l+j'} \in B_s$. Then by the definition of composition of poset matrices, $1 = a_{rs} \in M_m$. Sine M_m represents $\mathbf{A}$, we have $x_r \leqslant_A x_s$. Then, by the definition of composition of posets, $y_i \leqslant_c y_j$.

For the converse, similarly, we show that $y_i \leqslant_c y_j$ implies $1 = q_{ij} \in Q_T$ for all $1 \leq i, j \leq T$. Hence the matrix Q_T represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. □

Below we prove the result that every composite poset is decomposable as an immediate corollary of Theorem 3.2.

Corollary 3.2. *Every composite poset is decomposable.*

Proof. Let $\mathbf{C}$ be any composite poset. Then there exist the nonsingleton posets $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} \cong \mathbf{A} \otimes \mathbf{B}$. Let $|A| = m$. To show that $\mathbf{C}$ is decomposable, we now show that the following isomorphism holds

$$\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A}[\underbrace{\mathbf{B}, \mathbf{B}, \ldots, \mathbf{B}}_{m \text{ times}}]. \tag{4}$$

Let M_m represent the poset $\mathbf{A}$ and N_n represent the poset $\mathbf{B}$. Then, by Theorem 3.1, $M_m \boxtimes N_n$ is a poset matrix and it represents the poset $\mathbf{A} \otimes \mathbf{B}$, and by Theorem 3.2, $M_m[\underbrace{N_n, N_n, ..., N_n}_{m \text{ times}}]$ is a poset matrix and it represents the poset $\mathbf{A}[\underbrace{\mathbf{B}, \mathbf{B}, \ldots, \mathbf{B}}_{m \text{ times}}]$. Therefore, the isomorphism in equation (4) holds by the equality in equation (3), as established in Lemma 3.1. □

4. Recognition of decomposable posets

We now define the property of transitive blocks of 1s in a poset matrix.

Definition 4.1. *A poset matrix Q is said to have the property of* transitive blocks of 1s *of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ if and only if there exists a block representation $Q = [M_{ij}], 1 \leq i, j \leq m$ such that for all $1 \leq i, j, k \leq m$, the following conditions hold:*

1. *$M_{ii} = N_{n_i}$, a poset matrix,*
2. *$M_{ij} = Z_{n_i,n_j}$ or O_{n_i,n_j} for $i < j$; and $M_{ij} = Z_{n_j,n_i}$ for $i > j$,*
3. *$M_{ij} = O_{n_i,n_j}$ and $M_{jk} = O_{n_j,n_k}$ implies $M_{ik} = O_{n_i,n_k}$.*

Note that if $n_1 = n_2 = \cdots = n_m = n$ (say) then we write shortly $\{m, n\}$ for the length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

We see that although the poset matrix N in the following example seems not to satisfy the property of the transitive blocks of 1s, the poset matrix N'' (Example 4.1), obtained by $(3, 4)$-relabeling of N and then $(2, 3)$-relabeling of N', satisfies the property of transitive blocks of 1s of length $\{3, \{2, 4, 3\}\}$.

Example 4.1.

$$N = \left[\begin{array}{cc|cccc|ccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\hline
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

$$\xrightarrow{\text{(3,4)-relabeling}} \left[\begin{array}{cc|cccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = N'$$

$$\xrightarrow{\text{(2,3)-relabeling}} \left[\begin{array}{cc|cccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = N''.$$

Theorem 4.1. *A matrix satisfies the property of transitive blocks of 1s if and only if it is obtained as the composition of some poset matrices.*

Proof. Let the matrix Q be obtained as the composition of the poset matrices M_m and $N_{n_i}, 1 \leq i \leq m$. Then, by the definition of the composition of poset matrices, we have $Q = M_m[N_{n_1}, N_{n_2}, ..., N_{n_m}]$, and by Theorem 3.2, Q is a block poset matrix. This shows that Q is upper triangular having the poset matrices $N_{n_i}, 1 \leq i \leq m$ as diagonal blocks satisfying the first two cases in Definition 4.1. Let $M_m = [a_{ij}]$ and $Q = [Q_{ij}], 1 \leq i, j \leq m$ with $Q_{ij} = O_{n_i,n_j}$ and $Q_{jk} = O_{n_j,n_k}$ for some $1 \leq i < j \leq m$. Then, again by the definition of the composition of poset matrices, we have $a_{ij} = a_{jk} = 1$. Since M_m is transitive, $a_{ik} = 1$. Thus, $Q_{ik} = O_{n_i,n_k}$ which satisfies the last case in Definition 4.1. This shows that Q satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

Conversely, we suppose that the matrix Q satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$ and show similarly that Q can be obtained as the composition of some poset matrices M_m and $N_{n_i}, 1 \leq i \leq m$. $\square$

We observe that the poset matrix N'', as given in Example 4.1, represents the decomposable poset $\mathbf{B}_{2,1}[\,\mathbf{C}_2, \mathbf{Z}_4, \mathbf{B}_{1,2}]$ shown in Figure 2. In the following, we establish this result in general where we give a matrix recognition of the decomposable posets.

Theorem 4.2. *Let the matrix Q represent the poset $\mathbf{D}$. Then $\mathbf{D}$ is decomposable if and only if Q can be relabeled in such a form that it satisfies the property of transitive blocks of 1s.*

Proof. Let $\mathbf{D}$ be a decomposable poset. There exist the posets $\mathbf{A}$ and $\mathbf{B}_i$, $1 \leq i \leq m$, where $m \geq 2$ and at least one $\mathbf{B}_i$ is nonsingleton, such that $\mathbf{D} \cong \mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m]$. Let M_m represent the poset $\mathbf{A}$ and N_{n_i} represent the poset $\mathbf{B}_i$ for every $1 \leq i \leq m$. Then, by Theorem 3.2, $M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$ is a poset matrix and it represents the poset $\mathbf{A}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m] \cong \mathbf{D}$. This shows that Q can be relabeled in such a form that $Q = M_m[N_{n_1}, N_{n_2}, \ldots, N_{n_m}]$. By Theorem 4.1, Q satisfies the property of transitive blocks of 1s of length $\{m, \{n_1, n_2, \ldots, n_m\}\}$.

Conversely, we suppose that the poset matrix Q can be relabeled in such a form that it satisfies the property of transitive blocks of 1s and show similarly that the poset $\mathbf{D}$ is decomposable. □

Acknowledgement

We are extremely grateful to the authorities of the Ministry of Science and Technology, Bangladesh, for the financial support through the NST Fellowship 2019-20 [Grant no. 39.00.0000.012.002.04.19-08].

References

[1] A. Blass, *Graphs with unique maximal clumpings*, J. Graph Theory, 2 (1978), 19-24.

[2] B. A. Davey, H.A. Priestly, *Introduction to lattices and order*, Second Edition, Cambridge University Press, Cambridge, 2002.

[3] S. M. Khamis, *Recognition of prime posets and one of its applications*, J. Egypt. Math. Soc., 14 (2006), 5-13.

[4] E. L. Lawler, *Sequencing jobs to minimize total weighted completion time subject to precedence constraints*, Ann. Discrete Math, 2 (1978), 75-90.

[5] H. S. Mehta, U. P. Acharya, *Adjacency matrix of product of graphs*, Kalpa Publications in Computing, 2 (2017), 158-165.

[6] S. U. Mohammad, M. R. Talukder, *Poset matrix and recognition of series-parallel posets*, International Journal of Mathematics and Computer Science, 15 (2020), 107-125.

[7] S. U. Mohammad, M. R. Talukder, *Interpretations of Kronecker product and ordinal product of poset matrices*, International Journal of Mathematics and Computer Science, 16 (2021), 1665-1681.

[8] R. H. Möhring, *Computationally tractable classes of ordered sets*, In I. Rival (ed.), Algorithm and Order, Kluwer Acad. Publ., Dordrecht, 1989, 105-194.

[9] R. L. Jr. Sielken, H. O. Hartley, *A new statistical approach to project scheduling*, In Decision Information, C. P. Tsokos and R. M. Thrall, eds. Academic Press, New York, 1979, 153-184.

[10] L. N. Shevrin, N. D. Filippov, *Partially ordered sets and their comparability graphs*, Siberian Math. J., 11 (1970), 497-509.

[11] M. Skandera, B. Reed, *Total nonnegativity and (3+1)-free posets*, Journal of Combinatorial Theory, Series A, 103 (2003), 237-256.

[12] A. C. Tucker, *Matrix characterizations of circular-arc graphs*, Pacific Journal of Mathematics, 39 (1971), 535-545.

Accepted: October 13, 2021

Generalized hesitant fuzzy N-soft sets and their applications

Admi Nazra*
Jenizon
Atia Khairuni Chan
Gandung Catur Wicaksono
Yola Sartika Sari
Zulvera
Department of Mathematics
Andalas University
Kampus UNAND Limau Manis, Padang-25163
Indonesia
nazra@sci.unand.ac.id
jenizon@sci.unand.ac.id
can@student.unand.ac.id
wicaksono@student.unand.ac.id
sari@student.unand.ac.id
zulvera@agr.unand.ac.id

Abstract. The N-soft Set as a generalization of the Soft Sets was introduced in 2018 by Fatimah et al. The concept of the N-soft Sets combined with the hesitant fuzzy sets is called hesitant fuzzy N-soft sets. On the other hand, the concept of fuzzy soft sets as a combination of soft sets and fuzzy sets was generalized by Majumdar and Samanta in 2010, called Generalized fuzzy soft sets, where many scholars have studied their properties and characteristics. This paper aims to extend the hesitant fuzzy N-soft set to a generalized hesitant fuzzy N-soft set that incorporates some characteristics of generalized fuzzy soft sets. Definition of the generalized hesitant fuzzy N-soft set, complements, and some of their operations are defined. Moreover, some of their properties, such as associative and distributive related to binary operations, are studied. Finally, we propose two algorithms for decision-making problems by extending the TOPSIS method to apply under generalized hesitant fuzzy N-soft set information.
Keywords: N-soft sets, hesitant fuzzy N-soft sets, generalized hesitant fuzzy soft sets, TOPSIS method.

1. Introduction

In real life, many uncertainty or ambiguity problems cannot be expressed by a crisp set, while decision-making is needed to obtain a possible result on a problem. In 1965, Zadeh [23] introduced a theory to solve this problem called the fuzzy set (FS). The FS theory is usually used to facilitate decision-making on uncertain or unclear problems by defining the degree of each object under

*. Corresponding author

consideration, called the membership value, in the interval [0,1]. In an FS, only one parameter is considered. In 1999 Molodtsov [16] introduced soft sets that associate objects with more than one parameter. A soft set (SS) is a set of ordered pairs of each parameter or attribute with related objects. Studies on Soft Sets have developed rapidly. Mostafa et al. [17], constructed codes by soft sets PU-valued functions. Zhan and Alcantud [24] reviewed some different algorithms of parameter reduction based on some types of (fuzzy) soft sets and compared these algorithms to emphasize their respective advantages and disadvantages. The methodologies and applications of soft set theory in Multi-attribute decision-making (MADM) have been studied by Khameneh and Kilicman [12] from 71 research papers published in 30 academic journals.

Based on the definition of the fuzzy set and the soft set, researchers have introduced hybrid models, their generalization, and their decision-making applications. Maji et al. [14] defined fuzzy soft sets (FSSs). Then, Roy and Maji [18] studied FSSs in a theoretic approach to decision making-problems. Majumdar and Samanta [15] have further generalized the concept of fuzzy soft sets introduced by Maji et al. [14] and have shown their application in decision-making and medical diagnosis problems. Wang et al. [20] extended the classical soft sets to hesitant fuzzy soft sets (HFSS) which are combined by the soft sets and hesitant fuzzy sets. In 2019, Wang and Qin [22] proposed an algorithm of fuzzy soft sets based on decision-making problems under incomplete information. Li et al. [13] proposed generalized hesitant fuzzy soft sets (GHFSS) by integrating generalized fuzzy soft sets with hesitant fuzzy sets and provided an effective approach to decision making. Recently, Karaaslan and Karamaz [11] defined the concept of hesitant fuzzy parameterized hesitant fuzzy soft (HFPHFSs) sets and set-theoretical operations of them and then developed two decision-making algorithms based on the proposed distance measure methods. An FSS is the collection of pairs between a parameter with an FS. However, a generalized fuzzy soft sets (GFSS) is an FSS, along with the degree of importance of each parameter. An HFSS is similar to the FSS, but the membership value of each object is some values in [0,1].

The definition of the SS was generalized to a new set called the N-soft set (NSS), which was introduced by Fatimah et al. [7]. In the same year, Akram et al. [1] introduced the fuzzy N-soft set (FNSS), and then in 2019, Akram et al. [2] generalized the definition of NSS or FNSS into a Hesitant fuzzy N-soft set (HFNSS) and developed new approaches to decision-making such as TOPSIS, choose value, L-choose value, etc. The Research related to decision-making using the approach of the N-soft set continues to grow. Akram et al. [3] extended the notion of parameter reduction to N-soft set theory and developed its application. On the other hand, Alcantud et al. [5] offered a fresh insight into rough set theory from the perspective of N-soft sets, and the applicability of the theoretical results is highlighted with a case study using real data regarding hotel rating. Fatimah and Alcantud [8] introduced a novel hybrid model called a multi-fuzzy N-soft set and designed an adjustable decision-making methodology for solving problems. Kamaci and Petchimuthu [10] proposed a bipolar extension of the N-soft set and set forth two outstanding algorithms to handle the decision-making problems under bipolar N-soft set environments. In 2021, Akram et al. [4] presented a new framework called bipolar fuzzy N-soft set as an extended model of [10] and proposed three algorithms to handle MADM problems. Newly, Alcantud [6] presented the first detailed analysis of the semantics of N-soft sets and designed three-way decision models with both a qualitative and a quantitative character. Another sophisticated hybrid model proposed recently is defined by Zhang et al. [25], where they proposed a q-rung orthopair fuzzy N-soft set (q-ROFNSS) and established two kinds of multiple-attribute group decision-making (MAGDM) methods.

In real life, a decision-maker sometimes needs to consider that the degree or contribution of each parameter in a decision-making problem is not necessarily the same. However, this problem cannot be solved using the HFNSS concept [2]. Therefore, it should be considered a new model in which the degree of each parameter is not the same. This degree is called the degree of preference.

This article constructs a new definition to generalize an HFNSS, called the generalized hesitant fuzzy N-soft set (GHFNSS). On the other hand, the GHFNSS is also a new hybrid model between the generalized fuzzy soft set (GFSS), HFSS and NSS. With this definition, the GHFNSS does not consider only the membership degrees (not necessarily unique for each object) and grades of objects but also the preference degree (the importance degree) of parameters. Furthermore, we can define some operations on GHFNSSs and prove the related properties. Finally, we apply the new TOPSIS algorithms for decision-making problems based on GHFNSS information.

We organize this paper as follows. Section 2 recalls the definitions and operations of SSs, FSs, FSSs, GFSSs, HFSSs, NSSs, FNSSs, and HFNSSs. Section 3 introduces a new model GHFNSS, some of its complements and examples. Then we propose some operations on GHFNSSs, and related to the operations, we derive some properties, such as associative and distributive, in Section 4. Section 5 proposes two algorithms by extending the TOPSIS method to apply under GHFNSS information and give a numerical example. Section 6 concludes the paper.

2. Preliminaries

In this section, the definitions introduced by previous scholars, such as soft sets, fuzzy sets, hesitant fuzzy sets, fuzzy soft sets, hesitant fuzzy soft sets, and N-soft sets, are recalled.

Definition 2.1 ([16]). *Suppose that U is a set of objects, $P(U)$ is the power set of U, and E is the set of parameters, $A \subseteq E$. A soft set (SS) F_A over U is a set, defined by a function f_A, that is represented as*

$$F_A = \{(e, f_A(e)) \mid e \in A, f_A(e) \in P(U)\}$$

Table 1: The soft set F_A

U \ A	e_2	e_3	e_4
u_1	0	1	1
u_2	1	1	0
u_3	0	1	0
u_4	1	1	0
u_5	1	0	1
u_6	0	0	1

where $f_A : A \to P(U)$.

Example 2.1. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a set of job applicants and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. Given the parameters "appearance" (e_1), "courtesy" (e_2), "public speaking" (e_3), "innovative" (e_4) and $A = \{e_2, e_3, e_4\}$. By a decision-maker, based on his/her monitoring, a relation between each parameter with objects is represented as $f(e_2) = \{u_2, u_4, u_5\}$, $f(e_3) = \{u_1, u_2, u_3, u_4\}$ and $f(e_4) = \{u_1, u_5, u_6\}$. By definition, it is obtained an SS F_A as follows.

$F_A = \{(e_2, \{u_2, u_4, u_5\}), (e_3, \{u_1, u_2, u_3, u_4\}), (e_4, \{u_1, u_5, u_6\})\}$.

The SS F_A can be represented as in Table 1.

Definition 2.2 ([23])**.** *Suppose that U is a set of objects. A Fuzzy Set (FS) F over U is defined as*

$$F = \{(u, \mu(u)) \mid u \in U\}$$

where $\mu : U \to [0, 1]$ is called the membership function of F over U and $\mu(u)$ is the membership value of u.

A membership value of u represents the degree of the trust of an object u over a valuation of a decision-maker. Membership values of objects in an FS over U represent membership in a vaguely defined set.

Definition 2.3 ([19])**.** *Suppose that U is a set of objects. A Hesitant Fuzzy Set (HFS) H over U is defined as*

$$H = \{(u, \mu(u)) \mid u \in U\},$$

where $\mu : U \to int[0, 1]$ is called the membership function of H over U and $\mu(u)$ is the set of membership values of u. Here, $int[0, 1]$ is the collection of all subsets of $[0, 1]$.

The concept of the HFS is almost the same as the FS, but an object u may have more than one membership value. This happens because a decision-maker hesitates to valuation for an object or more than one decision-maker evaluates objects.

Definition 2.4 ([18]). *Suppose that U is a set of objects, E is the set of parameters and $A \subseteq E$. A Fuzzy Soft Set (FSS) G_A over U is a set*

$$G_A = \{(e, g_A(e)) \mid e \in A, g_A(e) \in I^U\},$$

where $g_A : A \to I^U$ and I^U is the collection of all FSs over U.

An FSS is the ordered pair of each parameter or attribute with an FS over U. This set provides more explanation than FSs and SSs to give more meaning to the assessment of objects.

Definition 2.5 ([15]). *Suppose that U is a set of objects, E is the set of parameters and I^U is the collection of all FSs over U. A Generalized Fuzzy Soft Set (GFSS) F_μ over U is defined as*

$$F_\mu = \{(e, F_\mu(e)) \mid e \in E\} = \{(e, (F(e), \mu(e))) \mid e \in E\}$$

where $F_\mu : E \longrightarrow I^U \times [0,1]$, $F\colon E \longrightarrow I^U$ is an FSS over U, $\mu\colon E \longrightarrow [0,1]$ is an FS over E, and $\mu(e)$ is called the degree of preference of $e \in E$ in F_μ .

Example 2.2. Suppose that a decision-maker interviews three candidates for agricultural extension workers, which are expressed in the set of objects $U = \{c_1, c_2, c_3\}$. Competencies (parameters) interviewed are e_1=Development of Farmer Participation and e_2=Development of Extension Programs. The candidate's ability to explain all the competencies tested will be assessed from the interview test. The results of this assessment are expressed as real numbers in [0,1], which are the membership values of each candidate for each parameter. Assume that a decision-maker defines the degrees of importance for each parameter as follows.

$$\mu(e_1) = 0.6; \mu(e_2) = 0.4.$$

Following are the results of the assessment of all candidates, which can be stated in the GFSS F_μ.

$$\begin{aligned} F_\mu &= \{(e_1, (F(e_1), \mu(e_1))), (e_2, (F(e_2), \mu(e_2)))\} \\ &= \{(e_1, (\{(c_1, 0.4), (c_2, 0.5), (c_3, 0.8)\}, 0.6)), \\ &\quad (e_2, (\{(c_1, 0.6), (c_2, 0.5), (c_3, 0.4)\}, 0.4))\}. \end{aligned}$$

Definition 2.6 ([21]). *Suppose that U is a set of objects, E is the set of parameters and $A \subseteq E$. A Hesitant Fuzzy Soft Set (HFSS) H_A over U is defined as*

$$H_A = \{(e, h_A(e)) \mid e \in A\},$$

where $h_A : A \to H^U$ and H^U is the collection of all HFSs over U.

As illustrated in Example 2.1, the SS can be represented as a matrix; their entries consist of 0 or 1. Fatimah et al. [7] generalized the concept of SSs called N-soft set as in the following definition.

Definition 2.7 ([7]). *Suppose that U is a set of objects, E is the set of parameters or attributes, and $A \subseteq E$. $R = \{0, 1, 2, ..., N-1\}$ is the set of grades where $N \in \{2, 3, ...\}$. An N-soft set (NSS) (F, A, N) over U is defined as*

$$(F, A, N) = \{(a, F(a)) \mid a \in A\},$$

where $F : A \to 2^{U \times R}$ such that $F(a) = \{(u, r_{au}) \mid u \in U, r_{au} \in R\}$. Here we also write $r_{au} = F(u)(a)$ as the grade of the object u related to the parameter a, and for each $a \in A$ and $u \in U$, there exists a unique $r_{au} \in R$.

Example 2.3. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of cinemas and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the set of medias that making valuation. Suppose that $A = \{e_1, e_3, e_5\}$. For $N = 4$, $R = \{0, 1, 2, 3\}$, suppose that

$$F(e_1) = \{(u_1, 3), (u_2, 1), (u_3, 0), (u_4, 2), (u_5, 2)\}$$
$$F(e_3) = \{(u_1, 2), (u_2, 1), (u_3, 3), (u_4, 1), (u_5, 3)\}$$
$$F(e_5) = \{(u_1, 0), (u_2, 3), (u_3, 1), (u_4, 2), (u_5, 3)\}.$$

Then, by definition, we obtain the NSS (F, A, N) as follows.

$$\begin{aligned}(F, A, N) =\{&(e_1, \{(u_1, 3), (u_2, 1), (u_3, 0), (u_4, 2), (u_5, 2)\}),\\ &(e_3, \{(u_1, 2), (u_2, 1), (u_3, 3), (u_4, 1), (u_5, 3)\}),\\ &(e_5, \{(u_1, 0), (u_2, 3), (u_3, 1), (u_4, 2), (u_5, 3)\})\}.\end{aligned}$$

The NSS (F, A, N) can be represented as in Table 2.

Table 2: The Representation Table of the NSS (F, A, N)

U \ A	e_1	e_3	e_5
u_1	3	2	0
u_2	1	1	3
u_3	0	3	1
u_4	2	1	2
u_5	2	3	3

Definition 2.8 ([7]). *Suppose that U is a set of objects, E is the set of parameters or attributes, $A \subseteq E$, $B \subseteq E$ and $A \cap B \neq \emptyset$. Let $R_1 = \{0, 1, 2, ..., N_1-1\}$ and $R_2 = \{0, 1, 2, ..., N_2-1\}$ be the sets of grades where $N_1, N_2 \in \{2, 3, ...\}$. The restricted intersection of (F, A, N_1) and (G, B, N_2) is defined as*

$$(F, A, N_1) \cap_{\Re} (G, B, N_2) = (J, A \cap B, \min(N_1, N_2))$$

where, for $e \in A \cap B$*,* $u \in U$*,* $(u, r_{eu}) \in J(e)$ *if and only if* $r_{eu} = \min(r_{eu}^{(1)}, r_{eu}^{(2)})$*, with* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$*.*

Definition 2.9 ([7])**.** *Suppose that* U *is a set of objects,* E *is the set of parameters or attributes,* $A \subseteq E$ *and* $B \subseteq E$*. Let* $R_1 = \{0, 1, 2, ..., N_1 - 1\}$ *and* $R_2 = \{0, 1, 2, ..., N_2 - 1\}$ *be the sets of grades where* $N_1, N_2 \in \{2, 3, ...\}$*. The extended intersection of* (F, A, N_1) *and* (G, B, N_2) *is defined as*

$$(F, A, N_1) \cap_{\mathcal{E}} (G, B, N_2) = (J, A \cup B, \max(N_1, N_2))$$

where, for $e \in A \cup B$*,* $u \in U$*,*

$$J(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ \{(u, r_{eu}) \mid u \in U\}, & \text{if } e \in A \cap B, \end{cases}$$

with $r_{eu} = \min(r_{eu}^{(1)}, r_{eu}^{(2)})$*, for* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$*.*

Definition 2.10 ([7])**.** *Suppose that* U *is a set of objects,* E *is the set of parameters or attributes,* $A \subseteq E$*,* $B \subseteq E$ *and* $A \cap B \neq \emptyset$*. Let* $R_1 = \{0, 1, 2, ..., N_1 - 1\}$ *and* $R_2 = \{0, 1, 2, ..., N_2 - 1\}$ *be the sets of grades where* $N_1, N_2 \in \{2, 3, ...\}$*. The restricted union of* (F, A, N_1) *and* (G, B, N_2) *is defined as*

$$(F, A, N_1) \cup_{\Re} (G, B, N_2) = (J, A \cap B, \max(N_1, N_2))$$

where, for $e \in A \cap B$*,* $u \in U$*,* $(u, r_{eu}) \in J(e)$ *if and only if* $r_{eu} = \max(r_{eu}^{(1)}, r_{eu}^{(2)})$*, with* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$*.*

Definition 2.11 ([7])**.** *Suppose that* U *is a set of objects,* E *is the set of parameters or attributes,* $A \subseteq E$ *and* $B \subseteq E$*. Let* $R_1 = \{0, 1, 2, ..., N_1 - 1\}$ *and* $R_2 = \{0, 1, 2, ..., N_2 - 1\}$ *be the sets of grades where* $N_1, N_2 \in \{2, 3, ...\}$*. The extended union of* (F, A, N_1) *and* (G, B, N_2) *is defined as*

$$(F, A, N_1) \cup_{\mathcal{E}} (G, B, N_2) = (J, A \cup B, \max(N_1, N_2))$$

where, for $e \in A \cup B$*,* $u \in U$*,*

$$J(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ \{(u, r_{eu}) \mid u \in U\}, & \text{if } e \in A \cap B, \end{cases}$$

with $r_{eu} = \max(r_{eu}^{(1)}, r_{eu}^{(2)})$ *for* $(u, r_{eu}^{(1)}) \in F(e)$ *and* $(u, r_{eu}^{(2)}) \in G(e)$*.*

Akram et al. [1] constructed a new hybrid model called fuzzy N-soft set as a suitable combination of FS theory with NSS.

Definition 2.12 ([1]). *Suppose that U is a set of objects, E is the set of parameters or attributes, $A \subseteq E$. A pair (μ, K), called a fuzzy N-soft set (FNSS) over U, with $K = (F, A, N)$ is an NSS over U, is defined as*

$$(\mu, K) = \{(a, \mu(a)) \mid a \in A\} = \left\{ \left(a, \left\{ \frac{(u, r_{au})}{m_{au}} \middle| u \in U \right\} \right) \middle| a \in A \right\},$$

where $\mu : A \to \bigcup_{a \in A} \mathcal{F}(F(a))$ with $\mathcal{F}(F(a))$ is the collection of all fuzzy sets over $F(a)$, $(u, r_{au}) \in F(a)$ and $m_{au} \in [0, 1]$ is the membership value of (u, r_{au}).

In 2019, Akram et al. [2] again introduced a novel model called hesitant fuzzy N-soft set as a hybrid of HFS and NSS.

Definition 2.13 ([2]). *Suppose that U is a set of objects, E is the set of parameters or attributes, $A \subseteq E$ and $N \in \{2, 3, ...\}$. A hesitant fuzzy N-soft set (HFNSS) $(\tilde{h}_f, A, N)$ over U is defined as*

$$(\tilde{h}_f, A, N) = \{((u, a), \tilde{h}_f(u, a)) \mid a \in A, u \in U\},$$

where $\tilde{h}_f : U \times A \to R \times \mathcal{P}^*([0, 1])$, with $\mathcal{P}^*([0, 1])$ denotes the set of non-empty subsets of real numbers in [0, 1]. Here $\tilde{h}_f(u, a) = (r_{au}, m_{au})$ with m_{au} and r_{au} denote the possible membership degrees and the grade of the element u related to parameter a, respectively, and for each $a \in A$ and $u \in U$, there exists a unique $r_{au} \in R$.

The HFNSS over U can also be represented by

$$(1) \quad (\hbar_f, A, N) = \{(a, \hbar_f(a)) \mid a \in A\} = \left\{ \left(a, \left\{ \frac{(u, r_{au})}{m_{au}} \middle| u \in U \right\} \right) \middle| a \in A \right\},$$

with $\hbar_f : A \longrightarrow \bigcup_{a \in A} \mathcal{H}(F(a))$, where $\mathcal{H}(F(a))$ is the collection of all HFSs over $F(a)$. Related to m_{au}, we defined $m^+_{au} = \max\{\gamma \mid \gamma \in m_{au}\}$ and $m^-_{au} = \min\{\gamma \mid \gamma \in m_{au}\}$.

The following definitions (Definitions 2.14-2.16) recall some complements of an HFNSS.

Definition 2.14 ([2]). *Given an HFNSS $(\hbar_f, A, N)$ over U as in the equation (1). The Hesitant Fuzzy Complement of $(\hbar_f, A, N)$ is defined as*

$$(\hbar^c_f, A, N) = \left\{ \left(a, \left\{ \frac{(u, r_{au})}{m^c_{au}} \middle| u \in U \right\} \right) \middle| a \in A \right\},$$

with

$$m^c_{au} = \bigcup_{\lambda \in m_{au}} \{1 - \lambda\}.$$

Definition 2.15 ([2]). *Given an HFNSS $(\tilde{h}_f, A, N)$ over U. The Top Weak Hesitant Fuzzy Complement $(\tilde{h}_f^T, A, N)$ of $(\tilde{h}_f, A, N)$ is defined as*

$$\tilde{h}_f^T(u,a) = \begin{cases} (N-1, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} < N-1, \\ (0, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} = N-1, \end{cases}$$

where $\tilde{h}_f(u,a) = (r_{au}, m_{au})$.

Definition 2.16 ([2]). *Given an HFNSS $(\tilde{h}_f, A, N)$ over U. The Bottom Weak Hesitant Fuzzy Complement $(\tilde{h}_f^B, A, N)$ of $(\tilde{h}_f, A, N)$ is defined as*

$$\tilde{h}_f^B(u,a) = \begin{cases} (0, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} > 0, \\ (N-1, \bigcup_{\lambda \in m_{au}} \{1-\lambda\}), & \text{if } r_{au} = 0, \end{cases}$$

where $\tilde{h}_f(u,a) = (r_{au}, m_{au})$.

Now, we review the fundamental set-theoretic operations on HFNSSs.

Definition 2.17 ([2]). *Given two HFNSSs over U $(\tilde{h}_{f_1}, A, N_1)$ and $(\tilde{h}_{f_2}, B, N_2)$. The restricted intersection $(\tilde{h}_f, C, N)$ of them is defined as*

$$(\tilde{h}_f, C, N) = (\tilde{h}_{f_1}, A, N_1) \cap_{\Re} (\tilde{h}_{f_2}, B, N_2) = (\tilde{h}_f, A \cap B, \min(N_1, N_2))$$

where, for $c \in A \cap B \neq \emptyset$ and $u \in U$, $(r_{cu}, m_{cu}) = \tilde{h}_f(u,c)$ if and only if $r_{cu} = \min(r_{cu}^{(1)}, r_{cu}^{(2)})$ and $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min({m_{cu}^{(1)}}^+, {m_{cu}^{(2)}}^+\}$ with $(r_{cu}^{(1)}, m_{cu}^{(1)}) = \tilde{h}_{f_1}(u,c)$, $(r_{cu}^{(2)}, m_{cu}^{(2)}) = \tilde{h}_{f_2}(u,c)$.

Definition 2.18 ([2]). *Given two HFNSSs over U $(\hbar_{f_1}, A, N_1)$ and $(\hbar_{f_2}, B, N_2)$. The extended intersection $(\hbar_f, C, N)$ of them is defined as*

$$(\hbar_f, C, N) = (\hbar_{f_1}, A, N_1) \cap_{\mathcal{E}} (\hbar_{f_2}, B, N_2) = (\hbar_f, A \cup B, \max(N_1, N_2))$$

where, for $c \in A \cup B$

$$\hbar_f(c) = \begin{cases} \hbar_{f_1}(c), & \text{if } c \in A - B, \\ \hbar_{f_2}(c), & \text{if } c \in B - A, \\ \left\{ \frac{(u, r_{cu})}{m_{cu}} \middle| u \in U \right\}, & \text{if } c \in A \cap B, \end{cases}$$

with $r_{cu} = \min(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min({m_{cu}^{(1)}}^+, {m_{cu}^{(2)}}^+)\}$, $\frac{(u, r_{cu}^{(1)})}{m_{cu}^{(1)}} \in \hbar_{f_1}(c)$ and $\frac{(u, r_{cu}^{(2)})}{m_{cu}^{(2)}} \in \hbar_{f_2}(c)$.

Definition 2.19 ([2]). *Let U be a set of objects. Suppose that $(\tilde{h}_{f_1}, A, N_1)$ and $(\tilde{h}_{f_2}, B, N_2)$ are two HFNSSs over U. The restricted union $(\tilde{h}_f, C, N)$ of them is defined as*

$$(\tilde{h}_f, C, N) = (\tilde{h}_{f_1}, A, N_1) \cup_{\Re} (\tilde{h}_{f_2}, B, N_2) = (\tilde{h}_f, A \cap B, \max(N_1, N_2))$$

where, for $c \in A \cap B \neq \emptyset$ *and* $u \in U$, $(r_{cu}, m_{cu}) = \tilde{h}_f(u,c)$ *if and only if* $r_{cu} = \max(r_{cu}^{(1)}, r_{cu}^{(2)})$ *and* $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \geq \max(m_{cu}^{(1)^-}, m_{cu}^{(2)^-})\}$ *with* $(r_{cu}^{(1)}, m_{au}^{(1)}) = \tilde{h}_{f_1}(u,c)$, $(r_{cu}^{(2)}, m_{cu}^{(2)}) = \tilde{h}_{f_2}(u,c)$.

Definition 2.20 ([2]). *Let* U *be a set of objects. Suppose that* $(\hbar_{f_1}, A, N_1)$ *and* $(\hbar_{f_2}, B, N_2)$ *are two HFNSSs over* U. *The extended union* $(\hbar_f, C, N)$ *of them is defined as*

$$(\hbar_f, C, N) = (\hbar_{f_1}, A, N_1) \cup_{\mathcal{E}} (\hbar_{f_2}, B, N_2) = (\hbar_f, A \cup B, \max(N_1, N_2))$$

where, for $c \in A \cup B$

$$\hbar_f(c) = \begin{cases} \hbar_{f_1}(c), & \text{if } c \in A - B, \\ \hbar_{f_2}(c), & \text{if } c \in B - A, \\ \left\{ \frac{(u, r_{cu})}{m_{cu}} \middle| u \in U \right\}, & \text{if } c \in A \cap B, \end{cases}$$

with $r_{cu} = \max(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \geq \max(m_{cu}^{(1)^-}, m_{cu}^{(2)^-})\}$, $\frac{(u, r_{cu}^{(1)})}{m_{cu}^{(1)}} \in \hbar_{f_1}(c)$ *and* $\frac{(u, r_{cu}^{(2)})}{m_{cu}^{(2)}} \in \hbar_{f_2}(c)$.

3. Generalized Hesitant Fuzzy N-Soft Sets, their complements and some further set-theoretic operations

This section will introduce a novel hybrid model called generalized hesitant fuzzy N-soft set as a hybrid model of HFNSS and GFSS. Furthermore, we construct some complements and operations related to the new model.

Definition 3.1. *Suppose that* U *is a set of objects and* E *is the set of parameters. Let* $A \subseteq E$, $N \in \{2, 3, ...\}$ *and* $R = \{0, 1, 2, \ldots, N-1\}$. *Let* $\mathcal{H} = (\hbar_f, A, N)$ *be an HFNSS over* U. *A Generalized Hesitant Fuzzy N-Soft Set (GHFNSS)* $(\mathcal{H}, \mu)$ *over* U *is defined by*

$$\begin{aligned} (\mathcal{H}, \mu) &:= ((\hbar_f, A, N), \mu) = \{(a, \hbar_f(a), \mu(a)) \mid a \in A\} \\ &= \left\{ \left(a, \left\{ \left(\frac{(u, r_{au})}{m_{au}} \right) \middle| u \in U \right\}, \mu(a) \right) \middle| a \in A \right\}, \end{aligned} \tag{2}$$

where $\hbar_f : A \to \bigcup_{a \in A} \mathcal{H}(F(a))$ *and* $\mu : A \to [0,1]$. *For all* $a \in A, u \in U$, $r_{au} \in R$, m_{au} *is a set of some values in [0,1] and* $\mu(a)$ *is a degree of preference of the parameter* $a \in A$.

A GHFNSS over U can be represented by a representation form.

Definition 3.2. *Suppose that* U *is a set of objects and* E *is the set of parameters. Let* $A \subseteq E$, $N \in \{2, 3, ...\}$ *and* $R = \{0, 1, 2, \ldots, N-1\}$. *A representation form of a GHFNSS* $(\mathcal{H}, \mu)$ *over* U *is defined by*

$$(\mathcal{H}, \mu) = \{((u_i, a_j), \tilde{h}_f(u_i, a_j)) \mid a_j \in A, u_i \in U\}, \tag{3}$$

where $\tilde{h}_f : U \times A \longrightarrow R \times \mathcal{P}^*[0,1] \times [0,1]$ *with* $\tilde{h}_f(u_i, a_j) := (r_{a_j u_i}, m_{a_j u_i}, \mu(a_j))$. *To simplify, we may write* $\tilde{h}_f(u_i, a_j) := (\frac{r_{a_j u_i}}{m_{a_j u_i}}, \mu(a_j))$.

The representation form of a GHFNSS can be presented by a table as in Table 3. Here $r_{ij} = F(u_i)(a_j) = r_{a_j u_i}$, and $m_{ij} = m_{a_j u_i}$.

Table 3: The table of a representation form of a GHFNSS $(\mathcal{H}, \mu)$ over U.

$u_i \diagdown a_j$	a_1	$\dots$	a_j	$\dots$	a_m
u_1	$(r_{11}, m_{11}, \mu(a_1))$	$\dots$	$(r_{1j}, m_{1j}, \mu(a_j))$	$\dots$	$(r_{1m}, m_{1m}, \mu(a_m))$
u_2	$(r_{21}, m_{21}, \mu(a_1))$	$\dots$	$(r_{2j}, m_{2j}, \mu(a_j))$	$\dots$	$(r_{2m}, m_{2m}, \mu(a_m))$
u_3	$(r_{31}, m_{31}, \mu(a_1))$	$\dots$	$(r_{3j}, m_{3j}, \mu(a_j))$	$\dots$	$(r_{3m}, m_{3m}, \mu(a_m))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_i	$(r_{i1}, m_{i1}, \mu(a_1))$	$\dots$	$(r_{ij}, m_{ij}, \mu(a_j))$	$\dots$	$(r_{im}, m_{im}, \mu(a_m))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_n	$(r_{n1}, m_{n1}, \mu(a_1))$	$\dots$	$(r_{nj}, m_{nj}, \mu(a_j))$	$\dots$	$(r_{nm}, m_{nm}, \mu(a_m))$

Example 3.1. Suppose that $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the degrees of preference of parameters in A, $\mu(e_1) = 0.5, \mu(e_2) = 0.6, \mu(e_3) = 0.5, \mu(e_4) = 0.7, \mu(e_5) = 0.6$ and $\mu(e_6) = 0.8$. Suppose that $A, B, C \subseteq E$ with $A = \{e_1, e_2, e_4\}$, $B = \{e_2, e_4, e_5\}$ and $C = \{e_1, e_5, e_6\}$. Given three GHFNSSs over U, $(\mathcal{H}_1, \mu) = ((\hbar_{f_1}, A, 5), \mu)$, $(\mathcal{H}_2, \mu) = ((\hbar_{f_2}, B, 4), \mu)$ and $(\mathcal{H}_3, \mu) = ((\hbar_{f_3}, C, 6), \mu)$ as follows

$$\text{a. } (\mathcal{H}_1, \mu) = \Bigg\{ \left(e_1, \left\{\frac{(u_1,4)}{\{0.7,0.8,0.85\}}, \frac{(u_2,2)}{\{0.4,0.55,0.6\}}, \frac{(u_3,3)}{\{0.5,0.55,0.65\}}\right\}, 0.5\right), \left(e_2, \left\{\frac{(u_1,1)}{\{0.3,0.4,0.45\}}, \frac{(u_2,2)}{\{0.5,0.55,0.65\}}, \frac{(u_3,3)}{\{0.5,0.6,0.65\}}\right\}, 0.6\right), \left(e_4, \left\{\frac{(u_1,2)}{\{0.55,0.6\}}, \frac{(u_2,4)}{\{0.75,0.8,0.85\}}, \frac{(u_3,2)}{\{0.45,0.5,0.6\}}\right\}, 0.7\right) \Bigg\}$$

$$\text{b. } (\mathcal{H}_2, \mu) = \Bigg\{ \left(e_2, \left\{\frac{(u_1,1)}{\{0.3,0.35,0.45\}}, \frac{(u_2,3)}{\{0.5,0.6,0.65\}}, \frac{(u_3,2)}{\{0.45,0.5,0.6\}}\right\}, 0.6\right), \left(e_4, \left\{\frac{(u_1,3)}{\{0.6,0.65,0.7\}}, \frac{(u_2,2)}{\{0.5,0.6,0.75\}}, \frac{(u_3,2)}{\{0.45,0.5,0.55\}}\right\}, 0.7\right), \left(e_5, \left\{\frac{(u_1,2)}{\{0.55,0.6\}}, \frac{(u_2,3)}{\{0.65,0.7,0.75\}}, \frac{(u_3,3)}{\{0.7,0.75,0.8\}}\right\}, 0.6\right) \Bigg\}$$

$$\text{c. } (\mathcal{H}_3, \mu) = \Bigg\{ \left(e_1, \left\{\frac{(u_1,4)}{\{0.6,0.65\}}, \frac{(u_2,3)}{\{0.5,0.55,0.6\}}, \frac{(u_3,5)}{\{0.7,0.75,0.8\}}\right\}, 0.5\right), \left(e_5, \left\{\frac{(u_1,4)}{\{0.65,0.7,0.75\}}, \frac{(u_2,5)}{\{0.8,0.85\}}, \frac{(u_3,3)}{\{0.6,0.65,0.75\}}\right\}, 0.6\right), \left(e_6, \left\{\frac{(u_1,3)}{\{0.55,0.6,0.7\}}, \frac{(u_2,2)}{\{0.45,0.55,0.65\}}, \frac{(u_3,4)}{\{0.6,0.7,0.75\}}\right\}, 0.8\right) \Bigg\}.$$

The GHFNSSs as in Example 3.1 can be presented as representation forms as in Table 4, Table 5 and Table 6 respectively.

Table 4: The repesentation form of the GHFNSS $(\mathcal{H}_1, \mu)$ over U

$u_i \diagdown a_j$	e_1	e_2	e_4
u_1	$(4, \{0.7, 0.8, 0.85\}, 0.5)$	$(1, \{0.3, 0.4, 0.45\}, 0.6)$	$(2, \{0.55, 0.6\}, 0.7)$
u_2	$(2, \{0.4, 0.55, 0.6\}, 0.5)$	$(2, \{0.5, 0.55, 0.65\}, 0.6)$	$(4, \{0.75, 0.8, 0.85\}, 0.7)$
u_3	$(3, \{0.5, 0.55, 0.65\}, 0.5)$	$(3, \{0.5, 0.6, 0.65\}, 0.6)$	$(2, \{0.45, 0.5, 0.6\}, 0.7)$

Table 5: The repesentation form of the GHFNSS $(\mathcal{H}_2, \mu)$ over U

$u_i \diagdown a_j$	e_2	e_4	e_5
u_1	$(1, \{0.3, 0.35, 0.45\}, 0.6)$	$(3, \{0.6, 0.65, 0.7\}, 0.7)$	$(2, \{0.55, 0.6\}, 0.6)$
u_2	$(3, \{0.5, 0.6, 0.65\}, 0.6)$	$(2, \{0.5, 0.6, 0.75\}, 0.7)$	$(3, \{0.65, 0.7, 0.75\}, 0.6)$
u_3	$(2, \{0.45, 0.5, 0.6\}, 0.6)$	$(2, \{0.45, 0.5, 0.55\}, 0.7)$	$(3, \{0.7, 0.75, 0.8\}, 0.6)$

Table 6: The repesentation form of the GHFNSS $(\mathcal{H}_3, \mu)$ over U

$u_i \diagdown a_j$	e_1	e_5	e_6
u_1	$(4, \{0.6, 0.65\}, 0.5)$	$(4, \{0.65, 0.7, 0.75\}, 0.6)$	$(3, \{0.55, 0.6, 0.7\}, 0.8)$
u_2	$(3, \{0.5, 0.55, 0.6\}, 0.5)$	$(5, \{0.8, 0.85\}, 0.6)$	$(2, \{0.45, 0.55, 0.65\}, 0.8)$
u_3	$(5, \{0.7, 0.75, 0.8\}, 0.5)$	$(3, \{0.6, 0.65, 0.75\}, 0.6)$	$(4, \{0.6, 0.7, 0.75\}, 0.8)$

Now, we introduce some definitions of the complement of a GHFNSS (Definitions 3.3-3.5).

Definition 3.3. *Suppose that $(\mathcal{H}, \mu)$ is a GHFNSS over U. We define some complements of such $(\mathcal{H}, \mu)$ as follows.*

a. A Weak Complement of $(\mathcal{H}, \mu)$ is

$$(\mathcal{H}^w, \mu) = \left\{ \left(a, \left\{ \left(\frac{(u, r^c_{au})}{m_{au}} \right) \middle| u \in U \right\}, \mu(a) \right) \middle| a \in A \right\}, \tag{4}$$

where $r^c_{au} \neq r_{au}$.

b. The Hesitant Fuzzy Complement of $(\mathcal{H}, \mu)$ is

$$(\mathcal{H}^f, \mu) = \left\{ \left(a, \left\{ \left(\frac{(u, r_{au})}{m^c_{au}} \right) \middle| u \in U \right\}, \mu(a) \right) \middle| a \in A \right\}, \tag{5}$$

where $m^c_{au} = \bigcup_{\lambda \in m_{au}} \{1 - \lambda\}$.

c. *The Preference Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H},\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r_{au})}{m_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\}, \tag{6}$$

where $\mu^c(a)=1-\mu(a)$.

d. *A Weak Hesitant Fuzzy Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^c,\mu)=\left\{\left(a,\left\{\left(\frac{(u,r^c_{au})}{m^c_{au}}\right)\middle|u\in U\right\},\mu(a)\right)\middle|a\in A\right\},$$

where r^c_{au} *and* m^c_{au} *are in equations (4) and (5) respectively.*

e. *A Weak Preference Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^w,\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r^c_{au})}{m_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\},$$

where r^c_{au} *and* $\mu^c(a)$ *are in equations (4) and (6) respectively.*

f. *The Hesitant Preference Fuzzy Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^f,\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r_{au})}{m^c_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\},$$

where m^c_{au} *and* $\mu^c(a)$ *are in equations (5) and (6) respectively.*

g. *A Weak Generalized Hesitant Fuzzy Complement of* $(\mathcal{H},\mu)$ *is*

$$(\mathcal{H}^c,\mu^c)=\left\{\left(a,\left\{\left(\frac{(u,r^c_{au})}{m^c_{au}}\right)\middle|u\in U\right\},\mu^c(a)\right)\middle|a\in A\right\},$$

where r^c_{au}, m^c_{au} *and* $\mu^c(a)$ *are in equations* $(4),(5)$ *and* (6) *respectively.*

It is clear that the complements b., c. and f. above are unique respectively, because of the definition of m^c_{ac} and $\mu^c(a)$.

Example 3.2. Based on Example 3.1, the Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$ is presented in Table 7.

Table 7: The representation form of the Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$.

$({\mathcal{H}_3}^c,\mu^c)$	e_1	e_5	e_6
u_1	$(3,\{0.35,0.4\},0.5)$	$(3,\{0.25,0.3,0.35\},0.4)$	$(2,\{0.3,0.4,0.45\},0.2)$
u_2	$(2,\{0.45,0.4,0.5\},0.5)$	$(4,\{0.15,0.2\},0.4)$	$(1,\{0.35,0.45,0.55\},0.2)$
u_3	$(4,\{0.2,0.25,0.3\},0.5)$	$(2,\{0.25,0.35,0.4\},0.4)$	$(3,\{0.25,0.3,0.4\},0.2)$

Definition 3.4. *Given a GHFNSS $(\mathcal{H}, \mu)$ over U. The following defines some special complements of such $(\mathcal{H}, \mu)$.*

a. *The Top Weak Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^T, \mu) = \{(a, \hbar_f(a)^T, \mu(a)) \mid a \in A\}, \quad \textit{with}$$

$$\hbar_f(a)^T = \begin{cases} \left\{ \frac{(u, N-1)}{m_{au}} \middle| u \in U \right\}, & \textit{if } r_{au} < N-1 \\ \left\{ \frac{(u, 0)}{m_{au}} \middle| u \in U \right\}, & \textit{if } r_{au} = N-1. \end{cases} \tag{7}$$

b. *The Top Weak Hesitant Fuzzy Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^{T^c}, \mu) = \{(a, \hbar_f(a)^{T^c}, \mu(a)) \mid a \in A\}, \quad \textit{with}$$

$$\hbar_f(a)^{T^c} = \begin{cases} \left\{ \frac{(u, N-1)}{\bigcup_{\lambda \in m_{au}} \{1-\lambda\}} \middle| u \in U \right\}, & \textit{if } r_{au} < N-1, \\ \left\{ \frac{(u, 0)}{\bigcup_{\lambda \in m_{au}} \{1-\lambda\}} \middle| u \in U \right\}, & \textit{if } r_{au} = N-1. \end{cases} \tag{8}$$

c. *The Top Weak Preference Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^T, \mu^c) = \{a, \hbar_f(a)^T, \mu^c(a) \mid a \in A\},$$

where $\hbar_f(a)^T$ is in equation (7), and $\mu^c(a) = 1 - \mu(a)$.

d. *The Top Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}, \mu)$ is defined by*

$$(\mathcal{H}^{T^c}, \mu^c) = \{a, \hbar_f(a)^{T^c}, \mu^c(a) \mid a \in A\}.$$

where $\hbar_f(a)^{T^c}$ is in equation (8).

Example 3.3. Based on Example 3.1, the Top Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3, \mu)$ is in Table 8.

Table 8: The representation form of the Top Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3, \mu)$.

$({\mathcal{H}_3}^{T^c}, {\mu_3}^c)$	e_1	e_5	e_6
u_1	$(5, \{0.35, 0.4\}, 0.5)$	$(5, \{0.25, 0.3, 0.35\}, 0.4)$	$(5, \{0.3, 0.4, 0.45\}, 0.2)$
u_2	$(5, \{0.45, 0.4, 0.5\}, 0.5)$	$(0, \{0.15, 0.2\}, 0.4)$	$(5, \{0.35, 0.45, 0.55\}, 0.2)$
u_3	$(0, \{0.2, 0.25, 0.3\}, 0.5)$	$(5, \{0.25, 0.35, 0.4\}, 0.4)$	$(5, \{0.25, 0.3, 0.4\}, 0.2)$

Definition 3.5. *Given a GHFNSS $(\mathcal{H}, \mu)$ over U. The following defines the other special complements of such $(\mathcal{H}, \mu)$.*

a. The Bottom Weak Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^B,\mu)=\{(a,\hbar_f(a)^B,\mu(a))\mid a\in A\},\quad \textit{with}$$

$$\hbar_f(a)^B=\begin{cases}\left\{\frac{(u,0)}{m_{au}}\middle|u\in U\right\}, & \textit{if } r_{au}>0,\\ \left\{\frac{(u,N-1)}{m_{au}}\middle|u\in U\right\}, & \textit{if } r_{au}=0.\end{cases}\tag{9}$$

b. The Bottom Weak Hesitant Fuzzy Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^{B^c},\mu)=\{(a,\hbar_f(a)^{B^c},\mu(a))\mid a\in A\},\quad \textit{with}$$

$$\hbar_f(a)^{B^c}=\begin{cases}\left\{\frac{(u,0)}{\bigcup_{\lambda\in m_{au}}\{1-\lambda\}}\middle|u\in U\right\}, & \textit{if } r_{au}>0,\\ \left\{\frac{(u,N-1)}{\bigcup_{\lambda\in m_{au}}\{1-\lambda\}}\middle|u\in U\right\}, & \textit{if } r_{au}=0.\end{cases}\tag{10}$$

c. The Bottom Weak Preference Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^B,\mu^c)=\{(a,\hbar_f(a)^B,\mu^c(a))\mid a\in A\},$$

where $\hbar_f(a)^B$ *is in equation (9) and* $\mu^c(a)=1-\mu(a)$.

d. The Bottom Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H},\mu)$ *is defined by*

$$(\mathcal{H}^{B^c},\mu^c)=\{(a,\hbar_f(a)^{B^c},\mu^c(a))\mid a\in A\}.$$

where $\hbar_f(a)^{B^c}$ *is in equation (10).*

Note that each complement in Definition 3.4 and Definition 3.5, is unique.

Example 3.4. Based on Example 3.1, the Bottom Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$ is in Table 9.

Table 9: The representation form of the Bottom Weak Generalized Hesitant Fuzzy Complement of $(\mathcal{H}_3,\mu)$.

$({\mathcal{H}_3}^{B^c},\mu^c)$	e_1	e_5	e_6
u_1	$(0,\{0.35,0.4\},0.5)$	$(0,\{0.25,0.3,0.35\},0.4)$	$(0,\{0.3,0.4,0.45\},0.2)$
u_2	$(0,\{0.45,0.4,0.5\},0.5)$	$(0,\{0.15,0.2\},0.4)$	$(0,\{0.35,0.45,0.55\},0.2)$
u_3	$(0,\{0.2,0.25,0.3\},0.5)$	$(0,\{0.25,0.35,0.4\},0.4)$	$(0,\{0.25,0.3,0.4\},0.2)$

Now, we propose some further set-theoretic operations in GHFNSSs.

Definition 3.6. *Suppose that U is a set of objects, E is the set of parameters, $A, B \subseteq E$ and $N_1, N_2 \in \{2, 3, ...\}$. Given two GHFNSSs $(\mathcal{H}_1, \mu_1)$ and $(\mathcal{H}_2, \mu_2)$ over U, as follows,*

$$
\begin{aligned}
(\mathcal{H}_1, \mu_1) =& ((\hbar_{f_1}, A, N_1), \mu_1) = \left\{((u, a), \tilde{h}_{f_1}(u, a)) | a \in A, u \in U\right\} \\
(\mathcal{H}_2, \mu_2) =& ((\hbar_{f_2}, B, N_2), \mu_2) = \left\{((u, b), \tilde{h}_{f_2}(u, b)) | b \in B, u \in U\right\}.
\end{aligned} \tag{11}
$$

Then, the restricted intersection $(\mathcal{H}, \mu)$ of such GHFNSSs is defined by

$$
\begin{aligned}
(\mathcal{H}, \mu) =& (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2) = ((\hbar_f, A \cap B, \min(N_1, N_2)), \mu) \\
=& \left\{((u, c), \tilde{h}_f(u, c)) | c \in C, u \in U\right\}
\end{aligned}
$$

where $\forall\, c \in C = A \cap B \neq \emptyset$, and $\forall\, u \in U$, $(r_{cu}, m_{cu}, \mu(c)) = \tilde{h}_f(u, c)$ if and only if

$$
\begin{aligned}
r_{cu} =& \min(r_{cu}^{(1)}, r_{cu}^{(2)}), m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min({m_{cu}^{(1)}}^+, {m_{cu}^{(2)}}^+)\} \\
\mu(c) =& \min(\mu_1(c), \mu_2(c))
\end{aligned}
$$

with ${m_{cu}^{(1)}}^+ = \max(m_{cu}^{(1)})$ and ${m_{cu}^{(2)}}^+ = \max(m_{cu}^{(2)})$ for $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u, c)$ and $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u, c)$.

Definition 3.7. *Suppose that U is a set of objects, E is the set of parameters, $A, B \subseteq E$ and $N_1, N_2 \in \{2, 3, ...\}$. Given two GHFNSSs $(\mathcal{H}_1, \mu_1)$ and $(\mathcal{H}_2, \mu_2)$ over U as in equation (11). Then the extended intersection $(\mathcal{H}, \mu)$ of such GHFNSSs is defined by*

$$
\begin{aligned}
(\mathcal{H}, \mu) =& (\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_2, \mu_2) = ((\hbar_f, A \cup B, \max(N_1, N_2)), \mu) \\
=& \left\{((u, c), \tilde{h}_f(u, c)) | c \in C, u \in U\right\}
\end{aligned}
$$

where $\forall\, c \in C = A \cup B$ and $\forall\, u \in U$

$$
\tilde{h}_f(u, c) = \begin{cases} \tilde{h}_{f_1}(u, c), & \text{if } c \in A - B, \\ \tilde{h}_{f_2}(u, c), & \text{if } c \in B - A, \\ (r_{cu}, m_{cu}, \mu(c)), & \text{if } c \in A \cap B, \end{cases}
$$

where $r_{cu} = \min(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda \in m_{cu}^{(1)} \cup m_{cu}^{(2)} \mid \lambda \leq \min({m_{cu}^{(1)}}^+, {m_{cu}^{(2)}}^+)\}$ and $\mu(c) = \min(\mu_1(c), \mu_2(c))$ with $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u, c)$ and $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u, c)$.

Definition 3.8. *Suppose that U is a set of objects, E is the set of parameters, $A, B \subseteq E$ and $N_1, N_2 \in \{2, 3, ...\}$. Given two GHFNSSs $(\mathcal{H}_1, \mu_1)$ and $(\mathcal{H}_2, \mu_2)$*

over U as in equation (11). Then the restricted union $(\mathcal{H},\mu)$ of such GHFNSSs is defined by

$$\begin{aligned}(\mathcal{H},\mu) =&(\mathcal{H}_1,\mu_1)\cup_{\Re}(\mathcal{H}_2,\mu_2) = ((\hbar_f, A\cap B, \max(N_1,N_2)),\mu)\\ &=\left\{((u,c),\tilde{h}_f(u,c))|c\in C, u\in U\right\}\end{aligned}$$

where $\forall\, c\in C = A\cap B\neq\emptyset$, $\forall\, u\in U$, $(r_{cu}, m_{cu}, \mu(c)) = \tilde{h}_f(u,c)$ if and only if

$$\begin{aligned} r_{cu} &= \max(r_{cu}^{(1)}, r_{cu}^{(2)}) m_{cu} = \{\lambda\in m_{cu}^{(1)}\cup m_{cu}^{(2)} \mid \lambda\geq \max(m_{cu}^{(1)^-}, m_{cu}^{(2)^-})\}\\ \mu(c) &= \max(\mu_1(c),\mu_2(c))\end{aligned}$$

with $m_{cu}^{(1)^-} = \min(m_{cu}^{(1)})$ and $m_{cu}^{(2)^-} = \min(m_{cu}^{(2)})$ for $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u,c)$ and $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u,c)$.

Definition 3.9. *Suppose that U is a set of objects, E is the set of parameters, $A, B\subseteq E$ and $N_1, N_2\in\{2,3,...\}$. Given two GHFNSSs $(\mathcal{H}_1,\mu_1)$ and $(\mathcal{H}_2,\mu_2)$ over U as in equation (11). Then the extended union $(\mathcal{H},\mu)$ of such GHFNSSs is defined by*

$$\begin{aligned}(\mathcal{H},\mu) =&(\mathcal{H}_1,\mu_1)\cup_{\mathcal{E}}(\mathcal{H}_2,\mu_2) = ((\hbar_f, A\cup B, \max(N_1,N_2)),\mu)\\ &=\left\{((u,c),\tilde{h}_f(u,c))|c\in C, u\in U\right\},\end{aligned}$$

where $\forall\, c\in C = A\cup B$, $\forall\, u\in U$, $(r_{cu}, m_{cu}, \mu(c)) \in \tilde{h}_f(u,c)$ if and only if

$$\tilde{h}_f(u,c) = \begin{cases}\tilde{h}_{f_1}(u,c), & \text{if } c\in A-B,\\ \tilde{h}_{f_2}(u,c), & \text{if } c\in B-A,\\ (r_{cu}, m_{cu}, \mu(c)), & \text{if } c\in A\cap B,\end{cases}$$

where $r_{cu} = \max(r_{cu}^{(1)}, r_{cu}^{(2)})$, $m_{cu} = \{\lambda\in m_{cu}^{(1)}\cup m_{cu}^{(2)} \mid \lambda\geq\max(m_{cu}^{(1)^-}, m_{cu}^{(2)^-})\}$ and $\mu(c) = \max(\mu_1(c),\mu_2(c))$ with $(r_{cu}^{(1)}, m_{cu}^{(1)}, \mu_1(c)) = \tilde{h}_{f_1}(u,c)$ and $(r_{cu}^{(2)}, m_{cu}^{(2)}, \mu_2(c)) = \tilde{h}_{f_2}(u,c)$.

4. Some properties of GHFNSSs

Referring to the operations in the previous section, we derive the following properties, such as associative and distributive. However, the commutative property of GHFNSSs is trivial.

Theorem 4.1 (Associative)**.** *Given three GHFNSSs $(\mathcal{H}_1,\mu_1)$, $(\mathcal{H}_2,\mu_2)$ and $(\mathcal{H}_3,\mu_3)$ over U, with $\mathcal{H}_1 = (\hbar_{f_1}, A, N_1)$, $\mathcal{H}_2 = (\hbar_{f_2}, B, N_2)$ and $\mathcal{H}_3 = (\hbar_{f_3}, C, N_3)$ are HFNSSs over U. Then*

1. $(\mathcal{H}_1,\mu_1)\cap_{\Re}((\mathcal{H}_2,\mu_2)\cap_{\Re}(\mathcal{H}_3,\mu_3)) = ((\mathcal{H}_1,\mu_1)\cap_{\Re}(\mathcal{H}_2,\mu_2))\cap_{\Re}(\mathcal{H}_3,\mu_3)$.

2. $(\mathcal{H}_1,\mu_1)\cap_{\mathcal{E}}((\mathcal{H}_2,\mu_2)\cap_{\mathcal{E}}(\mathcal{H}_3,\mu_3)) = ((\mathcal{H}_1,\mu_1)\cap_{\mathcal{E}}(\mathcal{H}_2,\mu_2))\cap_{\mathcal{E}}(\mathcal{H}_3,\mu_3)$.

3. $(\mathcal{H}_1,\mu_1)\cup_{\Re}((\mathcal{H}_2,\mu_2)\cup_{\Re}(\mathcal{H}_3,\mu_3)) = ((\mathcal{H}_1,\mu_1)\cup_{\Re}(\mathcal{H}_2,\mu_2))\cup_{\Re}(\mathcal{H}_3,\mu_3)$.

4. $(\mathcal{H}_1,\mu_1)\cup_{\mathcal{E}}((\mathcal{H}_2,\mu_2)\cup_{\mathcal{E}}(\mathcal{H}_3,\mu_3)) = ((\mathcal{H}_1,\mu_1)\cup_{\mathcal{E}}(\mathcal{H}_2,\mu_2))\cup_{\mathcal{E}}(\mathcal{H}_3,\mu_3)$.

Proof. We only give the proof of *2*. The others are similar. Suppose that $(\mathcal{H}_4,\mu_4) = (\mathcal{H}_2,\mu_2)\cap_{\mathcal{E}}(\mathcal{H}_3,\mu_3)$ and $D = B\cup C$. By using Definition 3.7

$$(\mathcal{H}_4,\mu_4) = ((\hbar_{f_4}, B\cup C, \max(N_2,N_3)),\mu_4). = \left\{((u,d),\tilde{h}_{f_4}(u,d))|d\in D, u\in U\right\},$$

where any $d\in D = B\cup C$, $\forall\, u\in U$,

$$\tilde{h}_{f_4}(u,d) = \begin{cases}\tilde{h}_{f_2}(u,d), & \text{if } d\in B-C,\\ \tilde{h}_{f_3}(u,d), & \text{if } d\in C-B,\\ (r_{cu}, m_{cu}, \mu(c)), & \text{if } d\in B\cap C,\end{cases}$$

where $r_{du} = \min(r_{du}^{(2)}, r_{du}^{(3)})$, $m_{du} = \{\lambda_4\in m_{du}^{(2)}\cup m_{du}^{(3)} \mid \lambda_4 \le \min({m_{du}^{(2)}}^+, {m_{du}^{(3)}}^+)\}$ and $\mu_4(d) = \min(\mu_2(d),\mu_3(d))$ with $(r_{du}^{(2)}, m_{du}^{(2)}, \mu_2(d)) = \tilde{h}_{f_2}(u,d)$ and $(r_{du}^{(3)}, m_{du}^{(3)}, \mu_3(d)) = \tilde{h}_{f_3}(u,d)$.

Suppose that $(\mathcal{H},\mu) = (\mathcal{H}_1,\mu_1)\cap_{\mathcal{E}}(\mathcal{H}_4,\mu_4)$ and $G = A\cup D$.
Based on Definition 3.7,

$$\begin{aligned}(\mathcal{H},\mu) =& ((\hbar_f, A\cup D, \min(N_1,N_4)),\mu)\\ =& ((\hbar_f, A\cup(B\cup C), \min(N_1,\min(N_2,N_3))),\mu)\\ =& ((\hbar_f, (A\cup B)\cup C, \min(\min(N_1,N_2),N_3))),\mu)\\ =& \left\{((u,d),\tilde{h}_f(u,d))|d\in G, u\in U\right\},\end{aligned}$$

where any $d\in A\cup D$, $\forall\, u\in U$,

$$\tilde{h}_f(u,d) = \begin{cases}\tilde{h}_{f_1}(u,d), & \text{if } d\in A-D,\\ \tilde{h}_{f_4}(u,d), & \text{if } d\in D-A,\\ (r_{du}, m_{du}, \mu(d)), & \text{if } d\in A\cap D,\end{cases}$$

where $r_{du} = \min(r_{du}^{(1)}, r_{du}^{(4)})$, $m_{du} = \{\lambda\in m_{du}^{(1)}\cup m_{du}^{(4)} \mid, \lambda\le \min({m_{du}^{(1)}}^+, {m_{du}^{(4)}}^+)\}$ and $\mu(d) = \min(\mu_1(d),\mu_4(d))$ with $(r_{du}^{(1)}, m_{du}^{(1)}, \mu_1(d)) = \tilde{h}_{f_1}(u,d)$ and $(r_{du}^{(4)}, m_{du}^{(4)}, \mu_4(d)) = \tilde{h}_{f_4}(u,d)$.

Since

$$\begin{aligned}r_{du} =& \min(r_{du}^{(1)}, \min(r_{du}^{(2)}, r_{du}^{(3)})) = \min(\min(r_{du}^{(1)}, r_{du}^{(2)}), r_{du}^{(3)})\\ m_{du} =& \{\lambda\in m_{du}^{(1)}\cup(m_{du}^{(2)}\cup m_{du}^{(3)}) \mid \lambda\le\min({m_{du}^{(1)}}^+, \min({m_{du}^{(2)}}^+, {m_{du}^{(3)}}^+))\}\\ =& \{\lambda\in(m_{du}^{(1)}\cup(m_{du}^{(2)})\cup m_{du}^{(3)} \mid \lambda\le\min(\min({m_{du}^{(1)}}^+, {m_{du}^{(2)}}^+), {m_{du}^{(3)}}^+)\} \text{ and}\\ \mu(d) =& \min(\mu_1(d), \min(\mu_2(d),\mu_3(d)) = \min(\min(\mu_1(d),\mu_2(d)),\mu_3(d)),\end{aligned}$$

then it is proved that $(\mathcal{H}_1,\mu_1)\cap_{\mathcal{E}}((\mathcal{H}_2,\mu_2)\cap_{\mathcal{E}}(\mathcal{H}_3,\mu_3)) = ((\mathcal{H}_1,\mu_1)\cap_{\mathcal{E}}(\mathcal{H}_2,\mu_2))\cap_{\mathcal{E}}(\mathcal{H}_3,\mu_3)$. □

Theorem 4.2. *(Distributive) Given three GHFNSSs* $(\mathcal{H}_1, \mu_1)$, $(\mathcal{H}_2, \mu_2)$ *and* $(\mathcal{H}_3, \mu_3)$ *over* U*, with* $\mathcal{H}_1 = (\hbar_{f_1}, A, N_1)$, $\mathcal{H}_2 = (\hbar_{f_2}, B, N_2)$ *and* $\mathcal{H}_3 = (\hbar_{f_3}, C, N_3)$ *are HFNSSs over* U*. Then*

1. $(\mathcal{H}_1, \mu_1) \cap_{\Re} ((\mathcal{H}_2, \mu_2) \cup_{\Re} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2)) \cup_{\Re} ((\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_3, \mu_3))$.
2. $(\mathcal{H}_1, \mu_1) \cup_{\Re} ((\mathcal{H}_2, \mu_2) \cap_{\Re} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cup_{\Re} (\mathcal{H}_2, \mu_2)) \cap_{\Re} ((\mathcal{H}_1, \mu_1) \cup_{\Re} (\mathcal{H}_3, \mu_3))$.
3. $(\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} ((\mathcal{H}_2, \mu_2) \cup_{\mathcal{E}} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_2, \mu_2)) \cup_{\mathcal{E}} ((\mathcal{H}_1, \mu_1) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3))$.
4. $(\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} ((\mathcal{H}_2, \mu_2) \cap_{\mathcal{E}} (\mathcal{H}_3, \mu_3)) =$
   $((\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} (\mathcal{H}_2, \mu_2)) \cap_{\mathcal{E}} ((\mathcal{H}_1, \mu_1) \cup_{\mathcal{E}} (\mathcal{H}_3, \mu_3))$.

Proof. Here, we give the proof of *1*. The others are similar. Suppose that $(\mathcal{H}_4, \mu_4) = (\mathcal{H}_2, \mu_2) \cup_{\Re} (\mathcal{H}_3, \mu_3)$, $D = B \cap C$ and $N_4 = \max(N_2, N_3)$. Based on Definition 3.8,

$$(\mathcal{H}_4, \mu_4) = ((\hbar_{f_4}, B \cap C, \max(N_2, N_3)), \mu_4) = \Big\{((u, d), \tilde{h}_{f_4}(u, d)) | d \in D, u \in U\Big\},$$

where, for any $d \in D = B \cap C$, $\forall\, u \in U$, $(r_{du}, m_{du}, \mu(d)) = \tilde{h}_{f_4}(u, d)$ if and only if

$$\begin{aligned} r_{du} =& \max(r_{du}^{(2)}, r_{du}^{(3)}), m_{du} = \{\lambda_4 \in m_{du}^{(2)} \cup m_{du}^{(3)} \mid \lambda_4 \geq \max({m_{du}^{(2)}}^{-}, {m_{du}^{(3)}}^{-})\} \\ \mu_4(d) =& \max(\mu_2(d), \mu_3(d)) \end{aligned}$$

for $(r_{du}^{(2)}, m_{du}^{(2)}, \mu_2(d)) = \tilde{h}_{f_2}(u, d)$ and $(r_{du}^{(3)}, m_{du}^{(3)}, \mu_3(d)) = \tilde{h}_{f_3}(u, d)$.

Suppose that $(\mathcal{H}, \mu) = (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_4, \mu_4)$ and $G = A \cap D$. By using Definition 3.6,

$$\begin{aligned} (\mathcal{H}, \mu) =& ((\hbar_f, A \cap D, \min(N_1, N_4)), \mu) \\ =& ((\hbar_f, A \cap (B \cap C), \min(N_1, \max(N_2, N_3))), \mu). \\ =& \Big\{((u, g), \tilde{h}_f(u, g)) | g \in G, u \in U\Big\}. \end{aligned}$$

where, for any $g \in G = A \cap D$, $\forall\, u \in U$, $(r_{gu}, m_{gu}, \mu(g)) = \tilde{h}_f(u, g)$ if and only if

$$\begin{aligned} r_{gu} =& \min(r_{gu}^{(1)}, r_{gu}^{(4)}) = \min(r_{gu}^{(1)}, \max(r_{gu}^{(2)}, r_{gu}^{(3)})) \\ m_{gu} =& \{\lambda \in m_{gu}^{(1)} \cup m_{gu}^{(4)} \mid \lambda \leq \min({m_{gu}^{(1)}}^{+}, {m_{gu}^{(4)}}^{+})\} \\ \mu(g) =& \min(\mu_1(g), \mu_4(g)) = \min(\mu_1(g), \max(\mu_2(g), \mu_3(g))) \end{aligned}$$

for $(r_{gu}^{(1)}, m_{gu}^{(1)}, \mu_1(g)) = \tilde{h}_{f_1}(u, g)$ and $(r_{gu}^{(4)}, m_{gu}^{(4)}, \mu_4(g)) = \tilde{h}_{f_4}(u, g)$.

Suppose that $(\mathcal{H}_5, \mu_5) = (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_2, \mu_2)$, $P = A \cap B$ and $N_5 = \min(N_1, N_2)$. Based on Definition 3.6

$$\begin{aligned}(\mathcal{H}_5, \mu_5) =& ((\hbar_{f_5}, A \cap B, \min(N_1, N_2)), \mu_5) \\ =& \left\{((u,p), \tilde{h}_{f_5}(u,p)) | p \in P, u \in U\right\}\end{aligned}$$

where, for any $p \in A \cap B$, $\forall\, u \in U$, $(r_{pu}, m_{pu}, \mu(p)) = \tilde{h}_{f_5}(u,p)$ if and only if

$$\begin{aligned} r_{pu} =& \min(r_{pu}^{(1)}, r_{pu}^{(2)}) \\ m_{pu} =& \{\lambda_5 \in m_{pu}^{(1)} \cup m_{pu}^{(2)} \mid \lambda_5 \leq \min({m_{pu}^{(1)}}^+, {m_{pu}^{(2)}}^+)\} \\ \mu_5(p) =& \min(\mu_1(p), \mu_2(p)). \end{aligned}$$

for $(r_{pu}^{(1)}, m_{pu}^{(1)}, \mu_1(p)) = \tilde{h}_{f_1}(u,p)$ and $(r_{pu}^{(2)}, m_{pu}^{(2)}, \mu_2(p)) = \tilde{h}_{f_2}(u,p)$.

Suppose that $(\mathcal{H}_6, \mu_6) = (\mathcal{H}_1, \mu_1) \cap_{\Re} (\mathcal{H}_3, \mu_3)$, $Q = A \cap C$ and $N_6 = \min(N_1, N_3)$. Based on Definition 3.6

$$\begin{aligned}(\mathcal{H}_6, \mu_6) =& ((\hbar_{f_6}, A \cap C, \min(N_1, N_3)), \mu_6) \\ =& \left\{((u,q), \tilde{h}_{f_6}(u,q)) | q \in Q, u \in U\right\}\end{aligned}$$

where, for any $q \in Q = A \cap C$, $\forall\, u \in U$, $(r_{qu}, m_{qu}, \mu(q)) = \tilde{h}_{f_6}(u,q)$ if and only if

$$\begin{aligned} r_{qu} =& \min(r_{qu}^{(1)}, r_{qu}^{(3)}), \\ m_{qu} =& \{\lambda_6 \in m_{qu}^{(1)} \cup m_{qu}^{(3)} \mid \lambda_6 \leq \min({m_{qu}^{(1)}}^+, {m_{cu}^{(3)}}^+)\}, \\ \mu_6(q) =& \min(\mu_1(q), \mu_3(q)) \end{aligned}$$

for $(r_{qu}^{(1)}, m_{qu}^{(1)}, \mu_1(q)) = \tilde{h}_{f_1}(u,q)$ and $(r_{qu}^{(3)}, m_{qu}^{(3)}, \mu_3(q)) = \tilde{h}_{f_3}(u,q)$.

Suppose that $(\mathcal{H}_7, \mu_7) = (\mathcal{H}_5, \mu_5) \cup_{\Re} (\mathcal{H}_6, \mu_6)$, $S = P \cap Q$ and $N_7 = \max(N_5, N_6)$. Based on Definition 3.8

$$\begin{aligned}(\mathcal{H}_7, \mu_7) =& ((\hbar_{f_7}, P \cap Q, \max(N_5, N_6)), \mu_7) \\ =& \left\{((u,s), \tilde{h}_{f_7}(u,s)) | s \in S, u \in U\right\},\end{aligned}$$

where, for any $s \in P \cap Q$, $\forall\, u \in U$, $(r_{su}, m_{su}, \mu(s)) = \tilde{h}_{f_7}(u,s)$ if and only if

$$\begin{aligned} r_{su} =& \max(r_{su}^{(5)}, r_{su}^{(6)}), \\ m_{su} =& \{\lambda_7 \in m_{su}^{(5)} \cup m_{su}^{(6)} \mid \lambda_7 \geq \max({m_{su}^{(5)}}^-, {m_{su}^{(6)}}^-)\}, \\ \mu_7(s) =& \max(\mu_5(s), \mu_6(s)), \end{aligned}$$

for $(r_{su}^{(5)}, m_{su}^{(5)}, \mu_5(s)) = \tilde{h}_{f_5}(u,s)$ and $(r_{su}^{(6)}, m_{su}^{(6)}, \mu_6(s)) = \tilde{h}_{f_6}(u,s)$.

