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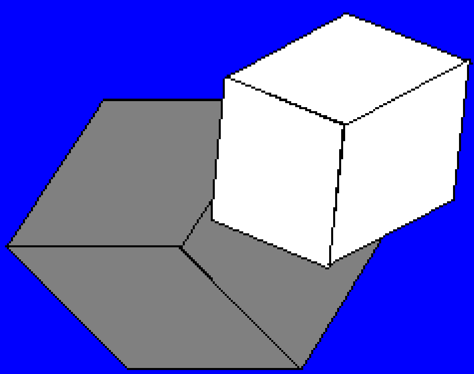

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Novel properties of neighbourly edge irregular interval-valued neutrosophic graphs

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Abstract. In this paper, some types of edge irregular interval-valued neutrosophic graphs such as neighbourly edge irregular interval-valued neutrosophic graphs and neighbourly edge totally irregular interval-valued neutrosophic graphs are introduced. A comparative study between neighbourly edge irregular interval-valued neutrosophic graphs and neighbourly edge totally irregular interval-valued neutrosophic graphs is done. Likewise some properties of them are studied.

Keywords: edge degree in IVNG, edge totall degree in IVNG, edge irregular IVNG, neighbourly edge irregular IVNG, neighbourly edge totally irregular IVNG.

1. Introduction

In 1736, Euler first introduced the concept of graph theory. In the history of mathematics, the solution given by Euler of the well known Konigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as logic, geometry, algebra, topology, analysis, number theory, information theory, artificial intelligence, operations research, optimization, neural networks, planning, computer science and etc [9, 10, 11, 13].

Fuzzy set theory, introduced by Zadeh in 1965, is a mathematical tool for handling uncertainties like vagueness, ambiguity and imprecision in linguistic variables [31]. Research on theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its application. Fuzzy set

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theory has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest.

Atanassov [3, 4] proposed the extended form of fuzzy set theory by adding a new component, called, intuitionistic fuzzy sets. Smarandache [23, 24] introduced the concept of neutrosophic sets by combining the non-standard analysis. In neutrosophic set, the membership value is associated with three components: truth-membership (T), indeterminacy-membership (I) and falsity-membership (F), in which each membership value is a real standard or non-standard subset of the non-standard unit interval $]0^-, 1^+[$ and there is no restriction on their sum. Smarandache [25] and Wang et al. [29] presented the notion of single valued neutrosophic sets to apply neutrosophic sets in real life problems more conveniently. In single valued neutrosophic sets, three components are independent and their values are taken from the standard unit interval $[0, 1]$. Wang et al. [30] presented the concept of interval-valued neutrosophic sets, which is more precise and more flexible than the single valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single valued neutrosophic set, in which three membership (T, I, F) functions are independent, and their values belong to the unit interval $[0, 1]$.

In 1975, Rosenfeld [19] introduced the concept of fuzzy graphs. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [12].

Later, Broumi et al. [5] presented the concept of single valued neutrosophic graphs by combining the single valued neutrosophic set theory and the graph theory, and defined different types of single valued neutrosophic graphs (SVNG). Recently, same authors [2, 6, 7, 8] introduced the concept of interval-valued neutrosophic graph as a generalization of fuzzy graph, intuitionistic fuzzy graph and single valued neutrosophic graph, and discussed some of their properties with examples. Moreover, Akram and Nasir [1] have introduced several concepts on interval-valued neutrosophic graphs.

A. Nagoorgani and K. Radha [15, 16] introduced the concept of regular fuzzy graphs and defined degree of a vertex in fuzzy graphs. A. Nagoorgani and S.R. Latha [14] introduced the concept of irregular fuzzy graphs, neighbourly irregular fuzzy graphs and highly irregular fuzzy graphs in 2008. S.P.Nandhini and E.Nandhini introduced the concept of strongly irregular fuzzy graphs and discussed about its properties [17].

K. Radha and N. Kumaravel [18] introduced the concept of edge degree, total edge degree in fuzzy graph and edge regular fuzzy graphs and discussed about the degree of an edge in some fuzzy graphs. N.R. Santhi Maheswari and C. Sekar introduced the concept of edge irregular fuzzy graphs and edge totally irregular fuzzy graphs and discussed about its properties [20]. Also, N.R. Santhi Maheswari and C. Sekar introduced the concept of neighbourly edge irregular fuzzy graphs, neighbourly edge totally irregular fuzzy graphs, strongly edge irregular fuzzy graphs and strongly edge totally irregular fuzzy graphs and discussed about its properties [21, 22]. Then we introduced this concepts on intuitionistic fuzzy graphs, single valued neutrosophic graphs and interval-valued neutrosophic graphs and discussed about their properties [26, 27, 28].

This is the background to introduce neighbourly edge irregular interval-valued neutrosophic graphs, neighbourly edge totally irregular interval-valued neutrosophic graphs and discussed some of their properties. Also neighbourly edge irregularity and strongly edge irregularity on some interval-valued neutrosophic graphs whose underlying crisp graphs are a path, a cycle and a star are studied.

2. Preliminaries

We present some known definitions and results for ready reference to go through the work presented in this paper.

Definition 2.1. *A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^*. A graph G^* is finite if its vertex set and edge set are finite.*

Definition 2.2. *The degree $d_{G^*}(v)$ of a vertex v in G^* or simply $d(v)$ is the number of edges of G^* incident with vertex v.*

Definition 2.3. *A Fuzzy graph denoted by $G : (\sigma, \mu)$ on the graph $G^* : (V, E)$ is a pair of functions (σ, μ) where $\sigma : V \to [0, 1]$ is a fuzzy subset of a non empty set V and $\mu : E \to [0, 1]$ is a symmetric fuzzy relation on σ such that for all u and v in V the relation $\mu(u, v) = \mu(uv) \leq min(\sigma(u), \sigma(v))$ is satisfied.*

Definition 2.4. *A single valued neutrosophic graph (SVNG) is of the form $G : (A, B)$ where $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ such that:*

(i) The functions $T_A : V \to [0, 1]$, $I_A : V \to [0, 1]$ and $F_A : V \to [0, 1]$ denote the degree of truth-membership, the degree of indeterminacy-membership and the degree of falsity-membership of the element $u \in V$, respectively, and $0 \leq T_A(u) + I_A(u) + F_A(u) \leq 3$ for every $u \in V$;

(ii) The functions $T_B : V \times V \to [0, 1], I_B : V \times V \to [0, 1]$ and $F_B : V \times V \to [0, 1]$ are the degree of truth-membership, the degree of indeterminacy-membership and the degree of falsity-membership of the edge $uv \in E$, respectively, such that $T_B(uv) \leq \min[T_A(u), T_A(v)]$, $I_B(uv) \geq \max[I_A(u), I_A(v)]$ and $F_B(uv) \geq \max[F_A(u), F_A(v)]$ and $0 \leq T_B(uv) + I_B(uv) + F_B(uv) \leq 3$ for every uv in E.

Definition 2.5. *Let $G : (A, B)$ be a SVNG on $G^* : (V, E)$. Then the degree of a vertex u is defined as $d_G(u) = (d_{T_A}(u), d_{I_A}(u), d_{F_A}(u))$ where $d_{T_A}(u) = \sum_{v \neq u} T_B(uv)$, $d_{I_A}(u) = \sum_{v \neq u} I_B(uv)$ and $d_{F_A}(u) = \sum_{v \neq u} F_B(uv)$.*

Definition 2.6. *Let $G : (A, B)$ be a SVNG on $G^* : (V, E)$. Then the total degree of a vertex u is defined by $td_G(u) = (td_{T_A}(u), td_{I_A}(u), td_{F_A}(u))$ where*

$td_{T_A}(u) = \sum_{v \neq u} T_B(uv) + T_A(u)$, $td_{I_A}(u) = \sum_{v \neq u} I_B(uv) + I_A(u)$ *and* $td_{F_A}(u) = \sum_{v \neq u} F_B(uv) + F_A(u)$.

Definition 2.7. *An interval-valued fuzzy graph (IVFG) is of the form* $G : (\sigma, \mu)$ *where* $\sigma = [\sigma^-, \sigma^+]$ *is an interval-valued fuzzy set in* V *and* $\mu = (\mu^-, \mu^+)$ *is an interval-valued fuzzy set in* $E \subseteq V \times V$ *such that* $\mu^-(uv) \leq min(\sigma^-(u), \sigma^-(v))$ *and* $\mu^+(uv) \leq min(\sigma^+(u), \sigma^+(v))$ *for every* uv *in* E.

Definition 2.8. *Let* $G : (\sigma, \mu)$ *be an IVFG on* $G^* : (V, E)$*. Then the degree of a vertex* u *is defined as* $d_G(u) = (d_{\sigma^-}(u), d_{\sigma^+}(u))$ *where* $d_{\sigma^-}(u) = \sum_{v \neq u} \mu^-(u, v)$ *and* $d_{\sigma^+}(u) = \sum_{v \neq u} \mu^+(u, v)$.

Definition 2.9. *Let* $G : (\sigma, \mu)$ *be an IVFG on* $G^* : (V, E)$*. Then the total degree of a vertex* u *is defined by* $td_G(u) = (td_{\sigma^-}(u), td_{\sigma^+}(u))$ *where* $td_{\sigma^-}(u) = \sum_{v \neq u} \mu^-(u, v) + \sigma^-(u)$ *and* $td_{\sigma^+}(u) = \sum_{v \neq u} \mu^+(u, v) + \sigma^+(u)$.

3. Interval-valued neutrosophic graphs (IVNGs)

Throughout this paper, we denote $G^* : (V, E)$ a crisp graph, and $G : (A, B)$ an interval-valued neutrosophic graph.

Definition 3.1. *By an interval-valued neutrosophic graph(IVNG) of a graph* $G^* : (V, E)$ *we mean a pair* $G : (A, B)$*, where* $A : (T_A, I_A, F_A) = ((T_A^-, T_A^+), (I_A^-, I_A^+), (F_A^-, F_A^+))$ *is an interval-valued neutrosophic set on* V*, and* $B : (T_B, I_B, F_B) = ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ *is an interval-valued neutrosophic relation on* E *satisfying the following condition:*

(i) $V = v_1, v_2, ..., v_n$ *such that* $T_A^- : V \to [0, 1]$*,* $T_A^+ : V \to [0, 1]$*,* $I_A^- : V \to [0, 1]$*,* $I_A^+ : V \to [0, 1]$*,* $F_A^- : V \to [0, 1]$ *and* $F_A^+ : V \to [0, 1]$ *denote the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element* $v_i \in V$*, respectively, and* $0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$ *for all* $v_i \in V, (i = 1, 2, ..., n)$.

(ii) The functions $T_B^- : V \times V \to [0, 1]$*,* $T_B^+ : V \times V \to [0, 1]$*,* $I_B^- : V \times V \to [0, 1]$*,* $I_B^+ : V \times V \to [0, 1]$*,* $F_B^- : V \times V \to [0, 1]$ *and* $F_B^+ : V \times V \to [0, 1]$ *are such that:*
$T_B^-(v_iv_j) \leq \min(T_A^-(v_i), T_A^-(v_j))$, $T_B^+(v_iv_j) \leq \min(T_A^+(v_i), T_A^+(v_j))$,
$I_B^-(v_iv_j) \geq \max(I_A^-(v_i), I_A^-(v_j))$, $I_B^+(v_iv_j) \geq \max(I_A^+(v_i), I_A^+(v_j))$,
$F_B^-(v_iv_j) \geq \max(F_A^-(v_i), F_A^-(v_j))$ *and* $F_B^+(v_iv_j) \geq \max(F_A^+(v_i), F_A^+(v_j))$
denotes the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $v_iv_j \in E$ *respectively, where* $0 \leq T_B(v_iv_j) + I_B(v_iv_j) + F_B(v_iv_j) \leq 3$ *for all* $v_iv_j \in E, (i, j = 1, 2, ..., n)$.

Definition 3.2. *Let* $G : (A, B)$ *be an interval-valued neutrosophic graph on* $G^* : (V, E)$*. Then the degree of a vertex* v_i *is defined as*
$d_G(v_i) = ((d_{T_A^-}(v_i), d_{T_A^+}(v_i)), (d_{I_A^-}(v_i), d_{I_A^+}(v_i)), (d_{F_A^-}(v_i), d_{F_A^+}(v_i)))$ *where*
$d_{T_A^-}(v_i) = \sum_{v_i \neq v_j} T_B^-(v_iv_j)$ $d_{T_A^+}(v_i) = \sum_{v_i \neq v_j} T_B^+(v_iv_j)$,

$d_{I_A^-}(v_i) = \sum_{v_i \neq v_j} I_B^-(v_iv_j)$, $d_{I_A^+}(v_i) = \sum_{v_i \neq v_j} I_B^+(v_iv_j)$,
$d_{F_A^-}(v_i) = \sum_{v_i \neq v_j} F_B^-(v_iv_j)$ *and* $d_{F_A^+}(v_i) = \sum_{v_i \neq v_j} F_B^+(v_i, v_j)$.

Definition 3.3. *Let* $G : (A, B)$ *be an interval-valued neutrosophic graph on* $G^* : (V, E)$*. Then the total degree of a vertex* v_i *is defined as*
$td_G(v_i) = ((td_{T_A^-}(v_i), td_{T_A^+}(v_i)), (td_{I_A^-}(v_i), td_{I_A^+}(v_i)), (td_{F_A^-}(v_i), td_{F_A^+}(v_i)))$ *where*
$td_{T_A^-}(v_i) = \sum_{v_i \neq v_j} T_B^-(v_iv_j) + T_A^-(v_i)$, $td_{T_A^+}(v_i) = \sum_{v_i \neq v_j} T_B^+(v_iv_j) + T_A^+(v_i)$,
$td_{I_A^-}(v_i) = \sum_{v_i \neq v_j} I_B^-(v_iv_j) + I_A^-(v_i)$, $td_{I_A^+}(v_i) = \sum_{v_i \neq v_j} I_B^+(v_iv_j) + I_A^+(v_i)$,
$td_{F_A^-}(v_i) = \sum_{v_i \neq v_j} F_B^-(v_iv_j) + F_A^-(v_i)$ *and* $td_{F_A^+}(v_i) = \sum_{v_i \neq v_j} F_B^+(v_i, v_j) + F_A^+(v_i)$.

Definition 3.4. *Let* $G : (A, B)$ *be an interval-valued neutrosophic graph on* $G^* : (V, E)$*. Then:*

(i) G is irregular, if there is a vertex which is adjacent to vertices with distinct degrees.

(ii) G is totally irregular, if there is a vertex which is adjacent to vertices with distinct total degrees.

Definition 3.5. *Let* $G : (A, B)$ *be a connected interval-valued neutrosophic graph on* $G^* : (V, E)$*. Then:*

(i) G is said to be a neighbourly irregular IVNG if every pair of adjacent vertices have distinct degrees.

(ii) G is said to be a neighbourly totally IVNG if every pair of adjacent vertices have distinct total degrees.

(iii) G is said to be a strongly irregular IVNG if every pair of vertices have distinct degrees.

(iv) G is said to be a strongly totally irregular IVNG if every pair of vertices have distinct total degrees.

(v) G is said to be a highly irregular IVNG if every vertex in G is adjacent to the vertices having distinct degrees.

(vi) G is said to be a highly totally irregular IVNG if every vertex in G is adjacent to the vertices having distinct total degrees.

Definition 3.6. *Let* $G : (A, B)$ *be an interval-valued neutrosophic graph on* $G^* : (V, E)$*. The degree of an edge* v_iv_j *is defined as*
$d_G(v_iv_j) = ((d_{T_B^-}(v_iv_j), d_{T_B^+}(v_iv_j)), (d_{I_B^-}(v_iv_j), d_{I_B^+}(v_iv_j)), (d_{F_B^-}(v_iv_j), d_{F_B^+}(v_iv_j)))$
where
$d_{T_B^-}(v_iv_j) = d_{T_A^-}(v_i) + d_{T_A^-}(v_j) - 2T_B^-(v_iv_j)$,
$d_{T_B^+}(v_iv_j) = d_{T_A^+}(v_i) + d_{T_A^+}(v_j) - 2T_B^+(v_iv_j)$,
$d_{I_B^-}(v_iv_j) = d_{I_A^-}(v_i) + d_{I_A^-}(v_j) - 2I_B^-(v_iv_j)$,
$d_{I_B^+}(v_iv_j) = d_{I_A^+}(v_i) + d_{I_A^+}(v_j) - 2I_B^+(v_iv_j)$,
$d_{F_B^-}(v_iv_j) = d_{F_A^-}(v_i) + d_{F_A^-}(v_j) - 2F_B^-(v_iv_j)$ *and*
$d_{F_B^+}(v_iv_j) = d_{F_A^+}(v_i) + d_{F_A^+}(v_j) - 2F_B^+(v_iv_j)$.

Definition 3.7. *Let $G:(A,B)$ be an interval-valued neutrosophic graph on $G^*:(V,E)$. The total degree of an edge v_iv_j is defined as*
$td_G(v_iv_j)=((td_{T_B^-}(v_iv_j),td_{T_B^+}(v_iv_j)),(td_{I_B^-}(v_iv_j),td_{I_B^+}(v_iv_j)),(td_{F_B^-}(v_iv_j),$
$td_{F_B^+}(v_iv_j)))$ *where*
$td_{T_B^-}(v_iv_j)=d_{T_A^-}(v_i)+d_{T_A^-}(v_j)-T_B^-(v_iv_j)=d_{T_B^-}(v_iv_j)+T_B^-(v_iv_j)$,
$td_{T_B^+}(v_iv_j)=d_{T_A^+}(v_i)+d_{T_A^+}(v_j)-T_B^+(v_iv_j)=d_{T_B^+}(v_iv_j)+T_B^+(v_iv_j)$,
$td_{I_B^-}(v_iv_j)=d_{I_A^-}(v_i)+d_{I_A^-}(v_j)-I_B^-(v_iv_j)=d_{I_B^-}(v_iv_j)+I_B^-(v_iv_j)$,
$td_{I_B^+}(v_iv_j)=d_{I_A^+}(v_i)+d_{I_A^+}(v_j)-I_B^+(v_iv_j)=d_{I_B^+}(v_iv_j)+I_B^+(v_iv_j)$,
$td_{F_B^-}(v_iv_j)=d_{F_A^-}(v_i)+d_{F_A^-}(v_j)-F_B^-(v_iv_j)=d_{F_B^-}(v_iv_j)+F_B^-(v_iv_j)$ *and*
$td_{F_B^+}(v_iv_j)=d_{F_A^+}(v_i)+d_{F_A^+}(v_j)-F_B^+(v_iv_j)=d_{F_B^+}(v_iv_j)+F_B^+(v_iv_j)$.

4. Neighbourly edge irregular interval-valued neutrosophic graphs and neighbourly edge totally irregular interval-valued neutrosophic graphs

In this section, neighbourly edge irregular interval-valued neutrosophic graphs and neighbourly edge totally irregular interval-valued neutrosophic graphs are introduced.

Definition 4.1. *Let $G:(A,B)$ be a connected interval-valued neutrosophic graph on $G^*:(V,E)$. Then G is said to be:*

(i) A neighbourly edge irregular interval-valued neutrosophic graph if every pair of adjacent edges have distinct degrees.

(ii) A neighbourly edge totally irregular interval-valued neutrosophic graph if every pair of adjacent edges have distinct total degrees.

Example 4.1. Graph which is both neighbourly edge irregular interval-valued neutrosophic graph and neighbourly edge totally irregular interval-valued neutrosophic graph.

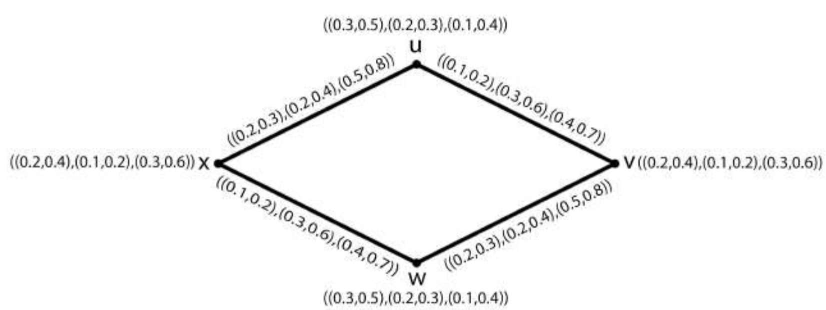

Figure 1: Both neighbourly edge irregular IVNG and neighbourly edge totally irregular IVNG

Consider $G^* : (V, E)$ where $V = \{u, v, w, x\}$ and $E = \{uv, vw, wx, xu\}$. From Figure 1, $d_G(u) = d_G(v) = d_G(w) = d_G(x) = ((0.3, 0.5), (0.5, 1.0), (0.9, 1.5))$.

Degrees of the edges are calculated as follows $d_G(uv) = d_G(wx) = ((0.4, 0.6), (0.4, 0.8), (1.0, 1.6)), d_G(vw) = d_G(xu) = ((0.2, 0.4), (0.6, 1.2), (0.8, 1.4))$.

It is noted that every pair of adjacent edges have distinct degrees. Hence, G is a neighbourly edge irregular interval-valued neutrosophic graph.

Total degrees of the edges are calculated as follows $td_G(uv) = td_G(wx) = ((0.5, 0.8), (0.7, 1.4), (1.4, 2.3)), td_G(vw) = td_G(xu) = ((0.4, 0.7), (0.8, 1.6), (1.3, 2.2))$.

It is observed that every pair of adjacent edges having distinct total degrees. So, G is a neighbourly edge totally irregular interval-valued neutrosophic graph.

Hence G is both neighbourly edge irregular interval-valued neutrosophic graph and neighbourly edge totally irregular interval-valued neutrosophic graph.

Example 4.2. Neighbourly edge irregular interval-valued neutrosophic graph don't need to be neighbourly edge totally irregular interval-valued neutrosophic graph.

Consider $G : (A, B)$ be an interval-valued neutrosophic graph such that $G^* : (V, E)$ is a star on four vertices.

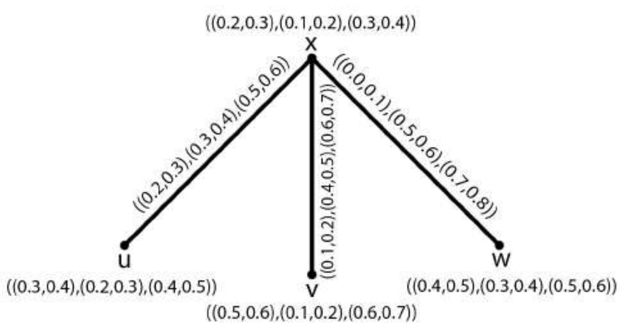

Figure 2: Neighbourly edge irregular IVNG, not neighbourly edge totally irregular IVNG

From Figure 2, $d_G(u) = ((0.2, 0.3), (0.3, 0.4), (0.5, 0.6)), d_G(v) = ((0.1, 0.2), (0.4, 0.5), (0.6, 0.7)), d_G(w) = ((0.0, 0.1), (0.5, 0.6), (0.7, 0.8)), d_G(x) = ((0.3, 0.6), (1.2, 1.5), (1.8, 2.1))$; $d_G(ux)=((0.1, 0.3), (0.9, 1.1), (1.3, 1.5)), d_G(vx)=((0.2, 0.4), (0.8, 1.0), (1.2, 1.4)), d_G(wx)=((0.3, 0.5), (0.7, 0.9), (1.1, 1.3))$; $td_G(ux)=td_G(vx) = td_G(wx) = ((0.3, 0.6), (1.2, 1.5), (1.8, 2.1))$.

Here, $d_G(ux) \neq d_G(vx) \neq d_G(wx)$. Hence G is a neighbourly edge irregular interval-valued neutrosophic graph. But G is not a neighbourly edge totally irregular interval-valued neutrosophic graph, since all edges have same total degrees.

Example 4.3. Neighbourly edge totally irregular interval-valued neutrosophic graphs don't need to be neighbourly edge irregular interval-valued neutrosophic graphs. Following shows this subject:

Consider $G : (A, B)$ be an interval-valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.

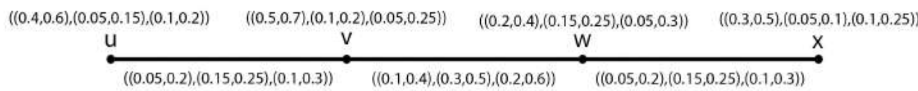

Figure 3: Neighbourly edge totally irregular IVNG, not neighbourly edge irregular IVNG

From Figure 3, $d_G(u) = d_G(x) = ((0.05, 0.20), (0.15, 0.25), (0.1, 0.3)), d_G(v) = d_G(w) = ((0.15, 0.60), (0.45, 0.75), (0.3, 0.9))$; $d_G(uv) = d_G(vw) = d_G(wx) = ((0.1, 0.4), (0.3, 0.5), (0.2, 0.6))$; $td_G(uv) = ((0.15, 0.60), (0.45, 0.75), (0.3, 0.9))$, $td_G(vw) = ((0.2, 0.8), (0.6, 1.0), (0.4, 1.2)), td_G(wx) = ((0.15, 0.60), (0.45, 0.75), (0.3, 0.9))$.

Here, $d_G(uv) = d_G(vw) = d_G(wx)$. Hence G is not a neighbourly edge irregular interval-valued neutrosophic graph. But G is a neighbourly edge totally irregular interval-valued neutrosophic graph, since $td_G(uv) \neq td_G(vw)$ and $td_G(vw) \neq td_G(wx)$.

Theorem 4.1. *Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$ and $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ a constant function. Then G is a neighbourly edge irregular interval-valued neutrosophic graph, if and only if G is a neighbourly edge totally irregular interval-valued neutrosophic graph.*

Proof. Assume that $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ is a constant function, let $B(uv) = C$, for all uv in E, where $C = ((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+))$ is constant.

Let uv and vw be pair of adjacent edges in E, then we have $d_G(uv) \neq d_G(vw) \Leftrightarrow d_G(uv)+C \neq d_G(vw)+C \Leftrightarrow ((d_{T_B^-}(uv), d_{T_B^+}(uv)), (d_{I_B^-}(uv), d_{I_B^+}(uv)), (d_{F_B^-}(uv), d_{F_B^+}(uv))) + ((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+)) \neq ((d_{T_B^-}(vw), d_{T_B^+}(vw)), (d_{I_B^-}(vw), d_{I_B^+}(vw)), (d_{F_B^-}(vw), d_{F_B^+}(vw))) + ((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+)) \Leftrightarrow ((d_{T_B^-}(uv) + C_T^-, d_{T_B^+}(uv) + C_T^+), (d_{I_B^-}(uv) + C_I^-, d_{I_B^+}(uv) + C_I^+), (d_{F_B^-}(uv) + C_F^-, d_{F_B^+}(uv) + C_F^+)) \neq ((d_{T_B^-}(vw) + C_T^-, d_{T_B^+}(vw) + C_T^+), (d_{I_B^-}(vw) + C_I^-, d_{I_B^+}(vw) + C_I^+), (d_{F_B^-}(vw)+C_F^-, d_{F_B^+}(vw)+C_F^+)) \Leftrightarrow ((d_{T_B^-}(uv)+T_B^-(uv), d_{T_B^+}(uv)+T_B^+(uv)), (d_{I_B^-}(uv)+I_B^-(uv), d_{I_B^+}(uv)+I_B^+(uv)), (d_{F_B^-}(uv)+F_B^-(uv), d_{F_B^+}(uv)+F_B^+(uv))) \neq ((d_{T_B^-}(vw)+T_B^-(vw), d_{T_B^+}(vw)+T_B^+(vw)), (d_{I_B^-}(vw)+I_B^-(vw), d_{I_B^+}(vw)+I_B^+(vw)),$

$(d_{F_B^-}(vw) + F_B^-(vw), d_{F_B^+}(vw) + F_B^+(vw))) \Leftrightarrow ((td_{T_B^-}(uv), td_{T_B^+}(uv)), (td_{I_B^-}(uv), td_{I_B^+}(uv)), (td_{F_B^-}(uv), td_{F_B^+}(uv))) \neq ((td_{T_B^-}(vw), td_{T_B^+}(vw)), (td_{I_B^-}(vw), td_{I_B^+}(vw)), (td_{F_B^-}(vw), td_{F_B^+}(vw))) \Leftrightarrow td_G(uv) \neq td_G(vw)$. Therefore, every pair of adjacent edges have distinct degrees if and only if have distinct total degrees. Hence G is a neighbourly edge irregular interval-valued neutrosophic graph if and only if G is a neighbourly edge totally irregular interval-valued neutrosophic graph. □

Remark 4.1. Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$. If G is both neighbourly edge irregular interval-valued neutrosophic graph and neighbourly edge totally irregular interval-valued neutrosophic graph, Then B don't need to be a constant function.

Example 4.4. Consider $G : (A, B)$ be an interval-valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.

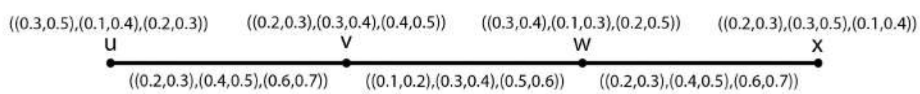

Figure 4: B is not a constant function.

From Figure 4,
$d_G(u) = d_G(x) = ((0.2, 0.3), (0.4, 0.5), (0.6, 0.7))$,
$d_G(v) = d_G(w) = ((0.3, 0.5), (0.7, 0.9), (1.1, 1.3))$;
$d_G(uv) = d_G(wx) = ((0.1, 0.2), (0.3, 0.4), (0.5, 0.6))$,
$d_G(vw) = ((0.4, 0.6), (0.8, 1.0), (1.2, 1.4))$;
$td_G(uv) = td_G(wx) = ((0.3, 0.5), (0.7, 0.9), (1.1, 1.3))$,
$td_G(vw) = ((0.5, 0.8), (1.1, 1.4), (1.7, 2.0))$.

Here, $d_G(uv) \neq d_G(vw)$ and $d_G(vw) \neq d_G(wx)$. Hence G is a neighbourly edge irregular interval-valued neutrosophic graph. Also, $td_G(uv) \neq td_G(vw)$ and $td_G(vw) \neq td_G(wx)$. Hence G is a neighbourly edge totally irregular interval-valued neutrosophic graph. But B is not constant function.

Theorem 4.2. *Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$ and $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ a constant function. If G is a strongly irregular interval-valued neutrosophic graph, then G is a neighbourly edge irregular interval-valued neutrosophic graph.*

Proof. Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$. Assume that $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ is a constant function, let $B(uv) = C$, for all uv in E, where $C = ((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+))$ is constant.

Let uv and vw be any two adjacent edges in G. Let us suppose that G is a strongly irregular interval-valued neutrosophic graph. Then, every pair of vertices in G having distinct degrees, and hence $d_G(u) \neq d_G(v) \neq d_G(w) \Rightarrow ((d_{T_A^-}(u), d_{T_A^+}(u)), (d_{I_A^-}(u), d_{I_A^+}(u)), (d_{F_A^-}(u), d_{F_A^+}(u))) \neq ((d_{T_A^-}(v), d_{T_A^+}(v)), (d_{I_A^-}(v), d_{I_A^+}(v)), (d_{F_A^-}(v), d_{F_A^+}(v))) \neq ((d_{T_A^-}(w), d_{T_A^+}(w)), (d_{I_A^-}(w), d_{I_A^+}(w)), (d_{F_A^-}(w), d_{F_A^+}(w))) \Rightarrow ((d_{T_A^-}(u), d_{T_A^+}(u)), (d_{I_A^-}(u), d_{I_A^+}(u)), (d_{F_A^-}(u), d_{F_A^+}(u))) + ((d_{T_A^-}(v), d_{T_A^+}(v)), (d_{I_A^-}(v), d_{I_A^+}(v)), (d_{F_A^-}(v), d_{F_A^+}(v))) - 2((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+)) \neq ((d_{T_A^-}(v), d_{T_A^+}(v)), (d_{I_A^-}(v), d_{I_A^+}(v)), (d_{F_A^-}(v), d_{F_A^+}(v))) + ((d_{T_A^-}(w), d_{T_A^+}(w)), (d_{I_A^-}(w), d_{I_A^+}(w)), (d_{F_A^-}(w), d_{F_A^+}(w))) - 2((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+)) \Rightarrow ((d_{T_A^-}(u)+d_{T_A^-}(v)-2C_T^-, d_{T_A^+}(u)+d_{T_A^+}(v)-2C_T^+), (d_{I_A^-}(u)+d_{I_A^-}(v)-2C_I^-, d_{I_A^+}(u)+d_{I_A^+}(v)-2C_I^+), (d_{F_A^-}(u)+d_{F_A^-}(v)-2C_F^-, d_{F_A^+}(u)+d_{F_A^+}(v)-2C_F^+)) \neq ((d_{T_A^-}(v)+d_{T_A^-}(w)-2C_T^-, d_{T_A^+}(v)+d_{T_A^+}(w)-2C_T^+), (d_{I_A^-}(v)+d_{I_A^-}(w)-2C_I^-, d_{I_A^+}(v)+d_{I_A^+}(w)-2C_I^+), (d_{F_A^-}(v)+d_{F_A^-}(w)-2C_F^-, d_{F_A^+}(v)+d_{F_A^+}(w)-2C_F^+)) \Rightarrow ((d_{T_A^-}(u)+d_{T_A^-}(v)-2T_B^-(uv), d_{T_A^+}(u)+d_{T_A^+}(v)-2T_B^-(uv)), (d_{I_A^-}(u)+d_{I_A^-}(v)-2I_B^-(uv), d_{I_A^+}(u)+d_{I_A^+}(v)-2I_B^-(uv)), (d_{F_A^-}(u)+d_{F_A^-}(v)-2F_B^-(uv), d_{F_A^+}(u)+d_{F_A^+}(v)-2F_B^-(uv))) \neq ((d_{T_A^-}(v)+d_{T_A^-}(w)-2T_B^-(vw), d_{T_A^+}(v)+d_{T_A^+}(w)-2T_B^-(vw)), (d_{I_A^-}(v)+d_{I_A^-}(w)-2I_B^-(vw), d_{I_A^+}(v)+d_{I_A^+}(w)-2I_B^-(vw)), (d_{F_A^-}(v)+d_{F_A^-}(w)-2F_B^-(vw), d_{F_A^+}(v)+d_{F_A^+}(w)-2F_B^-(vw))) \Rightarrow ((d_{T_B^-}(uv), d_{T_B^+}(uv)), (d_{I_B^-}(uv), d_{I_B^+}(uv)), (d_{F_B^-}(uv), d_{F_B^+}(uv))) \neq ((d_{T_B^-}(vw), d_{T_B^+}(vw)), (d_{I_B^-}(vw), d_{I_B^+}(vw)), (d_{F_B^-}(vw), d_{F_B^+}(vw))) \Rightarrow d_G(uv) \neq d_G(vw)$.