Now, we will prove $(\mathcal{H},\mu) = (\mathcal{H}_7,\mu_7)$. Consider that

$$
\begin{aligned}
(\mathcal{H}_7,\mu_7) =& ((\hbar_{f_7}, P\cap Q, \max(N_5,N_6)),\mu_7)\\
=& ((\hbar_{f_7}, (A\cap B)\cap(A\cap C), \max(\min(N_1,N_2),\min(N_1,N_3)),\mu_7)\\
=& ((\hbar_{f_7}, A\cap(B\cap C), \min(N_1,\max(N_2,N_3)),\mu_7) = (\mathcal{H},\mu),
\end{aligned}
$$

where, for any $s\in A\cap(B\cap C)$, $\forall\, u\in U$, $(r^{(7)}_{su}, m^{(7)}_{su}, \mu_7(s)) = \tilde{h}_{f_7}(u,s)$ if and only if

$$
\begin{aligned}
r^{(7)}_{su} =& \max(r^{(5)}_{su}, r^{(6)}_{su}) = \max(\min(r^{(1)}_{su}, r^{(2)}_{su}), \min(r^{(1)}_{su}, r^{(3)}_{su}))\\
=& \min(r^{(1)}_{su}, \max(r^{(2)}_{su}, r^{(3)}_{su})) = r_{su},\\
m^{(7)}_{su} =& \{\lambda_7 \in m^{(5)}_{su}\cup m^{(6)}_{su} \mid \lambda_7 \geq \max({m^{(5)}_{su}}^-, {m^{(6)}_{su}}^-)\}\\
=& \{\lambda_7 \in m^{(1)}_{su}\cup (m^{(2)}_{su}\cup m^{(3)}_{su}) \mid \lambda_7 \leq \min({m^{(1)}_{su}}^+, {m^{(4)}_{su}}^+)\} = m_{su}\\
\mu_7(s) =& \max(\mu_5(s),\mu_6(s)) = \max(\min(\mu_1(s),\mu_2(s)), \min(\mu_1(s),\mu_3(s)))\\
=& \min(\mu_1(s), \max(\mu_2(s),\mu_3(s))) = \mu(s).
\end{aligned}
$$

Therefore $(\mathcal{H}_1,\mu_1)\cap_{\Re}((\mathcal{H}_2,\mu_2)\cup_{\Re}(\mathcal{H}_3,\mu_3)) = ((\mathcal{H}_1,\mu_1)\cap_{\Re}(\mathcal{H}_2,\mu_2))\cup_{\Re}((\mathcal{H}_1,\mu_1)$ $\cap_{\Re}(\mathcal{H}_3,\mu_3))$. □

5. Application of GHFNSSs

Hwang and Yoon, in 1981 [9] introduced an algorithm for decision-making problems concerning parameters or attributes. This algorithm is called TOPSIS *(Technique for Order Preference by Similarity to Ideal Solution)*. Under HFNSS information, Akram et al. [2] have extended this method. When a decision-maker wants to rank objects to obtain the best performance, the chosen alternative has the shortest distance from the positive ideal solution (PIS) and the longest distance from the negative ideal solution (NIS).

We propose the two following algorithms by extending the TOPSIS method to apply under GHFNSS information. Algorithm 1 could apply for a condition that the number of elements of m_{ij} is not necessary the same for all i and j, while in Algorithm 2, that is the same. Algorithm 2 is a new extended method based on GHFNSSs as a generalization of the method introduced by Akram et al. [2]. In our method, we use the information on the preference degree of parameters. The sum of all the preference degrees does not need equal to one as in the definition of the weight of the parameters. On the other hand, in determining the ranking order of objects in choosing the best one, Akram et al. [2] refer to pairs of values called relative adjacency to ideal solution. It is impossible to determine the ranking order of a collection of pairs of values (a_i, a_j) for $i, j \in \mathbb{N}$, except in the condition that $a_i > a_j$ and $b_i > b_j$ for $i \neq j$. Because of this, in Algorithm 2, we give a modification of the Akram's method.

Algorithm 1

1. Input a subset A of a parameter set E. Given a set of objects $U = \{u_1, u_2, \ldots, u_p\}$ and the set of parameters or attributes $A = \{a_1, a_2, \ldots, a_q\}$.
2. Represent a GHFNSS in the representation form.
3. The matrix of the representation form of the corresponding GHFNSS over U is

$$D = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2q} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ij} & \cdots & b_{iq} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pq} \end{pmatrix} = [b_{ij}],$$

   where $b_{ij} = (\frac{r_{ij}}{m_{ij}}, \mu(a_j))$, with r_{ij} is the grade, $m_{ij} = \{\lambda^1_{ij}, \lambda^2_{ij}, \ldots, \lambda^{k_{ij}}_{ij}\}$ is the set of membership values of u_i with respect to the parameter a_j, and $\mu(a_j)$ is the degree of preference of the parameter a_j.
4. Transform the matrix $D = [b_{ij}]$ to be the matrix $D' = [b'_{ij}]$ where $b'_{ij} = (\frac{r_{ij}}{m'_{ij}}, \mu(a_j))$, with $m'_{ij} = \frac{1}{k_{ij}} \sum_{l=1}^{k_{ij}} \lambda^l_{ij}$, $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$.
5. Transform matrix D' to be normalized decision matrix $V = [(\frac{V_{ij}}{v_{ij}}, \sigma_j)]$ by using

$$V_{ij} = \frac{r_{ij}}{\sqrt{\sum_{i=1}^p {r_{ij}}^2}}, \; v_{ij} = \frac{m'_{ij}}{\sqrt{\sum_{i=1}^p {m'_{ij}}^2}} \text{ and } \sigma_j = \frac{\mu(e_j)}{\sum_{j=1}^q \mu(e_j)}.$$

6. Define matrix $W = [\frac{W_{ij}}{w_{ij}}]$ by $W_{ij} = V_{ij}\sigma_j$ and $w_{ij} = v_{ij}\sigma_j$.
7. Find the positive ideal solution D^+ and the negative ideal solution D^- defined by

$$\begin{aligned} D^+ =& \{(\frac{\max_i(W_{ij})}{\max_i(w_{ij})} \mid j \in J), (\frac{\min_i(W_{ij})}{\min_i(w_{ij})} \mid j \in J')\} \\ =& \{\frac{{W_j}^+}{{w_j}^+} \mid j = 1, 2, \ldots, q\} \\ D^- =& \{(\frac{\min_i(W_{ij})}{\min_i(w_{ij})} \mid j \in J), (\frac{\max_i(W_{ij})}{\max_i(w_{ij})} \mid j \in J')\} \\ =& \{\frac{{W_j}^-}{{w_j}^-} \mid j = 1, 2, \ldots, q\}, \end{aligned}$$

   where $J = \{j \mid j \text{ is a supporting parameter}\}$, $J' = \{j \mid j \text{ is not a supporting parameter}\}$, and $\mid J \mid + \mid J' \mid = q$.

8. Calculate separation measures $({S_i}^+, {s_i}^+)$ and $({S_i}^-, {s_i}^-)$

$$
\begin{aligned}
({S_i}^+, {s_i}^+) &= \left(\sqrt{\sum_{j=1}^{q}(W_{ij} - {W_j}^+)^2}, \sqrt{\sum_{j=1}^{q}(w_{ij} - {w_j}^+)^2}\right), \\
&\quad i = 1, 2, \dots, p \\
({S_i}^-, {s_i}^-) &= \left(\sqrt{\sum_{j=1}^{q}(W_{ij} - {W_j}^-)^2}, \sqrt{\sum_{j=1}^{q}(w_{ij} - {w_j}^-)^2}\right), \\
&\quad i = 1, 2, \dots, p.
\end{aligned}
\tag{12}
$$

9. Calculate relative adjacency to ideal solution

$$
\begin{aligned}
(C_i, c_i) &= \left(\frac{{S_i}^-}{{S_i}^+ + {S_i}^-}, \frac{{s_i}^-}{{s_i}^+ + {s_i}^-}\right) \\
&0 < C_i < 1, 0 < c_i < 1, i = 1, 2, \dots, p.
\end{aligned}
\tag{13}
$$

10. Form matrix $E = [E_i]$ with $E_i = \frac{C_i + c_i}{2}$, $i = 1, 2, \dots, p$.

11. The best choice is an object u_t such that $E_t \geq E_j$ for all $j \neq t$.

Algorithm 2.

1. Repeat steps 1-3 of Algorithm 1.

2. Using matrix D in Algorithm 1, determine positive ideal solution B^+ and negative ideal solution B^-

$$
\begin{aligned}
B^+ =& \{({r_j}^+, \{({\lambda_j}^1)^+, ({\lambda_j}^2)^+, \dots, ({\lambda_j}^k)^+\})\ j = 1, 2, \dots, q\}, \\
B^- =& \{({r_j}^-, \{({\lambda_j}^1)^-, ({\lambda_j}^2)^-, \dots, ({\lambda_j}^k)^-\})\ j = 1, 2, \dots, q\}
\end{aligned}
$$

where,

$$
\begin{aligned}
&{r_j}^+ = \max_i(r_{ij}),\ \ {r_j}^- = \min_i(r_{ij}),\ \ \lambda_{ij}^1 \leq \lambda_{ij}^2 \leq \cdots \leq \lambda_{ij}^k, \\
&\text{and for each i, j} \\
&({\lambda_j}^1)^+ = \max_i({\lambda_{ij}}^1),\ ({\lambda_j}^1)^- = \min_i({\lambda_{ij}}^1), \\
&({\lambda_j}^2)^+ = \max_i({\lambda_{ij}}^2),\ ({\lambda_j}^2)^- = \min_i({\lambda_{ij}}^2), \\
&\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&({\lambda_j}^k)^+ = \max_i({\lambda_{ij}}^k),\ ({\lambda_j}^k)^- = \min_i({\lambda_{ij}}^k).
\end{aligned}
$$

3. Calculate separation measures ${S_i}^+$ and ${S_i}^-$,

$${S_i}^+ = ({R_i}^+, {M_i}^+), \quad i = 1, 2, \ldots, p$$

where

$${R_i}^+ = \sum_{j=1}^{q} \sigma_j \mid r_{ij} - r_j^+ \mid, \ \ {M_i}^+ = \sum_{j=1}^{q} \sigma_j \sqrt{\frac{1}{k} \sum_{l=1}^{k} \mid {\lambda_{ij}}^l - ({\lambda_j}^l)^+ \mid^2}, \text{ and}$$

$${S_i}^- = ({R_i}^-, {M_i}^-), \quad j = 1, 2, \ldots, p$$

where

$${R_i}^- = \sum_{j=1}^{q} \sigma_j \mid r_{ij} - r_j^- \mid, \ \ {M_i}^- = \sum_{j=1}^{q} \sigma_j \sqrt{\frac{1}{k} \sum_{l=1}^{k} \mid {\lambda_{ij}}^l - ({\lambda_j}^l)^- \mid^2}, \text{ and}$$

$$\sigma_j = \frac{\mu(e_j)}{\sum_{j=1}^{q} \mu(e_j)}.$$

4. Calculate relative adjacency to ideal solution

$$(C_i, c_i) = \left(\frac{{R_i}^-}{{R_i}^+ + {R_i}^-}, \frac{{M_i}^-}{{M_i}^+ + {M_i}^-} \right),$$
$$0 < C_i < 1, \ 0 < c_i < 1, i = 1, 2, \ldots, p.$$

5. Form matrix $E = [E_i]$ with $E_i = \frac{C_i + c_i}{2}$, $i = 1, 2, \ldots, p$.

6. The best choice is an object u_t such that $E_t \geq E_j$, for all $j \neq t$..

Example 5.1. An Educational institution assesses several universities in order to choose the best university. Let $U = \{u_1, u_2, u_3, u_4\}$ be a set of universities and $A = \{e_1, e_2, e_3, e_4, e_5\}$ is the set of assessment criteria, namely, e_1=Teacher Credibility, e_2=facility, e_3=accreditation, e_4=research and $e_5 = alumni$. The assessment was carried out by two trusted teams and provided an assessment of the university in terms of 5 parameters. The assessment is expressed in the form of membership values. On the other hand, the assessment is also carried out by members of the university who concern about conditions in the university and the assessment is expressed in the form of grades. On the other hand, the Educational institution assumes that degree of important of parameters are 0.8, 0.6, 0.7, 0.7, and 0.6 for e_1, e_2, e_3, e_4 and e_5 respectively. The evaluation results by evaluators is given in Table 10.

We use Algorithm 1 to determine the best university by the following steps.
1. Input the evaluation result in matrix D below (or see Table 10).

$$D = \begin{pmatrix} \left(\frac{3}{\{0.7,0.8\}}, 0.8\right) & \left(\frac{3}{\{0.7,0.75\}}, 0.6\right) & \left(\frac{2}{\{0.6,0.7\}}, 0.7\right) & \left(\frac{2}{\{0.55,0.65\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.75\}}, 0.6\right) \\ \left(\frac{2}{\{0.65,0.75\}}, 0.8\right) & \left(\frac{2}{\{0.6,0.75\}}, 0.6\right) & \left(\frac{2}{\{0.6,0.7\}}, 0.7\right) & \left(\frac{3}{\{0.75,0.8\}}, 0.7\right) & \left(\frac{2}{\{0.6,0.75\}}, 0.6\right) \\ \left(\frac{2}{\{0.65,0.75\}}, 0.8\right) & \left(\frac{1}{\{0.55,0.65\}}, 0.6\right) & \left(\frac{2}{\{0.65,0.8\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.75\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.85\}}, 0.6\right) \\ \left(\frac{3}{\{0.65,0.75\}}, 0.8\right) & \left(\frac{2}{\{0.6,0.7\}}, 0.6\right) & \left(\frac{1}{\{0.55,0.7\}}, 0.7\right) & \left(\frac{2}{\{0.6,0.75\}}, 0.7\right) & \left(\frac{3}{\{0.7,0.85\}}, 0.6\right) \end{pmatrix}.$$

Table 10: Assessment data from several universities

$U_i \backslash e_j$	e_1	e_2	e_3	e_4	e_5
u_1	$\left(\frac{3}{\{0.7,0.8\}}, 0.8\right)$	$\left(\frac{3}{\{0.7,0.75\}}, 0.6\right)$	$\left(\frac{2}{\{0.6,0.7\}}, 0.7\right)$	$\left(\frac{2}{\{0.55,0.65\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.75\}}, 0.6\right)$
u_2	$\left(\frac{2}{\{0.65,0.75\}}, 0.8\right)$	$\left(\frac{2}{\{0.6,0.75\}}, 0.6\right)$	$\left(\frac{2}{\{0.6,0.7\}}, 0.7\right)$	$\left(\frac{3}{\{0.75,0.8\}}, 0.7\right)$	$\left(\frac{2}{\{0.6,0.75\}}, 0.6\right)$
u_3	$\left(\frac{2}{\{0.65,0.75\}}, 0.8\right)$	$\left(\frac{1}{\{0.55,0.65\}}, 0.6\right)$	$\left(\frac{2}{\{0.65,0.8\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.75\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.85\}}, 0.6\right)$
u_4	$\left(\frac{3}{\{0.65,0.75\}}, 0.8\right)$	$\left(\frac{2}{\{0.6,0.7\}}, 0.6\right)$	$\left(\frac{1}{\{0.55,0.7\}}, 0.7\right)$	$\left(\frac{2}{\{0.6,0.75\}}, 0.7\right)$	$\left(\frac{3}{\{0.7,0.85\}}, 0.6\right)$

2. Transform the matrix D to be the matrix D'

$$D' = \begin{pmatrix} \left(\frac{3}{0.75}, 0.8\right) & \left(\frac{3}{0.725}, 0.6\right) & \left(\frac{2}{0.65}, 0.7\right) & \left(\frac{2}{0.6}, 0.7\right) & \left(\frac{3}{0.725}, 0.6\right) \\ \left(\frac{2}{0.7}, 0.8\right) & \left(\frac{2}{0.675}, 0.6\right) & \left(\frac{2}{0.65}, 0.7\right) & \left(\frac{3}{0.775}, 0.7\right) & \left(\frac{2}{0.675}, 0.6\right) \\ \left(\frac{2}{0.7}, 0.8\right) & \left(\frac{1}{0.6}, 0.6\right) & \left(\frac{2}{0.725}, 0.7\right) & \left(\frac{3}{0.725}, 0.7\right) & \left(\frac{3}{0.775}, 0.6\right) \\ \left(\frac{3}{0.7}, 0.8\right) & \left(\frac{2}{0.65}, 0.6\right) & \left(\frac{1}{0.625}, 0.7\right) & \left(\frac{2}{0.625}, 0.7\right) & \left(\frac{3}{0.775}, 0.6\right) \end{pmatrix}.$$

3. Transform matrix D' to be normalized decision matrix V.

$$V = \begin{pmatrix} \left(\frac{0.5882}{0.5245}, 0.24\right) & \left(\frac{0.7143}{0.5451}, 0.18\right) & \left(\frac{0.5556}{0.4887}, 0.2\right) & \left(\frac{0.3922}{0.4380}, 0.2\right) & \left(\frac{0.5357}{0.4899}, 0.18\right) \\ \left(\frac{0.3922}{0.4895}, 0.24\right) & \left(\frac{0.4762}{0.5075}, 0.18\right) & \left(\frac{0.5556}{0.4887}, 0.2\right) & \left(\frac{0.5882}{0.5657}, 0.2\right) & \left(\frac{0.3571}{0.4561}, 0.18\right) \\ \left(\frac{0.3922}{0.4895}, 0.24\right) & \left(\frac{0.2381}{0.4511}, 0.18\right) & \left(\frac{0.5556}{0.5451}, 0.2\right) & \left(\frac{0.5882}{0.5292}, 0.2\right) & \left(\frac{0.5357}{0.5237}, 0.18\right) \\ \left(\frac{0.5882}{0.4895}, 0.24\right) & \left(\frac{0.4762}{0.4887}, 0.18\right) & \left(\frac{0.2778}{0.4699}, 0.2\right) & \left(\frac{0.3922}{0.4562}, 0.2\right) & \left(\frac{0.5357}{0.5237}, 0.18\right) \end{pmatrix}.$$

4. Calculate matrix W

$$W = \begin{pmatrix} \left(\frac{0.1412}{0.1259}\right) & \left(\frac{0.1286}{0.0981}\right) & \left(\frac{0.1111}{0.0977}\right) & \left(\frac{0.0784}{0.0876}\right) & \left(\frac{0.0964}{0.0882}\right) \\ \left(\frac{0.0941}{0.1175}\right) & \left(\frac{0.0857}{0.0914}\right) & \left(\frac{0.1111}{0.0977}\right) & \left(\frac{0.1176}{0.1131}\right) & \left(\frac{0.0643}{0.0821}\right) \\ \left(\frac{0.0941}{0.1175}\right) & \left(\frac{0.0429}{0.0812}\right) & \left(\frac{0.1111}{0.1090}\right) & \left(\frac{0.1176}{0.1058}\right) & \left(\frac{0.0964}{0.0943}\right) \\ \left(\frac{0.1412}{0.1175}\right) & \left(\frac{0.0857}{0.0880}\right) & \left(\frac{0.0556}{0.0940}\right) & \left(\frac{0.0784}{0.0912}\right) & \left(\frac{0.0964}{0.0943}\right) \end{pmatrix}.$$

5. Find the positive ideal solution D^+ and the negative ideal solution D^-

$$\begin{aligned} D^+ &= \left(\left(\frac{0.1412}{0.1259}\right) \left(\frac{0.1286}{0.0981}\right) \left(\frac{0.1111}{0.1090}\right) \left(\frac{0.1176}{0.1131}\right) \left(\frac{0.0964}{0.0943}\right)\right) \\ D^- &= \left(\left(\frac{0.0941}{0.1175}\right) \left(\frac{0.0429}{0.0812}\right) \left(\frac{0.0556}{0.0940}\right) \left(\frac{0.0784}{0.0876}\right) \left(\frac{0.0643}{0.0821}\right)\right). \end{aligned}$$

Here, we assume that all parameters are supporting ones.

6. Calculate separation measures

$$\begin{aligned} ({S_1}^+, {s_1}^+) =& (0.0387, 0.0030), ({S_2}^+, {s_2}^+) = (0.0707, 0.0224), \\ ({S_3}^+, {s_3}^+) =& (0.0975, 0.0224), ({S_4}^+, {s_4}^+) = (0.0800, 0.0300), \\ ({S_1}^-, {s_1}^-) =& (0.1131, 0.0220), ({S_2}^-, {s_2}^-) = (0.0800, 0.0283), \\ ({S_3}^-, {s_3}^-) =& (0.0748, 0.0265), ({S_4}^-, {s_4}^-) = (0.0640, 0.0173). \end{aligned}$$

7. Calculate relative adjacency to ideal solution

$$\begin{aligned} (C_1, c_1) =& (0.7451, 0.4231), (C_2, c_2) = (0.5309, 0.5582), \\ (C_3, c_3) =& (0.4341, 0.5419), (C_4, c_4) = (0.4444, 0.3658). \end{aligned}$$

8. Find E_i, $E_1 = 0.5841, E_2 = 0.5445, E_3 = 0.4880, E_4 = 0.4051$. We obtain $E_4 < E_3 < E_2 < E_1$.

9. The order of universities from the best is u_1, u_2, u_3 and u_4.

If we apply Algorithm 2, for Example 5.1, we will get separation measures as follows (see Table 11).

Table 11: Separation measures

U_i	R_i^+	M_i^+	R_i^-	M_i^-	E_i
u_1	0.2	0.012	0.98	0.011	0.64
u_2	0.6	0.011	0.58	0.011	0.49
u_3	0.6	0.009	0.58	0.012	0.53
u_4	0.4	0.014	0.78	0.008	0.51

Based on Table 11, we obtain that the ranking order of E_i is $E_1 > E_3 > E_4 > E_2$. Hence the best university is u_1.

We see that when the problem in Example 5.1 was solved by the two algorithms above, we obtained a different conclusion. This clearly can happen because the two algorithms use different approaches, especially in using membership values in the calculation and formulation of the Separation Measures.

6. Conclusion

In this article, we proposed the concept of Generalized Hesitant Fuzzy N-Soft sets (GHFNSSs) and defined some of their complements and operations, such as restricted and extended intersections and restricted and extended unions of two GHFNSSs. Based on the operations, we prove some properties, such as associative and distributive laws. Lastly, we propose two algorithms for decision-making problems by extending the TOPSIS method to apply under GHFNSS information. Since the GHFNSS is a generalization of Generalized Hesitant Fuzzy Soft sets, there are many further studies for scholars on the issue of studying NSSs, such as a generalization of Hesitant Intuitionistic Fuzzy Soft Sets and Interval-valued Hesitant Intuitionistic Fuzzy Soft Sets.

Acknowledgements

This research is supported by research fund from Universitas Andalas in accordance with contract of Professor's acceleration research cluster scheme (Batch II), T/25/UN.16.17/PP.Energi-PDU-KRP2GB-Unand/ 2021.

References

[1] M. Akram, A. Adeel, J.C.R. Alcantud, *Fuzzy N-soft sets: a novel model with applications*, Journal of Intelligent and Fuzzy Systems, 35 (2018), 4757-4771.

[2] M. Akram, A. Adeel, J.C.R. Alcantud, *Hesitant fuzzy N-soft sets: a new model with applications in decision-making*, Journal of Intelligent and Fuzzy Systems, 36 (2019), 6113-6127.

[3] M. Akram, G. Ali, J.C.R. Alcantud, F. Fatimah, *Parameter reductions in N-soft sets and their applications in decision making*, Expert Systems. 38 (2020), 1-15.

[4] M. Akram, U. Amjad, B. Davvaz, *Decision-making analysis based on bipolar fuzzy N-soft information*, Comp. Appl. Math., 40 (2021).

[5] J.C.R. Alcantud, F. Feng, R.R. Yager, *An N-soft set approach to rough sets*, IEEE Transactions on Fuzzy Systems, 28 (2020), 2996-3007.

[6] J.C.R. Alcantud, *The semantics of N-soft sets, their applications, and a coda about three-way decision*, Information Sciences, 606 (2022), 837-852.

[7] F. Fatimah, D. Rosadi, R.B.F. Hakim, J.C.R. Alcantud, *N-soft sets and their decision making algorithms*, Soft Computing, 22 (2018), 3829-3842.

[8] F. Fatimah, J.C.R. Alcantud, *The multi-fuzzy N-soft set and its applications to decision-making*, Neural Comput & Applic, 33 (2021), 11437-11446.

[9] C.L. Hwang, K. Yoon, *Multiple attribute decision making, methods and applications, a state-of-the-art survey*, Lecture Notes in Economics and Mathematical Systems, Springer Verlag Berlin Heidelberg New York 186, 1981.

[10] H. Kamaci, S. Petchimuthu, *Bipolar N-soft set theory with applications*, Soft Comput, 24 (2020), 16727-16743.

[11] F. Karaaslan, F. Karamaz, *Hesitant fuzzy parameterized hesitant fuzzy soft sets and their applications in decision-making*, International Journal of Computer Mathematics, 99 (2022), 1868-1889.

[12] A.Z. Khameneh, A. Kilicman, *Multi-attribute decision-making based on soft set theory: a systematic review*, Soft Comput, 23 (2019), 6899-6920.

[13] C. Li, D. Li, and J. Jin, *Generalized hesitant fuzzy soft sets and its application to decision making*, International Journal of Pattern Recognition and Artificial Intelligence, 33 (2019), 1950019.

[14] P.K. Maji, R. Biswas, A.R. Roy, *Fuzzy soft sets*, Journal of Fuzzy Mathematics, 9 (2001), 589-602.

[15] P. Majumdar, S.K. Samanta, *Generalised fuzzy soft sets*, Computers and Mathematics with Applications, 59 (2010), 1425-1432.

[16] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, 37 (1999), 19-31.

[17] S.M. Mostafa, F.F. Kareem, H.A. Jad *Brief review of soft set and its application in coding theory*, Journal of New Theory, 33 (2020), 95-106.

[18] A.R. Roy, P.K. Maji, *A fuzzy soft set theoretic approach to decision-making problems*, Journal of Computational and Applied Mathematics, 203 (2007), 412-418.

[19] V. Torra, *Hesitant fuzzy set*, International Journal of Intelligent Systems, 25 (2010), 529-539.

[20] F. Wang, X. Li, X. Chen *Hesitant fuzzy soft set and its applications in multicriteria decision making*, Journal of Applied Mathematics 2014: Article ID 643785, 10 pages.

[21] J.Q. Wang, X.E. Li, X.H. Chen *Hesitant fuzzy soft sets with application in multicriteria group decision making problems*, The Scientific World Journal 2015: Article ID 806983, 14 pages.

[22] L. Wang, K. Qin, *Incomplete fuzzy soft sets and their application to decision-making*, Symmetry, 11 (2019), 535.

[23] L.A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

[24] J. Zhan, J.C.R. Alcantud, *A survey of parameter reduction of soft sets and corresponding algorithms*, Artif Intell Rev, 52 (2019), 1839-1872.

[25] H. Zhang, T. Nan, Y. He, *q-Rung orthopair fuzzy N-soft aggregation operators and corresponding applications to multiple-attribute group decision making*, Soft Computing, 26 (2022), 6087–6099.

Accepted: February 6, 2023

A note on k-zero-divisor hypergraphs of some commutative rings

Elham Mehdi-Nezhad
Department of Mathematics and Applied Mathematics
University of the Western Cape, Private Bag X17,
Bellville 7535, Cape Town
South Africa
emehdinezhad@uwc.ac.za

Amir M. Rahimi*
School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran
Iran
amrahimi@ipm.ir

Abstract. The main object of this paper is to study and characterize the connectedness, diameter, dominating sets and domination number of the k-zero-divisor hypergraph $H_k(R)$ of a finite direct product of integral domains and a class of commutative Artinian rings R, respectively. We will show that the k-zero-divisor hypergraph associated to the direct product of $k \geq 3$ integral domains (resp., commutative Artinian rings which are the direct product of $k \geq 3$ local rings) are connected with diameter at most 3 and domination number at most 2 (resp., connected with diameter at most 4 and domination number at most $2k$). We will also provide some examples related to these results.

Keywords: k-uniform hypergraph, k-zero-divisor, dominating set, domination number, connectedness, diameter, Artinian ring.

1. Introduction and definitions

The main goal of this paper is to study and characterize *the connectedness, diameter, dominating sets and domination number* of the k-zero-divisor hypergraphs $H_k(R)$ of two well-known classes of commutative rings R; namely, *a finite direct product of integral domains and a class of commutative Artinian rings*, respectively. Through out this work, all rings are *commutative with identity* $1 \neq 0$, $J(R)$ denotes the Jacobson radical of R, and a *local ring* is a ring with only one maximal ideal.

In this section we recall some definitions together with some references and will discuss the main results in the next section. We will show (Theorem 2.1) that the k-zero-divisor hypergraph associated to $k \geq 3$ direct product of integral

∗. Corresponding author

domains (e.g., $R/J(R)$ of a semilocal ring (Corollary 2.1), finite reduced rings (Corollary 2.2)), respectively commutative Artinian rings which are the direct product of $k \geq 3$ local rings (Theorem 2.2), e.g., ring of integers modulo n Corollary 2.3 are connected with diameter at most 3 and domination number at most 2 (resp., connected with diameter at most 4 and domination number at most $2k$).

The concept of the *zero-divisor graph* of a commutative ring has been studied extensively by many authors, and the *k-zero-divisor hypergraph* of a commutative ring R, denoted by $H_k(R)$, is a nice abstraction of this concept which was first introduced by Eslahchi and Rahimi [6]. In their work, they studied some ring-theoretic properties of the k-zero-divisors of R and graph-theoretic properties of $H_k(R)$ and investigated the interplay between the ring-theoretic properties of R and the graph-theoretic properties of its associated *k-uniform hypergraph* $H_k(R)$. Specially, in Section 3, they discussed the connectedness and completeness of $H_3(R)$ and showed that its (diameter, girth) is bounded above by (4, 9) and also found a lower bound for its clique number. Furthermore, the research on this subject continued and extended by other authors as well (e.g., [14], [15], [16]).

We now define the zero-divisor graph of a commutative ring.

The *zero-divisor graph* of a commutative ring R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is an empty graph if and only if R is an integral domain. Beck in [4] introduced the concept of a zero-divisor graph of a commutative ring, but this work was mostly connected with colorings of zero-divisor of rings. The above definition first appeared in the work of D.F. Anderson and Livingston [2], which contains several fundamental results concerning $\Gamma(R)$. This definition, unlike the earlier work of D.D. Anderson and Naseer [1] and Beck [4], does not take zero to be a vertex of $\Gamma(R)$.

We now recall the following two definitions, i.e., the k-zero-divisor and k-zero-divisor hypergraph of a ring, respectively from [6].

Definition 1.1. *Let R be a commutative ring and $k \geq 2$ a fixed integer. A nonzero non unit element a_1 in R is said to be a k-zero-divisor in R if there exist $k-1$ distinct non unit elements $a_2, a_3, \ldots, a_k$ in R different from a_1 such that $a_1a_2a_3 \cdots a_k = 0$ and the product of no elements of any proper non-singleton subset of $A = \{a_1, a_2, \ldots, a_k\}$ is zero.*

Definition 1.2. *Let R be a commutative ring (with $1 \neq 0$) and let $Z(R, k)$ be the set of all k-zero-divisors in R. We associate a k-uniform hypergraph $H_k(R)$ to R with vertex set $Z(R, k)$, and for distinct elements $x_1, x_2, \ldots, x_k$ in $Z(R, k)$, the set $\{x_1, x_2, \ldots, x_k\}$ is an edge of $H_k(R)$ if and only if $x_1x_2 \cdots x_k = 0$ and the product of elements of no $(k-1)$-subset of $\{x_1, x_2, \ldots, x_k\}$ is zero.*

Remark 1.1. It is not difficult to show that the statement "the product of no elements of any proper (nonsingleton) subset of A is zero" or the statement "the product of no elements of any $(k-1)$-subset of A is zero" can be used in Definition 1.1 equivalently. Clearly, from Definition 1.1, every element of the set $\{a_2, a_3, \ldots, a_k\}$ is a k-zero-divisor in R. It is clear that every k-zero-divisor in R is also a zero-divisor in R, but, the converse is not true in general. For example, the element 2 is a zero-divisor, but not a 3-zero-divisor in $\mathbb{Z}_{10}$ and 2 in $\mathbb{Z}_4$ is a zero-divisor but not a 2-zero-divisor.

We now review some basic graph-theoretic definitions and notions used throughout to keep this paper as self contained as possible; and for the necessary definitions and notations of graphs and hypergraphs, we refer the reader to standard texts of graph theory such as [17] and [5].

A hypergraph is a pair (V, E) of disjoint sets, where the elements of E are non-empty subsets (of any cardinality) of V. The elements of V are the vertices, and the elements of E are the edges of the hypergraph. The hypergraph $H = (V, E)$ is called k-uniform whenever every edge e of H consists of k vertices. A k-uniform hypergraph H is called complete if every k-subset of the vertices is an edge of H. An r-coloring of a hypergraph $H = (V, E)$ is a map $c : V \to \{1, 2, \cdots, r\}$ such that for every edge e of H, there exist at least two vertices x and y in e with $c(x) \neq c(y)$. The smallest integer r such that H has an r-coloring is called the chromatic number of H and is denoted by $\chi(H)$. A path in a hypergraph H is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k$ such that v_i, v_{i+1} is in e_i for all $1 \leq i \leq k-1$. The number of edges of a path is its length. The distance between two vertices x and y of H, denoted by $d_H(x, y)$, is the length of the shortest path from x to y. If no such path between x and y exists, we set $d_H(x, y) = \infty$. The greatest distance between any two vertices in H is called the diameter of H and is denoted by $\mathrm{diam}(H)$. The hypergraph H is said to be connected whenever $\mathrm{diam}(H) < \infty$. A cycle in a hypergraph H is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$ such that v_i, v_{i+1} is in e_i for all $1 \leq i \leq k-1$ with $v_k, v_1 \in e_k$. The girth of a hypergraph H containing a cycle, denoted by $\mathrm{gr}(H)$, is the smallest size of the length of cycles of H.

We now define the notion of the dominating set and domination number of a hypergraph and for a detailed study of the dominating sets and domination number of the zero-divisor graph of a commutative ring (resp., with respect to an ideal), see [13] and [10], respectively (see, also, [7]).

Definition 1.3. *Let $H = (V, E)$ be a hypergraph with vertex set$V$ and edge set E. A nonempty set $S \subseteq V$ is a dominating set of H if every vertex in V is either in S or is adjacent to a vertex in S. That is, for every $v \in V \setminus S$, there exists an edge $e \in E$ such that $v \in e$ and the intersection of e and S is nonempty. The domination number of H, denoted by $\gamma(H)$, is the minimum cardinality among all dominating sets of H.*

We end this section with a brief general overview related to graphs associated to some algebras.

The area of research on assigning a graph to an algebra (algebraic structure) has been very active (specially) since last two decades and there are many papers which apply combinatorial methods (using graph-theoretic properties and parameters such as *connectedness, planarity, clique number, chromatic number, independence number, domination number*, and so on) to obtain algebraic results and vice versa. For instance, there are many papers on this interdisciplinary subject and for a short list of them, see for example [11] and [12] (covering many different cases using *commutator theory*) and also see the work of Mehdi-Nezhad and Rahimi in [9] for some other references and a *brief historical note* on some graphs associated to some algebraic structures.

2. Main results

We begin this section with a lemma using for Theorem 2.1 and provide some examples and corollaries as an application to this theorem (e.g., $R/J(R)$ of a semilocal ring (Corollary 2.1), finite reduced rings (Corollary 2.2) to show that their corresponding k-zero-divisor hypergraphs are connected with diameter (resp., domination number) at most 3 (resp., 2). Then, we continue to show that $H_k(R)$ is connected with diameter at most 4 and domination number at most $2k$ (Theorem 2.2), where R is an Artinian ring which is the direct product of $k \geq 3$ local rings (see also Corollary 2.3 as an application to this theorem).

Lemma 2.1. *Let $k \geq 3$ be a fixed integer and $R = R_1 \times R_2 \times \cdots \times R_k$ the direct product of k integral domains. Then, $(a_1, a_2, \ldots, a_k) \in R$ is a vertex in $H_k(R)$ if and only if exactly one of its components is zero. That is,*

$$Z(R, k) = \{(a_1, a_2, \ldots, a_k) \in R \mid exactly\ one\ of\ the\ a_i's\ is\ zero\ for\ 1 \leq i \leq k\}.$$

Proof. The sufficient part follows directly from definition. For example, let $x_1 = (a_1, a_2, \ldots, a_k) \in R$ such that exactly one and only one of the components is zero. Without loss of generality, assume that $a_1 = 0$. Let $x_i = (1, 1, \ldots, 1, 0, 1, 1, \ldots, 1)$, where the ith component is the only zero component of x_i for each $2 \leq i \leq k$. Now, it is obvious that $\{x_1, x_2, \ldots, x_k\} \in E(H_k(R))$.

For the necessary part, it is obvious that any k-zero-divisor of R must have at least one zero component. Now, let $x_1 = (a_{11}, a_{12}, \ldots, a_{1k})$ be a k-zero-divisor (vertex in $H_k(R)$) with at least two zero components. Without loss of generality, assume that $a_{11} = a_{12} = 0$. Consequently, there exist $x_2, x_3, \ldots, x_k \in V(H_k(R))$ such that $\{x_1, x_2, \ldots, x_k\} \in E(H_k(R))$, where $x_i = (a_{i1}, a_{i2}, \ldots, a_{ik})$ for all $1 \leq i \leq k$. Thus, $\prod_{i \geq 1} a_{ij} = 0$ for each $j \geq 3$. Now, since R_j is an integral domain, then for each fixed $j \geq 3$, there exists at least one i_j with $1 \leq i \leq k$ such that $a_{i_j j} = 0$. Let I be the set of all i_j's such that $a_{i_j j} = 0$ for the smallest i in the set $\{1, 2, \ldots, k\}$. Thus, we have $x_1 \prod_{i \in I} x_i = 0$ and since $|I| \leq k - 2$, we have a contradiction and the proof is complete. □

Theorem 2.1. *For any fixed integer $k \geq 3$, there exists a ring R whose k-zero-divisor hypergraph is connected with diameter at most 3 and domination number at most 2.*

Proof. Let $R = R_1 \times R_2 \times \cdots \times R_k$ be the direct product of k integral domains. Now, the proof is straight forward by using the above lemma. For instance, $D = \{x_1, x_2\}$ is a dominating set in $H_k(R)$, where $x_1 = (0, 1, 1, \ldots, 1)$ and $x_2 = (1, 0, 1, \ldots, 1)$. Note that $e = \{x_1, x_2, \ldots, x_k\}$ is an edge in $H_k(R)$, where x_i is a k-tuple with ith component 0 and jth component 1 for each $1 \leq i \neq j \leq k$. □

We now provide some examples as an application to the above theorem.

Example 2.1. For any fixed integer $k \geq 3$, we have the following:

(a) Let R be the direct product of k factors of the ring $\mathbb{Z}_2$. Clearly, $H_k(R)$ has only one edge and hence is connected and its domination number is 1 since the singleton set of each vertex is a dominating set. Note that the chromatic number of this hypergraph is 2.

(b) Let R be the direct product of k factors of the ring $\mathbb{Z}_p$ for some prime $p \geq 2$. Then, by the above theorem, $H_k(R)$ is a connected k-zero-divisor hypergraph with diameter at most 3 and domination number at most 2.

(c) let $n = p_1 \cdots p_k$ for distinct primes $p_1, \ldots, p_k$. Then, $H_k(\mathbb{Z}_n)$ is a connected k-zero-divisor hypergraph with diameter at most 3 and domination number at most 2. The proof follows directly from the above theorem and the fact that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$.

We now apply the above theorem to a semilocal ring.

Corollary 2.1. *For a semilocal ring R with $k \geq 3$ maximal ideals $M_1, M_2, \ldots, M_k$, there exists a connected k-zero-divisor hypergraph associated to $R/J(R)$ whose diameter is bounded above by 3 and its domination number is at most 2, where $J(R)$ is the Jacobson radical of R.*

Proof. The proof is an immediate consequence of the above theorem since (by Chinese Remainder Theorem) $R/J(R) \cong F_1 \times F_2 \times \ldots \times F_k$, where $F_i = R/M_i$ for each $1 \leq i \leq k$. □

We next apply the above theorem when R is a reduced or finite reduced ring, i.e., a direct product of finitely many finite fields.

Corollary 2.2. *Let R be a reduced* (*resp., finite reduced*) *commutative ring* (*which is not an integral domain*) *with at least $k \geq 3$ minimal prime ideals and nil(R) the ideal of nilpotent elements of R. Then, there exists a ring whose k-zero-divisor hypergraph is connected with diameter at most 3 and domination number at most 2* (*resp., $H_k(R)$ satisfies the mentioned properties*).

Proof. Let $P_1, \ldots, P_k$ be the minimal prime ideals of R. Then, $P_1 \cap \cdots \cap P_k = \text{nil}(R) = \{0\}$ since R is reduced. Thus there is a monomorphism from R to $T = R/P_1 \times \cdots \times R/P_k$. Now, the proof follows from the above theorem and for the finite case, R is isomorphic to T, by Chinese Remainder Theorem, since prime ideals are maximal in a finite ring. □

We next discuss the results of Theorem 2.1 for commutative Artinian rings which are the direct product of $k \geq 3$ local rings. Recall that any commutative Artinian ring is a finite direct product of Artinian local rings ([3, Theorem 8.7]).

Theorem 2.2. *Let R be a commutative Artinian ring (in particular, R could be a finite commutative ring) which is the direct product of $k \geq 3$ Artinian local rings, where k is a fixed integer. Then, $H_k(R)$, the k-zero-divisor hypergraph of R, is connected with diameter at most 4 and domination number at most $2k$.*

Proof. Let $R = R_1 \times R_2 \times \cdots \times R_k$, where R_i is an Artinian local ring with maximal ideal M_i and assume $M_i \neq 0$ for each $i = 1, 2, \ldots, k$. By [8, Theorem 82], suppose $M_i = \text{ann}(m_i)$ for some nonzero $m_i \in M_i$ and each $i = 1, 2, \ldots, k$. We now construct a dominating set S of size $2k$ for $H_k(R)$. Let $S = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$, where for each $1 \leq i \leq k$, y_i is a k-tuple whose ith component is 0 and other components are 1's; and for each $1 \leq i \leq (k-1)$, x_i is a k-tuple whose ith component is m_i, its kth component is 0, and the other components are all 1's, and $x_k = (1, 1, \ldots, 1, 0, m_k)$. Further, we take $m_i = 1$ whenever $M_i = (0)$ for any $1 \leq i \leq k$. Note that a nonzero element $(a_1, a_2, \ldots, a_k)$ is a vertex in $H_k(R)$ (k-zero-divisor in R) provided that at most one of its components can be 0 and at least one of its components; must belong to its corresponding maximal ideal. □

We now end the paper by applying the above theorem to $\mathbb{Z}_n$, the ring of integers modulo n.

Corollary 2.3. *For any fixed integer $k \geq 3$, let $n = p_1^{t_1} \cdots p_k^{t_k}$ for distinct primes $p_1, \ldots, p_k$ and positive integers $t_1, \ldots, t_k$. Then, $H_k(\mathbb{Z}_n)$ is a connected k-zero-divisor hypergraph with diameter at most 4 and domination number at most $2k$.*

Proof. The proof follows directly from Theorem 2.2 and the fact that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{t_1}} \times \cdots \times \mathbb{Z}_{p_k^{t_k}}$. Note that for any prime $p \geq 2$ and integer $t \geq 2$, $\mathbb{Z}_p^t$ is a local ring. □

Acknowledgement

The research of the second author was in part supported by grant no. 1400130011 from IPM.

References

[1] D.D. Anderson, M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, 159 (1993), 500-514.

[2] D.F. Anderson, P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (1999), 434-447.

[3] M.F. Atiyah, I.G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969.

[4] I. Beck, *Coloring of commutative rings*, J. Algebra, 116 (1988), 208-226.

[5] C. Berge, *Graphs and hypergraphs*, North-Holland Publishing Company, London, 2003.

[6] Ch. Eslahchi, A.M. Rahimi, *The k-zero-divisor hypergraph of a commutative ring*, Int. J. Math. Math. Sci. 2007, Art. ID 50875, 15 pp.

[7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.

[8] I. Kaplansky, *Commutative rings*, The University of Chicago Press, Chicago, Ill.-London, 1974.

[9] E. Mehdi-Nezhad, A. M Rahimi, *Comaximal submodule graphs of unitary modules*, Lib. Math., 38 (2018), 75-98.

[10] E. Mehdi-Nezhad, A. M Rahimi, *Dominating sets of the comaximal and ideal-based zero-divisor graphs of commutative rings*, Quaestiones Mathematicae, 38 (2015), 17pages.

[11] E. Mehdi-Nezhad, A. M Rahimi, *The annihilation graphs of commutator posets and lattices with respect to an element*, Journal of Algebra and Its Applications, Volume 16, Issue 06, (2017), 20 pages.

[12] E. Mehdi-Nezhad, A. M Rahimi, *The annihilation graphs of commutator posets and lattices with respect to an ideal*, Journal of Algebra and its applications, J. Algebra Appl. 17, Issue 06, (2018), 23 pages.

[13] D. A. Mojdeh, A. M. Rahimi, *Dominating sets of some graphs associated to commutative rings*, Communications in Algebra, 40 (2012), 3389-3396

[14] K. Selvakumar, V. Ramanathan, *On the genus of the k-annihilating-ideal hypergraph of commutative rings*, Indian Journal of Pure and Applied Mathematics, (2019), 461-475

[15] K. Selvakumar, V. Ramanathan, *Classification of nonlocal rings with genus one 3-zero-divisor hypergraphs*, Comm. Algebra, 45 (2016), 275-284.

[16] T. Tamizh Chelvam, K. Selvakumar, V. Ramanathan, *On the planarity of the k-zero-divisor hypergraphs*, AKCE Inter. J. Graphs and Combin., 12 (2015), 169-179.

[17] D.B. West, *Introduction to graph theory*, (Second Edition), Prentice Hall, USA, 2001.

Accepted: March 18, 2022

Relative averaging operators and trialgebras

Li Qiao
School of Mathematics and Statistics
Southwest University
Chongqing 400715
China
qiaoliswu@swu.edu.cn

Jun Pei*
School of Mathematics and Statistics
Southwest University
Chongqing 400715
China
peitsun@swu.edu.cn

Abstract. In this paper, the relative averaging operator is introduced as a relative generalization of the averaging operator. We explicitly determine all averaging operators on the 2-dimensional complex associative algebra. The results show that not every dialgebra can be derived from an averaging algebra. We then generalize the construction of dialgebras and trialgebras from averaging operators to a construction from relative averaging operators. It is shown that this construction from relative averaging operators gives all dialgebras and trialgebras.
Keywords: averaging operator, relative averaging operator, dialgebra, trialgebra.

1. Introduction

There are two seemingly unrelated objects, namely averaging operators (resp., of weight λ) and dialgebras (resp., trialgebras). This paper shows that there is a close tie between them, generalizing and strengthening a previously established connection from averaging algebras to dialgebras [1, 12, 13].

Let $\mathbf{k}$ be a unitary commutative ring and A a $\mathbf{k}$-algebra. If a $\mathbf{k}$-linear map $P : A \to A$ satisfies the averaging relations:

$$(1) \qquad P(x \cdot P(y)) = P(x) \cdot P(y) = P(P(x) \cdot y), \quad \forall x, y \in A,$$

then P is called an averaging operator and (A, P) is called an averaging algebra.

Averaging operator was implicitly studied in the famous paper of O. Reynolds [15] in connection with the theory of turbulence and explicitly defined by Kolmogoroff and Kampé de Fériet [7]. It later attracted the attentions of other well-known mathematicians including G. Birkhoff [4] and Rota with motivation from quantum physics and combinatorics. It has found diverse applications in

∗. Corresponding author

many areas of pure and applied mathematics, such as the theory of turbulence, probability, function analysis, and information theory [8, 15, 16, 17, 18, 19].

Recently, averaging operators have been studied for many algebraic structures [1, 6, 12, 13]. In [14], we studied the averaging operators from an algebraic point of view and built a connection between averaging operators and large Schröder numbers. We also defined a related new operator, called averaging operator of weight λ in [13]. For a fixed $\lambda \in \mathbf{k}$. An *averaging operator of weight* λ on A is a $\mathbf{k}$-linear map $P : A \longrightarrow A$ such that Eq. (1) holds and

$$P(x) \cdot P(y) \quad = \quad \lambda P(x \cdot y), \quad \forall x, y \in A. \tag{2}$$

By definition, if P is an averaging operator of weight 1, then λP is an averaging operator of weight λ. We note that an averaging operator of weight zero is not an averaging operator. So we can't give a uniform definition for the averaging operator as in the case of Rota-Baxter operators of weight λ.

On the other hand, motivated by the study of the periodicity in algebraic K-theory, J.-L. Loday [9] introduced the concept of Leibniz algebra thirty years ago as a non-skew-symmetric generalization of Lie algebra. He then defined dialgebra [10] as the enveloping algebra of Leibniz algebra by analogy with associative algebra as the enveloping algebra of Lie algebra.

Definition 1.1. A *dialgebra* is a $\mathbf{k}$-module D with two associative bilinear operations $\dashv$ and $\vdash$ such that

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{3}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{4}$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \tag{5}$$

for all $x, y, z \in D$.

M. Aguiar showed the following connection from averaging algebras to dialgebras.

Theorem 1.1 ([1])**.** *Let (A, P) be an averaging $\mathbf{k}$-algebra. Define two new operations on A by*

$$x \dashv y = xP(y), \quad x \vdash y = P(x)y, \quad \forall x, y \in A. \tag{6}$$

Then $(A, \dashv, \vdash)$ is a dialgebra.

Theorem 1.1 gives a functor from the category of averaging algebras to the category of dialgebras. The relationship between averaging algebras and dialgebras is generalized in [13] in two directions. In one direction, the relationship is generalized from associative algebras to other algebraic structures. In the other direction, the averaging operator of weight λ is introduced to give trialgebra.

The former studies told us that there is a close tie between averaging algebra (resp., of weight λ) and dialgebra (resp., trialgebra). Then it is natural to ask whether every dialgebra (resp., trialgebra) could be derived from an averaging algebra (resp., of weight λ) by a construction like Eq. (6). As Section 2 shows, the answer is no.

Interestingly, there is an analogous phenomenon that a Rota-Baxter algebra gives a dendriform or tridendriform algebra, depending on the weight. The problem that whether every dendriform algebra and tridendriform algebra could be derived from a Rota-Baxter algebra was solved by C. Bai, L. Guo and X. Ni [3]. They found there is a generalization of the concept of a Rota-Baxter operator that could derive all the dendriform algebras and tridendriform algebras. In this paper, we turn to consider the recovering problem for dialgebras from averaging algebras. Inspired by their observation, we define the concept of relative averaging operator (resp., of weight λ) as a generalization of averaging operator (resp., of weight λ) and show that every dialgebra (resp., trialgebra) can be recovered from a relative averaging operator (resp., of weight λ).

This paper is organized as follows. In the next section, we first determine all averaging operators on the 2-dimensional complex associative algebra and then list the dialgebras induced by these averaging operators. In Section 3 , the definitions of relative averaging operator and relative averaging operator of weight λ are given. Finally, we prove that every dialgebra (resp., trialgebra) can be derived from relative averaging algebra (resp., of weight λ).

2. Averaging operators on the complex 2-dimensional associative algebra

In this section, we determine all averaging operators on 2-dimensional complex associative algebras. Then we find all dialgebras induced by averaging operators on the 2-dimensional complex associative algebra. The results show that not every dialgebra can be derived from an averaging algebra.

There are six associative algebras structures on the 2-dimensional vector space $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ except the trivial one, two of them are non-commutative and the other four are commutative [2, 5]. We list their characteristic matrices in the following and denote the corresponding algebra by $(A_i, \bullet_i)$, $1 \leq i \leq 6$, respectively:

$\bullet_1$	e_1	e_2
e_1	0	e_1
e_2	0	e_2

$\bullet_2$	e_1	e_2
e_1	0	0
e_2	e_1	e_2

$\bullet_3$	e_1	e_2
e_1	e_1	0
e_2	0	0

$\bullet_4$	e_1	e_2
e_1	e_2	0
e_2	0	0

$\bullet_5$	e_1	e_2
e_1	e_1	0
e_2	0	e_2

$\bullet_6$	e_1	e_2
e_1	0	e_1
e_2	e_1	e_2

.

A linear operator $P : A_i \to A_i$ is determined by

$$\begin{pmatrix} P(e_1) \\ P(e_2) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{7}$$

where $a_{ij} \in \mathbb{C}, 1 \le i,j \le 2$. P is an averaging operator on A_i if the above matrix $(a_{ij})_{2\times 2}$ satisfies Eq. (1) for $x, y \in \{e_1, e_2\}$.

In order to show P is an averaging operator, we only need to check

$$P(e_i)P(e_j) = P(e_iP(e_j)) = P(P(e_i)e_j), \quad 1 \le i,j \le 2. \tag{8}$$

It is clear that the zero operator is an averaging operator on A_i. Furthermore, it follows from a direct check that P is an averaging operator if and only if λP is an averaging operator for $0 \neq \lambda \in \mathbb{C}$. Thus, the set $AV(A_i)$ of averaging operators on A_i carries an action of $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ by scalar multiplication. To determine all the averaging operators on A_i, we only need to give a complete set of representatives of $AV(A_i)$ under this action.

We only give the sketch of process for determining averaging operators on A_1 here. The others discussions are the same as A_1.

By direct computation, we have

$$\begin{aligned}
&P(e_1)P(e_1) = a_{11}a_{12}e_1 + a_{12}^2e_2, \quad P(e_1P(e_1)) = a_{11}a_{12}e_1 + a_{12}^2e_2,\\
&P(P(e_1)e_1) = 0, \quad P(e_1)P(e_2) = a_{11}a_{22}e_1,\\
&P(e_1P(e_2)) = a_{11}a_{22}e_1, \quad P(P(e_1)e_2) = a_{11}^2e_1,\\
&P(e_2)P(e_1) = 0, \quad P(e_2P(e_1)) = 0, \quad P(P(e_2)e_1) = 0,\\
&P(e_2)P(e_2) = a_{21}a_{22}e_1 + a_{22}^2e_2, \quad P(e_2P(e_2)) = a_{21}a_{22}e_1 + a_{22}^2e_2,\\
&P(P(e_2)e_2) = (a_{11}a_{21} + a_{21}a_{22})e_1 + a_{22}^2e_2.
\end{aligned}$$

By Eq. (8) and comparing the corresponding coefficients of e_1 and e_2, we have

$$a_{11}a_{12} = 0, \quad a_{12}^2 = 0, \quad a_{11}^2 = a_{11}a_{22}, \quad a_{11}a_{21} = 0.$$

Hence, the averaging operators on A_1 are given by a complete set of representatives of $AV(A_1)$ under the action of $\mathbb{C}^*$ by scalar product consists of the 5 averaging operators whose linear transformation matrices with respect to the basis e_1, e_2 are listed below, where a are non-zero complex numbers:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 2.1. *1. The non-zero averaging operators on A_1 and A_2 are given by*

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a \neq 0.$$

2. The non-zero averaging operators on A_3 are given by

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad a \neq 0.$$

The non-zero averaging operators on A_4 are given by

$$\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \begin{pmatrix}0&0\\0&1\end{pmatrix},\quad \begin{pmatrix}0&1\\0&a\end{pmatrix},\quad \begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \begin{pmatrix}1&a\\0&1\end{pmatrix},\quad a\neq 0.$$

3. *The non-zero averaging operators on A_5 are given by, $a \neq 0$,*

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&a\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\a&a\end{pmatrix}.$$

4. *The non-zero averaging operators on A_6 are given by, $a \neq 0$,*

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&a\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\a&1\end{pmatrix}.$$

By Theorem 1.1 and Theorem 2.1, after a direct computation, we have

Corollary 2.1. *Let $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $(V, \dashv, \vdash)$ be a dialgebra which is induced by the averaging operators on A_1—A_6 and the trivial 2-dimensional complex associative algebra A_0. Then either $(V, \dashv) \cong A_i$, $(V, \vdash) \cong A_i$, $0 \leq i \leq 6$, or one of the following items holds:*

(1) $(V, \dashv) \cong A_0$, $(V, \vdash) \cong A_4$;

(2) $(V, \dashv) \cong A_1$, $(V, \vdash) \cong A_3$;

(3) $(V, \dashv) \cong A_3$, $(V, \vdash) \cong A_2$;

(4) $(V, \dashv) \cong A_1$, $(V, \vdash) \cong A_2$;

(5) $(V, \dashv) \cong A_5$, $(V, \vdash) \cong A_2$.

Remark 2.1. Let $\dashv$ be the zero multiplication and $\vdash = \bullet_i$, $i = 1, 2, 3, 5, 6$. For each i, the multiplications $\dashv$ and $\vdash$ give a dialgebra structure on $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. By Corollary 2.1, the above dialgebras can't be derived from a 2-dimensional complex averaging algebra.

3. Relative averaging operators, dialgebras and trialgebras

In this section we study the relationship between relative averaging operators (resp., of weight λ) and dialgebras (resp., trialgebras) on the domains of these operators. First, we give some related concepts. Then we show that relative averaging operators recover all dialgebras and trialgebras on the domains of the operators.

3.1 A-bimodule k-algebras and relative averaging operators

First, we recall a generalization of the well-known concept of bimodules in [3].

Definition 3.1. Let $(A,*)$ be a $\mathbf{k}$-algebra with multiplication $*$ and $(R,\circ)$ be a $\mathbf{k}$-algebra with multiplication $\circ$. Let $\ell, r : A \longrightarrow End_{\mathbf{k}}(R)$ be two linear maps. We call $(R,\circ,\ell,r)$ or simply R an A-*bimodule* $\mathbf{k}$-*algebra* if (R,ℓ,r) is an A-bimodule that is compatible with the multiplication $\circ$ on R. More precisely, forall $x,y\in A, v,w\in R$, we have

$$\ell(x*y)v = \ell(x)(\ell(y)v), \quad \ell(x)(v\circ w) = (\ell(x)v)\circ w, \tag{9}$$

$$vr(x*y) = (vr(x))r(y), \quad (v\circ w)r(x) = v\circ(wr(x)), \tag{10}$$

$$(\ell(x)v)r(y) = \ell(x)(vr(y)), \quad (vr(x))\circ w = v\circ(\ell(x)w). \tag{11}$$

Note that an A-bimodule (V,ℓ,r) becomes an A-bimodule $\mathbf{k}$-algebra if V is regarded as an algebra with the zero multiplication. For a $\mathbf{k}$-algebra $(A,*)$ and $x\in A$, define the left and right actions $L(x): A\longrightarrow A$, $L(x)y = x*y$; $R(x): A\longrightarrow A$, $yR(x) = y*x$, $y\in A$. For $x\in A$, define

$$L = L_A : A \longrightarrow End_{\mathbf{k}}(A), x\longmapsto L(x); R = R_A : A\longrightarrow End_{\mathbf{k}}(A), x\longmapsto R(x).$$

Then (A,L,R) is an A-bimodule and $(A,*,L,R)$ is an A-bimodule $\mathbf{k}$-algebra.

Now, we can define our generalization of the averaging operator .

Definition 3.2. Let $(A,*)$ be a $\mathbf{k}$-algebra.

1. Let V be an A-bimodule. A linear map $Q: V\longrightarrow A$ is called a *relative averaging operator* on the module V if Q satisfies

$$Q(u)*Q(v) = Q(\ell(Q(u)v)) = Q(ur(Q(v))), \quad u,v\in V. \tag{12}$$

2. Let $(R,\circ,\ell,r)$ be an A-bimodule $\mathbf{k}$-algebra and $\lambda\in\mathbf{k}$. A linear map $Q: R\longrightarrow A$ is called a *relative averaging operator of weight* λ on the algebra R if Q satisfies

$$Q(u)*Q(v) = Q(\ell(Q(u))v) = Q(ur(Q(v))) = \lambda Q(u\circ v), \quad u,v\in R. \tag{13}$$

When V is taken to be the A-bimodule (A,L,R) associated to the algebra A, a relative averaging operator (resp., of weight λ) on the module is just an averaging operator (resp., of weight λ).

3.2 Averaging algebras, dialgebras and trialgebras

The concept of a trialgebra was introduced by Loday and Ronco as a generalization of a dialgebra.

Definition 3.3 ([11]). *A trialgebra is a* **k***-module* T *with three associative bilinear operations* $\dashv$, $\vdash$ *and* $\perp$ *such that*

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \quad (x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{14}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \quad (x \dashv y) \dashv z = x \dashv (y \perp z), \tag{15}$$

$$(x \perp y) \dashv z = x \perp (y \dashv z), \quad (x \dashv y) \perp z = x \perp (y \dashv z), \tag{16}$$

$$(x \vdash y) \perp z = x \vdash (y \perp z), \quad (x \perp y) \vdash z = x \vdash (y \vdash z), \tag{17}$$

for all $x, y, z \in T$.

The Corollary 4.9 in [13] generalized Theorem 1.1 and showed that if $(A, \circ, P)$ is an averaging algebra of weight $\lambda \neq 0$, then the multiplications

$$x \dashv_P y := x \circ P(y), \quad x \vdash_P y := P(x) \circ y, \quad x \perp_P y := \lambda x \circ y, \quad \forall x, y \in A, \tag{18}$$

define a trialgebra $(A, \dashv_P, \vdash_P, \perp_P)$.

For a given **k**-module V, define $\mathcal{AV}(V)$ (resp., $\mathcal{AV}_\lambda(V)$) to be the set of all averaging algebras (resp., of weight λ) on V. Let $\mathcal{AD}(V)$ (resp., $\mathcal{AT}(V)$) be the set of all dialgebras (resp., trialgebras) on V.

Then Eqs. (6) and (18) induce two maps

$$\Phi : \mathcal{AV}(V) \longrightarrow \mathcal{AD}(V), \tag{19}$$

$$\Phi_\lambda : \mathcal{AV}_\lambda(V) \longrightarrow \mathcal{AT}(V). \tag{20}$$

Thus deriving all dialgebras (resp., trialgebras) on V from averaging operators (resp., of weight λ) on V amounts to the surjectivity of Φ (resp., Φ_λ). Unfortunately, by Remark 2.1, these maps are not surjective. Next, we will consider the case of relative averaging operators.

3.3 From relative averaging operators to dialgebras and trialgebras

Theorem 3.1. *Let* $(A, *)$ *be an associative algebra.*

(a) *Let* $(R, \circ, \ell, r)$ *be an* A*-bimodule* **k***-algebra. Let* $Q : R \longrightarrow A$ *be a relative averaging operator of weight* λ *on the algebra* R. *Then the multiplications*

$$u \dashv_Q v := ur(Q(v)), u \vdash_Q v := \ell(Q(u))v, u \perp_Q v := \lambda u \circ v, \quad \forall u, v \in R, \tag{21}$$

define a trialgebra $(R, \dashv_Q, \vdash_Q, \perp_Q)$.

(b) *Let* (V, ℓ, r) *be an* A*-bimodule. Let* $Q : V \longrightarrow A$ *be a relative averaging operator on the module* V. *Then the multiplications*

$$u \dashv_Q v := ur(Q(v)), u \vdash_Q v := \ell(Q(u))v, \quad \forall u, v \in V, \tag{22}$$

define a dialgebra $(V, \dashv_Q, \vdash_Q)$.

Proof. (a) For any $x, y, z \in R$, by the definitions of $\dashv_Q$, $\vdash_Q$ and $\perp_Q$ and A-bimodule $\mathbf{k}$-algebra, we have

$$(x \dashv_Q y) \dashv_Q z = \big(xr(Q(y))\big)r(Q(z)) = xr(Q(y) * Q(z)).$$

Since $Q(y) * Q(z) = Q(\ell(Q(y))z) = Q(yr(Q(z))) = \lambda Q(y \circ z)$, we have

$$(x \dashv_Q y) \dashv_Q z = x \dashv_Q (y \vdash_Q z) = x \dashv_Q (y \dashv_Q z) = x \dashv_Q (y \perp_Q z).$$

It follows from $x \vdash_Q (y \vdash_Q z) = \ell(Q(x))\big(\ell(Q(y))z\big) = \ell(Q(x) * Q(y))z$ and $Q(x) * Q(y) = Q(\ell(Q(x))y) = Q(xr(Q(y))) = \lambda Q(x \circ y)$ that

$$x \vdash_Q (y \vdash_Q z) = (x \vdash_Q y) \vdash_Q z = (x \dashv_Q y) \vdash_Q z = (x \perp_Q y) \vdash_Q z.$$

We also, have

$$\begin{aligned}
&(x \vdash_Q y) \dashv_Q z = \big(\ell(Q(x))y\big)r(Q(z)) = \ell(Q(x))(yr(Q(z))) = x \vdash_Q (y \dashv_Q z),\\
&(x \perp_Q y) \dashv_Q z = (\lambda x \circ y)r(Q(z)) = \lambda x \circ (yr(Q(z))) = x \perp_Q (y \dashv_Q z),\\
&(x \dashv_Q y) \perp_Q z = \lambda(xr(Q(y)) \circ z = x \circ \big(\ell(Q(y))z\big) = x \perp_Q (y \dashv_Q z),\\
&(x \vdash_Q y) \perp_Q z = \lambda(\ell(Q(x))y) \circ z = \ell(Q(x))(\lambda y \circ z) = x \vdash_Q (y \perp_Q z),\\
&(x \perp_Q y) \perp_Q z = \lambda(\lambda x \circ y) \circ z = \lambda(x \circ (\lambda y \circ z)) = x \perp_Q (y \perp_Q z).
\end{aligned}$$

The above relations for $\dashv_Q$, $\vdash_Q$ and $\perp_Q$ coincide with the axioms of trialgebra in Definition 3.3.
(b) By the definitions of $\dashv_Q$, $\vdash_Q$ and bimodule, similar to the proof of (a), $(V, \dashv_Q, \vdash_Q)$ is a dialgebra. □

For a $\mathbf{k}$-algebra A and an A-bimodule $\mathbf{k}$-algebra $(R, \circ)$, denote

$$\begin{aligned}
&\mathcal{RA}_\lambda^{alg}(R, A)\\
&:= \{Q : R \to A | Q \text{ is a relative averaging operator of weight } \lambda \text{ on algebra } R\}.
\end{aligned}$$

By (a) of Theorem 3.1, we obtain a map

$$\Phi_{\lambda,R,A}^{alg} : \mathcal{RA}_\lambda^{alg}(R, A) \longrightarrow \mathcal{AT}(R_{mod}), \tag{23}$$

where R_{mod} denotes the underlying $\mathbf{k}$-module of R.

Now let V be a $\mathbf{k}$-module. Let $\mathcal{AV}_\lambda(V, -)$ be the set of relative averaging operators of weight λ on algebra $(V, \circ)$, where $\circ$ is an associative product on V. In other words,

$$\mathcal{AV}_\lambda(V, -) := \coprod_{R,A} \mathcal{AV}_\lambda^{alg}(R, A), \tag{24}$$

where the disjoint union runs through all pairs (R, A) where A is a $\mathbf{k}$-algebra and R is an A-bimodule $\mathbf{k}$-algebra such that $R_{mod} = V$. Then from the map $\Phi_{\lambda,V,A}^{alg}$ in Eq. (23), we have

$$\Phi_{\lambda,V}^{alg} := \coprod_{R,A} \Phi_{\lambda,V,A}^{alg} : \mathcal{AV}_\lambda^{alg}(V, -) \longrightarrow \mathcal{AT}(V). \tag{25}$$

Similarly, for a **k**-module V and **k**-algebra A, denote

$$\mathcal{RA}^{mod}(V,A)$$
$$:= \{Q : V \to A | Q \text{ is a relative averaging operator on the module } V\},$$

By (b) of Theorem 3.1, we obtain a map

$$\Phi^{alg}_{V,A} : \mathcal{AV}^{mod}(V,A) \longrightarrow \mathcal{AD}(V) \tag{26}$$

Let $\mathcal{AV}^{mod}(V,-)$ be the set of relative averaging operators on the module V. In other words, $\mathcal{AV}^{mod}(V,-) := \coprod_A \mathcal{AV}^{mod}(V,A)$, where A runs through all the **k**-algebras. Then we have

$$\Phi^{mod}_{V} := \coprod_A \Phi^{mod}_{V,A} : \mathcal{AV}^{mod}(V,-) \longrightarrow \mathcal{AD}(V). \tag{27}$$

Theorem 3.2. *Let V be a* **k***-module. The maps $\Phi^{alg}_{1,V}$ and Φ^{mod}_{V} are surjective.*

Proof. We first prove the surjectivity of $\Phi^{alg}_{1,V}$. Let $(V, \dashv, \vdash, \perp)$ be a trialgebra. Define two linear maps

$$L_{\vdash}, R_{\dashv} : V \longrightarrow End_{\mathbf{k}}(V), L_{\vdash}(x)(y) = x \vdash y, R_{\dashv}(x)(y) = y \dashv x, \forall x, y \in V. \tag{28}$$

Let I be the ideal generated by the set $\{u \dashv v - u \vdash v \mid u, v \in V\} \cup \{u \dashv v - u \perp v \mid u, v \in V\}$. Let $\widetilde{V} := V/I$, then we have $\dashv=\vdash=\perp$ in $\widetilde{V}$. Furthermore, $\widetilde{V}$ can be regarded as an associative algebra with an operation $* :=\dashv=\vdash=\perp$.

By comparing the trialgebra axioms and the axioms of $(V, *)$-bimodule **k**-algebra, we have that if we replace the operation $*$ in Eq. (9) and (10), by any of $\dashv, \vdash, \perp$, the equations still hold. Hence, $(V, \perp, L_{\vdash}, R_{\dashv})$ is a $(\widetilde{V}, *)$-bimodule **k**-algebra.

Let Q be the natural projection from V to $\widetilde{V}$. Then we have

$$Q(x) = x, \quad Q(x \dashv y) = Q(x \vdash y) = Q(x \perp y) = Q(x) * Q(y).$$

Hence,

$$Q(x) * Q(y) = Q(Q(x) \vdash y) = Q(x \dashv Q(y)) = Q(x \perp y),$$

and then

$$Q(x) * Q(y) = Q(L_{\vdash}(Q(x))y) = Q(xR_{\dashv}(Q(y))) = Q(x \perp y).$$

That is Q is a relative averaging operator of weight 1 on the algebra $(V, \perp)$.

To prove the surjective of Φ^{mod}_{V}, let $(V, \dashv, \vdash)$ be a dialgebra. Let I be the ideal generated by the set $\{u \dashv v - u \vdash v \mid u, v \in V\}$. Define Q be the natural projection from V to V/I. Similar to the proof for $\Phi^{alg}_{1,V}$, we get Q is a relative averaging operator on bimodule $(V, L_{\vdash}, R_{\dashv})$. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 11901472, 12071377), the Natural Science Foundation of Chongqing (cstc2020jcyj-msxmX0364), Fundamental Research Funds for Central Universities (SWU-XDJH202305).

References

[1] M. Aguiar, *Pre-Poisson algebras,* Lett. Math. Phys., 54 (2000), 263-277.

[2] H.H. An, C.M. Bai, *From Rota-Baxter algebras to pre-Lie algebras,* J. Phys. A: Math. Theor., 41 (2008) 145–150.

[3] C.M. Bai, L. Guo, X. Ni, *Relative Rota-Baxter operators and tridendriform algebras,* J. Algebra Appl., 12 (2013), 1350027.

[4] G. Birkhoff, *Lattices in applied mathematics, II Averaging operators,* Proc. Symp. Pure Math., 2 (1961), 163-184.

[5] D. Burde, *Simple left-symmetric algebras with solvable Lie algebra,* Manuscr. Math., 95 (199), 397-411.

[6] W. Cao, *An algebraic study of averaging operators,* Ph.D. Thesis, Rutgers University at Newark, 2000.

[7] K. Fériet, *Letat actuel du problems de la turbulaence (I and II),* La Sci. Aerienne, 3 (1934), 9-34.

[8] S.A. Kilic, *Averaging operators on Orlicz spaces,* Bull. Inst. Math. Acad. Sinica., 23 (1995), 67-77.

[9] J.-L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz,* Enseign. Math., 39 (1993), 269-293.

[10] J.-L. Loday, *Dialgebras and related operads,* Lecture Notes in Math., 1763 (2002), 7-66.

[11] J.-L. Loday, M. Ronco, *Trialgebras and families of polytopes in homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory,* Comtep. Math., 346 (2004), 369-398.

[12] J. Pei, C.M. Bai, L. Guo, X. Ni,*Disuccessors and duplicators of operads, manin products and operators, In "Symmetries and Groups in Contemporary Physics",* Nankai Series in Pure, Applied Mathematics and Theoretical Physics, 11 (2013), 191-196.

[13] J. Pei, C.M. Bai, L. Guo, X. Ni, *Replicators, Manin white product of binary operads and average operators,* New Trends in Algebras and Combinatorics, 2020, 317-353.

[14] J. Pei, L. Guo, *Averaging algebras, Schröder numbers and rooted trees,* J. Algebr. Comb., 12 (2015), 73-109.

[15] O. Reynolds, *On the dynamic theory of incompressible viscous fluids,* Phil. Trans. Roy. Soc. A, 136 (1895), 123-164.

[16] G.C. Rota, *On the representation of averaging operator,* Rendiconti del Seminario Matematico della Universita di Padova, 30 (1960), 52-64.

[17] S.-T.C. Moy, *Characterizations of conditional expectation as a transformation on function Spaces,* Pacific J. Math., 4 (1954), 47-63.

[18] A. Triki, *Extensions of positive projections and averaging operators,* J. Math. Anal., 153 (1990), 486-496.

[19] H. Umegaki, *Representations and extremal properties of averaging operators and their applications to information channels,* J. Math. Anal. Appl., 25 (1969), 41-73.

Accepted: October 17, 2022

New sequences of processing times for Johnson's algorithm in PFSP

Shahriar Farahmand Rad
Department of Mathematics
Payame Noor University
P.O. Box. 19395-3697, Tehran
Iran
sh_fmand@pnu.ac.ir

Abstract. There are many researches about converting n job m machine problem to a n job 2 machine one, and finally using Johnson's rule for minimizing makespan. In one case, this converting leads to the inner product of processing times by Pascal numbers. In this paper, it is shown that there are other suitable numerical sequences with a triangle pattern or without it, producing better makespans in several cases. The quality of results is checked by the benchmark of Taillard in permutation flow shop scheduling problem.

Keywords: flow shop, scheduling, NEH algorithm, stirling numbers, Fibonacci numbers, Bell's numbers, Pascal numbers.

1. Introduction

In flow shop scheduling, the issue is to determine the best sequence of n jobs that are processed on m machines in the same order. Let t_{ij} denote deterministic processing time of job j at machine i, which is a positive integer. It is assumed that all jobs process on every single machine. Makespan or $C_{\max}$ refers to the total time for complete processing of all jobs.

It is usually supposed that all jobs are independent and available. No matter when, each machine processes at most one job and each job is processed only by one machine. No preemption is allowed. Set up times are included in the processing times. Infinite storage buffer between machines is also assumed and machines are available. There are job permutations, which change from machine to machine. Therefore, $(n!)^m$ schedules can be obtained. Having the same permutation for all machines is supposed; hence, $n!$ schedules are possible. The resulting problem is known as the permutation flow shop scheduling problem (PFSP), denoted by Fm/prmu/ $C_{\max}$ Graham et al. [7]. Only the F2/prmu/$C_{\max}$ problem is polynomially, solvable and proposed by Johnson [8]; for $m \geq 3$, the problem is NP-complete Garey et al. [6].

It seems that after several papers in 1950s and then the widespread concern about expansion complexity theory by Karp [9], the great numerical growth of papers was stopped in 1990s. Now, there are few papers about adequate heuristic algorithm for solving deterministic flow shop scheduling problems by minimizing makespan criterion.

For the reason that Johnson's algorithm is exact, authors have hardly tried to convert each arbitrary $n \times m$ PFSP to a 2-machine problem.

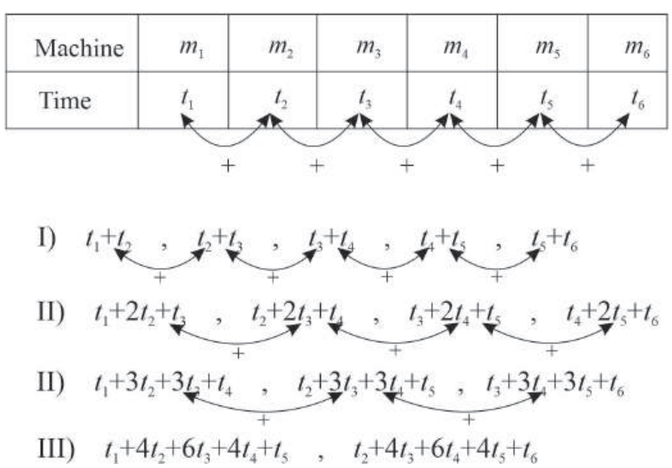
Bonney et al. [4] expressed a job in terms of the slopes of the comulative start and times. It converted the original n jab m machine to a n jab 2 machine problem.

Semanco et al. [11] employed Johnson's rule to present a good initial solution for improving heuristic and the proposed algorithm called MOD. Wei Jia et al. [13] proposed a new algorithm. Firstly, the proposed algorithm normalized the matrix A of processing times. Then it transformed the original problem containing m machines into a 2 machine one that is solved by Johnson's rule. Fernandez- Viagas et al. [5] presented two constructive heuristics based on Johnson's algorithm. Belabid et al. [2] studied the resolution of PFSP that their first method was based on Johnson's rule.

2. The extended Johnson's algorithm

Before explaining the presented algorithm, it is better to make an example about the process of reducing m machine problem into a 2 machine one.

An illustration of this is that $m = 6$ and one job must be processed in $m = 6$ machines with processing times t_1 to t_6. By adding the first two processing times, it is assigned to the first hypothetical machine. It is also continued in a similar way for all jobs and finally the problem was transformed into a $m = 2$ machine, Baskar et al. [1].



It is observed that for m number of machines, the coefficients are the members of Pascal's Triangle for $\binom{m-2}{k}$; $k = 0, 1, 2, \ldots, n$. Indeed, the dot products of these numbers with the original times are obtained. Also at the end, the terms including last and first processing times i.e t_6 , t_1 are respectively omitted.

3. The presented algorithm

In general, suppose the deterministic times for PFSP are t_{ij}; $1 \leq i \leq n$ and $1 \leq j \leq m$. This problem is transformed to a two machine one. The job order obtained from Johnson's algorithm is used to find initial order and calculate the makespan.

Let the time martrix of the initial problem be $M = [t_{ij}]_{n\times m}$ and the time matrix after using Johnson's algorithm be $N = [T_{pq}]_{n\times 2}$. Then, the following relations can be given

$$q{=}1 \Rightarrow T_{p1}{=}(t_{p1}, t_{p2}, t_{p3}, \ldots, t_{pm}) \bullet \left(\binom{m{-}2}{0}, \binom{m{-}2}{1}, \binom{m{-}2}{2}, \ldots, 0\right)$$

$$= \sum_{k=1}^{m} \binom{m-2}{k-1} t_{pk}$$

$$q{=}2 \Rightarrow T_{p2}{=}(t_{p1}, t_{p2}, t_{p3}, \ldots, t_{pm}) \bullet \left(0, \binom{m-2}{0}, \binom{m-2}{1}, \ldots, \binom{m-2}{m-2}\right)$$

$$= \sum_{k=1}^{m} \binom{m-2}{k-2} t_{pk}.$$

The optimal permutation is resulted from Johnson's algorithm on N. This permutation is applied to M. Utilizing Belman et al.'s theorem [3] leads to the minimum makespan.

As was mentioned, inner products of t_{pk}s by Pascal's triangle elements are equal to T_{p1} and T_{p2}. The algorithm is executed on Taillard's problems [12] by Pascal numbers. The triangular neutrality of Pascal's numbers draws attention to the scalar products of the times of Taillard's problems by first and second kind Stirling numbers, Bell's numbers and Fibonacci numbers. These sequences of numbers have triangular pattern du to the next equations.

1) $s_{n+1,k} = s_{nk-1} - ns_{n,k}$, $s_{n,k}$ is a number of first kind Stirling numbers (St 1) that is in n'th row and k'th column in the triangle;

2) $S_{n+1,k} = S_{n,k-1} + kS_{n,k}$, second kind of Stirling numbers (St 2);

3) $f_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots + \binom{n-k}{k}, k = \left[\frac{n}{2}\right]$, f_{n+1} is $n + 1$'th term in Fibonacci sequence (Fibo);

4) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, k \geq 1$, Pascal numbers (Pasc);

5) $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, B_0 = 1, B_{n+1}$ is $n + 1$'th term in Bell's sequence (Bell).

Now, the presented algorithm is divided into three simple steps:

1. Select the first m (m is the number of machines) elements of the above numerical sequences i.e.

i) $s_{m-1,k}, k = 0, 1, 2, \ldots, m-1$ (St 1);

ii) $S_{m-1,k}, k = 0, 1, 2, \ldots, m-1$ (St 2);

iii) f_{m-1} (Fibo);

iv) $\binom{m-1}{k}, k = 0, 1, 2, \ldots, m-1$ (Pasc);

v) $B_k, k = 0, 1, 2, \ldots, m-1$ (Bell).

For each stage, inner products of the above elements with the times of original problem are obtained. First, the term concluding t_{pm}, and then the term concluding t_{p_1} are clearly omitted.

2. Johnson's algorithm is applied to give job order from artificial n job and two machine problems with (i),(ii),. . . ,(v) sequences.

3. The job order obtained in previous step is used to find initial order and compute the makespan in original problem.

The algorithm implemented in Visual Basic and carried out all tests on Pentium IV computer at 3.2 GHz with 2 GBytes of RAM memory.

For the statistical analysis, the well known standard benchmark set of Taillard [12] was used. This set includes 120 instances divided into 12 groups with 10 replicates each. The sizes range from 20 jobs, 5 machines to 500 jobs, 20 machines. In the flowshop scheduling literature, this benchmark has been extensively used in the past years. For each instance, a very tight lower bound and upper bound are known. All 10 instances in the 50×20 set , nine in 100×20, six in 200×20 and three in 500×20 are open. For all other instances, the optimum solution is already known.

The applied performance measure that was used, is the Relative parcentage Deviation (RPD) over the optimum or the best solution (upper bound) for each instance:

$$\text{Relative Perentage Deviation (RPD)} = \frac{Heu_{sol} - Best_{sol}}{Best_{sol}} \times 100,$$

where Heu_{sol} is the solution given by any of the tested heuristic for a given instance and $Best_{sol}$ is the optimum solution or the lowest known upper bound for Taillard's instances.

The solutions of presented algorithm are compared with the results of NEH and Taillard's benchmark. NEH was made-up by Nawas et. al. [10], that is the best heuristic that have ever been proposed for solving PFSP [5].

In the following tables, the summary of these comparisons and the results of Talllard's problems are shown. The tables also display the results of NEH.

Table 1. The least RPD for Heuristic Algorithm

Size of Problems	Least RPD Obtained With				NEH reaults	Taillard Upper Bounds
	Prob.No	Sequence	Best Makespan	RPD		
20×5	2	St 2	1422	4.6	1365	1359
20×10	9	St 2	1848	16	1639	1593
20×20	8	St 2	2473	12.4	2249	2200
50×5	6	Fibo	3093	9.3	2835	2829
50×10	6	Pasc	3728	24	3148	3006
50×20	1	St 2	4762	26.2	4006	3771
100×5	6	Pasc	5740	11.7	5154	5135
100×10	9	St 1	6973	18.7	6016	5871
100×20	10	St 2	7994	24.8	6680	6434
200×10	4	St 2	12880	18.2	11057	10889
200×20	8	St 2	14327	21.1	11824	11334
500×20	5	Pasc	31706	20.3	26928	26334

Table 2. The greatest RPD for Heuristic Algorithm

Size of Problems	Greatest RPD Obtained With				NEH results	Taillard Upper Bounds
	Prob.No	Sequence	Best Makespan	RPD		
20×5	3	St 2	1349	24.7	1132	1081
20×10	4	Fibo	1856	34.7	1416	1377
20×20	4	Pasc	2749	23.6	2257	2223
50×5	3	St 2	3209	22.4	2650	2621
50×10	3	Pasc	3878	36.5	2994	2839
50×20	4	Pasc	4874	34	3953	3723
100×5	2	Pasc	6171	17.1	5284	5268
100×10	2	Pasc	6795	27	5466	5349
100×20	7	St 1	8135	31.5	6578	6268
200×10	2	St 2	13142	25.4	10677	10480
200×20	3	Pasc	14693	30.2	11724	11281
500×20	10	Pasc	32782	23.9	27103	26457

It is seen that the least RPD is 4.6 which is obtained in Taillard's $20 \times 5 - 2$ problem after the inner product of second kind Stirling numbers. The greatest RPD is 36.5 that is resulted in $50 \times 10 - 3$ problem after the dot product of Pascal numbers.

The best makespan in each instance is shown in the table 1 after the dot product of sequences and comparing with each other. For example in the first section, 7 times second kind Strirling numbers, 1 time Fibonacci numbers, and only 3 times Pascal numbers are resulted the best makespan! In this research, Bell numbers are not resulted this.

The best solutions in the light of quality are those obtained from the inner product of seoond kind Stirling numbers.

For more researches, the Bank of numerical sequences is chosen. This Bank i.e oeis.org includes "The On-Line Encyclopedia Of Integer Sequences" found by N.J.A. Sloane. He has worked the sustainable collection of these sequences since 1964.

At another time, the algorithm implemented in Python and carried out all tests on a Quad-Core Intel Core i7 computer at 2.6 GHz with 16 GBytes of RAM memory. In 10 hours, first 100000 sequences were chosen and scalar products of them were determined. Then Johnson's algorithm found initial order of jobs. The average of ten obtaining makespans in each package of Taillard instances was calculated, afterward 100000 solutions in them were collected in following figures.

In these figures, l is the average of lower bounds or the average of solution; and u is the average of upper bounds in each package of Taillard's instances.

Additionally, X-axis shows the number of sequences, and the result of each correspondent sequence is a point in the direction of Y-axis. It is regarded that the specified average points have not good situation with respect to l and u.

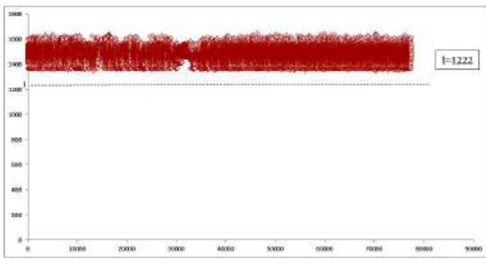


Figure 1. average results of 20 × 5

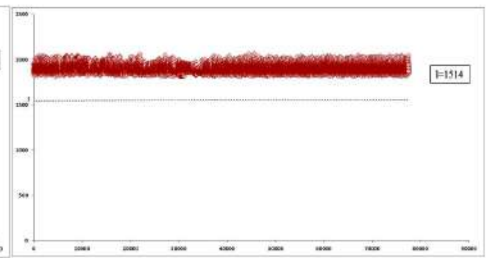


Figure 2. average results of 20 × 10

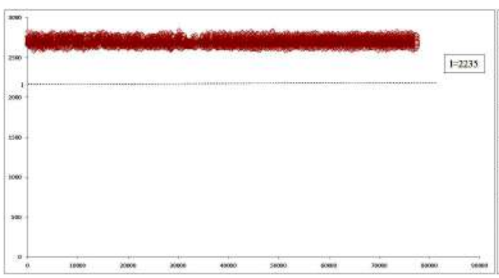


Figure 3. average results of 20 × 20

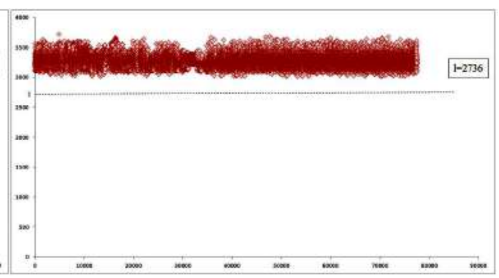


Figure 4. average results of 50 × 5

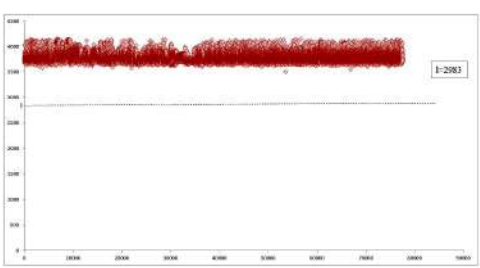


Figure 5. average results of 50 × 10

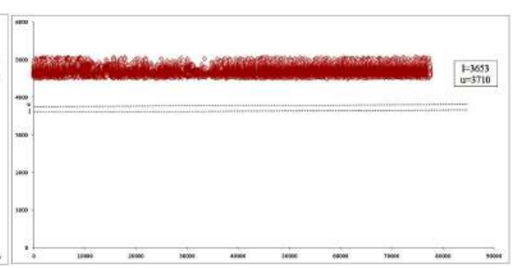


Figure 6. average results of 50 × 20

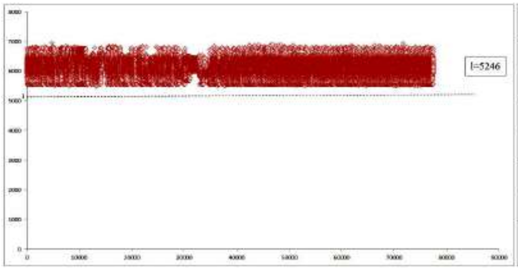


Figure 7. average results of 100 × 5

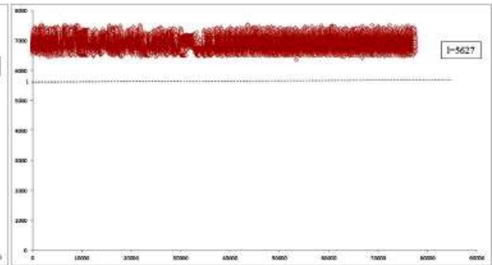


Figure 8. average results of 100 × 10

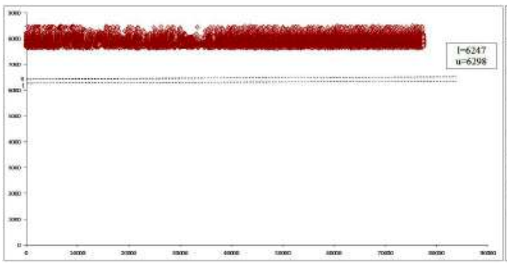


Figure 9. average results of 100×20

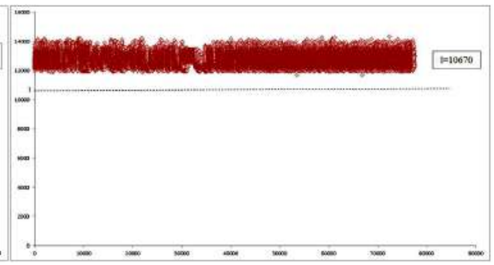


Figure 10. average results of 200×10

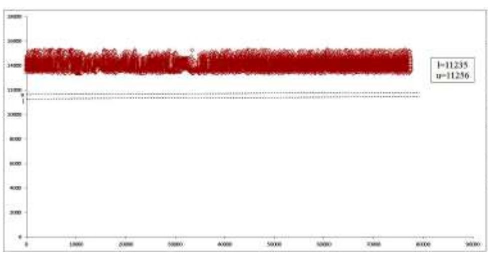


Figure 11. average results of 200×20

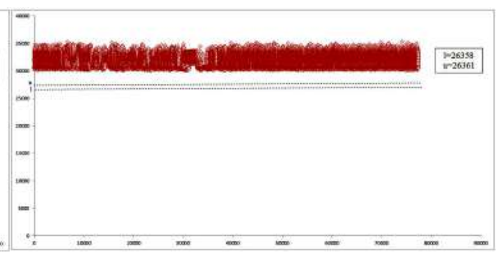


Figure 12. average results of 500×20

In all 120 instances of Taillard's and 120 × 100000 cases, there is only one result that is the same of Taillard's solution after fixing all steps and running Johnson's algorithm.

This is $100 \times 5 - 1$ problem and the solution is 5493. The result is consequent of the inner product of the original processing times by the sequence A088661 with the general term:

$$a_n = \sum_{k=1}^{8} \left[\frac{P_{n,k}}{P_{n-1,k}} \right]$$

when

$$P_{n,k} = \frac{\sum_{i=1}^{n} \log i}{\sum_{i=n-\left[\frac{3n}{4^k}\right]}^{n-\left[\frac{n}{4^k}\right]} \log i}$$

namely "A log based cantor self similar sequence" (bracket is floor). The author is Roger L. Bagula, Nov 21 2003. For $n = 3, 4, \ldots, 107$ the terms of this sequence are $8, 8, 7, 6, 7, 8, 8, 7, 6, 8, 8, 7, 7, 8, 8, 7, 7, 8, 8, 5, 7, 8, 8, 7, 6, 8, 8, 7, 7, 8, 8, 7, 7,$ $8, 8, 6, 7, 8, 8, 7, 5, 8,$ $8, 7, 7, 8, 8, 7, 7, 8, 8, 6, 7, 8, 8, 7, 6, 8, 8,$ $7, 7, 8, 8, 7, 7, 8, 8, 6, 7, 8, 8, 7, 6, 8, 8, 7, 7,$ $8, 8, 7, 7, 8, 8, 4, 7, 8, 8, 7, 6, 8, 8, 7, 7, 8, 8, 7, 7, 8, 8, 6, 7, 8, 8, 7, 5$.

A brief research is denoted that after each 4-term section, four numbers 7, 8, 8, 7 are repeated.

For the intuitive perception of sequence's behaviour, its pin plot and scatter plot are designed in the following.

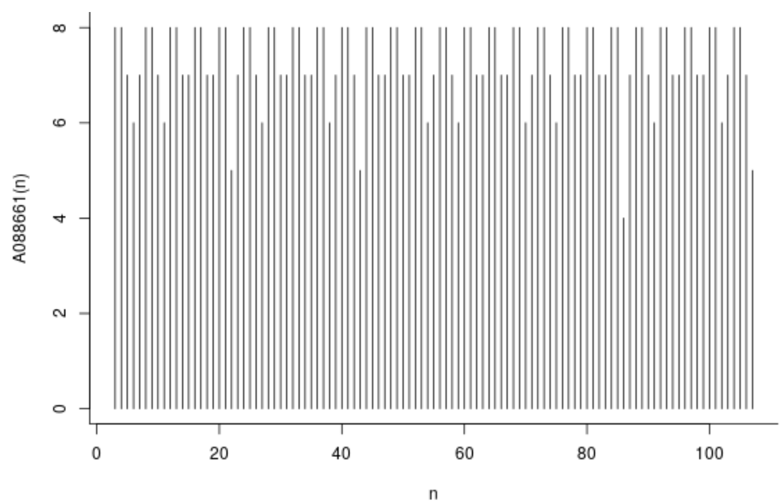


Figure 13. Pin plot of A088661(n)

4. Conclusions

In this paper, it was tried to transform the $n \times m$ problem to a $2 \times m$ problem after obtaining the inner product of proessing times in PFSP by the most famous numerical sequences. Johnson's algorithm was used and the results were compared with those of the Taillard's 120 problem solutions. The least relative percentage deviation was obtained in $20 \times 5 - 2$ Taillard's problem with the dot product of Stirling second kind numbers. Moreover, the greatest RPD

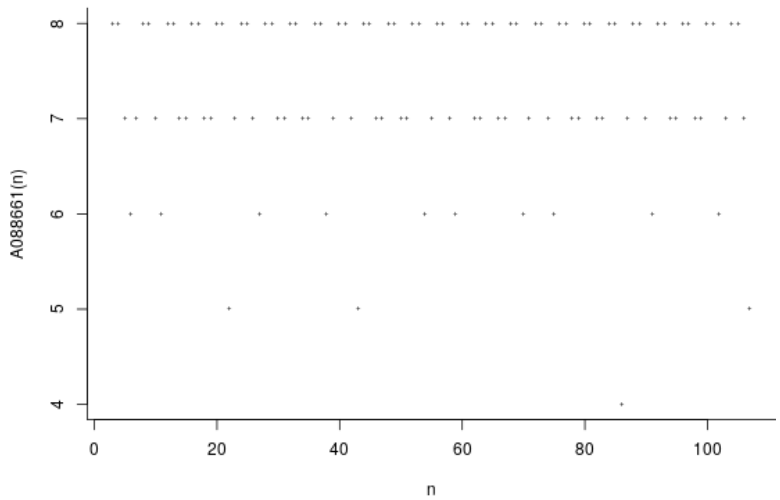


Figure 14. Scatterplot of A088661(n)

was resulted in $50 \times 10 - 3$ Taillard's problem. All these arguments lead to the conclusion that Baskar's ideas [1] about good solutions of the inner product of Pascal numbers have been invalidated.

After obtaining the dot product of 100000 different numerical sequences in processing times for $100 \times 5 - 1$ instance, Johnson's rule results optimum solutions.

References

[1] A. Baskar, M.A. Xavior, *A new Heuristic algorithm using Pascal's triangle to determine more than one sequence having optimal/near optimal make span in flow shop scheduling problems*, International Journal of Computer Applications, 975 (2012), 8887.

[2] J. Belabid, S. Aqil, K. Allali, *Solving permutation flow shop scheduling problem with sequence-independent setup time*, Journal of Applied Mathematics, 2020.

[3] R. Bellman, A.O. Esogbue, I. Nabeshima, Mathematical aspects of scheduling and applications: modern applied mathematics and computer science, Elsevier, vol. 4, 2014.

[4] M.C. Bonney, S.W. Gundry, *Solutions to the constrained flowshop sequencing problem*, Journal of the Operational Research Society, 27 (1976), 869-883.

[5] V. Fernandez-Viagas, J.M. Molina-Pariente, J.M. Framinan, *New efficient constructive heuristics for the hybrid flowshop to minimise makespan: a*

computational evaluation of heuristics, Expert Systems with Applications, 114 (2018), 345-356.

[6] M.R. Garey, D.S. Johnson, R. Sethi, *The complexity of flowshop and jobshop scheduling*, Mathematics of Operations Research, 1 (1976), 117-129.

[7] R.L. Graham, E.L. Lawler, J.K. Lenstra, A.R. Kan, *Optimization and approximation in deterministic sequencing and scheduling: a survey*, In Annals of Discrete Mathematics, Elsevier, 5 (1979), 287-326.

[8] S.M. Johnson, *Optimal two and three stage production schedules with setup times included*, Naval Research Logistics Quarterly, 1 (1954), 61-68.

[9] R.M. Karp, *Reducibility among combinatorial problems*, In R. E. Miller and J. W. Thatcher (eds.) Complexity of computer computations, Plenum Press, New York, 1972, 85-104.

[10] M. Nawaz, E.E. Enscore, I. Ham, *A heuristic algorithm for the m-machine, n-job flow-shop sequencing problem*, Omega, 11 (1983), 91-95.

[11] P. Semančo, V. Modrák, *A comparison of constructive heuristics with the objective of minimizing makespan in the flow-shop scheduling problem*, Acta Polytechnica Hungarica, 9 (2012), 177-190.

[12] E. Taillard, *Benchmarks for basic scheduling problems*, European Journal of Operational Research, 64 (1993), 278-285.

[13] J.Y. Wei, Y.B. Qin, D.Y. Xu, *DRPFSP algorithm for solving permutation flow shop scheduling problem*, Computer Science, 2015.

Accepted: February 10, 2021

Approximate solution of Fredholm type fractional integro-differential equations using Bernstein polynomials

Azhaar H. Sallo
Department of Mathematics
College of Science
University of Duhok
Kurdistan Region
Iraq
azhaar.sallo@uod.ac

Alias B. Khalaf*
Department of Mathematics
College of Science
University of Duhok
Kurdistan Region
Iraq
aliasbkhalaf@uod.ac

Shazad S. Ahmed
Department of Mathematics
College of Science
University of Sulaimani
Kurdistan Region
Iraq
shazad.ahmed@suluniv.edu.krd

Abstract. The main goal of this paper is to find an approximate solution for a certain type of Fredholm fractional integro-differential equation by using Bernstein polynomials. In the last section, some examples have been presented to compare their approximate and exact solutions.

Keywords: Caputo derivative, fractional integro-differential equations, Bernstein polynomials.

1. Introduction

Fractional differential equations have been implemented to model various problems in several fields, [2], [3], [4], [6] and [10]. Any system containing fractional derivatives is more practical than the regular system because of the non-locality of the fractional derivative. Recently, mathematicians have shown a lot of interest in studying new types of equations having non-local fractional derivatives. The study of any type of fractional integro-differential equation depends on the

∗. Corresponding author

type of the fractional derivative. Therefore, many researchers have shown great interest in studying new types of the Caputo fractional differential equations and their applications, see [12] and [13]. Fractional integro-differential equations of Fredholm type have been studied by many researchers to find their approximate solutions using many types of methods and polynomials, see [1], [5], [12], [14], [18] and [19]. The Bernstein polynomials [7] is one of the methods for computing the approximate solution of fractional equation, see [13], [14], [17]. In [8], a solution of a special type of fractional integro-differential equations using Jacobi wavelet operational matrix of fractional integration presented and the same authors in [9], discussed numerical Solution of a Fredholm Fractional Integro-differential equation. Recently, Mansouri and Azimzadeh in [11], introduced an approximate solution of fractional delay Volterra integro-differential equations by Bernstein polynomials . Also, in [16], numerical solution method for multi-term variable order fractional differential equations by shifted Chebyshev polynomials of the third kind is given.

In this article, we study how to find approximate solutions to a class of Fredholm fractional integro-differential equations that contains the Caputo fractional derivative of order $n-1 < \alpha \leq n$. Finally, some examples are given to find their approximate solutions.

2. Preliminaries

In this section, we present some necessary definitions and results which will be used in other sections. We start with the definition and main properties of the fractional derivative. For more details on the subject see [15] and [4].

Definition 2.1 ([15])**.** Let $y = f(x)$ be a function, then the fractional derivative of y in Caputo sense of order $\alpha > 0$ is defined as:

$$
{}_a^cD_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1+n}} dt, & n-1 < \alpha < n, \quad n \in N, \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n \in N. \end{cases}
$$

If $f(x)$ is a constant function, then ${}_a^cD_x^\alpha f(x) = 0$.

The Caputo derivative of $f(x) = (x-a)^j$ is defined as: (see [15])

$$
{}_a^cD_x^\alpha (x-a)^j = \begin{cases} 0, & \text{for} \quad j \in N \cup \{0\} \quad \text{and} \quad j < \lceil \alpha \rceil, \\ \dfrac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (x-a)^{j-\alpha}, & \text{for} \quad j \in N \quad \text{and} \quad j \geq \lceil \alpha \rceil \\ & \text{or} \quad j \notin N \quad \text{and} \quad j > \lfloor \alpha \rfloor. \end{cases}
$$

Here, $\lceil \alpha \rceil$ is denoted to be the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ is the largest integer less than or equal to α.

Lemma 2.1 ([15])**.** *The Caputo fractional differentiation is a linear operation, that is for any two constants a_1, a_2 and any two functions y_1, y_2, we have*

$$ {}_a^cD_x^\alpha(a_1y_1 + a_2y_2) = a_1({}_a^cD_x^\alpha(y_1)) + a_2({}_a^cD_x^\alpha(y_2)). $$

Definition 2.2 ([7])**.** The Bernstein polynomials of degree n are denoted by $B_{i,n}(x)$ and defined as:

$$ B_{i,n}(x) = \frac{\binom{n}{i}(x-a)^i(b-x)^{n-i}}{(b-a)^n}, \quad x \in [a,b] \subseteq \mathbb{R}, \quad i = 0,1,2,...,n. \tag{1} $$

Particularly, if $x \in [0,1]$ then $B_{i,n}(x)$ are defined as:

$$ B_{i,n}(x) = \binom{n}{i} x^i(1-x)^{n-i}, \quad i = 0,1,2,...,n. $$

Since $(b-x)^{n-i} = [(b-a)-(x-a)]^{(n-i)}$, equation (1) can be written as:

$$ B_{i,n}(x) = \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{i} \binom{n-i}{j-i} (x-a)^j. \tag{2} $$

Hence,

$$ B_{i,n}(x) = \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} (x-a)^j. \tag{3} $$

Lemma 2.2 ([7])**.** *The derivatives of Bernstein polynomials of degree n can be written as a linear combination of Bernstein polynomials of degree $n-1$ which is given by:*

$$ \frac{d}{dx} B_{i,n}(x) = n(B_{i-1,n-1}(x) - B_{i,n-1}(x)). \tag{4} $$

Lemma 2.3. The fractional derivative of order $0 < \alpha \in \mathbb{R} \setminus \mathbb{N}$ of the Bernstein polynomials of degree n in the Caputo sense is given by:

$$ {}_a^cD_x^\alpha B_{i,n}(x) = \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} {}_a^cD_x^\alpha (x-a)^j. \tag{5} $$

Since ${}_a^cD_x^\alpha (x-a)^j = 0$ for each $j < \alpha$, we have

$$ {}_a^cD_x^\alpha B_{i,n}(x) = \sum_{j=\lceil\alpha\rceil}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (x-a)^{j-\alpha}. \tag{6} $$

Proof. Follows from applying Definition 2.1 to equation (3). □

3. Approximation method

In this section, we propose the following fractional integro-differential equation and provide approximate solutions to this equation:

$$
\begin{aligned}
&{}_a^cD_x^{\alpha} y(x) + \sum_{k=2}^{n} g_k(x)\ {}_a^cD_x^{(\frac{\alpha}{k})} y(x) + g_0(x)y(x) \\
&= f(x) + \sum_{m=1}^{n} \int_a^b K_m(x,t)\ {}_a^cD_t^{\frac{\beta}{m}} y(t)dt, \qquad (7)
\end{aligned}
$$

where $n-1 < \alpha \leq n$, $\beta \leq \alpha$ and $a \leq t, x \leq b$. Subject to the conditions $y^{(i)}(a) = \lambda_i$, $i = 0, 1, 2, ..., n-1$.

The solution of equation (7) is the function $y(x)$ which is a continuous function and its approximate solution can be expressed in terms of n^{th}-degree of Bernstein polynomial

$$
y_n(x) = \sum_{i=0}^{n} c_i B_{i,n}(x). \qquad (8)
$$

From the initial condition, we have $\lambda_0 = y_n(a) = \sum_{i=0}^{n} c_i B_{i,n}(a)$, which implies that

$$
c_0 = \lambda_0. \qquad (9)
$$

Again, from equation (3), we have

$$
y_n'(x) = \sum_{i=0}^{n} c_i B_{i,n}'(a) = \sum_{i=0}^{n} c_i \sum_{j=i}^{n-i} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} j(x-a)^{j-1}.
$$

This implies that all the terms are zero at $x = a$ except when $j = 1$. Hence, we obtain that

$$
\lambda_1 = y_n'(a) = \sum_{i=0}^{n} c_i \frac{(-1)^{1-i}}{(b-a)} \binom{n}{1} \binom{1}{i}.
$$

Therefore,

$$
\lambda_1 = \frac{-n}{b-a} c_0 + \frac{n}{b-a} c_1.
$$

Hence,

$$
c_1 = \lambda_0 + \frac{(b-a)\lambda_1}{n}. \qquad (10)
$$

Thus, in general, if $n \geq m \in N$ we have

$$
y_n^{(m)}(x) = \sum_{i=0}^{n} c_i B_{i,n}^{m}(x) = \sum_{i=0}^{n} c_i \sum_{j=i}^{n} \frac{(-1)^{j-i}}{(b-a)^j} \binom{n}{j} \binom{j}{i} m! \binom{j}{m} (x-a)^{j-m},
$$

when $x = a$ all the terms are zero except $j = m$. Hence,

$$\lambda_m = y_n^{(m)}(a) = \sum_{i=0}^{m} c_i \frac{(-1)^{m-i} m!}{(b-a)^m} \binom{n}{m} \binom{m}{i}. \tag{11}$$

From equation (11) and solving for the coefficients c_i, $i = 0, 1, ..., m$, we obtain that:

$$c_i = \sum_{k=0}^{i} \frac{\binom{i}{k}}{\binom{n}{k}} \times \frac{(b-a)^i \;\, y^{(i)}(a)}{k!}. \tag{12}$$

Now, by substituting equations (2), (4), (12) in equation (7), we get an algebraic equation with unknown constants c_i, $i = m+1, m+2, ..., n$ and by a suitable way we can find a matrix equation of the form $AC = B$, where A ia an $(n-m) \times (n-m)$ matrix and $C^T = [c_{m+1}, c_{m+2}, ..., c_n]$. Then $C = A^{-1}B$. Substituting the c_i's in equation (8) we get the approximate solution of equation (7).

4. Illustrative examples

In this section, we discuss the approximate solution of some examples for distinct fractional derivatives α and β, where $n-1 < \alpha \leq n$ and $\beta \leq \alpha$ and compare them with their exact solutions. We start with the following example:

Example 4.1. Consider the integro-differential equation

$${}_1^cD_x^\alpha y(x) = f(x) + 3\int_1^2 (xt)\ {}_1^cD_t^\beta y(t)dt, \tag{13}$$

where $f(x) = \frac{2}{\Gamma(3-\alpha)}(x-1)^{2-\alpha} - \frac{6x(\beta-3)(2\beta-9)}{\Gamma(5-\beta)}$, $1 < \alpha \leq 2$, $\beta \leq \alpha$ and $1 \leq t, x \leq 2$. Subject to the conditions $y(1) = y'(1) = 2$.

Using Bernstein polynomials of degree $n = 3$, we approximate the solution as:

$$y(x) = \sum_{i=0}^{3} c_i B_{i,3}(x). \tag{14}$$

From equations (9), (10), we obtain that $c_0 = 2$ and $c_1 = \frac{8}{3}$.

Applying equation (6) on $y(x)$ and substituting in equation (13), we get

$${}_1^cD_x^\alpha \sum_{i=0}^{3} c_i B_{i,3}(x) = f(x) + 3\int_1^2 (xt)\ {}_1^cD_t^\beta \sum_{i=0}^{3} c_i B_{i,3}(t)dt. \tag{15}$$

Hence

$$\sum_{i=0}^{3} c_i \{ {}_1^cD_x^\alpha B_{i,3}(x) - 3\int_1^2 (xt)\ {}_1^cD_t^\beta B_{i,3}(t)dt \} = f(x). \tag{16}$$

Applying equation (6) , we get

$$\sum_{i=0}^{3} c_i\{ \sum_{j=\lceil\alpha\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j}\binom{3}{j}\binom{j}{i}\frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)}(x-1)^{j-\alpha}$$
$$-3\int_1^2 (xt) \sum_{j=\lceil\beta\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j}\binom{3}{j}\binom{j}{i}\frac{\Gamma(j+1)}{\Gamma(j+1-\beta)}(x-1)^{j-\beta}dt\} = f(x).$$

As a particular case, if we take $\alpha = 2$ and $\beta = 1$ the exact solution of equation (13) is $y(x) = x^2 + 1$. After integrating and simplifying the above equation, we get the following equation:

$$\begin{aligned} & c_0\Big[12-6x\Big] + c_1\Big[-30+18x\Big] + c_2\Big[24-18x\Big] + c_3\Big[-6+6x\Big] \\ & -3x\int_1^2 \{c_0\Big[-12t+12t^2-3t^3\Big] + 3c_1\Big[8t-10t^2+3t^3\Big] \\ & -3c_2\Big[-5t+8t^2-3t^3\Big] + c_3\Big[3t-6t^2+3t^3\Big]\}dt = 2-14x. \end{aligned} \tag{17}$$

Integrating the last equation and substituting for c_0 and c_1 and simplifying, we get

$$c_2[24 - \frac{69}{4}x] + c_3[-6 + \frac{3}{4}x] = 58 - \frac{119}{2}x.$$

Solving for c_2 and c_3, we obtain that $c_2 = 3.666$ and $c_3 = 4.997$. The approximate solution of equation (13) is

$$y(x) \approx 2(2-x)^3 + 8(x-1)(2-x)^2 + 3 \times (3.66)(x-1)^2(2-x) + 4.997(x-1)^3.$$

The following table describes the relation between the exact and approximate solution of some selected values of x, where $n = 3$, $\alpha = 2$ and $\beta = 1$.

Table 1: Exact and approximate solution when $\alpha = 2$ and $\beta = 1$

x	y_{Approx}	y_{Exact}
1.1	2.20998	2.21
1.2	2.43991	2.44
1.3	2.68979	2.69
1,4	2.95962	2.95999999999999
1.5	3.24938	3.25
1.6	3.55906	3.55999999999999
1.7	3.88868	3.88999999999999
1.8	4.23821	4.24
1.9	4.60765	4.60999999999999
2	4.9971	4.99999999999999

Now, if we take $\alpha = \frac{3}{2}$ and $\beta = 0.5$, we have

$$\sum_{i=0}^{3} c_i \{ \sum_{j=\lceil\alpha\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j} \binom{3}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j-\frac{1}{2})} (x-1)^{j-\frac{3}{2}}$$
$$- 3 \int_1^2 (xt) \sum_{j=\lceil\beta\rceil}^{3} \frac{(-1)^{j-i}}{(b-a)^j} \binom{3}{j} \binom{j}{i} \frac{\Gamma(j+1)}{\Gamma(j+\frac{1}{2})} (x-1)^{j-\frac{1}{2}} dt \}$$
$$= \frac{4}{\sqrt{\pi}} (\sqrt{x-1} - \frac{64x}{7}).$$

Substituting and simplifying, we get

$$c_2 \left[\frac{6}{\Gamma(\frac{3}{2})} (x-1)^{\frac{1}{2}} - \frac{18}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} - \frac{64x}{35\sqrt{\pi}} \right] + c_3 \left[\frac{6}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} - \frac{64x}{7\Gamma(\frac{7}{2})} \right]$$
$$= -\frac{320x}{21\sqrt{\pi}} + \frac{256x}{105\sqrt{\pi}} + \frac{24}{\Gamma(\frac{3}{2})} (x-1)^{\frac{1}{2}} - \frac{36}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} - \frac{48}{\Gamma(\frac{3}{2})} (x-1)^{\frac{1}{2}}$$
$$- \frac{48}{\Gamma(\frac{5}{2})} (x-1)^{\frac{3}{2}} + \frac{4}{\sqrt{\pi}} (\sqrt{x-1} - \frac{64x}{7}).$$

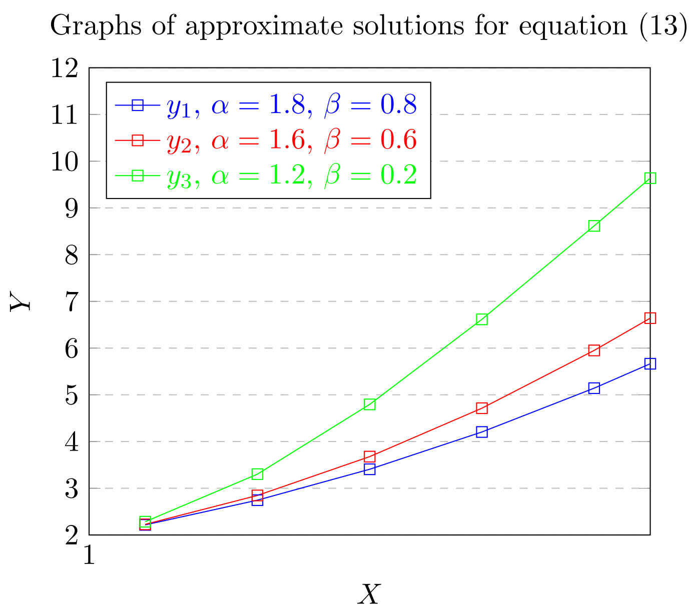
We get $1.0316c_2 + 2.7511c_3 = 9.2489$ and $8.8335c_2 - 0.98876c_3 = -2.4978$. Solving for c_2 and c_3, we get $c_2 = 0.0897$ and $c_3 = 3.3283$.

The following table describes the approximate solution of equation (13) for some selected values of n, α and β. Here, y_1, y_2 and y_3 represent the approximate solution when $n = 3$, ($\alpha = 1.8$, $\beta = 0.8$), ($\alpha = 1.6$, $\beta = 0.6$) and ($\alpha = 1.2$, $\beta = 0.2$), respectively. While y_4, y_5 and y_6 represent the approximate solution when $n = 7$, ($\alpha = 1.8$, $\beta = 0.8$), ($\alpha = 1.6$, $\beta = 0.6$) and ($\alpha = 1.2$, $\beta = 0.2$).

Table 2: Approximate solution when ($n = 3$) and($n = 7$)

x	y_1	y_2	y_3	y_4	y_5	y_6
1.1	2.215946531	2.227593549	2.284126883	2.216914203	2.233433219	2.337681516
1.2	2.464096576	2.509845878	2.724168086	2.466686882	2.524408268	2.836704064
1.3	2.744915811	2.84596451	3.301614439	2.748712324	2.865521894	3.420408092
1.4	3.058869912	3.235156969	3.997956773	3.063062995	3.254315643	4.07293886
1.5	3.406424555	3.676630777	4.794685919	3.410119352	3.690280056	4.793196618
1.6	3.788045417	4.169593457	5.673292706	3.790382296	4.173485633	5.579198956
1.7	4.204198174	4.713252534	6.615267966	4.204405053	4.704224102	6.429009677
1.8	4.655348502	5.306815529	7.602102528	4.652781212	5.283043554	7.344388516
1.9	5.141962077	5.949489967	8.615287224	5.136125705	5.910560971	8.323316056
2	5.664504577	6.640483371	9.636312884	5.654985475	6.586435719	9.327548172

The following graphs represents the approximate solution of equation (13), for $n = 3$ and some selective α and β.



Example 4.2. Consider the following integro-differential equation:

$$ {}_2^cD_x^{\alpha} y(x) + g_1(x)\ {}_2^cD_x^{\frac{\alpha}{2}} y(x) + g_2(x)\ {}_2^cD_x^{\frac{\alpha}{3}} y(x) = f(x) + \int_2^4 K(x,t)\ {}_2^cD_t^{\beta} y(t) dtm, \tag{18}$$

where $g_1(x) = -\Gamma(4-\frac{\alpha}{2})(x-2)^{\frac{\alpha}{2}}$, $g_2(x) = \Gamma(4-\frac{\alpha}{3})(x-2)^{\frac{\alpha}{3}}$, $f(x) = \frac{72(x-2)^{3-\alpha}}{\Gamma(4-\alpha)} + 16(x-2) - 6\alpha(x-2)^2 - x[10 - \frac{6}{2-\beta}]$, $K(x,t) = \frac{\Gamma(2-\beta)}{16} x(t-2)^{\beta}$, $2 < \alpha \leq 3$, $\beta \leq \alpha$ and $2 \leq t, x \leq 4$. Subject to the conditions $y(2) = 0$, $y'(2) = 8$, and $y''(2) = -36$.

By using Bernstein polynomials of degree $n = 5$, we approximate the solution as:

$$y(x) = \sum_{i=0}^{5} c_i B_{i,5}(x). \tag{19}$$

From equations (9), (10), we obtain that $c_0 = 0$, $c_1 = 3.2$ and $c_2 = -0.8$.

For a particular case, if we take $\alpha = 3$ and $\beta = 1$, the exact solution of equation (18) is $y(x) = 12x^3 - 90x^2 + 224x - 184$. Applying equation (6) on $y(x)$ and substituting in equation (18), we obtain a system of equations and solving for c_i's we obtain that $c_3 = -2.4$, $c_4 = 8$ and $c_5 = 40$. The approximate solution of equation (18) is

$$y(x) \approx 3.2 \times 5(x-2)(4-x)^4 - 0.8 \times 10(x-2)^2(4-x)^3$$
$$-2.4 \times 10(x-2)^3(4-x)^2 + 8 \times 5(x-2)^4(4-x) + 40(x-2)^5.$$

Table 3: Exact and approximate solution of equation (18) when $\alpha = 3$ and $\beta = 1$

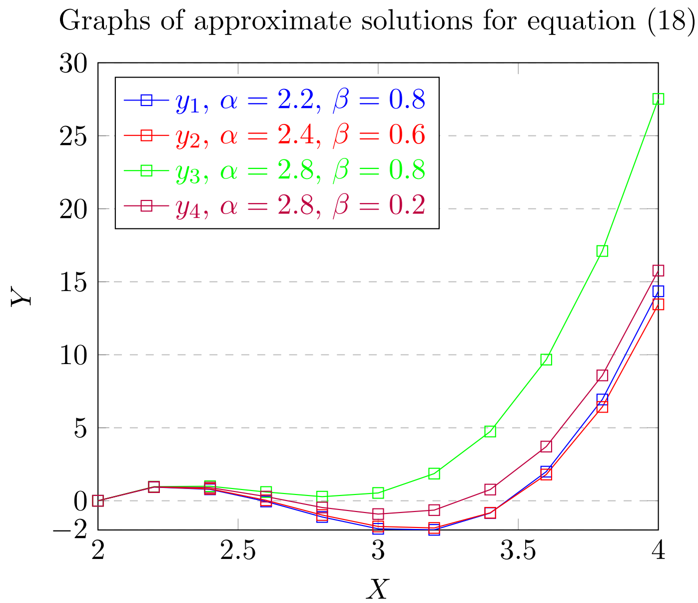
x	y_{Exact}	y_{Approx}
2	0	0
2.2	0.976	0.976
2.4	1.088	1.088
2.6	0.912	0.912000000000003
2.8	1.024	1.024
3	2	2.00000000000001
3.2	4.416	4.41600000000002
3.4	8.848	8.84800000000002
3.6	15.872	15.872
3.8	26.064	26.064
4	40	40

Table (3), describes the relation between the exact and approximate solution of some selected values of x when $n = 5$, $\alpha = 3$ and $\beta = 1$.

In Table 4, the approximate solution of equation (18) for some selected values of n, α and β is given. Where (y_1, y_2, y_3 and y_4) represent the approximate solution when $n = 5$, ($\alpha = 2.2$, $\beta = 0.8$), ($\alpha = 2.4$, $\beta = 0.6$), ($\alpha = 2.8$, $\beta = 0.8$) and ($\alpha = 2.8$, $\beta = 0.2$) respectively.

Table 4: Approximate solution of equation (18) when ($n = 5$)

x	y_1	y_2	y_3	y_4
2	0	0	0	0
2.2	0.935068639	0.940240031	0.964551823	0.952827794
2.4	0.783006127	0.815156792	0.996094846	0.902813899
2.6	-0.04904965	0.03156834	0.600239489	0.286683772
2.8	-1.109884972	-0.977248379	0.280076751	-0.461247938
3	-1.920849711	-1.761022338	0.535063774	-0.91120986
3.2	-1.992496767	-1.86195325	1.859909408	-0.640106363
3.4	-0.841221493	-0.824176374	4.743459784	0.766349975
3.6	1.994098875	1.796772669	9.667583876	3.711505903
3.8	6.93546579	6.424493083	17.10605907	8.58563503
4	14.34911996	13.45225408	27.52345674	15.76380536



Example 4.3. Consider the integro differential equation

$$ {}_0^cD_x^{\alpha}y(x) - {}_0^cD_x^{\frac{\alpha}{2}}y(x) = f(x) + \int_0^1 e^x \;\; y(t)dt, \tag{20}$$

where $f(x) = e^x(1-e)$, $1 < \alpha \leq 2$, and $0 \leq t, x \leq 1$.

Subject to the conditions $y(0) = y'(0) = 1$.

By using Bernstein polynomials of degree $n = 5$, we approximate the solution as:

$$ y(x) = \sum_{i=0}^{5} c_i B_{i,5}(x). \tag{21}$$

From equations (9), (10), we obtain that $c_0 = 1$ and $c_1 = 1.2$.

For a particular case, if we take $\alpha = 1.5$, the exact solution of equation (20) is $y(x) = e^x$. Applying equation (6) on $y(x)$ and substituting in equation (20), we obtain a system of equations and solving for c_i's we obtain that $c_2 = 1.4499$, $c_3 = 1.766749$, $c_4 = 2.1746$ and $c_5 = 2.71818$. The approximate solution of equation (18) is

$$ y(x) \approx (1-x)^5 + 1.2 \times 5x(1-x)^4 + 1.4499 \times 10x^2(1-x)^3 $$
$$ +1.766749 \times 10x^3(1-x)^2 + 2.1746 \times 5x^4(1-x) + 2.71818x^5. $$

Table 5, describes the relation between the exact and approximate solution of some selected values of x when $n = 5$ and $\alpha = 1.5$.

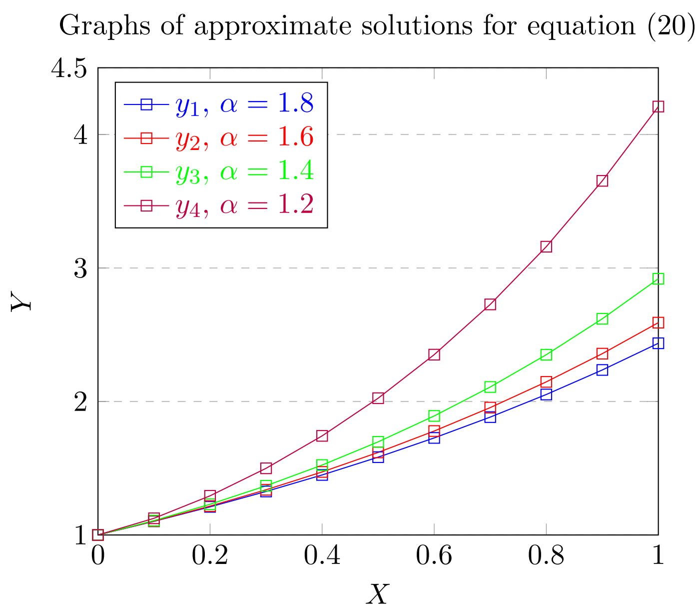
Table 5: Exact and approximate solution of equation (20) when $\alpha = 1.5$

x	y_{Exact}	y_{Approx}	Error
0	1	1	0
0.1	1.105170918	1.10516730537358	0.36127E-07
0.2	1.221402758	1.22139337801439	0.938015E-07
0.3	1.349858808	1.34984418528478	0.146223E-06
0.4	1.491824698	1.49180461512623	0.200825E-06
0.5	1.648721271	1.64869423914596	0.270316E-06
0.6	1.8221188	1.82208307570373	0.357247E-06
0.7	2.013752707	2.01370735299845	0.453545E-06
0.8	2.225540928	2.22548527215494	0.556563E-06
0.9	2.459603111	2.45953277031058	0.703408E-06
1	2.718281828459050	2.71817928370205	0.102545E-03

The following table describes the approximate solution of equation (20) when $n = 5$ and for some selected values of α. Where y_1, y_2, y_3 and y_4 represent the approximate solution when ($\alpha = 1.8$), ($\alpha = 1.6$), ($\alpha = 1.4$) and ($\alpha = 1.2$) respectively.

Table 6: Approximate solution of equation (20) when ($n = 5$)

x	y_1	y_2	y_3	y_4
0	1	1	1	1
0.1	1.10240044295794	1.1038346323492	1.10747314761792	1.12496938543169
0.2	1.21058024191943	1.21626456566409	1.2300657682098	1.29333323551814
0.3	1.32590023707549	1.33863237800476	1.36843578414082	1.49945853949894
0.4	1.44961437376952	1.47225158750577	1.52372197489588	1.74235948212542
0.5	1.58291419217209	1.61844778803726	1.69748145554922	2.02453402379607
0.6	1.72697331695556	1.77859978486614	1.891627155234	2.35080048069196
0.7	1.88299194696886	1.95418073031716	2.10836529561194	2.72713410491211
0.8	2.05224134491214	2.14679925943408	2.35013286934289	3.15950366460895
0.9	2.23610832701151	2.35824062564082	2.61953511855447	3.65270802412371
1	2.43613975269372	2.59050783640254	2.91928301331163	4.20921272412183



Example 4.4. Consider the integro-differential equation:

$$\begin{aligned} &{}_2^cD_x^{\alpha} y(x) + \frac{1}{6}\sum_{k=2}^{n} g_k(x)\ {}_2^cD_x^{\frac{\alpha}{k}} y(x) + g_0(x)\ \ y(x) \\ &= f(x) + \frac{1}{64}\int_2^6 \sum_{m=1}^{2} K_m(x,t)\ {}_2^cD_t^{\frac{\beta}{m}} y(t)dt, \end{aligned} \tag{22}$$

where $g_0(x) = -5$, $g_k(x) = \Gamma(4 - \frac{\alpha}{k})(x-2)^{\frac{\alpha}{k}}$, $k = 2, 3, 4, 5, 6$,

$$K_m(x,t) = 6\Gamma\left(4 - \frac{\beta}{m}\right)(x-2)^2(t-2)^{\frac{\beta}{m}}, m = 1, 2,$$

$$f(x) = \left(6 - 12\beta + \frac{57\alpha}{30}\right)(x-2)^2 + \left(\frac{(12-\alpha)(18-\alpha)}{24} - 45\right)(x-2) - 10,$$

$5 < \alpha \leq 6$, $\beta \leq \alpha$ and $2 \leq t, x \leq 6$.

Subject to the conditions $y(2) = 2$, $y'(2) = 9$, $y''(2) = -12$, $y'''(2) = 6$, $y^{(4)}(2) = y^{(5)}(2) = 0$.

By using Bernstein polynomials of degree $n = 8$, the approximate solution is:

$$y(x) = \sum_{i=0}^{8} c_i B_{i,8}(x). \tag{23}$$

From equations (9), (10), we obtain that $c_0 = 2$, $c_1 = 6.5$, $c_2 = 7.571428571$, $c_3 = 6.357142857$, $c_4 = 4$, $c_5 = 1.642857143$.

Applying equation (6) on $y(x)$ and substituting in equation (22). For a particular case, if we take $\alpha = 6$ and $\beta = 3$, then the exact solution is $y(x) = x^3 - 12x^2 + 45x - 48$. After simplifying, we obtain a system of equations and solving for $c_i's$ we obtain that $c_6 = 0.428571429$, $c_7 = 1.5$ and $c_8 = 6$.

In the following table, we clarify the relation between the exact and approximate solution of some selected values of x when $n = 6$, $\alpha = 6$ and $\beta = 3$.

Table 7: Exact and approximate solution of equation (22) when $n = 6$, $\alpha = 6$ and $\beta = 3$

x	y_{Exact}	y_{Approx}
2	2	2
2.2	3.568	3.568
2.4	4.704	4.704
2.6	5.456	5.45599999999999
2.8	5.872	5.872
3.2	5.888	5.88799999999999
3.6	5.136	5.13599999999999
3.8	4.592	4.59199999999999
4.2	3.408	3.40799999999999
4.6	2.416	2.416
4.8	2.112	2.112
5	2	2
5.2	2.128	2.128
5.4	2.544	2.544
5.6	3.296	3.296
5.8	4.432	4.43200000000001
6	6	6.00000000000002

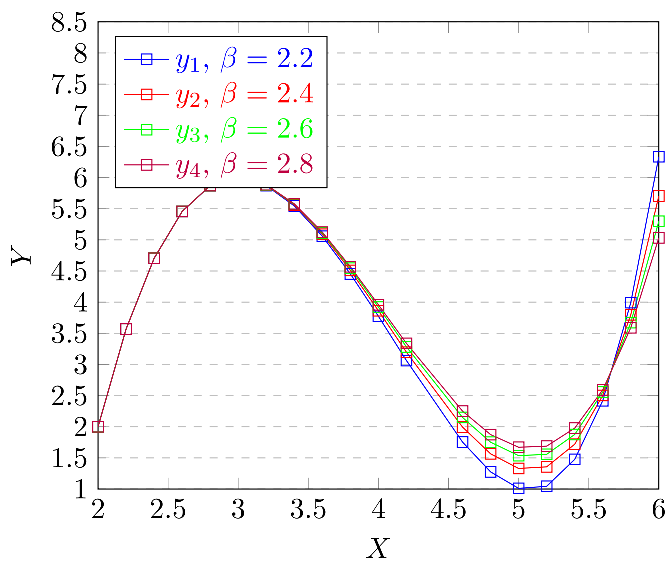
Table 8 describes the approximate solution of equation (22) for some selected values of n, α and β. y_1, y_2, y_3 and y_4 represent the approximate solution when $n = 8$, ($\alpha = 5.2$, $\beta = 2.2$), ($\alpha = 5.2$, $\beta = 2.4$), ($\alpha = 5.2$, $\beta = 0.6$) and ($\alpha = 5.2$, $\beta = 2.8$) respectively.

Table 8: Approximate solution for equation (22) when ($n = 8$)

x	y_1	y_2	y_3	y_4
2.2	3.567999399	3.567999659	3.567999826	3.567999935
2.6	5.455632344	5.455789229	5.455890186	5.455956281
2.8	5.870120127	5.870916513	5.871428998	5.871764511
3.2	5.870551725	5.877816333	5.882491207	5.885551739
3.6	5.058221496	5.08987522	5.110244822	5.12358033
3.8	4.45331972	4.50895321	4.54475412	4.568192147
4	3.772873431	3.862434539	3.920068334	3.957799846
4.2	3.06214361	3.195683913	3.28161892	3.33787858
4.8	1.275625972	1.565429718	1.751922382	1.874014815
5.2	1.044082921	1.356223109	1.557089579	1.688592209
5.4	1.475720031	1.718932302	1.875442716	1.977906463
5.8	3.992755454	3.797248767	3.671437547	3.589071852
6	6.335131668	5.704188236	5.298167531	5.032355169

In the following graphs, the approximate solution of equation (22) is drawn with distinct given β.

Graphs of approximate solutions for equation (22) when $n = 8$ and $\alpha = 5.2$



5. Conclusion

In this paper, an approximate solution of certain types of Fredholm integro-differential equations of fractional order $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ is given by using the general form of Bernstein polynomials of various degrees. It is noted that the approximate solution of such equations is very close to the exact one.

References

[1] A. Dascioglu, D. V. Bayram, *Solving fractional Fredholm integro-differential equations by Laguerre polynomials*, Sains Malaysiana, 48 (2019), 251-257.

[2] A. Freed, K. Diethelm, Yu. Luchko, *Fractional-order viscoelasticity (FOV), Constitutive development using the fractional calculus*, NASA's Glenn Research Center, Ohio, 2002.

[3] J. F. Gómez-Aguila, *Space-time fractional diffusion equation using a derivative with nonsingular and regular kernel*, Physica A: Statistical Mechanics and its Applications, 465 (2017), 562-572.

[4] R. Gorenflo, F. Mainardi, *Fractional calculus, integral and differential equations of fractional order*, in A. Carpinteri and F. Mainardi (Editors), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien (1997), 223-276.

[5] M. Gülsu, Y. Öztürk, A.Anapali, *Numerical approach for solving fractional Fredholm integro-differential equation*, International Journal of Computer Mathematics, 90 (2013), 1413-1434.

[6] R. Hilfer (Ed.), *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[7] K. I. Joy, *Bernstein polynomial*, On-Line Geometric Modeling Notes, 13 (2000), 1-13.

[8] J. R. Loh, C. Phang, *Jacobi wavelet operational matrix of fractional integration for solving fractional integro-differential equation*, Journal of Physics: Conference Series, 693 (2016), 012002.

[9] J. R. Loh, C. Phang, *Numerical solution of Fredholm fractional integro-differential equation with right-sided Caputo's derivative using Bernoulli polynomials operational matrix of fractional derivative*, Mediterr. J. Math., 28 (2019).

[10] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity*, Imperial College Press, London, 2010.

[11] L. Mansouri, Z. Azimzadeh, *Numerical solution of fractional delay Volterra integro-differential equations by Bernstein polynomials*, Mathematical Sciences (2022), 1-12.

[12] D.Sh. Mohammed, *Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial*, Mathematical Problems in Engineering, 2014 (2014), 5-10.

[13] J.A. Nanware1, P.I M. Goud, T.L. Holambe, *Solution of fractional integro-differential equations by Bernstein polynomials*, Malaya Journal of Matematik, Vol. S, 1 (2020), 581-586.

[14] T. Oyedopo, O. A. Taiwo, J. U. Abubakar, Z. O. Ogunwobi, *Numerical studies for solving fractional integro differential equations by using least squares method and Bernstein polynomials*, Fluid Mech. Open Access, 3 (2016), 1-7.

[15] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, Calif, USA, 198 (1999).

[16] T. Polat, S. Nergis, A. T. Dincel, *Numerical solution method for multi-term variable order fractional differential equations by shifted Chebyshev polynomials of the third kind*, Alexandria Engineering Journal, 61.7 (2022), 5145-5153.

[17] A. Saadatmandi, *Bernstein operational matrix of fractional derivatives and its applications*, Applied Mathematical Modelling, 38 (2014), 1365-1372.

[18] S. C. Shiralashetti, S. Kumbinarasaiah,*CAS wavelets analytic solution and Genocchi polynomials numerical solutions for the integral and integro-differential equations*, Journal of Interdisciplinary Mathematics, 22 (2019), 201-218.

[19] L. Tabharit, Z. Dahmani, *Integro-differential equations of arbitrary orders involving convergent series*, Journal of Interdisciplinary Mathematics, 23 (2020), 935-953.

Accepted: January 12, 2023

Applications of β-open sets

Shallu Sharma[*]
Department of Mathematics
University of Jammu
Jammu 180006
India
shallujamwal09@gmail.com

Tsering Landol
Department of Mathematics
University of Jammu
Jammu 180006
India
tseringlandol09@gmail.com

Sahil Billawria
Department of Mathematics
University of Jammu
Jammu 180006
India
sahilbillawria2@gmail.com

Abstract. In this paper, we establish the validity of the β-open sets. We introduce and study topological properties of β-limit point, β-derived set, β-interior points, β-border, β-frontier and β-exterior. The existence of their relation is also investigated with examples and counter examples.

Keywords: β-open sets, β-interior points, β-derived set, β-boundary, β-frontier and β-exterior.

1. Introduction

Generalized open sets play a vital role in General Topology and are now the research topics of many topologists worldwide. N. Levine [6] in 1863, introduced the notion of semi-open sets and T.M. Nour [10] in 1998 presented the concept of semi-closure, semi-interior, semi-frontier and semi-exterior. Njastad [9] presented the notion of α-open sets and Caldas [4] further developed the topological properties of α-open sets [11]. One of the generalized forms of open sets is the pre-open set which is given by Mashhour et. al. [8] in 1983. It gave an inspiration to Youngbae Jun et. al. [5] to further generalized the properties of pre-open set. Abd El-Monsef et. al. [1] gave the concept of β-open sets and β-continuity in topological spaces. The concept of nearly open set played a

∗. Corresponding author

significant role in expansions of some advance theories of topological structures such as fuzzy set theory, soft rough set theory, probability theory and are widely research these days due to its wide application.

In this paper, we investigate the fundamental properties of β-limit points, β-derived sets, β-closure of a set, β-interior points, β-border, β-frontier and β-exterior with numerous examples. Moreover, the relation between the properties and existing properties are studied.

2. Preliminaries

Throughout this paper, (X, τ) (or simply X) means topological space. For $A \subseteq X$, closure of A is denoted by Cl(A) and interior of A is denoted Int(A).

Definition 2.1. *Let X be a topological space, then $A \subseteq X$ is called:*
(a) *semi-open [6] if $A \subseteq Cl(Int(A))$;*
(b) *α-open [9] if $A \subseteq Int(Cl(Int(A)))$;*
(c) *pre-open [8] if $A \subseteq Int(Cl(A))$;*
(d) *β-open [1] if $A \subseteq Cl(Int(Cl(A)))$.*

The complement of β-open(resp.α-open, semi-open, pre-open) set is called β-closed set(resp.α-closed set, semi-closed set, pre-closed set). The intersection of all β-closed sets(resp.α-closed sets, semi-closed sets, pre-closed sets) in X containing a subset A in X is called β-closure(resp. α-closure, semi-closure, pre-closure) and is denoted by $Cl_\beta(A)$(resp.$Cl_\alpha(A)$, sCl(A), $Cl_p(A)$). It is well known fact that the set $B \subseteq X$ is β-closed iff $B = Cl_\beta(A)$.

We denote the family of β-open(resp. α-open, pre-open) sets by τ^β(resp. τ^α, τ^p). But τ^β need not be a topology which is explained in Example 3.3.

Example 2.1. (a) Consider a topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ on set $X = \{a, b, c\}$. Then the family of β-open sets, α-open sets and pre-open sets are equal with topology τ on X i.e. $\tau^\beta = \tau^\alpha = \tau = \tau^p$.

(b) Consider a topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ on a set $X = \{a, b, c\}$. Then, $\tau^\beta = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $\tau^\alpha = \tau = \tau^p$.

3. Applications of β-open sets

Definition 3.1. *Let B be a subset of a topological space (X, τ). A point $b \in B$ is said to be β-limit point of B if $\forall A \in \tau^\beta$ containing b, $A \cap B \setminus \{b\} \neq \emptyset$.*

The set of β-limit points of B is called β-derived set of B and is denoted by $D_\beta(B)$. Note that $D_p(B)$ [5], $D_\alpha(B)$ [4] and $D(B)$ denotes derived set of pre-open set, α-open set and derived set of B respectively.

Example 3.1. (a) Let (X, τ) be the topological space which is described in Example 2.1[a]. Let $A = \{a, b\}$. Then, $D_\beta(A) = \{c\} = D_p(A) = D_\alpha(A) = D(A)$.

(b) Let (X, τ) be the topological space which is described in Example 2.1[b]. Let $A = \{a, b\}$. Then, $D_p(A) = D_\alpha(A) = D(A) = \{c\} = D_\beta(A)$.

Theorem 3.1. *Let B be a subset of X and $b \in X$. Then the following are equivalent:*

(*i*) *For $b \in A$ and $\forall A \in \tau^\beta$, $B \cap A \neq \emptyset$.*

(*ii*) *$b \in Cl_\beta(B)$.*

Proof. If $b \notin Cl_\beta(B)$, then there exist β-closed set C such that $B \subseteq C$ and $b \notin C$. Hence, $X \setminus C$ is β-open set containing b and $B \cap X \setminus C \subseteq B \cap X \setminus B = \emptyset$, which is a contradiction to (i). Hence, $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ is straightforward. □

Corollary 3.1. *For any subset B of X, we have $D_\beta(B) \subseteq Cl_\beta(B)$.*

Proof. Suppose $b \in D_\beta(B)$, then there exists a β-open set A such that $A \cap B \setminus \{b\} \neq \emptyset$ which implies $A \cap B \neq \emptyset$. Hence, $b \in Cl_\beta(B)$. □

Theorem 3.2. *For any subset B of X, $Cl_\beta(B) = B \cup D_\beta(B)$.*

Proof. Let $b \in Cl_\beta(B)$. Assume that $b \notin B$ and let $G \in \tau^\beta$ with $b \in G$. Then $G \cap B \setminus \{b\} \neq \emptyset$ and so $b \in D_\beta(B)$. Hence, $Cl_\beta(B) \subseteq B \cup D_\beta(B)$. For the reverse inclusion, $B \subseteq Cl_\beta(B)$ and by Corollary 3.1, $B \cup D_\beta(B) \subseteq Cl_\beta(B)$. Hence, the proof. □

Corollary 3.2. *A subset B is β-closed set iff it contains the set of β-limit points.*

Lemma 3.1. *If $\{A_i : i \in \Delta\}$ is a family of β-open sets in X, then $\bigcup_{i \in \Delta} A_i$ is a β-open set in X, where Δ is any index set.*

Proof. Straightforward □

Example 3.2. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}\, \{a, b\}\}$. Then, $\tau^\beta = \tau \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$.

So, $\{a, d\} \cap \{b, d\} = \{d\} \notin \tau^\beta$ which means that the intersection of two β-open set is not β-open in general.

Remark 3.1. For any topology τ on a set X, τ^β may not be topology on X.

Example 3.3. Let $X = \{a, b, c, d\}$ be a set with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$. Then, $\tau^\beta = \tau \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Clearly τ^β is not a topology as $\{b, c\}, \{a, c\} \in \tau^\beta$ but $\{b, c\} \cap \{a, c\} = \{c\} \notin \tau^\beta$. Another reason for τ^β not being topology is explained in Example 3.5.

Theorem 3.3. *Let B_1 and B_2 be subsets of X. If $B_1 \in \tau^\beta$ and τ^β is a topology on X, then $B_1 \cap Cl_\beta(B_2) \subseteq Cl_\beta(B_1 \cap B_2)$.*

Proof. Let $b \in B_1 \cap Cl_\beta(B_2)$. Then, $b \in B_1$ and $b \in Cl_\beta(B_2) = B_2 \cup D_\beta(B_2)$. If $b \in B_2$, then $b \in B_1 \cap B_2 \subseteq Cl_\beta(B_1 \cap B_2)$. If $b \notin B_2$, then $b \in D_\beta(B_2)$ and for all β-open set G containing b, $G \cap B_2 \neq \emptyset$. Since $B_1 \in \tau^\beta$, so $G \cap B_1$ is also a β-open set containing b.

Hence, $G \cap (B_1 \cap B_2) = (G \cap B_1) \cap B_2 \neq \emptyset$ and consequently $b \in D_\beta(B_1 \cap B_2) \subseteq Cl_\beta(B_1 \cap B_2)$. Therefore, $B_1 \cap Cl_\beta(B_2) \subseteq Cl_\beta(B_1 \cap B_2)$. □

The converse of the above theorem is not true in general as seen in the following example.

Example 3.4. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{c\}, \{c, d\}, \{a, b, c\}\}$ be a topology on X and $\tau^\beta = \tau \cup \{\{a, c\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}$ is a topology on X. Let $B_1 = \{c, d\}$, $B_2 = \{b, c\} \in \tau^\beta$ and $B_1 \cap B_2 = \{c\}$. Then, $B_1 \cap Cl_\beta(B_2) = \{c, d\} \cap X = \{c, d\}$ and $Cl_\beta(B_1 \cap B_2) = X$. Therefore, converse is not true in general.

Example 3.5. Let (X, τ) be the topological space and τ^β be same as described in Example 3.3. Let $B_1 = \{b, c, d\}$, $B_2 = \{a, b, c\}$ and $B_1 \cap B_2 = \{b, c\}$. Then, $B_1 \cap Cl_\beta(B_2) = \{b, c, d\}$ and $Cl_\beta(B_1 \cap B_2) = \{b, c\}$. Therefore, $B_1 \cap Cl_\beta(B_2) = \{b, c, d\} \nsubseteq \{c, d\} = Cl_\beta(B_1 \cap B_2)$, which implies τ^β is not a topology.

Corollary 3.3. *If B_1 is β-closed in Theorem 3.3, then equality holds i.e. $B_1 \cap Cl_\beta(B_2) = Cl_\beta(B_1 \cap B_2)$.*

Proof. The first implication $B_1 \cap Cl_\beta(B_2) \subseteq Cl_\beta(B_1 \cap B_2)$ is same as in Theorem 3.3. For the other way, $Cl_\beta(B_1) = B_1$ since B_1 is β-closed so, $Cl_\beta(B_1 \cap B_2) \subseteq Cl_\beta(B_1) \cap Cl_\beta(B_2) = B_1 \cap Cl_\beta(B_2)$, which is the desired result. □

Theorem 3.4 (Properties of β-Derived set)**.** *For any subset B_1 and B_2 of topological space (X, τ), the following assertions hold:*

1. *If $B_1 \subseteq B_2$, then $D_\beta(B_1) \subseteq D_\beta(B_2)$.*
2. *$D_\beta(B_1) \cup D_\beta(B_2) \subseteq D_\beta(B_1 \cup B_2)$ and $D_\beta(B_1 \cap B_2) \subseteq D_\beta(B_1) \cap D_\beta(B_2)$.*
3. *$D_\beta(D_\beta(B)) \setminus B \subseteq D_\beta(B)$.*
4. *$D_\beta(B \cup D_\beta(B)) \subseteq B \cup D_\beta(B)$.*

Proof. 1. Let $b \in D_\beta(B_1)$. Then $U \cap B_1 \setminus \{b\} \neq \emptyset$, for any β-open set U containing b. Since $B_1 \subseteq B_2$, $U \cap B_2 \setminus \{b\} \neq \emptyset$, which implies $b \in D_\beta(B_2)$.

2. Follows directly from (1).

3. Let $b \in D_\beta(D_\beta(B)) \setminus B$, then $U \cap D_\beta(B) \setminus \{b\} \neq \emptyset$, for any β-open set U containing b. Let $c \in U \cap D_\beta(B) \setminus \{b\}$. Then, $c \in U$ and $c \in D_\beta(B)$ which implies $U \cap B \setminus \{c\} \neq \emptyset$. Let $d \in U \cap B \setminus \{c\}$. Thus, $d \neq b$, for $d \in B$ and $b \notin B$. Hence, $U \cap B \setminus \{b\} \neq \emptyset$. Hence, $b \in D_\beta(B)$.

4. Let $b \in D_\beta(B \cup D_\beta(B))$. If $b \in B$, the result is obvious. Suppose $b \notin B$, then $G \cap (B \cup D_\beta(B)) \setminus \{b\} \neq \emptyset$, for all $G \in \tau^\beta$ with $b \in G$. Hence, $G \cap B \setminus \{b\} \neq \emptyset$ or $G \cap D_\beta(B) \setminus \{b\} \neq \emptyset$. This implies $b \in D_\beta(B)$ for the first case.

If $G \cap D_\beta(B) \setminus \{b\} \neq \emptyset$, then $b \in D_\beta(D_\beta(B))$. Since, $b \notin B$, it follows from (3) that $b \in D_\beta(D_\beta(B)) \setminus B \subseteq D_\beta(B)$. Hence, the proof. □

Example 3.6. Let $X = \{a, b, c, d, e\}$ with

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Then, $\tau^\beta = \{X, \emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, b, d\}, \{a, c, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$. Consider $B_1 = \{a, c\}$ and $B_2 = \{d, e\}$. Then, $D_\beta(B_1) = \emptyset = D_\beta(B_2)$ and so $D_\beta(B_1) \cup D_\beta(B_2) = \emptyset \subset D_\beta(B_1 \cup B_2) = \{b, e\}$. Hence, converse is not true in the case of Theorem 3.4(2).

Example 3.7. Let $X = \{a, b, c, d\}$ be a set with topology $\tau = \{X, \emptyset, \{c\}, \{c, d\}, \{a, b, c\}\}$. Then, $\tau^\beta = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}\}$. Let $B = \{a, b, c\}$ be a subset of X. Then, $D_\beta(B) = \{a, b, d\}$ and so $D_\beta(D_\beta(B)) = \emptyset$, which implies converse of part (3) of the Theorem 3.4 need not be true in general. Similarly, $B \cup D_\beta(B) = \{a, b, c, d\}$ and so $D_\beta(B \cup D_\beta(B)) = \{a, b, d\}$. Hence, $B \cup D_\beta(B) \nsubseteq D_\beta(B \cup D_\beta(B))$ which implies the converse of part (4) of the above theorem is not true in general.

Definition 3.2. *Let A be a subset of a topological space X. A point $p \in A$ is called pre-interior point [5] of A if there exists a pre-open set P containing p such that $P \subseteq A$. The set of all pre-interior points of A is known as pre-interior points of A and it is denoted by $Int_p(B)$*

Definition 3.3. *Let B be a subset of a topological space X. A point $b \in B$ is called β-interior point of B if there exists a β-open set G containing b such that $G \subseteq B$. The set of all β-interior points of B is called β-interior points of B and is denoted by $Int_\beta(B)$.*

Theorem 3.5. *Let B be a subset of X. Then, every pre-interior point of B is β-interior point of B, i.e. $Int_p(B) \subseteq Int_\beta(B)$.*

Proof. Let $b \in Int_p(B)$. Then, there exist pre-open set P containing b such that $P \subseteq B$. Every pre-open set is β-open, thus we get a β-open set P containing b such that $P \subseteq B$. It follows that $b \in Int_\beta(B)$.

The converse of this theorem is not true in general given by following example. □

Example 3.8. Let $X=\{a, b, c, d, e\}$ with topology $\tau=\{X, \emptyset, \{b\}, \{d, e\}, \{b, d, e\}\}$. Then, $\tau^p=\tau\cup\{\{d\}, \{e\}, \{b, d\}, \{b, e\}, \{a, b, d\}, \{a, b, e\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$ and $\tau^\beta = \tau^p\cup\{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, c, e\}, \{c, d, e\}, \{a, c, d, e\}\}$.

(i) Consider a subset $B = \{a, c, d\}$. Then, we have $Int_p(B) = \{d\}$ and $Int_\beta(B) = \{a, c, d\}$.

(ii) Consider a subset $B = \{a, c, d, e\}$. Then, we have $Int_p(B) = \{d, e\}$ and $Int_\beta(B) = \{a, c, d, e\}$.

(iii) Consider a subset $B = \{a, b\}$. Then, we have $Int_p(B) = \{b\}$ and $Int_\beta(B) = \{a, b\}$.

Theorem 3.6 (Properties of β-interior)**.** *For subsets $B,\ B_1,\ B_2$ of a topological space X, the following hold:*

(1) *$Int_\beta(B)$ is the largest β-open set contained in B.*
(2) *B is β-open iff $B = Int_\beta(B)$.*
(3) *$Int_\beta(Int_\beta(B)) = Int_\beta(B)$.*
(4) *$Int_\beta(B) = B \setminus D_\beta(X \setminus B)$.*
(5) *$X \setminus Int_\beta(B) = Cl_\beta(X \setminus B)$.*
(6) *$Int_\beta(X \setminus B) = X \setminus Cl_\beta(B)$.*
(7) *If $B_1 \subseteq B_2$, then $Int_\beta(B_1) \subseteq Int_\beta(B_2)$.*
(8) *$Int_\beta(B_1) \cup Int_\beta(B_2) \subseteq Int_\beta(B_1 \cup B_2)$.*
(9) *$Int_\beta(B_1 \cap B_2) \subseteq Int_\beta(B_1) \cap Int_\beta(B_2)$.*

Proof. (1), (2) are straightforward.

(3) Trivially by (1) and (2).

(4) If $b \in B \setminus D_\beta(X \setminus B)$, then $b \notin D_\beta(X \setminus B)$ which implies there exists β-open set U containing b such that $U \cap (X \setminus B) = \emptyset$. Hence, $b \in U \subseteq B$ and $b \in Int_\beta(B)$. On the other hand, if $b \in Int_\beta(B) \subseteq B$ and $Int_\beta(B)$ is β-open set and $Int_\beta(B) \cap (X \setminus B) = \emptyset$. Hence, $b \notin D_\beta(X \setminus B)$. Therefore, $Int_\beta(B) = B \setminus D_\beta(X \setminus B)$.

(5) Using Theorem 3.2 and above part,

$$\begin{aligned} X \setminus Int_\beta(B) &= X \setminus (B \setminus D_\beta(X \setminus B)) \\ &= (X \setminus B) \cup D_\beta(X \setminus B) \\ &= Cl_\beta(X \setminus B). \end{aligned}$$

Hence, the proof.

(6) We have,

$$\begin{aligned} Int_\beta(X \setminus B) &= (X \setminus B) \setminus D_\beta(B) \\ &= (X \setminus B) \cap (D_\beta(B))^c \\ &= (X \setminus B) \cap (X \setminus D_\beta(B)) \\ &= X \setminus (B \cup D_\beta(B)) \\ &= X \setminus Cl_\beta(B). \end{aligned}$$

Hence, the proof.

(7) Let $b \in Int_\beta(B_1)$. Then, by definition, there exists β-open set U such that $b \in U \subseteq B_1$. Since $B_1 \subseteq B_2$ implies $b \in U \subseteq B_2$. Hence, $b \in Int_\beta(B_2)$. Hence, the proof.

(8) Since $B_1 \subseteq B_1 \cup B_2$ therefore, $Int_\beta(B_1) \subseteq B_1 \subseteq B_1 \cup B_2$. Similarly, $Int_\beta(B_2) \subseteq B_2 \subseteq B_1 \cup B_2$. We have, $Int_\beta(B_1) \cup Int_\beta(B_2) \subseteq B_1 \cup B_2$. Now,

$Int_\beta(B_1) \cup Int_\beta(B_2)$ is β-open subset of $B_1 \cup B_2$. As $Int_\beta(B_1 \cup B_2)$ is largest β-open subset of $B_1 \cup B_2$, we have $Int_\beta(B_1) \cup Int_\beta(B_2) \subseteq Int_\beta(B_1 \cup B_2)$. Hence, the proof.

(9) is same as in (8).

Converse of (7), (8) and (9) is not true in general as seen in the following example. □

Example 3.9. 1. Consider a set $X = \{a, b, c, d, e\}$ with same topology $\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ and τ^β as in Example 3.6. Let $B_1 = \{a, b, e\}$ and $B_2 = \{a, c, e\}$ be a subset of X. Then $Int_\beta(B_1) = \{a\}$ and $Int_\beta(B_2) = \{a, c, e\}$ which implies $Int_\beta(B_1) \subseteq Int_\beta(B_2)$ while $B_1 \nsubseteq B_2$. Again, let $B_1 = \{b, e\}$ and $B_2 = \{c, d\}$ be a subset of X, then $Int_\beta(B_1) = \emptyset$ and $Int_\beta(B_2) = \{c, d\}$. Hence $Int_\beta(B_1 \cup B_2) = \{b, c, d, e\} \nsubseteq \{c, d\} = Int_\beta(B_1) \cup Int_\beta(B_2)$.

2. Let $X = \{a, b, c, d\}$ be a set with topology $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$. Then $\tau^\beta = \tau \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, c, d\}, \{b, c, d\}\}$ as in Example 3.3. Consider a subset $B_1 = \{b, c\}$ and $B_2 = \{a, c, d\}$ of X. Then $Int_\beta(B_1) \cap Int_\beta(B_2) = \{c\}$ while $Int_\beta(B_1 \cap B_2) = \emptyset$ which proves that $Int_\beta(B_1) \cap Int_\beta(B_2) \nsubseteq Int_\beta(B_1 \cap B_2)$.

Definition 3.4 ([5]). *For any subset A of X, the set*

$$b_p(A) = A \setminus Int_p(A)$$

is called the pre-border of A, and the set

$$Fr_p(A) = Cl_p(A) \setminus Int_p(A)$$

is called the pre-frontier of A.

Definition 3.5. *For any subset B of X, the set,*

$$b_\beta(B) = B \setminus Int_\beta(B)$$

is called the β-border of B, and the set

$$Fr_\beta(B) = Cl_\beta(B) \setminus Int_\beta(B)$$

is called the β-frontier of B.