Therefore, every pair of adjacent edges have distinct degrees, hence G is a neighbourly edge irregular interval-valued neutrosophic graph. □

Similar to the above theorem can be considered the following theorem:

Theorem 4.3. *Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$ and $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ a constant function. If G is a strongly irregular interval-valued neutrosophic graph, then G is a neighbourly edge totally irregular interval-valued neutrosophic graph.*

Remark 4.2. Converse of the above theorems don't need to be true.

Example 4.5. Consider $G : (A, B)$ be an interval-valued neutrosophic graph such that $G^* : (V, E)$ is a path on four vertices.

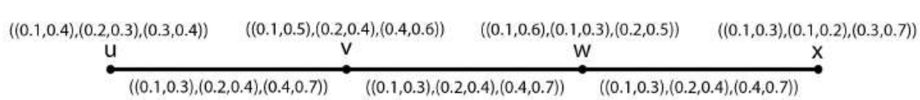

Figure 5: Both neighbourly edge irregular IVNG and neighbourly edge totally irregular IVNG, not strongly irregular IVNG

From Figure 5,
$d_G(u) = d_G(x) = ((0.1, 0.3), (0.2, 0.4), (0.4, 0.7))$,
$d_G(v) = d_G(w) = ((0.2, 0.6), (0.4, 0.8), (0.8, 1.4))$.

Here, G is not a strongly irregular interval-valued neutrosophic graph.

$d_G(uv) = d_G(wx) = ((0.1, 0.3), (0.2, 0.4), (0.4, 0.7))$,
$d_G(vw) = ((0.2, 0.6), (0.4, 0.8), (0.8, 1.4))$;

$td_G(uv) = td_G(wx) = ((0.2, 0.6), (0.4, 0.8), (0.8, 1.4))$,
$td_G(vw) = ((0.3, 0.9), (0.6, 1.2), (1.2, 2.1))$.

It is noted that $d_G(uv) \neq d_G(vw)$ and $d_G(vw) \neq d_G(wx)$. And also, $td_G(uv) \neq td_G(vw)$ and $td_G(vw) \neq td_G(wx)$. Hence G is both neighbourly edge irregular interval-valued neutrosophic graph and neighbourly edge totally irregular interval-valued neutrosophic graph. But G is not a strongly irregular interval-valued neutrosophic graph.

Theorem 4.4. *Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$ and $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ a constant function. Then G is a highly irregular interval-valued neutrosophic graph if and only if G is a neighbourly edge irregular interval-valued neutrosophic graph.*

Proof. Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$. Assume that $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ is a constant function, let $B(uv) = C$, for all uv in E, where $C = ((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+))$ is constant.

Let uv and vw be any two adjacent edges in G. Then, we have $d_G(u) \neq d_G(w) \Leftrightarrow ((d_{T_A^-}(u), d_{T_A^+}(u)), (d_{I_A^-}(u), d_{I_A^+}(u)), (d_{F_A^-}(u), d_{F_A^+}(u))) \neq ((d_{T_A^-}(w), d_{T_A^+}(w)), (d_{I_A^-}(w), d_{I_A^+}(w)), (d_{F_A^-}(w), d_{F_A^+}(w))) \Leftrightarrow ((d_{T_A^-}(u), d_{T_A^+}(u)), (d_{I_A^-}(u), d_{I_A^+}(u)), (d_{F_A^-}(u), d_{F_A^+}(u))) + ((d_{T_A^-}(v), d_{T_A^+}(v)), (d_{I_A^-}(v), d_{I_A^+}(v)), (d_{F_A^-}(v), d_{F_A^+}(v))) - 2((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+)) \neq ((d_{T_A^-}(v), d_{T_A^+}(v)), (d_{I_A^-}(v), d_{I_A^+}(v)), (d_{F_A^-}(v), d_{F_A^+}(v))) + ((d_{T_A^-}(w), d_{T_A^+}(w)), (d_{I_A^-}(w), d_{I_A^+}(w)), (d_{F_A^-}(w), d_{F_A^+}(w))) - 2((C_T^-, C_T^+), (C_I^-, C_I^+), (C_F^-, C_F^+)) \Leftrightarrow ((d_{T_A^-}(u) + d_{T_A^-}(v) - 2C_T^-, d_{T_A^+}(u) + d_{T_A^+}(v) - 2C_T^+), (d_{I_A^-}(u) + d_{I_A^-}(v) - 2C_I^-, d_{I_A^+}(u) + d_{I_A^+}(v) - 2C_I^+), (d_{F_A^-}(u) + d_{F_A^-}(v) - 2C_F^-, d_{F_A^+}(u) + d_{F_A^+}(v) - 2C_F^+)) \neq ((d_{T_A^-}(v) + d_{T_A^-}(w) - 2C_T^-, d_{T_A^+}(v) + d_{T_A^+}(w) - 2C_T^+), (d_{I_A^-}(v) + d_{I_A^-}(w) - 2C_I^-, d_{I_A^+}(v) + d_{I_A^+}(w) - 2C_I^+), (d_{F_A^-}(v) + d_{F_A^-}(w) - 2C_F^-, d_{F_A^+}(v) + d_{F_A^+}(w) - 2C_F^+)) \Leftrightarrow ((d_{T_A^-}(u) + d_{T_A^-}(v) - 2T_B^-(uv), d_{T_A^+}(u) + d_{T_A^+}(v) - 2T_B^-(uv)), (d_{I_A^-}(u) + d_{I_A^-}(v) - 2I_B^-(uv), d_{I_A^+}(u) + d_{I_A^+}(v) - 2I_B^-(uv)), (d_{F_A^-}(u) + d_{F_A^-}(v) - 2F_B^-(uv), d_{F_A^+}(u) + d_{F_A^+}(v) - 2F_B^-(uv))) \neq ((d_{T_A^-}(v) + d_{T_A^-}(w) - 2T_B^-(vw), d_{T_A^+}(v) + d_{T_A^+}(w) - 2T_B^-(vw)), (d_{I_A^-}(v) + d_{I_A^-}(w) - 2I_B^-(vw), d_{I_A^+}(v) + d_{I_A^+}(w) - 2I_B^-(vw)), (d_{F_A^-}(v) + d_{F_A^-}(w) - 2F_B^-(vw), d_{F_A^+}(v) + d_{F_A^+}(w) - 2F_B^-(vw))) \Leftrightarrow$

$((d_{T_B^-}(uv), d_{T_B^+}(uv)), (d_{I_B^-}(uv), d_{I_B^+}(uv)), (d_{F_B^-}(uv), d_{F_B^+}(uv))) \neq ((d_{T_B^-}(vw), d_{T_B^+}(vw)), (d_{I_B^-}(vw), d_{I_B^+}(vw)), (d_{F_B^-}(vw), d_{F_B^+}(vw))) \Leftrightarrow d_G(uv) \neq d_G(vw)$.

Therefore, every pair of adjacent edges have distinct degrees, if and only if every vertex adjacent to the vertices having distinct degrees. Hence G is a highly irregular interval-valued neutrosophic graph, if and only if G is a neighbourly edge irregular interval-valued neutrosophic graph. □

Theorem 4.5. *Let $G : (A, B)$ be a connected interval-valued neutrosophic graph on $G^* : (V, E)$ and $B : ((T_B^-, T_B^+), (I_B^-, I_B^+), (F_B^-, F_B^+))$ a constant function. Then G is highly irregular interval-valued neutrosophic graph if and only if G is neighbourly edge totally irregular interval-valued neutrosophic graph.*

Proof. Proof is similar to the above Theorem 4.4. □

Theorem 4.6. *Let $G : (A, B)$ be an interval-valued neutrosophic graph on $G^* : (V, E)$, a star $K_{1,n}$. Then G is a totally edge regular interval-valued neutrosophic graph. Also, if the degrees of truth-membership, indeterminacy-membership and falsity-membership of no two edges are same, then G is a neighbourly edge irregular interval-valued neutrosophic graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices adjacent to the vertex x. Let $e_1, e_2, e_3, \ldots, e_n$ be the edges of a star G^* in that order having the degrees of truth-membership $p_1, p_2, p_3, \ldots, p_n$, the degrees of indeterminacy-membership $q_1, q_2, q_3, \ldots, q_n$ and the degrees of falsity-membership $r_1, r_2, r_3, \ldots, r_n$ where $p_i = (p_i^-, p_i^+)$, $q_i = (q_i^-, q_i^+)$ and $r_i = (r_i^-, r_i^+)$ for $i = 1, 2, ..., n$ such that $0 \leq p_i + q_i + r_i \leq 3$, for every $1 \leq i \leq n$. Then, $td_G(e_i) = ((td_{T_B^-}(e_i), td_{T_B^+}(e_i)), (td_{I_B^-}(e_i), td_{I_B^+}(e_i)), (td_{F_B^-}(e_i), td_{F_B^+}(e_i))) = ((d_{T_B^-}(e_i) + T_B^-(e_i), d_{T_B^+}(e_i) + T_B^+(e_i)), (d_{I_B^-}(e_i) + I_B^-(e_i), d_{I_B^+}(e_i) + I_B^+(e_i)), (d_{F_B^-}(e_i) + F_B^-(e_i), d_{F_B^+}(e_i) + F_B^+(e_i))) = ((\sum_{k=1}^n p_k^- - p_i^- + p_i^-, \sum_{k=1}^n p_k^+ - p_i^+ + p_i^+), (\sum_{k=1}^n q_k^- - q_i^- + q_i^-, \sum_{k=1}^n q_k^+ - q_i^+ + q_i^+), (\sum_{k=1}^n r_k^- - r_i^- + r_i^-, \sum_{k=1}^n r_k^+ - r_i^+ + r_i^+)) = ((\sum_{k=1}^n p_k^-, \sum_{k=1}^n p_k^+), (\sum_{k=1}^n q_k^-, \sum_{k=1}^n q_k^+), (\sum_{k=1}^n r_k^-, \sum_{k=1}^n r_k^+))$.

All edges e_i , $(1 \leq i \leq n)$, having same total degrees. Hence G is a totally edge regular interval-valued neutrosophic graph.

Now, if $p_i^- \neq p_j^-$, $p_i^+ \neq p_j^+$, $q_i^- \neq q_j^-$, $q_i^+ \neq q_j^+$, $r_i^- \neq r_j^-$ and $r_i^+ \neq r_j^+$, for every $1 \leq i, j \leq n$ then, we have $d_G(e_i) = ((d_{T_B^-}(e_i), d_{T_B^+}(e_i)), (d_{I_B^-}(e_i), d_{I_B^+}(e_i)), (d_{F_B^-}(e_i), d_{F_B^+}(e_i))) = ((d_{T_A^-}(x) + d_{T_A^-}(v_i) - 2T_B^-(xv_i), d_{T_A^+}(x) + d_{T_A^+}(v_i) - 2T_B^+(xv_i)), (d_{I_A^-}(x) + d_{I_A^-}(v_i) - 2I_B^-(xv_i), d_{I_A^+}(x) + d_{I_A^+}(v_i) - 2I_B^+(xv_i)), (d_{F_A^-}(x) + d_{F_A^-}(v_i) - 2F_B^-(xv_i), d_{F_A^+}(x) + d_{F_A^+}(v_i) - 2F_B^+(xv_i))) = ((\sum_{k=1}^n p_k^- + p_i^- - 2p_i^-, \sum_{k=1}^n p_k^+ + p_i^+ - 2p_i^+), (\sum_{k=1}^n q_k^- + q_i^- - 2q_i^-, \sum_{k=1}^n q_k^+ + q_i^+ - 2q_i^+), (\sum_{k=1}^n r_k^- + r_i^- - 2r_i^-, \sum_{k=1}^n r_k^+ + r_i^+ - 2r_i^+)) = ((\sum_{k=1}^n p_k^- - p_i^-, \sum_{k=1}^n p_k^+ - p_i^+), (\sum_{k=1}^n q_k^- - q_i^-, \sum_{k=1}^n q_k^+ - q_i^+), (\sum_{k=1}^n r_k^- - r_i^-, \sum_{k=1}^n r_k^+ - r_i^+))$ for every $1 \leq i \leq n$.

Therefore, all edges e_i ,$(1 \leq i \leq n)$, having distinct degrees. Hence G is a neighbourly edge irregular interval-valued neutrosophic graph. □

Theorem 4.7. *Let $G:(A,B)$ be an interval-valued neutrosophic graph such that $G^*:(V,E)$ is a path on $2m(m>1)$ vertices. If the degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges e_i, $i=1,3,5,...,2m-1$, are $p_1=(p_1^-,p_1^+)$, $q_1=(q_1^-,q_1^+)$ and $r_1=(r_1^-,r_1^+)$, respectively, and the degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges e_i, $i=2,4,6,...,2m-2$, are $p_2=(p_2^-,p_2^+)$, $q_2=(q_2^-,q_2^+)$ and $r_2=(r_2^-,r_2^+)$, respectively, such that $p_1\neq p_2$ and $p_2\neq 2p_1$ and $q_1\neq q_2$ and $q_2\neq 2q_1$ and $r_1\neq r_2$ and $r_2\neq 2r_1$, then G is both neighbourly edge irregular interval-valued neutrosophic graph and neighbourly edge totally irregular interval-valued neutrosophic graph.*

Proof. Let $G:(A,B)$ be an interval-valued neutrosophic graph on $G^*:(V,E)$, a path on $2m(m>1)$ vertices.

Let $e_1,e_2,e_3,\ldots,e_{2m-1}$ be the edges of path G^*. If the alternate edges have the same degrees of truth-membership, indeterminacy-membership and falsity-membership, such that

$$\begin{aligned}B(e_i)&=(T_B(e_i),I_B(e_i),F_B(e_i))\\&=((T_B^-(e_i),T_B^+(e_i)),(I_B^-(e_i),I_B^+(e_i)),(F_B^-(e_i),F_B^+(e_i)))\\&=\begin{cases}(p_1,q_1,r_1), & \text{if } i \text{ is odd},\\(p_2,q_2,r_2), & \text{if } i \text{ is even}\end{cases}=\begin{cases}((p_1^-,p_1^+),(q_1^-,q_1^+),(r_1^-,r_1^+)), & \text{if } i \text{ is odd},\\((p_2^-,p_2^+),(q_2^-,q_2^+),(r_2^-,r_2^+)), & \text{if } i \text{ is even}\end{cases}\end{aligned}$$

where $0\le p_i+q_i+r_i\le 3$ for $i=1,2$ and $(p_1^-,p_1^+)\neq(p_2^-,p_2^+)$ and $(p_2^-,p_2^+)\neq 2(p_1^-,p_1^+)$ and $(q_1^-,q_1^+)\neq(q_2^-,q_2^+)$ and $(q_2^-,q_2^+)\neq 2(q_1^-,q_1^+)$ and $(r_1^-,r_1^+)\neq(r_2^-,r_2^+)$ and $(r_2^-,r_2^+)\neq 2(r_1^-,r_1^+)$, then $d_G(e_1)=(((p_1^-)+(p_1^-+p_2^-)-2p_1^-,(p_1^+)+(p_1^++p_2^+)-2p_1^+),((q_1^-)+(q_1^-+q_2^-)-2q_1^-,(q_1^+)+(q_1^++q_2^+)-2q_1^+),((r_1^-)+(r_1^-+r_2^-)-2r_1^-,(r_1^+)+(r_1^++r_2^+)-2r_1^+))=((p_2^-,p_2^+),(q_2^-,q_2^+),(r_2^-,r_2^+))=(p_2,q_2,r_2)$

for $i=3,5,7,\ldots,2m-3$; $d_G(e_i)=(((p_1^-+p_2^-)+(p_1^-+p_2^-)-2p_1^-,(p_1^++p_2^+)+(p_1^++p_2^+)-2p_1^+),((q_1^-+q_2^-)+(q_1^-+q_2^-)-2q_1^-,(q_1^++q_2^+)+(q_1^++q_2^+)-2q_1^+),((r_1^-+r_2^-)+(r_1^-+r_2^-)-2r_1^-,(r_1^++r_2^+)+(r_1^++r_2^+)-2r_1^+))=((2p_2^-,2p_2^+),(2q_2^-,2q_2^+),(2r_2^-,2r_2^+))=(2p_2,2q_2,2r_2)$

for $i=2,4,6,\ldots,2m-2$; $d_G(e_i)=(((p_1^-+p_2^-)+(p_1^-+p_2^-)-2p_2^-,(p_1^++p_2^+)+(p_1^++p_2^+)-2p_2^+),((q_1^-+q_2^-)+(q_1^-+q_2^-)-2q_2^-,(q_1^++q_2^+)+(q_1^++q_2^+)-2q_2^+),((r_1^-+r_2^-)+(r_1^-+r_2^-)-2r_2^-,(r_1^++r_2^+)+(r_1^++r_2^+)-2r_2^+))=((2p_1^-,2p_1^+),(2q_1^-,2q_1^+),(2r_1^-,2r_1^+))=(2p_1,2q_1,2r_1)$ $d_G(e_{2m-1})=(((p_1^-+p_2^-)+(p_1^-)-2p_1^-,(p_1^++p_2^+)+(p_1^+)-2p_1^+),((q_1^-+q_2^-)+(q_1^-)-2q_1^-,(q_1^++q_2^+)+(q_1^+)-2q_1^+),((r_1^-+r_2^-)+(r_1^-)-2r_1^-,(r_1^++r_2^+)+(r_1^+)-2r_1^+))=((p_2^-,p_2^+),(q_2^-,q_2^+),(r_2^-,r_2^+))=(p_2,q_2,r_2)$.

We observe that the adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular interval-valued neutrosophic graph. Also, we have $td_G(e_1)=(p_1+p_2,q_1+q_2,r_1+r_2)$ $td_G(e_i)=(2p_1+p_2,2q_1+q_2,2r_1+r_2)$ for $i=2,4,6,\ldots,2m-2$, $td_G(e_i)=(p_1+2p_2,q_1+2q_2,r_1+2r_2)$ for $i=3,5,7,\ldots,2m-3$ $td_G(e_{2m-1})=(p_1+p_2,q_1+q_2,r_1+r_2)$.

Therefore, the adjacent edges have distinct total degrees, hence G is a neighbourly edge totally irregular interval-valued neutrosophic graph. □

Theorem 4.8. *Let $G : (A, B)$ be an interval-valued neutrosophic graph such that $G^* : (V, E)$ is an even cycle of length $2m$. If the alternate edges have the same degrees of truth-membership, the same degrees of indeterminacy-membership and the same degrees of falsity-membership , then G is both neighbourly edge irregular interval-valued neutrosophic graph and neighbourly edge totally irregular interval-valued neutrosophic graph.*

Proof. Let $G : (A, B)$ be an interval-valued neutrosophic graph on $G^* : (V, E)$, an even cycle of length $2m$. Let $e_1, e_2, e_3, \dots, e_{2m}$ be the edges of cycle G^*. If the alternate edges have the same degrees of truth-membership, the same degrees of indeterminacy-membership and the same degrees of falsity-membership, such that

$$\begin{aligned}B(e_i) &= (T_B(e_i), I_B(e_i), F_B(e_i))\\ &= ((T_B^-(e_i), T_B^+(e_i)), (I_B^-(e_i), I_B^+(e_i)), (F_B^-(e_i), F_B^+(e_i)))\\ &= \begin{cases}(p_1, q_1, r_1), & \text{if } i \text{ is odd,}\\ (p_2, q_2, r_2), & \text{if } i \text{ is even}\end{cases} = \begin{cases}((p_1^-, p_1^+), (q_1^-, q_1^+), (r_1^-, r_1^+)), & \text{if } i \text{ is odd,}\\ ((p_2^-, p_2^+), (q_2^-, q_2^+), (r_2^-, r_2^+)), & \text{if } i \text{ is even}\end{cases}\end{aligned}$$

where $0 \le p_i + q_i + r_i \le 3$ for $i = 1, 2$ and $p_1 \ne p_2$ and $q_1 \ne q_2$ and $r_1 \ne r_2$, then for $i = 1, 3, 5, 7, \dots, 2m-1$: $d_G(e_i) = (((p_1^- + p_2^-) + (p_1^- + p_2^-) - 2p_1^-, (p_1^+ + p_2^+) + (p_1^+ + p_2^+) - 2p_1^+), ((q_1^- + q_2^-) + (q_1^- + q_2^-) - 2q_1^-, (q_1^+ + q_2^+) + (q_1^+ + q_2^+) - 2q_1^+), ((r_1^- + r_2^-) + (r_1^- + r_2^-) - 2r_1^-, (r_1^+ + r_2^+) + (r_1^+ + r_2^+) - 2r_1^+)) = ((2p_2^-, 2p_2^+), (2q_2^-, 2q_2^+), (2r_2^-, 2r_2^+)) = (2p_2, 2q_2, 2r_2)$,
for $i = 2, 4, 6, \dots, 2m$: $d_G(e_i) = (((p_1^- + p_2^-) + (p_1^- + p_2^-) - 2p_2^-, (p_1^+ + p_2^+) + (p_1^+ + p_2^+) - 2p_2^+), ((q_1^- + q_2^-) + (q_1^- + q_2^-) - 2q_2^-, (q_1^+ + q_2^+) + (q_1^+ + q_2^+) - 2q_2^+), ((r_1^- + r_2^-) + (r_1^- + r_2^-) - 2r_2^-, (r_1^+ + r_2^+) + (r_1^+ + r_2^+) - 2r_2^+)) = ((2p_1^-, 2p_1^+), (2q_1^-, 2q_1^+), (2r_1^-, 2r_1^+)) = (2p_1, 2q_1, 2r_1)$.

We observe that the adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular interval-valued neutrosophic graph. Also, we have $td_G(e_i) = (p_1 + 2p_2, q_1 + 2q_2, r_1 + 2r_2)$, for $i = 1, 3, 5, 7, \dots, 2m-1$, $td_G(e_i) = (2p_1 + p_2, 2q_1 + q_2, 2r_1 + r_2)$, for $i = 2, 4, 6, \dots, 2m$.

Therefore, the adjacent edges have distinct total degrees, hence G is a neighbourly edge totally irregular interval-valued neutrosophic graph. □

Theorem 4.9. *Let $G : (A, B)$ be an interval-valued neutrosophic graph on $G^* : (V, E)$, a cycle on $m (m \ge 4)$ vertices. If the degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges $e_1, e_2, e_3, \dots, e_m$ are $p_1, p_2, p_3, \dots, p_m$ such that $p_1 < p_2 < p_3 < \dots < p_m$ and $q_1, q_2, q_3, \dots, q_m$ such that $q_1 > q_2 > q_3 > \dots > q_m$ and $r_1, r_2, r_3, \dots, r_m$ such that $r_1 > r_2 > r_3 > \dots > r_m$, respectively, then G is both neighbourly edge irregular interval-valued neutrosophic graph and neighbourly edge totally irregular interval-valued neutrosophic graph.*

Proof. Let $G : (A, B)$ be an interval-valued neutrosophic graph on $G^* : (V, E)$, a cycle on $m(m \geq 4)$ vertices. Let $e_1, e_2, e_3, \dots, e_m$ be the edges of cycle G^* in that order. Let degrees of truth-membership, indeterminacy-membership and falsity-membership of the edges $e_1, e_2, e_3, \dots, e_m$ are $p_1, p_2, p_3, \dots, p_m$ such that $p_1 < p_2 < p_3 < ... < p_m$ and $q_1, q_2, q_3, \dots, q_m$ such that $q_1 > q_2 > q_3 > ... > q_m$ and $r_1, r_2, r_3, \dots, r_m$ such that $r_1 > r_2 > r_3 > ... > r_m$, respectively, where $p_i = (p_i^-, p_i^+)$ and $q_i = (q_i^-, q_i^+)$ and $r_i = (r_i^-, r_i^+)$ for $i = 1, 2, ..., m$, then $d_G(v_1) = ((p_1^- + p_m^-, p_1^+ + p_m^+), (q_1^- + q_m^-, q_1^+ + q_m^+), (r_1^- + r_m^-, r_1^+ + r_m^+)) = (p_1 + p_m, q_1 + q_m, r_1 + r_m)$, for $i = 2, 3, 4, 5, \dots, m$: $d_G(v_i) = ((p_{i-1}^- + p_i^-, p_{i-1}^+ + p_i^+), (q_{i-1}^- + q_i^-, q_{i-1}^+ + q_i^+), (r_{i-1}^- + r_i^-, r_{i-1}^+ + r_i^+)) = (p_{i-1} + p_i, q_{i-1} + q_i, r_{i-1} + r_i)$, $d_G(e_1) = ((p_2^- + p_m^-, p_2^+ + p_m^+), (q_2^- + q_m^-, q_2^+ + q_m^+), (r_2^- + r_m^-, r_2^+ + r_m^+)) = (p_2 + p_m, q_2 + q_m, r_2 + r_m)$, for $i = 2, 3, 4, 5, \dots, m-1$: $d_G(e_i) = ((p_{i-1}^- + p_{i+1}^-, p_{i-1}^+ + p_{i+1}^+), (q_{i-1}^- + q_{i+1}^-, q_{i-1}^+ + q_{i+1}^+), (r_{i-1}^- + r_{i+1}^-, r_{i-1}^+ + r_{i+1}^+)) = (p_{i-1} + p_{i+1}, q_{i-1} + q_{i+1}, r_{i-1} + r_{i+1})$, $d_G(e_m) = ((p_1^- + p_{m-1}^-, p_1^+ + p_{m-1}^+), (q_1^- + q_{m-1}^-, q_1^+ + q_{m-1}^+), (r_1^- + r_{m-1}^-, r_1^+ + r_{m-1}^+)) = (p_1 + p_{m-1}, q_1 + q_{m-1}, r_1 + r_{m-1})$.

We observe that the adjacent edges have distinct degrees. Hence G is a neighbourly edge irregular interval-valued neutrosophic graph.

$td_G(e_1) = (p_1 + p_2 + p_m, q_1 + q_2 + q_m, r_1 + r_2 + r_m)$, for $i = 2, 3, 4, 5, \dots, m-1$, $td_G(e_i) = (p_{i-1} + p_i + p_{i+1}, q_{i-1} + q_i + q_{i+1}, r_{i-1} + r_i + r_{i+1})$ for $i = 2, 3, 4, 5, \dots, m-1$, $td_G(e_m) = (p_1 + p_{m-1} + p_m, q_1 + q_{m-1} + q_m, r_1 + r_{m-1} + r_m)$.

We note that the adjacent edges have distinct total degrees. Hence G is a neighbourly edge totally irregular interval-valued neutrosophic graph. □

5. Conclusion

It is well known that graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in computer science, physical, biological and social systems. In general graphs theory has a wide range of applications in diverse fields. IVNG is an extended structure of a fuzzy graph which gives more precision, flexibility, and compatibility to the system when compared with the classical, fuzzy and neutrosophic models.

In this paper, we defined degree of an edge and total degree of an edge. Also, we introduced some types of edge irregular interval-valued neutrosophic graphs and properties of them.

A comparative study between neighbourly edge irregular interval-valued neutrosophic graphs and neighbourly edge totally irregular interval-valued neutrosophic graphs did. Also some properties of neighbourly edge irregular interval-valued neutrosophic graphs and neighbourly edge totally irregular interval-valued neutrosophic graphs studied.

In our future work, we will introduce strongly edge irregular interval-valued neutrosophic graphs and highly edge irregular interval-valued neutrosophic graphs. Also, we will study some properties of them.

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Strongly regular relation and n-Bell groups derived from it

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Abstract. A new strongly regular relation θ_n^* is defined on polygroup P such that the quotient P/θ_n^*, the set of all equivalence classes, is a Bell group for $n \in \{2, 3\}$.
Keywords: hypergroup, polygroup, regular and strongly regular equivalence relations, n-Bell, n-Engel, n-Kappe, n-Levi and n-Abelian groups.

1. Introduction

Hyperstructure theory was first initiated by Marty [15] in 1934. Let H be a non-empty set and $o : H \times H \longrightarrow P^*(H)$ be a hyperopration where $P^*(H)$ is the family of non-empty subset of H. The couple (H,o) is called a hypergroupoid. For any two non-empty subset A and B of H and $x \in H$, we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b, A \circ x = A \circ \{x\}$ and $B\ o\ x = B\ o\ \{x\}$. A hypergroupoid (H,o) is called semihypergroup if for all $a, b, c \in H$, we have $(a \circ b) \circ c = a \circ (b \circ c)$ which means that $\bigcup_{u\in a\circ b} u \circ c = \bigcup_{v\in b\circ c} a \circ v$ and hypergrupoid (H,o) is called qusihypergroup if for all a of H, we have $a \circ H = H \circ a = H$, which is called reproduction axiom. This axiom means that for any $x, y \in H$, there exist $u, v \in H$ such that $y \in x \circ u, y \in v \circ x$. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a qusihypergroup is called hypergroup.

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Definition 1.1 ([6])**.** *A polygroup is a hypergroup $\langle P, \cdot, e, ^{-1} \rangle$ where $e \in P, ^{-1}$ is a unitary operation on P, and the following axiom hold for all $x, y, z \in P$*

(*i*) $e \cdot x = x \cdot e = x$;

(*ii*) $x \in y \cdot z \Longrightarrow y \in x \cdot z^{-1} \Longrightarrow z \in y^{-1} \cdot x$.

Definition 1.2 ([5])**.** *Let $(H, \cdot)$ be a hypergroup and $\rho \subseteq H \times H$ be an equivalence relation. For non-empty subset A and B of H, we define $A \ \bar{\bar{\rho}} \ B$ if and only if $a \ \rho \ b$, for all $a \in A$ and $b \in B$. The relation ρ is called strongly regular on the left (on the right) if $x \ \rho \ y$, then $a \circ x \ \bar{\bar{\rho}} \ a \circ y$ ($x \circ a \ \bar{\bar{\rho}} \ y \circ b$, respectively), for all $x, y, a \in H$.*

Moreover, ρ is called strongly regular if it is strongly regular on the left and on the right.

Theorem 1.3 ([4])**.** *If $(H, \cdot)$ is a hypergroup and ρ is a strongly regular relation on H, then H/ρ is a group under the operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \quad \forall z \in x \cdot y.$$

For all $n \geq 1$, we define the relation β_n on a semihypergroup H, as follows, $a \ \beta_n \ b$, if and only if there exists $(x_1, \ldots, x_n)$ in H^n such that $\{a, b\} \subseteq \prod_{i=1}^{n} x_i$ and $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x); x \in H\}$, is the diagonal relation on H. This relation was introduced by Koskas [14]. Suppose that β^* is the transitive closure of β, the relation β^* is a strongly regular relation [4].

In [11], $\gamma = \bigcup_{n \geq 1} \gamma_n$, where γ_1 is the diagonal relation and for every integer $n > 1$, γ_n is the relation defined as follows, $x \ \gamma_n \ y$ if and only if there exists $(z_1, \cdots, z_n)$ in H^n and $\tau \in S_n$ such that $x \in \prod_{i=1}^{n} z_i$ and $y \in \prod_{i=1}^{n} z_{\tau(i)}$, where S_n is the symmetric group of order n. Suppose that γ^* is the transitive closure of γ. The relation γ^* is a strongly regular relation [11].

The relation β^* is the least equivalence relation on hypergroup H such that the quotient H/β^* is a group, while γ^* is the least equivalence relation on hypergroup H, such that the quotient H/γ^* is an abelian group.

In [12], $\tau_n = \bigcup_{m \geq 1} \tau_{m,n}$, where $\tau_{1,n}$ is the diagonal relation and for every integer $m > 1$, $\tau_{m,n}$ is the relation defined as follows, $x \ \tau_{m,n} \ y$ if and only if there exists $(z_1, \cdots, z_m)$ in H^m, and $\sigma \in S_m$ such that $\sigma(i) = i$, if $z_i \notin H^{(n)}$ such that $x \in \prod_{i=1}^{m} z_i$ and $y \in \prod_{i=1}^{m} z_{\sigma(i)}$, where

(1) $H^{(0)} = H$;

(2) $H^{(k+1)} = \{h \in H^{(k)} \mid xy \cap hyx \neq \emptyset \ ; \ x, y \in H^{(k)}\}$.

Clearly, for every integer $n \geq 1$, the relation τ_n is reflexive and symmetric.

Now, suppose that τ_n^* is the transitive closure of τ_n. The relation τ_n^* is strongly regular such that the quotient H/τ_n^* is a solubale group of the class at most $n + 1$.