Theorem 3.7 (Properties of β-Boundary). *For any subset B of X, the following statements hold:*

(1) $b_\beta(B) \subseteq b_p(B)$.
(2) $B = Int_\beta(B) \cup b_\beta(B)$ *and* $Int_\beta(B) \cap b_\beta(B) \neq \emptyset$.
(3) B *is β-open set* $\Leftrightarrow b_\beta(B) = \emptyset$.
(4) $b_\beta(Int_\beta(B)) = \emptyset$.
(5) $Int_\beta(b_\beta(B)) = \emptyset$.
(6) $b_\beta(b_\beta(B)) = b_\beta(B)$.
(7) $b_\beta(B) = B \cap Cl_\beta(X \setminus B)$.
(8) $b_\beta(B) = B \cap D_\beta(X \setminus B)$.

Proof. (1) Since $Int_p(B) \subseteq Int_\beta(B)$, we have $b_\beta(B) = B \setminus Int_\beta(B) \subseteq B \setminus Int_p(B)$, which implies $b_\beta(B) \subseteq b_p(B)$.

Converse of above is not true which is explained in Example 3.10.

(2) Straightforward.

(3) Since $Int_\beta(B) \subseteq B$ and B is β-open $\Leftrightarrow B = Int_\beta(B) \Leftrightarrow b_\beta(B) = B \setminus Int_\beta(B) \Leftrightarrow b_\beta(B) = \emptyset$.

(4) Since $Int_\beta(B)$ is β-open implies directly from (3) that $b_\beta(Int_\beta(B)) = \emptyset$.

(5) Let $b \in Int_\beta(b_\beta(B))$, then $b \in b_\beta(B) \subseteq B$ and so $b \in Int_\beta(B)$ since $Int_\beta(b_\beta(B)) \subseteq Int_\beta(B)$. Thus, $b \in Int_\beta(B) \cap b_\beta(B)$, which is a contradiction as per (2) of Theorem 3.7. Hence, $Int_\beta(b_\beta(B)) = \emptyset$.

(6) Since $b_\beta(b_\beta(B)) = b_\beta(B) \setminus Int_\beta(b_\beta(B)) = b_\beta(B)$, using part (5) Theorem 3.7. Hence, the proof.

(7) Since $b_\beta(B) = B \setminus Int_\beta(B) = B \setminus (X \setminus Cl_\beta(X \setminus B)) = B \cap (X \setminus Cl_\beta(X \setminus B))^c = B \cap Cl_\beta(X \setminus B)$, using part(6) of Theorem 3.6.

(8) By using Theorem 3.2 and above part,

$$\begin{aligned} b_\beta(B) &= B \cap Cl_\beta(X \setminus B) \\ &= B \cap ((X \setminus B) \cup D_\beta(X \setminus B)) \\ &= (B \cap X \setminus B) \cup (B \cap D_\beta(X \setminus B)) \\ &= \emptyset \cup (B \cap D_\beta(X \setminus B)) \\ &= B \cap D_\beta(X \setminus B). \end{aligned}$$

Hence, the proof. □

Example 3.10. Let $X = \{a, b, c, d, e\}$ be a set with topology $\tau = \{X, \emptyset, \{b\}, \{d, e\}, \{b, d, e\}\}$. Then $\tau^p = \tau \cup \{\{d\}, \{e\}, \{b, d\}, \{b, e\}, \{a, b, d\}, \{a, b, e\}, \{b, c, d\}, \{b, c, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$ and $\tau^\beta = \tau^p \cup \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, c\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, c, e\}, \{c, d, e\}, \{a, c, d, e\}$. Consider a subset $B = \{a, c, d\}$. Then $b_p(B) = \{a, c\}$ and $b_\beta(B) = \emptyset$ which implies that the converse of Theorem 3.7(1) is not true in general.

Lemma 3.2. *Let B be a subset of topological space X, then B is β-closed if and only if $Fr_\beta(B) \subseteq B$.*

Proof. Let B be β-closed. Then, $Fr_\beta(B) = Cl_\beta(B) \setminus Int_\beta(B) = B \setminus Int_\beta(B) \subseteq B$. Conversely, suppose $Fr_\beta(B) \subseteq B$. Then, $Cl_\beta(B) \setminus Int_\beta(B) \subseteq B$ and so $Cl_\beta(B) \subseteq B$. Hence, $B = Cl_\beta(B)$ and so B is β-closed, which completes the proof. □

Theorem 3.8 (Properties of β-Frontier)**.** *Let B be a subset of X, then the following assertions hold:*

(1) $Fr_\beta(B) \subseteq Fr_p(B)$.

(2) $Cl_\beta(B) = Int_\beta(B) \cup Fr_\beta(B)$ *and* $Int_\beta(B) \cup Fr_\beta(B) = \emptyset$.

(3) $b_\beta(B) \subseteq Fr_\beta(B)$.

(4) $Fr_\beta(B) = b_\beta(B) \cup (D_\beta(B) \setminus Int_\beta(B))$.

(5) *B is β-open* $\Leftrightarrow Fr_\beta(B) = b_\beta(X \setminus B)$.
(6) $Fr_\beta(B) = Cl_\beta(B) \cap Cl_\beta(X \setminus B)$.
(7) $Fr_\beta(B) = Fr_\beta(X \setminus B)$.
(8) *$Fr_\beta(B)$ is β-closed.*
(9) $Int_\beta(B) = B \setminus Fr_\beta(B)$.
(10) $Fr_\beta(Fr_\beta(B)) \subseteq Fr_\beta(B)$.
(11) $Fr_\beta(Int_\beta(B)) \subseteq Fr_\beta(B)$.
(12) $Fr_\beta(Cl_\beta(B)) \subseteq Fr_\beta(B)$.

Proof. (1) Since $Fr_\beta(B) = Cl_\beta(B) \setminus Int_\beta(B) \subseteq Cl_p(B) \setminus Int_\beta(B) \subseteq Cl_p(B) \setminus Int_p(B) = Fr_p(B)$.

(2) The first part is direct. For the next, we have $Int_\beta(B) \cup F_\beta(B) = Int_\beta(B) \cup (Cl_\beta(B) \setminus Int_\beta(B)) = \emptyset$ (Obviously).

(3) Since $B \subseteq Cl_\beta(B)$ and $b_\beta(B) = B \setminus Int_\beta(B) \subseteq Cl_\beta(B) \setminus Int_\beta(B) = Fr_\beta(B)$.

(4) By using the definition of β-boundary of B and Theorem 3.2, we have

$$\begin{aligned} Fr_\beta(B) &= Cl_\beta(B) \setminus Int_\beta(B) \\ &= (B \cup D_\beta(B)) \setminus Int_\beta(B) \\ &= (B \cup D_\beta(B)) \cap (X \setminus Int_\beta(B)) \\ &= (B \cap (X \setminus Int_\beta(B)) \cup (D_\beta(B) \cap (X \setminus Int_\beta(B)) \\ &= (B \setminus Int_\beta(B)) \cup (D_\beta(B) \setminus Int_\beta(B)) \\ &= b_\beta(B) \cup (D_\beta(B) \setminus Int_\beta(B)), \end{aligned}$$

which completes the proof.

(5) Suppose B is β-open. Then,

$$\begin{aligned} Fr_\beta(B) &= b_\beta(B) \cup (D_\beta(B) \setminus Int_\beta(B)) \\ &= \emptyset \cup (D_\beta(B) \setminus B) \\ &= D_\beta(B) \setminus B \\ &= D_\beta(B) \cap (X \setminus B) \\ &= b_\beta(X \setminus B), \end{aligned}$$

using part (3) and (8) of Theorem 3.7.

Conversely, suppose $Fr_\beta(B) = b_\beta(X \setminus B)$. Then

$$\begin{aligned} \emptyset &= Fr_\beta(B) \setminus b_\beta(X \setminus B) \\ &= (Cl_\beta(B) \setminus Int_\beta(B)) \setminus (X \setminus B \setminus Int_\beta(B)) \\ &= B \setminus Int_\beta(B), \end{aligned}$$

which implies $B \subseteq Int_\beta(B)$. In general, $Int_\beta(B) \subseteq B$. Hence, $Int_\beta(B) = B$.

(6) Using the part (5) of Theorem 3.6, we have

$$\begin{aligned} Cl_\beta(B) \cap Cl_\beta(X \setminus B) &= Cl_\beta(B) \cap (X \setminus Int_\beta(B)) \\ &= Cl_\beta(B) \cap (Int_\beta(B))^c \\ &= Cl_\beta(B) \setminus Int_\beta(B) \\ &= Fr_\beta(B), \end{aligned}$$

which complete the proof.

(7)Same as (6).

(8) We need to show that $Cl_\beta(Fr_\beta(B)) = Fr_\beta(B)$. Clearly, $Fr_\beta(B) \subseteq Cl_\beta(Fr_\beta(B))$. Next, we shall show that $Cl_\beta(Fr_\beta(B) \subseteq Fr_\beta(B)$. We have,

$$\begin{aligned} Cl_\beta(Fr_\beta(B)) &= Cl_\beta(Cl_\beta(B) \cap Cl_\beta(X \setminus B)) \\ &\subseteq Cl_\beta(Cl_\beta(B)) \cap Cl_\beta(Cl_\beta(X \setminus B)) \\ &= Cl_\beta(B) \cap Cl_\beta(X \setminus B) \\ &= Fr_\beta(B), \end{aligned}$$

which implies $Fr_\beta(B)$ is closed set.

(9) Using the definition of β-frontier of B and basic property of set theory, we have

$$\begin{aligned} B \setminus Fr_\beta(B) &= B \setminus (Cl_\beta(B) \setminus Int_\beta(B)) \\ &= (B \setminus Cl_\beta(B)) \cup (B \cap Cl_\beta(B) \cap Int_\beta(B)) \\ &= (B \setminus Cl_\beta(B)) \cup Int_\beta(B) \\ &= \emptyset \cup Int_\beta(B) \\ &= Int_\beta(B). \end{aligned}$$

This completes the proof.

(10) Since $Fr_\beta(B)$ is β-closed and so by Lemma 3.2, $Fr_\beta(Fr_\beta(B)) \subseteq Fr_\beta(B)$.

(11) We have,

$$\begin{aligned} Fr_\beta(Int_\beta(B)) &= Cl_\beta(Int_\beta(B)) \setminus Int_\beta(Int_\beta(B)) \\ &\subseteq Cl_\beta(B) \setminus Int_\beta(B) \\ &= Fr_\beta(B). \end{aligned}$$

(12)We have,

$$\begin{aligned} Fr_\beta(Cl_\beta(B)) &= Cl_\beta(Cl_\beta(B)) \setminus Int_\beta(Cl_\beta(B)) \\ &\subseteq Cl_\beta(B) \setminus Int_\beta(B) \\ &= Fr_\beta(B). \end{aligned}$$

Hence, the proof. □

Example 3.11. Let $X = \{a, b, c, d\}$ be a set with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$. Then $\tau^p = \tau$ and $\tau^\beta = \tau^p \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, c, d\} \{a, c, d\}\}$.

Consider a subset $A = \{c, d\}$ and $B = \{a, c\}$ of X, then $Fr_\beta(A) = \{c, d\} = Fr_p(A)$. Also, $Fr_\beta(B) = \emptyset$ while $Fr_p(B) = \{c, d\}$ which implies equality in Theorem 3.8(1) may not hold.

Example 3.12. Consider $X = \{a, b, c, d\}$ with same topology τ and τ^β as in Example 3.2. Let $B = \{a, b, c\}$, then $b_\beta(B) = \emptyset$ while $Fr_\beta(B) = \{d\}$, which shows that the converse of Theorem 3.8(3) is not true in general.

Definition 3.6. *Let B be a subset of X, $Ext_\beta(B) = Int_\beta(X \setminus B)$ is said to be β-exterior of B.*

We denote $Ext_p(B)$ to be pre-exterior [5] of B.

Example 3.13. Let $X = \{a, b, c, d, e\}$ with

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

Then, $\tau^\beta = \{X, \emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, c, d\}, \{a, d, e\}, \{a, b, d\}, \{a, c, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{b, c, d, e\}\}$. Consider a subset $A = \{b, c, d\}$ and $B = \{a, c, d, e\}$ of set X , then $Ext_\beta(A) = Int_\beta(X \setminus A) = \{a\}$and $Ext_\beta(B) = Int_\beta(X \setminus B) = \emptyset$.

Theorem 3.9. *For a subset B, B_1, B_2 of X, the following assertion are valid.*

(1) $Ext_p(B) \subseteq Ext_\beta(B)$.
(2) $Ext_\beta(B)$ *is a* β*-open.*
(3) $Ext_\beta(B) = X \setminus Cl_\beta(B)$.
(4) $Ext_\beta(Ext_\beta(B)) = Int_\beta(Cl_\beta(B)) \supseteq Int_\beta(B)$.
(5) *If* $B_1 \subseteq B_2$, *then* $Ext_\beta(B_1) \subseteq Ext_\beta(B_2)$.
(6) $Ext_\beta(B_1 \cup B_2) \subseteq Ext_\beta(B_1) \cap Ext_\beta(B_2)$.
(7) $Ext_\beta(B_1) \cup Ext_\beta(B_2) \subseteq Ext_\beta(B_1 \cap B_2)$.
(8) $Ext_\beta(X) = \emptyset, Ext_\beta(\emptyset) = X$.
(9) $Ext_\beta(B) = Ext_\beta(X \setminus Ext_\beta(B))$.
(10) $B = Int_\beta(B) \cup Ext_\beta(B) \cup Fr_\beta(B)$.

Proof. (1) Clearly by Theorem 3.5, $Int_p(B) \subseteq Int_\beta(B)$, we have $Ext_p(B) = Int_\beta(X \setminus B) \subseteq Int_\beta(X \setminus B) = Ext_\beta(B)$.

(2) Straightforward.

(3) By part(6) of Theorem 3.6, $X \setminus Cl_\beta(B) = Int_\beta(X \setminus B) = Ext_\beta(X \setminus B)$.

(4) By Theorem 3.5 and part (5) of Theorem 3.6,

$$\begin{aligned} Ext_\beta(Ext_\beta(B)) &= Ext_\beta(Int_\beta(X \setminus B)) \\ &= Int_\beta(X \setminus Int_\beta(X \setminus B)) \\ &= Int_\beta(Cl_\beta(X \setminus (X \setminus B))) \\ &= Int_\beta(Cl_\beta(B)) \supseteq Int_\beta(B). \end{aligned}$$

(5) Let $B_1 \subseteq B_2$. Then, $Ext_\beta(B_2) = Int_\beta(X \setminus B_2) \subseteq Int_\beta(X \setminus B_1) = Ext_\beta(B_1)$.

(6) By using part (9) of Theorem 3.6, we have

$$\begin{aligned} Ext_\beta(B_1 \cup B_2) &= Int_\beta(X \setminus (B_1 \cup B_2)) \\ &= Int_\beta((X \setminus B_1) \cap (X \setminus B_2)) \\ &\subseteq Int_\beta(X \setminus B_1) \cap Int_\beta(X \setminus B_2) \\ &= Ext_\beta(B_1) \cap Ext_\beta(B_2), \end{aligned}$$

which completes the proof.

(7) By using part (8) of Theorem 3.6, we have

$$\begin{aligned} Ext_\beta(B_1) \cup Ext_\beta(B_2) &= Int_\beta(X \setminus B_1) \cup Int_\beta(X \setminus B_2) \\ &\subseteq Int_\beta((X \setminus B_1) \cup (X \setminus B_2)) \\ &= Int_\beta(X \setminus (B_1 \cap B_2)) \\ &= Ext_\beta(B_1 \cap B_2), \end{aligned}$$

hence the proof.

(8) Straightforward.

(9) By using the definition of β-exterior of B, we have

$$\begin{aligned} Ext_\beta(X \setminus Ext_\beta(B)) &= Ext_\beta(X \setminus Int_\beta(X \setminus B)) \\ &= Int_\beta(Int_\beta(X \setminus B)) \\ &= Int_\beta(X \setminus B) \\ &= Ext_\beta(B). \end{aligned}$$

Hence, the proof.

(10) Trivial. □

Example 3.14. Let (X, τ) be a topological space same as given in Example 3.13. Consider $B_1 = \{b, c, d\}$ and $B_2 = \{b, c, e\}$, then $Ext_\beta(B_1) = Int_\beta(X \setminus B_1) = \{a\}$ and $Ext_\beta(B_2) = Int_\beta(X \setminus B_2) = \{a, d\}$, which implies $Ext_\beta(B_1) \subseteq Ext_\beta(B_2)$ but $B_1 \nsubseteq B_2$. This shows that the converse of Theorem 3.9(5) is not true.

Example 3.15. Let (X, τ) be a topological space same as given in Example 3.13. Let $B_1 = \{d, e\}$ and $B_2 = \{c\}$. Then, $Ext_\beta(B_1 \cup B_2) = \{a\} \neq \{a, b\} = \{a, b, c\} \cap \{a, b, d, e\} = Ext_\beta(B_1) \cap Ext_\beta(B_2)$, which implies that the equality in the Theorem 3.9(6) is not true.

Example 3.16. Let (X, τ) be a topological space same as given in Example 3.13. Let $B_1 = \{a, c, d\}$ and $B_2 = \{b, e\}$. Then, $Int_\beta(X \setminus B_1) = \emptyset$ and $Int_\beta(X \setminus B_2) = \{a, c, d\}$. Hence, $Ext_\beta(B_1) \cup Ext_\beta(B_2) = \emptyset \cup \{a, c, d\} = \{a, c, d\} \subseteq Ext_\beta(B_1 \cap B_2) = X$ which shows that the equality in Theorem 3.9(7) is not valid.

4. Conclusion

This paper begins with a brief survey of the notion of β-open sets and β-continuity introduced by Abd El-Monsef et al. [1]. We also recall some other generalized open sets in topological spaces, like semi-open sets [6], pre-open sets [8] and α-open sets [9] so as to compare these sets to β-open sets.

The authors studied β-limit points and β-derived sets in topological spaces and proved many results on β-derived sets. Some characteristics of β-interiors and β-closures of sets are also investigated.

Moreover, β-exterior, β-frontier and β-boundary of sets are also studied. Several examples are given to indicate the connections among these concepts. Some properties of these concepts are also discussed which will open the way for more applications of β-open sets in real-life problems. Also, all these properties of β-open sets in topological spaces can be very handy for studying compactness, connectedness, separation axioms via β-open sets.

References

[1] M. E. Abd El-Monsef, S. N. El-deeb, R. A. Mahmoud, *$\beta-$open sets and $\beta-$continuous mappings*, Bull. Fac. Sci. Assiut Univ., 12 (1983), 77-90.

[2] M. E. Abd El-Monsef, A. N. Geaisa, R. A. Mahmoud, *β-regular spaces*, Proc. Math. Phys. Soc. Egypt, 60 (1985), 47-52.

[3] Y. A. Abou-Elwan, *Some properties of β-continuous mappings, β-open mappings and β-homeomorphism,* Middle-East J. of Sci. Res., 19 (2014), 1722-1728.

[4] M. Caldas, *A note on some application of α-sets*, Int. J. Math. & Math. Sci, 2 (2003), 125-130.

[5] Y. B. Jun, S. W. Jeong, H. J. Lee, J. W. Lee, *Application of pre-open sets*, Applied General Topology, 9 (2008), 213-228.

[6] N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly, 70 (1963), 36-41.

[7] R. A. Mahmoud and M. E. Abd El-Monsef, *β-irresolute and β-topological invariant,* Proc. Pakistan Acad. Sci., 27 (1990), 285-296

[8] A.S. Mashour, M.E. Abd El-Monsef, S.N. El-Deeb, *On pre-continuous and weak pre-continuous mappings*, Proc. Math. Phys. Soc. Egypt., 53 (1982), 47-53.

[9] O. Najastad, *On some classes of nearly open sets*, Pacific J. Math., 15 (1965), 961-970.

[10] T. M. Nour, *A note on some application of semi-open sets,* Internat. J. Math. & Math. Sci., 21 (1998), 205-207.

[11] I. L. Reilly, M. K. Vamanamurthy, *On α-sets in topological spaces,* Tamkang J. Math., 16 (1985), 7-11.

[12] S. Sharma and M. Ram, *On β-topological vector spaces*, J. Linear. Topol. Algeb., 08 (2019), 63-70.

Accepted: May 16, 2022

A unified generalization of some refinements of Jensen's inequality

Shou-Hua Shen
Department of Mathematics
Longyan University
Longyan Fujian 364012
People's Republic of China
shouhuashen@163.com

Shan-He Wu*
Department of Mathematics
Longyan University
Longyan Fujian 364012
People's Republic of China
shanhewu@gmail.com

Abstract. In this paper, we establish a unified generalization of three refinements of Jensen's inequality by introducing several parameters. As applications, we illustrate that the improved Jensen's inequality can generate some new inequalities for special means such as arithmetic mean, geometric mean and logarithmic mean.

Keywords: Jensen's inequality, convex function, generalization, refinement, special means.

1. Introduction and main result

Let f be a convex function on $[a, b] \subset \mathbb{R}$. The classical Jensen's inequality reads as

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The Jensen's inequality, which was first proposed by Jensen in 1905, is one of the most important inequalities in pure and applied mathematics (see [1, 2]). For over 100 years, this celebrated inequality has generated lots of extensions and applications, see [3, 4, 5, 6, 7, 8, 9] and references cited therein. Besides these, there are some papers dealing with refinements of Jensen's inequality, the most famous of which is the Hermite-Hadamard inequality below:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

∗. Corresponding author

In [10], Wu provided two refinements of Jensen's inequality, as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}, \tag{3}$$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2}{(b-a)}\int_a^b f(x)dx \\ &- \frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{4}$$

where f is a convex function and F is a differentiable function such that $F''(x) = f(x)$ on $[a,b]$.

Inspired by inequalities (2), (3) and (4) above, a natural and interesting problem is whether we can establish a unified generalization of these refined Jensen's inequalities. In this paper we address this issue. Specifically, we shall construct a new inequality by introducing several parameters. Moreover, we will apply the inequality obtained to establish some inequalities for special means involving arithmetic mean, geometric mean, logarithmic mean and generalized logarithmic mean.

Our main result is stated in the following theorem.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, and let F be a differentiable function such that $F''(x) = f(x)$ on $[a,b]$. Then, for $\mu \geq \max\{0,\lambda\}$ and $\lambda \in \mathbb{R}$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2\mu-2\lambda}{(2\mu-\lambda)(b-a)}\int_a^b f(x)dx \\ &+ \frac{8\lambda}{(2\mu-\lambda)(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{5}$$

Remark 1.1. As a direct consequence of Theorem 1.1, if we put $\lambda = 0,\ \mu = 1$ in (5), we acquire the Hermite-Hadamard inequality; if we take $\lambda = 1,\ \mu = 1$ in (5), we obtain inequality (3); if we choose $\lambda = -1,\ \mu = 0$ in (5), we get inequality (4).

2. Proof of Theorem 1.1

Let us first transcribe a lemma that we will need in the proof of Theorem 1.1.

Lemma 2.1 ([10])**.** *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, let g be a nonnegative, integrable function on $[a,b]$ and let*

$$\eta = \Big(\int_a^b (b-x)g(a+b-x)dx\Big) \Big/ \Big(\int_a^b (b-x)g(x)dx\Big).$$

Then

$$f\left(\frac{a+\eta b}{1+\eta}\right) \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq \frac{f(a)+\eta f(b)}{1+\eta}. \tag{6}$$

Proof of Theorem 1.1. Choosing a function $g:\ [a,b]\to\mathbb{R}$ defined by

$$\begin{aligned} g(x) &:= (\mu-\lambda)\left|2x-a-b\right|+\mu(b-a-\left|2x-a-b\right|) \\ &= \begin{cases} (\mu-\lambda)(a+b-2x)+2\mu(x-a), & a\le x\le \dfrac{a+b}{2} \\ (\mu-\lambda)(2x-a-b)+2\mu(b-x), & \dfrac{a+b}{2}<x\le b. \end{cases} \end{aligned}$$

In view of the assumption $\mu\ge\max\{0,\lambda\}$, it is easy to verify that $g(x)$ is nonnegative and integrable on $[a,b]$. Then, one has

$$\begin{aligned} \eta &= \frac{\int_a^b (b-x)g(a+b-x)dx}{\int_a^b (b-x)g(x)dx} \\ &= \frac{\int_a^b (b-x)[(\mu-\lambda)\left|a+b-2x\right|+\mu(b-a-\left|a+b-2x\right|)]dx}{\int_a^b (b-x)[(\mu-\lambda)\left|2x-a-b\right|+\mu(b-a-\left|2x-a-b\right|)]dx} \\ &= 1 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} &= \frac{\int_a^b f(x)[(\mu-\lambda)\left|2x-a-b\right|+\mu(b-a-\left|2x-a-b\right|)]dx}{\int_a^b [(\mu-\lambda)\left|2x-a-b\right|+\mu(b-a-\left|2x-a-b\right|)]dx} \\ &= \frac{\int_a^b f(x)[\mu(b-a)-\lambda\left|2x-a-b\right|]dx}{\int_a^b [\mu(b-a)-\lambda\left|2x-a-b\right|]dx} \\ &= \frac{\mu(b-a)\int_a^b f(x)dx-\lambda\int_a^b f(x)\left|2x-a-b\right|dx}{\mu(b-a)^2-\lambda\int_a^b \left|2x-a-b\right|dx}. \end{aligned} \tag{8}$$

Note that

$$\begin{aligned} \int_a^b \left|2x-a-b\right|dx &= \int_a^{\frac{a+b}{2}}(a+b-2x)dx+\int_{\frac{a+b}{2}}^b (2x-a-b)dx \\ &= \frac{(b-a)^2}{2} \end{aligned} \tag{9}$$

and

$$\begin{aligned} \int_a^b f(x)\left|2x-a-b\right|dx &= \int_a^{\frac{a+b}{2}} f(x)(a+b-2x)dx+\int_{\frac{a+b}{2}}^b f(x)(2x-a-b)dx \\ &= \int_a^{\frac{a+b}{2}}(a+b-2x)dF'(x)+\int_{\frac{a+b}{2}}^b (2x-a-b)dF'(x) \end{aligned}$$

$$\begin{aligned}
&= -(b-a)F'(a) + 2\int_a^{\frac{a+b}{2}} F'(x)dx + (b-a)F'(b) - 2\int_{\frac{a+b}{2}}^b F'(x)dx \\
&= (b-a)[F'(b)-F'(a)] + 2\int_a^{\frac{a+b}{2}} dF(x) - 2\int_{\frac{a+b}{2}}^b dF(x) \\
&= (b-a)[F'(b)-F'(a)] + 2F(\frac{a+b}{2}) - 2F(a) - 2F(b) + 2F(\frac{a+b}{2}) \\
&= (b-a)\int_a^b f(x)dx + 4F(\frac{a+b}{2}) - 2[F(a)+F(b)]. \qquad (10)
\end{aligned}$$

Applying equalities (9) and (10) to (8), we obtain

$$\begin{aligned}
\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} &= \frac{2\mu-2\lambda}{(2\mu-\lambda)(b-a)}\int_a^b f(x)dx \\
&+ \frac{8\lambda}{(2\mu-\lambda)(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right]. \qquad (11)
\end{aligned}$$

Combining (6), (7) and (11) leads to the desired inequality (5). The proof of Theorem 1.1 is complete. □

3. Some applications

A growing number of inequalities for special means have been found significant applications in theory and practice (see [11, 12, 13, 14, 15, 16, 17, 18, 19]). To demonstrate usefulness of Theorem 1.1, in this section, we derive some inequalities for special means via the inequalities of Theorem 1.1.

Let us recall the arithmetic mean, geometric mean, logarithmic mean and generalized logarithmic mean for positive numbers α and β which are defined as follows:

$$\begin{aligned}
A(\alpha,\beta) &= \frac{\alpha+\beta}{2}, && \text{arithmetic mean,} \\
G(\alpha,\beta) &= \sqrt{\alpha\beta} && \text{geometric mean,} \\
L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln\beta-\ln\alpha}, && \text{logarithmic mean,} \\
L_p(\alpha,\beta) &= \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, \ p\neq -1,\ 0, && \text{generalized logarithmic mean.}
\end{aligned}$$

We have the following results:

Theorem 3.1. *Let a, b be positive real numbers, $\mu \geq \max\{0, \lambda\}$ and $\lambda \in \mathbb{R}$. Then, for $p \geq 1$ or $p < 0$ $(p \neq -1, -2)$, the following inequalities hold*

$$\begin{aligned} (A(a,b))^p &- \left(\frac{2\mu - 2\lambda}{2\mu - \lambda}\right)(L_p(a,b))^p \\ &\leq \frac{8\lambda}{(2\mu - \lambda)(p+1)(p+2)(b-a)^2}\left[A(a^{p+2}, b^{p+2}) - (A(a,b))^{p+2}\right] \\ &\leq A(a^p, b^p) - \left(\frac{2\mu - 2\lambda}{2\mu - \lambda}\right)(L_p(a,b))^p. \end{aligned} \tag{12}$$

Furthermore, inequality (12) *is reversed for* $0 < p < 1$.

Proof of Theorem 3.1. It is clear that inequality (12) is symmetric with respect to variable a and b. Without loss of generality we assume that $b > a > 0$. Note that the function $f(x) = x^p$ is convex on $(0, +\infty)$ for $p \geq 1$ or $p < 0$, and the function $f(x) = -x^p$ is convex on $(0, +\infty)$ for $0 < p < 1$. We obtain immediately inequality (12) and its reverse version by applying these functions to Theorem 1.1. This completes the proof of Theorem 3.1. □

As a consequence of Theorem 3.1, taking $\lambda = 0,\ \mu = 1$; $\lambda = 1,\ \mu = 1$ and $\lambda = -1,\ \mu = 0$ respectively in (12), we obtain the following inequalities.

Corollary 3.1. *If a, b are positive real numbers, $p \geq 1$ or $p < 0$ $(p \neq -1, -2)$, then we have*

$$(A(a,b))^p \leq (L_p(a,b))^p \leq A(a^p, b^p), \tag{13}$$

$$\begin{aligned} (A(a,b))^p &\leq \frac{8}{(p+1)(p+2)(b-a)^2}\left[A(a^{p+2}, b^{p+2}) - (A(a,b))^{p+2}\right] \\ &\leq A(a^p, b^p), \end{aligned} \tag{14}$$

$$\begin{aligned} (A(a,b))^p - 2(L_p(a,b))^p &\leq \frac{8}{(p+1)(p+2)(b-a)^2}\left[(A(a,b))^{p+2} - A(a^{p+2}, b^{p+2})\right] \\ &\leq A(a^p, b^p) - 2(L_p(a,b))^p. \end{aligned} \tag{15}$$

All of the above inequalities are reversed for $0 < p < 1$.

Theorem 3.2. *Let a, b be positive real numbers, $\mu \geq \max\{0, \lambda\}$ and $\lambda \in \mathbb{R}$. Then the following inequalities hold*

$$\begin{aligned} G(a,b) - \left(\frac{2\mu - 2\lambda}{2\mu - \lambda}\right)L(a,b) &\leq \left(\frac{8\lambda}{2\mu - \lambda}\right)\left(\frac{L(a,b)}{b-a}\right)^2[A(a,b) - G(a,b)] \\ &\leq A(a,b) - \left(\frac{2\mu - 2\lambda}{2\mu - \lambda}\right)L(a,b). \end{aligned} \tag{16}$$

Proof of Theorem 3.2. Without loss of generality we assume that $b > a > 0$. Note that $f(x) = e^x$ is convex on $[\ln a, \ln b]$. Using Theorem 1.1 with $f(x) = e^x, x \in [\ln a, \ln b]$, we acquire inequality (16) described in Theorem 3.2. □

As a consequence of Theorem 3.2, putting $\lambda = 0,\ \mu = 1$; $\lambda = 1,\ \mu = 1$ and $\lambda = -1,\ \mu = 0$ respectively in (16), we get the following inequalities.

Corollary 3.2. *If a, b are positive real numbers, then we have*

$$G(a,b) \le L(a,b) \le A(a,b), \tag{17}$$

$$G(a,b) \le 8\Big(\frac{L(a,b)}{b-a}\Big)^2 [A(a,b) - G(a,b)] \le A(a,b), \tag{18}$$

$$G(a,b) - 2L(a,b) \le 8\Big(\frac{L(a,b)}{b-a}\Big)^2 [G(a,b) - A(a,b)] \le A(a,b) - 2L(a,b). \tag{19}$$

Acknowledgements

This work was supported by the Natural Science Foundation of Fujian Province of China under Grant No. 2020J01365.

References

[1] J. L. W. V. Jensen, *Om konvexe funktioner og uligheder mellem middelvaerdier*, Nyt. Tidsskr. Math., B., 16 (1905), 49-69.

[2] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings and statistical applications*, Academic Press, New York, 1992.

[3] A. W. Roberts, D. E. Varberg, *Convex functions,* Academic Press, New York, 1973.

[4] A. M. Marshall, I. Olkin, *Inequalities: the theory of majorization and its application,* Academic Press, New York, 1979.

[5] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new inequalities in analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[6] C.-Y. Luo, T.-S. Du, C. Zhou, T.-G. Qin, *Generalizations of Simpson-type inequalities for relative semi-(h, α, m)-logarithmically convex mappings*, Italian J. Pure Appl. Math., 45 (2021), 498-520.

[7] T.-S. Du, Y.-J. Li, Z.-Q. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions*, Appl. Math. Comput., 293 (2017), 358-369.

[8] Z.-Q. Yang, Y.-J. Li, T.-S. Du, *A generalization of Simpson type inequality via differentiable functions using (s, m)-convex functions*, Italian J. Pure Appl. Math., 35 (2015), 327-338.

[9] S.-H. Wu, L. Debnath, *Inequalities for convex sequences and their applications*, Comput. Math. Appl., 54 (2007), 525-534.

[10] S.-H. Wu, *On the weighted generalization of the Hermite-Hadamard inequality and its applications*, Rocky Mountain J. Math., 39 (2009), 1741-1749.

[11] P. S. Bullen, D. S. Mitrinvić, P. M. Vasić, *Means and their inequalities*, Kluwer Academic Publishers, Dordrecht, 1988.

[12] P. S. Bullen, *Handbook of means and their inequalities*, Kluwer Academic Publishers, Dordrecht, 2003.

[13] K. B. Stolarsky, *Generalizations of the Logarithmic mean*, Math. Mag., 48 (1975), 87-92.

[14] K. B. Stolarsky, *The power and generalized logarithmic means*, Amer. Math. Monthly, 87 (1980), 545-548,

[15] B. C. Carlson, *The logarithmic mean*, Amer. Math. Monthly, 79 (1972), 615-618.

[16] S.-H. Wu, L. Debnath, *Inequalities for differences of power means in two variables*, Anal. Math., 37 (2011), 151-159.

[17] S.-H. Wu, *Generalization and sharpness of power means inequality and their applications*, J. Math. Anal. Appl., 312 (2005), 637-652.

[18] D.-S. Wang, C.-R. Fu, H.-N Shi, *Schur-m power convexity of Cauchy means and its application*, Rocky Mountain J. Math., 50 (2020), 1859-1869.

[19] H.-N. Shi, *Schur convex functions and inequalities*, Harbin Institute of Technology Press, Harbin, 2017 (Chinese).

Accepted: April 7, 2022

Improvements of Hölder's inequality via Schur convexity of functions

Huan-Nan Shi
Institute of Applied Mathematics
Longyan University
Longyan Fujian 364012
People's Republic of China
and
Department of Electronic Information
Beijing Union University
Beijing 100011
People's Republic of China
sfthuannan@buu.edu.cn

Shan-He Wu*
Department of Mathematics
Longyan University
Longyan Fujian 364012
People's Republic of China
shanhewu@gmail.com

Dong-Sheng Wang
Basic Courses Department
Beijing Polytechnic
Beijing 100176
People's Republic of China
wds000651225@sina.com

Bing Liu
Basic Courses Department
Beijing Commercial School
Beijing 102209
People's Republic of China
liubing6612@sohu.com

Abstract. In this paper, we study the Schur convexity of some functions associated with Hölder's inequality, the results obtained are then used to establish the refined versions of Hölder's inequality under certain specified conditions. At the end of the paper, applications to inequalities for special means are given.

Keywords: Hölder inequality, Schur convexity, majorization, special means.

∗. Corresponding author

1. Introduction and main results

The discrete Hölder inequality states that if $a_k \geq 0$, $b_k \geq 0, k = 1, 2, \ldots, n$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leq \Big(\sum_{k=1}^{n} a_k^p \Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{n} b_k^q \Big)^{\frac{1}{q}}. \tag{1}$$

Correspondingly, the integral version of Hölder's inequality can be formulated as

$$\int_a^b f(x)g(x)dx \leq \Big(\int_a^b f^p(x)dx \Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx \Big)^{\frac{1}{q}}, \tag{2}$$

where $f(x)$ and $g(x)$ are nonnegative integrable on $[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Hölder's inequality is one of the most important foundational inequality in analysis, it also plays a key role in dealing with various problems of pure and applied mathematics, see [1] for background information on Hölder's inequality. In the past more than 100 years, this classical inequality has been paid considerable attention, there have been a large number of literature focusing on its improvements, extensions and applications. For example, some refinements and generalizations of Hölder's inequality were established by Yang in the references [2] and [3], respectively. A sharpened version was given by Hu [4]. A complementary version of sharpening Hölder's inequality related to the work [4] was provided by Wu [5]. A generalization of the result of Hu [4] was obtained by Wu [6]. A further generalization and refinement of Hölder's inequality was proposed by Qiang and Hu in [7]. For more results regarding different improvements of Hölder's inequality can be found in monograph [8] and references therein.

In recent years, application of Schur convexity and majorization properties to establish and improve various inequalities has been a hot research topic. For details about the applications of Schur convexity of functions, we refer the reader to the references [9-13].

In this paper, we provide a novel method to study the improvements and variants of Hölder's inequality. More specifically, we will construct some functions associated with Hölder's inequality, and then we use Schur convexity of these functions to derive the refined versions of Hölder's inequality under certain specified conditions.

We denote the n-dimensional real vector by $\boldsymbol{V} = (v_1, v_2, \ldots, v_n)$, and let

$$\mathbb{R}^n = \{(v_1, v_2, \ldots, v_n) : v_i \in \mathbb{R}, i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{(v_1, v_2, \ldots, v_n) : v_i \geq 0, i = 1, 2, \ldots, n\}.$$

Our main results are as follows:

Theorem 1.1. *Let* $\boldsymbol{a} = (a_1, a_2, \ldots, a_n), \boldsymbol{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n_+$, *and let* p, q *be two non-zero real numbers*

$$H_1(\boldsymbol{a}) = \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}}.$$

If $p \geq 1$, *then for fixed* $\boldsymbol{b}$, $H_1(\boldsymbol{a})$ *is Schur-convex on* $\mathbb{R}^n_+$. *If* $p \leq 1$, *then for fixed* $\boldsymbol{b}$, $H_1(\boldsymbol{a})$ *is Schur-concave on* $\mathbb{R}^n_+$.

Theorem 1.2. *Let* $\boldsymbol{a} = (a_1, a_2, \ldots, a_n), \boldsymbol{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n_+$, *and let* p, q *be two non-zero real numbers,* $A_n(\boldsymbol{a}) = \frac{1}{n}\sum_{k=1}^{n} a_k$,

$$H_2(\boldsymbol{b}) = n^{\frac{1}{p}} \Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}} A_n(\boldsymbol{a}).$$

If $q \geq 1$, *then for fixed* $\boldsymbol{a}$, $H_2(\boldsymbol{b})$ *is Schur-convex on* $\mathbb{R}^n_+$. *If* $q \leq 1$, *then for fixed* $\boldsymbol{a}$, $H_2(\boldsymbol{b})$ *is Schur-concave on* $\mathbb{R}^n_+$.

Theorem 1.3. *Let* $f(x), g(x)$ *be two nonnegative and continuous functions on* I, *let* $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, *for any* $a, b \in I$ $(a \neq b)$, $p, q \in \mathbb{R}$, *and let*

$$H_3(a,b) = \begin{cases} \Big(\frac{\int_a^b (g(x))^q dx}{\int_a^b f(x)g(x)dx}\Big)^p \Big(\frac{\int_a^b (f(x))^p dx}{\int_a^b f(x)g(x)dx}\Big)^q, & a \neq b, \\ [f(a)g(a)]^{pq-p-q}, & a = b. \end{cases} \tag{3}$$

Then, $H_3(a,b)$ *is Schur-convex (Schur-concave) on* I^2 *if and only if*

$$\frac{q(f^p(b) + f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b) + g^q(a))}{\int_a^b g^q(x)dx} \geq (\leq) \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}. \tag{4}$$

2. Preliminaries

In this section, we introduce some essential definitions and lemmas.

Definition 2.1 ([14])**.** *Let* $\boldsymbol{U} = (u_1, u_2, \ldots, u_n)$ *and* $\boldsymbol{V} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$.

(*i*) *The vector* $\boldsymbol{U}$ *is said to be majorized by the vector* $\boldsymbol{V}$, *symbolized as* $\boldsymbol{U} \prec \boldsymbol{V}$, *if* $\sum_{i=1}^{\ell} u_{[i]} \leq \sum_{i=1}^{\ell} v_{[i]}$ *for* $\ell = 1, 2, \ldots, n-1$ *and* $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i$, *where* $u_{[1]} \geq u_{[2]} \cdots \geq u_{[n]}$ *and* $v_{[1]} \geq v_{[2]} \cdots \geq v_{[n]}$ *are rearrangements of* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *in a descending order.*

(*ii*) *Let* $\Omega \subset \mathbb{R}^n$, Ψ: $\Omega \to \mathbb{R}$ *is said to be Schur-convex function on* Ω *if* $\boldsymbol{U} \prec \boldsymbol{V}$ *on* Ω *implies* $\Psi(\boldsymbol{U}) \leq \Psi(\boldsymbol{V})$, *while* Ψ *is said to be Schur-concave function on* Ω *if and only if* $-\Psi$ *is Schur-convex function.*

Lemma 2.1 ([14]). *Suppose that $\Omega \subset \mathbb{R}^n$ is a convex set and has a nonempty interior set Ω°, suppose also that $\Psi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°. Then Ψ is the Schur-convex (or Schur-concave) function, if and only if it is symmetric on Ω and*

$$(v_1 - v_2)\left(\frac{\partial \Psi}{\partial v_1} - \frac{\partial \Psi}{\partial v_2}\right) \geq 0 \ (or \leq 0)$$

holds, for any $\boldsymbol{V} = (v_1, v_2, \ldots, v_n) \in \Omega^\circ$.

Lemma 2.2 ([15], Chebyshev inequality). *Let $a_k \geq 0$, $b_k \geq 0, k = 1, 2, \ldots, n$.*

(*i*) *If $\{a_k\}$ and $\{b_k\}$ $(k = 1, 2, \ldots, n)$ have opposite monotonicity, then*

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \geq n \sum_{k=1}^{n} a_k b_k \tag{5}$$

(*ii*) *If $\{a_k\}$ and $\{b_k\}$ $(k = 1, 2, \ldots, n)$ have same monotonicity, then*

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k \leq n \sum_{k=1}^{n} a_k b_k \tag{6}$$

Lemma 2.3 ([15], Hermite-Hadamard inequality). *If $f(x)$ is a convex function on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{7}$$

If $f(x)$ is a concave function on $[a, b]$, then inequality (7) is reversed.

Lemma 2.4 ([16]). *If $a \leq b$, $u(t) = tb + (1-t)a$, $v(t) = ta + (1-t)b$, $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$ or $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$, then*

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b) \tag{8}$$

Lemma 2.5 ([16]). *Let $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n_+$, $A_n(\boldsymbol{a}) = \frac{1}{n}\sum_{i=1}^{n} a_i$. Then*

$$\mathbf{u} = \Big(\underbrace{A_n(\boldsymbol{a}), A_n(\boldsymbol{a}), \cdots, A_n(\boldsymbol{a})}_{n}\Big) \prec (a_1, a_2, \cdots, a_n) = \boldsymbol{a}. \tag{9}$$

3. Proof of main results

Proof of Theorem 1.1. It is obvious that $H_1(\boldsymbol{a})$ is symmetric about $a_1, a_2, \ldots, a_n$ on $\mathbb{R}^n_+$, without loss of generality, we may assume that $a_1 \geq a_2$.

Differentiating $H_1(\boldsymbol{a})$ with respect to a_1 and a_2 respectively, we obtain

$$\frac{\partial H_1}{\partial a_1} = \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}-1}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}} a_1^{p-1}, \quad \frac{\partial H_1}{\partial a_2} = \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}-1}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}} a_2^{p-1}.$$

Hence, we have

$$\Delta_1 := (a_1 - a_2)\Big(\frac{\partial H_1}{\partial a_1} - \frac{\partial H_1}{\partial a_2}\Big) = (a_1 - a_2)\Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}-1}\Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}}(a_1^{p-1} - a_2^{p-1}).$$

It is easy to see that $\Delta_1 \geq 0$ for $p \geq 1$, and $\Delta_1 \leq 0$ for $p \leq 1$. By Lemma 2.1, it follows that $H_1(\boldsymbol{a})$ is Schur-convex on $\mathbb{R}_+^n$ for $p \geq 1$, and $H_1(\boldsymbol{a})$ is Schur-concave on $\mathbb{R}_+^n$ for $p \leq 1$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Using the same arguments as that described in the proof of Theorem 1.1, we can easily carry out the proof of Theorem 1.2. □

Proof of Theorem 1.3. Note that

$$H_3(a,b) = \Big(\frac{\int_a^b (g(x))^q dx}{\int_a^b f(x)g(x)dx}\Big)^p \Big(\frac{\int_a^b (f(x))^p dx}{\int_a^b f(x)g(x)dx}\Big)^q = \frac{\big(\int_a^b f^p(x)dx\big)^q \big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{p+q}}.$$

Since $H_3(a,b)$ is symmetric about a, b on $\mathbb{R}_+^2$, we may assume that $b \geq a$. Differentiating $H_3(\boldsymbol{a})$ with respect to b and a respectively gives

$$\begin{aligned}
\frac{\partial H_3}{\partial b} =& \frac{q\big(\int_a^b f^p(x)dx\big)^{q-1} f^p(b)\big(\int_a^b g^q(x)dx\big)^p\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}} \\
&+ \frac{p\big(\int_a^b g^q(x)dx\big)^{p-1} g^q(b)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}} \\
&- \frac{(p+q)\big(\int_a^b f(x)g(x)dx\big)^{p+q-1} f(b)g(b)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}, \\
\frac{\partial H_3}{\partial a} =& - \frac{q\big(\int_a^b f^p(x)dx\big)^{q-1} f^p(a)\big(\int_a^b g^q(x)dx\big)^p\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}} \\
&- \frac{p\big(\int_a^b g^q(x)dx\big)^{p-1} g^q(a)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b f(x)g(x)dx\big)^{p+q}}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}} \\
&+ \frac{(p+q)\big(\int_a^b f(x)g(x)dx\big)^{p+q-1} f(a)g(a)\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\Delta_2 :=& (b-a)\Big(\frac{\partial H_3}{\partial b} - \frac{\partial H_3}{\partial a}\Big) \\
=& \frac{b-a}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\Big[q\Big(\int_a^b f^p(x)dx\Big)^{q-1}\Big(\int_a^b g^q(x)dx\Big)^p
\end{aligned}$$

$$
\begin{aligned}
&\times \Big(\int_a^b f(x)g(x)dx\Big)^{p+q}\big(f^p(b)+f^p(a)\big)+p\Big(\int_a^b g^q(x)dx\Big)^{p-1}\\
&\times \Big(\int_a^b f^p(x)dx\Big)^q\Big(\int_a^b f(x)g(x)dx\Big)^{p+q}\big(g^q(b)+g^q(a)\big)-(p+q)\\
&\times \Big(\int_a^b f(x)g(x)dx\Big)^{p+q-1}\Big(\int_a^b f^p(x)dx\Big)^q\Big(\int_a^b g^q(x)dx\Big)^p\big(f(b)g(b)+f(a)g(a)\big)\Big]\\
&=\frac{b-a}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\Big(\int_a^b f(x)g(x)dx\Big)^{p+q-1}\Big(\int_a^b f^p(x)dx\Big)^{q-1}\\
&\times \Big(\int_a^b g^q(x)dx\Big)^{p-1}\Big[\Big(\int_a^b f(x)g(x)dx\Big)\Big(q\int_a^b g^q(x)dx(f^p(b)+f^p(a))\\
&+p\int_a^b f^p(x)dx(g^q(b)+g^q(a))\Big)\\
&-(p+q)\Big(\int_a^b f^p(x)dx\int_a^b g^q(x)dx\Big)(f(b)g(b)+f(a)g(a))\Big].
\end{aligned}
$$

Using the assumption condition of Theorem 1.3 and the non-negativity of

$$
\frac{b-a}{\big(\int_a^b f(x)g(x)dx\big)^{2(p+q)}}\Big(\int_a^b f(x)g(x)dx\Big)^{p+q-1}\Big(\int_a^b f^p(x)dx\Big)^{q-1}\Big(\int_a^b g^q(x)dx\Big)^{p-1},
$$

we deduce that $\Delta_2 \geq (\leq)\ 0$ if and only if

$$
\begin{aligned}
&\Big(\int_a^b f(x)g(x)dx\Big)\Big[q\int_a^b g^q(x)dx(f^p(b)+f^p(a))+p\int_a^b f^p(x)dx(g^q(b)+g^q(a))\Big]\\
&\geq (\leq)\Big(\int_a^b f^p(x)dx\int_a^b g^q(x)dx\Big)(f(b)g(b)+f(a)g(a))(p+q)\\
&\Longleftrightarrow \frac{q(f^p(b)+f^p(a))}{\int_a^b f^p(x)dx}+\frac{p(g^q(b)+g^q(a))}{\int_a^b g^q(x)dx}\geq(\leq)\frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}.
\end{aligned}
$$

Hence, $H_3(a,b)$ is Schur-convex (Schur-concave) on I^2 if and only if

$$
\frac{q(f^p(b)+f^p(a))}{\int_a^b f^p(x)dx}+\frac{p(g^q(b)+g^q(a))}{\int_a^b g^q(x)dx}\geq(\leq)\frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx}.
$$

This completes the proof of Theorem 1.3. □

4. Some corollaries

In this section, we give some consequences of Theorem 1.3.

Corollary 4.1. *Let $f(x), g(x)$ be two nonnegative convex functions on I, $f''g+g''f+2f'g' \leq 0$, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a,b \in I$ $(a \neq b)$. If $p \geq 1, q \geq 1$, then $H_3(a,b)$ is Schur-convex on I^2.*

Proof. Direct computation gives
$(f^p)'' = pf^{p-2}[(p-1)(f')^2 + ff'']$, $(g^q)'' = qg^{q-2}[(q-1)(g')^2 + gg'']$,
$(fg)'' = f''g + g''f + 2f'g'$.

Since $f(x), g(x)$ are convex function on I, $p \geq 1, q \geq 1$, we have $(f^p(x))'' \geq 0$, $(g^q(x))'' \geq 0$ for $x \in I$, so $f^p(x), g^q(x)$ are convex functions on I. In addition, form the assumption $f''g+g''f+2f'g' \leq 0$, we conclude that $f(x)g(x)$ is concave function on I.

By using Lemma 2.3 (Hermite-Hadamard inequality), we obtain

$$\begin{aligned}
&\frac{q(f^p(b)+f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b)+g^q(a))}{\int_a^b g^q(x)dx} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\
&\geq \frac{2q}{b-a} + \frac{2p}{b-a} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\
&= (p+q)\Big[\frac{2}{b-a} - \frac{(f(b)g(b)+f(a)g(a))}{\int_a^b f(x)g(x)dx}\Big] \geq 0.
\end{aligned}$$

We deduce from Theorem 1.3 that $H_3(a,b)$ is Schur-convex on I^2. The proof of Corollary 4.1 is complete. □

Corollary 4.2. *Let $f(x), g(x)$ be two nonnegative and opposite monotonicity concave functions, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p < 0, q < 0$, then $H_3(a,b)$ is Schur-concave on I^2.*

Proof. In light of

$$\begin{aligned}
&(f^p)'' = p[(p-1)(f')^2 + ff'']f^{p-2}, (g^q)'' = q[(q-1)(g')^2 + gg'']g^{q-2}, \\
&(fg)'' = f''g + g''f + 2f'g',
\end{aligned}$$

we conclude that $(f^p(x))'' \geq 0$, $(g^q(x))'' \geq 0$, so $f^p(x)$ and $g^q(x)$ are convex functions on I. Since $f(x), g(x)$ are opposite monotonicity concave functions, which implies that $f(x)g(x)$ is concave function on I. By Hermite-Hadamard inequality, we have

$$\begin{aligned}
&\frac{q(f^p(b)+f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b)+g^q(a))}{\int_a^b g^q(x)dx} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\
&\leq \frac{2q}{b-a} + \frac{2p}{b-a} - \frac{(f(b)g(b)+f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\
&= (p+q)\Big[\frac{2}{b-a} - \frac{(f(b)g(b)+f(a)g(a))}{\int_a^b f(x)g(x)dx}\Big] \leq 0.
\end{aligned}$$

It follows from Theorem 1.3 that $H_3(a,b)$ is Schur-concave on I^2. Corollary 4.2 is proved. □

Corollary 4.3. *Let $f(x), g(x)$ be two nonnegative and opposite monotonicity concave functions, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p < 0$, $0 < q \leq 1$ and $p + q \geq 0$, then $H_3(a, b)$ is Schur-convex on I^2.*

Proof. In view of

$$(f^p)'' = p[(p-1)(f')^2 + ff'']f^{p-2}, (g^q)'' = q[(q-1)(g')^2 + gg'']g^{q-2},$$
$$(fg)'' = f''g + g''f + 2f'g',$$

we deduce that $f^p(x)$ is convex function for $p < 0$, $g^q(x)$ is concave function for $0 < q \leq 1$, $f(x)g(x)$ is concave function on I. By using Hermite-Hadamard inequality, we obtain

$$\begin{aligned} &\frac{q(f^p(b) + f^p(a))}{\int_a^b f^p(x)dx} + \frac{p(g^q(b) + g^q(a))}{\int_a^b g^q(x)dx} - \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\ &\geq \frac{2q}{b-a} + \frac{2p}{b-a} - \frac{(f(b)g(b) + f(a)g(a))(p+q)}{\int_a^b f(x)g(x)dx} \\ &= (p+q)\Big[\frac{2}{b-a} - \frac{(f(b)g(b) + f(a)g(a))}{\int_a^b f(x)g(x)dx}\Big] \geq 0. \end{aligned}$$

We deduce from Theorem 1.3 that $H_3(a, b)$ is Schur-convex on I^2. Corollary 4.3 is proved. □

5. Applications to inequalities of Hölder type

Firstly, we establish two discrete Hölder-type inequality involving power mean and arithmetic mean.

Theorem 5.1. *Let $a_k \geq 0, b_k \geq 0, k = 1, 2, \ldots, n$, and let p, q be two non-zero real numbers.*

(i) If $p \geq 1, q \geq 1$, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}+\frac{1}{q}} A_n(\boldsymbol{a})A_n(\boldsymbol{b}); \tag{10}$$

(ii) If $p \leq 1, q \leq 1$, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \leq n^{\frac{1}{p}+\frac{1}{q}} A_n(\boldsymbol{a})A_n(\boldsymbol{b}), \tag{11}$$

where $A_n(\boldsymbol{a}) = \frac{1}{n}\sum_{k=1}^n a_k$, $A_n(\boldsymbol{b}) = \frac{1}{n}\sum_{k=1}^n b_k$.

Proof. (i) By Lemma 2.5, one has the majorization relationship

$$(a_1, a_2, \cdots, a_n) \succ (A_n(\boldsymbol{a}), A_n(\boldsymbol{a}), \cdots, (A_n(\boldsymbol{a}))\,.$$

From Theorem 1.1, we know that, for $p \geq 1$, $H_1(\boldsymbol{a})$ is Schur-convex on $\mathbb{R}^n_+$. It follows from Definition 2.1 that $H_1(\boldsymbol{a}) \geq H_1(A_n(\boldsymbol{a}))$ for $p \geq 1$.
Hence

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq (n(A_n(\boldsymbol{a}))^p)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} = n^{\frac{1}{p}}A_n(\boldsymbol{a})\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}}.$$

On the other hand, by Theorem 1.2, we obtain that, for $q \geq 1$, $H_2(\boldsymbol{b})$ is Schur-convex on $\mathbb{R}^n_+$. Now, from the majorization relation

$$(b_1, b_2, \cdots, b_n) \succ \Big(A_n(\boldsymbol{b}), A_n(\boldsymbol{b}), \cdots, A_n(\boldsymbol{b})\Big),$$

we have $H_2(\boldsymbol{b}) \geq H_2(A_n(\boldsymbol{b}))$ for $q \geq 1$, that is

$$n^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}}A_n(\boldsymbol{a}) \geq n^{\frac{1}{p}}A_n(\boldsymbol{a})(n(A_n(\boldsymbol{b}))^q)^{\frac{1}{q}} = n^{\frac{1}{p}+\frac{1}{q}}A_n(\boldsymbol{a})A_n(\boldsymbol{b}).$$

Hence, we get

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}}A_n(\boldsymbol{a})\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}+\frac{1}{q}}A_n(\boldsymbol{a})A_n(\boldsymbol{b}),$$

which implies the required inequality (10).

(ii) By the same way as the proof of inequality (10), we can prove the inequality (11). This completes the proof of Theorem 5.1. □

Nextly, we provide two refined versions of discrete Hölder-type inequality under certain specified conditions.

Theorem 5.2. *Let $a_k \geq 0$, $b_k \geq 0, k = 1, 2, \ldots, n$, p, q be two non-zero real numbers.*

(i) If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\{a_k\}, \{b_k\}$ $(k = 1, 2, \ldots, n)$ have opposite monotonicity, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) \geq \sum_{k=1}^n a_k b_k; \tag{12}$$

(ii) If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\{a_k\}, \{b_k\}$ $(k = 1, 2, \ldots, n)$ have same monotonicity, then

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \leq nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) \leq \sum_{k=1}^n a_k b_k. \tag{13}$$

Proof. (i) For $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, by utilizing Theorem 1.1, we have

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq n^{\frac{1}{p}+\frac{1}{q}}A_n(\boldsymbol{a})A_n(\boldsymbol{b}) = nA_n(\boldsymbol{a})A_n(\boldsymbol{b}).$$

Moreover, using Lemma 2.2 (Chebyshev inequality) gives

$$nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) = \frac{\sum_{k=1}^n a_k \sum_{k=1}^n b_k}{n} \geq \sum_{k=1}^n a_k b_k.$$

Hence, we have

$$\Big(\sum_{k=1}^n a_k^p\Big)^{\frac{1}{p}}\Big(\sum_{k=1}^n b_k^q\Big)^{\frac{1}{q}} \geq nA_n(\boldsymbol{a})A_n(\boldsymbol{b}) \geq \sum_{k=1}^n a_k b_k,$$

which implies the required inequality (12).

(ii) In the same way as the proof of inequality (12), we can verify the validity of inequality (13). The proof of Theorem 5.2 is complete. □

In Theorems 5.3, 5.4 and 5.5 below, we will give some refined versions of integral Hölder-type inequality under certain specified conditions.

Theorem 5.3. *Let $f(x), g(x)$ be two integrable and nonnegative functions on $[a, b]$, and let p, q be two non-zero real numbers.*

(i) If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f(x), g(x)$ have opposite monotonicity, then

$$\begin{aligned}&\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}\\ &\geq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \geq \int_a^b f(x)g(x)dx.\end{aligned} \tag{14}$$

(ii) If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f(x), g(x)$ have same monotonicity, then

$$\begin{aligned}&\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}\\ &\leq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx.\end{aligned} \tag{15}$$

Proof. (i) If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_k \geq 0, b_k \geq 0$ and $\{a_k\}, \{b_k\}$ $(k = 1, 2, \ldots, n)$ have opposite monotonicity, then by Theorem 5.2, we obtain

$$\begin{aligned}&\Big(\frac{b-a}{n}\sum_{k=1}^n f^p\Big(a+\frac{k(b-a)}{n}\Big)\Big)^{\frac{1}{p}}\Big(\frac{b-a}{n}\sum_{k=1}^n g^q\Big(a+\frac{k(b-a)}{n}\Big)\Big)^{\frac{1}{q}}\\ &\geq \frac{1}{b-a}\left(\frac{b-a}{n}\sum_{k=1}^n f\Big(a+\frac{k(b-a)}{n}\Big)\right)\left(\frac{b-a}{n}\sum_{k=1}^n g\Big(a+\frac{k(b-a)}{n}\Big)\right)\\ &\geq \frac{b-a}{n}\sum_{k=1}^n f\Big(a+\frac{k(b-a)}{n}\Big)g\Big(a+\frac{k(b-a)}{n}\Big).\end{aligned}$$

Letting $n \to \infty$ in both sides of the above inequalities, we obtain

$$\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \geq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \geq \int_a^b f(x)g(x)dx,$$

which is the desired inequality (14).

(ii) By the same way as the proof of inequality (14), one can prove the inequality (15). This completes the proof of Theorem 5.3. □

Obviously, inequalities (12), (13), (14), (15) are the sharpened versions of Hölder's inequality under some specified conditions.

Theorem 5.4. *Let $f(x), g(x)$ be two nonnegative convex functions on I, $f''g + g''f + 2f'g' \leq 0$, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$(i) \quad \int_a^b f(x)g(x)dx \leq \frac{\int_{u(t)}^{v(t)} f(x)g(x)dx}{\big(\int_{u(t)}^{v(t)} f^p(x)dx\big)^{\frac{1}{p}} \big(\int_{u(t)}^{v(t)} g^q(x)dx\big)^{\frac{1}{q}}}$$

$$\times \Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \leq \Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}, \tag{16}$$

where $u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$, $0 \leq t \leq 1, t \neq \frac{1}{2}$.

$$(ii) \quad \Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \geq \frac{1}{b-a} \int_a^b f(x)dx \int_a^b g(x)dx$$

$$\geq f\big(\frac{a+b}{2}\big) g\big(\frac{a+b}{2}\big)(b-a) \geq \int_a^b f(x)g(x)dx. \tag{17}$$

Proof. (i) Since $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(x)$ are nonnegative convex functions with $f''g + g''f + 2f'g' \leq 0$ on I, it follows from Corollary 4.1 that $H_3(a, b)$ is Schur-convex on I^2. Additionally, from Lemma 2.4, one has, for $0 \leq t \leq 1, t \neq \frac{1}{2}$, the relation $\big(\frac{a+b}{2}, \frac{a+b}{2}\big) \prec (u(t), v(t)) \prec (a, b)$. Hence, we obtain

$$H_3(a,b) \geq H_3(u(t), v(t)) \geq H_3\Big(\frac{a+b}{2}, \frac{a+b}{2}\Big) = \Big(f\big(\frac{a+b}{2}\big) g\big(\frac{a+b}{2}\big)\Big)^{pq-p-q} = 1,$$

which implies that

$$\frac{\big(\int_a^b f^p(x)dx\big)^q \big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{p+q}} \geq \frac{\big(\int_{u(t)}^{v(t)} f^p(x)dx\big)^q \big(\int_{u(t)}^{v(t)} g^q(x)dx\big)^p}{\big(\int_{u(t)}^{v(t)} f(x)g(x)dx\big)^{p+q}} \geq 1$$

$$\Longleftrightarrow$$

$$\begin{aligned}
&\big(\int_a^b f(x)g(x)dx\big)^{p+q} \\
&\leq \frac{\big(\int_{u(t)}^{v(t)} f(x)g(x)dx\big)^{p+q}}{\big(\int_{u(t)}^{v(t)} f^p(x)dx\big)^q \big(\int_{u(t)}^{v(t)} g^q(x)dx\big)^p} \big(\int_a^b f^p(x)dx\big)^q \big(\int_a^b g^q(x)dx\big)^p \\
&\leq \big(\int_a^b f^p(x)dx\big)^q \big(\int_a^b g^q(x)dx\big)^p.
\end{aligned}$$

It follows from $\frac{1}{p}+\frac{1}{q}=1$ that $p+q=pq$, taking the $\frac{1}{pq}$ power of two sides in the above inequalities, we derive the desired inequality (16).

(ii) Using Hölder's inequality (2) gives

$\big(b-a\big)^{\frac{1}{q}}\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}} \geq \int_a^b f(x)dx,\ \big(b-a\big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \geq \int_a^b g(x)dx.$

Hence, we have

$$\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \geq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx.$$

In addition, from the assumption conditions, we find that $f(x), g(x)$ are convex on I, $f(x)g(x)$ is concave on I, thus we deduce from the Hermite-Hadamard inequality that

$$\frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx \geq f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)(b-a) \geq \int_a^b f(x)g(x)dx.$$

The proof of Theorem 5.4 is complete. □

It is worth noting that inequalities (16) and (17) are the refined versions of Hölder's inequality under a specified condition.

Theorem 5.5. *Let $f(x), g(x)$ be two nonnegative and opposite monotonicity concave functions, and let $\int_a^b f(x)g(x)dx \neq 0$, $\int_a^b (f(x))^p dx \neq 0$, $\int_a^b (g(x))^q dx \neq 0$, for any $a, b \in I$ $(a \neq b)$. If $p<0$, $q<0$, then*

$$\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \leq \Big(f\big(\frac{a+b}{2}\big)g\big(\frac{a+b}{2}\big)\Big)^{1-\frac{1}{p}-\frac{1}{q}}\Big(\int_a^b f(x)g(x)dx\Big)^{\frac{1}{p}+\frac{1}{q}}. \tag{18}$$

Proof. By the aid of Corollary 4.2, we observe that $H_3(a,b)$ is Schur-concave on I^2, in addition, from Lemma 2.5, one has $\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (a,b)$. We thus have

$$H_3(a,b) \leq H_3\Big(\frac{a+b}{2}, \frac{a+b}{2}\Big) = \Big(f\big(\frac{a+b}{2}\big)g\big(\frac{a+b}{2}\big)\Big)^{pq-p-q}$$

$$\Longleftrightarrow$$

$$\frac{\big(\int_a^b f^p(x)dx\big)^q\big(\int_a^b g^q(x)dx\big)^p}{\big(\int_a^b f(x)g(x)dx\big)^{p+q}} \leq \Big(f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\Big)^{pq-p-q},$$

taking the $\frac{1}{pq}$ power of the two-sides inequality above, we obtain the required inequality (18). Theorem 5.5 □

6. Applications to inequalities for special means

Let $b > a > 0$, the Stolarsky mean is defined as follows (see [12])

$$L_p(a,b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \neq -1, 0.$$

The arithmetic mean, geometric mean and logarithmic mean are respectively defined by

$$A(a,b) = \frac{a+b}{2}, \;\; G(a,b) = \sqrt{ab}, \;\; L(a,b) = \frac{b-a}{\log b - \log a}.$$

Theorem 6.1. *Let $b > a > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) If $p > 1$, then

$$L_p(a,b) \geq A(a,b)L(a,b)L_{-q}(a,b) \geq L_{-q}(a,b). \tag{19}$$

(ii) If $0 < p < 1$, then

$$L_p(a,b) \leq (A(a,b))^2(L_q(a,b))^{-1} \leq (L_2(a,b))^2(L_q(a,b))^{-1}. \tag{20}$$

Proof. Note that

$$\Big(\frac{1}{b-a}\int_a^b x^p dx\Big)^{\frac{1}{p}} = \Big(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\Big)^{\frac{1}{p}} = L_p(a,b),$$

$$\Big(\frac{1}{b-a}\int_a^b x^{-q} dx\Big)^{\frac{1}{q}} = \Big(\frac{b^{-q+1} - a^{-q+1}}{(-q+1)(b-a)}\Big)^{\frac{1}{q}} = (L_{-q}(a,b))^{-1}.$$

(i) For $p > 1$, by Theorem 5.3, we have

$$\begin{aligned}&\Big(\frac{1}{b-a}\int_a^b f^p(x)\Big)^{\frac{1}{p}}\Big(\frac{1}{b-a}\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}\\ \geq&\Big(\frac{1}{b-a}\Big)^2\int_a^b f(x)dx\int_a^b g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)g(x)dx.\end{aligned}$$

Taking $f(x) = x, g(x) = x^{-1}$ in the above inequality, it follows that

$$L_p(a,b)(L_{-q}(a,b))^{-1} \geq \frac{1}{(b-a)^2}\int_a^b xdx\int_a^b x^{-1}dx \geq \frac{1}{b-a}\int_a^b dx,$$

that is

$$L_p(a,b) \geq A(a,b)L(a,b)L_{-q}(a,b) \geq L_{-q}(a,b).$$

(ii) For $0 < p < 1$, by Theorem 5.3, we have

$$\begin{aligned}&\Big(\frac{1}{b-a}\int_a^b f^p(x)\Big)^{\frac{1}{p}}\Big(\frac{1}{b-a}\int_a^b g^q(x)dx\Big)^{\frac{1}{q}}\\ \leq&\Big(\frac{1}{b-a}\Big)^2\int_a^b f(x)dx\int_a^b g(x)dx \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx.\end{aligned}$$

Taking $f(x) = x, g(x) = x$, we obtain

$$L_p(a,b) \le (A(a,b))^2(L_q(a,b))^{-1} \le (L_2(a,b))^2(L_q(a,b))^{-1}.$$

The proof of Theorem 6.1 is complete. □

Theorem 6.2. *Let* $b > a > 0$, $u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$, $0 \le t \le 1$, $t \ne \frac{1}{2}$. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$L_{-q}(a,b) \le \frac{L_{-q}(u(t),v(t))}{L_p(u(t),v(t))} L_p(a,b) \le L_p(a,b). \tag{21}$$

Proof. Using Theorem 5.4 with a substitution of $f(x) = x,\ g(x) = x^{-1}$ in inequality (16), we obtain

$$\begin{aligned}\int_a^b dx \le & \frac{\int_{u(t)}^{v(t)} dx}{\big(\int_{u(t)}^{v(t)} x^p dx\big)^{\frac{1}{p}} \big(\int_{u(t)}^{v(t)} (x^{-1})^q dx\big)^{\frac{1}{q}}} \Big(\int_a^b x^p dx\Big)^{\frac{1}{p}} \Big(\int_a^b (x^{-1})^q dx\Big)^{\frac{1}{q}} \\ \le & \Big(\int_a^b x^p dx\Big)^{\frac{1}{p}} \Big(\int_a^b (x^{-1})^q dx\Big)^{\frac{1}{q}},\end{aligned}$$

that is

$$\begin{aligned}(b-a) \le & \frac{(v(t)-u(t))(b-a)^{\frac{1}{p}+\frac{1}{q}} L_p(a,b)(L_{-q}(a,b))^{-1}}{(v(t)-u(t))^{\frac{1}{p}+\frac{1}{q}} L_p(u(t),v(t))(L_{-q}(u(t),v(t)))^{-1}} \\ \le & (b-a)^{\frac{1}{p}+\frac{1}{q}} L_p(a,b)(L_{-q}(a,b))^{-1},\end{aligned}$$

which leads to the desired inequality

$$L_{-q}(a,b) \le \frac{L_{-q}(u(t),v(t))}{L_p(u(t),v(t))} L_p(a,b) \le L_p(a,b).$$

This completes the proof of Theorem 6.2. □

7. Conclusion

In this work, we provided a new approach to refine Hölder's inequality. Firstly, we constructed some functions associated with Hölder's inequality and verified their Schur convexities, meanwhile, in Theorems 1.1 and 1.2, we proved the Schur convexity of functions associated with discrete Hölder's inequality, we derived the Schur convexity of function connected to integral Hölder's inequality in Theorem 1.3. Nextly, with the help of the Schur convexity of functions, in Theorem 5.1 we acquired two discrete Hölder-type inequality involving power mean and arithmetic mean; in Theorem 5.2 we provided two refined versions of discrete Hölder-type inequality; in Theorems 5.3, 5.4 and 5.5, we offered some refined versions of integral Hölder-type inequality. Finally, we illustrated the applications of the obtained Hölder-type inequalities, some novel comparison inequalities for Stolarsky mean, arithmetic mean, geometric mean and logarithmic mean are derived respectively in Theorems 6.1 and 6.2.

Acknowledgements

This work was supported by the Natural Science Foundation of Fujian Province of China under Grant No. 2020J01365. All authors contributed equally and significantly in writing this paper.

References

[1] D. S. Mitrinović, P. M. Vasić, *Analytic inequalities*, Springer-Verlag, New York, 1970.

[2] X. j. Yang, *Hölder inequality*, Appl. Math. Lett., 16 (2003), 897-903.

[3] X. j. Yang, *A generalization of Hölder inequality*, J. Math. Anal. Appl., 247 (2000), 328-330.

[4] K. Hu, *On an inequality and its applications*, Sci. Sinica., 24 (1981), 1047-1055.

[5] S. H. Wu, *A new sharpened and generalized version of Hölder's inequality and its applications*, Appl. Math. Comput., 197 (2008), 708-714.

[6] S. H. Wu, *Generalization of a sharp Hölder's inequality and its application*, J. Math. Anal. Appl., 332 (2007), 741-750.

[7] H. Qiang, Z. C. Hu, *Generalizations of Hölder's and some related inequalities*, Computers Math. Appl., 61 (2011), 392-396.

[8] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[9] H. N. Shi, S. H. Wu, *Majorized proof and refinement of the discrete Steffensen's inequality*, Taiwanese J. Math., 11 (2007), 1203-1208.

[10] L. L. Fu, B. Y. Xi, H. M. Srivastava, *Schur-convexity of the generalized Heronian means involving two positive numbers*, Taiwanese J. Math., 15 (2011), 2721-2731.

[11] Y. M. Chu, X. M. Zhang, G. D. Wang, *The Schur geometrical convexity of the extended mean values*, J. Convex Anal., 15 (2008), 707-718.

[12] Z. H. Yang,*Schur power convexity of Stolarsky means*, Publ. Math. Debrecen, 80 (2012), 43-66.

[13] Z. H. Yang, *Schur power convexity of Gini means*, Bull. Korean Math. Soc., 50 (2013), 485-498.

[14] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: theory of majorization and its application,* 2nd ed., Springer, New York, 2011.

[15] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, partial orderings and statistical applications*, Academic Press, New York, 1992.

[16] B. Y. Wang, *Foundations of majorization inequalities*, Beijing Normal University Press, Beijing, 1990 (Chinese).

Accepted: February 3, 2023

On nodal filter theory of EQ-algebras

Jie Qiong Shi
School of Mathematics and Statistics
Shaanxi Normal University
Xi'an, 710119
P.R. China
jieqiong6912@126.com

Xiao Long Xin*
School of Mathematics
Northwest University
Xi'an, 710127
P.R. China
and
School of Science
Xi'an Polytechnic University
Xi'an, 710048
P.R. China
xlxin@nwu.edu.cn

Abstract. In this article, we mainly focus on a new kind of filter on EQ-algebras. At first, we introduce some new concepts of seminodes, nodes and nodal filters (n-filters, for short) on EQ-algebras and investigate the relationships among them and some other elements. Also, we investigate their lattice structures and obtain that the set $\mathcal{SN}(E)$ of all seminodes on an EQ-algebra is a Hertz-algebra and a Heyting-algebra under some conditions. Then, we discuss the properties of n-filters and show that there is a one-to-one correspondence between nodal principle filter and node element in an idempotent EQ-algebra. Furthermore, the relationships among it and other filters are presented. It is proved that each obstinate filter or each (positive) implicative filter is an n-filter under some conditions. At last, we introduce the algebraic structures and topological structures of the set of all n-filters on EQ-algebras and prove that $(NP(E), \tau)$ is a compact T_0 space. Moreover, we set up the connections from the set $NF(E)$ of all n-filters on an EQ-algebra to other algebraic structures, like BCK-algebras, Hertz algebras and so on.
Keywords: EQ-algebra, seminode, node, nodal filter, topological space.

1. Introduction

As we all know, logic is not only an important tool in mathematics and information science, but also a basic technology. Non-classical logic consists of fuzzy logic and multi-valued logic, they deal with uncertain information such as

*. Corresponding author

fuzziness and randomness. Therefore, all kinds of fuzzy logic algebras are widely introduced and studied, such as residuated lattices, BL-algebras, MV-algebras, which play a very important role in fuzzy logic algebra system. In [11], Goguen put forward a new point of view, which is that the algebraic structure of many-valued logic may be a residuated lattice satisfying some additional conditions. This view has been widely recognized by scholars at home and abroad. However, since the publication of Hájek's book [12] in 1998, fuzzy logic has been developed into different formal systems, and each one is based on a residuated lattice. With the passage of time, propositional logic and first-order logic have been widely developed. For this reason, in order to develop the higher-order fuzzy logic as a correspondence of the classical higher-order logic. Novák and De Bates [17] came out with a new algebra, which is called an EQ-algebra, for the first time. An EQ-algebra has three operations, which are fuzzy equality, multiplication and meet. By replacing the basic conjunction fuzzy equality with implication, EQ-algebras open up a new filed for another development of many-valued fuzzy logic and a possibility for developing a fuzzy logic with non commutative connection but only one implication. Since then, EQ-algebras have been widely concerned and many significant properties and conclusions have been proved [1], [10], [14][17], [21], [26].

Filter theory is of great significance to study the completeness of different logical systems and their matching logical algebras. Start with a logical viewpoint, we can use the filters to represent the provable formula sets in relevant reasoning systems. Also, the characters of filters is closely related to the structure properties of algebras. Hence, there are numerous researches on filter theory. In [17], Novák and De Bates introduced filters on EQ-algebra for the first time. In [10], M. El-Zekey and V. Novák proposed the concepts of (prime) prefilters on EQ-algebras. Moreover, their related properties were stated and proved. And then, in [14], implicative and positive implicative prefilters (filters) in EQ-algebra were proposed by Liu and Zhang and they also represented some related conclusions of them. Also, they discussed the properties of quotient algebras, which is induced by the positive implicative filters. Furthermore, they discussed the relationships between these two prefilters and concluded that in good IEQ-algebras positive implicative prefilters and implicative prefilters coincided.