In [1], $\nu_n = \bigcup_{m\geq 1} \nu_{m,n}$, where $\nu_{1,n}$ is the diagonal relation and for every integer $m > 1$, $\nu_{m,n}$ is the relation defined as follows, $x\ \nu_{m,n}\ y$ if and only if, there exists $(z_1, \cdots, z_m)$ in H^m and $\sigma \in S_m$ such that $\sigma(i) = i$, if $z_i \notin L_n(H)$ such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\sigma(i)}$, where

(1) $L_0(H) = H$;

(2) $L_{k+1}(H) = \{h \mid xy \cap hyx \neq \emptyset\ ;\ x \in L_k(H),\ y \in H\}$.

Clearly, for every integer $n \geq 1$, the relation ν_n is reflexive and symmetric.

Now, suppose that ν_n^* is the transitive closure of ν_n. The relation ν_n^* is strongly regular such that the quotient H/ν_n^* is a nilpotent group of the class at most $n+1$.

In [2], $\xi_{n,s} = \bigcup_{m\geq 1} \xi_{m,n,s}$, where $\xi_{1,n,s}$ is the diagonal relation and for every integer $m \geq 1$, $\xi_{m,n,s}$ is the relation defined as follows:

$x\ \xi_{m,n,s}\ y$ if and only if, there exists $(z_1, \cdots, z_m)$ in H^m and $\delta \in S_m$ such that $\delta(i) = i$ if $z_i \notin L_{n,s}(H)$ such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\delta(i)}$, where

(1) $L_{0,s}(H) = H$;

(2) $L_{k+1,s}(H) = \{h \mid xs \cap hsx \neq \emptyset\ ;\ x \in L_{k,s}(H)\}, \forall k \geq 0$,

for fix element $s \in H$.

Obviously, for every $n \geq 1$, the relation $\xi_{n,s}$ is reflexive and symmetric. Now let $\xi_{n,s}^*$ be the transitive closure of $\xi_{n,s}$.

In [2], the authors proved that the relation $\xi_{n,s}^*$ is strongly regular such that the quotient $H/\xi_{n,s}^*$ is an n-Engel group.

Let $n \neq 0, 1$ be an integer. A group G is said to be n-Bell if $[x^n, y] = [x, y^n]$ for all x and y in G, where $[x, y]$ is the commutator of x and y. The study of n-Bell groups was introduced by Kappe and Brandl in [3], [13] and it was also the subject of several papers, see for instance [8], [9], [10] and [18]. For example all of groups of finite exponent dividing n, groups of finite exponent dividing $n-1$, 2-Engel groups and n-Levi groups, are n-Bell groups (see, [9]).

In this paper, we define a new relation θ_n on a polygroup and then we show that θ_n^* is a strongly regular relation. In continue, we bring some results related to θ_n^* and one of the main result of this paper is about the relation of θ_n^* and n-Bell groups for $n = 2$ and 3. Also, if we set $\theta^* = \bigcap_{n\geq 1} \theta_n^*$, then we show that P/θ^* is a Bell group for any finite polygoup P.

In a polygroup P, the commutator of two elements x, y in P is defined by $[x, y] = \{t \mid t \in xyx^{-1}y^{-1}\}$. If $A \subseteq P$, then $[A, y] = \{t \mid t \in AyA^{-1}y^{-1}\}$.

Theorem 1.4 ([2], Theorem 2.2)**.** *Let P be a polygroup. Then, for all $x, y, h, \in P$, $\{h \mid xy \cap hyx \neq \emptyset\} = \{h \mid h \in xyx^{-1}y^{-1}\}$.*

Remark 1.5. Let P be a polygroup. Then, for all $x, y, h, \in P$ and $n \in N$, $\{h \mid x^n y \cap hyx^n \neq \emptyset\} = \{h \mid h \in x^n yx^{-n}y^{-1}\}$.

Theorem 1.6 ([2], Theorem 2.10)**.** *$H/\xi_{n,s}^*$ is an n-Engel group.*

Theorem 1.7 ([1], Theorem 2.9)**.** *H/ν_n^* is a nilpotent group of the class at most $n+1$.*

2. New strongly regular relation θ_n^*

Now, we introduce a new strongly regular relation θ_n^* on a polygroup P.

In the whole of this paper, P is a polygroup and S_n is symmetric group.

Definition 2.1. *Let P be a polygroup. For fix elements $x, y \in P$, we define:*

(1) $L_{0,x,y}(P) = P$;

(2) $L_{n+1,x,y}(P) = \{h \mid h \in L_{n,x,y}(P), x^{n+1}y \cap hyx^{n+1} \neq \emptyset\}$.

Let $\theta_n = \bigcup_{m \geq 1} \theta_{m,n}$ where $\theta_{1,n}$ is diagonal relation and for every integer $m \geq 1, \theta_{m,n}$ is relation defined as follows:

$x\ \theta_{m,n}\ y$ if and only if, there exists $(z_1, \cdots, z_m)$ in P^m and $\zeta \in S_m$ if, $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$, then $\zeta(i) = i$ and $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\zeta(i)}$.

Clearly, θ_n is reflexive and symmetric. Let θ_n^* be the transitive closure of θ_n.

Theorem 2.2. *For every $n \in \mathbb{N}$, the relation θ_n^* is strongly regular relation.*

Proof. Suppose that $n \in \mathbb{N}$. Clearly, θ_n^* is an equivalence relation. In order to prove that it is strongly regular. First we have to show that if $x\ \theta_n\ y$, then $x \cdot z\ \overline{\overline{\theta}}_n\ y \cdot z$, $z \cdot x\ \overline{\overline{\theta}}_n\ z \cdot y$, for every $z \in P$. Suppose that $x\theta_n y$. Then, there exists $m \in \mathbb{N}$ such that $x\ \theta_{m,n}\ y$. Hence, there exists $(z_1, \cdots, z_m) \in P^m,\ \zeta \in S_m$ with $\zeta(i) = i$ if $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$ such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\zeta(i)}$. Suppose that $z \in P$. We have $x \cdot z \ \subseteq (\prod_{i=1}^m z_i) \cdot z, y \cdot z \subseteq (\prod_{i=1}^m z_{\zeta(i)}) \cdot z$. Now, suppose that $z_{m+1} = z$ and we define the permutation $\zeta' \in S_{m+1}$ as follows:

$$\begin{cases} \zeta'(i) = \zeta(i), & \text{for all } 1 \leq i \leq m, \\ \zeta'(m+1) = m+1. \end{cases}$$

Thus, $x \cdot z \subseteq \prod_{i=1}^{m+1} z_i, y \cdot z \subseteq \prod_{i=1}^{m+1} z_{\zeta'(i)}$. such that $\zeta'(i) = i$ if $z_i \notin L_{n,x,y}(P)$ and $z_i^{-1} \notin L_{n,y,x}(P)$. Therefore, $x \cdot z\ \overline{\overline{\theta}}_n\ y \cdot z$. Similary, we have $z \cdot x\ \overline{\overline{\theta}}_n\ z \cdot y$. Now, if $x\ \theta_n^*\ y$, then, there exists $k \in \mathbb{N}$ and $(x = u_0, u_1, \cdots, u_k = y) \in P^{k+1}$ such that $x = u_0\ \theta_n\ u_1\ \theta_n \cdots \theta_n\ u_{k-1}\ \theta_n\ u_k = y$. Hence, we obtain $x \cdot z = u_0 \cdot z\ \overline{\overline{\theta}}_n^*\ u_1 \cdot z\ \overline{\overline{\theta}}_n^*\ u_2 \cdot z\ \overline{\overline{\theta}}_n^* \cdots \overline{\overline{\theta}}_n^*\ u_{k-1} \cdot z\ \overline{\overline{\theta}}_n^*\ u_k \cdot z = y \cdot z$ and so $x \cdot z\ \overline{\overline{\theta}}_n^*\ y \cdot z$. Similarly, we can prove that $z \cdot x\ \overline{\overline{\theta}}_n^*\ z \cdot y$, therefore $\overline{\overline{\theta}}_n^*$ is strongly regular relation on P. □

Proposition 2.3. *For every $n \in \mathbb{N}$, we have $\theta_{n+1}^* \subseteq \theta_n^*$.*

Proof. Let $x\ \theta_{n+1}\ y$, so, there exists $m \in \mathbb{N}$ and $(z_1, \cdots, z_m) \in P^m$ and $\zeta \in S_m$ such that $\zeta(i) = i$ if $z_i \notin L_{n+1,x,y}(P)$ and $z_i^{-1} \notin L_{n+1,y,x}(P)$, such that $x \in \prod_{i=1}^m z_i$ and $y \in \prod_{i=1}^m z_{\zeta(i)}$. Now, let $\zeta_1 = \zeta$, since $L_{n+1,x,y}(P) \subseteq L_{n,x,y}(P)$ and $L_{n+1,y,x}(P) \subseteq L_{n,y,x}(P)$, we have $x\ \theta_n\ y$. □

Corollary 2.4. *If P is a commutative hypergroup, then $\beta^* = \theta_n^* = \xi_n^* = \nu_n^* = \gamma^*$.*

Definition 2.5 ([13])**.** *Let G be a group and n be an integer. The n-Bell center of G denoted by B_n and defined as follows:*

$$B_n = B(G,n) = \{x \in G \mid [x^n, y] = [x, y^n], ; \ \forall y \in G\}.$$

Clearly, $B(G,0) = B(G,1) = G$, and easy to see that $B(G,2)$ and $B(G,3)$ are subgroup of G.

Remark 2.6. For every integer n, a group is n-Bell if $B(G,n) = G$.

Theorem 2.7. *If P is a polygroup and ρ is a strongly regular relation on P, then for fix elements $x, y \in P$;*

$$L_{n+1,\bar{x},\bar{y}}(\frac{P}{\rho}) = \{[\bar{x}^{n+1}, \bar{y}]\},$$

where $\bar{x}, \bar{y}$ are the classes of x,y with respect to ρ.

Proof. The proof follows from definition of commutator of two elements in a polygroup, Theorem 1.4 and Remark 1.5. $\square$

3. n-Bell groups derived from polygroups for $n \in \{2,3\}$

In this section, we obtain an n-Bell group derived from polygroup for $n = 2, 3$, and then we propose an open problem related to n-Bell groups.

Theorem 3.1. *Let P be a polygroup. Then, for $n \in \{2,3\}$, P/θ_n^* is an n-Bell group.*

Proof. Let $G = P/\theta_n^*$. For $n \in \{2,3\}$, we have $B(G,n) \leq G$. By Remark 2.6, it is enough to prove that $G \leq B(G,n)$. For this we should show that for every $\bar{h} \in L_{n,\bar{x},\bar{y}}(G)$ we have $\bar{h}^{-1} \in L_{n,\bar{y},\bar{x}}(G)$ and is obvious by Theorem 2.7. $\square$

Definition 3.2 ([16])**.** *A group G is called an Engel group if, for each ordered pair (x, y) of elements in G, there is a positive integer $n = n(x, y)$, such that $[x,_n y] = 1$.*

Theorem 3.3 ([17])**.** *Let G be a group. Then*

(a) *G is a n-Bell group if and only if G is a $(1-n)$-Bell group;*

(b) *G is a 2-Bell group if and only if G is a 2-Engel group;*

(c) *G is a 3-Bell group if and only if G is a 3-Engel group satisfying the identity $[x, y, y]^3 = 1$, for all $x, y \in G$. In addition G has nilpotent of class at most 4.*

Definition 3.4. *Let H_1 and H_2 be two hypergroups (polygroups), and ρ_1 and ρ_2 be two strongly regular relations. If H_1/ρ_1 and H_2/ρ_2 are isommorphism groups, then we say that ρ_1 is "the same" property to ρ_2.*

Remark 3.5. According to the above definition, θ_2^* is the same property to $\xi_{2,s}^*$, by Theorem 3.1, 3.3 and apply Theorem 1.6, and θ_3^* is the same property to $\xi_{3,s}^*$ and ν_3^*, by Theorem 3.1, 3.3 and apply Theorem 1.6 and 1.7.

Example 3.6 ([2])**.** Let H be $\{e, a, b, c, d, f, g\}$. Consider the non-commutative polygroup $(H, \cdot)$, defined on H as follows: It is easy to see that $H/\beta^* \cong S_3$ (for

.	e	a	b	c	d	f	g
e	e	a	b	c	d	f,g	f,g
a	a	e	d	f,g	b	c	c
b	b	f,g	e	d	c	a	a
c	c	d	f,g	e	a	b	b
d	d	c	a	b	f,g	e	e
f	f,g	b	c	a	e	d	d
g	f,g	b	c	a	e	d	d

more details, see [7]). Since S_3 is not nilpotent, we conclude that $\beta^* \neq \nu_n^*$, hence H/ν_n^* is an abelian group of order less than 6 and the class of nilpotency of H/ν_n^* is one for all $n \in \mathbb{N}$ [1], besides, S_3 is not Engel and $H/\xi_{n,s}^* \subseteq H/\beta^* \cong S_3$, then it concluded $H/\xi_{n,s}^*$ is an abelian group of order less than 6 and H/ξ^* is 1-Engel group. Then, H/θ_2^* is not 2-Bell or 3-Bell group, by apply the Remark 3.5.

Remark 3.7. We know that $B(G, n)$ is called the n-Bell center of G. It is open problem whether the n-Bell center always forms a subgroup. But, it is shown that $B(G, 2)$ is characteristic subgroup of all right 2-Engel elements and $B(G, 3)$ is characteristic subgroup of G which is nilpotent of class at most 4 (see, [13]).

Hence, according to above remark, we can put the following open problem:

Open Problem 3.8. *Let H be non-commutative polygroup, for all $n \geq 4$, is H/θ_n^* a n-Bell group?*

4. On Bell groups derived from finite polygroup

In this section, we introduce a strongly regular relation θ^* on finite polygroup P such that P/θ^* is a Bell group.

Definition 4.1. *Let P be a finite polygroup. Then, we define the relation θ^* on P by $\theta^* = \bigcap_{n\geq 1} \theta_n^*$.*

Definition 4.2. *An equivalence relation ρ on a finite polygroup P, is called Bell if and only if its derived group P/ρ is a Bell group.*

Example 4.3. θ_2^* and θ_3^* are Bell relations. By using the Remark 3.5, and Example 3.3 in [2], Bell relations θ_2^* and θ_3^* are the same with Engel relations $\xi_{2,s}^*$ and $\xi_{3,s}^*$.

Theorem 4.4. (*a*) *The relation θ^* is a strongly regular relation on a finite polygroup P.*

(*b*) *P/θ^* is a Bell group.*

Proof. (a) Since $\theta^* = \bigcap_{n\geq 1}\theta_n^*$, it is easy to see that θ^* is strongly regular relation on P.

(b) By using Proposition 2.3, we conclude that there exists $k \in \mathbb{N}$ $(k \geq 1)$ such that $\theta_{k+1}^* = \theta_k^*$ and so $\theta^* = \theta_m^*$ for some $m \in \mathbb{N}$. □

5. Transitivity of θ^*

Definition 5.1. *Let X be a non-empty subset of P and x, y are fix elements of P. Then, we say that X is a θ-part of P if for every $t \in \mathbb{N}$, $(z_1, \cdots, z_t) \in P^t$ and for every $\zeta \in S_t$ if $z_i \notin \bigcup_{n\geq 1} L_{n,x,y}(P)$, $z_i^{-1} \notin \bigcup_{n\geq 1} L_{n,y,x}(P)$, then $\zeta(i) = i$, then*

$$\prod_{i=1}^{t} z_i \cap X \neq \emptyset \Longrightarrow \prod_{i=1}^{t} z_{\zeta(i)} \subseteq X.$$

Theorem 5.2. *Let X be a non-empty subset of a polygroup P. Then the following conditions are equivalent:*

(1) *X is a θ-part of P.*

(2) $x \in X,\ x\ \theta\ y \Longrightarrow y \in X.$

(3) $x \in X,\ x\ \theta^*\ y \Longrightarrow y \in X.$

Proof. (1) $\Longrightarrow$ (2) : If $(x, y) \in P^2$ is a pair, such that $x \in X,\ x\ \theta\ y$, then there exist $(z_1, \cdots, z_t) \in P^t$ such that $x \in \prod_{i=1}^t z_i \cap X,\ y \in \prod_{i=1}^t z_{\zeta(i)}$ and $\zeta(i) = i$ if $z_i \notin \bigcup_{n\geq 1} L_{n,x,y}(P),\ z_i^{-1} \notin \bigcup_{n\geq 1} L_{n,y,x}(P)$. Since X is a θ-part of P, we have $\prod_{i=1}^t z_{\zeta(i)} \subseteq X$ and so $y \in X$.

(2) $\Longrightarrow$ (3) : Suppose that $(x, y) \in P^2$ is a pair, such that $x \in X,\ x\ \theta^*\ y$, then there exist $(z_1, \cdots, z_t) \in P^t$ such that $x = z_0\ \theta\ z_1\ \theta\ \cdots\ \theta\ z_t = y$. Now, by using (2) "t" times iterated then, we obtain $y \in X$.

(3) $\Longrightarrow$ (1) : Suppose that $x \in \prod_{i=1}^t z_i \cap X$ and $\zeta \in S_t$ such that $\zeta(i) = i$ if $z_i \notin \bigcup_{n\geq 1} L_{n,x,y}(P),\ z_i^{-1} \notin \bigcup_{n\geq 1} L_{n,y,x}(P)$. Let $y \in \prod_{i=1}^t z_{\zeta(i)}$. Since $x\ \theta\ y$ by (3), we have $y \in X$. Consequently, $\prod_{i=1}^t z_{\zeta(i)} \subseteq X$ and so X is a θ-part. □

Theorem 5.3. *The following conditions are equivalent:*

(1) *For every $a \in P$, $\theta(a)$ is a θ-part of P.*

(2) *θ is transitive.*

Proof. (1) $\Longrightarrow$ (2) : Suppose that $x\ \theta^*\ y$. Then, there is $(z_1, \cdots, z_t) \in P^t$ such that $x = z_0\ \theta\ z_1\ \theta\ \cdots\ \theta\ z_t = y$. Since $\theta(z_i)$, for all $0 \leq i \leq t$, is a θ-part, we have $z_i \in \theta(z_{i-1})$, for all $0 \leq i \leq t$, thus $y \in \theta(x)$, which means that $x\ \theta\ y$.

(2) $\Longrightarrow$ (1) : Suppose that $x \in P$, $z \in \theta(x)$ and $z\ \theta\ y$. By transitivity of θ, we have $y \in \theta(x)$. Now, according to Theorem 5.2, $\theta(x)$ is a θ-part of P. □

Definition 5.4. *Let A be a non-empty subset of a polygroup P. The intersection of all θ-part, which contain A is called θ-closure of A in P and it will be denoted by $K(A)$.*

In follows, we determine the set $Z(A)$.

Assume that $Z_1(A) = A$ and $Z_{n+1}(A) = \{x \in P | \exists (z_1, \cdots, z_t) \in P^t,\ x \in \prod_{i=1}^t z_i,\ \exists \zeta \in S_t$ if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P),\ \&\ z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$ then, $\zeta(i) = i$ and $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(A) \neq \emptyset\}$.

We denote $Z(A) = \bigcup_{n \geq 1} Z_n(A)$.

Theorem 5.5. *For any non-empty subset A of P, the following statements hold:*

(1) *$Z(A) = K(A)$;*

(2) *$K(A) = \bigcup_{a \in A} K(a)$.*

Proof. (1) It is enough to prove that:

(a) $Z(A)$ is a θ-part.

(b) If $A \subseteq B$ and B is a θ-part, then $Z(A) \subseteq B$.

In order to (a), suppose that $\prod_{i=1}^t z_i \cap Z(A) \neq \emptyset$ and $\zeta \in S_t$ such that $\zeta(i) = i$ if $z_i \notin \bigcup_{n \geq 1} L_{n,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{n \geq 1} L_{n,y,x}(P)$. Therefore, there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^t z_i \cap Z(A) \neq \emptyset$, where it follows that $\prod_{i=1}^t z_{\zeta(i)} \subseteq Z_{n+1}(A) \subseteq Z(A)$.

Now, we prove (b) by induction on n. We have $Z_1(A) = A \subseteq B$.

Suppose that $Z_n(A) \subseteq B$. We prove that $Z_{n+1}(A) \subseteq B$. If $z \in Z_{n+1}(A)$, then $z \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$, and also $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(A) \neq \emptyset$. Therefore, $\prod_{i=1}^t z_{\zeta(i)} \cap B \neq \emptyset$ and hence $z \in \prod_{i=1}^t z_i \subseteq B$.

(2) It is clear that for all $a \in A$, $K(a) \subseteq K(A)$. By part (1), we have $K(A) = \bigcup_{n \geq 1} Z_n(A)$ and $Z_1(A) = A = \bigcup_{a \in A} a$. It is enough to prove that $Z_n(A) = \bigcup_{a \in A} Z_n(a)$, for all $n \in \mathbb{N}$. We follow by induction on n. Suppose it is true for n. We prove that $Z_{n+1}(A) = \bigcup_{a \in A} Z_{n+1}(a)$. If $z \in Z_{n+1}(A)$, then $z \in \prod_{i=1}^t z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s \geq 1} L_{s,x,y}(P)$ and $z_i^{-1} \notin \bigcup_{s \geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(A) \neq \emptyset$. By the hypothesis of induction $\prod_{i=1}^t z_{\zeta(i)} \cap Z_n(a') \neq \emptyset$, for some $a' \in A$. therefore, $z \in Z_{n+1}(a')$, and so $Z_{n+1}(A) \subseteq \bigcup_{a \in A} Z_{n+1}(a)$. Hence, $K(A) = \bigcup_{a \in A} K(a)$. □

Theorem 5.6. *The following relation is equivalence relation on P,*

$$x \ Z \ y \Longleftrightarrow x \in Z(y),$$

for every $(x, y) \in P^2$, where $Z(y) = Z(\{y\})$.

Proof. It is easy to see that Z is reflexive and transitive. For the proof of symmetric of relation Z, it is enough that we prove the following statements:

(1) For all $n \geq 2$ and $x \in H$, $Z_n(Z_2(x)) = Z_{n+1}(x)$.

(2) $x \in Z_n(y)$ if and only if $y \in Z_n(x)$.

We prove (1) by induction on n. Suppose that $z \in Z_2(Z_2(x))$. Then, $z \in \prod_{i=1}^{t} z_i$ and there is $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s\geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_2(x) \neq \emptyset$, thus $z \in Z_3(x)$. if $z \in Z_{n+1}(Z_2(x))$, then $z \in \prod_{i=1}^{t} z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s\geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_n(Z_2(x)) \neq \emptyset$. By hypothesis of induction, we have $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_{n+1}(x) \neq \emptyset$ and so $z \in Z_{n+2}(x)$.

Now, we prove (2) by induction on n, too. It is clear that $x \in Z_2(y)$ if and only if $y \in Z_2(x)$. Then $x \in \prod_{i=1}^{t} z_i$ and there exists $\zeta \in S_t$ such that $\zeta(i) = i$, if $z_i \notin \bigcup_{s\geq 1} L_{s,x,y}(P)$, $z_i^{-1} \notin \bigcup_{s\geq 1} L_{s,y,x}(P)$ and also $\prod_{i=1}^{t} z_{\zeta(i)} \cap Z_n(y) \neq \emptyset$. Suppose that $b \in \prod_{i=1}^{t} z_{\zeta(i)} \cap Z_n(y)$, then, we have $y \in Z_n(b)$. From $x \in \prod_{i=1}^{t} z_i \cap Z_1(x)$ and $b \in \prod_{i=1}^{t} Z_{\zeta(i)}$ we conclude that $b \in Z_2(x)$. Therefore, $y \in Z_n(Z_2(x)) = Z_{n+1}(x)$. □

6. Conclusion

In this paper, we have introduced a new strongly regular relation θ_n^* on a polygroup P and we have shown that P/θ_n^* is a n-Bell group for $n = 2, 3$.

We defined the same relation structure between two strongly regular relations on a hypergroup (polygroup), and we bring an open problem relate to n-Bell group of P/θ_n^*. In continue, we obtained some results related to θ_n^*. We try to answer the mention open problem and in this regard, for the other research work.

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Applications of extended Hadamard K-fractional integral

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Abstract. In this paper, we use the extended Hadamard k-fractional integral to obtain some new fractional integral inequalities by introducing the new parameters s and k. These extended fractional integral inequalities also hold true for usual Hadamard fractional integral when we substitute k is equal to one and s is equal to zero.

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1. Introduction

Fractional calculus has been extensively studied and investigated in the last two decades. Its new results and their applications have emerged as a very effective and powerful tool for many mathematical problems of science and engineering. Recently, fractional derivatives and integrals have employed in many fluid problems to get more accurate and valid results. These fractional operators are used in finance, biophysics, electrochemistry, computed tomography, engineering, control theory, geological surveying, thermodynamics, hydrology, electric conductance of biological systems, statistical mechanics, astrophysics, mathematical physics, and also used for the mathematical modelling of viscoelastic materials.

Diaz and Pariguan [13] give new direction to fractional calculus by introducing k-gamma function and k-beta function which are the extensions of classical gamma and beta functions. So, k-fractional calculus version was introduced. Many results of fractional calculus were extended. Farid and Habibullah [14] defined the Hadamard k-fractional integral. Azam et. al. [3], introduced the extended Hadamard k-fractional integrals of order α.

Let f be continuous on $[0, \infty]$ and α, kϵ , sϵ. Then, $\forall x > a > 0$

$$
{}_k^s I_H^\alpha [f(x)] = \frac{1}{k\Gamma_k(\alpha)} \int_a^x [\log \frac{x}{\tau}]^{\frac{\alpha}{k}-1} (\frac{\tau}{x})^s f(\tau) \frac{d\tau}{\tau}. \tag{1}
$$

The objective of this work is to extend some existing fractional integral inequalities by using extended Hadamard k-fractional integral [3]. New parameter s and k are introduced. A few mathematicians have devoted their efforts to generalize and refine the fractional integral inequalities in the recent years due to their applications in different fields of science and technology. We may refer the interested reader to [1, 3, 4, 6, 10, 14, 15, 16, 17, 18].

2. Our some new results and discussions

Now, extension of some fractional integral inequalities using the equation (1) are given below

Theorem 2.1. *Let* $(g_i)_{i=1,2,,...n}$ *be positive increasing function on* $[1, \infty)$, *and* α, k ,s. *Then* $\forall z > 1$

$$
{}_k^s I_H^\alpha ({}_{i=1}\prod^n g_i) \geq [{}_k^s I_H^\alpha (I)(z)]^{1-\alpha} {}_{i=1}\prod^n {}_k^s I_H^\alpha (g_i)(z). \tag{2}
$$

Proof. We prove this by induction. For $n = 1$

$$
{}_k^s I_H^\alpha g_1(z) \geq_k^s I_H^\alpha g_1(z), \tag{3}
$$

which is true. For $n = 2$

$$ {}^{s}_{k}I^{\alpha}_{H}(g_1 g_2)(z) \geq [{}^{s}_{k}I^{\alpha}_{H}(l)(z)]^{-1}\,{}^{s}_{k}I^{\alpha}_{H}g_1(z)\,{}^{s}_{k}I^{\alpha}_{H}g_2(z), \tag{4} $$

which is also true.

Suppose that the statement is true for $n = k - 1$

$$ {}^{s}_{k}I^{\alpha}_{H}({}_{i=1}\prod^{n-1} g_i)(z) \geq [{}^{s}_{k}I^{\alpha}_{H}(l)(z)]^{2-n}\,{}_{i=1}\prod^{n-1}{}^{s}_{k}I^{\alpha}_{H}(g_i)(z). \tag{5} $$

Now, $({}_{i=1}\prod^{n-1} g_i)(z)$ is an increasing function on $[1, \infty]$ because of $(g_i)_{i=1,2,\ldots,n}$. So, we can get

$$ {}^{s}_{k}I^{\alpha}_{H}({}_{i=1}\prod^{n} g_i)(z) \geq {}^{s}_{k}I^{\alpha}_{H}({}_{i=1}\prod^{n} g_i g_n)(z)[{}^{s}_{k}I^{\alpha}_{H}(l)(z)]^{-1}\,{}^{s}_{k}I^{\alpha}_{H} $$
$$ ({}_{i=1}\prod^{n-1} g_i)(z)\,{}^{s}_{k}I^{\alpha}_{H}(g_n)(z). $$

using (5), we get

$$ {}^{s}_{k}I^{\alpha}_{H}({}_{i=1}\prod^{n} g_i)(z) \geq [{}^{s}_{k}I^{\alpha}_{H}(l)(z)]^{-1}[{}^{s}_{k}I^{\alpha}_{H}(l)(z)]^{2-n}({}_{i=1}\prod^{n-1} g_i)(z)\,{}^{s}_{k}I^{\alpha}_{H}({}_{i=1}(g_n)(z). $$

Hence, we get

$$ {}^{s}_{k}I^{\alpha}_{H}({}_{i=1}\prod^{n} g_i)(z) \geq [{}^{s}_{k}I^{\alpha}_{H}(l)(z)]^{1-n}\,{}_{i=1}\prod^{n}{}^{s}_{k}I^{\alpha}_{H}(g_i)(z). \tag{6} $$ □

Theorem 2.2. *For integrable function g on $[1, \infty]$. Assume that:*

A_1, There exist two integrable functions Ψ_1 and Ψ_2 on $[1, \infty]$ such that

$$ \Psi_2(r) \geq g(r) \geq \Psi_1(r), \qquad , \forall r \epsilon [1, \infty]. \tag{7} $$

Then, for $r > 1$ $s\epsilon$ and $\alpha, \beta, k\epsilon$,

$$ {}^{s}_{k}I^{\beta}_{H}\Psi_1(r)\,{}^{s}_{k}I^{\alpha}_{H}g_r + {}^{s}_{k}I^{\alpha}_{H}\Psi_2(r)\,{}^{s}_{k}I^{\beta}_{H}g_r \geq {}^{s}_{k}I^{\alpha}_{H}\Psi_2(r)\,{}^{s}_{k}I^{\beta}_{H}\Psi_1(r) + {}^{s}_{k}I^{\alpha}_{H}g(r)\,{}^{s}_{k}I^{\beta}_{H}g_r. \tag{8} $$

Proof. From A_1, $\forall, p \geq 1, q \geq 1$, we have

$$ [\Psi_2(p) - g(p)][g(q) - \Psi_1(q)] \geq 0. \tag{9} $$

Therefore,

$$ \Psi_2(p)g(q) + g(p)\Psi_1(q) \geq \Psi_1(q)\Psi_2(p) + g(p)g(q). \tag{10} $$

Multiplying (10), by

$$ \frac{1}{k\Gamma_k(\alpha)}[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^{s}\frac{1}{p}, \tag{11} $$

and integrating w.r.t. p on $[1,\infty]$

$$
\begin{aligned}
&g(q)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s\Psi_2(p)\frac{dp}{p}]\\
&\qquad+\Psi_1(q)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s g(p)\frac{dp}{p}]\\
&\qquad\geq\Psi_2(w)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s\Psi_2(p)\frac{dp}{p}]\\
&\qquad+g(w)\frac{1}{k\Gamma_k(\alpha)}\int_1^r[[\log\frac{r}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{r}]^s g(p)\frac{dp}{p}].
\end{aligned}\tag{12}
$$

Using the result (1),

$$
g(q)^s_kI^\alpha_H\Psi_2(r)+\Psi_1(q)^s_kI^\alpha_H g_r\geq\Psi_1(q)^s_kI^\alpha_H\Psi_2(r)+g(q)^s_kI^\alpha_H g(r).\tag{13}
$$

Multiplying (13) by

$$
\frac{1}{k\Gamma_k(\beta)}[\log\frac{r}{p}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\frac{1}{q},\tag{14}
$$

and integrating w.r.t. q on $[1,\infty]$,

$$
\begin{aligned}
&{}^s_kI^\alpha_H\Psi_2(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s g(q)\frac{dq}{q}]\\
&+{}^s_kI^\alpha_H g(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\Psi_1(q)\frac{dq}{q}]\\
&\geq{}^s_kI^\alpha_H\Psi_2(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\Psi_1(q)\frac{dq}{q}]\\
&+{}^s_kI^\alpha_H g(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s\Psi_1(q)\frac{dq}{q}]\\
&+{}^s_kI^\alpha_H g(r)\frac{1}{k\Gamma_k(\beta)}\int_1^r[[\log\frac{r}{q}]^{\frac{\beta}{k}-1}[\frac{q}{r}]^s g(q)\frac{dq}{q}].
\end{aligned}\tag{15}
$$

Using the equation (1), we get (8). □

Theorem 2.3. *Let f,g and h are positive valued and continuous functions on $[0.\infty]$ such that*

$$
[g(v)-g(w)](\frac{f(w)}{h(w)}-\frac{f(v)}{h(v)})\geq 0,\tag{16}
$$

for all $v,w\epsilon(0,z)$. Then $\forall z>0, s\epsilon,\alpha,\beta,k\epsilon,$

$$
\frac{{}^s_kI^\alpha_H[gh](z){}^s_kI^\beta_H[f](z)+{}^s_kI^\alpha_H[f](z){}^s_kI^\beta_H[gh](z)}{{}^s_kI^\alpha_H[fg](z){}^s_kI^\beta_H[h](z)+{}^s_kI^\alpha_H[h](z){}^s_kI^\beta_H[fg](z)}\geq 1.\tag{17}
$$

Proof. Multiplying

$$[g(v)-g(w)]\left(\frac{f(w)}{h(w)}-\frac{f(v)}{h(v)}\right)\geq 0, \tag{18}$$

by $h(v)h(w)$, we can get

$$g(v)f(w)h(v)-g(v)f(v)h(w)-g(w)f(w)h(w)+g(w)f(v)h(w)\geq 0. \tag{19}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,\infty]$

$$\begin{aligned}\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(v)f(w)h(v)\frac{dv}{v}\\ -\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(v)f(v)h(w)\frac{dv}{v}\\ -\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(w)f(w)h(v)\frac{dv}{v}\\ +\frac{1}{k\Gamma_k(\alpha)}\int_1^z[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s g(w)f(v)h(w)\frac{dv}{v}\geq 0.\end{aligned} \tag{20}$$