Now, in this paper, we introduce a new kind of filter to EQ-algebras, which is said to be a nodal filter. Originally, Balbes and Horn [2] put forward the concept of nodes in a lattice. In [22], the definition of a nodal filter was introduced by Varlet in the (implicative) semilattice. Afterward, T. Khorami and B. Saeid [13] presented the concepts of nodes and nodal filters on BL-algebra and the congruence relations induced by nodal filters on BL-algebra is stated and proved. In [6], Bakhshi presented the concept of nodal filters in residuated lattices and obtained that the set of all nodal filters forms a Heyting algebra. Namdar and Borzooei [18] researched nodal filters theory in hoop algebras. Next, X. Xun and X.L. Xin [24] introduced it in equality algebras. Now, we introduce this concept to EQ-algebras, here is the outline of this paper: In the next Section, we recollect some basic definitions and properties of EQ-algebras. In Section 3, we introduce the concepts of seminodes and nodes on EQ-algebras and investigate the related properties of them. We obtain that the set $\mathcal{SN}(E)$ of all seminodes is a Hertz-algebra and a Heyting-algebra under some conditions. Moreover, we consider the relationships among seminodes, nodes and some other elements on an EQ-algebra. In Section 4, we present the notion of nodal filter (for short, n-filter) in an EQ-algebra and investigate their related properties. Furthermore, we discuss the relationships between nodal filters and node elements, as well as their relationships with other filters. In Section 5, we study the algebraic structures of $NF(E)$ and topological structures of $NP(E)$ on EQ-algebras.

2. Preliminaries

In this section, we present some basic concepts and conclusions relevant to EQ-algebras.

Definition 2.1 ([17]). *An algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is said to be an EQ-algebra, if for all $x, y, p, q \in E$, it satisfies the following axioms:*

$(EQ1)$ $< E, \wedge, 1 >$ *is a commutative idempotent monoid.*

$(EQ2)$ $< E, \otimes, 1 >$ *is a monoid and $\otimes$ is isotone w.r.t. " $\leq$ ", where $x \leq y$ is defined as $x \wedge y = x$.*

$(EQ3)$ $x \sim x = 1$.

$(EQ4)$ $((x \wedge y) \sim p) \otimes (q \sim x) \leq p \sim (q \wedge y)$.

$(EQ5)$ $(x \sim y) \otimes (p \sim q) \leq (x \sim p) \sim (y \sim q)$.

$(EQ6)$ $(x \wedge y \wedge p) \sim x \leq (x \wedge y) \sim x$.

$(EQ7)$ $x \otimes y \leq x \sim y$.

An EQ-algebra $\mathcal{E}$ is bounded if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. And we define the unary operation: $x' = x \to 0$, for all $x \in E$. If $x^2 = x$, for all $x \in E$, then $\mathcal{E}$ is called an idempotent EQ-algebra. For any $x \in E$, x is called:

(1) dense if $x' = 0$.

(2) atom if x is the minimal element in $E \setminus \{0\}$.

(3) co-atom if x is the maximal element in $E \setminus \{1\}$.

(4) involutive if $x'' = x$.

Definition 2.2 ([17]). *Let $\mathcal{E}$ be an EQ-algebra and $x, y, z \in E$. Then, it is called*

(1) *good if* $x \sim 1 = x$ *for each* $x \in E$.

(2) *prelinear if* 1 *is the unique upper bound of the set* $\{x \to y, y \to x\}$ *in* E, *for all* $x, y \in E$.

(3) *residuated if for each* $x, y, z \in E$, $(x \odot y) \wedge z = (x \odot y)$ *if and only if* $x \wedge ((y \wedge z) \sim y) = x$.

(4) *lattice-ordered if it has a lattice reduct.*

(5) *distributively lattice-ordered if the lattice reduct is distributive.*

Proposition 2.3 ([9, 10, 17])**.** *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra, and let* $x \to y := (x \wedge y) \sim x$ *and* $\bar{x} = x \sim 1$. *Then, for all* $x, y, w \in E$ *the following properties hold:*

(1) $x \otimes y \leq x, y$, $x \otimes y \leq x \wedge y$.

(2) $x \sim y \leq x \to y$, $x \sim y = y \sim x$.

(3) $x \leq \bar{x} \leq y \to x$, $\bar{1} = 1$.

(4) $x \to y \leq (w \to x) \to (w \to y)$, $x \to y \leq (y \to w) \to (x \to w)$.

(5) $x \to x \wedge y = x \to y$.

(6) *if* $x \leq y$, *then* $x \sim y = y \to x$, $w \to x \leq w \to y$ *and* $y \to w \leq x \to w$.

(7) $x \to y \leq (x \wedge w) \to (y \wedge w)$, $w \to (x \wedge y) \leq (w \to x) \wedge (w \to y)$.

(8) *if* $x \vee y$ *exists, then* $(x \vee y) \to w = (x \to w) \wedge (y \to w)$.

Proposition 2.4 ([9])**.** *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra. Then,* $\mathcal{E}$ *is residuated iff* $\mathcal{E}$ *is good and* $x \to y \leq (x \otimes z) \to (y \otimes z)$, *for all* $x, y, z \in E$.

Definition 2.5 ([17])**.** *Let* $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra. Then, a subset* H *of* E *is called a prefilter provided that, for all* $x, y, z \in E$, *the following conditions hold:*

$(F1)$ $1 \in H$.

$(F2)$ *If* $x, y \in H$, *then* $x \otimes y \in H$.

$(F3)$ *If* $x, x \to y \in H$, *then* $y \in H$.

A prefilter H *is called a filter provided that, for all* $x, y, z \in E$, *the following condition holds:*

$(F4)$ *If* $x \to y \in H$, *then* $(x \otimes z) \to (y \otimes z) \in H$.

The set of all filters of $\mathcal{E}$ is denoted by $\mathcal{F}(E)$.

Theorem 2.6 ([16]). *Let $\mathcal{E}$ be an EQ-algebra.*

(1) *For any $\emptyset \neq X \subseteq E$, the prefilter generated by X is written as $\langle X\rangle = \{x \in E \mid x_1 \to (x_2 \to (x_3 \to \cdots (x_n \to x)\cdots)) = 1$ for some $x_i \in X$ and $n \geq 1\}$. If $X = \{a\}$, then the prefilter $\langle a\rangle$ generated by $\{a\}$ is called a principal prefilter.*

(2) *If $\mathcal{E}$ is residuated, then $\langle X\rangle$ is a filter.*

(3) *$\langle x\rangle \cap \langle y\rangle = \langle x \vee y\rangle$, for all $x, y \in E$, where $\langle x\rangle$ denotes the principal prefilter generated by x.*

Definition 2.7 ([14]). *Let H be a filter of an EQ-algebra. Then:*

(1) *H is called an implicative filter if $z \to ((x \to y) \to x) \in H$ and $z \in H$ imply $x \in H$ for any $x, y, z \in E$.*

(2) *H is called a positive implicative filter if $x \to (y \to z) \in H$ and $x \to y \in H$, then $x \to z \in H$ for any $x, y, z \in E$.*

(3) *H is called an obstinate filter of $\mathcal{E}$ if, for all $x, y \in E$, $x, y \notin H$ implies $x \to y \in H$ and $y \to x \in H$.*

For any filter H of an EQ-algebra and $x, y \in E$, we define a relation $\approx_H$ on $\mathcal{E}$ as follows:

$$x \approx_H y \text{ iff } x \sim y \in H$$

In [17], we know that $\approx_H$ is a congruence relation on E. Define the factor algebra $\mathcal{E}/H = (E/H, \wedge, \odot, \sim_H, 1)$ as follows: $E/H = \{[x] \mid x \in E\}$, the operation $\wedge$ is defined by $[x] \wedge [y] = [x \wedge y]$, and similarly for the other operations. The ordering in $\mathcal{E}/H$ is defined by:

$$[x] \leq [y] \text{ iff } [x] \wedge [y] = [x] \text{ iff } x \wedge y \approx_H x \text{ iff } x \wedge y \sim x = x \to y \in H$$

Definition 2.8 ([4]). *An algebra $(E, \wedge, \to, 1)$ of type $(2, 2, 0)$ is called a Hertz-algebra provided that, for all $x, y, w \in E$, the following axioms hold:*

$(HE1)$ $x \to x = 1$.

$(HE2)$ $y \wedge (x \to y) = y$.

$(HE3)$ $x \wedge (x \to y) = x \wedge y$.

$(HE4)$ $x \to (y \wedge w) = (x \to y) \wedge (x \to w)$.

Definition 2.9 ([15]). *A BCK-algebra $(A, \to, 1)$ is an algebra of type $(2, 0)$, which satisfies the following conditions for any $x, y, w \in E$:*

$(B1)$ $(y \to w) \to ((w \to x) \to (y \to x)) = 1$.

$(B2)$ $y \to ((y \to x) \to x) = 1$.

$(B3)$ $x \to x = 1$.

$(B4)$ $x \to 1 = 1$.

$(B5)$ *If* $x \to y = 1$, $y \to x = 1$, *then* $x = y$.

Definition 2.10 ([8]). *An algebra* $(H, \to, 1)$ *of type* $(2, 0)$ *is said to be a Hilbert algebra, if for all* $x, y, w \in E$*, we have:*

$(HL1)$ $x \to (y \to x) = 1$.

$(HL2)$ $(x \to (y \to w)) \to ((x \to y) \to (x \to w)) = 1$.

$(HL3)$ *If* $x \to y = y \to x = 1$, *then* $x = y$.

Definition 2.11 ([3]). *If* $(E, \vee, \wedge, 1)$ *is a lattice, which satisfies* $x \leq y \to z$ *iff* $x \wedge y \leq z$ *for any* $x, y, z \in E$*, then the algebra* $(E, \vee, \wedge, \to, 1)$ *is said to be a Heyting-algebra.*

Definition 2.12 ([19, 20]). *If* $(L, \vee, \wedge, 0, 1)$ *is a distributive lattice satisfying* $0' = 1$, $1' = 0$, *and* $(x \wedge y)'' = x'' \wedge y''$, $(x \vee y)' = x' \wedge y'$ *and* $x''' = x'$ *hold for any* $x, y \in L$*. Then, the algebra* $(L, \vee, \wedge, ', 0, 1)$ *of type* $(2, 2, 1, 0, 0)$ *is said to be a semi-De Morgan algebra.*

3. Seminodes and nodes on EQ-algebras

In this section, we present the concepts of seminodes and nodes on EQ-algebras and study their related properties. Moreover, we consider the relationships among seminodes, nodes and some other elements on an EQ-algebra.

Definition 3.1. *Let* $\mathcal{E}$ *be an EQ-algebra and* $x \in E$*. Then,* x *is called a:*

(1) *seminode, if the set* $\{x \to y, y \to x\}$ *has a unique upper bound* 1*, for all* $y \in E$*;*

(2) *node, if either* $x \leq y$ *or* $y \leq x$ *for any* $y \in E$*.*

Let us denote the set of all seminodes of an EQ-algebra by $\mathcal{SN}(E)$ and the set of all nodes of an EQ-algebra by $\mathcal{ND}(E)$. Since $1 \in \mathcal{SN}(E)$ and $1 \in \mathcal{ND}(E)$, it readily follows that $\mathcal{SN}(E)$ and $\mathcal{ND}(E)$ are nonempty.

Example 3.2 ([5]). (1) Assume that $E = \{0, u, v, w, 1\}$ with $0 < u < v < w < 1$. Then, one can check that $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the two operations $\otimes$ and $\sim$ are given by:

$\otimes$	0	u	v	w	1
0	0	0	0	0	0
u	0	0	0	0	u
v	0	0	0	0	v
w	0	0	u	u	w
1	0	u	v	w	1

$\sim$	0	u	v	w	1
0	1	w	v	v	0
u	w	1	w	w	u
v	v	w	1	w	v
w	v	w	w	1	w
1	0	u	v	w	1

Obviously, $\mathcal{SN}(E) = \mathcal{ND}(E) = \{0, u, v, w, 1\}$. But the element u is not a co-atom and dense element, w is not a dense element and a atom and v is not a dense element. Moreover, the involutive elements are $\{0, v, 1\}$.

(2) Suppose that $E = \{0, u, v, p, q, 1\}$ with $0 < u < v < p, q < 1$. Then, $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the operations $\otimes$ and $\sim$ are given by the next tables:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	0	0	u	0	u
v	0	0	v	v	v	v
p	0	u	v	p	v	p
q	0	0	v	v	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	q	u	0	0	0
u	q	1	u	u	u	u
v	u	u	1	q	p	v
p	0	u	q	1	v	p
q	0	u	p	v	1	q
1	0	u	v	p	q	1

One can check that $\mathcal{SN}(E) = \{0, u, v, p, q, 1\}$ and $\mathcal{ND}(E) = \{0, u, v, 1\}$. Although p and q are not node elements, they are dense elements and co-atoms. In addition, the involutive elements are $\{0, 1\}$.

(3) Let $E = \{0, u, v, p, q, 1\}$ satisfies $0 < u, v < p < q < 1$. Then, $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra with respect to the following operations $\otimes$ and $\sim$:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	0	0	0	u	u
v	0	0	0	0	v	v
p	0	0	0	0	p	p
q	0	u	v	p	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	p	p	p	0	0
u	p	1	p	p	u	u
v	p	p	1	p	v	v
p	p	p	p	1	p	p
q	0	u	v	p	1	1
1	0	u	v	p	1	1

It is apparent that $\mathcal{SN}(E) = \{0, u, v, p, q, 1\}$ and $\mathcal{ND}(E) = \{0, p, q, 1\}$. Although u and v are atoms, they are not dense elements and nodes. Moreover, 0 and p are involutive elements, but they are not atoms and dense elements.

According to the above example, we see immediately that seminodes and nodes are different from dense elements, (co-)atoms and involutive elements in an EQ-algebra. In addition, they have the following properties:

Remark 3.3. Suppose $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra.

(1) If E is a chain, then each element of E is a node.

(2) If E has at most one node u, then $u = 1$. Therefore, it is neither an atom nor a co-atom.

(3) Each node of E is a seminode of E. But the converse is not true. In fact, by definitions of nodes and seminodes, we can easily check that each node is a seminode. Also, by Example 3.2 (3), we know that u and v are seminodes, but not nodes. Therefore, we conclude that a seminode element is more general than a node.

In general EQ-algebra, we can only obtain that $(q_1 \wedge q_2) \to q_3 \geq (q_1 \to q_3) \vee (q_2 \to q_3)$ and $q_1 \to (q_2 \wedge q_3) \leq (q_1 \to q_2) \wedge (q_1 \to q_3)$ hold. But when we define q_1, q_2, q_3 in the set $\mathcal{SN}(E)$, we shall prove the equations hold.

Proposition 3.4. *Let $\mathcal{E}$ be a lattice-ordered EQ-algebra. Then, the following hold, for all $q_1, q_2 \in \mathcal{SN}(E)$ and $q_3 \in E$:*

(1) $(q_1 \wedge q_2) \to q_3 = (q_1 \to q_3) \vee (q_2 \to q_3)$.

(2) $q_1 \to (q_2 \wedge q_3) = (q_1 \to q_2) \wedge (q_1 \to q_3)$.

Proof. (1) From the Proposition 2.3 (5) and (4), we get $q_1 \to q_2 = q_1 \to (q_1 \wedge q_2) \leq ((q_1 \wedge q_2) \to q_3) \to (q_1 \to q_3) \leq ((q_1 \wedge q_2) \to q_3) \to ((q_1 \to q_3) \vee (q_2 \to q_3))$. Similarly, we obtain that $q_2 \to q_1 \leq ((q_1 \wedge q_2) \to q_3) \to ((q_1 \to q_3) \vee (q_2 \to q_3))$. Since $q_1 \in \mathcal{SN}(E)$, it implies that $(q_1 \to q_2) \vee (q_2 \to q_1) = 1$, and so $((q_1 \wedge q_2) \to q_3) \to ((q_1 \to q_3) \vee (q_2 \to q_3)) = 1$. Thus, we obtain $((q_1 \wedge q_2) \to q_3) \leq ((q_1 \to q_3) \vee (q_2 \to q_3))$. In addition, because $q_1 \wedge q_2 \leq q_1, q_2$, we have $q_1 \to q_3$, $q_2 \to q_3 \leq (q_1 \wedge q_2) \to q_3$. Thus, it readily follows that $(q_1 \to q_3) \vee (q_2 \to q_3) \leq (q_1 \wedge q_2) \to q_3$. Therefore, we see immediately that $(q_1 \wedge q_2) \to q_3 = (q_1 \to q_3) \vee (q_2 \to q_3)$.

(2) By Proposition 2.3 (5) and (4), we obtain $q_2 \to q_3 = q_2 \to (q_2 \wedge q_3) \leq (q_1 \to q_2) \to (q_1 \to (q_2 \wedge q_3)) \leq ((q_1 \to q_2) \wedge (q_1 \to q_3)) \to (q_1 \to (q_2 \wedge q_3))$. Analogously, $q_3 \to q_2 \leq ((q_1 \to q_2) \wedge (q_1 \to q_3)) \to (q_1 \to (q_2 \wedge q_3))$ holds. Since $q_2 \in \mathcal{SN}(E)$, we obtain $(q_2 \to q_3) \vee (q_3 \to q_2) = 1$, and then $((q_1 \to q_2) \wedge (q_1 \to q_3)) \to (q_1 \to (q_2 \wedge q_3)) = 1$. Thus, it follows that $(q_1 \to q_2) \wedge (q_1 \to q_3) \leq q_1 \to (q_2 \wedge q_3)$. In addition, since $q_2 \wedge q_3 \leq q_2, q_3$, it readily implies $q_1 \to (q_2 \wedge q_3) \leq q_1 \to q_2, q_1 \to q_3$, and so $q_1 \to (q_2 \wedge q_3) \leq (q_1 \to q_2) \wedge (q_1 \to q_3)$. Therefore, it readily follows $q_1 \to (q_2 \wedge q_3) = (q_1 \to q_2) \wedge (q_1 \to q_3)$. □

Theorem 3.5. *Let $\mathcal{E}$ be a lattice-latticed EQ-algebra. Then, the following conclusions hold:*

(1) *Denote $BL(E) = \{u \in E \mid u \vee m = 1,\ u \wedge m = 0$ for some $m \in E\}$. Then, $\mathcal{ND}(E) \cap BL(E) = \{0, 1\}$.*

(2) *If $\mathcal{E}$ is distributive, then $(\mathcal{SN}(E), \wedge, \vee)$ is a distributive lattice.*

(3) *$(\mathcal{ND}(E), \vee, \wedge)$ is a distributive lattice, too.*

Proof. (1) It is clear that $\{0,1\} \subseteq \mathcal{ND}(E) \cap BL(E)$. Conversely, for any $u \in \mathcal{ND}(E) \cap BL(E)$, we have $u \in \mathcal{ND}(E)$ and $u \in BL(E)$. From $u \in \mathcal{ND}(E)$, we know that either $u \leq m$ or $m \leq u$ for any $m \in E$. Moreover, it follows from $u \in BL(E)$ that $u \vee m = 1$ and $u \wedge m = 0$ for some $m \in E$, which implies that $u \vee m = m$, $u \wedge m = u$ or $u \vee m = u$, $u \wedge m = m$. Hence, $u = 0$ or $u = 1$, and so $u \in \{0,1\}$. Therefore, we obtain $\mathcal{ND}(E) \cap BL(E) = \{0,1\}$.

(2) Firstly, we prove $((u \wedge m) \to w) \vee (w \to (u \wedge m)) = 1$ for any $w \in E$ and $u, m \in \mathcal{SN}(E)$. In fact, by Proposition 3.4, we have $((u \wedge m) \to w) \vee (w \to (u \wedge m)) = ((u \to w) \vee (m \to w)) \vee ((w \to u) \wedge (w \to m)) = [(u \to w) \vee (m \to w) \vee (w \to u)] \wedge [(u \to w) \vee (m \to w) \vee (w \to m)] \geq [(u \to w) \vee (w \to u)] \wedge [(m \to w) \vee (w \to m)] = 1$. Thus, it readily follows that $u \wedge m \in \mathcal{SN}(E)$.

Now, we shall prove that $((u \vee m) \to w) \vee (w \to (u \vee m)) = 1$ for any $w \in E$. Indeed, by Proposition 2.3 (8), we obtain $((u \vee m) \to w) \vee (w \to (u \vee m)) = ((u \to w) \wedge (m \to w)) \vee (w \to (u \vee m)) = ((u \to w) \vee (w \to (u \vee m))) \wedge ((m \to w) \vee (w \to (u \vee m))) \geq [(u \to w) \vee (w \to u)] \wedge [(m \to w) \vee (w \to m)] = 1$ Therefore, we get that $u \vee m \in \mathcal{SN}(E)$, and so $(\mathcal{SN}(E), \wedge, \vee)$ is a distributive lattice.

(3) Let $u, m \in \mathcal{ND}(E)$. It suffices to show that $u \vee m$, $u \wedge m \in \mathcal{ND}(E)$. Assume that $w \in E$. If $w \leq u, m$, then $w \leq u \wedge m$. And, if $u \leq w \leq m$ or $m \leq w \leq u$, then $u \wedge m \leq u \leq w$ or $u \wedge m \leq m \leq w$. Thus $u \wedge m \in \mathcal{ND}(E)$. Analogously, $u \vee m \in \mathcal{ND}(E)$ also holds. Therefore, $(\mathcal{ND}(E), \vee, \wedge, 0, 1)$ is a lattice. By definition of $\mathcal{ND}(E)$, we see immediately that it is a distributive lattice. □

Theorem 3.6. *Let $\mathcal{E}$ be an EQ-algebra. If for any $x, y \in \mathcal{SN}(E)$, $x \wedge (x \to y) = x \wedge y$ holds and $\mathcal{SN}(E)$ is closed with the operator $\to$. Then, $(\mathcal{SN}(E), \wedge, \vee, \to, 1)$ is a Hertz-algebra and a Heyting-algebra.*

Proof. Firstly, we prove that it is a Hertz-algebra. Obviously, (HE1) holds. By Proposition 2.3 (3), we know that (HE2) holds. By hypothesis, the (HE3) is valid. Moreover, from Proposition 3.4 (2), it implies that (HE4) holds. Hence, $(\mathcal{SN}(E), \wedge, \vee, \to, 1)$ is a Hertz-algebra.

Now, we show that it is a Heyting-algebra. For any $x, y, w \in \mathcal{SN}(E)$, if $x \leq y \to w$, then $x \wedge y \leq y \wedge (y \to w) = y \wedge w \leq w$, i.e. $x \wedge y \leq w$. Conversely, if $x \wedge y \leq w$, then it follows that $x \leq y \to x = 1 \wedge (y \to x) = (y \to y) \wedge (y \to x) = y \to (y \wedge x) \leq y \to w$ by Proposition 2.3 (3) and Proposition 3.4 (2). Therefore, the conclusion holds. □

4. Nodal filters on EQ-algebras

In this section, we introduce the notion of an nodal filter on EQ-algebras and give the equivalent characterization of it. Furthermore, the relationships between nodal filters and node elements, as well as between nodal filters and other filters are discussed.

Definition 4.1. *Let H be a filter of an EQ-algebra. If H is a node in poset $(\mathcal{F}(E), \subseteq)$, then it is said to be an nodal filter (for short, n-filter).*

Let us denote the set of all n-filters of $\mathcal{E}$ by $NF(E)$ in the sequel.

Example 4.2 ([16])**.** Let $E = \{0, u, v, p, q, 1\}$ such that $0 < u, v < p < 1$, $0 < v < q < 1$. Then, $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the operations $\otimes$ and $\sim$ are given by the following tables:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	u	0	u	0	u
v	0	0	0	0	v	v
p	0	u	0	u	v	p
q	0	0	v	v	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	q	p	v	u	0
u	q	1	v	p	0	u
v	p	v	1	q	p	v
p	v	p	q	1	v	p
q	u	0	p	v	1	q
1	0	u	v	p	q	1

It is easy for us to check that $\mathcal{F}(E) = \{\{1\}, \{q, 1\}, \{u, p, 1\}, \{u, v, p, q, 1\}, E\}$, but $\mathcal{NF}(E) = \{\{1\}, \{u, v, p, q, 1\}, E\}$.

Example 4.3 ([7])**.** Suppose that $E = \{0, u, v, p, q, 1\}$ with $0 < u < v, p < q < 1$. Then, we can verify that $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra, where the operations $\otimes$ and $\sim$ are given by the next tables:

$\otimes$	0	u	v	p	q	1
0	0	0	0	0	0	0
u	0	0	0	0	0	u
v	0	0	0	0	0	v
p	0	0	0	0	0	p
q	0	0	0	0	q	q
1	0	u	v	p	q	1

$\sim$	0	u	v	p	q	1
0	1	1	u	u	u	u
u	1	1	u	u	u	u
v	u	u	1	p	p	p
p	u	u	p	1	p	p
q	u	u	p	p	1	q
1	u	u	p	p	q	1

Obviously, $\mathcal{F}(E) = \{\{1\}, \{q, 1\}, \{v, q, 1\}, \{u, p, q, 1\}, \{u, v, p, q, 1\}, E\}$, but $NF(E) = \{\{1\}, \{q, 1\}, \{u, v, p, q, 1\}, E\}$.

From the above Examples, we see immediately that $n-$filters are distinct from filters of EQ-algebras.

Theorem 4.4. *Let H be a filter of an idempotent and good EQ-algebra. Then, H is an n-filter if and only if $u \in H$ and $v \notin H$ imply $v < u$ for any $u, v \in E$.*

Proof. $(\Rightarrow)$ Assume that $u \in H$ and $v \notin H$ for any $u, v \in E$. Then, it follows from H is an n-filter that $\langle u \rangle \subseteq H$ and $H \subseteq \langle v \rangle$, which implies $u \in \langle v \rangle$. Hence, $v^n \leq u$ for some $n \in N$. Moreover, by assumption, we get $v = v^n$. If $v = u$, then $v \in H$, which is a contradiction. Hence, it readily follows that $v < u$.

$(\Leftarrow)$ Suppose that $v < u$, for all $u \in H$ and $v \notin H$. If there exists a filter J such that H and J are incomparable. Then, $u \in H \setminus J$ and $v \in J \setminus H$ for some

$u, v \in E$. Now, since J is a filter and $v < u$, it implies that $u \in J$, which is impossible. Hence, either $H \subseteq J$ or $J \subseteq H$ for any filter J of $\mathcal{E}$. Therefore, we obtain that H is an n-filter. □

Corollary 4.5. *If $\mathcal{E}$ is linearly ordered, then each filter is an n-filter.*

Proof. For any filter H such that $u \in H$ and $v \notin H$. Since $u \in \mathcal{ND}(E)$, we get $v < u$. Indeed, if $u \leq v$, then $v \in H$ as H is a filter. Hence, by the Theorem above, we obtain that H is an n-filter. □

Proposition 4.6. *Let H be a filter of a good EQ-algebra. If $u \in H$ is a node, then H is an n-filter. Especially, the filter $\langle u \rangle$ generated by u is also an n-filter.*

Proof. Assume H is a filter of $\mathcal{E}$ and $v \notin H$. If $u \in \mathcal{ND}(E)$, then either $u \leq v$ or $v \leq u$. If $u \leq v$, then $v \in H$, which is a contradiction. Thus, it readily follows that $v < u$. By Theorem 4.4, we obtain that H is an n-filter. □

Remark 4.7. In Example 4.2, we obtain that $\{u, v, p, q, 1\}$ is an n-filter of $\mathcal{E}$, but $v \notin \mathcal{ND}(E)$, which implies that the converse of Proposition 4.6 may not hold, in general.

Proposition 4.8. *Let $\mathcal{E}$ be an idempotent and good EQ-algebra.*

(1) *If $\langle u \rangle \in NF(E)$, then $u \in \mathcal{ND}(E)$.*

(2) *If E has n node elements, then it has at least n n-filters.*

Proof. (1) For any $v \in E$, then either $v \in \langle u \rangle$ or $v \notin \langle u \rangle$. If $v \notin \langle u \rangle$, then we obtain that $v < u$ by Theorem 4.4. If $v \in \langle u \rangle$, then $u^n = u \leq v$ for some $n \in N$. Hence, u is a node element.

(2) Let $u \in \mathcal{ND}(E)$. Then, it follows that $\langle u \rangle$ is a nodal filter by Proposition 4.6. Now, assume u and v are two nodes of E. If $\langle u \rangle = \langle v \rangle$, then $u \in \langle v \rangle$ and $v \in \langle u \rangle$. Since $u^2 = u$ and $v^2 = v$, we obtain $u \geq v$ and $v \geq u$, which implies that $u = v$. Therefore, we see immediately that it has at least n n-filters. □

Combining Proposition 4.6 and Proposition 4.8, we know that there is a one-to-one correspondence between nodal principle filters and node elements in an idempotent EQ-algebra.

Proposition 4.9. *Suppose that H is an n-filter of a residuated EQ-algebra. Then, for any $u \in \mathcal{ND}(E)$, $H(u) = \langle H \cup \{u\} \rangle$ is an n-filter.*

Proof. If $u \in H$, then $H(u) = H$. Thus, it readily implies that $H(u)$ is an n-filter of $\mathcal{E}$. By the above Proposition, we obtain that $\langle u \rangle$ is an n-filter. Now, suppose that $J \in \mathcal{F}(E)$ and $J \nsubseteq H(u)$. Note that if $J \subseteq H$ or $J \subseteq \langle u \rangle$, then $J \subseteq H(u)$, which is contradiction. Hence, we get H, $\langle u \rangle \subseteq J$. If $v \in H(u)$, then $u \to_n v \in H \subseteq J$ for some $n \in N$. Thus, we know that $v \in J$ as J is a filter. Hence, $H(u) \subseteq J$, which readily follows that $H(u)$ is an n-filter. □

Example 4.10. In Example 4.2, we know that $H = \{1\}$ is an n-filter. And, one can check that $p \notin \mathcal{ND}(E)$ and $H(p) = \{u, p, 1\} \notin NF(E)$. Moreover, $J = \{q, 1\} \notin NF(E)$ but $J(v) = H(v) = E \in NF(E)$. That is to say, the converse of Proposition 4.9 may not hold, in general.

Proposition 4.11. *Assume that E_1 and E_2 are two idempotent and good EQ-algebras and $g : E_1 \to E_2$ is a homomorphism.*

(1) *If g is injective and $H \in NF(E_2)$, then $g^{-1}(H) = \{a \in E_1 \mid g(a) \in H\} \in NF(E_1)$.*

(2) *If g is surjective and $H \in NF(E_1)$, then $g(H) \in NF(E_2)$.*

Proof. (1) Firstly, we show that $g^{-1}(H)$ is a filter. Since $g(1_{E_1}) = 1_{E_2} \in H$, we get $1_{E_1} \in g^{-1}(H)$, i.e. (F1) holds. For any $a, b \in g^{-1}(H)$, it implies that $g(a), g(b) \in H$. And, because $H \in NF(E_2)$, we obtain $g(a \otimes b) = g(a) \otimes g(b) \in H$, which implies $(a \otimes b) \in g^{-1}(H)$, i.e. (F2) holds. For any $a, b \in E_1$, assume $a, a \to b \in g^{-1}(H)$. Then, $g(a), g(a \to b) \in H$, i.e. $g(a), g(a) \to g(b) \in H$. Thus, $g(b) \in H$ and so $b \in g^{-1}(H)$, i.e. (F3) holds. Let $a \to b \in g^{-1}(H)$. Then, $g(a) \to g(b) = g(a \to b) \in H$, which readily follows that $(g(a) \otimes g(c)) \to (g(b) \otimes g(c)) \in H$, where $c \in E_1$ and $g(c) \in H$, i.e. $g((a \otimes c) \to (b \otimes c)) \in H$. Hence $(a \otimes c) \to (b \otimes c) \in g^{-1}(H)$, i.e. (F4) holds. Therefore, we see immediately that $g^{-1}(H)$ is a filter.

Now, we shall prove that $g^{-1}(H)$ is an n-filter. Let $a \in g^{-1}(H)$ and $b \notin g^{-1}(H)$. Then, $g(a) \in H$ and $g(b) \notin H$. Since H is an n-filter and $a^2 = a$ holds, for all $a \in E_1$, we have $g(b) < g(a)$ by Theorem 4.4, which implies that $g(b \to a) = g(b) \to g(a) = 1_{E_2}$. Moreover, since $g(1_{E_1}) = 1_{E_2}$ and g is injective, we obtain that $b \to a = 1_{E_1}$ and so $b \leq a$. If $b = a$, then $g(b) = g(a)$, which generates a contradiction, and so $b < a$. Now, by Theorem 4.4, we see immediately that $g^{-1}(H)$ is an n-filter.

(2) Analogously, we show that $g(H)$ is a filter firstly. Since $1_{E_2} = g(1_{E_1}) \in g(H)$, it implies that (F1) holds. Let $a, b \in g(H)$. Since g is surjective, there exist $a_1, b_1 \in H$ such that $g(a_1) = a$, $g(b_1) = b$. Hence $a \otimes b = g(a_1) \otimes g(b_1) = g(a_1 \otimes b_1) \in g(H)$, i.e. (F2) holds. Now, let a, $a \to b \in g(H)$, i.e. $g(a_1)$, $g(a_1) \to g(b_1) = g(a_1 \to b_1) \in g(H)$. Thus, we get a_1, $a_1 \to b_1 \in H$, and so $b_1 \in H$. Hence, we obtain that $b = g(b_1) \in g(H)$, i.e. (F3) holds. Moreover, let $a \to b \in g(H)$. Then, $g(a_1) \to g(b_1) = g(a_1 \to b_1) \in g(H)$, i.e. $a_1 \to b_1 \in H$. Hence, $(a_1 \otimes c_1) \to (b_1 \otimes c_1) \in H$, where $c_1 \in E_1$, and so $(g(a_1) \otimes g(c_1)) \to (g(b_1) \otimes g(c_1)) = g((a_1 \otimes c_1) \to (b_1 \otimes c_1)) \in H$, i.e. (F4) holds. Therefore, we see immediately that $g(H)$ is a filter.

Now, we prove $g(H)$ is an n-filter. Let $a \in g(H)$ and $b \notin g(H)$. Since g is surjective, there exists $a_1 \in H$ such that $g(a_1) = a$. But there is no $b_1 \in H$ such that $g(b_1) = b$. Moreover, because $b_1 \notin H$, then we get $b_1 < a_1$ and so $b_1 \to a_1 = 1$. Thus, it implies $g(b_1) \to g(a_1) = 1$, i.e. $g(b_1) \leq g(a_1)$. If $g(b_1) = g(a_1)$, i.e. $a = b$, which is a contradiction. Hence, $g(b_1) < g(a_1)$, i.e.

$b < a$. Therefore, we see immediately that $g(H)$ is an n-filter by Theorem 4.4. □

In what follows, we will prove the relationships among n-filters, (positive) implicative filters, prime filters and obstinate filters, in genaral. Futhermore, we discuss the relationships among them.

Definition 4.12 ([10]). *Let H be a proper filter of an EQ-algebra. Then, H is called prime if $x \to y \in H$ or $y \to x \in H$ for any $x, y \in E$.*

Example 4.13. (1) In Example 4.2, we obtain that $H_1 = \{1\}$ is an n-filter. Now, since $p \vee q = 1 \in \{1\}$, but $p, q \notin \{1\}$, we obtain that it is not a prime filter. Moreover, $H_2 = \{u, p, 1\} \notin NF(E)$, but it is a implicative filter and a prime filter. Furthermore, $H_3 = \{q, 1\}$ is a obstinate filter, but $H_3 \notin NF(E)$.

(2) In Example 4.3, although $H_3 = \{1\} \in NF(E)$, it is not a positive implicative filter as $p \to (1 \to v) = 1 \in \{1\}$, $p \to 1 = 1 \in \{1\}$, but $p \to v = u \in \{1\}$. Also, $H_2 = \{u, p, q, 1\}$ is a positive implicative and obstinate filter, but it is not an n-filter.

Lemma 4.14. *Let H be a filter of a prelinear and lattice-orderd EQ-algebra. Then, H is a prime filter iff for any $x, y \in E$, $x \vee y \in H$ implies $x \in H$ or $y \in H$.*

Proof. (⇒) Let $x \to y \in H$ and $x \vee y \in H$. Since $(x \vee y) \leq (x \to y) \to y$, we have $(x \to y) \to y \in H$, and so $y \in H$. As to another case, we can immediately obtain that $x \in H$.

(⇐) Let $x, y \in E$. Since $(x \to y) \vee (y \to x) = 1 \in H$, we have $x \to y \in H$ or $y \to x \in H$ by assumption. Therefore, it readily follows that H is a prime filter. □

Proposition 4.15. *Each non principal n-filter H is a prime filter of a prelinear EQ-algebra.*

Proof. Suppose there are $x, y \in E$ satisfying $x \vee y \in H$ but $x \notin H$, $y \notin H$. Then, we know that $\langle x \vee y \rangle \subseteq H$, $\langle x \rangle \nsubseteq H$ and $\langle y \rangle \nsubseteq H$. And, by the fact that H is a nodal filter, it follows that $H \subseteq \langle x \rangle$ and $H \subseteq \langle y \rangle$. Thus, by Theorem 2.6 (3), we obtain $H \subseteq \langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$. For this reason, we get that $H = \langle x \vee y \rangle$, which is a contradiction. Hence, we obtain that $x \in H$ or $y \in H$, and so H is a prime filter. □

Proposition 4.16. *Let H be an obstinate filter of a bounded EQ-algebra. If $(x \otimes y') \leq y$ for any $x, y \in E$, then H is an n-filter.*

Proof. Assume H is not an n-filter. Then, we get $J \nsubseteq H$ and $H \nsubseteq J$ for some $J \in \mathcal{F}(E)$. Thus, there are $u, v \in E$ such that $u \in H/J$ and $v \in J/H$. It follows from H is an obstinate filter that $v' = v \to 0 \in H$, and so $u \otimes v' \in H$. Moreover, since $(u \otimes v') \leq v$, we get $v \in H$, which generates a contradiction. Hence, we see immediately that H is an n-filter. □

Proposition 4.17. *Suppose H is an implicative filter of a good EQ-algebra. If d is a dense element for any $d \in E$, then H is an n-filter.*

Proof. Suppose H is not an n-filter. Firstly, we show that $d'' \to d \in H$ for any $d \in E$. Since $d' \to 0 \leq d' \to d$, we have $d'' \to (d' \to d) = 1 \in H$. And, because $d \leq d'' \to d$, we get $d' \to d \leq (d'' \to d)' \to d$. Thus $d'' \to (d' \to d) \leq d'' \to [(d'' \to d)' \to d] \in H$, which implies that $1 \to [(d'' \to d)' \to (d'' \to d)] = (d'' \to d)' \to (d'' \to d) = d'' \to [(d'' \to d)' \to d] \in H$. By definition of an implicative filter, we know that $d'' \to d \in H$.

Assume H is not an n-filter of $\mathcal{E}$. Then, $J \nsubseteq H$ and $H \nsubseteq J$ for some $J \in \mathcal{F}(E)$. Thus, $v \in J/H$ for some $v \in E$. By the conclusion above, we obtain that $v'' \to v \in H$. Since v is a dense element, we have $v'' \to v = v \in H$, which generates a contradiction. Hence, we see immediately that H is an n-filter. □

Proposition 4.18. *Assume H is a positive implicative filter of a residuated EQ-algebra. If $y \to (x \odot y) = x \to y$ holds for any $x, y \in E$, then H is an n-filter.*

Proof. Assume that H is not an n-filter. Firstly, we shall prove that for any $x \in E$, $x \to x^2 \in H$. Since $x \to (x \to x^2) = x^2 \to x^2 = 1 \in H$ and $x \to x = 1 \in H$. Then, by definition of a positive implicative filter, we get $x \to x^2 \in H$. If H is not an n-filter, then there is $J \in \mathcal{F}(E)$ satisfying $J \nsubseteq H$ and $H \nsubseteq J$. Moreover, assume $x \in H/J$ and $y \in J/H$. By the conclusion above, it follows that $y \to y^2 \in H$. Then, $x \otimes (y \to y^2) \in H$. And, because $x \otimes (y \to y^2) \leq y \to (x \otimes y^2) \leq y \to (x \otimes y) = x \to y$, we have $x \to y \in H$, and so $y \in H$, which is a contradiction. Hence, we obtain that H is an n-filter. □

Proposition 4.19. *Let H be a non principal n-filter of an EQ-algebra $\mathcal{E}$. Then, $(E/H, \wedge, \odot, \sim_H, 1)$ is linearly ordered.*

Proof. Let x/H, $y/H \in E/H$ and $x/H \nleq y/H$. Then, we can obtain that $x \to y \notin H$. Moreover, because H is a non principal n-filter, then from Theorem 4.15 that we get H is a prime filter. Hence, it readily follows that $y \to x \in H$, and so $[y] \leq [x]$. Thus, we see immediately that E/H is a chain. □

Lemma 4.20 ([9])**.** *Assume θ is a congruence relation on a separated EQ-algebra. Then, $F = [1]_\theta = \{a \in E \mid a\theta 1\}$ is a filter.*

Theorem 4.21. *Assume $\mathcal{E}$ is an EQ-algebra. Then, $[1]_\theta$ is an n-filter iff θ is a node of $Con(E)$, where $Con(E)$ denotes the set of all congruence relation of E.*

Proof. Note that the mapping $\theta \mapsto F_\theta$ of $Con(E)$ on to $NF(E)$ is an isomorphism and F_θ is an n-filter iff it is a node of $NF(E)$. □

5. The structures of the set of all nodal filters on EQ-algebras

In this section, we study the algebraic properties $NF(E)$ and topological properties of $NP(E)$ on EQ-algebras.

Let $O, J \in NF(E)$. Define five operations as follows:

$$O \sqcap J := O \cap J, O \sqcup J := \langle O \cup J \rangle, O \to J := \{a \in E \mid O \cap \langle a \rangle \subseteq J\},$$
$$O \otimes J := \{o \otimes j \mid o \in O, j \in J\}, O' := O \to \{1\}.$$

Proposition 5.1. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $O, J \in NF(E)$, the following properties hold:*

(1) $O \sqcap J$, $O \sqcup J \in NF(E)$.

(2) $O \to J \in NF(E)$.

(3) $O \otimes J \in NF(E)$ *and* $O \otimes J = O \cup J$.

Proof. (1) For any $K \in \mathcal{F}(E)$. If $O, J \subseteq K$, then $O \sqcup J = \langle O \cup J \rangle \subseteq K$. And, if $K \subseteq O, J$, we have $K \subseteq O \subseteq \langle O \cup J \rangle = O \sqcup J$. Now, if $O \subseteq K \subseteq J$ or $J \subseteq K \subseteq O$, we obtain that $K \subseteq \langle O \cup J \rangle = O \sqcup J$. Thus, it readily follows that $O \sqcup J \in NF(E)$. Analogously, we can prove that $O \sqcap J \in NF(E)$ hold.

(2) If $O = J$, we can get that $O \to J = E \in NF(E)$. Now, if $O \neq J$. Suppose that $O \subseteq J$. Then, $O \cap \langle a \rangle \subseteq O \subseteq J$ for any $a \in E$, which implies that $O \to J = E$. If $J \subseteq O$, we shall prove that $O \to J = J$. In fact, for any $a \in O \to J$, if $a \in J$, then $O \to J \subseteq J$. And, if $a \notin J$ and $a \in O$, we get $\langle a \rangle \subseteq O$. Thus, $\langle a \rangle = O \cap \langle a \rangle \subseteq J$, which is a contradiction. Suppose that $a \notin J$ and $a \notin O$. Then, we have $O \subseteq \langle a \rangle$, which means $O = O \cap \langle a \rangle \subseteq J$. Moreover, because $J \subseteq O$, we get that $O = J$, which is a contradiction. Hence, $O \to J \subseteq J$. Conversely, for any $a \in J$, we can easily get $\langle a \rangle \subseteq J$, which implies $O \cap \langle a \rangle \subseteq \langle a \rangle \subseteq J$, that is $a \in O \to J$. Hence $J \subseteq O \to J$, and so $O \to J = J$.

(3) If $O \subseteq J$, then $O \otimes J = \{o \otimes j \mid o \in O, j \in J\} = J \in NF(E)$. Similarly, if $J \subseteq O$, then $O \otimes J = O \in NF(E)$. In any cases, $O \otimes J = O$ or J holds. Thus, we see immediately that $O \otimes J = O \cup J$. □

Remark 5.2. In particular, we know that $H' := H \to \{1\} \in NF(E)$ for any $H \in NF(E)$.

Proposition 5.3. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $O, J, K \in NF(E)$, the following properties hold:*

(1) $E \to O = O$, $O \to O = E$, $O \to E = E$, $\{1\} \to O = E$.

(2) $O' = \{1\}$, $O'' = E$, *for* $O \neq \{1\}$.

(3) $O \to J' = J \to O'$ *for* $O, J \neq \{1\}$.

(4) $O \subseteq J$ *implies* $J \to K \subseteq O \to K$, $K \to O \subseteq K \to J$.

(5) $O \subseteq J$ *iff* $O \to J = E$.

(6) $O \subseteq J \to O$ *and* $O, J \subseteq O \otimes (O \to J)$.

(7) $O \otimes (J \otimes K) = (O \otimes J) \otimes K$.

Proof. (1) By definition, we have $E \to O = \{a \in E \mid E \cap \langle a\rangle \subseteq O\} = \{a \in E \mid \langle a\rangle \subseteq O\} = O$. Similarly, we can prove other equations hold.

(2) By definition, it readily implies $O' = O \to \{1\} = \{a \in E \mid O \cap \langle a\rangle \subseteq \{1\}\}$. Now, let $a \in O'$ and $a \neq 1$. If $a \in O$, then $O \cap \langle a\rangle = \langle a\rangle \nsubseteq \{1\}$, which is a contradiction. Thus $a = 1$, and so $O' = O \to \{1\} = \{1\}$. Furthermore, by (1), we see immediately that $O'' = O' \to \{1\} = \{1\} \to \{1\} = E$.

(3) By (2), we get that $O' = J' = \{1\}$. Then, $O \to J' = O \to \{1\} = O' = \{1\}$. Similarly, we can obtain $J \to O' = \{1\}$. Hence, we obtain that $O \to J' = J \to O'$.

(4) For any $a \in J \to K$, we get $J \cap \langle a\rangle \subseteq K$. And, since $O \subseteq J$, it readily follows that $O \cap \langle a\rangle \subseteq J \cap \langle a\rangle \subseteq K$. Thus $a \in O \to K$. That is $J \to K \subseteq O \to K$. Analogously, we can obtain that $K \to O \subseteq K \to J$.

(5) By definition, we know that $O \subseteq J$ iff $\langle a\rangle \cap O \subseteq J$ holds for any $a \in E$ iff $O \to J = E$.

(6) By the proof of Proposition 5.1, we obtain that if $J \subseteq O$, then $O \otimes (O \to J) = O \otimes J = O$ and $J \to O = E$. And, if $O \subseteq J$, then $O \otimes (O \to J) = E$ and $J \to O = J$. Therefore, in any case, we have $O \subseteq J \to O$ and $O, J \subseteq O \otimes (O \to J)$.

(7) The proof is clear. □

Proposition 5.4. *Let $\mathcal{E}$ be an EQ-algebra. Then, $(NF(E), \sqcup, \sqcap)$ is a bounded distributive lattice.*

Proof. By Proposition 5.1 (1), we know that $(NF(E), \sqcup, \sqcap)$ is a lattice. Next we shall show that $O \cap \langle J \cup K\rangle = \langle\langle O \cap J\rangle \cup \langle O \cap K\rangle\rangle$ holds for any $O, J, K \in NF(E)$. Let us consider the following six cases:

Case 1. Assume $O \subseteq J \subseteq K$. Then, $O \cap \langle J \cup K\rangle = O \cap K = O = \langle O \cup O\rangle = \langle\langle O \cap J\rangle \cup \langle O \cap K\rangle\rangle$.

Case 2. Assume $O \subseteq K \subseteq J$. Then, $O \cap \langle J \cup K\rangle = O \cap J = O = \langle O \cup O\rangle = \langle\langle O \cap J\rangle \cup \langle O \cap K\rangle\rangle$.

Case 3. Assume $K \subseteq O \subseteq J$. Then, $O \cap \langle J \cup K\rangle = O \cap J = O = \langle O \cup K\rangle = \langle\langle O \cap J\rangle \cup \langle O \cap K\rangle\rangle$.

Case 4. Assume $K \subseteq J \subseteq O$. Then, $O \cap \langle J \cup K\rangle = O \cap J = J = \langle J \cup K\rangle = \langle\langle O \cap J\rangle \cup \langle O \cap K\rangle\rangle$.

Case 5. Assume $J \subseteq K \subseteq O$. Then, $O \cap \langle J \cup K\rangle = O \cap K = K = \langle J \cup K\rangle = \langle\langle O \cap J\rangle \cup \langle O \cap K\rangle\rangle$.

Case 6. Assume $J \subseteq O \subseteq K$. Then, $O \cap \langle J \cup K\rangle = O \cap K = O = \langle J \cup O\rangle = \langle\langle O \cap J\rangle \cup \langle O \cap K\rangle\rangle$.

Hence, we obtain that $(NF(E), \sqcup, \sqcap)$ is a bounded distributive lattice. □

Theorem 5.5. *Assume that $\mathcal{E}$ is an EQ-algebra. Then, $(NF(E), \sqcap, \to, E)$ is a Hertz-algebra.*

Proof. It is apparent that (HE1) is valid. By Proposition 5.3 (6), we know that (HE2) holds. For (HE3), if $O \subseteq J$, then $O \sqcap (O \to J) = O \sqcap E = O = O \sqcap J$. And, if $J \subseteq O$, then $O \sqcap (O \to J) = O \sqcap J$. Hence, it implies that (HE3) holds. Now, we prove that (HE4) is valid and we consider the following scenarios:

Case 1. Suppose that $O \subseteq J \subseteq K$. Then, $O \to (J \sqcap K) = O \to J = E = E \sqcap E = (O \to J) \sqcap (O \to K)$.

Case 2. If $O \subseteq K \subseteq J$, it follows that $O \to (J \sqcap K) = O \to K = E = E \sqcap E = (O \to J) \sqcap (O \to K)$.

Case 3. If $J \subseteq O \subseteq K$, we conclude that $O \to (J \sqcap K) = O \to J = J = J \sqcap E = (O \to J) \sqcap (O \to K)$.

Case 4. Suppose $J \subseteq K \subseteq O$, we obtain that $O \to (J \sqcap K) = O \to J = J = J \sqcap K = (O \to J) \sqcap (O \to K)$.

Case 5. If $K \subseteq O \subseteq J$, it implies that $O \to (J \sqcap K) = O \to K = K = E \sqcap K = (O \to J) \sqcap (O \to K)$.

Case 6. If $K \subseteq J \subseteq O$, we have $O \to (J \sqcap K) = O \to K = K = J \sqcap K = (O \to J) \sqcap (O \to K)$.

Hence, $(HE4)$ holds. Therefore, we obtain that $(NF(E), \sqcap, \to, E)$ is a Hertz-algebra. □

Theorem 5.6. *Let $\mathcal{E}$ be an EQ-algebra. Then, the following properties hold:*

(1) $(NF(E), \otimes, \{1\})$ *is a commutative monoid.*

(2) $(NF(E), \to, E)$ *is a Hilbert algebra.*

(3) $(NF(E), \sqcup, \sqcap, \to, E)$ *is a Heyting algebra.*

(4) $(NF(E), \to, E)$ *is a BCK-algebra.*

Proof. (1) If $O \subseteq J$, then $O \otimes J = J = J \otimes O$. And, if $J \subseteq O$, we get $O \otimes J = O = J \otimes H$. Moreover, because $O \otimes \{1\} = O = \{1\} \otimes O$, we see immediately that $(NF(E), \otimes, \{1\})$ is a commutative monoid.

(2) Firstly, we show that (HL1) is valid. If $O \subseteq J$, then we obtain $O \to (J \to O) = O \to O = E$ by Proposition 5.1 and Proposition 5.3 (1). Similarly, if $J \subseteq O$, it follows that $O \to (J \to O) = O \to E = E$. Hence, we conclude that (HL1) holds.

Next, we shall prove that (HL2). If $O \subseteq J \subseteq K$, then $[O \to (J \to K)] \to [(O \to J) \to (O \to K)] = (O \to E) \to (E \to E) = E \to E = E$. And, if $O \subseteq K \subseteq J$, then $[O \to (J \to K)] \to [(O \to J) \to (O \to K)] = (O \to K) \to (E \to E) = E$. Moreover, if $K \subseteq O \subseteq J$ or $K \subseteq J \subseteq O$ or $J \subseteq K \subseteq O$ or $J \subseteq O \subseteq K$, we can prove it in a similar way. Thus, we obtain that (HL2) holds.

Finally, by Proposition 5.3 (5), we can easily check that (HL3) holds. Therefore, $(NF(E), \to, E)$ is a Hilbert algebra.

(3) By Proposition 5.4, we know that $(NF(E), \sqcup, \sqcap)$ is a bounded distributive lattice. Now, for any $O, J, K \in NF(E)$, we shall prove that $O \cap K \subseteq J$ iff $K \subseteq O \to J$. Let us take the following six cases into account:

Case 1. If $O \subseteq J \subseteq K$, then $O \cap K = O \subseteq J$ iff $K \subseteq E = O \to J$.

Case 2. If $O \subseteq K \subseteq J$, then $O \cap K = O \subseteq J$ iff $K \subseteq E = O \to J$.

Case 3. If $K \subseteq O \subseteq J$, then $O \cap K = K \subseteq J$ iff $K \subseteq E = O \to J$.

Case 4. If $K \subseteq J \subseteq O$, then $O \cap K = K \subseteq J$ iff $K \subseteq E = O \to J$.

Case 5. If $J \subseteq O \subseteq K$, then $O \cap K = O \nsubseteq J$ iff $K \nsubseteq J = O \to J$.

Case 6. If $J \subseteq K \subseteq O$, then $O \cap K = K \nsubseteq J$ iff $K \nsubseteq J = O \to J$.

Hence, we obtain that $(NF(E), \sqcup, \sqcap, \to, E)$ is a Heyting algebra.

(4) Firstly, we show that (B1) holds. Let us consider the following six scenarios:

Case 1. Assume $O \subseteq J \subseteq K$. Then, $(J \to K) \to [(K \to O) \to (J \to O)] = E \to (O \to O) = E \to E = E$.

Case 2. If $O \subseteq K \subseteq J$, then $(J \to K) \to [(K \to O) \to (J \to O)] = K \to (O \to O) = K \to E = E$.

Case 3. If $K \subseteq O \subseteq J$, then $(J \to K) \to [(K \to O) \to (J \to O)] = K \to (E \to O) = K \to O = E$.

Case 4. Suppose $K \subseteq J \subseteq O$, then $(J \to K) \to [(K \to O) \to (J \to O)] = K \to (E \to E) = K \to E = E$.

Case 5. If $J \subseteq K \subseteq O$, then $(J \to K) \to [(K \to O) \to (J \to O)] = E \to (E \to E) = E$.

Case 6. If $J \subseteq O \subseteq K$, then $(J \to K) \to [(K \to O) \to (J \to O)] = E \to (O \to E) = E$.

Hence, we obtain that $(B1)$ holds.

As for (B2), if $O \subseteq J$, then it implies that $J \to ((J \to O) \to O) = J \to (O \to O) = J \to E = E$ by Proposition 5.3 (1). Similarly, if $J \subseteq O$, we can get that $J \to ((J \to O) \to O) = J \to (E \to O) = J \to O = E$. Hence, we conclude that (B2) holds. Moreover, from Proposition 5.3 (1) and (5), we can easily check that (B3), (B4) and (B5) hold. Therefore, we obtain that $(NF(E), \to, E)$ is a BCK-algebra. □

Theorem 5.7. *Suppose that $\mathcal{E}$ is an EQ-algebra. If for any $\{1\} \neq O, J \in NF(E)$, $O \cap J \neq \{1\}$, then $(NF(E), \sqcup, \sqcap, ', \{1\}, E)$ is a semi-De Morgan algebra.*

Proof. Similar to above, it follows that it is a bounded distributive lattice by Theorem 5.4. Now, for any $O, J \in NF(E)$, we shall show that $(O \sqcup J)' = O' \sqcap J'$, $(O \sqcap J)'' = O'' \sqcap J''$ and $O' = O'''$. If $O = J = \{1\}$, since $O' = E$ and $J' = E$, we get $(O \sqcup J)' = \langle O \cup J \rangle \to \{1\} = \{1\} \to \{1\} = E = E \cap E = O' \cap J' = O' \sqcap J'$,

$(O \sqcap J)'' = ((O \sqcap J)')' = E' = \{1\} = \{1\} \cap \{1\} = O'' \cap J'' = O'' \sqcap J''$ and $O''' = E = O'$. Now, assume $O = \{1\}$ and $J \neq \{1\}$. Because $O' = E$, it follows that $(O \sqcup J)' = \langle O \cup J \rangle \to \{1\} = J \to \{1\} = J' = J' \cap E = J' \cap O' = J' \sqcap O'$ and $(O \sqcap J)'' = O'' = \{1\} = \{1\} \cap J'' = O'' \cap J'' = O'' \sqcap J''$ and $O' = E = O'''$ by Proposition 5.3 (2). Finally, assume $O \neq \{1\}$ and $J \neq \{1\}$. Since $O' = J' = \{1\}$, we obtain $(O \sqcup J)' = \langle O \cup J \rangle \to \{1\} = \{1\} = \{1\} \cap \{1\} = O' \cap J' = O' \sqcap J'$, $(O \sqcap J)'' = ((O \sqcap J) \to \{1\})' = \{1\}' = E = E \cap E = O'' \cap J'' = O'' \sqcap J''$ and $O''' = \{1\}'' = E' = \{1\} = O'$. Hence, the conclusion holds. □

In the following, some topological properties of $NF(E)$ will be stated and proved. By Proposition 4.15, we know that each non principal nodal filter is prime. Let us call this kind of filter nodal prime filter and denote the set of all nodal prime filters by $NP(E)$.

Proposition 5.8. *Suppose H is a prime filter of an EQ-algebra.*

(1) *If H_1 is a proper filter with $H \subseteq H_1$, then H_1 is a prime filter.*

(2) *If $\{H_i \mid i \in I\} \subseteq \mathcal{F}(E)$ satisfying $H \subseteq \bigcap_{i\in I} H_i$, then $\{H_i \mid i \in I\}$ is a chain.*

Proof. (1) It follows from H is a prime that either $a \to b \in H \subseteq H_1$ or $b \to a \in H \subseteq H_1$ for any $a, b \in E$. Thus, we obtain that H_1 is a prime filter.

(2) Let $H_1, H_2 \in \{H_i \mid i \in I\}$. When $H_1 = E$ or $H_2 = E$, the proof is obvious. Now, let $H_1 \neq E$, $H_2 \neq E$ and $H_1 \nsubseteq H_2$, $H_2 \nsubseteq H_1$. Then, $u \in H_1 \setminus H_2$ and $v \in H_2 \setminus H_1$ for some $u, v \in E$. Since $H \subseteq \bigcap_{i\in I} H_i \subseteq H_1 \cap H_2$, we know that $H_1 \cap H_2$ is prime. Moreover, since $\langle u \rangle \in H_1$ and $\langle v \rangle \in H_2$, it follows that $\langle u \rangle \cap \langle v \rangle \subseteq H_1 \cap H_2$, and so $u \in \langle u \rangle \subseteq H_1 \cap H_2$ or $v \in \langle v \rangle \subseteq H_1 \cap H_2$, which generates a contradiction. Therefore, $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$, it turns out that $\{H_i \mid i \in I\}$ is a chain. □

Theorem 5.9. *Let H be a filter of an EQ-algebra and $\emptyset \neq I \subseteq E$ with $I \cap H = \emptyset$. Then, there is a prime filter J satisfying $H \subseteq J$ and $I \cap J = \emptyset$.*

Proof. Denote $\Gamma = \{K \in \mathcal{N}(F) \mid H \subseteq K \text{ and } I \cap K = \emptyset\}$. It follows from $H \in \Gamma$ that Γ is non-empty. Assume $\{K_i \mid i \in I\} \subseteq \Gamma$ is a chain. Then, $J = \bigcup_{i\in I} K_i$ is a maximal element in Γ by Zorn's Lemma, and so we shall show that J is a filter. Obviously, $1 \in J$. For any $u \in J$ and $u \leq v$, we get $u \in K_{i_1}$ for some $i_1 \in I$. And, since K_{i_1} is a filter, we obtain that $v \in K_{i_1} \subseteq J$. Suppose that $x, y \in J$. Then, there are $i, j \in I$ such that $x \in K_i, y \in K_j$. If $K_i \subseteq K_j$, then we get $x \otimes y \in K_i \subseteq J$. Otherwise, we obtain that $x \otimes y \in K_j \subseteq J$. Now, for any $u \to v \in J$, there exists $i_2 \in I$ such that $u \to v \in K_{i_2}$. Thus, it follows from K_{i_2} is a filter that $u \odot w \to v \odot w \in K_{i_2} \subseteq J$ for any $w \in E$. Hence, we obtain that J is a filter. By Proposition 5.8, we know that J is a prime filter. Therefore, we see immediately that J is what we want. □

Corollary 5.10. *Let H be a filter of an EQ-algebra and $x \notin H$. Then, there is a prime filter J satisfying $H \subseteq J$ and $x \notin J$.*

For any $A \subseteq E$, denote $T(A) = \{H \in NP(E) \mid A \nsubseteq H\}$. Next, we will present the properties of $T(A)$ and the topology space induced by it.

Proposition 5.11. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $M, N \subseteq E$, the following properties hold:*

(1) *If $M \subseteq N$, then $T(M) \subseteq T(N)$.*

(2) *$T(\{0\}) = NP(E)$, $T(\emptyset) = \emptyset$.*

(3) *If $\langle M\rangle = E$, then $T(M) = NP(E)$.*

(4) *$T(M) = T(\langle M\rangle)$.*

(5) *$T(M) = T(N)$ iff $\langle M\rangle = \langle N\rangle$.*

(6) *$T(M) \cap T(N) = T(\langle M\rangle \cap \langle N\rangle)$.*

(7) *Let $\{M_i \mid i \in I\} \subseteq E$. Then, $T(\bigcup_{i\in I} M_i) = \bigcup_{i\in I} T(M_i)$.*

Proof. (1) For any $H \in T(M)$, we get $M \nsubseteq H$. And, by assumption, it follows that $N \nsubseteq H$, which means $H \in T(N)$. Thus, we obtain that $T(M) \subseteq T(N)$.

(2) Let $H \in NP(E)$. Since H is a prime filter, it implies that H is proper, which means $0 \in H$, that is $\{0\} \subseteq H$. Thus, we obtain that $H \in T(\{0\})$, and it readily follows that $T(\{0\}) = NP(E)$. Obviously, $T(\emptyset) = \emptyset$ holds.

(3) If $\langle M\rangle = E$, we know that E is the smallest filter containing M by definition. Then, for any $H \in NP(E)$, it readily follows that $M \nsubseteq H$. Thus $H \in T(M)$ holds, and then $NP(E) \subseteq T(M)$. Hence, we obtain that $T(M) = NP(E)$.

(4) Since $M \subseteq \langle M\rangle$, we get $T(M) \subseteq T(\langle M\rangle)$ by (1). Conversely, let $H \in T(\langle M\rangle)$. Then, $\langle M\rangle \nsubseteq H$. If $M \subseteq H$, it follows from the definition of $\langle M\rangle$ that $\langle M\rangle \subseteq H$, which generates a contradiction. Hence, $M \nsubseteq H$, and so $H \in T(M)$. Therefore, we see immediately that $T(M) = T(\langle M\rangle)$.

(5) Assume $\langle M\rangle = \langle N\rangle$. Then, we get $T(\langle M\rangle) = T(\langle N\rangle)$, and so $T(M) = T(N)$ by (4). Conversely, let $T(M) = T(N)$. If $\langle M\rangle \neq \langle N\rangle$, then we obtain that there is a prime filter H satisfying $\langle M\rangle \subseteq H$ and $\langle N\rangle \nsubseteq H$ by Proposition 5.9. Thus, $H \notin T(M)$ and $H \in T(N)$, which contradict to $T(M) = T(N)$. Therefore, $\langle M\rangle = \langle N\rangle$ holds.

(6) By (4), it suffices to show that $T(\langle M\rangle) \cap T(\langle N\rangle) = T(\langle M\rangle \cap \langle N\rangle)$. Obviously, $\langle M\rangle \cap \langle N\rangle \subseteq \langle M\rangle, \langle N\rangle$, which implies that $T(\langle M\rangle \cap \langle N\rangle) \subseteq T(\langle M\rangle)$, $T(\langle N\rangle)$, and so $T(\langle M\rangle \cap \langle N\rangle) \subseteq T(\langle M\rangle) \cap T(\langle N\rangle)$. Conversely, for any $H \in T(\langle M\rangle) \cap T(\langle N\rangle)$, we obtain that $\langle M\rangle \nsubseteq H$ and $\langle N\rangle \nsubseteq H$. Hence, there are $a \in \langle M\rangle$ and $b \in \langle N\rangle$ satisfying $a \notin H$ and $b \notin H$. Now, we show that $\langle M\rangle \cap \langle N\rangle \nsubseteq H$. Otherwise, it follows from $a \vee b \in \langle M\rangle \cap \langle N\rangle$ that $a \vee b \in H$.

By the fact that H is prime, we obtain that $a \in H$ or $b \in H$, which generates a contradiction. Hence, it follows that $\langle M\rangle \cap \langle N\rangle \nsubseteq H$, and so $H \in T(\langle M\rangle \cap \langle N\rangle)$.

(7) Since $M_i \subseteq \bigcup_{i\in I} M_i$ for any $i \in I$, we get $T(M_i) \subseteq T(\bigcup_{i\in I} M_i)$ for any $i \in I$, that is $\bigcup_{i\in I} T(M_i) \subseteq T(\bigcup_{i\in I} M_i)$. Conversely, assume $H \in T(\bigcup_{i\in I} M_i)$, we have $\bigcup_{i\in I} M_i \nsubseteq H$ by definition. Hence, there is M_{i_1} satisfying $H \in T(M_{i_1})$, and so $M_{i_1} \nsubseteq H$. It follows that $\bigcup_{i\in I} M_{i_1} \nsubseteq H$ and $H \in T(\bigcup_{i\in I} M_i)$. Hence, we obtain that $T(\bigcup_{i\in I} M_i) = \bigcup_{i\in I} T(M_i)$. □

Proposition 5.12. *Let H, J be two filters of an EQ-algebra. Then, the equations $T(H \sqcup J) = T(H) \cup T(J)$ and $T(H \cap J) = T(H) \cap T(J)$ hold.*

Proof. Let $K \in T(H) \cup T(J)$. Then, $H \nsubseteq K$ or $J \nsubseteq K$. Now, because $H, J \subseteq H \sqcup J$, we get $H \sqcup J \nsubseteq K$, that is $K \in T(H \sqcup J)$. Conversely, for any $K \in T(H \sqcup J)$, it readily implies that $H \sqcup J \nsubseteq K$. Assume that $H \subseteq K$ and $J \subseteq K$. Then, $H \sqcup J \subseteq K$, which is a contradiction. Thus, we get $H \nsubseteq K$ or $J \nsubseteq K$, it follows that $K \in T(H)$ or $K \in T(J)$, that is $K \in T(H) \cup T(J)$. Hence, $T(H \sqcup J) = T(H) \cup T(J)$ holds.

Now, we prove that $T(H \cap J) = T(H) \cap T(J)$ holds. Obviously, $T(H \cap J) \subseteq T(H) \cap T(J)$ is valid. Conversely, for any $K \in T(H) \cap T(J)$, it implies that $H \nsubseteq K$ and $J \nsubseteq K$. Thus, $u \in H$ and $u \notin K$ for some $u \in E$. If $K \nsubseteq T(H \cap J)$, we get $H \cap J \subseteq K$, and then $u \vee v \in H \cap J \subseteq K$ for some $v \in J$. Moreover, since K is prime and $u \notin K$, it follows that $v \in K$, and so $J \subseteq K$, which is a contradiction. Hence, $K \in T(H \cap J)$, which implies $T(H \cap J) = T(H) \cap T(J)$. □

Especially, if $A = \{u\}$, then we denote $T(u) = \{H \in NP(E) \mid u \notin H\}$. Analogously, we have the following properties:

Proposition 5.13. *Assume $\mathcal{E}$ is an EQ-algebra. Then, for any $x, y \subseteq E$, the following properties hold:*

(1) *If $x \leq y$, then $T(y) \leq T(x)$.*

(2) *$T(0) = NP(E)$, $T(1) = \emptyset$.*

(3) *If $\langle x\rangle = E$, then $T(x) = NP(E)$.*

(4) *$T(x) = T(\langle x\rangle)$.*

Proposition 5.14. *Let $\mathcal{E}$ be an EQ-algebra. Then, for any $x, y \subseteq E$, the following properties hold:*

(1) *$\bigcup_{x\in E} T(x) = NP(E)$.*

(2) *If $x \vee y$ exists, then $T(x) \cap T(y) = T(x \vee y)$.*

(3) *$T(x) \cup T(y) = T(x \wedge y) = T(x \otimes y)$.*

Proof. (1) It follows from Proposition 5.11 (2).

(2) Let $H \in T(x) \cap T(y)$. Then, we have $H \in T(x)$ and $H \in T(y)$, which implies $x \notin H$, $y \notin H$. If $x \vee y \in H$, then by the fact that H is prime, we get $x \in H$ or $y \in H$, which generates a contradiction. Thus, we get $x \vee y \notin H$, which means $H \in T(x \vee y)$. Hence, it follows that $T(x) \cap T(y) \subseteq T(x \vee y)$. Conversely, for any $H \in T(x \vee y)$, it implies that $x \vee y \notin H$. If $x \in H$ or $y \in H$, then we get $x \vee y \in H$ by $x, y \leq x \vee y$, which generates a contradiction. Hence, it follows that $x \notin H$ and $y \notin H$, that is $H \in T(x)$ and $H \in T(y)$, and so $H \in T(x) \cap T(y)$. Therefore, we obtain that $T(x) \cap T(y) = T(x \vee y)$.

(3) For any $H \in T(x) \cup T(y)$, it implies that $H \in T(x)$ or $H \in T(y)$, which means $x \notin H$ or $y \notin H$. Now, since H is a filter, we get $x \wedge y \notin H$, that is $H \in T(x \wedge y)$, and so $T(x) \cup T(y) \subseteq T(x \wedge y)$. Conversely, for any $H \in T(x \wedge y)$, we have $x \wedge y \notin H$. If $x, y \in H$, then $x \otimes y \in H$, and so $x \wedge y \in H$, which generates a contradiction. Hence, $x \notin H$ or $y \notin H$, that is $H \in T(x) \cup T(y)$. Therefore, $T(x) \cup T(y) = T(x \wedge y)$. Analogously, $T(x) \cup T(y) = T(x \otimes y)$ also holds. □

Let $\mathcal{E}$ be an EQ-algebra and $\tau = \{T(M) \mid M \subseteq E\}$. Then, by the above Proposition, we have:

(1) $\emptyset$, $NP(E) \in \tau$.

(2) If $T(M)$, $T(N) \in \tau$, then $T(M) \cap T(N) \in \tau$.

(3) If $\{T(M_i) \mid i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} T(M_i) \in \tau$.

Hence, τ is a topology on $NP(E)$ and $(NP(E), \tau)$ is a topological space of nodal prime filters.

Proposition 5.15. *Assume that $\mathcal{E}$ is an EQ-algebra. Then, $\{T(m) \mid m \in E\}$ is a topological base of $(NP(E), \tau)$.*

Proof. Let $T(M) \in \tau$. Then, we get $T(M) = T(\bigcup_{i \in I} m_i) = \bigcup_{i \in I} T(m_i)$, that is to say each element in τ can be expressed by the union of elements in subset of $\{T(m) \mid m \in E\}$. Hence, $\{T(m) \mid m \in E\}$ is a topological base of $(NP(E), \tau)$. □

Proposition 5.16. *Suppose that $\mathcal{E}$ is an EQ-algebra. Then, $(NP(E), \tau)$ is a compact T_0 space.*

Proof. Firstly, we show that $T(u)$ is compact set in $(NP(E), \tau)$ for any $u \in E$. By definition of compact, we shall prove that each open covering of $T(u)$ has a finite open covering. Assume $T(u) = \bigcup_{i \in I} T(u_i) = T(\bigcup_{i \in I} u_i)$. Then, from Proposition 5.11 (5), we obtain that $\langle u \rangle = \langle \bigcup_{i \in I} u_i \rangle$, and so $u \in \langle \bigcup_{i \in I} u_i \rangle$. Hence, there are finite $u_{i_1}, u_{i_2}, \cdots, u_{i_n}$ satisfying $u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n} \leq u$, which implies $T(u) \leq T(u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}) = T(u_{i_1}) \cup T(u_{i_2}) \cup \cdots \cup T(u_{i_n}) \subseteq \bigcup_{i \in I} T(u_i) = T(u)$. Therefore, it follows that $(NP(E), \tau)$ is compact.

Next, we show that $(NP(E), \tau)$ is a T_0 space. Assume that $H, J \in NP(E)$ with $H \neq J$. Then, we get $H \nsubseteq J$ or $J \nsubseteq H$. If $H \nsubseteq J$, then there exists a such that $a \in H$ but $a \notin J$. Let $U = T(a)$. Then, it implies that $J \in U$ and $H \nsubseteq U$. If $J \nsubseteq H$, the proof is similar. Hence, the conclusion holds. □

6. Conclusion

In this article, we presented the definitions of seminodes, nodes and nodal filters in EQ-algebras and their related properties are stated and proved. At first, we exemplify that the seminodes and nodes are different with other specific elements and show that the set $\mathcal{ND}(E)$ is a distributive lattice and the set $\mathcal{SN}(E)$ is a Hertz-algebra and a Heyting-algebra under some conditions. Then, we introduced the concept of n-filters, we studied it with the help of node elements and obtained that there is a one-to-one correspondence between nodal principle filters and node elements in an idempotent EQ-algebra. Furthermore, the relationships among it and other filters were given. It was turned out that each obstinate filter and each (positive) implicative filter is an n-filter under some conditions. Finally, we investigated the algebraic structures of $NF(E)$ and topological structures of $NP(E)$ on EQ-algebras and set up the connections from the set $NF(E)$ of all nodal filters in an EQ-algebra $\mathcal{E}$ to other algebraic structures, like BCK$-$algebras, Hertz algebras and so on. In addition, we concluded that $(NP(E), \tau)$ is a compact T_0 space.

Acknowledgments

This research is supported by a grant of National Natural Science Foundation of China (11971384) and Foreign Expert Program of China (Grant No. DL20230410021).

References

[1] N. Akhlaghinia, R.A. Borzooei, M. Aaly Kologani, *Preideals in EQ-algebras*, Soft Computing, 25 (2021), 12703-12715.

[2] R. Balbes, A. Horn, *Injective and projective Heyting algebras*, Transactions of the American Mathematical Society, 148 (1970), 549-559.

[3] V. Boicescu, A. Filipoiu, G. Georgescu, S. Rudeanu, *Lukasiewicz-moisil algebras*, Annals of Discrete Mathematics, 49 (1991), 201-212.

[4] D. Busneag, *Hertz algebras of fractions and maximal hertz algebra of quotients*, Math Japon., 39 (1993), 461-469.

[5] R.A. Borzooei, B.G. Saffar, *States on EQ-algebras*, Journal of Intelligent and Fuzzy Systems, 29 (2015), 209-221.

[6] M. Bakhshi, *Nodal filters in residuated lattice*, Journal of Intelligent and Fuzzy Systems, 30 (2016), 2555-2562.

[7] X.Y. Cheng, M. Wang, W. Wang, J.T. Wang, *Stabilizer in EQ-algebras*, Open Mathematics, 17 (2019), 998-1013.

[8] A. Diego, *Sur les algebras dde Hilbert*, Collection de Logiue Mathematique Séerie, 21 (1966), 1-54.

[9] M. El-Zekey, *Representable good EQ-algebra*, Soft Computing, 14 (2010), 1011-1023.

[10] M. El-Zekey, V. Novák, R. Mesiar, *On good EQ-algebras*, Fuzzy Sets and System, 178 (2011), 1-23.

[11] J.A. Goguen, *The logic of inexact concepts*, Synthese, 19 (1968), 325-373.

[12] P. Hájek, *Metamathmatics of fuzzy logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[13] R.T. Khorami, A.B. Saeid, *Nodal filters of BL-algebras*, Journal of Intelligent and Fuzzy Systems, 28 (2015), 1159-1167.

[14] L.Z. Liu, X.Y. Zhang, *Implicative and positive implicative prefilters of EQ-algebras*, Journal of Intelligent and Fuzzy Systems, 26 (2014), 2087-2097.

[15] J. Meng, Y.B. Jun, *BCK-algebra*, Kyungmoonsa Co, Seoul, 1994.

[16] N. Mohtashamnia, L. Torkzadeh, *The lattice of prefilters of an EQ-algebra*, Fuzzy Sets System, 311 (2016), 86-98.

[17] V. Novák, B.De Baets, *EQ-algebras*, Fuzzy Sets and Systems, 160 (2009), 2956-2978.

[18] A. Namdar, R.A. Borzooei, *Nodal filters in hoop algebras*, Soft Computing, 22 (2018), 7119-7128.

[19] D. Piciu, *Algebras of fuzzy logic*, Edition Universitaria, Craiova, 2007.

[20] H.P. Sankappanavar, *Semi-De Morgan algebras*, Journal of Symbolic Logic, 52 (1987), 712-724.

[21] J.Q. Shi, X.L. Xin, *Ideal theory on EQ-algebras*, AIMS Mathematics, 6 (2021), 11686-11707.

[22] J.C. Varlet, *Nodal filters of semilattices*, Commentationes Mathematicae Universitatis Carolinae, 14 (1973), 263-277.

[23] W. Wang, X.L. Xin, J.T. Wang, *EQ-algebras with internal states*, Soft Computing, 22 (2018), 2825-2841.

[24] X. Xun, X.L. Xin, *Nodal filters and seminodes in equality algebras*, Journal of Intelligent and Fuzzy Systems, 37 (2019), 1457-1466.

[25] Y.Q. Zhu, Y. Xu, *On filter theory of residuated lattices*, Information Sciences, 180 (2010), 3614-3632.

[26] B. Zhao, W. Wang, *Prime spectrums of EQ-algebras*, Journal of logic and computation, (2023).