Using the equation (1),

$$\begin{aligned}&f(w)^s_kI^\alpha_H[gh](z)-h(w)^s_kI^\alpha_H[fg](z)-f(w)^s_kI^\alpha_H[h](z)\\ &+h(w)g(w)^s_kI^\alpha_H[f](z)\geq 0.\end{aligned} \tag{21}$$

Multiplying by $\frac{1}{k\Gamma_k(\beta)}[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s\frac{1}{w}$, and integrating w.r.t. w on $[1,\infty]$

$$\begin{aligned}{}^s_kI^\alpha_H[gh](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s f(w)\frac{dw}{w}\\ -{}^s_kI^\alpha_H[fg](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s h(w)\frac{dw}{w}\\ -{}^s_kI^\alpha_H[h](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s f(w)g(w)\frac{dw}{w}\\ +{}^s_kI^\alpha_H[f](z)\frac{1}{k\Gamma_k(\beta)}\int_1^z[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s h(w)g(w)\frac{dw}{w}\geq 0.\end{aligned} \tag{22}$$

Using the equation (1), we get

$$\begin{aligned}&{}^s_kI^\alpha_H[gh](z)^s_kI^\beta_H[f](z)-{}^s_kI^\alpha_H[fg](z)^s_kI^\beta_H[h](z)\\ &-{}^s_kI^\alpha_H[h](z)^s_kI^\beta_H[fg](z)+{}^s_kI^\alpha_H[f](z)^s_kI^\beta_H[gh](z)\geq 0.\end{aligned} \tag{23}$$

Which gives (17). □

Corollary 2.1. *Let f, g and h are positive valued and continuous functions on $[0.\infty]$ such that*

$$[g(v)-g(w)]\left(\frac{f(w)}{h(w)}-\frac{f(v)}{h(v)}\right)\geq 0, \tag{24}$$

for all $v, w\epsilon(0,z)$. *Then* $\forall z>0, s\epsilon, \alpha, k\epsilon$,

$$\frac{{}_{k}^{s}I_{H}^{\alpha}[f](z)}{{}_{k}^{s}I_{H}^{\beta}[h](z)} \geq \frac{{}_{k}^{s}I_{H}^{\alpha}[fg](z)}{{}_{k}^{s}I_{H}^{\beta}[gh](z)}. \tag{25}$$

Proof. By substituting $\beta=\alpha$ in (17), we get

$$\frac{{}_{k}^{s}I_{H}^{\alpha}[f](z){}_{k}^{s}I_{H}^{\alpha}[gh](z)}{{}_{k}^{s}I_{H}^{\beta}[h](z){}_{k}^{s}I_{H}^{\alpha}[fg](z)} \geq 1. \tag{26}$$

Which gives (25). □

Theorem 2.4. *For integrable function* g *on* $[1,\infty]$ *and constants* $l \geq m \geq 0$, $l \neq 0$. *Let* (A_1) *holds. Then, for any* $z>1$, $s\epsilon$ *and* $\alpha, \beta, k, q\epsilon$, *we have*

$$\begin{aligned} &{}_{k}^{s}I_{H}^{\alpha}(\Psi_2-g)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_{k}^{s}I_{H}^{\alpha}g(z) \\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}{}_{k}^{s}I_{H}^{\alpha}\Psi_2(z)+\frac{m-l}{l}q^{\frac{m}{l}}{}_{k}^{s}I_{H}^{\alpha}I(z), \end{aligned} \tag{27}$$

$$\begin{aligned} &{}_{k}^{s}I_{H}^{\alpha}(g-\Psi_1)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_{k}^{s}I_{H}^{\alpha}\Psi_1(z) \\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}{}_{k}^{s}I_{H}^{\alpha}g(z)+\frac{l-m}{l}q^{\frac{m}{l}}{}_{k}^{s}I_{H}^{\alpha}I(z). \end{aligned} \tag{28}$$

Proof. By the condition (A_1) holds and for $l \geq m \geq 0$, $l \neq 0$, we have

$$[\Psi_2(w)-g(w)]^{\frac{m}{l}} \leq \frac{m}{l}q^{\frac{m-l}{l}}[\Psi_2(w)-g(w)]+\frac{l-m}{l}q^{\frac{m}{l}}, \tag{29}$$

multiplying this by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{w}]^{\frac{\alpha}{k}-1}[\frac{w}{z}]^s\frac{1}{w}$, and integrating w.r.t. w on $[1,z]$, we have

$$\begin{aligned} &\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{w})^{\frac{\alpha}{k}-1}(\frac{w}{z})^s[\Psi_2(w)-g(w)]^{\frac{m}{l}}\frac{dw}{w} \\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{w})^{\frac{\alpha}{k}-1}(\frac{w}{z})^s[\Psi_2(w)-g(w)]\frac{dw}{w} \\ &+\frac{l-m}{l}q^{\frac{m}{l}}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{w})^{\frac{\alpha}{k}-1}(\frac{w}{z})^s\frac{dw}{w}. \end{aligned} \tag{30}$$

Using equation (1), we get

$$\begin{aligned} &{}_{k}^{s}I_{H}^{\alpha}[\Psi_2-g]^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_{k}^{s}I_{H}^{\alpha}g(z) \leq \frac{m}{l}q^{\frac{m-l}{l}}{}_{k}^{s}I_{H}^{\alpha}\Psi_2(z) \\ &+\frac{l-m}{l}q^{\frac{m}{l}}{}_{k}^{s}I_{H}^{\alpha}(I)(z). \end{aligned} \tag{31}$$

For (14), we can use similar steps. □

Theorem 2.5. *For integrable functions g and h on $[1,\infty]$. Let (A_1) holds and also suppose the following:*

(A_2) *There exist the integrable functions ϕ_1 and ϕ_2 on $[1,\infty]$ such that*

$$\phi_1(r) \leq h(r) \leq \phi_2(r), \qquad \forall r\epsilon[1,\infty]. \tag{32}$$

Then, for any $r>1$, $s\epsilon$ and $\lambda,\gamma,\kappa\epsilon$,

$$\begin{aligned} &{}^s_kI^\alpha_H\phi_1(r){}^s_kI^\alpha_Hg(r) + {}^s_kI^\alpha_H\Psi_2(r){}^s_kI^\alpha_Hh(r) \\ &\qquad \geq {}^s_kI^\alpha_H\Psi_2(r){}^s_kI^\alpha_H\phi_1(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_Hh(r), \end{aligned} \tag{33}$$

$$\begin{aligned} &{}^s_kI^\alpha_H\Psi_1(r){}^s_kI^\alpha_Hh(r) + {}^s_kI^\alpha_H\phi_2(r){}^s_kI^\alpha_Hg(r) \\ &\qquad \geq {}^s_kI^\alpha_H\phi_2(r){}^s_kI^\alpha_H\Psi_1(r) + {}^s_kI^\alpha_Hh(r){}^s_kI^\alpha_Hg(r), \end{aligned} \tag{34}$$

$$\begin{aligned} &{}^s_kI^\alpha_H\phi_2(r){}^s_kI^\alpha_H\Psi_2(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_Hh(r) \\ &\qquad \geq {}^s_kI^\alpha_H\Psi_2(r){}^s_kI^\alpha_Hh(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_H\phi_2(r), \end{aligned} \tag{35}$$

$$\begin{aligned} &{}^s_kI^\alpha_H\Psi_1(r){}^s_kI^\alpha_H\phi_1(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_Hh(r) \\ &\qquad \geq {}^s_kI^\alpha_H\Psi_1(r){}^s_kI^\alpha_Hh(r) + {}^s_kI^\alpha_Hg(r){}^s_kI^\alpha_H\phi_1(r). \end{aligned} \tag{36}$$

Proof. From (A_1) and (A_2), $\forall p\geq 1, q\geq 1$, we have

$$[\Psi_2(p)-g(p)][h(q)-\phi_1(q)] \geq 0. \tag{37}$$

Therefore,

$$\Psi_2(p)h(q)+\phi_1(q)g(p) \geq \phi_1(p)\Psi_2(p)+g(q)h(q). \tag{38}$$

Multiplying by $\frac{1}{k\Gamma_k(\lambda)}[\log\frac{r}{p}]^{\frac{\lambda}{k}-1}[\frac{p}{r}]^s\frac{1}{p}$ and integrating w.r.t. p on $[1,z]$, we have

$$\begin{aligned} &h(q)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^s\Psi_2(p)\frac{dp}{p} \\ &+\phi_1(q)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^sg(p)\frac{dp}{p} \\ &\geq \phi_1(q)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^s\Psi_2(p)\frac{dp}{p} \\ &+h(p)\frac{1}{k\Gamma_k(\lambda)}\int_1^r(\log\frac{r}{p})^{\frac{\lambda}{k}-1}(\frac{p}{r})^sg(p)\frac{dp}{p}. \end{aligned} \tag{39}$$

Using equation (1)

$$h(q){}^s_kI^\lambda_H\Psi_2(r)+\phi_1(q){}^s_kI^\lambda_Hg(r) \geq \phi_1(p){}^s_kI^\lambda_H\Psi_2(r)+h(q){}^s_kI^\lambda_Hh(r). \tag{40}$$

Multiplying by $\frac{1}{k\Gamma_k(\gamma)}[\log\frac{r}{q}]^{\frac{\gamma}{k}-1}[\frac{q}{z}]^s\frac{1}{q}$ and integrating w.r.t. q on $[1,z]$, using the equation (1), we get

$$\begin{aligned}&{}_k^sI_H^{\lambda}\phi_1(r){}_k^sI_H^{\lambda}g(r)+{}_k^sI_H^{\lambda}\Psi_2(r){}_k^sI_H^{\lambda}h(r)\\ &\quad\geq {}_k^sI_H^{\lambda}\Psi_2(r){}_k^sI_H^{\lambda}\phi_1(r)+{}_k^sI_H^{\lambda}g(r){}_k^sI_H^{\lambda}h(r).\end{aligned}\tag{41}$$

For (34), (35) and (36), we can use similar steps. □

Corollary 2.2. *For integrable functions g and h on $[1,\infty]$ and constants $l\geq m\geq 0$, $l\neq 0$. Let (A_1) holds. Then for any $z>1$, $s\epsilon$ and $\alpha,\beta,k,q\epsilon$,*

$$\begin{aligned}&{}_k^sI_H^{\alpha}(\Psi_2-g)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}\Psi_2h(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}g\phi_2(z)\\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}\Psi_2\phi_2(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}gh(z)+\frac{l-m}{l}q^{\frac{m}{l}}{}_k^sI_H^{\alpha}I(z).\end{aligned}\tag{42}$$

$$\begin{aligned}&{}_k^sI_H^{\alpha}(g-\Psi_1)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}\Psi_1h(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}g\phi_1(z)\\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}\Psi_1\phi_1(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}gh(z)+\frac{l-m}{l}q^{\frac{m}{l}}{}_k^sI_H^{\alpha}I(z).\end{aligned}\tag{43}$$

$$\begin{aligned}&{}_k^sI_H^{\alpha}(g-\Psi_1)^{\frac{m}{l}}(z){}_k^sI_H^{\beta}(h-\phi_1)^{\frac{m}{l}}(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}g(z){}_k^sI_H^{\beta}\phi_1(z)\\ &\leq \frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}\Psi_1(z){}_k^sI_H^{\alpha}\phi_1(z)+\frac{m}{l}q^{\frac{m-l}{l}}{}_k^sI_H^{\alpha}g(z){}_k^sI_H^{\beta}h(z)\\ &+\frac{l-m}{l}q^{\frac{m}{l}}{}_k^sI_H^{\alpha+\beta}I(z).\end{aligned}\tag{44}$$

Theorem 2.6. *Let g and h are positive valued and continuous functions on $[0.\infty]$ such that $g\leq h$. If g is increasing and $\frac{g}{h}$ is decreasing on $[0,\infty]$, then for any $q\geq 0$, $s\epsilon$, $\alpha,\beta,k,z\epsilon$,*

$$\frac{{}_k^sI_H^{\alpha}[g](z){}_k^sI_H^{\beta}[h^q](z)+{}_k^sI_H^{\alpha}[H^q](z){}_k^sI_H^{\beta}[g](z)}{{}_k^sI_H^{\alpha}[h](z){}_k^sI_H^{\beta}[g^q](z)+{}_k^sI_H^{\alpha}[g^q](z){}_k^sI_H^{\beta}[h](z)}\geq 1\tag{45}$$

Proof. Using $g\leq h$, we can get

$$hg^{q-1}(z)\leq h^q(z).\tag{46}$$

Multiplying (46) by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,z]$,

$$\begin{aligned}&\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s hg^{q-1}(v)\frac{dv}{v}\\ &\leq \frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s h^q(v)\frac{dv}{v}.\end{aligned}\tag{47}$$

Using the equation (1)

$$ {}_k^sI_H^{\alpha} hg^{q-1}(z) \leq {}_k^sI_H^{\alpha} h^q(z). \tag{48}$$

Multiplying by ${}_k^sI_H^{\beta}[g](z)$

$$ {}_k^sI_H^{\alpha} hg^{q-1}(z) {}_k^sI_H^{\beta}[g](z) \leq {}_k^sI_H^{\beta}[g](z) {}_k^sI_H^{\alpha} h^q(z). \tag{49}$$

Multiplying (46) by $\frac{1}{k\Gamma_k(\beta)}[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s\frac{1}{w}$ and integrating w.r.t. w on $[1,z]$,

$$\begin{aligned} &\frac{1}{k\Gamma_k(\beta)}\int_1^z (\log\frac{z}{w})^{\frac{\beta}{k}-1}(\frac{w}{z})^s hg^{q-1}(w)\frac{dw}{w} \\ &\leq \frac{1}{k\Gamma_k(\beta)}\int_1^z (\log\frac{z}{w})^{\frac{\beta}{k}-1}(\frac{w}{z})^s h^q(w)\frac{dw}{w}. \end{aligned} \tag{50}$$

Using the equation (1), we get

$$ {}_k^sI_H^{\beta} hg^{q-1}(z) \leq {}_k^sI_H^{\alpha} h^q(z). \tag{51}$$

Multiplying by ${}_k^sI_H^{\alpha}[g](z)$

$$ {}_k^sI_H^{\beta} hg^{q-1}(z) {}_k^sI_H^{\alpha}[g](z) \leq {}_k^sI_H^{\alpha}[g](z) {}_k^sI_H^{\beta} h^q(z). \tag{52}$$

Adding (49) and (52), then simplifying we get

$$ \frac{{}_k^sI_H^{\alpha}[g](z){}_k^sI_H^{\beta}[h^q](z) + {}_k^sI_H^{\beta}[g](z){}_k^sI_H^{\alpha}[H^q](z)}{{}_k^sI_H^{\beta}[hg^{q-1}](z){}_k^sI_H^{\alpha}[g](z) + {}_k^sI_H^{\alpha}[hg^{q-1}](z){}_k^sI_H^{\beta}[g](z)} \geq 1. \tag{53}$$

Substituting $g = g^{q-1}$ and $f = g$ in Theorem 2.3, we get

$$ \frac{{}_k^sI_H^{\alpha}[hg^{q-1}](z){}_k^sI_H^{\beta}[g](z) + {}_k^sI_H^{\alpha}[g](z){}_k^sI_H^{\beta}[hg^{q-1}](z)}{{}_k^sI_H^{\alpha}[g^q](z){}_k^sI_H^{\beta}[h](z) + {}_k^sI_H^{\alpha}[h](z){}_k^sI_H^{\beta}[g^q](z)} \geq 1. \tag{54}$$

(27) and (28) give (23). □

Corollary 2.3. *Let g and h are positive valued and continuous functions on $[0.\infty]$ such that $g \leq h$. If g is increasing and $\frac{g}{h}$ is decreasing on $[0,\infty]$, then for any $q \geq 0$, $s\epsilon$, $\alpha, k, z\epsilon$,*

$$ \frac{{}_k^sI_H^{\alpha}[g](z)}{{}_k^sI_H^{\alpha}[h](z)} \geq \frac{{}_k^sI_H^{\alpha}[g^q](z)}{{}_k^sI_H^{\alpha}[h]^q(z)}. \tag{55}$$