Accepted: January 13, 2021

Multiset group and its generalization to (A, B)-multiset group

Suma P*
Department of Mathematics
National Institute of Technology Calicut
Calicut 673601, Kerala
India
sumamuraleemohan@gmail.com

Sunil Jacob John
Department of Mathematics
National Institute of Technology Calicut
Calicut 673601, Kerala
India
sunil@nitc.ac.in

Abstract. Multiset groups are multisets with its elements taken from a group and the characteristic function of the multiset satisfying certain conditions. Apart from the definition and examples of multiset groups, we try to explain some properties, that a multiset should satisfy in order to become a multiset group. From this point, we broaden the concept of multiset group to a new scenario, (A, B)- multiset group, where A and B are non negative real numbers. The multiplicity of the identity element e has its own importance in an (A, B)- multiset group. The count value of the elements depends largely on the values of A and B. We have also delved upon the peculiarities of an (A, B)- multiset group drawn from a cyclic group and defined and explored an (A, B)- multiset normal group and cosets of (A, B)- multiset group.
Keywords: multiset, characteristic function, root set, multiset group, multiset subgroup, level set, (A, B)- multiset group, (A, B)-multiset normal group

1. Introduction

The limitations of classical set theory is what led to the other forms of sets, such as fuzzy set or multiset. Many researchers contributed in the development of these generalized sets. Looking to the case of multisets (also, known as Bags), D. E. Knuth pointed out the essentialness of such a set ([1]). Chris Brink in his studies explained the relations and operations with multisets [2]. Later Wayne D. Blizard developed some of the fundamental structures in multiset background ([3]). C. S. Calude [4], N.J. Wildberger [5], D. Singh [6] are some of the persons who were put milestones in this journey. K.P. Girish and S.J. John [7] explores the relations and functions in multiset context.

The algebraic structures, group, ring, ideal etc. with fuzzy set context are being applied in subjects like computer science, physics and so on. Some of the

∗. Corresponding author

research work in this area are done by Azriel Rosenfield [8], Sabu Sebastian and T. V. Ramakrishnan [9], and Yuying Li et all [10]. The structure with multiset base are yet to be used and implemented widely. Multiset groups (shortly mset groups) and some of its properties have been studied by the authors like A.M. Ibrahim and P.A. Ajegwa [11], Binod Chandra Tripathy [12], A.A. Johnson [13], P.A. Ejegwa [14], S.K. Nazmul [15], Tella [16]. Suma P. and Sunil J. John [17] extended this to ring and ideal structures.

This paper is an attempt to extend the properties of multiset group to a generalized form (A, B)- multiset group. Here, A and B are non negative real numbers with $A < B$. Section 3 is a discussion of multiset group and some of the properties of mset normal groups and cosets of mset groups. In section 4, these properties are analysed in (A, B)- mset group.

2. Preliminaries

In this section, we will be revisiting some of the fundamental properties of Multiset that have been developed by several researchers, which are necessary for this paper.

A *Multiset* (shortly *mset*) T drawn (or derived) from a set U is represented by a function $C_T : U \to N$, where N is the set of non negative integers. $C_T(u)$ represents the number of occurrences of the element u in the multiset T. The function C_T is known as *Characteristic function* or *Count Function* and $C_T(u)$ is the *Count value* of u in T (see, Girish and John (2009)).

Let T be an mset drawn from U, and let $\{u_1, u_2, \cdots, u_n\}$ be a subset of T, with u_1 appearing k_1 times, u_2 appearing k_2 times and so on. Then T is written as

$$T = \{k_1|u_1, k_2|u_2, \cdots, k_n|u_n\}.$$

The subset $S = \{u_1, u_2, \cdots, u_n\}$ of U is called the *Root Set* of T.

Operations of multisets:-

1. Let T_1 and T_2 be two msets drawn from a set U. T_1 is a *submultiset* of T_2, $(T_1 \subseteq T_2)$ if $C_{T_1}(u) \leq C_{T_2}(u)$ for all u in U.

2. Two msets T_1 and T_2 are *equal* if $T_1 \subseteq T_2$ and $T_2 \subseteq T_1$.

3. The *intersection* of T_1 and T_2 is a multiset, $T = T_1 \cap T_2$, with the count function $C_T(u) = \min\{C_{T_1}(u), C_{T_2}(u)\}$, for every $u \in U$.

4. The *union* of T_1 and T_2 is a multiset, $T = T_1 \cup T_2$, with the count function $C_T(u) = \max\{C_{T_1}(u), C_{T_2}(u)\}$, for every $u \in U$.

   More details in [7].

3. Multiset group

Consider the group (G, *) and a multiset T drawn from G. Then, T is said to be a *multiset group* or shortly *mset group* if the characteristic function satisfies the following properties:

(1) $C_T(g * h) \geq \min\{C_T(g), C_T(h) : g, h \in G\}$;

(2) $C_T(g) = C_T(g^{-1})$ for all $g \in G$ where g^{-1} is the inverse of g in G.

Let T be an mset group. A subset P of T is an mset subgroup, if P itself is an mset group on G ([15]).

Example 3.1. Let $G = \{1, -1, i, -i\}$. Then $(G, *)$ is a group, where $*$ is the usual multiplication of real numbers. Consider the multiset $T = \{5|1, 3|-1, 4|i, 4|-i\}$. Here T is a multiset group.

Theorem 3.1. *Let T be a multiset group derived from a group (G, *) and let S be the root set of T. Then S is a subgroup of G.*

Proof. Let $g, h \in S$. Then $C_T(g) > 0$ and $C_T(h) > 0$, $C_T(g * h^{-1}) \geq \min\{C_T(g), C_T(h^{-1})\} = \min\{C_T(g), C_T(h)\} > 0$ means that $g * h^{-1} \in S$. □

Proposition 3.1. *Consider a group (G, *) with identity element e and a multiset group T drawn from G. Then:*

(1) $C_T(e) \geq C_T(g), \forall g \in G$;

(2) $C_T(g^n) \geq C_T(g), \forall g \in G$, and all natural number n. Here, g^n means $g * g * \cdots n$ times.

Proof. (1) Since $e = g * g^{-1}, \forall g \in G$, $C_T(e) \geq \min\{C_T(g), C_T(g^{-1})\} = C_T(g)$;

(2) Applying mathematical induction on n. For $n = 1, C_T(g) = C_T(g)$, and hence the result is true. Assume the result is true for $n - 1$ i.e., $C_T(g^{n-1}) \geq C_T(g)$.

Now, $C_T(g^n) = C_T(g^{n-1} * g) \geq \min\{C_T(g^{n-1}),\ C_T(g)\} = C_T(g)$, by induction hypothesis. □

Theorem 3.2. *If T is an mset derived from a group G, then T is an mset group if and only if $C_T(g * h^{-1}) \geq \min\{C_T(g), C_T(h)\}, \forall g, h \in G$.*

Proof. First assume that T is an mset group. Then

$$C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h^{-1})\} = \{\min\{C_T(h),\ C_T(h)\}.$$

Conversely, suppose $C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h)\}, \forall g, h \in G$.

Now, $C_T(e) = C_T(g * g^{-1}), \forall g \in G. \geq \min\{C_T(g),\ C_M(g)\}$, by assumption. So, $C_T(e) \geq C_T(g), \forall g \in G$. Now, $C_T(g^{-1}) = C_T(e*g^{-1}) \geq \min\{C_T(e),\ C_T(g)\} \geq C_T(g)$. Similarly, $C_T(g) = C_T(e * g) \geq \min\{C_T(e),\ C_T(g^{-1})\} \geq C_T(g^{-1})$. Hence, we get $C_T(g) = C_T(g^{-1}), \forall g \in G$, which is the second e condition of Mset group. Now, to show the first condition, take two arbitrary elements g and h from G.

$$\begin{aligned} C_T(g * h) &= C_T(g * (h^{-1})^{-1}) \geq \min\{C_T(g),\ C_T(h^{-1})\}, \text{ by assumption} \\ &= \min\{C_T(g),\ C_T(h)\}. \end{aligned}$$

□

Theorem 3.3. *Let $(G, *)$ be a group with identity e and T be an mset group derived from G. If $E = \{g \in G : C_T(g) = C_T(e)\}$, then E is a subgroup of G.*

Proof. Take g and h from E. Then, $C_T(g) = C_T(h) = C_T(e)$. $C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h)\}$, by Theorem 3.4 $= C_T(e)$. Therefore, $g * h^{-1} \in E$. Hence, E is a subgroup of G. □

Definition 3.1. *Let T be an mset drawn from a group G. The subset $\{g : C_T(g) \geq r\}$ of G is known as the* Level Set of T, *denoted by T_r. Here, r is a non negative real number.*

Theorem 3.4. *If T is an mset group drawn from a group $(G, *)$ having identity element e, then the level sets T_r are all subgroups of G.*

Proof. If $T_r = \phi$, then T_r is a subgroup.

If T_r is a singleton set, then $T_r = \{e\}$, which is also a subgroup of G. Otherwise, Let $g, h \in T_r$. Then, $C_T(g) \geq r$ and $C_T(h) \geq r$. Now, $C_T(g * h^{-1}) \geq \min\{C_T(g),\ C_T(h)\} \geq r$. So, T_r is a subgroups of G for all positive real number r. □

Theorem 3.5. *Let T be an mset group drawn from a group $(G, *)$ having identity element e. If $C_T(g * h^{-1}) = C_T(e)$, for some g and h in* G, *then $C_T(g) = C_T(h)$.*

Proof. $C_T(g) = C_T(g * e) = C_T(g * (h^{-1} * h)) = C_T((g * h^{-1}) * h) \geq \min\{C_T(g * h^{-1}),\ C_T(h)\}, = C_T(h)$.

Similarly, starting from $C_T(h)$, we can show that $C_T(h) \geq C_T(g)$. □

Definition 3.2. *An mset group T drawn from a group G is said to be an* Mset Normal group, *if $C_T(g * h * g^{-1}) \geq C_T(h), \forall g, h$ in G.*

Proposition 3.2. *If T is an mset normal group, then $C_T(g * h) = C_M(h * g)$, for every g and h in G.*

Proof. Suppose T is an mset normal group derived from G. Then $C_T g * h * g^{-1}) \geq C_T(h), \forall g, h$ in G. Replacing h by $h * g$, $C_T(g * (h * g) * g^{-1}) \geq C_T(h * g)$.

By associativity $C_T(g * h) \geq C_T(h * g)$. Interchanging the role of g and h, $C_T(h * g) \geq C_T(g * h)$. □

Proposition 3.3. *Let T an mset group drawn from a group G. If T is an mset normal group, then T_r is a normal subgroup of G, for every $r > 0$.*

Proof. Take an mset normal group T derived from G and r a positive real number. Then $C_T(g * h * g^{-1}) \geq C_T(h), \forall g, h$ in G. Choose a $h \in T_r$. Then, $C_T(h) \geq r$. For any $g \in G$, $C_T(g * h * g^{-1}) \geq C_T(h) \geq r$, $g * h * g^{-1} \in T_r$. T_r is a normal subgroup of G. □

Theorem 3.6. *Let T be an mset group drawn from a cyclic group G with generator a. Then $C_T(g) \geq C_T(a), \forall x \in G$.*

Proof. Let $g \in G$. Then $g = a^n$ for some non negative integer n and $C_T(g) \geq C_T(a)$, by Proposition 3.3. □

Corollary 3.1. *Let T be an mset group drawn from a cyclic group G with generators a and b.Then $C_T(a) = C_T(b)$.*

Proof. since a is a generator, and $b \in G$, by above theorem $C_T(a) \leq C_T(b)$. By interchanging the roles of a and b, $C_T(b) \leq C_T(a)$. □

Corollary 3.2. *Let T be an mset group drawn from a group G of prime order. Then $C_T(g)$ are all equal for all $g \in G$ other than the identity element.*

Proof. Being prime order, G is cyclic and every element other than the identity element of G are generators. The proof is then straight forward from above theorem and corollary. □

Definition 3.3. *Let T be an mset group drawn from a group G and $g \in G$ such that $C_T(g) = 0$. The* Left Coset *gM is defined as $C_{gT}(x) = C_T(g * x)$, for $x \in G$.*

Similarly, the Right Coset *Tg is $C_{Tg}(x) = C_T(x * g)$, for $x \in G$.*

Proposition 3.4. *If T is an mset group drawn from G, and $g, h \in G$, then*

(a) $eT = Te = T$.

(b) $g(hT) = (g * h)T$

(c) $(Tg)h = T(g * h)$.

(d) $gT = hT \Leftrightarrow T = (g^{-1} * h)T \Leftrightarrow T = (h^{-1} * g)T$

(e) $Tg = Th \Leftrightarrow T = T(h * g^{-1}) \Leftrightarrow T = T(g * h^{-1})$.

Proposition 3.5. *Let T and R are two mset groups drawn from the same group G, and $g, h \in G$*

(a) $gT = hR \Leftrightarrow T = (g^{-1} * h)R \Leftrightarrow (h^{-1} * g)T = R$.

(b) $Tg = Rh \Leftrightarrow T = R(h * g^{-1}) \Leftrightarrow T(g * h^{-1}) = R$.

4. (A, B)-multiset group

Definition 4.1. *Let M be an mset drawn from a group G, and A, B are two real numbers with $0 \leq A < B$. Then M is called an (A, B)-* multiset group *if the characteristic function satisfies the following conditions.*

1. $\max\{C_M(x * y), A\} \geq \min\{C_M(x), C_M(y), B\}$;
2. $\max\{C_M(x^{-1}), A\} \geq \min\{C_M(x), B\}$,

for every x and y in G.

Notation 4.1. *An (A, B)- mset group is denoted by M_{AB}.*

Proposition 4.1. *If M is an mset group derived from a group G, then it is an (A, B)- mset group for every real number A and B with $0 \leq A < B$.*

Proof. M is an mset group means $C_M(x * y) \geq \min\{C_M(x), C_M(y)\}$, for every x and y in G. For $0 \leq A < B$,

$$\begin{aligned}\max\{C_M(x * y), A\} &\geq C_M(x * y)\\ &\geq \min\{C_M(x), C_M(y)\}\\ &\geq \min\{C_M(x), C_M(y), B\}\end{aligned}$$

M is an (A, B)- mset group. □

Proposition 4.2. *If an mset M derived from a group G is a $(0, N)$- mset group, where $N = \max\{C_M(x) : x \in G\}$, then it is an mset group.*

Proof. For any $x, y \in G, \max\{C_{0N}(x * y), 0\} \geq \min\{C_{M_{0N}}(x), C_{M_{0N}}(y), N\}$

$$C_{M_{0N}}(x * y) \geq \min\{C_{M_{0N}}(x), C_{M_{0N}}(y)\},$$

since $N \geq C_M(x)$ and $N \geq C_M(y)$.

Similarly, by the second condition of (A, B)- mset group

$$\begin{aligned}&\max\{C_{M_{0N}}(x^{-1}, 0)\} \geq \min\{C_{M_{0N}}(x), N\},\\ &C_{M_{0N}}(x^{-1}) \geq C_{M_{0N}}(x).\end{aligned}$$

Hence, the two conditions of mset group is satisfied by M_{0N}. □

Note 4.1. *If an mset drawn from a group G, is not an (A, B) mset group for all A and B with $0 \leq A < B$, then M need not be an mset group.*

Example 4.1. Consider the group $G = \{1, -1, i, -i\}$ with usual multiplication and the mset $M = \{3|1, 4| - 1\}$. Here, M is a (5,6)- mset group,because both th conditions of the definition of (A, B)-mset group is satisfied. But M is not a (1,5)- mset group. Taking $x = y = -1$, LHS of condition (1) of definition is $\max\{C_M(-1 * -1), A\} = \max\{3, 1\} = 3$.

RHS becomes $\min\{C_M(-1), C_M(-1), 5\} = \min\{4, 4, 5\} = 4$. We get LHS=3 and RHS=4, so that the first condition is not satisfied and hence not a $(1, 5)$-mset group. Note that M is not an mset group.

Example 4.2. Consider the group $G = \{1, -1, i, -i\}$ with usual multiplication and the mset $M = \{3|1, 3| - 1, 2|i, 2| - i\}$. Here, M is an (A, B) mset group for all A and B. M is an mset group also.

Definition 4.2. *Let M_{AB} be an (A, B) mset group drawn from a group G. The subset $\{x \in G : C_{M_{AB}}(x) \geq r\}$ of G is known as* level set *of M_{AB} and is denoted by M_r, where r is any positive number.*

The following theorem gives some of the properties of the count value of the identity element e in an (A, B)- mset group.

Theorem 4.1. *If G is a group with identity element e, and M_{AB}- is an (A, B)-mset group drawn from G, then:*

(a) $\max\{C_{M_{AB}}(e), A\} \geq \min\{C_{M_{AB}}(x), B\}, \forall x \in G$.

(b) *If $C_{M_{AB}}(x) \geq B$, for some $x \in G$, then $C_{M_{AB}}(e) \geq B$.*

(c) *If $C_{M_{AB}}(x) < B, \forall x \in G$, and $C_{M_{AB}}(x) > A$, for atleast one $x \in G$, then $C_{M_{AB}}(e) = \max\{C_{M_{AB}}(x) : x \in G\}$.*

(d) *If $C_{M_{AB}}(e) \leq A$, then $C_{M_{AB}}(x) \leq A, \forall x \in G$.*

(e) *If $A < C_{M_{AB}}(e) < B$, then $C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G$.*

Proof. (a) In condition 1 of the definition of (A, B)- mset group, taking $y = x^{-1}$, we get

$$\max\{C_{M_{AB}}(x * x^{-1}), A\} \geq \min\{C_{M_{AB}}(x), C_{M_{AB}}(x^{-1}), B\} \text{ i.e.}$$
$$\max\{C_{M_{AB}}(e), A\} \geq \min\{C_{M_{AB}}(x), C_{M_{AB}}(x^{-1}), B\}$$
$$\geq \min\{C_{M_{AB}}(x), B\}.$$

(b) Suppose there is an $x_0 \in G$ with $C_{M_{AB}}(x_0) \geq B$. By part (a)

$$\max\{C_{M_{AB}}(e), A\} \geq \min\{C_{M_{AB}}(x_0), B\} = B,$$

since $C_{M_{AB}}(x_0) \geq B C_{M_{AB}}(e) \geq B$, because $A < B$.

(c) If $C_{M_{AB}}(x) < B, \forall x \in G$, $\min\{C_{M_{AB}}(x), B\} = C_{M_{AB}}(x), \forall x \in G$. So, by part (a),

$$\max\{C_{M_{AB}}(e), A\} \geq C_{M_{AB}}(x), \forall x \in G. \tag{1}$$

Suppose, there is an $x_0 \in G$ with

$$C_{M_{AB}}(x_0) \geq A.$$

For this particular x_0, (4.1) becomes $\max\{C_{M_{AB}}(e), A\} \geq C_{M_{AB}}(x_0)$, $C_{M_{AB}}(e) \geq C_{M_{AB}}(x_0)$. Since, x_0 is arbitrary, $C_{M_{AB}}(e) = \max\{C_{M_{AB}}(x) : x \in G\}$.

(d) If $C_{M_{AB}}(e) \leq A$, $\max\{C_{M_{AB}}(e), A\} = A$. Then, by part (a),

$$A \geq \min\{C_{M_{AB}}(x), B\}, \forall x \in G$$

$A \geq C_{M_{AB}}(x), \forall x \in G$, since $A < B$.

(e) If possible, let $C_{M_{AB}}(x_0) \geq B$ for some $x_0 \in G$. Then, by part (b), $C_{M_{AB}}(e) \geq B$, which is not the case. Therefore, $C_{M_{AB}}(x) \leq B, \forall x \in G$.

Since $C_{M_{AB}}(e) > A$, by part (c), $C_{M_{AB}}(e) = \max\{C_{M_{AB}}(x) : x \in G\}$. i.e. $C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G$.

□

Corollary 4.1. *If* $A < C_{M_{AB}}(e) < B$*, then* $C_{M_{AB}}(x) = C_{M_{AB}}(e), \forall x \in M_k$*, where* $k = C_{M_{AB}}(e)$

Proof. For $x \in M_k,\ C_{M_{AB}}(x) \geq k, C_{M_{AB}}(x) \geq C_{M_{AB}}(e)$. By Theorem 4.9 (e),

$$C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G.$$

Hence, for $x \in M_k,\ C_{M_{AB}}(x) = C_{M_{AB}}(e)$. □

Theorem 4.2. *Let* M *be an mset drawn from a group* G*. If* M *is an* (A, B)*-mset group, then the level set* M_r *is a subgroup of* G *for* $A < r \leq B$*.*

Proof. If $M_r = \phi$, then it is a subgroup trivially.

If M_r has exactly one element say x, then, by Theorem 4.9 (a), $x = e$, the identity element of G and is a subgroup of G.

Otherwise, take two element x and y from M_r, for a particular r. $C_{M_{AB}}(x) \geq r$ and $C_{M_{AB}}(y) \geq r$ and $A < r \leq B$, will give $\min\{C_{M_{AB}}(x), C_{M_{AB}}(y), B\} \geq r$.

By definition, $C_{M_{AB}}(x * y^{-1}) \geq r$.

$\implies x * y^{-1} \in M_r$, completes the proof. □

Corollary 4.2. *If* $C_{M_{AB}}(x) \geq B$ *and* $C_{M_{AB}}(y) \geq B$ *for* $x \in G, y \in G$*, then* $C_{M_{AB}}(x * y) \geq B$*.*

Proof. $x \in M_B$, $y \in M_B$ and M_B is a subgroup will imply $x * y \in M_B$. □

Example 4.3. In Example 4.7, $M_r = G$, if $r \leq 2$, $M_r = \{1, -1\}$, if $2 < r \leq 3$, and $M_r = \phi$, if $r > 3$.

In all cases, M_r is a subgroup of G.

Theorem 4.3. *If* $A < C_{M_{AB}}(x) < B$*, for* $x \in G$*, then* $C_{M_{AB}}(x * y) = C_{M_{AB}}(x), \forall y \in G$ *with* $C_{M_{AB}}(y) > C_{M_{AB}}(x)$*.*

Proof. By the definition of M_{AB} mset group

$$\max\{C_{M_{AB}}(x * y), A\} \geq \min\{C_{M_{AB}}(x), C_{M_{AB}}(y), B\} = C_{M_{AB}}(x),$$

since both B and $C_{M_{AB}}(y)$ are greater than $C_{M_{AB}}(x)$.

$$\therefore C_{M_{AB}}(x * y) \geq C_{M_{AB}}(x). \tag{2}$$

If $C_{M_{AB}}(x * y) > C_{M_{AB}}(x)$, let $r_0 = \min\{C_{M_{AB}}(x * y), C_{M_{AB}}(y), B\}$. Then $r_0 > C_{M_{AB}}(x)$. Also, $A < r_0 \leq B$ and hence M_{r_0} is a subgroup of G.

$$x * y \in M_{r_0}, y \in M_{r_0} \implies (x * y) * y^{-1} \in M_{r_0} \implies x \in M_{r_0},$$

i.e. $C_{M_{AB}}(x) \geq r_0 > C_{M_{AB}}(x)$, a contradiction and this completes the proof. □

Theorem 4.4. *If $C_{M_{AB}}(x) \leq A$ and $C_{M_{AB}}(y) > A$, for x, y in G, then $C_{M_{AB}}(x* y) \leq A$.*

Proof. If possible, let $C_{M_{AB}}(x*y) > A$. Take $r_0 = \min\{C_{M_{AB}}(x*y), C_{M_{AB}}(y), B\}$. Then $A < r_0 \leq B$ and hence M_{r_0} is a subgroup of G

$$x * y \in M_{r_0}, y \in M_{r_0} \implies (x * y) * y^{-1} \in M_{r_0} \implies x \in M_{r_0},$$

i.e. $C_{M_{AB}}(x) \geq r_0 > A$, a contradiction. □

Theorem 4.5. *If $A < C_{M_{AB}}(x) < B$, then $C_{M_{AB}}(x^n) \geq C_{M_{AB}}(x)$, for a positive integer n.*

Proof. By definition

$$\max\{C_M(x * x), A\} \geq \min\{C_M(x), C_M(x), B\},$$
$$\max\{C_{M_{AB}}(x^2), A\} \geq \min\{C_{M_{AB}}(x), B\},$$
$$C_{M_{AB}}(x^2) \geq C_{M_{AB}}(x),$$

since $A < C_{M_{AB}}(x) < B$. By the same argument $C_{M_{AB}}(x^3) \geq C_{M_{AB}}(x^2) \geq C_{M_{AB}}(x)$. Proceeding like this, $C_{M_{AB}}(x^n) \geq C_{M_{AB}}(x)$. □

Proposition 4.3. *If G is a group and M_{AB} is an (A, B)- mset group drawn from G, then*

(a) *If $C_{M_{AB}}(x) \leq A$, for some $x \in G$, then $C_{M_{AB}}(x^{-1}) \leq A$, for those x.*

(b) *If $A < C_{M_{AB}}(x) < B$, for some $x \in G$, then $C_{M_{AB}}(x) = C_{M_{AB}}(x^{-1})$.*

(c) *If $C_{M_{AB}}(x) \geq B$, for some $x \in G$, then $C_{M_{AB}}(x^{-1}) \geq B$.*

Proof. (a) Suppose $C_{M_{AB}}(x_0) \leq A$, for $x_0 \in G$. If possible, let $C_{M_{AB}}(x_0^{-1}) > A$. Let $r_0 = \min\{C_{M_{AB}}(x_0^{-1}), B\}$. Then $r_0 > A$, $x_0^{-1} \in (M_{AB})_{r_0}$ and being $(M_{AB})_{r_0}$ is a subgroup of G, $x_0 \in (M_{AB})_{r_0}$. Therefore, $C_{M_{AB}}(x_0) \geq r_0 > A$, a contraduction.

(b) choose x_0 from G such that

$$A < C_{M_{AB}}(x_0) < B. \tag{3}$$

By condition 2 of the definition of (A, B)-mset group,

$$\max\{C_{M_{AB}}(x_0^{-1}), A\} \geq \min\{C_{M_{AB}}(x_0), B\},$$

$$\max\{C_{M_{AB}}x_0^{-1}, A\} \geq C_{M_{AB}}(x_0), \text{ by (4.1)}$$

since, $A < C_{M_{AB}}(x_0)$,

$$C_{M_{AB}}(x_0^{-1}) \geq C_{M_{AB}}(x_0). \tag{4}$$

Again by applying condition 2 of the definition of (A, B)-mset group to the point (x_0^{-1})

$$\max\{C_{M_{AB}}(x_0), A\} \geq \min\{C_{M_{AB}}(x_0^{-1}), B\}.$$

In view of equation (4.4), this can be reduced to

$$C_{M_{AB}}(x_0) \geq C_{M_{AB}}(x_0^{-1}) \tag{5}$$

the required result is obtained from the equations (4.4) and (4.5).

(c) Choose an x_0 from G such that $C_{M_{AB}}(x_0) \geq B$.

Consider M_B. $x_0 \in M_B$. Since M_B is a subgroup of G, $x_0^{-1} \in M_B$, which gives $C_{M_{AB}}(x_0^{-1}) \geq B$. □

4.1 M_{AB} drawn from a cyclic group G

Theorem 4.6. *Let G be a cyclic group with generator a, and M_{AB} be an (A, B)-mset group drawn from G.*

If $A < C_{M_{AB}}(a) < B$, then $C_{M_{AB}}(x) \geq C_{M_{AB}}(a), \forall x \in G$.

Proof. By an above theorem, $C_{M_{AB}}(x) \leq C_{M_{AB}}(e), \forall x \in G$. So, $C_{M_{AB}}(a) \leq C_{M_{AB}}(e)$.

Now, for $x \neq e, x = a^n$, for some positive integer n. Again, by a previous theorem, $C_{M_{AB}}(a^n) \geq C_{M_{AB}}(a)$ i.e. $C_{M_{AB}}(x) \geq C_{M_{AB}}(a)$. □

Theorem 4.7. *Let G be a cyclic group with generator a, and M_{AB} be an (A, B)-mset group drawn from G. If $C_{M_{AB}}(a) \geq B$, then $G = M_B$.*

Proof. M_B is a subgroup of G. Now to show $G \subseteq M_B$.

Let $x \in G$. Then $x = a^n$ for a positive integer n. Given, $C_{M_{AB}}(a) \geq B \implies a \in M_B \implies a^n \in M_B \implies x \in M_B$. Hence, $G = M_B$. □

Theorem 4.8. *Let G be a cyclic group with two generators a and b and M_{AB} be an (A, B)- mset group drawn from G. If $A < C_{M_{AB}}(a) < B$, then $C_{M_{AB}}(a) = C_{M_{AB}}(b)$.*

Proof. By Theorem 4.18,

$$C_{M_{AB}}(b) \geq C_{M_{AB}}(a). \tag{6}$$

If possible, let $C_{M_{AB}}(b) \geq B$. Then, by Theorem 4.14, $G = M_B$ and so $a \in M_B$

$$\implies C_{M_{AB}}(a) \geq B,$$

a contradiction. Therefore,

$$C_{M_{AB}}(b) < B. \tag{7}$$

From (4.6) and (4.7), $A < C_{M_{AB}}(b) < B$. By Theorem 4.13

$$C_{M_{AB}}(a) \geq C_{M_{AB}}(b) \tag{8}$$

(4.6) and (4.8) together provides the requirement. □

Corollary 4.3. *If G is a cyclic group of prime order with generator a and identity element e, then $C_{M_{AB}}(x) = C_{M_{AB}}(a), \forall x \neq e$ of G.*

Proof. For a cyclic group of prime order, every element other than e, is a generator, and hence the result is obtained by above theorem. □

4.2 (A, B)- Mset normal group

Definition 4.3. *An (A, B)- mset group drawn from a group G is said to be an (A, B)- mset Normal group if* $\max\{C_{M_{AB}}(x * y * x^{-1}), A\} \geq \min\{C_{M_{AB}}(y), B\}$, *for every x and y in G.*

Proposition 4.4. *If an (A, B)- mset group is an (A, B) mset normal group, then* $\max\{C_{M_{AB}}(x * y), A\} \geq \min\{C_{M_{AB}}(y * x), B\}$, *for every x and y in G.*

Proof. Replacing y by $y * x$ in the definition of (A, B)- mset normal group, we get this proposition. □

Corollary 4.4. *For an abelian group G, M_{AB} is normal iff $A < C_{M_{AB}}(x) < B$ for all x in G.*

Proposition 4.5. *$If M_{AB}$ is an mset normal group drawn from a group G, then M_r is a normal subgroup of G, for $A < r \leq B$.*

Proof. Choose r such that $A < r \leq B$. If $M_r = \phi$, is a normal subgroup of G. If M_r is a singleton set, then $m_r = \{e\}$, again a subgroup of G.

On the other hand, if M_r contains more than one element. Take two arbitrary elenemts x and y from M_r. Then, $C_{M_{AB}}(x) \geq r$ and $C_{M_{AB}}(y) \geq r$. Therefore, $\min\{C_{M_{AB}}(y), B\} = r$. From the definition of (A, B)- mset normal group $\max\{C_{M_{AB}}(x * y * x^{-1}, A\} \geq r$.

$C_{M_{AB}}(x * y * x^{-1} \geq r$, since $A < r \leq B$.

$\implies x * y * x^{-1} \in M_r$, proving that M_r is a normal subgroup of G. □

Proposition 4.6. *M_{AB} is an (A, B)- mset normal group drawn from a group G, and x, y elements of G.*

(a) *If $C_{M_{AB}}(x) \geq B$, then $C_{M_{AB}}(y * x * y^{-1}) \geq B$.*

(b) *If $A < C_{M_{AB}}(x) < B$, then $C_{M_{AB}}(y * x * y^{-1}) = C_{M_{AB}}(x)$.*

(c) *If $C_{M_{AB}}(x * y) \leq A$, then $C_{M_{AB}}(y * x) \leq A$.*

(d) *if $A < C_{M_{AB}}(x * y) < B$, then $C_{M_{AB}}(x * y) = C_{M_{AB}}(y * x)$.*

(e) *If $C_{M_{AB}}(x * y) \geq B$, then $C_{M_{AB}}(y * x) \geq B$.*

Proof. The poof is straight forward from the definition of (A, B)- mset normal group. □

4.3 Cosets of (A, B)- mset group

Definition 4.4. *Let M_{AB} be an (A, B)- mset group drawn from a group G and let $g \in G$. The* left coset *gM_{AB} is defined as $C_{gM_{AB}}(x) = \min\{\max(C_{M_{AB}}(g^{-1} * x), A), B\}, \forall x \in G$. The* right coset *$M_{AB}g$ is $C_{M_{AB}g}(x) = \min\{\max(C_{M_{AB}}(x * g^{-1}), A), B\}, \forall x \in G$.*

Proposition 4.7. *If M_{AB} is an (A, B)- mset group drawn from a group G with identity element e, then $eM_{AB} = M_{AB}e$.*

Proof. By Definition,

$$\begin{aligned}
C_{eM_{AB}}(x) &= \min\{\max(C_{M_{AB}}(e^{-1} * x), A), B\}, \forall x \in G \\
&= \min\{\max(C_{M_{AB}}(e * x), A), B\}, \forall x \in G \\
&= \min\{\max(C_{M_{AB}}(x), A), B\}, \forall x \in G \\
&= \min\{\max(C_{M_{AB}}(x * e), A), B\}, \forall x \in G \\
&= \min\{\max(C_{M_{AB}}(x * e^{-1}), A), B\}, \forall x \in G \\
&= C_{M_{AB}e}(x).
\end{aligned}$$ □

Proposition 4.8. (a) *$C_{eM_{AB}}(x) = A$ if $C_{M_{AB}}(x) \leq A$.*

(b) *If $A < C_{M_{AB}}(x) < B$, then $C_{eM_{AB}}(x) = C_{M_{AB}}(x)$.*

(c) *$C_{eM_{AB}}(x) = B$ if $C_{M_{AB}}(x) \geq B$.*

Proof. The proof is obtained directly from the definition of left coset. □

Corollary 4.5. *$eM_{AB} = M_{AB}$ if $A \leq C_{M_{AB}}(x) \leq B, \forall x \in G$.*

Note 4.2. *Similar results hold for right cosets also.*

Proposition 4.9. (a) *If M_{AB}- is an (A, B) mset group, then both eM_{AB} and $M_{AB}e$ are (A, B)- mset groups.*

(b) *If M_{AB} is an (A, B)- mset normal group, then both eM_{AB} and $M_{AB}e$ are (A, B)- mset normal groups.*

Theorem 4.9. *If M_{AB} is an (A, B)- mset group drawn from a group G with identity element e. Suppose $C_{M_{AB}}(e) \geq B$. An element $a \neq e \in M_B$, if and only if $aM_{AB} = eM_{AB}$.*

Similar result hold for right cosets also.

Proof. Let $a \neq e \in M_B$. Then $a^{-1} \in M_B$.
Case 1. For $x \in G$ with $C_{M_{AB}}(x) \geq B$,

$$\begin{aligned}
x \in M_B &\implies a^{-1} * x \in M_B \\
&\implies C_{M_{AB}}(a^{-1} * x) \geq B \\
&\implies C_{aM_{AB}}(x) = B,
\end{aligned}$$

by definition of left coset. For the same x, $C_{eM_{AB}}(x) = \min\{\max(C_{M_{AB}}(x), A), B\}$ $= B$. So, $C_{aM_{AB}}(x) = C_{eM_{AB}}(x)$.

Case 2. For $x \in G$ with $A < C_{M_{AB}}(x) < B$,

$$\begin{aligned} C_{M_{AB}}(a^{-1} * x) &= C_{M_{AB}}(x), \text{ by Theorem } 4.7 \\ &= C_{M_{AB}}(e^{-1} * x) \\ \therefore C_{aM_{AB}}(x) &= C_{eM_{AB}}(x) \end{aligned}$$

Case 3 : For $x \in G$ with $C_{M_{AB}}(x) \leq A$,

$$\begin{aligned} C_{M_{AB}}(a^{-1} * x) &\leq A, \text{ by Theorem 4.8} \\ \therefore C_{aM_{AB}}(x) &= A \\ &= C_{eM_{AB}}(x). \end{aligned}$$

Hence, in all the three cases, $C_{aM_{AB}}(x) = C_{eM_{AB}}(x)$ and this completes one part of the proof.

Conversely, assume that $aM_{AB} = eM_{AB}$ for some $a \in G$. $C_{aM_{AB}}(x) = C_{eM_{AB}}(x), \forall x \in G$ i.e., $\min\{\max(C_{M_{AB}}(a^{-1} * x), A), B\} = \min\{\max(C_{M_{AB}}(e^{-1} * x), A), B\}, \forall x \in G$. Taking $x = a$,

$$\begin{aligned} \min\{\max(C_{M_{AB}}(a^{-1} * a), A), B\} &= \min\{\max(C_{M_{AB}}(e^{-1} * a), A), B\} \\ i.e. \min\{\max(C_{M_{AB}}(e), A), B\} &= \min\{\max(C_{M_{AB}}(a), A), B\} \\ &\implies B \\ &= \min\{\max(C_{M_{AB}}(e^{-1} * a), A), B\} \\ &\implies C_{M_{AB}}(a) \geq B \\ &\implies a \in M_B. \end{aligned}$$ □

Corollary 4.6. *Let M_{AB} is an (A, B)- mset group drawn from a group G with identity element e. If $a \in M_B$, then $aM_{AB} = M_{AB}a = eM_{AB} = M_{AB}e$.*

Proof. if $a \in M_B$, then by above theorem $aM_{AB} = eM_{AB}$ and $aM_{AB} = eM_{AB}$. But by Proposition 4.24, $eM_{AB} = M_{AB}e$. □

Corollary 4.7. *Let M_{AB} is an (A, B)- mset group drawn from a group G and let $a, b \in G$. $aM_B = bM_B$ if and only if $aM_{AB} = bM_{AB}$.*
Similarly for right cosets.

Proof.

$$\begin{aligned} &aM_B = bM_B \\ &\Leftrightarrow a^{-1}b \in M_B \\ &\Leftrightarrow (a^{-1}b)M_{AB} = eM_{AB} \\ &\Leftrightarrow bM_{AB} = aM_{AB}. \end{aligned}$$ □

Theorem 4.10. *Let M_{AB} is an (A, B)- mset group drawn from a group G with identity element e and suppose $A < C_{M_{AB}}(e) < B$. Then, for an element $a \in G$, $C_{M_{AB}}(a) = C_{M_{AB}}(e)$ if and only if $aM_{AB} = eM_{AB}$*

Proof. Assume first that $C_{M_{AB}}(a) = C_{M_{AB}}(e)$. Choose an $x \in G$.
Case 1. $C_{M_{AB}}(x) \leq A$. Then $C_{M_{AB}}(a^{-1} * x) \leq A$, by Theorem 4.8 and Proposition 4.10 (b). Hence, by definition of left coset and Proposition 4.25 $C_{aM_{AB}}(x) = A = C_{eM_{AB}}(x)$.
Case 2. $A < C_{M_{AB}}(x) < C_{M_{AB}}(e)$ then, $C_{M_{AB}}(a^{-1} * x) = C_{M_{AB}}(x) = C_{M_{AB}}(e^{-1} * x)$, by Theorem 4.7 and Proposition 4.10 (b) i.e. $C_{aM_{AB}}(x) = C_{eM_{AB}}(x)$.
Case 3. $C_{M_{AB}}(x) \geq C_{M_{AB}}(e)$. Let $C_{M_{AB}}(e) = m$. $C_{M_{AB}}(x) = m$, by Theorem 4.11 (e).

Here, $a \in M_m$, by assumption and M_m being a subgroup, $a^{-1} \in M_m$. Also, $x \in M_m \implies (a^{-1} * x) \in M_m \implies C_{M_{AB}}(a^{-1} * x) = m$.

$$\begin{aligned}
\therefore C_{aM_{AB}}(x) &= \min\{\max(C_{M_{AB}}(a^{-1} * x), A), B\} \\
&= \min\{\max(m, A), B\} \\
&= \min\{\max(C_{M_{AB}}(e^{-1} * x), A), B\} \\
&= C_{eM_{AB}}(x).
\end{aligned}$$

From the above three cases, $aM_{AB} = eM_{AB}$. Conversely, assume that $aM_{AB} = eM_{AB}$

$$\begin{aligned}
C_{aM_{AB}}(x) &= C_{eM_{AB}}(x), \forall x \in G \\
C_{aM_{AB}}(a) &= C_{eM_{AB}}(a) \\
\min\{\max(C_{M_{AB}}(a^{-1} * a), A), B\} &= \min\{\max(C_{M_{AB}}(e^{-1} * a), A), B\} \\
\min\{\max(C_{M_{AB}}(e), A), B\} &= \min\{\max(C_{M_{AB}}(a), A), B\} \\
C_{M_{AB}}(a) &= C_{M_{AB}}(e).
\end{aligned}$$
□

5. Conclusion and future work

We have broadened the group structure in multiset context to a new scenario , (A, B) multiset group. Here both A and B are non negative real numbers and the (A, B) multiset group depends on A, B and the count value of the elements. Hence, in practical situations, it will be more adequate to apply (A, B) multiset groups, rather than multiset groups, and in this way, we are providing a novel path for research.

References

[1] D. E. Knuth, *The art of computer programming*, Semi numerical Algorithms, Adison- Wesley, 2 (1981).

[2] Chris Brink, *Multisets and the algebra of relevance logic*, Non-Classical Logic, 5 (1988), 75-95.

[3] Wayne D. Blizard, *The developement of multiset theory*, Modern logic, 1 (1991), 319-352.

[4] C. S. Calude, G. Paun, G. Rozenberg, A. Salomaa, *Mathematics of Multisets*, Springer Verlag, (2001), 347-358.

[5] N.J. Wildberger, *A new look at multisets*, School of Mathematics UNSW Sydney 2053, (2003).

[6] D. Singh, A. M. Ibrahim, T. Yohanna, J. N. SINGH, *An overview of the application of multiset*, Novi Sad J. Math, 37 (2007), 73-92.

[7] K. P. Girish, Sunil Jacob John, *Relations and functions in multiset context*, Information sciences, 179 (2009), 758-768.

[8] Azriel Rosenfield, *Fuzzy groups*, Journal of Mathematical Analysis and Applications, 35 (1971), 512-517.

[9] Sabu Sebastian, T. V. Ramakrishnan, *Multi-fuzzy sets: an expansion of fuzzy sets*, Fuzzy Information and Engineering, 1 (2011), 35-43.

[10] Yuying Li, Xuzhu Wang, Liqiong Yang, *A study of (λ, μ)-fuzzy subgroups*, Hindawi Publishing Corporation, Jounal of Applied Mathematics, vol. 2013 Article ID 485768 (2013).

[11] A.M. Ibrahim, P.A. Ejegwa, *A survey on the concept of multigroups*, Journal of the Nigerian Association of Mathematical Physics, 38 (2016), 1-8.

[12] Binod Chandra Tripathy, Shyamal Debnath, Debjani Rakshit, *On multiset group*, Proyecciones Journal of Mathematics, 33 (2018), 479-489.

[13] Johnson Aderemi Awolola, Adeku Musa Ibrahim, *Some results on multigroups*, Quasigroups and Related Systems, 24 (2016), 169-177.

[14] P.A. Ejegwa, *Upper and lower cuts of multigroups*, Prajna International Journal of Mathematical Sciences and Applications, 1 (2017), 19-26.

[15] S. K. Nazmul, P. Majumdar, S.K. Samanta, *On multisets and multigroups*, Annals of Fuzzy Mathematics and Informatics, 3 (2013), 643-656.

[16] Y. Tella, S. Daniel, *A study of group theory in the context of multiset theory*, International Journal of Science and Technology, 8 (2013), 609-615.

[17] Suma P, Sunil Jacob John, *Handbook of research on emerging applications of fuzzy algebraic structures*, IGI Global., Chapter 5, Multiset Approach to Algebraic Structures, (2019), 78-90.

[18] I. Deli, *Convex and concave sets based on soft sets and fuzzy soft sets*, Journal of New Theory, 29 (2019), 101-110.

[19] F. Karaaslan, I. Deli, *On soft neutrosophic classical sets*, Plaestine Journal of Mathematics, 9/1 (2020), 312-326.

[20] I. Deli, M. A. Keles, *Distance measures on trapezoidal fuzzy multi-numbers and application to multi-criteria decision-making problems*, Soft Computing, `https://doi.org/10.1007/s00500-021-05588-6`DOI s00500-021-05588-6 (2021).

Accepted: September 20, 2021

Sensitivity analysis of interest rate derivatives in a normal inverse Gaussian Lévy market

A. M. Udoye*
Department of Mathematics
Federal University Oye-Ekiti
Ekiti
Nigeria
adafavour2@yahoo.co.uk

G. O. S. Ekhaguere
Department of Mathematics
University of Ibadan
Ibadan
Nigeria
gose676@gmail.com

Abstract. Abrupt happenings in financial markets contribute to jumps of different magnitudes that invariably affect interest rate derivatives. Many of the existing interest rate models do not capture jumps, leading to inaccurate prediction of option prices and sensitivity analysis in the markets. To incorporate jumps in interest rate derivatives, we extend the Vasicek model with a Brownian motion as an underlying process to a model driven by a normal inverse Gaussian process, which is a subordinated Lévy process, use the extended model to obtain an expression for the price of an interest rate derivative called a zero-coupon bond. We employ Malliavin calculus to compute the greeks *delta* and *vega* of the derived price, which are important risk quantifiers in the interest rate derivative markets driven by a normal inverse Gaussian process.
Keywords: interest rate derivatives, Lévy process, Malliavin calculus, normal inverse Gaussian process, Vasicek model.

1. Introduction

Investing in an interest rate derivative market requires a good understanding of how to minimize risks. This may be achieved by formulating a model which incorporates sudden or rare occurrences that may lead to jumps in a market. Such occurrences often arise from changes in monetary policy, inflation, natural disaster, abrupt information, economic recession, presence of a pandemic, etc.

In the literature, many models of interest rate derivatives do not consider jumps and heavy tails. The present paper bridges this gap by adopting a subordinated Lévy process called a *normal inverse Gaussian (NIG) process* to derive an extended Vasicek interest rate model and use the extended model to derive an expression for the price of an interest rate derivative called a *zero-coupon*

∗. Corresponding author

bond and compute its sensitivity to some of its parameters using *Malliavin calculus*. These will assist an investor and risk manager to make the right decision and minimize risks in an NIG interest rate derivative market.

The NIG process was introduced by Barndorf-Nielsen [2] to generate good models for log-return process of prices and exchange rates [7]. Using the NIG process allows jumps and heavy tails to be captured. Examples of NIG markets include (i) volatile markets such as an electricity market, whose forward price has a return distribution with excess kurtosis and heavy tails [1]; and (ii) stock market prices [19]. Núñez [15] introduced the process as a replacement of the Gaussian assumption of underlying asset returns since it takes care of the heavy tails found in returns data series. Dhull and Kumar [9] emphasized the usefulness of the process in modelling various real-life time-series data. Lahcene [13] discussed an extension of the process in modelling and analyzing statistical data with emphasis on extensive sets of observations and some applications. Pintoux and Privault [18] discussed an interest rate derivative *zero-coupon bond price* using the Dothan model driven by a Wiener process while Yin et al. [22] emphasized that non-Gaussian Ornstein-Uhlenbeck process based on a negative/positive subordinated Lévy process fits and provides a better economic interpretation of the associated time series. Sabino [20] considered how to price energy derivatives for spot prices driven by a tempered stable Ornstein-Uhlenbeck process, while Hainaut [12] discussed an interest rate model driven by a mean reverting Lévy process with a sub-exponential memory of sample path achieved by considering an Ornstein–Uhlenbeck process in which the exponential decaying kernel is replaced by a Mittag–Leffler function. We adopt the Vasicek model since it has the property of mean-reversion and possibility of a negative interest rate. Research has shown that a good model should take care of negative interest rates that now occur in the current market environment as observed by Orlando et al. [16].

Bavouzet-Morel and Messaoud [3] discussed the Malliavin calculus for jump processes while Petrou [17] extended the theory of the calculus adding some tools for the computation of sensitivities. Bayazit and Nolder [4] applied the calculus to the sensitivities of an option whose underlying is driven by an exponential Lévy process. This work extends Bayazit and Nolder [4] to the sensitivity analysis of interest rate derivatives in a normal inverse Gaussian Lévy market.

In the next section, we discuss important mathematical tools to be employed in our results. In Section 3, we derive an extended Vasicek model driven by the NIG process and derive an equivalent expression for the zero-coupon bond price. In Section 4, we compute the greeks of the derived price using the Malliavin calculus, and discuss sensitivity analysis of the interest rate derivatives. In a previous publication [21], we derived expressions for certain greeks in a model involving the variance gamma process.

2. Foundational notion

In this section, we discuss important mathematical tools employed for the success of the paper.

2.1 The normal inverse Gaussian process

The inverse Gaussian process is a random process with infinite number of jumps for each finite period. The NIG process is a subordinated Lévy process.

Remark 2.1. 1. Let X be a random variable with an NIG distribution denoted $X \sim \text{NIG}(x;\alpha,\beta,\mu,\delta)$, then its probability density function is given by

$$f_{NIG}(x) = \frac{\alpha\delta\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu))}{\pi\cdot\sqrt{\delta^2+(x-\mu)^2}}K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})$$

where $\alpha > 0$, $|\beta| < \alpha$, $\delta > 0$, and $K_1(x)$ is the modified Bessel function of the third kind with index λ given by

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty t^{\lambda-1}\exp\Big(-\frac{1}{2}x\big(t+\frac{1}{t}\big)\Big)dt,\ x > 0.$$

2. The parameters α, β, δ and μ are for tail heaviness, symmetry, scale and location, respectively.

3. The characteristic function of the NIG process is given by

$$\phi_t(u) = \exp\big(-\delta t((\alpha^2-(\beta+iu)^2)^{\frac{1}{2}}-(\alpha^2-\beta^2)^{\frac{1}{2}})\big).$$

4. In what follows, we discuss the Malliavin calculus to be employed in the computation of greeks.

2.2 The Malliavin calculus for Lévy processes

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $X_i, i=1,\dots,n$ be a sequence of random variables with piecewise differentiable probability density functions. Let $C^p(\mathbb{R}^n)$ where $p,n \geq 1$, be the space of p times continuously differentiable functions. The following basic definitions will be utilized in the sequel.

Definition 2.1. *Let $L^0(\Omega,\mathbb{R})$ be the linear space of all $\mathbb{R}$-valued random variables on $(\Omega,\mathcal{B},\mathbb{P})$. A map $F : (L^0(\Omega,\mathbb{R}))^n \to L^0(\Omega,\mathbb{R})$, $n \in \mathbb{N}$ is defined as* (n,p)-simple functional *of the n random variables if there exists an $\mathbb{R}$-valued function $\widehat{F} \in C^p(\mathbb{R}^n)$ where*

$$F(X_1,\dots,X_n)(\omega) = \widehat{F}(X_1(\omega),\dots,X_n(\omega)),\ \omega\in\Omega,\ X_1,\dots,X_n \in L^0(\Omega,\mathbb{R}).$$

An (n,p)-simple process *of length n is a sequence of random variables $U = (U_i)_{i\leq n}$ such that $U_i(\omega) = u_i(X_1(\omega),\dots,X_n(\omega))$ where $u_i \in C^p(\mathbb{R}^n)$, $X_1,\dots,X_n \in L^0(\Omega,\mathbb{R})$ and $\omega\in\Omega$.*

We write $S_{(n,p)}$ for the space of all (n,p)-simple functionals and $P_{(n,p)}$ for the space of all (n,p)-simple processes.

Definition 2.2. *Let $F \in S_{(n,1)}$, where $F(X_1,\dots,X_n)(\omega)=\widehat{F}(X_1(\omega),\dots,X_n(\omega))$, $\omega\in\Omega$, $\widehat{F} \in C^1(\mathbb{R}^n)$, and $X_1,\dots,X_n \in L^0(\Omega,\mathbb{R})$. Define the operator $D : S_{(n,1)} \to (P_{(n,0)})^n$ called the* Malliavin derivative operator *by $DF = (D_iF)_{i\le n}$ where*

$$D_iF(X_1,\dots,X_n)(\omega) = \left(\frac{\partial \widehat{F}}{\partial x_i}\right)(X_1(\omega),\dots,X_n(\omega)),$$

$$D_iF(X)(\omega) = \left(\frac{\partial \widehat{F}}{\partial x}\right)(X(\omega)), \quad \text{when } n = 1. \tag{1}$$

Definition 2.3. *Let $F=(F_1,\dots,F_d)$ be a d-dimensional vector of simple functionals where $F_i\in S_{(n,1)}$. The matrix $\mathcal{M} = (\mathcal{M}(F)_{i,j})$ defined by*

$$\mathcal{M}(F)_{i,j} = \langle DF_i, DF_j\rangle_n = \sum_{m=1}^{n} D_mF_iD_mF_j$$

is called the Malliavin covariance matrix *of F [4]. This implies that if $n = 1$,*

$$\mathcal{M}(F)_{i,j} = \langle DF_i, DF_j\rangle = DF_iDF_j. \tag{2}$$

Definition 2.4. *Define the operator $\widetilde{\delta} : P_{(n,1)} \to S_{(n,0)}$ called the* Skorohod integral operator *for a simple process $U = (U_i)_{i=1,\dots,n} \in P_{(n,1)}$, $U_i(\omega) = u_i(X_1(\omega), \dots, X_n(\omega))$, $\omega \in \Omega$ by*

$$\begin{aligned}\widetilde{\delta}(U)(X_1,\dots,X_n) &= \sum_{i=1}^{n} \widetilde{\delta}_i(U)(X_1,\dots,X_n)\\ &= -\sum_{i=1}^{n}[D_iu_i(X_1,\dots,X_n) + u_i(X_1,\dots,X_n)\varphi_i(\mathbf{x})],\end{aligned}$$

where $\varphi_i(\mathbf{x}) = \frac{\partial \ln f_X(\mathbf{x})}{\partial x_i} = \frac{f'_{X_i}(\mathbf{x})}{f_X(\mathbf{x})}, f_X(\mathbf{x}) \neq 0,\ 1 \le i \le n,\ \mathbf{x} = x_1,\dots,x_n$ and $f_X(x)$ is the density function of the random variable X.

Definition 2.5. *The* Ornstein-Uhlenbeck (O-U) operator *$L : S_{(n,2)} \to S_{(n,0)}$ is defined as*

$$(LF)(X_1,\dots,X_n) = -\sum_{i=1}^{n}[(\partial^2_{ii}\widehat{F})(X_1,\dots,X_n) + \varphi_i(\mathbf{x})(\partial_i\widehat{F})(X_1,\dots,X_n)],$$

where $F \in S_{(n,2)}$, $X_1,\dots,X_n \in L^0(\Omega,\mathbb{R})$ and $\varphi_i(\mathbf{x})$ is given by Definition 2.4. For $n = 1$,

$$LF(X) = -[DD\widehat{F}(X) + \varphi(x)D\widehat{F}(X)] \tag{3}$$

where

$$\varphi(x) = \frac{\partial \ln f_X(x)}{\partial x} = \frac{f'_X(x)}{f_X(x)}, \text{ and } f_X(x) \neq 0. \tag{4}$$

2.2.1 Malliavin integration by parts theorem

To compute the greeks of the interest rate derivative, we need the integration by parts theorem of the Malliavin calculus stated below.

Proposition 2.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space;* $X_1, \dots, X_n$, *a sequence of real-valued random variables on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $P = (P_1, \dots, P_d) \in (S_{(n,2)})^d$, $Q \in S_{(n,1)}$. *Let* $\mathcal{M} = (\mathcal{M}_{ij}(P))_{1\leq i\leq n, 1\leq j\leq n}$ *be an invertible Malliavin covariance matrix with inverse given by* $\mathcal{M}(P)^{-1} = (\mathcal{M}(P)_{ij})^{-1}_{1\leq i\leq n,1\leq j\leq n}$. *Suppose that* $\mathbb{E}[\det \mathcal{M}(P)^{-1}]^p < \infty$, $p \geq 1$, *and* $\Phi : \mathbb{R}^d \to \mathbb{R}$ *represents a smooth bounded function with bounded derivative. Then,*

$$\mathbb{E}[\partial_i \Phi(P)Q] = \mathbb{E}[\Phi(P)H_i(P,Q)] \text{ where } \mathbb{E}[H_i(P,Q)] < \infty,\ i = 1, 2, \dots, n;$$

and the Malliavin weight is given by

$$H_i(P,Q) = \sum_{j=1}^{n} Q\mathcal{M}(P)^{-1}_{ij} LP_j - \mathcal{M}(P)^{-1}_{ij}\langle DP_j, DQ\rangle - Q\langle DP_j, D\mathcal{M}(P)^{-1}_{ij}\rangle.$$

Remark 2.2. For $d = n = 1$, the Malliavin weight is given by

$$H(P,Q) = Q\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle.$$

We proceed to the next section and derive our results.

3. The Short rate model under the NIG process

In this section, we extend the Vasicek short rate model to a market driven by the NIG process and derive an expression for the price of an interest rate derivative called a *zero-coupon bond.*

The Vasicek (1977) interest rate model satisfies the stochastic differential equation given by

$$dr_t = a(b - r_t)dt + \sigma dX_t \tag{5}$$

where $X_t = X(t)$, b, a and σ denote the Lévy process, long-term mean rate, speed of mean reversion and volatility of the interest rate, respectively.
Integrating equation (5) by using Itô's formula, we obtain

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dX_s. \tag{6}$$

We adopt the NIG model given by $X_t = \mathbf{w}t + \beta\delta^2 IG_t + \delta W(IG_t)$ [11] where $\mathbf{w}$ is the cumulant generating function given by

$$\mathbf{w} = -\frac{1}{t}\ln(\phi_t(-i)) = \delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}).$$

The parameters α, β and δ control the behaviour of the tail, skewness and scale of the distribution, respectively. $IG_t = IG(t)$ denotes the inverse Gaussian process. We represent the standard Brownian motion $W(t)$ as the process $W(t)-W(s) = \sqrt{|\, t-s\,|}Z$, $t, s \geq 0$, where Z is a $N(0,1)$ Gaussian random variable. Then, $W(t) = \sqrt{t}Z$ and $\mathbb{E}(W(t)W(s)) = \min(t,s)$, $t, s \geq 0$. Thus,

$$\begin{aligned} X_t &= \mathbf{w}t + \delta\sqrt{IG(t)}Z + \beta\delta^2 IG(t), \\ &\Longrightarrow dX_t = \mathbf{w}dt + \delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t). \end{aligned} \tag{7}$$

Substituting equation (7) into (6) and evaluating, we have

$$\begin{aligned} r_t &= r_0 e^{-at} + b(1-e^{-at}) + \frac{\sigma\mathbf{w}}{a}(1-e^{-at}) + \sigma\delta\Big(\sum_{0\leq s\leq t}(\Delta\sqrt{IG(s)}Z \\ &\quad + \beta\delta\Delta IG(s))e^{-a(t-s)}\Big). \end{aligned} \tag{8}$$

We adopt the above expression (8) to derive an expression for the zero-coupon bond price driven by the NIG process.

3.1 Expression for a zero-coupon bond price with a Vasicek short rate model under the NIG process

The dynamics of the zero-coupon bond price under a risk neutral measure is given by

$$dP = r_t P dt + \sigma P dX_t. \tag{9}$$

Applying Itô's lemma to equation (9), we obtain

$$\begin{aligned} d\ln P &= r_t dt + \sigma\mathbf{w}dt + \sigma(\delta\Delta\sqrt{IG(t)}Z + \beta\delta^2\Delta IG(t)) - \frac{1}{2}\sigma^2(\delta\Delta\sqrt{IG(t)}Z \\ &\quad + \beta\delta^2\Delta IG(t))^2. \end{aligned} \tag{10}$$

Integrating equation (10), we get

$$\begin{aligned} \ln P(t,T) &= -\Big(\int_t^T r_u du + \sigma\mathbf{w}\int_t^T du \\ &\quad + \sigma\Big(\sum_{0\leq u\leq T}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u)) \\ &\quad - \sum_{0\leq u\leq t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))\Big) - \frac{1}{2}\sigma^2\Big(\sum_{0\leq u\leq T}(\delta\Delta\sqrt{IG(u)}Z \\ &\quad + \beta\delta^2\Delta IG(u))^2 - \sum_{0\leq u\leq t}(\delta\Delta\sqrt{IG(u)}Z + \beta\delta^2\Delta IG(u))^2\Big)\Big). \end{aligned} \tag{11}$$

By equation (8), it follows that

$$
\begin{aligned}
\int_t^T r_u du &= \frac{-r_0}{a}(e^{-aT} - e^{-at}) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\
&+ \frac{\sigma \mathbf{w}}{a}\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\
&+ \sigma\delta\big(\sum_{0\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(u-s)}\big) \\
&- \sigma\delta\big(\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s))e^{-a(u-s)}\big).
\end{aligned} \tag{12}
$$

Substituting equation (12) into (11) and evaluating, we obtain the zero-coupon bond price driven by the NIG process as

$$
\begin{aligned}
P(t,T) = \exp\Bigg(&- \Bigg[\frac{-r_0}{a}(e^{-aT} - e^{-at}) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\
&+ \frac{\sigma \mathbf{w}}{a}\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) + \sigma\delta\big(\sum_{0\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z \\
&+ \beta\delta\Delta IG(s))e^{-a(u-s)}\big) - \sigma\delta\big(\sum_{0\le u\le t}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}Z + \beta\delta\Delta IG(s)) \\
&\cdot e^{-a(u-s)}\big) + \sigma\mathbf{w}[T-t] + \sigma\delta\Bigg(\sum_{0\le u\le T}(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)) \\
&- \sum_{0\le u\le t}(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u))\Bigg) - \frac{1}{2}\sigma^2\delta^2\Bigg(\sum_{0\le u\le T}(\Delta\sqrt{IG(u)}Z \\
&+ \beta\delta\Delta IG(u))^2 - \sum_{0\le u\le t}(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u))^2\Bigg)\Bigg]\Bigg).
\end{aligned} \tag{13}
$$

Besides being a function of t and T, the expression on the right hand side of equation (13) also depends on $r_0, \beta, \delta, \sigma, \mathbf{w}$ and Z. Thus, in the sequel, we shall regard P as a function of $t, T, r_0, \beta, \delta, \sigma, \mathbf{w}$ and Z.

The price of the zero-coupon bond driven by the NIG Lévy process given by equation (13) can be written as

$$
\begin{aligned}
P(t,T) = \exp\Bigg(&- \Bigg(\frac{-r_0}{a}(e^{-aT} - e^{-at}) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\
&+ \frac{\sigma \mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \mathbf{w}\sigma[T-t] \\
&+ \sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z + \beta\delta\Delta IG(s)e^{-a(u-s)}\big) \\
&+ \sigma\delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\
&- \frac{\sigma^2\delta^2}{2}\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Bigg)\Bigg)\Bigg).
\end{aligned} \tag{14}
$$

We state the necessary lemmas for the computation of the delta which measures the sensitivity of a bond option price driven by the NIG process to changes in the initial interest rate and vega which measures the sensitivity of the bond option price with respect to changes in the volatility of the short rate model.

Lemma 3.1. *Let P be the price of a zero-coupon bond driven by the NIG process. Then, the Malliavin derivative on P is given by*

$$\begin{aligned} DP = -\Big[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P. \end{aligned} \tag{15}$$

Proof. By equation (1) of Definition 2.2 and the zero-coupon price given by equation (14), we get the Malliavin derivative

$$\begin{aligned} DP = -\Big[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \frac{\sigma^2\delta^2}{2}\Big(2\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P. \end{aligned}$$

Hence, the result follows. □

Lemma 3.2. *Let P be the price of the zero-coupon bond driven by the NIG process. Then, the Ornstein Uhlenbeck operator L on P is given by*

$$\begin{aligned} LP = -\Big[&\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big) + \Big(\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) \\ &+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)^2 \\ &- \varphi(z)\Big(\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)\Big]P,\ \varphi(z) = -z. \end{aligned} \tag{16}$$

Proof. By equation (15) of Lemma 3.1, it follows that

$$\begin{aligned} DDP = \sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)P + \Big[&\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) \\ + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Big(&\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \delta\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^2 P. \end{aligned}$$

By equations (3) and (4) of Definition 2.5, we obtain

$$LP = -[DDP + \varphi(z)DP]$$

where

$$\varphi(z) = \frac{\partial \ln f_{\mathcal{N}}(z)}{\partial z} = \frac{\partial \ln\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\right)}{\partial z} = -z.$$

Substituting DDP and equation (15) of Lemma 3.1 into LP yields the desired result. □

Lemma 3.3. *Let P be the price of the zero-coupon bond driven by the NIG process and $\mathcal{M}(P)$, its Malliavin covariance matrix. Then,*

$$\begin{aligned} M(P)^{-1} = \Big[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-2} P^{-2}. \end{aligned} \tag{17}$$

Proof. By equation (2) of Definition 2.3, $\mathcal{M}(P) = \langle DP, DP\rangle$. Thus, by equation (15), it follows that

$$\begin{aligned} M(P) = \Big[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{2} P^{2}. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{M}(P)^{-1} = \Big(\Big[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]P\Big)^{-2} \end{aligned}$$

which gives equation (17). □

Lemma 3.4. *Let P be the price of the zero-coupon bond driven by the NIG process. Then, the Malliavin derivative of the inverse Malliavin covariance matrix of P is given by*

$$\begin{aligned} D\mathcal{M}(P)^{-1} = 2\Big[&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\left(\Delta\sqrt{IG(s)}e^{-a(u-s)}\right) + \sigma\delta \sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-3} P^{-2} \end{aligned} \tag{18}$$

$$\Big[\Big(\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)^2$$
$$+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\Big].$$

Proof. Applying Malliavin derivative to equation (17) gives

$$D\mathcal{M}(P)^{-1}=2\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^{-3}P^{-2}$$
$$\cdot\Big[\Big[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big)+\sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})$$
$$-\sigma^2\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u)+\Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Big]^2$$
$$+\sigma^2\delta^2\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\Big]$$

which yields the desired result. □

4. The greeks of the zero-coupon bond price driven by the NIG Lévy process

The greeks serve as risk quantifiers. They give insight on various dimensions of insecurity involved in grabbing a bond option's position. Investors and risk managers use the greeks to predict future price and hedge risks. Some of the greeks are delta, vega, gamma and Theta. We shall concentrate on the delta and vega.

Remark 4.1. The price of a call option, with P as the underlying is given by

$$\mathbb{V}=e^{-r_0T}\mathbb{E}[\Phi(P)]$$

where $\Phi(P)=\max(P-K,0)$ is the payoff with strike price K.

A greek is computed using the formula

$$\frac{\partial\mathbb{V}}{\partial\varsigma}=\frac{\partial\big(e^{-r_0T}\mathbb{E}[\Phi(P)]\big)}{\partial\varsigma}$$

where ς represents a parameter of the bond price whose effect is to be determined.

4.1 Computation of *delta* for NIG-driven interest rate derivatives

The greek *delta* measures the sensitivity of the zero-coupon bond option price to changes in its initial interest rate. It helps investors and portfolio managers by indicating the extent to which the bond option's price will move when the initial interest rate increases by a unit currency. This is very important because movements in the underlying, that is, the initial interest rate can change the worth of their investment [8].

Let P be the zero-coupon bond price given by equation (14), $\Phi(P)$ be the payoff function and $Q = \frac{\partial P}{\partial r_0}$. Then, by Proposition 2.1,

$$\triangle_{NIG} = \frac{\partial}{\partial r_0}[e^{-r_0 T}\mathbb{E}(\Phi(P))] = -Te^{-r_0 T}\mathbb{E}(\Phi(P)) + e^{-r_0 T}\mathbb{E}[\Phi(P)H(P,Q)].$$

Next, we establish Lemmas 4.1 - 4.4 using Lemmas 3.1-3.4, to obtain the Malliavin weight $H(P,Q)$.

Lemma 4.1. *Let P be the zero-coupon bond price driven by the NIG process and $Q = \frac{\partial P}{\partial r_0}$. Then the following hold:*

$$Q = \frac{1}{a}(e^{-aT} - e^{-at})P \tag{19}$$

and

$$\begin{aligned} DQ = -\frac{1}{a}(e^{-aT} - e^{-at})\Big(&\sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) \\ &+ \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &- \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big)\Big)P. \end{aligned} \tag{20}$$

Proof. Applying partial derivative to equation (14) yields equation (19). Moreover, the Malliavin derivative

$$DQ = \frac{1}{a}(e^{-aT} - e^{-at})DP.$$

Substituting DP from equation (15) into the above equation yields equation (20). □

Lemma 4.2. *Let P be the zero-coupon bond price driven by the NIG process and L, the Ornstein-Uhlenbeck operator. Then,*

$$\begin{aligned} &Q\mathcal{M}(P)^{-1}LP \\ &= -\frac{1}{a}(e^{-aT} - e^{-at})\Big[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\widehat{K}^{-2} + 1 - \varphi(z)\widehat{K}^{-1}\Big], \end{aligned} \tag{21}$$

where $\varphi(z) = -z$ *and* $\widehat{K}$ *is given by*

$$\begin{aligned}\widehat{K} &= \sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\ &\quad - \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\big).\end{aligned} \tag{22}$$

Proof. The result follows from Lemmas 3.2, 3.3 and 4.1 by substituting Q from equation (19) of Lemma 4.1, $\mathcal{M}(P)^{-1}$ from equation (17) of Lemma 3.3, and LP from equation (16) of Lemma 3.2 into $Q\mathcal{M}(P)^{-1}LP$. □

Lemma 4.3. *Let* P *be the zero-coupon bond price driven by the NIG process. Then,*

$$\mathcal{M}(P)^{-1}\langle DP, DQ\rangle = \frac{1}{a}(e^{-aT} - e^{-at}). \tag{23}$$

$$\begin{aligned}&Q\langle DP, D\mathcal{M}(P)^{-1}\rangle \\ &= -2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\Bigg[\frac{\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)}{\widehat{K}^2} + 1\Bigg],\end{aligned} \tag{24}$$

where $\widehat{K}$ *is given by equation* (22).

Proof. The result in equation (23) follows from Lemmas 3.1, 3.3 and 4.1 by substituting $\mathcal{M}(P)^{-1}$ from equation (17) of Lemma 3.3, DP from equation (15) of Lemma 3.1 and DQ from equation (20) of Lemma 4.1 into $\mathcal{M}(P)^{-1}\langle DP, DQ\rangle$; while the result in equation (24) follows from Lemmas 3.1, 3.4 and 4.1 by substituting Q from equation (19) of Lemma 4.1, DP from equation (15) of Lemma 3.1 and $D\mathcal{M}(P)^{-1}$ from equation (18) of Lemma 3.4 into $Q\langle DP, D\mathcal{M}(P)^{-1}\rangle$. □

Lemma 4.4. *Let* P *be the zero-coupon bond price driven by the NIG process and its payoff function be given by* $\Phi(P) = \max(P(t,T) - K, 0)$. *Then,*

$$\begin{aligned}\mathbb{E}[\Phi(P)] &= \int_K^\infty\int_K^\infty (p(t,T,y,z) - K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \\ &\quad\cdot\Bigg(\frac{t(y(u))^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}} \\ &\quad\cdot\exp\big(-\frac{1}{2}\big(\frac{t^2}{y(u)} + (\delta\sqrt{\alpha^2-\beta^2})^2 y(u)\big)\big)\cdot \mathbf{1}_{y>0}\Bigg)dydz\end{aligned}$$

where K *is the strike price and from equation* (14), $p(t,T) = p(t,T,y,z)$ *is given by*

$$\begin{aligned}p(t,T) = \exp\Bigg(-\Bigg[&\frac{-r_0}{a}(e^{-aT} - e^{-at}) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\ &+ \frac{\sigma\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big]\end{aligned} \tag{25}$$

$$+ \mathbf{w}\sigma[T-t] + \sigma\delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\sqrt{y(s)}e^{-a(u-s)}z + \beta\delta y(s)e^{-a(u-s)}\big)$$
$$+ \sigma\delta\sum_{t\le u\le T}\big(\sqrt{y(u)}z+\beta\delta y(u)\big) - \frac{\sigma^2\delta^2}{2}\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)^2\Big]\Big).$$

Proof. Let $f_{\mathcal{N}(z;0,1)}$ and $f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})$ be the probability density functions for a Gaussian random variable and an inverse Gaussian random variable, respectively. Then,

$$\begin{aligned}\mathbb{E}[\Phi(P)] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(p)\cdot f_{\mathcal{N}(z;0,1)} f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\big)dzdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \max(p(t,T)-K,0)\cdot f_{\mathcal{N}(z;0,1)} f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})\big)dzdy.\end{aligned}$$

where K is a constant, $f_{\mathcal{N}(z;0,1)} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ and

$$\begin{aligned}&f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2}) \\ &= \frac{ty^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\big(\frac{t^2}{y}+(\delta\sqrt{\alpha^2-\beta^2})^2 y\big)\Big)\cdot \mathbf{1}_{y>0}.\end{aligned}$$

Substituting the expression for $f_{\mathcal{N}(z;0,1)}$ and $f_{IG}(y;t,\delta\sqrt{\alpha^2-\beta^2})$ into $\mathbb{E}[\Phi(P)]$ gives the desired result. □

Lemma 4.5. *Let P be the zero-coupon bond price driven by the NIG process and let $\mathbb{E}[\Phi(P)H(P,Q)] = \mathbb{E}[\Phi(P)H(P,\frac{\partial P}{\partial r_0})]$. Then,*

$$\begin{aligned}&\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big] \\ &= \int_K^\infty\int_K^\infty (p(t,T,y,z)-K)H\Big(p,\frac{\partial p}{\partial r_0}\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2} \\ &\cdot\Big(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\big(-\frac{1}{2}\big(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2 y(u)\big)\big)\cdot \mathbf{1}_{y>0}\Big)dydz\end{aligned}$$

and the Malliavin weight for the delta satisfies

$$H\Big(p,\frac{\partial p}{\partial r_0}\Big) = \frac{1}{a}(e^{-aT}-e^{-at})\Big(\sigma^2\delta^2\sum_{t\le u\le T}(\sqrt{y(u)})^2\widehat{K}^{*^{-2}} - z\widehat{K}^{*^{-1}}\Big) \tag{26}$$

where $\widehat{K}^*$ is obtained from $\widehat{K}$ given by equation (22) *as*

$$\begin{aligned}\widehat{K}^* &= \sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\sqrt{y(s)}e^{-a(u-s)}\big) + \sigma\delta\sum_{t\le u\le T}(\sqrt{y(u)}) \\ &- \sigma^2\delta^2\big(\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)\sqrt{y(u)}\big).\end{aligned} \tag{27}$$

Proof. From Proposition 2.1, the Malliavin weight becomes

$$H(P,Q)=H\Big(P,\frac{\partial P}{\partial r_0}\Big)=Q\mathcal{M}(P)^{-1}LP-\mathcal{M}(P)^{-1}\langle DP,DQ\rangle-Q\langle DP,D\mathcal{M}(P)^{-1}\rangle.$$

Substituting equation (21) from Lemma 4.2 for $Q\mathcal{M}(P)^{-1}LP$, equations (23) and (24) from Lemma 4.3 for $\mathcal{M}(P)^{-1}\langle DP,DQ\rangle$ and $Q\langle DP,D\mathcal{M}(P)^{-1}\rangle$, respectively into $H(P,Q)$, we obtain the expression in (26) from

$$H(P,Q)=\frac{1}{a}(e^{-aT}-e^{-at})\big(\sigma^2\delta^2\sum_{t\leq u\leq T}(\Delta\sqrt{IG(u)})^2\widehat{K}^{-2}+\varphi(z)\widehat{K}^{-1}\big),$$

where $\varphi(z)=-z$ and $\widehat{K}$ is given by equation (22). Hence, the result follows. □

Theorem 4.1. *Let P be the zero-coupon bond price driven by the NIG process and $Q=\frac{\partial P}{\partial r_0}$, then*

$$\begin{aligned}\triangle_{NIG}&=e^{-r_0T}\bigg(-T\int_K^\infty\int_K^\infty(p(t,T,y,z)-K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\\&\cdot\bigg(\frac{t(y(u))^{-3/2}\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\\&\cdot\exp\big(-\frac{1}{2}(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2y(u))\big)\cdot\mathbf{1}_{y>0}\bigg)dydz\\&+\int_K^\infty\int_K^\infty(p(t,T,y,z)-K)H\Big(p,\frac{\partial p}{\partial r_0}\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2}\\&\cdot\bigg(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\big(-\frac{1}{2}(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2y(u))\big)\cdot\mathbf{1}_{y>0}\bigg)dydz\bigg)\end{aligned}$$

where $H(p,\frac{\partial p}{\partial r_0})$ is given by Lemma 4.5.

Proof. The greek *delta* is given by

$$\triangle_{NIG}=\frac{\partial}{\partial r_0}e^{-r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}(-T\mathbb{E}[\Phi(P)]+\mathbb{E}[\Phi(P)H(P,Q)]).$$

Substituting $\mathbb{E}[\Phi(P)]$ given by equation (25) of Lemma 4.4 and $\mathbb{E}[\Phi(P)H(P,Q)]$ given by Lemma 4.5 into $\triangle_{NIG}$, gives the desired result. □

4.2 Computation of *vega* for the NIG-driven interest rate derivative

The greek *vega* $\mathcal{V}$ measures the sensitivity of the zero-coupon bond option price with respect to changes in its volatility. High vega implies that the bond option's value is very sensitive to little shift in volatility [6]. It presents uncertainty in future prices for the underlying contract [5]. It is given by

$$\mathcal{V}=\frac{\partial}{\partial\sigma}e^{r_0T}\mathbb{E}[\Phi(P)]=e^{-r_0T}\mathbb{E}\Big[\Phi'(P)\frac{\partial P}{\partial\sigma}\Big]=e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial\sigma}\Big)\Big].$$

Lemma 4.6. *Let P be the zero-coupon bond price driven by the NIG process and $Q_\sigma = \frac{\partial P}{\partial \sigma}$. Then,*

$$
\begin{aligned}
Q_\sigma = -\Bigg[&\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] \\
&+ \mathbf{w}[T-t] + \delta \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z \\
&+ \beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\
&- \sigma\delta^2\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Bigg]P,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
DQ_\sigma = -\Bigg[&\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\
&- 2\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]P + \widetilde{\Lambda}\widehat{K}P,
\end{aligned}
\tag{29}
$$

where $\widehat{K}$ is given by equation (22) and

$$
\begin{aligned}
\widetilde{\Lambda} = &\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \mathbf{w}[T-t] \\
&+ \delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z \\
&+ \beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta\sum_{t\le u\le T}\big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\
&- \sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Big).
\end{aligned}
\tag{30}
$$

Proof. Applying partial derivative to equation (14) yields equation (28). Hence, the Malliavin derivative

$$
\begin{aligned}
DQ_\sigma = -\Bigg[&\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\
&- \sigma\delta^2\Big(2\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Big)\Bigg]P \\
&+ \Bigg(-\Bigg[\frac{\mathbf{w}}{a}\big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big] + \mathbf{w}[T-t] \\
&+ \delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}Z
\end{aligned}
$$

$$
\begin{aligned}
&+ \beta\delta\Delta IG(s)e^{-a(u-s)}\big) + \delta \sum_{t\le u\le T} \big(\Delta\sqrt{IG(u)}Z + \beta\delta\Delta IG(u)\big) \\
&- \sigma\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)^2\Bigg)\Bigg] \\
&\cdot\Bigg(-\Bigg[\sigma\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\Delta\sqrt{IG(s)}e^{-a(u-s)}\big) \\
&+ \sigma\delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - \sigma^2\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) \\
&+ \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg]\Bigg)\Bigg)P
\end{aligned}
$$

which yields equation (29). □

Lemma 4.7. *Let P be the zero-coupon bond price driven by the NIG process. The following holds concerning the sensitivity with respect to σ:*

$$
\begin{aligned}
&Q_\sigma\mathcal{M}(P)^{-1}LP \\
&= \widetilde{\Lambda}\Bigg[\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\widehat{K}^{-2} + 1 - \varphi(z)\widehat{K}^{-1}\Bigg],\ \varphi(z) = -z,
\end{aligned}
\tag{31}
$$

where $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) *and* (22)*, respectively.*

Proof. The result follows from Lemmas 3.2, 3.3 and 4.6. Substituting equation (28) of Lemma 4.6 for Q_σ, equation (17) of Lemma 3.3 for $\mathcal{M}(P)^{-1}$, and equation (16) of Lemma 3.2 for LP into $Q_\sigma\mathcal{M}(P)^{-1}LP$ yields the expression in equation (31). □

Lemma 4.8. *Let P be the zero-coupon bond price driven by the NIG process. Then,*

$$
\begin{aligned}
&\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle \\
&= \widehat{K}^{-1}\Bigg[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\Delta\sqrt{IG(s)}e^{-a(u-s)}) + \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) \\
&- 2\sigma\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) + \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg] - \widetilde{\Lambda},
\end{aligned}
\tag{32}
$$

where $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) *and* (22)*, respectively.*

Proof. The result follows from Lemmas 3.1, 3.3 and 4.6 by substituting equation (17) of Lemma 3.3 for $\mathcal{M}(P)^{-1}$, equation (15) of Lemma 3.1 for DP and equation (29) of Lemma 4.6 for DQ_σ into $\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle$. □

Lemma 4.9. *Let P denote the zero-coupon bond price driven by the NIG process. Then, the following holds:*

$$Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle = 2\widetilde{\Lambda}\Big[1+\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\widehat{K}^{-2}\Big], \tag{33}$$

where $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) *and* (22)*, respectively.*

Proof. The result follows from Lemmas 3.1, 3.4 and 4.6 by substituting equation (28) of Lemma 4.6 for Q_σ, equation (15) of Lemma 3.1 for DP and equation (18) of Lemma 3.4 for $D\mathcal{M}(P)^{-1}$ into $Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle$. □

Lemma 4.10. *Let P be the zero-coupon bond price driven by the NIG process. Then, the Malliavin weight for the greek* vega *is given by*

$$\begin{aligned} H\Big(p,\frac{\partial p}{\partial\sigma}\Big) &= z\widetilde{\Lambda}\widehat{K}^{*^{-1}} - \widetilde{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\sqrt{y(u)})^2\big)\widehat{K}^{-2}\\ &- \widehat{K}^{-1}\Big[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}(\sqrt{y(s)}e^{-a(u-s)})\\ &+\delta\sum_{t\le u\le T}(\sqrt{y(u)}) - 2\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)\sqrt{y(u)}\Big)\Big], \end{aligned} \tag{34}$$

where $\widehat{K}^*$ is given by equation (27) *and*

$$\begin{aligned} \widetilde{\Lambda}^* &= \frac{\mathbf{w}}{a}\big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big]+\mathbf{w}[T-t]\\ &+\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\big(\sqrt{y(s)}e^{-a(u-s)}z\\ &+\beta\delta y(s)e^{-a(u-s)}\big)+\delta\sum_{t\le u\le T}\big(\sqrt{y(u)}z+\beta\delta y(u)\big)\\ &-\sigma\delta^2\Big(\sum_{t\le u\le T}(\beta\delta y(u)+\sqrt{y(u)}z)^2\Big). \end{aligned} \tag{35}$$

Proof. The Malliavin weight $H(P,Q_\sigma)$ for the sensitivity with respect to volatility, is obtained by substituting equation (31) of Lemma 4.6 for $Q_\sigma\mathcal{M}(P)^{-1}LP$, equation (32) of Lemma 4.7 for $\mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle$ and equation (33) of Lemma 4.8 for $Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle$ into $H(P,Q_\sigma)$. Thus,

$$\begin{aligned} H(P,Q_\sigma) &= H\Big(P,\frac{\partial P}{\partial\sigma}\Big)\\ &= Q_\sigma\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_\sigma\rangle - Q_\sigma\langle DP, D\mathcal{M}(P)^{-1}\rangle \end{aligned}$$

$$
\begin{aligned}
&= -\widetilde{\Lambda}\varphi(z)\widehat{K}^{-1} - \widetilde{\Lambda}\sigma^2\delta^2\big(\sum_{t\le u\le T}(\Delta\sqrt{IG(u)})^2\big)\widehat{K}^{-2} \\
&- \widehat{K}^{-1}\Bigg[\delta\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{IG(s)}e^{-a(u-s)} \\
&+ \delta\sum_{t\le u\le T}(\Delta\sqrt{IG(u)}) - 2\sigma\delta^2\Bigg(\sum_{t\le u\le T}(\beta\delta\Delta IG(u) \\
&+ \Delta\sqrt{IG(u)}Z)\Delta\sqrt{IG(u)}\Bigg)\Bigg],
\end{aligned}
$$

where $\varphi(z) = -z$; $\widetilde{\Lambda}$ and $\widehat{K}$ are given by equations (30) and (22), respectively. Hence, the result follows. □

Theorem 4.2. *Let P be the zero-coupon bond price driven by the NIG process. Then, the greek* vega *is given by*

$$
\begin{aligned}
\mathcal{V} = &\int_K^\infty\int_K^\infty (p(t,T,y,z)-K)H\Big(p,\frac{\partial p}{\partial\sigma}\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2} \\
&\cdot\Bigg(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\big(-\frac{1}{2}(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2y(u))\big)\cdot\mathbf{1}_{y>0}\Bigg)dydz,
\end{aligned}
$$

where the Malliavin weight $H(p,\frac{\partial p}{\partial\sigma})$ is given by equation (34) *of Lemma* 4.10.

Proof. Recall that $\mathcal{V} = e^{-r_0T}\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial\sigma}\Big)\Big]$. Thus,

$$
\begin{aligned}
\mathcal{V} = &\int_{\mathbb{R}}\int_{\mathbb{R}}\max(p(t,T,y,z)-K,0)H\Big(p,\frac{\partial p}{\partial\sigma}\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\cdot t(y(u))^{-3/2} \\
&\cdot\Bigg(\frac{\exp(t(\delta\sqrt{\alpha^2-\beta^2}))}{\sqrt{2\pi}}\exp\big(-\frac{1}{2}(\frac{t^2}{y(u)}+(\delta\sqrt{\alpha^2-\beta^2})^2y(u))\big)\cdot\mathbf{1}_{y>0}\Bigg)dydz.
\end{aligned}
$$

Hence, the result follows. □

5. Discussion and conclusion

In this paper, we have extended the work of Bavouzet-Morel & Messaoud [3] and Bayazit & Nolder [4] to the sensitivity analysis of an interest rate derivative market driven by a subordinated Lévy process. The Vasicek interest rate model was extended by considering the normal inverse Gaussian subordinated Lévy process. This was used to derive an expression for the price of a zero-coupon bond. The new model is important for transactions in a Lévy market situation where the prices of financial derivatives may experience jumps of different sizes. The greeks, namely: *delta* $\triangle_{NIG}$ and *vega* $\mathcal{V}$ were computed using the Malliavin integration by parts formula. The greeks assist an investor or decision maker to evaluate certain risks and predict the possibility of making money in a particular investment. Vega is important since an increase in volatility will increase the bond option price while a decrease in volatility will lead to a decrease in the bond option value. It helps investors to quantify the risk in the interest rate derivative Lévy market as the volatility changes. An investor or portfolio manager requires an adequate understanding of these greeks in order to predict future worth of a bond option so as to minimize risks. The work provided a better modelling of the interest rate derivative and understanding of sensitivities in a market driven by a normal inverse Gaussian process.

Appendix

Itô formula for semi-martingale [7]

Let $Y = (Y_t)_{0\leq t\leq T}$ be a semi-martingale and $f : [0,T]\times\mathbb{R}\to\mathbb{R}$, a $C^{1,2}$ function, then

$$\begin{aligned} f(t,Y_t) &= f(0,Y_0) + \int_0^t \frac{\partial f}{\partial s}(s,Y_s)ds + \int_0^t \frac{\partial f}{\partial y}(s,Y_{s_-})dY_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(s,Y_s)d[Y,Y]_s^c + \sum_{0\leq s\leq t,\Delta Y_s\neq 0}[f(s,Y_s) - f(s,Y_{s_-}) \\ &- \Delta Y_s \frac{\partial f}{\partial y}(s,Y_{s_-})], \end{aligned}$$

where $[Y,Y]_s^c$ is the continuous part of the quadratic variation of Y and $\Delta Y_s = Y_s - Y_{s_-}$.

References

[1] A. Andresen, S. Koekebakker, S. Westgaard, *Modelling electricity forward prices using the multivariate normal inverse Gaussian distribution*, The Journal of Energy Markets, 3 (2010), 1-13.

[2] O.E. Barndorf-Nielson, *Processes of NIG type*, Working paper no. 1, Centre for Analytic Finance, Aarhus University, 1997.

[3] M.-P. Bavouzet-Morel, M. Messaoud, *Computation of greeks using Malliavin's calculus in jump type market models*, Electronic Journal of Probability, 11 (2006), 276-300.

[4] D. Bayazit, C. A. Nolder, *Malliavin calculus for Lévy markets and new sensitivities*, Quantitative Finance, 13 (2013), 1257-1287.

[5] A. Carol, *Pricing, hedging and trading instrument. Market risk analysis*, vol. III, John Wiley & Sons Ltd., 2008.

[6] D. Chorafas, *Introduction to derivative financial instrument: Bonds, Swaps, Options and Hedging*, McGraw-Hill Education, 2008.

[7] R. Cont, P. Tankov, *Financial modelling with jump processes,* Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, BocaRaton, FL, 2004.

[8] H. Corb, *Interest rate swaps and other derivatives*, Columbia University Press, 2012.

[9] M. S. Dhull, A. Kumar, *Normal inverse Gaussian autoregressive model using EM algorithm*, arXiv:2105.14502v3[stat.ME], 16 pages (2021).

[10] G. O. S. Ekhaguere, *Lecture notes on financial mathematics* (unpublished), University of Ibadan, Nigeria, 2010.

[11] N. Gabrielli, *What is an affine process*, `https://nicolettagabrielli.weebly.com`, 2011.

[12] D. Hainaut, *Lévy interest rate models with a long memory*, Risks, 10 (2022).

[13] B. Lahcene, *On extended normal inverse Gaussian distribution: theory, methodology, properties and applications*, American Journal of Applied Mathematics and Statistics, 7 (2019), 224-230.

[14] S. Lang, R. Signer, K. Spremann, *The choice of interest rate models and its effect on bank capital requirements regulation and financial stability*, International Journal of Economics and Finance, 10 (2018), 74-92.

[15] J. A. Núñez, M. I. Contreras-Valdez, A. Ramírez-García, E. Sánchez-Ruenes, *Underlying assets distribution in derivatives: the BRIC case*, Theoretical Economics Letters, 8 (2018), 502-513.

[16] G. Orlando, R. M. Mininni, M. Bufalo, *On the calibration of short-term interest rates through a CIR model*, www.arXiv:1806.03683v1 (2018).

[17] E. Petrou, *Malliavin calculus in Lévy Spaces and applications to finance*, Electronic Journal of Probability, 13 (2008), 852-879.

[18] C. Pintoux, N. Privault, *The Dothan pricing model revisited*, Mathematical Finance, 21 (2011), 355-363.

[19] O. Rubenis, A. Matvejevs, *Increments of normal inverse Gaussian process as logarithmic returns of stock price*, Information Technology and Management Science, 21 (2018), 93-97.

[20] P. Sabino, *Pricing energy derivatives in markets driven by tempered stable and CGMY processes of Ornstein-Uhlenbeck type*, Risks, 148 (2022).

[21] A. M. Udoye and G. O. S. Ekhaguere, *Sensitivity analysis of a class of interest rate derivatives in a Lévy market*, Palestine Journal of Mathematics, 11 (2022), 159-176.

[22] Y. Yin, S. Li, J. L. Roca, *OU models based on positive and negative subordinate processes applying SHIBOR time series analysis and derivative pricing - through discrete differential method*, Journal of Differential Equations and Applications, 2019.

Accepted: January 24, 2023

Real hypersurfaces in nonflat complex space forms with Lie derivative of structure tensor fields

Wenjie Wang
School of Mathematics
Zhengzhou University of Aeronautics
Zhengzhou 450046, Henan
P. R. China
wangwj072@163.com

Abstract. In this paper, we obtain some non-existence theorems for real hypersurfaces in nonflat complex space forms such that the structure tensor fields are of Lie Codazzi, Lie Killing or Lie recurrent type.
Keywords: real hypersurface, complex space form, structure tensor field, Lie derivative.

1. Introduction

Let $M^n(c)$ be a complete and simply connected complex space form which is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$;
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$;
- a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$,

where c is the constant holomorphic sectional curvature. Let M be a real hypersurface of real dimension $2n-1$ immersed in $M^n(c)$, $n \geq 2$. On M there exists a natural almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure on $M^n(c)$ and the normal vector field, respectively, where ξ and ϕ are called the structure vector field and the structure tensor field, respectively. If the structure vector field ξ on real hypersurfaces is principal at each point, then the hypersurface is said to be Hopf. In geometry of real hypersurface, the structure tensor field ϕ plays important roles in classification and characterization of Hopf hypersurfaces (see, many references in [2, 17]). Before stating our main study, we exhibit some well known results in this field.

A Hopf hypersurface in $\mathbb{C}P^n(c)$ has constant principal curvatures if and only if it is locally congruent to a type (A_1), (A_2), (B), (C), (D) or (E) hypersurfaces (see, [9, 21]). A Hopf hypersurface in $\mathbb{C}H^n(c)$ has constant principal curvatures if and only if it is locally congruent to a type (A_0), $(A_{1,0})$, $(A_{1,1})$, (A_2) or (B) hypersurfaces (see, [1]). All type (A_0), (A_1), $(A_{1,0})$, $(A_{1,1})$ and (A_2) hypersurfaces are referred to collectively as type (A).

Maeda and Udagawa in [16] first considered the Lie derivative of the structure tensor field ϕ and proved that the structure vector field ξ of a real hypersurface in $\mathbb{C}P^n$ is an infinitesimal automorphism of the structure tensor field ϕ if and only if the hypersurface is of type (A). Such a conclusion is still true even when the restriction was weakened to some other geometric conditions and this was first considered by Kwon and Suh in [12, Theorem] for a real hypersurface of dimension ≥ 5. Results in [16] have been generalized by Lim [13] by considering the coincidence of the Lie derivative and covariant derivative of the structure tensor field along ξ. Very recently, a new operator generated by the Lie derivative of the structure tensor field ϕ along the structure vector field ξ was extensively studied by Okumura in [18, 19] (see, also Cho [3, 4]). Nonexistence of the real hypersurfaces with a Killing type structure tensor field was proved by Cho in [5]. Some other results on the Lie derivative of the structure tensor field along ξ can also be found in [8, 11, 14, 15]. In 2013, Kaimakamis and Panagiotidou in [6, pp. 2091] proposed that it would be an interesting question for studying the Lie recurrency of the structure tensor field. In the present paper, we study the Lie derivative of the structure tensor field for real hypersurfaces in nonflat complex space forms $M^n(c)$, $c \neq 0$, and solved the question posed in [6].

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of the metric g, respectively. Then, the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \overline{\nabla}_X N = -AX, \tag{1}$$

for any $X, Y \in \mathfrak{X}(M)$, where A denotes the shape operator of M in $M^n(c)$. For any vector field $X \in \mathfrak{X}(M)$, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi. \tag{2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \tag{4}$$

for any $X, Y \in \mathfrak{X}(M)$. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$ at each point, where $\alpha = \eta(A\xi)$, then M is called a Hopf hypersurface and α is called Hopf principal curvature.

Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla}J = 0$) of $M^n(c)$ and using (1), (2) we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{5}$$

$$\nabla_X\xi = \phi AX, \tag{6}$$

for any $X, Y \in \mathfrak{X}(M)$.

3. Non-existence results

We denote by $\mathcal{L}$ the Lie derivative of a real hypersurface in a nonflat complex space form $M^n(c)$, $c \neq 0$, $n \geq 2$.

Definition 3.1. *The structure tensor field of a real hypersurface is called Lie Killing if*

$$(\mathcal{L}_X\phi)Y + (\mathcal{L}_Y\phi)X = 0, \tag{7}$$

for any vector fields X, Y.

Obviously, the above condition (7) is a generalization of the Lie parallelism of the structure tensor field, i.e., $\mathcal{L}_X\phi = 0$, for any $X \in \mathfrak{X}(M)$.

Theorem 3.1. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is of Lie Killing type.*

Proof. By applying (5), we have

$$(\mathcal{L}_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi - \nabla_{\phi Y}X + \phi\nabla_Y X, \tag{8}$$

for any vector fields X, Y. Now suppose that the structure tensor field of a real hypersurface M is Lie Killing. From (7) and (8) we get

$$\eta(Y)AX - 2g(AX, Y)\xi - \nabla_{\phi Y}X + \phi\nabla_Y X + \eta(X)AY - \nabla_{\phi X}Y + \phi\nabla_X Y = 0, \tag{9}$$

for any vector fields X, Y. Taking the inner product of (9) with ξ, we obtain

$$\eta(Y)\eta(AX) - 2g(AX, Y) - \eta(\nabla_{\phi Y}X) + \eta(X)\eta(AY) - \eta(\nabla_{\phi X}Y) = 0, \tag{10}$$

for any vector fields X, Y. In (10), selecting $Y = \xi$ we obtain

$$A\xi = \eta(A\xi)\xi.$$

This means that M is a Hopf hypersurface. In (10), selecting $X, Y \in \ker\eta$, with the help of (3), (6) and $A\xi = \eta(A\xi)\xi := \alpha\xi$, we get

$$AX - \phi A\phi X = 0 \ (\Leftrightarrow A\phi X + \phi AX = 0), \tag{11}$$

for any $X \in \ker\eta$. On the other hand, recall that, for any Hopf hypersurfaces, we have (see, [17, Lemma 2.2]):

$$A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - \frac{c}{4}\phi = 0. \tag{12}$$

Substituting $A\phi X + \phi AX = 0$ (for any $X \in \ker\eta$) into equality (12), then we obtain $A\phi AX = \frac{c}{4}\phi X$, for any $X \in \ker\eta$. Now, let X be a unit eigenvector field of A with eigenfunction λ orthogonal to ξ, then ϕX is also a unit eigenvector field of A with eigenfunction $c/(4\lambda)$. Notice that λ is nowhere vanishing. Otherwise we shall arrive at a contradiction (i.e., $c = 0$) according to $A\phi AX = \frac{c}{4}\phi X$, for any $X \in \ker\eta$. Therefore, with the aid of the second equality in (11), the inner product of $A\phi AX = \frac{c}{4}\phi X$ with ϕX gives

$$\frac{c}{4} = g(A\phi AX, \phi X) = g(\phi AX, A\phi X) = -|A\phi X|^2 = -\frac{c^2}{16\lambda^2}.$$

In view of the above equality, one sees that this situation occurs only for a real hypersurface in the complex hyperbolic space, and the two (distinct) principal curvatures λ and ν of the shape operator on the holomorphic distribution $\ker\eta$ are

$$\lambda = \frac{\sqrt{-c}}{2} \text{ and } \nu = -\frac{\sqrt{-c}}{2}. \tag{13}$$

Recall that the Hopf principal curvature for any Hopf hypersurface is a constant (see, [17, Theorem 2.1]). Thus, M is a Hopf hypersurface in $\mathbb{C}H^n(c)$ with constant principal curvatures. According to [1], M is locally congruent to a

- type (A_2) hypersurface whose two principal curvatures on holomorphic distribution $\ker\eta$ are $\frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$ and $\frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$; or a
- type (B) hypersurface whose two principal curvatures on holomorphic distribution $\ker\eta$ are $\frac{\sqrt{-c}}{2}\tanh(\frac{\sqrt{-c}}{2}r)$ and $\frac{\sqrt{-c}}{2}\coth(\frac{\sqrt{-c}}{2}r)$.

Notice that the summation of the two principal curvatures of M on holomorphic distribution $\ker\eta$ in (13) vanishes, but by the above table this is impossible for type (A_2) or (B) hypersurfaces in $\mathbb{C}H^n(c)$. □

Corollary 3.1. *There are no real hypersurfaces in nonflat complex space forms with Lie parallel structure tensor field.*

Definition 3.2. *The structure tensor field of a real hypersurface is called Lie Codazzi if*

$$(\mathcal{L}_X\phi)Y = (\mathcal{L}_Y\phi)X, \tag{14}$$

for any vector fields X, Y.

Obviously, the above condition (14) is also a generalization of Lie parallelism of the structure tensor field.

Theorem 3.2. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is of Lie Codazzi type.*

Proof. If the structure tensor field ϕ of a real hypersurface M in nonflat complex space forms is Lie Codazzi, from (8) and (14) we get

$$\eta(Y)AX - \nabla_{\phi Y}X + \phi\nabla_Y X = \eta(X)AY - \nabla_{\phi X}Y + \phi\nabla_X Y, \tag{15}$$

for any vector fields X, Y. Taking the inner product of (15) with ξ gives

$$\eta(Y)\eta(AX) - \eta(\nabla_{\phi Y}X) = \eta(X)\eta(AY) - \eta(\nabla_{\phi X}Y),$$

for any vector fields X, Y. In the above equality, replacing Y by ξ gives

$$A\xi = \eta(A\xi)\xi.$$

This means that M is a Hopf hypersurface. We may write $A\xi = \eta(A\xi)\xi := \alpha\xi$, and replacing Y by ξ in (15), we obtain

$$2AX + \phi\nabla_\xi X = 2\alpha\eta(X)\xi - \phi A\phi X, \tag{16}$$

for any vector field X. With the aid of $A\xi = \alpha\xi$, the operation of ϕ on (16) gives

$$2\phi AX - \nabla_\xi X + \eta(\nabla_\xi X)\xi = A\phi X,$$

for any vector field X. On the other hand, with the aid of $A\xi = \alpha\xi$, replacing X by ϕX in the above equality we have

$$2\phi A\phi X - \nabla_\xi \phi X = -AX + \alpha\eta(X)\xi,$$

for any vector field X. Thus, adding the above equality to (16), with the aid of (5), we get

$$AX + \phi A\phi X = \alpha\eta(X)\xi,$$

for any vector field X, where we have applied $\nabla_\xi\phi = 0$ which is obtained from (5) and $A\xi = \alpha\xi$. With the aid of $A\xi = \alpha\xi$, the operation of ϕ on the above equality gives

$$A\phi = \phi A. \tag{17}$$

In general, the above relation implies that M is a type (A) hypersurface. However, in our case, there are no real hypersurfaces satisfying the above relation. In fact, with the aid of (5), using (17) in (16) we get

$$AX = -\phi\nabla_\xi X + \alpha\eta(X)\xi, \tag{18}$$

for any vector field X. The operation of ϕ on (18) gives

$$\phi AX = \nabla_\xi X - \eta(\nabla_\xi X)\xi.$$

With the aid of (17) and $A\xi = \alpha\xi$, the operation of A on (18) gives

$$A^2X = -\nabla_\xi X + \eta(\nabla_\xi X)\xi + \alpha^2\eta(X)\xi,$$

for any vector field X. Eliminating $\nabla_\xi X$, according to the above relation and the previous one we get

$$A^2X + \phi AX = \alpha^2\eta(X)\xi,$$

for any vector field X. From the above equality, we conclude that all principal curvatures of the shape operator on $\ker\eta$ are zero. For any Hopf hypersurfaces, if $AU = \lambda U$ and $A\phi U = \nu\phi U$ for certain $U \in \ker\eta$, from [17, Corollary 2.3] we have

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{19}$$

As all principal curvatures are zero on $\ker\eta$, applying this in (19) implies $c = 0$, a contradiction. □

Definition 3.3. *The structure tensor field of a real hypersurface is called Lie recurrent if*

$$(\mathcal{L}_X\phi)Y = \omega(X)\phi Y, \tag{20}$$

for any vector fields X, Y, and certain one-form ω.

Obviously, the above condition (20) is also a generalization of Lie parallelism of the structure tensor field.

Theorem 3.3. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is Lie recurrent.*

Proof. If the structure tensor field ϕ of a real hypersurface M in nonflat complex space forms is Lie recurrent, from (8) and (20) we get

$$\eta(Y)AX - g(AX, Y)\xi - \nabla_{\phi Y}X + \phi\nabla_Y X = \omega(X)\phi Y, \tag{21}$$

for any vector fields X, Y. Taking the inner product of (21) with ξ gives

$$\eta(Y)\eta(AX) - g(AX, Y) - \eta(\nabla_{\phi Y}X) = 0, \tag{22}$$

for any vector fields X, Y. In (22), replacing X by ξ we see $A\xi = \eta(A\xi)\xi$, and hence M is Hopf. In (21), with the aid of $A\xi = \alpha\xi$, replacing X by ξ we obtain

$$-\phi A\phi Y - AY + \alpha\eta(Y)\xi = \omega(\xi)\phi Y, \tag{23}$$

for any vector field Y. With the aid of $A\xi = \alpha\xi$, the operation of ϕ on (23) gives

$$A\phi Y - \phi AY = \omega(\xi)\phi^2 Y.$$

In (23), with the aid of $A\xi = \alpha\xi$, replacing Y by ϕY gives

$$\phi AY - A\phi Y = \omega(\xi)\phi^2 Y.$$

Subtracting the last equality from the previous one gives $A\phi = \phi A$. Making use of this, with the aid of $A\xi = \alpha\xi$, selecting $X \in \ker\eta$ in (22), we obtain

$$AX = 0,$$

for any $X \in \ker\eta$. As seen in proof of Theorem 3.2, this is impossible because of (19). □

Remark 3.1. Corollary 3.1 is also a direct corollary of Theorems 3.2 and 3.3.

Remark 3.2. It has been proved in [6, Main Theorem] that there exist no real hypersurfaces in $M^n(c)$, $c \neq 0$, $n \geq 2$, whose structure Jacobi operator l is of Lie recurrent type, i.e., $\mathcal{L}_X l = \omega(X)l$, for any vector field X and certain one-form ω. This conclusion is still valid when the structure Jacobi operator l is replaced by the shape operator (see, [2, Theorem 8.116]) or the structure tensor field ϕ (see, Theorem 3.3).

We remark that it was proposed in [6] that how about if we weaken condition (20) to Lie $\mathcal{D}$-recurrent? Before closing this paper, we also answer this question and obtain again a nonexistence theorem. Next we denote by $\mathcal{D}$ the holomorphic distribution $\ker\eta$.

Definition 3.4. *The structure tensor field of a real hypersurface is called Lie $\mathcal{D}$-recurrent if*

$$(\mathcal{L}_X\phi)Y = \omega(X)\phi Y, \tag{24}$$

for any vector field Y and $X \in \mathcal{D}$, and certain one-form ω.

Obviously, condition (24) is much weaker than Lie parallelism (i.e., $\mathcal{L}_X\phi = 0$). Next, we extend Theorem 3.3 to the following form.

Theorem 3.4. *There exist no real hypersurfaces in nonflat complex space forms such that the structure tensor field is Lie $\mathcal{D}$-recurrent.*

Proof. By Definition 3.4, equalities (21) and (22) are still valid, for any vector field Y and $X \in \mathcal{D}$. Considering $X \in \mathcal{D}$ and $Y = \xi$ in (21), we get

$$AX - \eta(AX)\xi + \phi\nabla_\xi X = 0. \tag{25}$$

Replacing X by ϕX in (25) gives

$$A\phi X - \eta(A\phi X)\xi + \phi\nabla_\xi \phi X = 0,$$

which is operated by ϕ yielding

$$\phi A\phi X - \nabla_\xi \phi X + \eta(\nabla_\xi \phi X)\xi = 0.$$

Notice that from (6) and (4) we have $\eta(\nabla_\xi \phi X) + \eta(AX) = 0$, for any $X \in \mathcal{D}$, which is substituted into the above equality giving

$$\phi A\phi X - \nabla_\xi \phi X - \eta(AX)\xi = 0,$$

for any $X \in \mathcal{D}$. Adding this to (25) gives

$$(\nabla_\xi \phi)X = \phi A\phi X + AX - 2\eta(AX)\xi, \tag{26}$$

for any $X \in \mathcal{D}$. Comparing (26) with (5) we obtain

$$\phi A\phi X + AX - \eta(AX)\xi = 0, \tag{27}$$

for any $X \in \mathcal{D}$. On the other hand, considering $X \in \mathcal{D}$ in (22), with the aid of (6), we get

$$\phi A\phi X - AX + \eta(AX)\xi = 0.$$

Consequently, eliminating $\phi A\phi X$, from the above equality and (27) we obtain $AX = \eta(AX)\xi$, for any $X \in \mathcal{D}$. This implies that $g(AX, Y) = 0$, for any vector fields $X, Y \in \mathcal{D}$, and now the hypersurface is a ruled one (see, [2, 10, 17]). On a ruled hypersurface, there exists a unit vector field $U \in \mathcal{D}$ such that

$$A\xi = \alpha\xi + \beta U,\ AU = \beta\xi,\ AX = 0, \tag{28}$$

for any $X \in \{\xi, U\}^\perp$, where β is a non-vanishing function. Moreover, according to [7, pp. 404] (see, also, [10, 20]) we have

$$\nabla_X U = \begin{cases} \frac{1}{\beta}(\beta^2 - \frac{c}{4})\phi X, & X = U, \\ 0, & X = \phi U, \end{cases} \tag{29}$$

and

$$\mathrm{d}\beta(X) = \begin{cases} 0, & X = U, \\ \beta^2 + \frac{c}{4}, & X = \phi U. \end{cases} \tag{30}$$

In (21), considering $X = Y = U$, with the aid of (28), we obtain from (29) that

$$\beta^2 - \frac{c}{4} = 0 \text{ and } \omega(U) = 0.$$

The first equality implies that β is a constant, and hence according to (30) we obtain $\beta^2 + \frac{c}{4} = 0$, which is compared with the above equality implying $c = 0$, a contradiction. □

Remark 3.3. By Theorem 3.4, the structure tensor field of a real hypersurface in nonflat complex space forms cannot be Lie $\mathcal{D}$-parallel, but it can be Lie Reeb-parallel (i.e., $\mathcal{L}_\xi \phi = 0$). In fact, it has been proved in [13, Theorem A] that the structure tensor field of a real hypersurface is Lie Reeb-parallel if and only if the hypersurface is of type (A) (see, also, [12, 16]).

Acknowledgement

This paper has been supported by the Youngth Scientific Research Program in Zhengzhou University of Aeronautics (No. 23ZHQN01010) and the Nature Science Foundation of Henan Province (No. 232300420359).

References

[1] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math., 395 (1989), 132-141.

[2] T. E. Cecil, P. J. Ryan, *Geometry of hypersurfaces*, Springer Monographs in Mathematics, Springer, New York, 2015.

[3] J. T. Cho, *Geometry of CR-manifolds of contact type*, Proc. Eighth International Workshop on Diff. Geom., 8 (2004), 137-155,

[4] J. T. Cho, *Contact metric hypersurfaces in cpmplex space forms*, Proc. Workshop on Differential Geometry of Submanifolds and its Related Topics, Saga, 2012.

[5] J. T. Cho, *Notes on real hypersurfaces in a complex space form*, Bull. Korean Math. Soc., 52 (2015), 335-344.

[6] G. Kaimakamis, K. Panagiotidou, *Real hypersurfaces in a non-flat complex space form with Lie recurrent structure Jacobi operator*, Bull. Korean Math. Soc. 50 (2013), 2089-2101.

[7] U.-H. Ki, N.-G. Kim, *Ruled real hypersurfaces of a complex space form*, Acta Math. Sinica New Ser. 10 (1994), 401-409.

[8] U.-H. Ki, S. J. Kim, S. B. Lee, *Some characterizations of a real hypersurface of type A*, Kyungpook Math. J., 31 (1991), 73-82.

[9] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc., 296 (1986), 137-149.

[10] M. Kimura, *Sectional curvatures of a holomorphic plane in* $P^n(C)$, Math. Ann., 276 (1987), 487-497.

[11] M. Kimura, S. Maeda, *Lie derivatives on real hypersurfaces in a complex projective space*, Czechoslovak Math. J. 45 (1995), 135-148.

[12] J. H. Kwon, Y. J. Suh, *Lie derivatives on homogeneous real hypersurfaces of type A in complex space forms*, Bull. Korean Math. Soc., 34 (1997), 459-468.

[13] D. H. Lim, *Characterizations of real hypersurfaces in a nonflat complex space form with respect to structure tensor field*, Far East. J. Math. Sci., 104 (2018), 277-284.

[14] D. H. Lim, Y. M. Park, H. S. Kim, *Geometrical study of real hypersurfaces with differentials of structure tensor field in a nonflat complex space form*, Global J. Pure Appl. Math., 14 (2018), 1251-1257.

[15] T. H. Loo, *Characterizations of real hypersurfaces in a complex space form in terms of Lie derivatives*, Tamsui Oxford J. Math. Sci., 19 (2003), 1-12.

[16] S. Maeda, S. Udagawa, *Real hypersurfaces of a complex projective space in terms of holomorphic distribution*, Tsukuba J. Math., 14 (1990), 39-52.

[17] R. Niebergall, P. J. Ryan, *Real hypersurfaces in complex space forms*, in Tight and Taut Submanifolds, Math. Sci. Res. Inst. Publ., Vol. 32, Cambridge Univ. Press, Cambridge, 1997, 233-305.

[18] K. Okumura, *A certain tensor on real hypersurfaces in a nonflat complex space form*, Czechoslovak Math. J., 70 (2020), 1059-1077.

[19] K. Okumura, *A certain η-parallelism on real hypersurfaces in a nonflat complex space form*, Math. Slovaca, 71 (2021), 1553-1564.

[20] Y. J. Suh, *Characterizations of real hypersurfaces in complex space forms in terms of Weingarten map*, Nihonkai Math., J. 6 (1995), 63-79.

[21] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math., 10 (1973), 495-506.

Accepted: July 4, 2022

On improved Heinz inequalities for matrices

Yaoqun Wang
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500
P.R. China

Xingkai Hu*
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500
P.R. China
huxingkai84@163.com

Yunxian Dai
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500
P.R. China

Abstract. In this paper, we improve some Heinz inequalities for matrices by using the convexity of function. Theoretical analysis shows that new inequalities are refinement of the result in the related literature.

Keywords: Heinz inequalities, convex function, positive semidefinite matrix.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The singular values $s_j(A)(j = 1, \ldots, n)$ of A are the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$. The Schatten p-norm $\|\cdot\|_p$ is defined as

$$\|A\|_p = (\sum_{j=1}^{n} s_j^p(A))^{\frac{1}{p}}, 1 \leq p < \infty$$

and the Ky Fan k-norm $\|\cdot\|_{(k)}$ is defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1, \ldots, n.$$

*. Corresponding author

It is well known that the Schatten p-norm $\|\cdot\|_p$ and the Ky Fan k-norm $\|\cdot\|_{(k)}$ are unitarily invariant [1].

Bhatia and Davis [2] have proved the following inequality

$$2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \le \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| \le \|AX+XB\|, 0\le v\le 1, \tag{1.1}$$

where $A,B,X\in M_n$ with A and B are positive semidefinite matrices.

Kittaneh [3] proved that if $A,B,X\in M_n$ such that A and B are positive semidefinite, then

$$\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| \le (1-2r_0)\|AX+XB\|+4r_0\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|, \tag{1.2}$$

where $0\le v\le 1, r_0=\min\{v,1-v\}$. The inequality (1.2) is a refinement of the second inequality in (1.1).

He et al. [4] proved that if $A,B,X\in M_n$ such that A and B are positive semidefinite, then

$$\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\|^2 \le (1-2r_0)\|AX+XB\|^2+8r_0\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2, \tag{1.3}$$

where $0\le v\le 1, r_0=\min\{v,1-v\}$.

Improvements of Heinz inequalities have been done by many researchers. We refer the reader to [5-8]. In this paper, we will improve the inequalities (1.2) and (1.3) using the convexity of function.

2. Main results

Applying the convexity of function, we obtain the following theorem.

Theorem 1. *Let $A,B,X\in M_n$ such that A and B are positive semidefinite. Then for every unitarily invariant norm*

$$\begin{aligned}\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| &\le (1-6r_0)\|AX+XB\| \\ &\quad +6r_0\left\|A^{\frac{1}{6}}XB^{\frac{5}{6}}+A^{\frac{5}{6}}XB^{\frac{1}{6}}\right\|, v\in[0,\frac{1}{6}]\cup(\frac{5}{6},1],\\ \left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| &\le (6r_0-1)\left\|A^{\frac{1}{3}}XB^{\frac{2}{3}}+A^{\frac{2}{3}}XB^{\frac{1}{3}}\right\| \\ &\quad +2(1-3r_0)\left\|A^{\frac{1}{6}}XB^{\frac{5}{6}}+A^{\frac{5}{6}}XB^{\frac{1}{6}}\right\|,\\ &\quad v\in(\frac{1}{6},\frac{1}{3}]\cup(\frac{2}{3},\frac{5}{6}]\end{aligned}$$

and

$$\begin{aligned}\left\|A^vXB^{1-v}+A^{1-v}XB^v\right\| &\le 4(3r_0-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| \\ &\quad +3(1-2r_0)\left\|A^{\frac{1}{3}}XB^{\frac{2}{3}}+A^{\frac{2}{3}}XB^{\frac{1}{3}}\right\|, v\in(\frac{1}{3},\frac{2}{3}],\end{aligned}$$

where $r_0=\min\{v,1-v\}$.

Proof. For $v = 0$, the Theorem 1 is obvious. For $0 < v \leq \frac{1}{6}$, since $f(v) = \left\| A^v XB^{1-v} + A^{1-v}XB^v \right\|$ is convex on $[0, 1]$, it follows by a slope argument that

$$\frac{f(v) - f(0)}{v - 0} \leq \frac{f(\frac{1}{6}) - f(0)}{\frac{1}{6} - 0},$$

and so

$$f(v) \leq (1 - 6v)f(0) + 6vf(\frac{1}{6}),$$

that is

$$\begin{aligned} \left\| A^v XB^{1-v} + A^{1-v}XB^v \right\| &\leq (1 - 6v) \left\| AX + XB \right\| \\ &+ 6v \left\| A^{\frac{1}{6}} XB^{\frac{5}{6}} + A^{\frac{5}{6}} XB^{\frac{1}{6}} \right\|. \end{aligned} \tag{2.1}$$

For $\frac{1}{6} < v \leq \frac{1}{3}$, similarly, we have

$$\frac{f(v) - f(\frac{1}{6})}{v - \frac{1}{6}} \leq \frac{f(\frac{1}{3}) - f(\frac{1}{6})}{\frac{1}{6}},$$

and so

$$f(v) \leq (6v - 1)f(\frac{1}{3}) + (2 - 6v)f(\frac{1}{6}),$$

that is

$$\begin{aligned} \left\| A^v XB^{1-v} + A^{1-v}XB^v \right\| &\leq (6v - 1) \left\| A^{\frac{1}{3}} XB^{\frac{2}{3}} + A^{\frac{2}{3}} XB^{\frac{1}{3}} \right\| \\ &+ 2(1 - 3v) \left\| A^{\frac{1}{6}} XB^{\frac{5}{6}} + A^{\frac{5}{6}} XB^{\frac{1}{6}} \right\|. \end{aligned} \tag{2.2}$$

For $\frac{1}{3} < v \leq \frac{1}{2}$, similarly, we have

$$\begin{aligned} \left\| A^v XB^{1-v} + A^{1-v}XB^v \right\| &\leq 4(3v - 1) \left\| A^{\frac{1}{2}} XB^{\frac{1}{2}} \right\| \\ &+ 3(1 - 2v) \left\| A^{\frac{1}{3}} XB^{\frac{2}{3}} + A^{\frac{2}{3}} XB^{\frac{1}{3}} \right\|. \end{aligned} \tag{2.3}$$

For $\frac{1}{2} < v \leq \frac{2}{3}$, it follows by applying (2.3) to $1 - v$ that

$$\begin{aligned} \left\| A^v XB^{1-v} + A^{1-v}XB^v \right\| &\leq 4(2 - 3v) \left\| A^{\frac{1}{2}} XB^{\frac{1}{2}} \right\| \\ &+ 3(2v - 1) \left\| A^{\frac{1}{3}} XB^{\frac{2}{3}} + A^{\frac{2}{3}} XB^{\frac{1}{3}} \right\|. \end{aligned}$$

For $\frac{2}{3} < v \leq \frac{5}{6}$, by applying (2.2) to $1 - v$, we have

$$\begin{aligned} \left\| A^v XB^{1-v} + A^{1-v}XB^v \right\| &\leq (5 - 6v) \left\| A^{\frac{1}{3}} XB^{\frac{2}{3}} + A^{\frac{2}{3}} XB^{\frac{1}{3}} \right\| \\ &+ 2(3v - 2) \left\| A^{\frac{1}{6}} XB^{\frac{5}{6}} + A^{\frac{5}{6}} XB^{\frac{1}{6}} \right\|. \end{aligned}$$

For $\frac{5}{6} < v \leq 1$, by applying (2.1) to $1-v$, we have

$$\begin{aligned}\left\|A^v XB^{1-v} + A^{1-v}XB^v\right\| &\leq (6v-5)\left\|AX + XB\right\| \\ &\quad + 6(1-v)\left\|A^{\frac{1}{6}}XB^{\frac{5}{6}} + A^{\frac{5}{6}}XB^{\frac{1}{6}}\right\|.\end{aligned}$$

This completes the proof. □

Remark 1. Theorem 1 is better than inequality (1.2). For $v \in [0, \frac{1}{6}] \cup (\frac{5}{6}, 1]$, we have

$$\begin{aligned}&(1-6r_0)\left\|AX + XB\right\| + 6r_0\left\|A^{\frac{1}{6}}XB^{\frac{5}{6}} + A^{\frac{5}{6}}XB^{\frac{1}{6}}\right\| \\ &\leq (1-6r_0)\left\|AX + XB\right\| + 6r_0(\frac{2}{3}\left\|AX + XB\right\| + \frac{2}{3}\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|) \\ &= (1-2r_0)\left\|AX + XB\right\| + 4r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

For $v \in (\frac{1}{6}, \frac{1}{3}] \cup (\frac{2}{3}, \frac{5}{6}]$, we have

$$\begin{aligned}&(6r_0-1)\left\|A^{\frac{1}{3}}XB^{\frac{2}{3}} + A^{\frac{2}{3}}XB^{\frac{1}{3}}\right\| + 2(1-3r_0)\left\|A^{\frac{1}{6}}XB^{\frac{5}{6}} + A^{\frac{5}{6}}XB^{\frac{1}{6}}\right\| \\ &\leq (6r_0-1)[\frac{1}{3}\left\|AX + XB\right\| + \frac{4}{3}\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|] \\ &+ 2(1-3r_0)[\frac{2}{3}\left\|AX + XB\right\| + \frac{2}{3}\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|] \\ &= (1-2r_0)\left\|AX + XB\right\| + 4r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

For $v \in (\frac{1}{3}, \frac{2}{3}]$, we have

$$\begin{aligned}&4(3r_0-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| + 3(1-2r_0)\left\|A^{\frac{1}{3}}XB^{\frac{2}{3}} + A^{\frac{2}{3}}XB^{\frac{1}{3}}\right\| \\ &\leq 4(3r_0-1)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\| + 3(1-2r_0)[\frac{1}{3}\left\|AX + XB\right\| + \frac{4}{3}\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|] \\ &= (1-2r_0)\left\|AX + XB\right\| + 4r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|.\end{aligned}$$

The following result implies that the inequality in Theorem 2 is a refinement of the inequality (1.3).

Theorem 2. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then for $0 \leq v \leq 1$ and for every unitarily invariant norm*

$$\begin{aligned}&\left\|A^v XB^{1-v} + A^{1-v}XB^v\right\|^2 \\ &+ 2r_0(\left\|A^v XB^{1-v} + A^{1-v}XB^v\right\| - 2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|)(\left\|AX + XB\right\| - 2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|) \\ &\leq (1-2r_0)\left\|AX + XB\right\|^2 + 8r_0\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|^2,\end{aligned}$$

where $r_0 = \min\{v, 1-v\}$.

Proof. For $v = 0, 1$, the result in Theorem 2 is obvious. For $0 < v \le \frac{1}{2}$, since $f(v) = \left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|$ is convex on $[0,1]$, it follows that

$$\frac{f(v) - f(0)}{v - 0} \le \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0},$$

and so

$$2v(f(0) - f(\frac{1}{2}))(f(0) + f(v)) \le f^2(0) - f^2(v),$$

that is

$$f^2(v) + 2v(f(v) - f(\frac{1}{2}))(f(0) - f(\frac{1}{2})) \le (1 - 2v)f^2(0) + 2vf^2(\frac{1}{2}). \tag{2.4}$$

For $\frac{1}{2} < v < 1$, similarly, we have

$$\begin{aligned} &f^2(v) + 2(1-v)(f(v) - f(\frac{1}{2}))(f(0) - f(\frac{1}{2})) \\ &\le (1 - 2(1-v))f^2(0) + 2(1-v)f^2(\frac{1}{2}). \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$f^2(v) + 2r_0(f(v) - f(\frac{1}{2}))(f(0) - f(\frac{1}{2})) \le (1 - 2r_0)f^2(0) + 2r_0 f^2(\frac{1}{2}),$$

that is

$$\begin{aligned} &\left\|A^v X B^{1-v} + A^{1-v} X B^v\right\|^2 \\ &+ 2r_0(\left\|A^v X B^{1-v} + A^{1-v} X B^v\right\| - 2\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|)(\|AX + XB\| - 2\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|) \\ &\le (1 - 2r_0)\|AX + XB\|^2 + 8r_0\left\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right\|^2. \end{aligned}$$

This completes the proof. □

Acknowledgements

This research is supported by the Fund for Fostering Talents in Kunming University of Science and Technology (No. KKZ3202007048).

References

[1] X. Zhan, *Matrix throry*, Higher Education Press, Beijing, 2008.

[2] R. Bhatia, C. Davis, *More matrix forms of the arithmetic-geometric mean inequality*, SIAM J. Matrix Anal. Appl., 14 (1993), 132-136.

[3] F. Kittaneh, *On the convexity of the Heinz means*, Integr. Equ. Oper. Theory, 68 (2010), 519-527.

[4] C. He, L. Zou, S. Qaisar, *On improved arithmetic-geometric mean and Heinz inequalities for matrices*, J. Math. Inequal., 6 (2012), 453-459.

[5] M. Lin, *Squaring a reverse AM-GM inequality*, Stud. Math., 215 (2013), 187-194.

[6] L. Zou, *Unification of the arithmetic-geometric mean and Hölder inequalities for unitarily invariant norms*, Linear Algebra Appl., 562 (2019), 154-162.

[7] C. Yang, F. Lu, *Inequalities for the Heinz mean of sector matrices involving positive linear maps*, Ann. Funct. Anal., 11 (2020), 866-878.

[8] A.G. Ghazanfari, *Refined Heinz operator inequalities and norm inequalities*, Oper. Matrices, 15 (2021), 239-252.

Accepted: July 18, 2022

Schur convexity of a function whose fourth-order derivative is non-negative and related inequalities

Yiting Wu*
Department of Mathematics
China Jiliang University
Hangzhou, 310018
People's Republic of China
yitingly@sina.com

Qing Meng
Department of Mathematics
China Jiliang University
Hangzhou, 310018
People's Republic of China
mengqing2233@163.com

Abstract. In this paper, we study the Schur convexity of a function containing variable upper and lower limit of integration, we prove that the function is Schur-convex if its fourth-order derivative is non-negative. Finally, we use the obtained result to derive an inequality of Hermite-Hadamard type.
Keywords: Schur-convex, majorization, fourth-order derivative, Hermite-Hadamard-type inequality.

1. Introduction

Schur convexity is an important notion in the theory of convex functions, which was introduced by Schur in 1923 (see [1]). Over the past half a century, Schur convexity has aroused the interest of many researchers due to its powerful applications in the theory of inequalities, we refer the reader to [2–19] and references cited therein.

In [20], Elezović and Pečarić proved the Schur convexity of the following function.

Claim 1.1. Suppose $f : I \to \mathbb{R}$ is a continuous function. Then, the function

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y,\ x,y \in I \\ f(x), & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if f is convex (concave) on I.

*. Corresponding author

In [21], Chu, Wang and Zhang showed the Schur convexity of the following two functions.

Claim 1.2. Suppose $f : I \to \mathbb{R}$ is a continuous function. Then, the function

$$M(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right), & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if f is convex (concave) on I, and the function

$$T(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if f is convex (concave) on I.

In [22], Franjić and Pečarić verified the Schur convexity of the function below.

Claim 1.3. Suppose $f : I \to \mathbb{R}$ is a continuous function. Then, the function

$$S(x,y) = \begin{cases} \frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y) - \frac{1}{y-x}\int_x^y f(t)dt, & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if $f^{(4)} \geq 0$ ($f^{(4)} \leq 0$) on I.

Inspired by the research results described in [20-22] above, in this paper we study the Schur convexity of a function which contains variable upper and lower limit of integration, i.e.,

$$U(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - \frac{f(x)+f(y)}{2} + \frac{1}{12}\left(f'(y) - f'(x)\right)(y-x), & x \neq y,\ x,y \in I \\ 0, & x = y,\ x,y \in I. \end{cases}$$

The remaining parts of this paper are organized as follows. In Section 2, we present some definitions and lemmas which are essential in the proof of the main results. In Sections 3 and 4, we give our main result and an application.

2. Preliminaries

Let us recall some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 2.1 ([2, 23])**.** *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.*

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. And φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is a Schur-convex function on Ω.

Definition 2.2 ([2, 23]). *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$.

(*i*) *A set* $\Omega \subset \mathbb{R}^n$ *is called a symmetric set, if* $\boldsymbol{x} \in \Omega$ *implies* $\boldsymbol{x}P \in \Omega$ *for every* $n \times n$ *permutation matrix* P.

(*ii*) *A function* $\varphi : \Omega \to \mathbb{R}$ *is called a symmetric function if for every permutation matrix* P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \Omega$.

Lemma 2.1 ([2, 23]). *Let* $\Omega \subset \mathbb{R}^n$ *be symmetric and have a nonempty interior convex set.* Ω° *is the interior of* Ω. $\varphi : \Omega \to \mathbb{R}$ *is continuous on* Ω *and differentiable in* Ω°. *Then,* φ *is Schur-convex on* Ω *if and only if* φ *is symmetric on* Ω *and*

$$(x_i - x_j)\left(\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j}\right) \geq 0 \qquad (i \neq j,\ i, j = 1, 2, \ldots, n) \tag{1}$$

for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$. *Furthermore,* φ *is Schur-concave on* Ω *if and only if the reversed inequality above holds.*

Lemma 2.2 ([24]). *Let* $x \leq y$, $u(t) = ty + (1-t)x$, $v(t) = tx + (1-t)y$, $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$ *or* $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$. *Then*

$$\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (x, y). \tag{2}$$

Lemma 2.3 ([25]). (Simpson formula) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *and* $x, y \in I$. *If* $f^{(4)}$ *is continuous on* I, *then*

$$\frac{1}{y-x}\int_x^y f(t)dt - \frac{1}{6}\left(f(x) + 4f\left(\frac{x+y}{2}\right) + f(y)\right) = -\frac{(y-x)^4}{2880} f^{(4)}(\xi), \tag{3}$$

where ξ *is some number between* x *and* y.

3. Main result

Our main result is stated in the following theorem.

Theorem 3.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a continuous function. If* $f^{(4)} \geq 0$ ($f^{(4)} \leq 0$) *on* I, *then the function*

$$U(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - \frac{f(x)+f(y)}{2} + \frac{1}{12}\left(f'(y) - f'(x)\right)(y-x), & x \neq y,\ x, y \in I \\ 0, & x = y,\ x, y \in I \end{cases}$$

is Schur-convex (*Schur-concave*) *on* I^2.

Proof. Note that, $U(x, y)$ is symmetric about x, y on I, without loss of generality, we may assume that $y \geq x$. Below we divide the proof into two cases.

Case 1. If $x = y$, it follows from the definition of derivative and L'Hopital's rule that, for any $t_0 \in I$,

$$\begin{aligned}\frac{\partial U}{\partial x}|_{(t_0,t_0)} &= \lim_{\Delta t\to 0}\frac{U(t_0+\Delta t,t_0)-U(t_0,t_0)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{-\frac{1}{\Delta t}\int_{t_0+\Delta t}^{t_0} f(t)dt - \frac{f(t_0+\Delta t)+f(t_0)}{2} - \frac{\Delta t}{12}\left(f'(t_0)-f'(t_0+\Delta t)\right)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{-\int_{t_0+\Delta t}^{t_0} f(t)dt - \frac{\Delta t}{2}\left(f(t_0+\Delta t)+f(t_0)\right)}{(\Delta t)^2}\\ &= -\lim_{\Delta t\to 0}\frac{\Delta t f''(t_0+\Delta t)}{4}\\ &=0.\end{aligned}$$

Similarly, we can obtain $\frac{\partial U}{\partial y}|_{(t_0,t_0)} = 0$. Hence we have, for any $x = y \in I$,

$$(y-x)\left(\frac{\partial U}{\partial y}-\frac{\partial U}{\partial x}\right)=0.$$

Case 2. If $x \neq y$, differentiating $U(x, y)$ with respect to y and x respectively gives

$$\begin{aligned}\frac{\partial U}{\partial y} &= -\frac{1}{(y-x)^2}\int_x^y f(t)dt + \frac{f(y)}{y-x} - \frac{f'(y)}{2} + \frac{f''(y)(y-x)+f'(y)-f'(x)}{12},\\ \frac{\partial U}{\partial x} &= \frac{1}{(y-x)^2}\int_x^y f(t)dt - \frac{f(x)}{y-x} - \frac{f'(x)}{2} - \frac{f''(x)(y-x)+f'(y)-f'(x)}{12}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&(y-x)\left(\frac{\partial U}{\partial y}-\frac{\partial U}{\partial x}\right)\\ &= -\frac{2}{y-x}\int_x^y f(t)dt + (f(x)+f(y)) - \frac{(y-x)}{3}(f'(y)-f'(x))\\ &+ \frac{(y-x)^2}{12}(f''(x)+f''(y)).\end{aligned} \tag{4}$$

Using the Simpson formula (Lemma 2.3) with $f^{(4)} \geq 0$, we obtain

$$\frac{1}{y-x}\int_x^y f(t)dt \leq \frac{1}{6}\left(f(x)+4f\left(\frac{x+y}{2}\right)+f(y)\right). \tag{5}$$

Combining (4) and (5), we acquire that

$$\begin{aligned}&(y-x)\left(\frac{\partial U}{\partial y}-\frac{\partial U}{\partial x}\right)\\ &\geq -\frac{4}{3}f\left(\frac{x+y}{2}\right) + \frac{2}{3}(f(x)+f(y)) - \frac{(y-x)}{3}(f'(y)-f'(x))\\ &+ \frac{(y-x)^2}{12}(f''(x)+f''(y))\\ &=: Q(x,y).\end{aligned} \tag{6}$$

It is enough to prove $Q(x,y) \geq 0$ for any $x, y \in I$. Differentiating $Q(x,y)$ with respect to y and x respectively, we obtain

$$\frac{\partial Q}{\partial y} = -\frac{2}{3} f'(\frac{x+y}{2}) + \frac{f'(x)+f'(y)}{3} + \frac{(y-x)(f''(x)-f''(y))}{6} + \frac{(y-x)^2 f'''(y)}{12},$$
$$\frac{\partial Q}{\partial x} = -\frac{2}{3} f'(\frac{x+y}{2}) + \frac{f'(x)+f'(y)}{3} + \frac{(y-x)(f''(x)-f''(y))}{6} + \frac{(y-x)^2 f'''(x)}{12},$$

Then, by $f^{(4)} \geq 0$, we have

$$(y-x)\left(\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{12}(y-x)^3 (f'''(y) - f'''(x)) \geq 0.$$

It follows from Lemma 2.1 that $Q(x,y)$ is Schur-convex on I^2. In addition, by Lemma 2.2, we have $\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (x,y)$. Hence, we deduce from Definition 2.1 that

$$Q(x,y) \geq Q\left(\frac{x+y}{2}, \frac{x+y}{2}\right) = 0. \tag{7}$$

Combining (6) and (7), we conclude that, for any $x, y \in I$, $x \neq y$,

$$(y-x)\left(\frac{\partial U}{\partial y} - \frac{\partial U}{\partial x}\right) \geq Q(x,y) \geq 0.$$

Hence, we derive from Lemma 2.1 that $U(x,y)$ is Schur-convex on I^2.

By the same way as the proof of Theorem 3.1 for $f^{(4)} \geq 0$ above, we can prove that the $U(x,y)$ is Schur-concave for $f^{(4)} \leq 0$. This completes the proof of Theorem 4. □

4. An application

Theorem 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function with $f^{(4)} \geq 0$ on I. Then, for $x \neq y$, $x, y \in I$, $0 \leq t_1 \leq t_2 < \frac{1}{2}$ or $\frac{1}{2} < t_2 \leq t_1 \leq 1$, we have the following inequalities*

$$\begin{aligned}
&\frac{1}{y-x}\int_x^y f(t)dt - \frac{f(x)+f(y)}{2} + \frac{1}{12}\left(f'(y)-f'(x)\right)(y-x) \\
&\geq \frac{1}{(1-2t_1)(y-x)}\int_{t_1 y+(1-t_1)x}^{t_1 x+(1-t_1)y} f(t)dt \\
&- \frac{f(t_1 y+(1-t_1)x) + f(t_1 x+(1-t_1)y)}{2} \\
&+ \frac{1}{12}\left(f'(t_1 x+(1-t_1)y) - f'((t_1 y+(1-t_1)x)\right)(1-2t_1)(y-x) \\
&\geq \frac{1}{(1-2t_2)(y-x)}\int_{t_2 y+(1-t_2)x}^{t_2 x+(1-t_2)y} f(t)dt \\
&- \frac{f(t_2 y+(1-t_2)x) + f(t_2 x+(1-t_2)y)}{2}
\end{aligned} \tag{8}$$

$$+\frac{1}{12}\left(f'\left(t_2x+(1-t_2)y\right)-f'((t_2y+(1-t_2)x)\right)(1-2t_2)(y-x)\geq \quad 0.$$

Each of the inequalities in (8) *is reverse for* $f^{(4)} \leq 0$ *on* I.

Proof. Since each of the inequalities in (8) is symmetric about x, y, without loss of generality, we can assume that $y > x$.

Using Lemma 2.2, we have

$$\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(u(t_2),v(t_2))\prec(u(t_1),v(t_1))\prec(x,y), \tag{9}$$

where $u(t) = ty + (1-t)x$, $v(t) = tx + (1-t)y$.

In addition, from Theorem 3.1, we find that

$$U(x,y)=\begin{cases}\frac{1}{y-x}\int_x^y f(t)dt-\frac{f(x)+f(y)}{2}+\frac{1}{12}\left(f'(y)-f'(x)\right)(y-x), & x\neq y, x,y\in I\\ 0, & x=y,\ x,y\in I\end{cases}$$

is Schur-convex on I^2 under the assumption that $f^{(4)} \geq 0$.

Thus, we derive from the Definition (2.1) that

$$U\left(\frac{x+y}{2},\frac{x+y}{2}\right)\leq U(u(t_2),v(t_2))\leq U(u(t_1),v(t_1))\leq U(x,y), \tag{10}$$

which implies the required inequalities in (8). Similarly, we can deduce the reversed inequalities of (8) under the assumption that $f^{(4)} \leq 0$. The proof of Theorem 4.1 is complete. □

As a direct consequence of Theorem 4.1, we obtain

Corollary 4.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a continuous function with* $f^{(4)} \geq 0$ *on* I. *Then, for* $x \neq y$, $x, y \in I$, *the following inequality holds.*

$$\frac{1}{y-x}\int_x^y f(t)dt\geq\frac{f(x)+f(y)}{2}-\frac{1}{12}\left(f'(y)-f'(x)\right)(y-x). \tag{11}$$

Inequality (11) *is reverse for* $f^{(4)} \leq 0$ *on* I.

Acknowledgements

This work is supported by the Natural Science Foundation of Zhejiang Province under Grant No. LY21A010016.

References

[1] I. Schur, *Über eine klasse von mittelbildungen mit anwendungen die determinanten*, Theorie Sitzungsber, Berlin. Math. Gesellschaft, 22 (1923), 9-20.

[2] A. W. Marshall, I. Olkin, *Inequalities: the theory of majorization and its applications*, Academic Press, New York, 1979.

[3] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, New York, 1992.

[4] H. N. Shi, S. H. Wu, *Majorized proof and improvement of the discrete Steffensen's inequality*, Taiwanese J. Math., 11 (2007), 1203-1208.

[5] S. H. Wu, L. Debnath, *Inequalities for convex sequences and their applications*, Comput. Math. Appl., 54 (2007), 525-534.

[6] S. H. Wu, H. N. Shi, *A relation of weak majorization and its applications to certain inequalities for means*, Math. Slovaca, 61 (2011), 561-570.

[7] H. N. Shi, S. H. Wu, *Schur m-power convexity of geometric Bonferroni mean*, Italian J. Pure Appl. Math., 38 (2017), 769-776.

[8] H. N. Shi, S. H. Wu, *A concise proof of a double inequality involving the exponential and logarithmic functions*, Italian J. Pure Appl. Math., 41 (2019), 284-289.

[9] S. H. Wu, H. N. Shi, D. S. Wang, *Schur convexity of generalized geometric Bonferroni mean involving three parameters*, Italian J. Pure Appl. Math., 42 (2019), 196-207.

[10] H. N. Shi, S. H. Wu, *Schur convexity of the dual form of complete symmetric function involving exponent parameter*, Italian J. Pure Appl. Math., 44 (2020), 836-845.

[11] H. N. Shi, S. H. Wu, F. Qi, *An alternative note on the Schur-convexity of the extended mean values*, Math. Inequal. Appl., 9 (2006), 219-224.

[12] H. N. Shi, Y. M. Jiang and W. D. Jiang, *Schur-convexity and Schur-geometrically concavity of Gini mean*, Comput. Math. Appl., 57 (2009), 266-274.

[13] H. N. Shi, J. Zhang, *Schur-convexity, Schur geometric and Schur harmonic convexities of dual form of a class symmetric functions*, J. Math. Inequal., 8 (2014), 349-358.

[14] D. Wang, C. R. Fu, H. N. Shi, *Schur m-power convexity for a mean of two variables with three parameters*, J. Nonlinear Sci. Appl., 9 (2016), 2298-2304.

[15] H. P. Yin, H. N. Shi, F. Qi, *On Schur m-power convexity for ratios of some means*, J. Math. Inequal., 9 (2015), 145-153.

[16] Z. H. Yang, *Schur power convexity of Stolarsky means*, Publ. Math. Debrecen, 80 (2012), 43-66.

[17] Z. H. Yang, *Schur power convexity of Gini means*, Bull. Korean Math. Soc., 50 (2013), 485-498.

[18] Z. H. Yang, *Schur power convexity of the Daróczy means*, Math. Inequal. Appl., 16 (2013), 751-762.

[19] W. Wang, S. Yang, *Schur m-power convexity of generalized Hamy symmetric function*, J. Math. Inequal., 8 (2014), 661-667.

[20] N. Elezović, J. Pečarić, *A note on Schur-convex functions*, Rocky Mountain J. Math., 30 (2000), 853-856.

[21] Y. Chu, G. Wang, X. Zhang, *Schur convexity and Hadamard's inequality*, Math. Inequal. Appl., 13 (2010), 725-731.

[22] I. Franjić, J. Pečarić, *Schur-convexity and the Simpson formula*, Appl. Math. Lett., 24 (2011), 1565-1568.

[23] H. N. Shi, *Schur-convex functions and inequalities*, Harbin Institute of Technology Press, Harbin, 2017 (Chinese).

[24] H. N. Shi, *Theory of majorization and analytic inequalities*, Harbin Industrial University Press, Harbin, 2012 (Chinese).

[25] Y. Li, *Schur convexity and the dual Simpson's formula*, J. Appl. Math. Phys., 4 (2016), 623-629.

Accepted: November 22, 2022

Finite groups of order p^3qr in which the number of elements of maximal order is p^4q

Qingliang Zhang
Jiangsu College of Engineering and Technology
Jiangsu 226006
P. R. China
qingliangzhang@jcet.edu.cn

Zhilin Qin*
Jiangsu College of Engineering and Technology
Jiangsu 226006
P. R. China
and
School of Sciences
Nantong University
Jiangsu 226019
P. R. China
zhilinqin2016@163.com
qin.zhl@ntu.edu.cn

Abstract. Suppose that G is a finite group. As is known to all, the order of G and the number of elements of maximal order in G are closely related to the structure of G. This topic involves Thompson's problem. In this paper we classify the finite groups of order p^3qr in which the number of elements of maximal order is p^4q, where $p < q < r$ are different primes.

Keywords: finite groups, group order, the number of elements of maximal order, isomorphic classification.

1. Introduction

All groups considered in our paper are finite. Let n be an integer. We denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Then, $\pi(|G|)$ is denoted by $\pi(G)$. The set of orders of elements of G is denoted by $\pi_e(G)$. We denote by $k(G)$ and $m(G)$ the maximal order of elements in G and the number of elements of order $k(G)$ in G, respectively. We write H char G if H is characteristic in G. $G = N \rtimes Q$ stands for the split extension of a normal subgroup N of G by a complement Q. By $M \lesssim G$ we denote M is isomorphic to a subgroup of G. And we denote by Z_n a cyclic group of order n. All unexplained notations are standard and can be found in [6].

*. Corresponding author

For a finite group G, $|G|$ and $m(G)$ have an important influence on the structure of G. The authors in [13, 3, 9] proved that finite groups G with $m(G) = lp$ are soluble, where $l = 2$, 4, or 18. In [8] it was proved that finite groups G with $m(G) = 2p^2$ are soluble. The authors in [2, 7] gave a classification of the finite groups G with $m(G) = 30$ and $m(G) = 24$. The authors in [10] showed that if G is a finite group which has $4p^2q$ elements of maximal order, where p, q are primes and $7 \leq p \leq q$, then either G is soluble or G has a section who is isomorphic to one of $L_2(7)$, $L_2(8)$ or $U_3(3)$. These studies are closely related to the following problem.

Thompson's problem. Let H be a finite group. For a positive integer d, define $H(d) = |\{x \in H||x| = d\}|$. Suppose that $H(d) = G(d)$ for $d = 1, 2, \ldots$, where G is a soluble group. Is it true that H is also necessarily soluble?

The problem we consider is also closely related to Thompson's problem. In this paper we classify the finite groups of order p^3qr in which the number of elements of maximal order is p^4q, where $p < q < r$ are primes (Let us denote this property by (*) for brevity). We find that this isomorphic classification problem is complex. Our results are:

Theorem 1.1. *A group G has property (*) if and only if one of the following statements holds:*

(1) $G \cong M \ltimes Z_r$ *and* $r - 1 = 16q$. *Moreover,* $C_M(Z_r) \cong Z_2$, $M/C_M(Z_r) \lesssim$ *Aut*(Z_r) *and* $|M/C_M(Z_r)| = 4q$*;*

(2) $G \cong K \ltimes Z_r$ *and* $r - 1 = 8q$. *Moreover,* $C_K(Z_r) \cong Z_4$, $K/C_K(Z_r) \lesssim$ *Aut*(Z_r) *and* $|K/C_K(Z_r)| = 2q$*;*

(3) $G \cong L \ltimes Z_r$ *and* $r - 1 = 8q$. *Moreover,* $C_L(Z_r) \cong D_8$, $L/C_L(Z_r) \lesssim$ *Aut*(Z_r) *and* $|L/C_L(Z_r)| = q$*;*

(4) $G \cong R \ltimes Z_r$ *and* $r - 1 = 4q$. *Moreover,* $C_R(Z_r) \cong Z_4 \times Z_2$, $R/C_R(Z_r) \lesssim$ *Aut*(Z_r) *and* $|R/C_R(Z_r)| = q$*;*

(5) $G \cong Z_q \ltimes Z_{8r}$ *and* $r - 1 = 4q$. *Moreover,* $C_{Z_q}(Z_{8r}) = 1$*;*

(6) $G \cong M \ltimes Z_7$ *and* $C_M(Z_7) \cong A_4$, *where* $M \cong A_4 \times Z_2$ *or* S_4*;*

(7) $G \cong Z_{168}$*;*

(8) $G \cong Q_8 \times Z_{15}$*;*

(9) $G \cong D_8 \times Z_{qr}$, *where* $q = 3$ *and* $r = 13$ *or* $q = 5$ *and* $r = 11$*;*

(10) $G \cong (Z_4 \times Z_2) \times Z_{21}$*;*

(11) $G \cong M \ltimes Z_{qr}$, $q = 3$ *and* $r = 13$ *or* $q = 5$ *and* $r = 11$, *where* M *is a group of order 8. Moreover,* $C_M(Z_{qr}) \cong Z_4$*;*

(12) $G \cong (A_4 \times Z_2) \times Z_7$*;*

(13) $G \cong SL_2(F_3) \times Z_7$*;*

(14) *G is a Frobenius group and* $G \cong Z_{8q} \ltimes Z_r$. *Moreover,* $r - 1 = 16q$*;*

(15) $G \cong L_2(7)$*;*

(16) *G is a 2-Frobenius group and* $G \cong Z_3 \ltimes (Z_7 \ltimes P)$, *where P is an elementary abelian 2-group of order 8,* $P \trianglelefteq G$ *and* $G/P \cong Z_3 \ltimes Z_7$. *Moreover,* $\pi_e(G) = \{1, 2, 3, 6, 7\}$.

Corollary 1.2. *All of the groups with property* $(*)$ *are of even order.*

Corollary 1.3. *Suppose that* G *is a non-soluble group with property* $(*)$*. Then,* $G \cong L_2(7)$*.*

Corollary 1.4. *The answer to Thompson's problem is yes for finite groups* (1)-(14) *and* (16) *of Theorem 1.1.*

2. Preliminaries

We need the following lemmas to prove our results.

Lemma 2.1 ([12])**.** *Let* G *be a finite group. Then, the number of elements whose orders are multiples of* n *is either zero, or a multiple of the greatest divisor of* $|G|$ *that is prime to* n*.*

Lemma 2.2 ([3])**.** *Let* G *be a finite group. We denote by* A_i $(1 \leq i \leq s)$ *a complete representative system of conjugate classes of cyclic subgroups of order* $k(G)$*, respectively. Then, we have:*

(1) $m(G) = \varphi(k(G))\sum n_i$*, where* $\varphi(k(G))$ *is Euler function,* $n_i = |G : N_G(A_i)|$ *and* $1 \leq i \leq s$*;*

(2) $|G| = |G : N_G(A_i)||N_G(A_i) : C_G(A_i)||C_G(A_i)|$*, where* $1 \leq i \leq s$*;*

(3) $|N_G(A_i) : C_G(A_i)||\varphi(k(G))$*, where* $1 \leq i \leq s$*;*

(4) $\pi(C_G(A_i)) = \pi(A_i)$*, where* $1 \leq i \leq s$*.*

Lemma 2.3 ([4])**.** *Let* G *be a soluble group of order* mn*, where* m *is prime to* n*. Then, the number of subgroups of* G *of order* m *may be expressed as a product of factors, each of which (i) is congruent to 1 modulo some prime factor of* m*, (ii) is a power of a prime and divides the order of some chief factor of* G*.*

Lemma 2.4 ([1])**.** *Let* H *be a finite group and* $\pi_e(H) = \{1, 2, 3, 4\}$*. Then,* $H = N \rtimes Q$ *and one of the following conclusions holds:*

(i) N *has exponent 4 and class* ≤ 2*,* $Q \cong Z_3$*.*

(ii) $N = {Z_2}^{2t}$ *and* $Q \cong S_3$*, where* ${Z_2}^{2t}$ *stands for the direct product of* $2t$ *copies of* Z_2*.*

(iii) $N = {Z_3}^{2t}$ *and* $Q \cong Z_4$ *or* Q_8 *and* H *is a Frobenius group, where* Q_8 *is the generalized quaternion group.*

Lemma 2.5 ([14])**.** *Let* G *be a finite group satisfying* $|G| = 2^3 \cdot 3 \cdot 7$ *and* $m(G) = 48$*.*

(1) *If* $k(G) = 42$*, then* $G \cong (A_4 \times Z_2) \times Z_7$ *or* $G \cong SL_2(F_3) \times Z_7$*.*

(2) *If* $k(G) = 21$*, then* $G \cong M \ltimes Z_7$ *and* $C_M(Z_7) \cong A_4$*, where* $M \cong A_4 \times Z_2$ *or* S_4*.*

Lemma 2.6 ([5])**.** *Let* G *be a finite simple group. If* $|\pi(G)| = 3$*, then we call* G *a simple* K_3*-group. If* G *is a simple* K_3*-group, then* G *is isomorphic to one of the following groups:* A_5*,* A_6*,* $L_2(7)$*,* $L_2(8)$*,* $L_2(17)$*,* $L_3(3)$*,* $U_3(3)$ *and* $U_4(2)$*.*

Lemma 2.7 ([15]). *Let G be a finite group. Then, $|G| = |L_2(7)|$ and $k(G) = k(L_2(7))$ if and only if $G \cong L_2(7)$ or G is a 2-Frobenius group, at this moment, $G \cong Z_3 \ltimes (Z_7 \ltimes P)$, where P is an elementary abelian 2-group of order 8, $P \trianglelefteq G$ and $G/P \cong Z_3 \ltimes Z_7$. Moreover, $\pi_e(G) = \{1, 2, 3, 6, 7\}$.*

3. Proof of the Results

Proof of Theorem 1.1

It is not hard to see that all the groups from items (1)-(16) of Theorem 1.1 have property (*).

Now, we assume that G has property (*). Namely, $|G| = p^3qr$ and $m(G) = p^4q$. From Lemma 2.1 we get that $\pi(G) \subseteq \pi(m(G)) \bigcup \pi(k(G))$. Then, $r \in \pi(k(G))$. Since $\varphi(k(G)) \mid m(G)$ by Lemma 2.2, we obtain that $\varphi(r) = r - 1 | p^4q$. From $2|r - 1$ it follows that $p = 2$. In the following we discuss four cases.

Case 1. If $\pi(k(G)) = \{2, r\}$, then $k(G) = 2r$, $4r$ or $8r$.

Suppose that $k(G) = 2r$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x\rangle = A$. It is clear that $x^2 \in Z(C_G(A))$ and so G has a Sylow r-subgroup P_r such that $P_r \leq Z(C_G(A))$. Therefore P_r char $C_G(A)$ and it follows that $P_r \trianglelefteq N_G(A)$ since $C_G(A) \trianglelefteq N_G(A)$. Therefore $N_G(A) \leq N_G(P_r)$ and thus $|G : N_G(P_r)|||G : N_G(A)|$. By Lemma 2.2 we get that $|G : N_G(A)||4q$. So $|G : N_G(P_r)||4q$.

If $P_r \ntrianglelefteq G$, then $|G : N_G(P_r)| = 2q$ or $4q$ by Sylow's theorem. If $|G : N_G(P_r)| = 4q$, then $|G : N_G(A)| = 4q$ and so $4q|n$ by Lemma 2.2, where n is the number of cyclic subgroups of order $k(G)$ in G. Note that $n = \frac{m(G)}{\varphi(2r)} = \frac{16q}{r-1}$, thus $r - 1 = 4$ and so $r = 5$. It follows that $q = 3$. Hence, $|G : N_G(P_5)| = 12$, which is contradict to Sylow's theorem. If $|G : N_G(P_r)| = 2q$, then $|N_G(P_r)| = 4r$ and $|C_G(P_r)| = 2^\alpha r$, where $1 \leq \alpha \leq 2$. Moreover, $C_G(P_r)$ contains exactly $\frac{m(G)}{2q} = 8$ elements of order $2r$. On the other hand, we get that $C_G(P_r) = H \times P_r$ by Schur-Zassenhaus's theorem since $P_r \leq Z(C_G(P_r))$, where H is a group satisfying $|H| = 2^\alpha$. It follows that $C_G(P_r)$ contains exactly $(2^\alpha - 1)(r - 1)$ elements of order $2r$. Thus, $(2^\alpha - 1)(r - 1) = 8$, which is impossible obviously since $1 \leq \alpha \leq 2$.

If $P_r \trianglelefteq G$, then $C_G(P_r)$ contains all the elements of order $k(G)$ in G since $A \leq C_G(A) \leq C_G(P_r)$. Note that $P_r \leq Z(C_G(P_r))$, thus $|C_G(P_r)| = 2^lr$, where $1 \leq l \leq 3$. Moreover, $C_G(P_r) = H_1 \times P_r$ by Schur-Zassenhaus's theorem, where H_1 is a group of order 2^l. If $l = 2$, then H_1 is an elementary abelian group of order 4. Thus, $3(r - 1) = 16q$ and it follows that $q = 3$ and $r = 17$. Since $|G/C_G(P_{17})|||Aut(P_{17})|$, we get that $6|16$, which is a contradiction. Similarly, we can show that $l \neq 3$. If $l = 1$, then $r - 1 = 16q$. Note that $P_r \cong Z_r$, then by Schur-Zassenhaus's theorem we get that $G \cong M \ltimes Z_r$. Moreover, $C_M(Z_r) \cong Z_2$, $M/C_M(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|M/C_M(Z_r)| = 4q$. Hence, (1) holds.

Suppose that $k(G) = 4r$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x\rangle = A$. Similar to the above, we can get that G has a Sylow r-

subgroup P_r such that $P_r \leq Z(C_G(A))$, $|G : N_G(P_r)||2q$ and $|G : N_G(P_r)| = |G : N_G(A)| = 2q$ by Sylow's theorem if $P_r \ntrianglelefteq G$. Hence, $C_G(P_r)$ contains exactly $\frac{m(G)}{2q} = 8$ elements of order $4r$. Note that $P_r \leq Z(C_G(P_r))$, thus $r-1|8$. It follows that $r = 5$ and so $q = 3$. Therefore $|G : N_G(P_5)| = 6$ and so $|N_G(P_5)| = |C_G(P_5)| = 20$. Hence, G is 5-nilpotent by Burnside's theorem. Then, G is soluble. By Lemma 2.3 it follows that $2 \equiv 1 (\mathrm{mod}\ 5)$ and $3 \equiv 1 (\mathrm{mod}\ 5)$, which is impossible.

If $P_r \trianglelefteq G$, then $C_G(P_r)$ contains all the elements of order $4r$ in G. Furthermore, $|C_G(P_r)| = 2^\alpha \cdot q^\beta \cdot r$, where $2 \leq \alpha \leq 3$ and $0 \leq \beta \leq 1$. Note that $P_r \leq Z(C_G(P_r))$, then by Schur-Zassenhaus's theorem we have $C_G(P_r) = H \times P_r$, where H is a group of order $2^\alpha \cdot q^\beta$.

Suppose that $\beta = 1$. Then, $q = 3$ since $k(G) = 4r$ is the maximal element order of G. If $\alpha = 2$, then H is a group of order 12 and $\pi_e(H) = \{1, 2, 3, 4\}$. It follows that $H \cong Z_4 \rtimes Z_3$ by Lemma 2.4. Hence, $2(r-1) = m(G) = 48$. It follows that $r = 25$, which is impossible. If $\alpha = 3$, then $C_G(P_r) = G$ and so $P_r \leq Z(G)$. Consequently, $G = M \times P_r$ by Schur-Zassenhaus's theorem, where M is a group of order 24. Note that $\pi_e(M) = \{1, 2, 3, 4\}$, thus $M \cong (Z_2 \times Z_2) \rtimes S_3$ or $N \rtimes Z_3$ by Lemma 2.4. If $M \cong (Z_2 \times Z_2) \rtimes S_3$, then $6(r-1) = m(G) = 48$ and thus $r = 9$, which is a contradiction. If $M \cong N \rtimes Z_3$, then the conjugate action of Z_3 on N is fixed-point-free. Thus, $|Z_3|||N| - 1$ and it follows that $3|7$, which is impossible.

Suppose that $\beta = 0$. Then, $|C_G(P_r)| = 4r$ or $8r$. If $|C_G(P_r)| = 4r$, then $C_G(P_r) \cong Z_4 \times Z_r$. It follows that $2(r-1) = m(G) = 16q$ and so $r - 1 = 8q$. Moreover, $G \cong K \ltimes Z_r$ by Schur-Zassenhaus's theorem, $C_K(Z_r) \cong Z_4$, $K/C_K(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|K/C_K(Z_r)| = 2q$. Hence, (2) holds. If $|C_G(P_r)| = 8r$, then H is isomorphic to the dihedral group D_8, the generalized quaternion group Q_8 or $Z_4 \times Z_2$ since $k(H) = 4$. If $H \cong Q_8$, then $6(r-1) = m(G) = 16q$ and so $r = 9$, which is a contradiction. If $H \cong D_8$, then $2(r-1) = m(G) = 16q$ and so $r - 1 = 8q$. Moreover, $G \cong L \ltimes Z_r$ by Schur-Zassenhaus's theorem, $C_L(Z_r) \cong D_8$, $L/C_L(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|L/C_L(Z_r)| = q$. Hence, (3) holds. If $H \cong Z_4 \times Z_2$, then $4(r-1) = m(G) = 16q$ and so $r - 1 = 4q$. Moreover, $G \cong R \ltimes Z_r$, $C_R(Z_r) \cong Z_4 \times Z_2$, $R/C_R(Z_r) \lesssim \mathrm{Aut}(Z_r)$ and $|R/C_R(Z_r)| = q$. Hence, (4) holds.

Suppose that $k(G) = 8r$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. It is clear that $x^8 \in Z(C_G(A))$ and so G has a Sylow r-subgroup P_r such that $P_r \leq Z(C_G(A))$. Since $A \leq C_G(A) \leq C_G(P_r)$, we have $|C_G(P_r)| = 8q^\gamma r$, where $0 \leq \gamma \leq 1$. Note that $P_r \leq Z(C_G(P_r))$, thus $C_G(P_r) = H \times P_r$ by Schur-Zassenhaus's theorem, where H is a group of order $8q^\gamma$ and $k(H) = 8$.

Suppose that $\gamma = 1$. Since $k(G) = 8r$, we have $q = 3$, 5 or 7. Note that the Sylow 2-subgroup P_2 of H is cyclic, thus H is 2-nilpotent and so the Sylow q-subgroup Q of H is normal in H. If $q = 5$ or 7, then the conjugate action of P_2 on Q is fixed-point-free since $k(H) = 8$. Therefore $8|q - 1$, which is impossible.

If $q = 3$, then H is a group of order 24 satisfying $k(H) = 8$. Now, we get a contradiction since such group H does not exist by [11].

Suppose that $\gamma = 0$. Then, $|C_G(P_r)| = 8r$. Since the Sylow 2-subgroup of G is cyclic, we get that G is 2-nilpotent. It follows that the subgroup of order qr of G is normal in G. Then, $P_r \trianglelefteq G$ by Sylow's theorem and so $C_G(P_r) \trianglelefteq G$. Hence, $C_G(P_r)$ contains all the elements of order $8r$ and $G \cong Z_q \ltimes Z_{8r}$ by Schur-Zassenhaus's theorem. Moreover, $4(r-1) = m(G) = 16q$ and $C_{Z_q}(Z_{8r}) = 1$. Hence, (5) holds.

Case 2. If $\pi(k(G)) = \{q, r\}$, then $k(G) = qr$.

Choose an arbitrary element x of order $k(G)$ in G and let $\langle x\rangle = A$. Similar to Case 1, we can get that G has a Sylow r-subgroup P_r such that $P_r \leq Z(C_G(A))$ and $|G : N_G(P_r)| = 1$ or 8.

If $|G : N_G(P_r)| = 1$, then $P_r \trianglelefteq G$ and $C_G(P_r)$ contains all the elements of order qr in G since $A \leq C_G(A) \leq C_G(P_r)$. Moreover, G is soluble. By Lemma 2.3 it follows that $|G : N_G(A)| = 1$, 4 or 8. If $|G : N_G(A)| = 8$, then $8(q-1)(r-1) = m(G) = 16q$. If follows that $q = 3$ and $r = 4$, which is a contradiction. If $|G : N_G(A)| = 4$, then $4 \equiv 1 \pmod q$ by Lemma 2.3. Therefore $q = 3$ and thus $4(3-1)(r-1) = m(G) = 48$. Hence, $r = 7$. Therefore by (2) of Lemma 2.5 we have $G \cong M \ltimes Z_7$ and $C_M(Z_7) \cong A_4$, where $M \cong A_4 \times Z_2$ or S_4. Hence, (6) holds. If $|G : N_G(A)| = 1$, then $(q-1)(r-1) = 16q$, which is impossible we can find by simple calculation.

If $|G : N_G(P_r)| = 8$, then $C_G(P_r)$ contains exactly $\frac{m(G)}{8} = 2q$ elements of order qr. On the other hand, we know that $A \leq C_G(A) \leq C_G(P_r)$, thus $C_G(P_r)$ contains at least $\varphi(qr) = (q-1)(r-1)$ elements of order qr. Now, we get a contradiction since $(q-1)(r-1) > 2q$.

Case 3. If $\pi(k(G)) = \{2, q, r\}$, then $k(G) = 8qr$, $4qr$ or $2qr$.

If $k(G) = 8qr$, then $\varphi(8qr) = 4(q-1)(r-1) = 16q$. Consequently, $\frac{q-1}{2} \cdot \frac{r-1}{2} = q$. Since $\frac{r-1}{2} > 1$, we have $\frac{q-1}{2} = 1$ and so $q = 3$. It follows that $r = 7$. Hence, $G \cong Z_{168}$ and thus (7) holds.

Suppose that $k(G) = 4qr$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x\rangle = A$. It is clear that $Z(C_G(A))$ contains elements of order qr, and so G has a subgroup H of order qr such that $H \leq Z(C_G(A))$. Therefore H char $C_G(A)$ and it follows that $H \trianglelefteq N_G(A)$ since $C_G(A) \trianglelefteq N_G(A)$. So $N_G(A) \leq N_G(H)$. Then, $|G : N_G(H)|\,|\,|G : N_G(A)|$. Note that $|G : N_G(A)| = 1$, thus $|G : N_G(H)| = 1$ and so $H \trianglelefteq G$. Therefore $C_G(H)$ contains all the elements of order $k(G)$ in G and so $|C_G(H)| = 2^\alpha qr$, where $2 \leq \alpha \leq 3$.

If $\alpha = 3$, then $C_G(H) = G$ and so $H \leq Z(G)$. Thus, $G = K \times H$ by Schur-Zassenhaus's theorem. Obviously, K is isomorphic to the dihedral group D_8, the generalized quaternion group Q_8 or $Z_4 \times Z_2$. If $K \cong Q_8$, then $6(q-1)(r-1) = m(G) = 16q$. Hence, $q = 3$ and $r = 5$. Therefore $G \cong Q_8 \times Z_{15}$. Hence, (8) holds. If $K \cong D_8$, then similarly we can get that $G \cong D_8 \times Z_{qr}$, where $q = 3$ and

$r = 13$ or $q = 5$ and $r = 11$. Hence, (9) holds. If $K \cong Z_4 \times Z_2$, then similarly we can get that $G \cong (Z_4 \times Z_2) \times Z_{21}$. Hence, (10) holds.

If $\alpha = 2$, then $C_G(H) \cong Z_4 \times Z_{qr}$. So $2(q-1)(r-1) = 16q$. It follows that $q = 3$ and $r = 13$ or $q = 5$ and $r = 11$. Furthermore, $G \cong M \ltimes Z_{qr}$ by Schur-Zassenhaus's theorem and $C_M(Z_{qr}) \cong Z_4$, where M is a group of order 8. Hence, (11) holds.

Suppose that $k(G) = 2qr$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. From the fact that $Z(C_G(A))$ contains elements of order qr we get that G has a cyclic subgroup H of order qr such that $H \leq Z(C_G(A))$. Similar to the above, we get that $|G : N_G(H)| = 1$, 2 or 4. Moreover, $|C_G(H)| = 2^\alpha qr$, where $1 \leq \alpha \leq 3$.

If $|G : N_G(H)| = 1$, then $H \trianglelefteq G$. It follows that $C_G(H)$ contains all elements of order $2qr$ since $A \leq C_G(A) \leq C_G(H)$. Since $H \leq Z(C_G(H))$, by Schur-Zassenhaus's theorem we have $C_G(H) = M \times H$, where M is an elementary abelian group of order 2^α. Hence, $(2^\alpha - 1)(q-1)(r-1) = m(G) = 16q$, which is impossible we can find by simple calculation. If $|G : N_G(H)| = 2$, then G is non-soluble by Lemma 2.3. Note that $N_G(H) \trianglelefteq G$, thus $N_G(H) \cong A_5$ by Lemma 2.6, which is a contradiction since $2qr \in \pi_e(N_G(H))$ and $2qr \notin \pi_e(A_5)$. If $|G : N_G(H)| = 4$, then $|G : N_G(A)| = 4$. Thus, $4|n$ by Lemma 2.2, where n is the number of the cyclic subgroups of order $2qr$ of G. Note that $n = \frac{m(G)}{\varphi(k(G)} = \frac{16q}{(q-1)(r-1)}$, thus $q = 3$ and $r = 7$. Therefore $G \cong (A_4 \times Z_2) \times Z_7$ or $G \cong SL_2(F_3) \times Z_7$ by (1) of Lemma 2.5. Hence, (12) and (13) hold.

Case 4. If $\pi(k(G)) = \{r\}$, then $k(G) = r$.

We know that the number n_r of Sylow r-subgroups of G is equal to 1, $2q$, $4q$, $8q$ or 8 by Sylow's theorem.

If $n_r = 1$, then the Sylow r-subgroup P_r of G is normal in G and $r - 1 = m(G) = 16q$. Moreover, G has an r-complement H of order $8q$ by Schur-Zassenhaus's theorem. Note that the conjugate action of H on P_r is fixed-point-free, thus G is a Frobenius group with Frobenius kernel P_r and Frobenius complement H. Note that $P_r \cong Z_r$ and H is a cyclic group since $H \lesssim \mathrm{Aut}(P_r)$, thus $G \cong Z_{8q} \ltimes Z_r$. Hence, (14) holds.

If $n_r = 2q$, then $2q(r-1) = m(G) = 16q$. It follows that $r = 9$, which is impossible.

If $n_r = 4q$, then $4q(r-1) = m(G) = 16q$. It follows that $r = 5$ and $q = 3$, which is contradict to Sylow's theorem.

If $n_r = 8q$, then $8q(r-1) = m(G) = 16q$. It follows that $r = 3$, which is impossible.

If $n_r = 8$, then $r = 7$ by Sylow's theorem and so $q = 3$ or 5. If $q = 5$, then $|N_G(P_7)| = 35$. Since $N_G(P_7)/C_G(P_7) \lesssim \mathrm{Aut}(P_7)$, we have $|N_G(P_7)/C_G(P_7)|$ divides $|Aut(P_7)|$. Note that $|C_G(P_7)| = 7$, thus $5|6$, which is a contradiction. If $q = 3$, then by Lemma 2.7 $G \cong L_2(7)$ or G is a 2-Frobenius group, at this moment, $G \cong Z_3 \ltimes (Z_7 \ltimes P)$, where P is an elementary abelian 2-group of order

8, $P \trianglelefteq G$ and $G/P \cong Z_3 \ltimes Z_7$. Moreover, $\pi_e(G) = \{1, 2, 3, 6, 7\}$. Hence, (15) and (16) hold.

Proof of Corollaries 1.2 and 1.3. It is evident by Theorem 1.1.

Proof of Corollary 1.4. Assume that G is a group, which is isomorphic to one of the finite groups (1-14) and (16) of Theorem 1.1. Suppose that H is a group satisfying $H(d) = G(d)$. Then, $|H| = |G|$ and $m(H) = m(G)$. Thus, H is soluble by Theorems 1.1. Hence, Corollary 1.4 holds.

Now, the proofs of our results are complete.

Acknowledgement

The authors would like to thank the referees with deep gratitude for pointing out some questions in the previous version of the paper. Their valuable suggestions help us improve the quality of our paper.

References

[1] R. Brandl, W.J. Shi, *Finite groups whose element orders are consecutive integers*, J. Algebra, 143 (1991), 388-400.

[2] G.Y. Chen, W.J. Shi, *Finite groups with 30 elements of maximal order*, Appl. Categor. Struct., 16 (2008), 239-247.

[3] X.L. Du, Y.Y. Jiang, *On finite groups with exact* $4p$ *elements of maximal order are soluble*, Chinese Ann. Math. Ser. A, 25 (2004), 607-612 (in chinese).

[4] M. Hall, *The theory of groups,* Macmillan Company, New York, 1959.

[5] M. Herzog, *On finite simple groups of order divisible by three primes only*, J. Algebra, 120 (1968), 383-388.

[6] B. Huppert, *Endliche Gruppen I.*, Die Grundlehren der mathematischen Wissenschafter in Einzeldarstellungen 134, Springer, Berlin, 1967 (in german).

[7] Q.H. Jiang, C.G. Shao, *Finite groups with 24 elements of maximal order*, Front. Math. China, 5 (2010), 665-678.

[8] Y.Y. Jiang, *Finite groups with* $2p^2$ *elements of maximal order are soluble*, Chinese Ann. Math. Ser. A, 21 (2000), 61-64 (in chinese).

[9] Y.Y. Jiang, *A theorem of finite groups with* $18p$ *elements having maximal order,* Algebra Collog., 15 (2008), 317-329.

[10] S.B Tan, G.Y. Chen, Y.X. Yan, *Finite groups with* $4p^2q$ *elements of maximal order*, Open Math., 19 (2021), 963-970.

[11] A.D. Thomas, G.V. Wood, *Group tables,* Shiva Publishing Limited, 1980.

[12] L. Weisner, *On the number of elements of a group which have a power in a given conjugate set*, Bull. Amer. Math. Soc., 31 (1925), 492-496.

[13] C. Yang, *Finite groups based on the numbers of elements of maximal order,* Chinese Ann. Math. Ser. A., 14 (1993), 561-567 (in chinese).

[14] Q.L. Zhang, *Finite groups with the same order and the same number of elements of maximal order as the projective special linear group* $L_2(q)$, to appear.

[15] Q.L. Zhang, W.J. Shi, *A new characterization of simple* K_3*-groups and some* $L_2(p)$, Algebra Collog., 20 (2013), 361-368.

Accepted: October 11, 2022

Derivative-based trapezoid rule for a special kind of Riemann-Stieltjes integral

Weijing Zhao*
College of Air Traffic Management
Civil Aviation University of China
Tianjin, 300300
P. R. China
693509394@qq.com

Zhaoning Zhang
College of Air Traffic Management
Civil Aviation University of China
Tianjin, 300300
P. R. China

Abstract. This paper adopts the concept of algebraic precision to construct the derivative-based trapezoid rule for a special kind of Riemann-Stieltjes integral, which uses two derivative values at the endpoints. This kind of quadrature rule obtains an increase of two orders of precision over the trapezoid rule for the Riemann-Stieltjes integral and the error term is investigated. Finally, some numerical examples indicate the numerical superiority of the proposed approach with respect to closed Newton-Cotes formulas.

Keywords: derivative, trapezoid rules, Riemann-Stieltjes integral, numerical integration, error term.

1. Introduction

Roughly speaking, the operation of integration is the reverse of differentiation. Definite integration is one of the most important and basic concepts in mathematics. The Riemann integral of a function f provides a continuous analog of the process of summation of numerical values $f(\xi_i)$, with each such value weighted by the width Δx_i of the interval $[x_{i-1}, x_i]$ from which ξ_i is selected. There are many reasons for generalizing this concept to allow for the weighting of the numerical values $f(\xi_i)$ by numbers different from Δx_i.

In mathematics, the Riemann-Stieltjes integral is a kind of generalization of the Riemann integral, named after Bernhard Riemann and Thomas Stieltjes. It is Stieltjes [1] that first gives the definition of this integral in 1894. The Riemann-Stieltjes integral allows for the replacement of Δx_i by $\Delta g_i = g(x_i) - g(x_{i-1})$, where g is a function of bounded variation [2, 3]. There are many reasons for making such an extension of the concept of the integral. It serves as an

∗. Corresponding author

instructive and useful precursor of the Lebesgue integral, and an invaluable tool in unifying equivalent forms of statistical theorems that apply to discrete and continuous probability.

The reason for introducing Riemann-Stieltjes integrals is to get a more unified approach to the theory of random variables, in particular for the expectation operator, as opposed to treating discrete and continuous random variables separately.

In probability theory, the interval $[a, b]$ might be the space of possible outcomes of a probabilistic experiment. Then $\Delta g_i = g(x_i) - g(x_{i-1})$ could represent the probability of the outcome landing in the interval $[x_{i-1}, x_i]$ of possibilities, and the function f could be the value in some sense of such an outcome [3]. In this illustration, $\int_a^b f(t)dg$ would be a probabilistically expected value to result from running the experiment [2, 3].

It is known that the Riemann-Stieltjes integral has wide applications in the field of stochastic process [4] and functional analysis [5], especially the spectral theorem for self-adjoint operators in a Hilbert space [2, 5] and in original formulation of F. Riesz's theorem [2, 5]. The Riesz's representation theorem establishes that every such bounded linear functional comes from a Riemann-Stieltjes integral with respect to a suitable function of bounded variation.

In several practical problems, we need to calculate integrals. As is known to all, as for $I = \int_a^b f(x)dx$, once the primitive function F of integrand f is known, the definite integral of f over the interval $[a, b]$ is given by Newton-Leibniz formula, i.e.,

$$\int_a^b f(x)dx = F(b) - F(a). \tag{1.1}$$

The need often arises for evaluating the definite integral of a function that has no explicit antiderivative $F(x)$ or whose antiderivative $F(x)$ is not easy to obtain, such as $e^{\pm x^2}$, $\cos x^2$, $\frac{\sin x}{x}$, etc.

Moreover, the integrand $f(x)$ is only available at certain points x_i, $i = 1, 2, \ldots, n$.

The problem of numerical evaluating definite integrals arises both in mathematics and beyond, in many areas of science and engineering. One of the most fruitful advances in the field of experimental mathematics has been the development of practical methods for very high-precision numerical integration. Beginning in the 1980s, researchers began to explore ways to extend some of the many known techniques to the realm of high precision numerical integration formulas-tens or hundreds of digits beyond the realm of standard machine precision [6].

The trapezoidal rule is the most well known numerical integration rules of this type. Trapezoidal rule for classical Riemann integral is

$$\int_a^b f(x)dx = \frac{b-a}{2}\left(f(a) + f(b)\right) - \frac{(b-a)^3}{12} f''(\xi), \tag{1.2}$$

where $\xi \in (a, b)$.

In spite of the many accurate and efficient methods for numerical integration being available in [7-9], recently Mercer [10] has obtained trapezoid rule for Riemann-Stielsjes integral which engenders a generalization of Hadamard's integral inequality. Trapezoidal rule with error term for Riemann-Stieltjes integral is

$$\int_a^b f(t)dg = [G - g(a)]\, f(a) + [g(b) - G]\, f(b) - \frac{(b-a)^3}{12} f''(\xi)g'(\eta), \tag{1.3}$$

where $G = \frac{1}{b-a}\int_a^b g(t)\,dt$, $\xi \in (a, b)$.

Then, Mercer develops Midpoint and Trapezoid rules for Riemann-Stielsjes integral in [11] by using the concept of relative convexity. The composite trapezoid rule for the Riemann-Stieltjes integral and its Richardson extrapolation formula is presented by Zhao, Zhang and Ye [12]. It is applied to the composite trapezoid rule to obtain high accuracy approximations with little computational cost. Burg [13] has proposed derivative-based closed Newton-Cotes numerical quadrature which uses both the function value and the derivative value on uniformly spaced intervals. Zhao and Li have proposed midpoint derivative-based closed Newton-Cotes quadrature [14] and numerical superiority has been shown. Then, the derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao and Zhang [15], which uses derivative values at the endpoints. The midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao, Zhang and Ye [16], which only uses derivative values at the midpoint. Recently, the Simpson's rule for the Riemann-Stieltjes integral is presented by Zhao and Zhang [17], which uses values instead of derivative values at the midpoint.

The exponential function is one of the most important functions in calculus. As we all know, the derivative of e^t is the exponential function e^t itself. This is one of the properties that makes the exponential function really important. Motivation for the research presented here lies in construction of derivative-based trapezoid rule for a kind of Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$, which is a generalization of the results in [10-17].

The remainder is organized into four sections. These new scheme is investigated in Section 2. Section 3 presents the error term. The numerical experiments results are shown in Section 4. Section 5 is the conclusion part.

2. Derivative-based trapezoid rule for the $\int_a^b f(t)d(e^t)$

In this section, by using the conclusions in [15], the derivative-based trapezoid rule for a kind of special Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$ is presented.

Theorem 2.1. *Suppose that f' is continuous on $[a, b]$ and $g(t) = e^t$ is obviously continuously differentiable and increasing there. Let T denote the derivative-*

based trapezoid rule for a kind of Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$. *Then*

$$\begin{aligned}\int_a^b f(t)d(e^t) \approx T \triangleq & \left(\frac{6}{(b-a)^2}\left(e^b+e^a\right)-\frac{12}{(b-a)^3}\left(e^b-e^a\right)-e^a\right)f(a)\\ & +\left(e^b-\frac{6}{(b-a)^2}\left(e^b+e^a\right)+\frac{12}{(b-a)^3}\left(e^b-e^a\right)\right)f(b)\\ & +\left(e^a+\frac{2}{b-a}\left(e^b+2e^a\right)-\frac{6}{(b-a)^2}\left(e^b-e^a\right)\right)f'(a)\\ & +\left(\frac{2}{b-a}\left(2e^b+e^a\right)-\frac{6}{(b-a)^2}\left(e^b-e^a\right)-e^b\right)f'(b).\end{aligned} \tag{2.1}$$

Proof. First of all, it is not difficult to obtain

$$\begin{cases}\int_a^b e^t dt = e^b-e^a,\\ \int_a^b\int_a^t e^t dx\,dt = \left(e^b-e^a\right)-(b-a)\,e^a,\\ \int_a^b\int_a^t\int_a^y e^t dx\,dy\,dt = \left(e^b-e^a\right)-(b-a)\,e^a-\frac{1}{2}(b-a)^2e^a,\\ \int_a^b\int_a^t\int_a^z\int_a^y e^t dx\,dy\,dz\,dt\\ \qquad = \left(e^b-e^a\right)-(b-a)\,e^a-\frac{1}{2}(b-a)^2e^a-\frac{1}{6}(b-a)^3e^a.\end{cases} \tag{2.2}$$

Looking for the derivative-based trapezoid rule for $\int_a^b f(t)d(e^t)$, we seek numbers a_0, a_1, b_0, b_1 such that

$$\int_a^b f(t)d(e^t) \approx a_0 f(a) + a_1 f(b) + b_0 f'(a) + b_1 f'(b)$$

is equality for $f(t) = 1, t, t^2, t^3$. That is

$$\begin{cases}\int_a^b 1\,d(e^t) = a_0+a_1,\\ \int_a^b t\,d(e^t) = a_0a+a_1b+b_0+b_1,\\ \int_a^b t^2\,d(e^t) = a_0a^2+a_1b^2+2b_0a+2b_1b,\\ \int_a^b t^3\,d(e^t) = a_0a^3+a_1b^3+3b_0a^2+3b_1b^2.\end{cases}$$

Therefore, by using the conclusions in [15] and a system of equations (2.2),

$$\begin{cases}a_0+a_1 = e^b-e^a,\\ a_0a+a_1b+b_0+b_1 = be^b-ae^a-\left(e^b-e^a\right),\\ a_0a^2+a_1b^2+2b_0a+2b_1b\\ = b^2e^b-a^2e^a-2b\left(e^b-e^a\right)+2\left[\left(e^b-e^a\right)-(b-a)\,e^a\right],\\ a_0a^3+a_1b^3+3b_0a^2+3b_1b^2\\ = b^3e^b-a^3e^a-3b^2\left(e^b-e^a\right)+6b\left[\left(e^b-e^a\right)-(b-a)\,e^a\right]\\ \quad -6\left[\left(e^b-e^a\right)-(b-a)\,e^a-\frac{1}{2}(b-a)^2e^a\right].\end{cases} \tag{2.3}$$

Solving simultaneous equations (2.3) for a_0, a_1, b_0, b_1, we obtain

$$\begin{cases} a_0 = \frac{6}{(b-a)^2}\left(e^b+e^a\right) - \frac{12}{(b-a)^3}\left(e^b - e^a\right) - e^a, \\ a_1 = e^b - \frac{6}{(b-a)^2}\left(e^b+e^a\right) + \frac{12}{(b-a)^3}\left(e^b - e^a\right), \\ b_0 = e^a + \frac{2}{b-a}\left(e^b + 2e^a\right) - \frac{6}{(b-a)^2}\left(e^b - e^a\right), \\ b_1 = \frac{2}{b-a}\left(2e^b + e^a\right) - \frac{6}{(b-a)^2}\left(e^b - e^a\right) - e^b. \end{cases}$$

So, we have the derivative-based trapezoid rule for the special Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$ as desired. □

We shall now deduce some consequences of Theorem 2.1.

Corollary 2.1. *The degree of precision of the derivative-based trapezoid rule for the special Riemann-Stieltjes integral* $\int_a^b f(t)d(e^t)$ *is 3. That is to say, the quadrature rule* (4) *is exact when* f *is any polynomial of degree 3 or less, but is not exact for some polynomial of degree 4.*

Proof. By looking at the construction of a_0, a_1, b_0, b_1, we know that the derivative-based trapezoidal rule for the Riemann-Stieltjes integral has degree of precision not less than 3.

In Section 3, Theorem 3.1, we can clearly see that the quadratue is not equality for $f(t) = t^4$. So the degree of precision of this method is 3. □

Remark 2.1. An integral $\int_a^b f(x)e^{kx}dx\,(k > 0)$ over an arbitrary $[a, b]$ can be transformed into an integral over $[\frac{a}{k}, \frac{b}{k}]$ by changing the variable via $t = kx$.

This permits Theorem 2.1 to be applied to any $\int_a^b f(x)e^{kx}dx\,(k > 0)$, because

$$\int_a^b f(x)e^{kx}dx = \int_{\frac{a}{k}}^{\frac{b}{k}} \frac{1}{k}f(\frac{t}{k})e^t dt = \frac{1}{k}\int_{\frac{a}{k}}^{\frac{b}{k}} f(\frac{t}{k})d(e^t).$$

3. The error term for the $\int_a^b f(t)d(e^t)$

In the previous section, the derivative-based trapezoid rule for a kind of Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$ is given in formula (2.1).

As is known to all, the most critical "indicator" of numerical integration, which compares the level of accuracy, is error term. In this section, we are now ready to establish the error term of the derivative-based trapezoid rule for $\int_a^b f(t)d(e^t)$.

Here, the error term for this quadrature rule has been obtained by using Generalized Rolle' s Theorem with Derivatives, the Weighted Mean Value Theorem for Integrals based on the concept of precision.

The error term is the difference between the exact value $\frac{1}{(p+1)!}\int_a^b x^{p+1}dx$ and the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$, where p is the precision of the quadrature formula.

Theorem 3.1. *Suppose that $f^{(4)}$ is continuous on $[a,b]$ and $g(t)=e^t$ is obviously continuously differentiable and increasing there. The derivative-based trapezoid rule for the Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$ with the error term is*

$$\begin{aligned}\int_a^b f(t)d(e^t) &= \left(\frac{6}{(b-a)^2}\left(e^b+e^a\right)-\frac{12}{(b-a)^3}\left(e^b-e^a\right)-e^a\right)f(a)\\ &+\left(e^b-\frac{6}{(b-a)^2}\left(e^b+e^a\right)+\frac{12}{(b-a)^3}\left(e^b-e^a\right)\right)f(b)\\ &+\left(e^a+\frac{2}{b-a}\left(e^b+2e^a\right)-\frac{6}{(b-a)^2}\left(e^b-e^a\right)\right)f'(a)\\ &+\left(\frac{2}{b-a}\left(2e^b+e^a\right)-\frac{6}{(b-a)^2}\left(e^b-e^a\right)-e^b\right)f'(b)\\ &+\left[-\left(\frac{5(b-a)^3}{24}+\frac{(b-a)^2}{2}+\frac{11(b-a)}{12}+1\right)e^a\right.\\ &+\left.\left(\frac{(b-a)^2}{12}-\frac{b-a}{12}+1\right)e^b\right]f^{(4)}(\xi)\,e^\eta,\end{aligned} \tag{3.1}$$

where $\xi,\eta\in(a,b)$. And the error term $R[f]$ of this method is

$$\begin{aligned}&\left[\left(\frac{(b-a)^2}{12}-\frac{b-a}{12}+1\right)e^b\right.\\ &\left.-\left(\frac{5(b-a)^3}{24}+\frac{(b-a)^2}{2}+\frac{11(b-a)}{12}+1\right)e^a\right]f^{(4)}(\xi)\,e^\eta.\end{aligned} \tag{3.2}$$

Proof. Let $f(t)=\frac{t^4}{4!}$. So

$$\begin{aligned}\frac{1}{4!}\int_a^b t^4 d(e^t) &= \frac{1}{24}\left(b^4-4b^3+12b^2-24b-24\right)e^b\\ &-\frac{1}{24}\left(a^4-4a^3+12a^2-24a-24\right)e^a.\end{aligned} \tag{3.3}$$

By the Theorem 2.1, we have

$$\begin{aligned}T &= \left(\frac{6}{(b-a)^2}\left(e^b+e^a\right)-\frac{12}{(b-a)^3}\left(e^b-e^a\right)-e^a\right)\frac{a^4}{24}\\ &+\left(e^b-\frac{6}{(b-a)^2}\left(e^b+e^a\right)+\frac{12}{(b-a)^3}\left(e^b-e^a\right)\right)\frac{b^4}{24}\\ &+\left(e^a+\frac{2}{b-a}\left(e^b+2e^a\right)-\frac{6}{(b-a)^2}\left(e^b-e^a\right)\right)\frac{a^3}{6}\end{aligned}$$

$$+\left(\frac{2}{b-a}\left(2e^{b}+e^{a}\right)-\frac{6}{(b-a)^{2}}\left(e^{b}-e^{a}\right)-e^{b}\right)\frac{b^{3}}{6}. \tag{3.4}$$

With the help of (3.3)-(3.4), we obtain

$$\begin{aligned}
&\frac{1}{4!}\int_{a}^{b}t^{4}d(e^{t})-T\\
&=\left[\left(\frac{(b-a)^{2}}{12}-\frac{b-a}{12}+1\right)\left(e^{b}-e^{a}\right)-\left(\frac{5\,(b-a)^{3}}{24}+\frac{5\,(b-a)^{2}}{12}+(b-a)\right)e^{a}\right]\\
&=\left[\left(\frac{(b-a)^{2}}{12}-\frac{b-a}{12}+1\right)e^{b}-\left(\frac{5\,(b-a)^{3}}{24}+\frac{(b-a)^{2}}{2}+\frac{11\,(b-a)}{12}+1\right)e^{a}\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
R[f]=\Bigg[&\left(\frac{(b-a)^{2}}{12}-\frac{b-a}{12}+1\right)e^{b}\\
&-\left(\frac{5\,(b-a)^{3}}{24}+\frac{(b-a)^{2}}{2}+\frac{11\,(b-a)}{12}+1\right)e^{a}\Bigg]f^{(4)}(\xi)\,e^{\eta}. \qquad \square
\end{aligned}$$

Remark 3.1. The method used in Theorem 3.1 does not only apply to special cases, but that one may select the precision p to calculate the difference between the exact value $\frac{1}{(p+1)!}\int_{a}^{b}x^{p+1}dx$ and the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$ and the similar conclusion will still hold.

Remark 3.2. The error term for the derivative-based trapezoid rule could also be obtained using Taylor series expansions, by making certain unverifiable assumptions about the higher order terms.

4. Numerical results

So far, we have proposed derivative-based trapezoid rule for a kind of Riemann-Stieltjes integral in Section 2 and demonstrate the error term in Section 3.

In this section, compared with the traditional Newton-Cotes quadrature, some numerical experiments are carried out to verify whether the novel methods are of high precision.

In order to compare the precision of Newton-Cotes quadrature and the proposed approach, we calculate the following integrals $\int_0^1 x^4 e^x\,dx$. The comparison results are shown in the following tables.

Let us define Absolute Error=|Exact value-Approximate value|.

In the following tables, the item Int. stands for the number of composite interval.

Exact value of $\int_0^1 x^4 e^x\,dx = 9e-24 \approx 0.4645$.

Int.	Trapezoidal rule		Int.	Derivative-based trapezoid rule	
	Approximate value	Absolute Error		Approximate value	Absolute Error
1	1.3591	0.8946	1	0.4086	0.0559
2	0.7311	0.2666	2	0.4610	0.0035
4	0.5342	0.0697			
8	0.4822	0.0177			

Table 1: Numerical comparison of the new method with the classical method

Int.	Simpson's rule		Int.	Derivative-based trapezoid rule	
	Approximate value	Absolute Error		Approximate value	Absolute Error
1	0.5217	0.0572	1	0.4086	0.0559
2	0.4686	0.0041	2	0.4610	0.0035

Table 2: Numerical comparison of the new method with the classical method

It can be seen from Table 1, Derivative-based trapezoid rule with Int.=1, 2 has a much higher accuracy than classical Trapezoidal rule with Int.=4, 8 respectively.

It can be seen from Table 2, Derivative-based trapezoid rule has a much higher accuracy than classical Simpson's rule with the same number of subintervals.

The efficiency of the proposed approach has been demonstrated.

5. Conclusions

The main contributions of this paper are highlighted as follows.

1) By using the concept of algebraic precision, the derivative-based trapezoid rule for a kind of Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$ is presented.

2) This kind of quadrature rule has 3 orders of algebraic precision.

3) The error term for Riemann-Stieltjes Simpson's rule is investigated. Some numerical examples are given to show the efficiency of the proposed approach. In future work, we will seriously consider the Simpson's rule for the kind of Riemann-Stieltjes integral $\int_a^b f(t)d(e^t)$.

It is hoped that the results in this paper will stimulate further research in this direction.

6. Acknowledgments

This work has been funded and supported by the Scientific Research Project of Tianjin Municipal Education Commission (No. 2019KJ129).

References

[1] R.A. Gordon, *The integrals of Lebesgue*, Denjoy, Perron and Henstock, American Mathematical Society, Providence, 1994.

[2] P. Billingsley, *Probability and measures*, John Wiley and Sons, Inc., New York, 1995.

[3] L. Richardson, *Advanced calculus: an introduction to linear analysis,* John Wiley and Sons, Inc., New Jersey, 2008.

[4] P.E. Kopp, *Martingales and stochastic integrals,* Cambridge University Press, Cambridge, 1984.

[5] W. Rudin, *Functional analysis*, McGraw Hill Science, McGraw, 1991.

[6] D.H. Bailey, J.M. Borwein, *High-precision numerical integration: Progress and challenges,* Journal of Symbolic Computation, 46 (2011), 741-754.

[7] K. Atkinson, *An introduction to numerical analysis*, second ed., Wiley, 1989.

[8] R. L. Burden, J. D. Faires, *Numerical analysis,* Brooks/Cole, Boston, Mass, USA, 9th edition, 2011.

[9] E. Isaacson, H. B. Keller, *Analysis of numerical methods*, John Wiley and Sons, New York, 1966.

[10] P. R. Mercer, *Hadamard's inequality and trapezoid rules for the Riemann-Stieltjes integral*, Journal of Mathematical Analysis and Applications, 344 (2008), 921-926.

[11] P. R. Mercer, *Relative convexity and quadrature rules for the Riemann-Stieltjes integral,* Journal of Mathematical Inequality, 6 (2012), 65-68.

[12] W. Zhao, Z. Zhang, Z. Ye, *Composite trapezoid rule for the Riemann-Stieltjes integral and its Richardson extrapolation formula*, Italian Journal of Pure and Applied Mathematics, 35 (2015), 311-318.

[13] O. E. Burg, *Derivative-based closed Newton-Cotes numerical quadrature,* Applied Mathematics and Computation, 218 (2012), 7052-7065.

[14] W. Zhao, H. Li, *Midpoint derivative-based closed Newton-Cotes quadrature,* Abstract and Applied Analysis, Article ID 492507, 2013.

[15] W. Zhao, Z. Zhang, *Derivative-based trapezoid rule for the Riemann-Stieltjes integral,* Mathematical Problems in Engineering, Article ID 874651, 2014.

[16] W. Zhao, Z. Zhang, Z. Ye, Midpoint, *Derivative-based trapezoid rule for the Riemann-Stieltjes integral,* Italian Journal of Pure and Applied Mathematics, 33 (2014), 369-376.

[17] W. Zhao, Z. Zhang, *Simpson's rule for the Riemann-Stieltjes integral*, Journal of Interdisciplinary Mathematics, 24 (2021), 1305-1314.

Accepted: March 15, 2022

A Mehrotra-type algorithm with logarithmic updating technique for $P_*(\kappa)$ linear complementarity problems

Yiyuan Zhou*
College of Science
China Three Gorges University
Yichang
P. R. China
Three Gorges Mathematical Research Center
China Three Gorges University
Yichang
P. R. China
zyy@ctgu.edu.cn

Mingwang Zhang
College of Science
China Three Gorges University
Yichang
P. R. China
zmwang@ctgu.edu.cn

Fangyan Huang
Wanzhou NO.2 Senior High School
Wanzhou
P. R. China
12huafa@163.com

Abstract. A Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ linear complementarity problems is presented. In this algorithm, the corrector step takes a new direction, and the barrier parameter is the smaller positive root of a logarithmic equation. The iteration complexity of the new algorithm matches the currently best-known results. Numerical results show that the algorithm is efficient.

Keywords: interior-point algorithm, linear complementarity problems, Mehrotra-type algorithm, iteration complexity.

1. Introduction

Mehrotra's predictor-corrector algorithm [1, 2] and its variants have become the backbones of some optimization solvers [3-7]. The superior practical perforance of Mehrotra-type predictor-corrector algorithms motivated scholars to explore their theoretical properties. Jarre and Wechs [8] investigated an interior point method in which the search direction is based on corrector directions of Mehrotra's algorithm. To avoid small steps, Salahi et al. [9] introduced a safeguard

*. Corresponding author

strategy for a Mehrotra-type algorithm. After that, Salahi and Terlaky [10] proposed a new variant of Mehrotra-type algorithm without any safeguards and proved the iteration complexity bound coincides with the result in [9]. Recently, Salahi [11] introduced a new adaptive updating technique of the barrier parameter in Mehrotra-type algorithm for linear optimization (LO), which allowed them to prove the polynomial iteration complexity without employing any safeguards. Infeasible versions of Mehrotra-type algorithm [12, 13] and second order Mehrotra-type algorithms [14, 15] are also studied by scholars. Since efficiency in computation, Mehrotra-type predictor-corrector algorithm are extended to linear complementarity problems (LCPs) [12, 16], semidefinite programming [17-19], nonlinear complementarity problems [20] and many other problems.

LCPs are closely associated with linear programming and quadratic programming. The class of $P_*(\kappa)$ LCP is an important branch of LCPs. Interior point algorithms for $P_*(\kappa)$ LCPs have been widely studied in the last few decades [21]. Large update technique [22], full-Newton step [23, 24] and interior point method based on kernel function [25] are also presented for $P_*(\kappa)$ LCPs.

In this paper, a new Mehrotra-type algorithm for $P_*(\kappa)$ LCPs is presented, in which it takes a different corrector search direction and an adaptive updating technique of the barrier parameter. It extends the algorithm in [11] for LO to $P_*(\kappa)$ LCPs. In $P_*(\kappa)$ LCPs, the search directions Δx and Δs are not orthogonal any more, while they are orthogonal in LO, this leads a different technique to analyze the iteration complexity. Taking a specific default value as the predictor step size, we prove that the algorithm stops after at most $O(\sqrt{(1+4\kappa)(1+2\kappa)}n\log((x^0)^T s^0/\epsilon))$ iterations. If $\kappa = 0$, the iteration bound coincides with the result of LO in [11].

The rest of this paper is organized as follows. In Section 2, we recall some basic concepts and state a new Mehrotra-type algorithm for $P_*(\kappa)$ LCPs. Section 3 includes several important technical results, and subsequently the iteration bound of this algorithm is derived. Two illustrative numerical results of this algorithm are presented in Section 4. Finally, conclusion and final remarks are shown in Section 5.

For simplicity, we use the following notations throughout the paper:

$$e = (1, 1, \cdots, 1)^T.$$
$$I = \{1, 2, \cdots, n\}, I_+ = \{i \in I | \Delta x_i^a \Delta s_i^a \geq 0\}, I_- = \{i \in I | \Delta x_i^a \Delta s_i^a < 0\}.$$
$$\mathcal{F} = \big\{(x, s) \in \mathbb{R}^n \times \mathbb{R}^n | s = Mx + q, (x, s) \geq 0\big\}.$$
$$\mathcal{F}^0 = \big\{(x, s) \in \mathcal{F} | (x, s) > 0\big\}.$$
$$X = \mathrm{diag}(x), S = \mathrm{diag}(s).$$
$$xs = Xs = (x_1 s_1, x_2 s_2, \cdots, x_n s_n)^T.$$

2. The algorithm

A matrix $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix if there is a constant $\kappa \geq 0$ such that

$$(1+4\kappa)\sum_{i\in I_+(x)} x_i(Mx)_i + \sum_{i\in I_-(x)} x_i(Mx)_i \geq 0,\ \forall\ x \in \mathbb{R}^n,$$

or equivalently

$$x^T Mx \geq -4\kappa \sum_{i\in I_+(x)} x_i(Mx)_i,\ \forall x \in \mathbb{R}^n,$$

where $I_+(x) = \{i|x_i(Mx)_i \geq 0, i \in I\}$ and $I_-(x) = \{i|x_i(Mx)_i < 0, i \in I\}$. Note that, M is a positive semidefinite matrix if $\kappa = 0$. Thus, the class of $P_*(\kappa)$ matrices includes positive semi-definite matrices. The goal of a $P_*(\kappa)$ LCP is to find solutions $(x, s) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$Mx + q = s, xs = 0, (x, s) \geq 0, \tag{1}$$

where M is a $P_*(\kappa)$ matrix, $q \in \mathbb{R}^n$ and $n \geq 2$.

To find an approximate solution of (1), a parameterized system is established as follows:

$$Mx + q = s, xs = \mu e, (x, s) \geq 0, \tag{2}$$

where $\mu > 0$. We assume that system (1) satisfies the interior point condition (IPC), i.e., there exists a point (x^0, s^0) such that

$$s^0 = Mx^0 + q,\quad x^0 > 0,\quad s^0 > 0.$$

For a given $\mu > 0$, if the IPC holds, then system (2) has a unique solution $(x(\mu), s(\mu))$, which is called the μ-center of (1). The set of all μ-centers is called the central path of (1). As μ goes to 0, the limit of $(x(\mu), s(\mu))$ exists and approaches the solution of (1).

In the following, a feasible version of Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ LCPs will be presented, which works in a negative infinity neighborhood defined as

$$\mathcal{N}_\infty^-(\gamma) = \{(x, s) \in \mathcal{F}^0|\ x_i s_i \geq \gamma\mu_g, \forall i \in I\},$$

where $\gamma \in (0, 1)$ is a constant independent of n. The neighborhood $\mathcal{N}_\infty^-(\gamma)$ is also widely used in the implementation of other interior point algorithms.

The predictor direction $(\Delta x^a, \Delta s^a)$ is determined by the following system:

$$\begin{aligned} M\Delta x^a &= \Delta s^a,\\ s\Delta x^a + x\Delta s^a &= -xs, \end{aligned} \tag{3}$$

and the predictor step size α_a is defined by

$$\alpha_a = \max\{\alpha|(x + \alpha\Delta x^a, s + \alpha\Delta s^a) \in \mathcal{F}, 0 < \alpha \leq 1\}. \tag{4}$$

However, the our algorithm does not take a predictor step right away. By using information about the predictor step, the algorithm derives the corrector direction from the following system:

$$\begin{aligned} M\Delta x &= \Delta s, \\ s\Delta x + x\Delta s &= \mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a. \end{aligned} \tag{5}$$

The corrector direction in system (5) is different from that in [9] where it is determined by the equations $M\Delta x = \Delta s$ and $s\Delta x + x\Delta s = \mu e - xs - \Delta x^a \Delta s^a$. Motivation of the modification is based on the following observation. Since $0 < \alpha_a \leq 1$, it can be found that $\alpha_a^2 |\Delta x^a \Delta s^a| \leq |\Delta x^a \Delta s^a|$, thus $\mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a$ is much closer to $\mu e - xs$ than $\mu e - xs - \Delta x^a \Delta s^a$.

In each iteration of a primal-dual interior point algorithm, the barrier parameter μ needs to be updated. In this paper, we focus on the updating technique in [11]. A classical logarithmic barrier proximity function is used to measure the distance from the current iterate to the central path, and it is defined as

$$\Phi(x, s, \mu) := \frac{x^T s}{2\mu} - \frac{n}{2} + \frac{n}{2}\log\mu - \frac{1}{2}\sum_{i=1}^{n} \log(x_i s_i). \tag{6}$$

Obviously, for given (x, s), the function $\Phi(x, s, \mu)$ is minimum if $\mu = \mu_g = \frac{x^T s}{n}$. We denote $\mu_h = \sqrt[n]{x_1 s_1 \cdots x_n s_n}$. From the Arithmetic Mean−Geometric Mean inequality, it is clear that $\mu_h \leq \mu_g$. We consider the following equation with respect to μ,

$$\Phi(x, s, \mu) = \frac{(\sigma - 1)n}{2}, \tag{7}$$

where the constant $\sigma > 4\kappa + 4$. From (6) and (7), it can be found that equation (7) is equivalent to

$$\frac{\mu_g}{\mu} + \log\frac{\mu}{\mu_h} - \sigma = 0. \tag{8}$$

Follows from Corollary 2.5 of [11], equation (8) has two positive roots. The smaller one is defined as the target barrier parameter denoted by μ_t.

The barrier parameter μ_t is used to compute the corrector search direction $(\Delta x, \Delta s)$ by the following equations

$$\begin{aligned} M\Delta x &= \Delta s, \\ s\Delta x + x\Delta s &= \mu_t e - xs - \alpha_a^2 \Delta x^a \Delta s^a. \end{aligned} \tag{9}$$

The new iterate is denoted as $(x(\alpha_c), s(\alpha_c)) = (x + \alpha_c \Delta x, s + \alpha_c \Delta s)$ where the corrector step size α_c is defined by

$$\alpha_c = \max\{\alpha | (x(\alpha), s(\alpha) \in \mathcal{N}_\infty^-(\gamma), 0 < \alpha \leq 1\}. \tag{10}$$

Based on the previous analysis, a new Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ LCP is stated as Algorithm 1.

Algorithm 1
Input:
A parameter $\sigma > 4\kappa + 4$, a starting point $(x^0, s^0) \in \mathcal{N}_\infty^-(\gamma)$ with $\gamma = \frac{1}{\sigma}$, an accuracy parameter $\epsilon > 0$.
begin
Set $x := x^0; s = s^0$;
while $x^T s \geq \epsilon$ **do**
begin **Predictor step**
Solve (3) and calculate the predictor step size α_a from (4);
end
begin **Corrector step**
Solve (8) to derive the smaller positive root μ_t;
Solve (9) and calculate the corrector step size α_c from (10);
Set $(x, s) := (x(\alpha_c), s(\alpha_c))$.
end
end

3. Complexity analysis

In this section, we establish the polynomial complexity for Algorithm 1. In the following, we give the bounds of μ_t, $\|\Delta x \Delta s\|$, $\Delta x^T \Delta s$ and step sizes of Algorithm 1. The bounds are important in the complexity analysis.

Lemma 3.1 ([11])**.** *For all iterates* (x, s) *of Algorithm* 1*, we have* $\sigma \leq \frac{\mu_g}{\mu_t} \leq 2\sigma$.

Lemma 3.2. *Let* $(\Delta x^a, \Delta s^a)$ *be the solution of* (3)*. Then:*

$$\text{(i)} \quad \Delta x_i^a \Delta s_i^a \leq \frac{x_i s_i}{4},\ i \in I_+; \quad -\Delta x_i^a \Delta s_i^a \leq \frac{1}{\alpha_a}(\frac{1}{\alpha_a} - 1) x_i s_i,\ i \in I_-;$$

$$\text{(ii)} \quad \sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{x^T s}{4}; \quad \sum_{i \in I_-} |\Delta x_i^a \Delta s_i^a| \leq \frac{4\kappa + 1}{4} x^T s;$$

$$\text{(iii)} \quad -\kappa x^T s \leq (\Delta x^a)^T \Delta s^a \leq \frac{x^T s}{4}.$$

Proof. (i) The proof is similar to that of Lemma A.1 and Proposition 4.1 in [9], and it is omitted here.

(ii) The first conclusion is a direct consequence of (i). We will prove the second conclusion in the following. Since M is a $P_*(\kappa)$ matrix, following from the first conclusion, we have

$$0 > \sum_{i \in I_-} \Delta x_i^a \Delta s_i^a \geq -(1 + 4\kappa) \sum_{i \in I_+} \Delta x_i^a \Delta s_i^a \geq -\frac{1 + 4\kappa}{4} x^T s,$$

that is $\sum_{i \in I_-} |\Delta x_i^a \Delta s_i^a| \leq \frac{1+4\kappa}{4} x^T s$.

(iii) From statement (ii), it follows that $(\Delta x^a)^T \Delta s^a \le \sum_{i\in I_+} \Delta x_i^a \Delta s_i^a \le \frac{x^T s}{4}$. Since $\Delta s^a = M\Delta x^a$ and M is a $P_*(\kappa)$ matrix, we get

$$(\Delta x^a)^T \Delta s^a \ge -4\kappa \sum_{i\in I_+} \Delta x_i^a \Delta s_i^a \ge -\kappa x^T s.$$

This completes the proof. □

Theorem 3.1 ([16])). *If the current iterate $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ and α_a is the predictor step size, then*

$$\alpha_a \ge \sqrt{\frac{\gamma}{(4\kappa+1)n}}.$$

In what follows, we consider the lower bound as a default value for predictor step size, that is

$$\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}}. \tag{11}$$

Lemma 3.3. *Let $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ and $(\Delta x, \Delta s)$ be the solution of (5) with $\mu > 0$, then*

$$||\Delta x \Delta s|| \le \sqrt{(\frac{1}{4}+\kappa)(\frac{1}{2}+\kappa)}||w||^2, \sum_{i\in I_+} \Delta x_i \Delta s_i \le \frac{1}{4}||w||^2,$$

where $w = (xs)^{-\frac{1}{2}}(\mu e - xs - \alpha_a^2 \Delta x^a \Delta s^a)$.

Proof. The proof is similar to that of Lemma 8 in [26], and we omit it here. □

Lemma 3.4. *Let $(x, s) \in \mathcal{N}_\infty^-(\gamma)$ and $(\Delta x, \Delta s)$ be the solution of (5) with $\mu > 0$, then*

$$||w||^2 \le \frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1) + 16}{16} n\mu_g.$$

Proof. From Lemma 3.3, one has

$$\begin{aligned} ||w||^2 =& \mu^2 \sum_{i\in I} \frac{1}{x_i s_i} + \sum_{i\in I} x_i s_i - 2n\mu + \alpha_a^4 \sum_{i\in I} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} \\ & - 2\alpha_a^2 \mu \sum_{i\in I} \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} + 2\alpha_a^2 \sum_{i\in I} \Delta x_i^a \Delta s_i^a. \end{aligned}$$

Due to $(x,s) \in \mathcal{N}_\infty^-(\gamma)$, we have $\mu^2 \sum_{i\in I} \frac{1}{x_i s_i} \leq \frac{n\mu^2}{\gamma\mu_g}$. Using (i) and (ii) in Lemma 3.2, we obtain

$$\begin{aligned}
\alpha_a^4 \sum_{i\in I} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} &= \alpha_a^4 \sum_{i\in I_+} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} + \alpha_a^4 \sum_{i\in I_-} \frac{(\Delta x_i^a \Delta s_i^a)^2}{x_i s_i} \\
&\leq \alpha_a^4 \sum_{i\in I_+} \frac{(\frac{x_i s_i}{4})^2}{x_i s_i} + \alpha_a^4 \sum_{i\in I_-} \frac{-\Delta x_i^a \Delta s_i^a}{x_i s_i} (-\Delta x_i^a \Delta s_i^a) \\
&\leq \alpha_a^4 \sum_{i\in I_+} \frac{x_i s_i}{16} + \frac{\alpha_a^4}{\alpha_a} (\frac{1}{\alpha_a} - 1) \sum_{i\in I_-} |\Delta x_i^a \Delta s_i^a| \\
&\leq \frac{\alpha_a^4}{16} x^T s + \alpha_a^2 (1-\alpha_a) \frac{4\kappa+1}{4} x^T s \\
&= \frac{\alpha_a^4 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1)}{16} n\mu_g,
\end{aligned}$$

and

$$-2\alpha_a^2 \mu \sum_{i\in I} \frac{\Delta x_i^a \Delta s_i^a}{x_i s_i} \leq 2\alpha_a^2 \mu \sum_{i\in I_-} \frac{|\Delta x_i^a \Delta s_i^a|}{x_i s_i} \leq \frac{2\alpha_a^2 \mu (4\kappa+1)}{4\gamma\mu_g} x^T s \leq \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma},$$

where the second inequality follows from $(x,s) \in \mathcal{N}_\infty^-(\gamma)$. Moreover,

$$2\alpha_a^2 \sum_{i\in I} \Delta x_i^a \Delta s_i^a \leq 2\alpha_a^2 \sum_{i\in I_+} \Delta x_i^a \Delta s_i^a \leq \frac{\alpha_a^2}{2} n\mu_g.$$

Combining the above results yields that

$$\begin{aligned}
&||w||^2 \\
\leq &\frac{n\mu^2}{\gamma\mu_g} + n\mu_g - 2n\mu + \frac{\alpha_a^4 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1)}{16} n\mu_g + \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma} + \frac{\alpha_a^2}{2} n\mu_g \\
= &\frac{n\mu^2}{\gamma\mu_g} - 2n\mu + \frac{(4\kappa+1)\alpha_a^2 n\mu}{2\gamma} + \frac{\alpha_a^4 + 8\alpha_a^2 + 4\alpha_a^2(1-\alpha_a)(4\kappa+1) + 16}{16} n\mu_g.
\end{aligned}$$

This completes the proof. □

Lemma 3.5. *Let $(x,s) \in \mathcal{N}_\infty^-(\gamma)$ and $(\Delta x, \Delta s)$ be the solution of (5) with $\mu = \mu_t$, then*

$$||\Delta x \Delta s|| \leq p_1 n\mu_g, \Delta x^T \Delta s \leq p_2 n\mu_g,$$

where $p_1 = \frac{37}{128}\sqrt{(1+4\kappa)(2+4\kappa)}$, $p_2 = \frac{37}{128}$.

Proof. Lemma 3.1 implies that $\frac{\gamma}{2} = \frac{1}{2\sigma} \leq \frac{\mu_t}{\mu_g} \leq \frac{1}{\sigma} = \gamma$. Following from Lemma 3.4, one has

$$\begin{aligned}
||w||^2 \leq& \frac{n\mu_t^2}{\gamma\mu_g} - 2n\mu_t + \frac{(4\kappa+1)\alpha_a^2 n\mu_t}{2\gamma} + \frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(1-\alpha_a)(4\kappa+1)+16}{16} n\mu_g \\
=& \Big[\frac{1}{\gamma}(\frac{\mu_t}{\mu_g})^2 - 2\frac{\mu_t}{\mu_g} + \frac{(4\kappa+1)\alpha_a^2}{2\gamma}\frac{\mu_t}{\mu_g} + \frac{\alpha_a^4+8\alpha_a^2+4\alpha_a^2(1-\alpha_a)(4\kappa+1)+16}{16}\Big] n\mu_g \\
\leq& \Big(\frac{1}{\gamma}\gamma^2 - 2\frac{\gamma}{2} + \frac{\gamma}{4\gamma}\gamma + \frac{6\gamma+16}{16}\Big) n\mu_g \\
\leq& \frac{37}{32} n\mu_g,
\end{aligned}$$

where the second inequality is due to $n \geq 2$, $\kappa \geq 0$ and $\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}} \leq 1$ by Theorem 3.1. The third inequality comes from $\gamma = \frac{1}{\sigma} < \frac{1}{4\kappa+4} \leq \frac{1}{4}$.

From Lemma 3.3, it follows that

$$||\Delta x \Delta s|| \leq \frac{37}{32}\sqrt{(\frac{1}{4}+\kappa)(\frac{1}{2}+\kappa)} n\mu_g = \frac{37}{128}\sqrt{(1+4\kappa)(2+4\kappa)} n\mu_g = p_1 n\mu_g,$$

and $\Delta x^T \Delta s \leq \frac{37}{128} n\mu_g$, which completes the proof. □

In order to simplify the analysis, we define

$$t = \max_{i \in I_+}\left\{\frac{\Delta x_i^a \Delta s_i^a}{x_i s_i}\right\}, \tag{12}$$

that is, $\Delta x_i^a \Delta s_i^a \leq t x_i s_i$ if $i \in I_+$. Since M is a $P_*(\kappa)$ matrix, one has $I_+ \neq \emptyset$ and $t \leq \frac{1}{4}$ from Lemma 3.2.

Theorem 3.2. *Let $(x, s) \in \mathcal{N}_\infty^-(\gamma)$, where $\gamma = \frac{1}{\sigma}$ and $\sigma > 4 + 4\kappa$. If $(\Delta x, \Delta s)$ is the solution of (5) with $\mu = \mu_t$ and α_c is the corrector step size, then*

$$\alpha_c \geq \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}. \tag{13}$$

Proof. The goal is to determine a maximum step size $\alpha \in (0, 1]$ in the corrector step such that

$$x_i(\alpha)s_i(\alpha) \geq \gamma\mu_g(\alpha), \quad \forall i \in I, \tag{14}$$

where $\mu_g(\alpha) = \frac{x(\alpha)^T s(\alpha)}{n}$ and

$$\begin{aligned}
x_i(\alpha)s_i(\alpha) &= x_i s_i + \alpha(x_i\Delta s_i + s_i\Delta x_i) + \alpha^2\Delta x_i\Delta s_i \\
&= x_i s_i + \alpha(\mu_t - x_i s_i - \alpha_a^2\Delta x_i^a\Delta s_i^a) + \alpha^2\Delta x_i\Delta s_i \\
&= (1-\alpha)x_i s_i + \alpha\mu_t - \alpha\alpha_a^2\Delta x_i^a\Delta s_i^a + \alpha^2\Delta x_i\Delta s_i.
\end{aligned}$$

Since we consider the lower bound of $\Delta x_i^a \Delta s_i^a$, we should give more focus on the case of $\Delta x_i^a \Delta s_i^a > 0$ than $\Delta x_i^a \Delta s_i^a \le 0$. Thus, we have to prove $x_i(\alpha)s_i(\alpha) \ge \gamma\mu_g(\alpha)$ for all $i \in I_+$. From Lemma 3.5 and equation (12), it follows that, for any $i \in I_+$,

$$\begin{aligned} x_i(\alpha)s_i(\alpha) &= (1-\alpha)x_is_i + \alpha\mu_t - \alpha\alpha_a^2\Delta x_i^a\Delta s_i^a + \alpha^2\Delta x_i\Delta s_i \\ &\ge [1-(1+\alpha_a^2 t)\alpha]x_is_i + \alpha\mu_t - \alpha^2 p_1 n\mu_g \\ &\ge [1-(1+\frac{\alpha_a^2}{4})\alpha]x_is_i + \frac{\alpha}{2}\gamma\mu_g - \alpha^2 p_1 n\mu_g, \end{aligned}$$

where the last inequality follows from $t \le \frac{1}{4}$ and $\frac{\mu_g}{\mu_t} \le 2\sigma$.

Since $(x,s) \in \mathcal{N}_\infty^-(\gamma)$, it is clear that $[1-(1+\frac{\alpha_a^2}{4})\alpha]x_is_i \ge [1-(1+\frac{\alpha_a^2}{4})\alpha]\gamma\mu_g$ if $\alpha \le \frac{4}{4+\alpha_a^2}$. Thus,

$$x_i(\alpha)s_i(\alpha) \ge [1-(1+\frac{\alpha_a^2}{4})\alpha]\gamma\mu_g + \frac{\alpha}{2}\gamma\mu_g - \alpha^2 p_1 n\mu_g \tag{15}$$

if $\alpha \le \frac{4}{4+\alpha_a^2}$.

On the other hand, we have

$$\mu_g(\alpha) = \frac{x(\alpha)^T s(\alpha)}{n} = \frac{x^T s + \alpha[n\mu_t - x^T s - \alpha_a^2(\Delta x^a)^T\Delta s^a] + \alpha^2\Delta x^T\Delta s}{n}.$$

From Lemma 3.1, 3.2 and 3.5, we get

$$\begin{aligned} \mu_g(\alpha) &\le \frac{x^T s + \alpha(\frac{n\mu_g}{\sigma} - x^T s + \alpha_a^2\kappa x^T s) + \alpha^2 n p_2\mu_g}{n} \\ &= (1-\alpha)\mu_g + \alpha\gamma\mu_g + \alpha\alpha_a^2\kappa\mu_g + \alpha^2 p_2\mu_g. \end{aligned} \tag{16}$$

Combining (15) and (16) yields that the new iterate is certainly in the neighborhood $\mathcal{N}_\infty^-(\gamma)$ if

$$\big[1-(1+\frac{\alpha_a^2}{4})\alpha\big]\gamma\mu_g + \frac{\alpha}{2}\gamma\mu_g - \alpha^2 p_1 n\mu_g \ge (1-\alpha)\gamma\mu_g + \alpha\gamma^2\mu_g + \alpha\alpha_a^2\kappa\gamma\mu_g + \alpha^2\gamma p_2\mu_g.$$

This is equivalent to $(\frac{1}{2} - \gamma - \frac{\alpha_a^2}{4} - \alpha_a^2\kappa)\gamma \ge (\gamma p_2 + np_1)\alpha$, that is,

$$\alpha \le \frac{(\frac{1}{2} - \gamma - \frac{\alpha_a^2}{4} - \alpha_a^2\kappa)\gamma}{\gamma p_2 + np_1}.$$

Furthermore,

$$\begin{aligned} \frac{\frac{1}{2} - \gamma - \frac{\alpha_a^2}{4} - \alpha_a^2\kappa}{\gamma p_2 + np_1} &= \frac{\frac{1}{2} - \gamma - \frac{\gamma}{4n}}{\frac{37}{128}\gamma + \frac{37}{128}n\sqrt{(1+4\kappa)(2+4\kappa)}} \\ &\ge \frac{\frac{7}{32}}{\frac{37}{64}n\sqrt{(1+4\kappa)(2+4\kappa)}} \\ &= \frac{14}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}, \end{aligned}$$

where the inequality follows from $\gamma < \frac{1}{4\kappa+4} \leq \frac{1}{4} < n\sqrt{(1+4\kappa)(2+4\kappa)}$ and $n \geq 2$. Therefore inequality (14) holds if $\alpha \leq \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}$. Thus, the maximal step size satisfies

$$\alpha \geq \min\Big\{\frac{4}{4+\alpha_a^2}, \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}\Big\}.$$

Since $\alpha_a \leq 1$, $\gamma < \frac{1}{4}$, $n \geq 2$ and $\kappa \geq 0$, we have $\frac{4}{4+\alpha_a^2} \geq \frac{4}{5} > \frac{14\gamma}{37n} > \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}$. Consequently, the corrector step size α_c satisfies

$$\alpha_c \geq \frac{14\gamma}{37n\sqrt{(1+4\kappa)(2+4\kappa)}}.$$

This completes the proof. □

The following theorem gives the upper bound of iteration number in which Algorithm 1 stops with an ϵ-approximate solution.

Theorem 3.3. *After at most*

$$O\Big(\sqrt{(1+4\kappa)(2+4\kappa)}n\log\frac{(x^0)^T s^0}{\epsilon}\Big)$$

iterations, Algorithm 1 stops with a solution for which $x^T s \leq \epsilon$.

Proof. After each iteration, the dual gap is $\mu_g(\alpha_c)$. From (16), it follows that

$$\begin{aligned}\mu_g(\alpha_c) &\leq \big[1-(1-\gamma-\alpha_a^2\kappa)\alpha_c + p_2\alpha_c^2\big]\mu_g\\ &\leq \Big[1-\Big(1-\frac{1}{4}-\frac{1}{32}\Big)\alpha_c + \frac{37}{128}\alpha_c\Big]\mu_g\\ &= \Big(1-\frac{55}{128}\alpha_c\Big)\mu_g\\ &\leq \Big[1-\frac{385\gamma}{2368n\sqrt{(1+4\kappa)(2+4\kappa)}}\Big]\mu_g,\end{aligned}$$

where the second inequality is due to $\alpha_a = \sqrt{\frac{\gamma}{(4\kappa+1)n}}$ and $\gamma < \frac{1}{4}$. This completes the proof by Theorem 3.2 of [27]. □

4. Numerical results

It is difficult to know the value of parameter κ of a $P_*(\kappa)$ matrix [26], however, it is well known that a positive semi-definite matrix is a $P_*(0)$ matrix. In the following, Algorithm 1 is applied to $P_*(0)$ LCPs.

Example 4.1. Let $M = (m_{ij})_{n\times n}$, $q = (q_i)_{n\times 1}$, where $q_i = n+1-i$ and

$$m_{ij} = \begin{cases} 2, & \text{if } i=j;\\ -1, & \text{if } |i-j|=1;\\ 0, & \text{else.}\end{cases}$$

Table 1: Iteration numbers of Example 4.1

	n=5	n=10	n=50	n=100	n=200	n=400	n=800	n=1000
$\sigma = 4.5$	11	11	14	15	16	17	18	18
$\sigma = 5$	10	11	13	14	15	16	17	17
$\sigma = 5.5$	10	11	13	14	15	15	16	17
$\sigma = 6$	10	10	12	13	14	15	16	16
$\sigma = 6.5$	9	10	12	13	14	15	16	16
$\sigma = 7$	9	10	12	13	14	14	15	16
$\sigma = 7.5$	9	10	12	13	13	14	15	16
$\sigma = 8$	9	10	12	12	13	14	15	15

Table 2: Average iteration numbers of Example 4.2

	n=5	n=10	n=50	n=100	n=200	n=400	n=800	n=1000
$\sigma = 4.5$	10.98	12.00	15.00	16.91	18.00	19.00	21.00	21.01
$\sigma = 5$	10.40	11.85	15.00	16.00	17.09	19.00	20.00	21.00
$\sigma = 5.5$	10.04	11.05	14.05	16.00	17.00	18.00	20.00	20.02
$\sigma = 6$	9.98	11.00	14.00	15.10	17.00	18.00	19.01	20.00
$\sigma = 6.5$	9.79	11.00	14.00	15.00	16.82	18.00	19.00	20.00
$\sigma = 7$	9.39	10.80	14.00	15.00	16.00	18.00	19.00	19.00
$\sigma = 7.5$	9.15	10.30	13.35	15.00	16.00	17.08	19.00	19.00
$\sigma = 8$	9.07	10.15	13.01	15.00	16.00	17.00	19.00	19.00

Example 4.2. Let $M = RR^T$, where $R = (r_{ij})_{n\times n}$ is randomly generated and $r_{ij} \in [0, 1]$. The vector $q = (q_i)_{n\times 1}$ is also randomly generated, where $q_i \in [0, 5]$.

In both examples, the accuracy parameter is set as $\epsilon = 10^{-6}$. Table 1 shows the iteration numbers to obtain an ϵ-solution for Example 4.1. In Example 4.2, for each n and every σ, one hundred random $P_*(0)$ LCPs are considered. Iteration numbers in Table 2 are the average iteration numbers of the one hundred LCPs. From Table 1 and Table 2, we can find that, for a given n, the iteration number decreases if σ increases. This is because that if σ is larger, then the neighborhood $N_\infty^-(\gamma)$ is bigger, and Algorithm 1 has a larger corrector step size and fewer steps. The numerical results show that Algorithm 1 is efficient.

5. Concluding remarks

In this paper, we present a modified Mehrotra-type predictor-corrector algorithm for $P_*(\kappa)$ LCPs and discuss the polynomial complexity of this algorithm. It should be pointed out that the corrector direction in our algorithm is different from other algorithms. The iteration bound is $O(\sqrt{(1+4\kappa)(2+4\kappa)}n \log \frac{(x^0)^T s^0}{\epsilon})$. If $\kappa = 0$, this bound coincides with the iteration bound for LO.

Acknowledgements

The authors are thankful to the referees for their valuable comments and suggestions. This work was supported by the Science Research Project of Education Department of Hubei Province (B2022034) and the Open Fund of Hubei Key Laboratory of Disaster Prevention and Mitigation, China Three Gorges University (2022KJZ19).

References

[1] S. Mehrotra, *On finding a vertex solution using interior-point methods*, Linear Alg. Appl., 152 (1991), 233-253.

[2] S. Mehrotra, *On the implementation of a primal-dual interior point method*, SIAM J. Optim., 2 (1992), 575-601.

[3] E. D. Andersen, K. D. Andersen, *The MOSEK interior point optimizer for linear programming: an implementation of the homogeneous algorithm.* In: H. Frenk, C. Roos, T. Terlaky, S. Zhang, eds. *High Performance Optimization, Applied Optimization*, Boston, MA: Springer, 2000.

[4] R. J. Vanderbei, *LOQO: an interior-point code for quadratic programming*, Optim. Method Softw., 11 (1999), 451-488.

[5] Y. Zhang, *Solving large-scale linear programs by interior point methods under the Matlab environment*, Optim. Method Softw., 10 (1998), 1-31.

[6] K. Fujisawa, K. Nakata, M. Yamashita, M. Fukuda, *SDPA project: solving large-scale semidefinite programs*, J. Oper. Res. Soc. Jpn., 50 (2007), 278-298.

[7] A. G. Pandala, Y. R. Ding, H. W. Park, *qpSWIFT: A real-time sparse quadratic program solver for robotic applications*, IEEE Robot. Autom. Lett., 4 (2019): 355-3362.

[8] F. Jarre, M. Wechs, *Extending Mehrotra's corrector for linear programs.* Adv. Model. Optim., 1 (1999), 38-60.

[9] M. Salahi, J. Peng, T. Terlaky, *On Mehrotra-type predictor-corrector algorithms*, SIAM J. Optim., 18 (2007), 1377-1397.

[10] M. Salahi, T. Terlaky, *Postponing the choice of the barrier parameter in Mehrotra-type predictor-corrector algorithms*, Eur. J. Oper. Res., 182(2007), 502-513.

[11] M. Salahi, *New barrier parameter updating technique in Mehrotra-type algorithm*, Bull. Iran Math. Soc., 36(2010), 99-108.

[12] X. Z. Liu, H. W. Liu, C. H. Liu, *Infeasible Mehrotra-type predictor-corrector interior-point algorithm for the cartesian $P_*(\kappa)$-LCP over symmetric cones*, Numer. Funct. Anal. Optim., 35 (2014), 588-610.

[13] X. M. Yang, H. W. Liu, X. L. Dong, *Polynomial convergence of Mehrotra-type prediction-corrector infeasible-IPM for symmetric optimization based on the commutative class directions*, Appl. Math. Comput., 230 (2014), 616-628.

[14] Y. Zhang, D. Zhang, *On polynomiality of the Mehrotra-type predictor-corrector interior point algorithms*, Math. Program., 68 (1995), 303-318.

[15] M. Salahi, N. Mahdavi-Amiri, *Polynomial time second order Mehrotra-type predictor-corrector algorithms*, Appl. Math. Comput., 183 (2006), 646-658.

[16] M. W. Zhang, Y. L. Lv, *New Mehrotra's second order predictor-corrector algorithm for $P_*(\kappa)$ linear complementarity problems*, J. Syst. Eng. Electron., 21 (2010), 705-712.

[17] K. C. Toh, *An inexact primal-dual path following algorithm for convex quadratic SDP*, Math. Program., 112 (2008), 221-254.

[18] M. H. Koulaei, T. Terlaky, *On the complexity analysis of a Mehrotra-type primal-dual feasible algorithm for semidefinite optimization*, Optim. Method Softw., 25 (2010), 467-485.

[19] X. M. Yang, Y. Q. Bai, *An adaptive infeasible-interior-point method with the one-norm wide neighborhood for semi-definite programming*, J. Sci. Comput., 78 (2019), 1790-1810.

[20] H. L. Zhao, H. W. Liu, *A new infeasible mehrotra-type predictor-corrector algorithm for nonlinear complementarity problems over symmetric cones*, J. Optim. Theory Appl., 176 (2018), 410-427.

[21] M. Kojima, N. Megiddo, T. Noma, A. Yoshise, *A unified approach to interior point algorithms for linear complementarity problems,* Lecture Notes in Computer Science, Berlin, Heidelberg: Springer, 1991.

[22] G. M. Cho, *A new large-update interior point algorithm for $P_*(\kappa)$ linear complementarity problems*, J. Comput. Appl. Math., 216 (2008), 265-278.

[23] S. Asadi, H. Mansouri, *Polynomial interior-point algorithm for $P_*(\kappa)$ horizontal linear complementarity problems*, Numer. Algorithms, 63 (2013), 385-398.

[24] G. Q. Wang, X. J. Fan, D. T. Zhu, D. Z. Wang, *New complexity analysis of a full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-LCP*, Optim. Lett., 9 (2015), 1105-1119.

[25] M. M. Li, M. W. Zhang, Z. W. Huang, *A new interior-point method for $P_*(\kappa)$ linear complementarity problems based on a parameterized kernel function*, Ital. J. Pure Appl. Math., 45 (2021), 323-348.

[26] T. Illés, M. Nagy, *A Mizuno-Todd-Ye type predictor-corrector algorithm for sufficient linear complementarity problems*, Eur. J. Oper. Res., 181 (2007), 1097-1111.

[27] S. J. Wright. *Primal-dual interior-point methods,* Philadelphia: SIAM, 1997.

Accepted: March 06, 2022

ITALIAN JOURNAL OF PURE AND APPLIED MATHEMATICS
INFORMATION FOR AUTHORS

- Before an article, that received a positive evaluation report of the Editorial Board, can be published in the Italian Journal of Pure and Applied Mathematics, the author is required to pay the publication fee, which has to be calculated with the following formula:

  fee = EUR (20 + 4n)

  where **n** is the number of pages of the article written in the journal's format

- The above amount needs to be payed through an international credit transfer in the following bank account:

Bank name:	**CREDIFRIULI – CREDITO COOPERATIVO FRIULI**
Bank branch:	**SUCCURSALE UDINE, VIA ANTON LAZZARO MORO 8**
IBAN code:	**IT 79 E 07085 12304 0000 000 33938**
SWIFT (BIC) code:	**ICRAITRRU50**
Account owner:	**FORMAZIONE AVANZATA RICERCA EDITORIA (FARE) SRL** **VIA LARGA 38** **33100 UDINE (ITALY)**

- All bank commissions **must be payed** by the author, adding them to the previous calculated net amount

- Include the following **mandatory** causal in the credit transfer transaction:

  CONTRIBUTO PUBBLICAZIONE ARTICOLO SULL'ITALIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

- Please, include also **First Name**, **Last Name** and **Paper Title** in the credit transfer Transaction

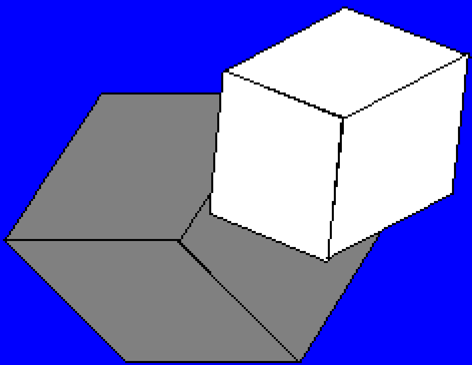
- After the transaction ends successfully, the author is requested to send an e-mail to the following addresses:

  italianjournal.math@gmail.com
  azamparo@forumeditrice.it

- This e-mail should contain the author's personal information (Last name, First Name, Postemail Address, City and State, PDF copy of the bank transfer), in order to allow Forum to create a confirmation of the payment for the author

- Payments, orders or generic fees will not be accepted if they refer to Research Institutes, Universities or any other public and private organizations

IJPAM
Italian Journal of Pure and Applied Mathematics
Issue n° 50-2023



FORUM EDITRICE UNIVERSITARIA UDINESE
FARE srl
Via Larga 38 - 33100 Udine
Tel: +39-0432-26001, Fax: +39-0432-296756
forum@forumeditrice.it

Rivista semestrale: Autorizzazione Tribunale di Udine n. 8/98 del 19.3.98
Direttore responsabile: Piergiulio Corsini
ISSN 2239-0227