Proof. By substituting $\beta = \alpha$ (45), we get

$$ \frac{{}_k^sI_H^{\alpha}[g](z){}_k^sI_H^{\alpha}[h]^q(z)}{{}_k^sI_H^{\alpha}[h](z){}_k^sI_H^{\alpha}[g^q](z)} \geq 1, \tag{56}$$

which gives (55). □

Theorem 2.7. *For integrable functions g and h on $[0.\infty]$ satisfying $\frac{1}{\theta_1}+\frac{1}{\theta_2}=1$, $\theta_1,\theta_2\epsilon(0,\infty)$. Let($A_1$ and (A_2 holds. Then, for $z>1$, $s\epsilon$, $\alpha,\beta,k\epsilon$, we have*

$$\begin{aligned}&\frac{1}{\theta_1}{}^s_kI^\beta_H(I)(z){}^s_kI^\alpha_H(\Psi_2-g)^{\theta_1}(z)\\ &+\frac{1}{\theta_2}{}^s_kI^\alpha_H(I)(z){}^s_kI^\alpha_H(\phi_2-h)^{\theta_2}(z)\geq {}^s_kI^\alpha_H(\Psi_2-g)(z){}^s_kI^\beta_H(\phi_2-h)(z).\end{aligned}\tag{57}$$

$$\begin{aligned}&\frac{1}{\theta_1}{}^s_kI^\beta_H(I)(z){}^s_kI^\alpha_H(g-\Psi_1)^{\theta_1}(z)+\frac{1}{\theta_2}{}^s_kI^\alpha_H(I)(z){}^s_kI^\beta_H(h-\phi_1)^{\theta_2}(z)\\ &\qquad\geq {}^s_kI^\alpha_H(g-\Psi_1)(z){}^s_kI^\beta_H(h-\phi_1)(z),\end{aligned}\tag{58}$$

$$\begin{aligned}&\frac{1}{\theta_1}{}^s_kI^\beta_H(\phi_2-h)^{\theta_1}(z){}^s_kI^\alpha_H(\Psi_2-g)^{\theta_1}(z)\\ &\qquad+\frac{1}{\theta_2}{}^s_kI^\alpha_H(\phi_2-g)^{\theta_2}(z){}^s_kI^\beta_H(\Psi_2-g)^{\theta_2}(z)\\ &\qquad\geq {}^s_kI^\alpha_H(\Psi_2-g)(\phi_2-h)(z){}^s_kI^\beta_H(\Psi_2-g)(\phi_2-h)(z),\end{aligned}\tag{59}$$

$$\begin{aligned}&\frac{1}{\theta_1}{}^s_kI^\beta_H(h-\phi_1)^{\theta_1}(z){}^s_kI^\alpha_H(g-\Psi_1)^{\theta_1}(z)\\ &\qquad+\frac{1}{\theta_2}{}^s_kI^\alpha_H(h-\phi_2)^{\theta_2}(z){}^s_kI^\beta_H(g-\Psi_1)^{\theta_2}(z)\\ &\qquad\geq {}^s_kI^\alpha_H(g-\Psi_1)(h-\phi_1)(z){}^s_kI^\beta_H(g-\Psi_1)(h-\phi_1)(z).\end{aligned}\tag{60}$$

Proof. By Young's inequality

$$\frac{1}{\theta_1}(x)^{\theta_1}+\frac{1}{\theta_2}(y)^{\theta_2}\geq xy,\qquad \forall x,y\epsilon[1,\infty],\theta_1,\theta_2\epsilon(0,\infty),\tag{61}$$

Also,

$$\frac{1}{\theta_1}+\frac{1}{\theta_2}=1,\qquad \theta_1,\theta_2\epsilon(0,\infty).\tag{62}$$

Let $x=\Psi_2(v)-g(v($ and $y=\phi_2(w)-h(w)$, $v,w\epsilon(0,\infty)$, we get

$$\frac{1}{\theta_1}[\Psi_2(v)-g(v)]^{\theta_1}+\frac{1}{\theta_2}(\phi_2(w)-h(w))^{\theta_2}\geq[\Psi_2(v)-g(v)][\phi_2(w)-h(w)].\tag{63}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,z]$,

$$\begin{aligned}&\frac{1}{\theta_1}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s[\Psi_2(v)-g(v)]^{\theta_1}\frac{dv}{v}\\ &+\frac{1}{\theta_2}[\phi_2(w)-h(w)]^{\theta_2}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s\frac{dv}{v}\\ &\geq[\phi_2(w)-h(w)]\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{v})^{\frac{\alpha}{k}-1}(\frac{v}{z})^s[\Psi_2(v)-g(v)]\frac{dv}{v}.\end{aligned}\tag{64}$$

Using the equation (1)

$$\frac{1}{\theta_1}{}^s_kI^{\alpha}_H[\Psi_2-g]^{\theta_1(z)}+\frac{1}{\theta_2}(\phi_2(w)-h(w))^{\theta_2}{}^s_kI^{\alpha}_H[I](z) \geq [\phi_2(w)-h(w)]^s_kI^{\alpha}_H[\Psi_2-g](z). \tag{65}$$

Multiplying by $\frac{1}{k\Gamma_k(\beta)}[\log\frac{z}{w}]^{\frac{\beta}{k}-1}[\frac{w}{z}]^s\frac{1}{w}$ and integrating w.r.t. w on $[1,\infty]$, and usingthe definition (1), we have

$$\frac{1}{\theta_1}{}^s_kI^{\beta}_H(I)(z){}^s_kI^{\alpha}_H(\Psi_2-g)^{\theta_1}(z)+\frac{1}{\theta_2}{}^s_kI^{\alpha}_H(I)(z){}^s_kI^{\beta}_H(\phi_2-h)^{\theta_2}(z) \geq {}^s_kI^{\alpha}_H(\Psi_2-g)(z){}^s_kI^{\beta}_H(\phi_2-h)(z). \tag{66}$$

For (58), (59) and (60), we can use similar steps. $\square$

Theorem 2.8. *For integrable function g on $[1.\infty]$ satisfying $\theta_1+\theta_2=1$, $\theta_1,\theta_2\epsilon(0,\infty)$. Let$(A_1$ holds. Then for $z>1$, $s\epsilon$, and $\alpha,\beta,k\epsilon$,*

$$\begin{aligned}\theta_1{}^s_kI^{\beta}_H(I)(z){}^s_kI^{\alpha}_H(\Psi_2(z))+\theta_2{}^s_kI^{\alpha}_H(I)(z){}^s_kI^{\beta}_Hg(z) &\geq {}^s_kI^{\alpha}_H(\Psi_2-g)^{\theta_1}(z){}^s_kI^{\beta}_H(g-\Psi_1)^{\theta_1}\\ &+\theta_1{}^s_kI^{\beta}_H(I)(z){}^s_kI^{\alpha}_Hg(z)+\theta_2{}^s_kI^{\alpha}_H(I)(z){}^s_kI^{\beta}_H\Psi_1(z).\end{aligned} \tag{67}$$

Proof. By the weighted AM-GM inequality,

$$\theta_1(x)^{\theta_1}+\theta_2(y)^{\theta_2}\geq x^{\theta_1}y^{\theta_2}, \qquad \forall x,y\epsilon[0,\infty],\theta_1,\theta_2\epsilon(0,\infty). \tag{68}$$

Also

$$\theta_1+\theta_2=1, \qquad \theta_1,\theta_2\epsilon(0,\infty). \tag{69}$$

Let $x=\Psi_2(p)-g(p)$ and $y=g(q)-\Psi_2(q)$, $p,q\epsilon(1,\infty)$, we get

$$\theta_1[\Psi_2(p)-g(p)]+\theta_2(g(q)-\Psi_1(q))\geq[\Psi_2(p)-g(p)]^{\theta_1}[g(q)-\Psi_1(q)]^{\theta_2}. \tag{70}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{p}]^{\frac{\alpha}{k}-1}[\frac{p}{z}]^s\frac{1}{p}$ and integrating w.r.t. p on $[1,\infty]$,

$$\begin{aligned}&\frac{\theta_1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{p})^{\frac{\alpha}{k}-1}(\frac{p}{z})^s[\Psi_2(p)-g(p)]\frac{dp}{p}\\ &+\theta_2[g(q)-\Psi_1(q)]\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{p})^{\frac{\alpha}{k}-1}(\frac{p}{z})^s\frac{dp}{p}\\ &\geq[g(q)-\Psi_1(q)]^{\theta_1}\frac{1}{k\Gamma_k(\alpha)}\int_1^z(\log\frac{z}{p})^{\frac{\alpha}{k}-1}(\frac{p}{z})^s[\Psi_2(p)-g(p)]^{\theta_2}\frac{dp}{p}.\end{aligned} \tag{71}$$

Using the equation (1)

$$\begin{aligned}&\theta_1{}^s_kI^{\alpha}_H[\Psi_2-g](z)+\theta_2(g(q)-\phi_1(q)){}^s_kI^{\alpha}_H(I)(z)\\ &\geq[g(q)-\Psi_1(q)]^{\theta_2}{}^s_kI^{\alpha}_H[\Psi_2-h]^{\theta_1}(z).\end{aligned} \tag{72}$$

Multiplying by $\frac{1}{k\Gamma_k(\beta)}[\log\frac{z}{q}]^{\frac{\beta}{k}-1}[\frac{q}{z}]^s\frac{1}{q}$ and integrating w.r.t. q on $[1,\infty]$,

$$\begin{aligned}&\theta_1{}^s_kIH^{\alpha}(\Psi_2-g)(z)\frac{1}{k\Gamma_k(\beta)}\int_1^z(\log\frac{z}{q})^{\frac{\beta}{k}-1}(\frac{q}{z})^s\frac{dq}{q}\\&+\theta_2{}^s_kI_H^{\alpha}(I)(z)\frac{1}{k\Gamma_k(\beta)}\int_1^z(\log\frac{z}{q})^{\frac{\beta}{k}-1}(\frac{q}{z})^s\frac{dq}{q}\\&\geq{}^s_kIH^{\alpha}(\Psi_2-g)(z)\frac{1}{k\Gamma_k(\beta)}\int_1^z(\log\frac{z}{q})^{\frac{\beta}{k}-1}(\frac{q}{z})^s[g(q)-\Psi_1(q)]^{\theta_2}\frac{dq}{q}.\end{aligned}\tag{73}$$

Using the equation (1)

$$\begin{aligned}&\theta_1{}^s_kI_H^{\beta}(I)(z){}^s_kI_H^{\alpha}(\Psi_2-g(z))(z)+\theta_2{}^s_kI_H^{\alpha}(I)(z){}^s_kI_H^{\beta}[g-\Psi_1](z)\\&\qquad\geq{}^s_kI_H^{\alpha}(\Psi_2-g)^{\theta_1}(z){}^s_kI_H^{\beta}(g-\Psi_1)^{\theta_1}(z).\end{aligned}\tag{74}$$

Due to linearity of integrals we get (67). ◻

Theorem 2.9. *For integrable functions g and h on $[1,\infty]$ satisfying $\theta_1+\theta_2=1$, $\theta_1,\theta_2\epsilon(0,\infty)$. Let$(A_1$. Let (A_1) and (A_2) hold. Then, for $r>1$, $s\epsilon$, $\lambda,\gamma,k\epsilon$,*

$$\begin{aligned}&\theta_1{}^s_kI_H^{\gamma}(I)(r){}^s_kI_H^{\lambda}(\Psi_2)(r)+\theta_2{}^s_kI_H^{\lambda}(I)(r){}^s_kI_H^{\gamma}[\phi_2](r)\\&\geq{}^s_kI_H^{\lambda}(\Psi_2-g)^{\theta_1}(r){}^s_kI_H^{\gamma}(\phi_2-h)^{\theta_2}(r)+\theta_1{}^s_kI_H^{\gamma}(I)(r){}^s_kI_H^{\lambda}(g)(r)\\&+\theta_2{}^s_kI_H^{\lambda}(I)(r){}^s_kI_H^{\gamma}(h)(r),\end{aligned}\tag{75}$$

$$\begin{aligned}&\theta_1{}^s_kI_H^{\gamma}(I)(r){}^s_kI_H^{\lambda}(g)(r)+\theta_2{}^s_kI_H^{\lambda}(I)(r){}^s_kI_H^{\gamma}(h)(r)\\&\geq{}^s_kI_H^{\lambda}(g-\Psi_1)^{\theta_1}(r){}^s_kI_H^{\gamma}(h-\phi_1)^{\theta_2}(r)+\theta_1{}^s_kI_H^{\gamma}(I)(r){}^s_kI_H^{\lambda}(\Psi_2)(r)\\&+\theta_1{}^s_kI_H^{\gamma}(I)(r){}^s_kI_H^{\gamma}(\phi_1)(r),\end{aligned}\tag{76}$$

$$\begin{aligned}&\theta_1{}^s_kI_H^{\gamma}(\phi1)(r){}^s_kI_H^{\lambda}(\Psi_2)(r)+\theta_1{}^s_kI_H^{\gamma}(h)(r){}^s_kI_H^{\gamma}g(r)\\&+\theta_2{}^s_kI_H^{\lambda}(\phi2)(r){}^s_kI_H^{\gamma}(\Psi_2)(r)+\theta_2{}^s_kI_H^{\gamma}(g)(r){}^s_kI_H^{\gamma}h(r)\\&\geq{}^s_kI_H^{\lambda}(\Psi_2-g)^{\theta_1}(\phi_2-h)^{\theta_1}(r){}^s_kI_H^{\gamma}(\phi_2-h)^{\theta_1}(\Psi_2-g)^{\theta_2}(r)\\&+\theta_1{}^s_kI_H^{\gamma}(h)(r){}^s_kI_H^{\lambda}(\Psi_2)(r)+\theta_2{}^s_kI_H^{\lambda}(\phi_2)(r){}^s_kI_H^{\gamma}(g)(r)\\&+\theta_2{}^s_kI_H^{\lambda}h(r){}^s_kI_H^{\gamma}(\Psi_2)(r),\end{aligned}\tag{77}$$

$$\begin{aligned}&\theta_1{}^s_kI_H^{\gamma}(h)(r){}^s_kI_H^{\lambda}(g)(r)+\theta_1{}^s_kI_H^{\gamma}(\phi_1)(r){}^s_kI_H^{\lambda}\Psi_2(r)\\&+\theta_2{}^s_kI_H^{\lambda}(\phi1)(r){}^s_kI_H^{\gamma}(\Psi_1)(r)+\theta_2{}^s_kI_H^{\gamma}(g)(r){}^s_kI_H^{\lambda}h(r)\\&\geq{}^s_kI_H^{\lambda}(g-\Psi_1)^{\theta_1}(h-\phi_2)^{\theta_2}(r){}^s_kI_H^{\gamma}(h-\phi_1)^{\theta_1}(g-\Psi_1)^{\theta_2}(r)\\&+\theta_1{}^s_kI_H^{\gamma}(\phi_1)(r){}^s_kI_H^{\lambda}(g)(r)+\theta_1{}^s_kI_H^{\gamma}(h)(r){}^s_kI_H^{\lambda}(\Psi_1)(r)\\&+\theta_2{}^s_kI_H^{\lambda}h(r){}^s_kI_H^{\gamma}(\Psi_1)(r)+\theta_2{}^s_kI_H^{\lambda}\phi_1(r){}^s_kI_H^{\gamma}(g)(r).\end{aligned}\tag{78}$$

Theorem 2.10. *Let g, Ψ_1 and Ψ_2 are integrable functions on $[1,\infty]$, assume that condition (A_1) holds. Then for $r>1$, $s\epsilon$, $\lambda, k\epsilon$,*

$$\begin{aligned}
&{}^s_kI^\lambda_H(I)(r){}^s_kI^\lambda_H g^2(r)-[{}^s_kI^\lambda_H g(r)]^2\\
&=[{}^s_kI^\lambda_H\Psi_2(r)-{}^s_kI^\lambda_H g(r)][{}^s_kI^\lambda_H g(r)-{}^s_kI^\lambda_H\Psi_1(r)]\\
&-{}^s_kI^\lambda_H(I)(r){}^s_kI^\lambda_H[\Psi_2-g](r)[g-\Psi_1(r)]+{}^s_kI^\lambda_H(I)(r){}^s_kI^\lambda_H\Psi_1 g(r)\\
&-{}^s_kI^\lambda_H(\Psi_1(r)){}^s_kI^\lambda_H g(r)+{}^s_kI^\lambda_H(\Psi_2)g(r){}^s_kI^\lambda_H(I)(r)-{}^s_kI^\lambda_H(\Psi_2)(r){}^s_kI^\lambda_H g(r)\\
&+{}^s_kI^\lambda_H(\Psi_1)(r){}^s_kI^\lambda_H\Psi_2(r)-{}^s_kI^\lambda_H(\Psi_1\Psi_2(r)){}^s_kI^\lambda_H(I)(r).
\end{aligned}\tag{79}$$

Proof. For $p>1, q>1$, we have

$$\begin{aligned}
&[\Psi_2(q)-g(q)][g(p)-\Psi_1(p)]\\
&+[\Psi_2(p)-g(p)][g(q)-\Psi_1(q)]-[\Psi_2(p)-g(p)],\\
&[g(p)-\Psi_1(p)]-[\Psi_2(q)-g(q)][g(q)-\Psi_1(q)]=g^2(p)+g^2(q)-2g(p)g(q)\\
&+\Psi_2(q)g(p)+\Psi_1(p)g(q)-\Psi_1(p)\Psi_2(q)+\Psi_2(p)g(q)+\Psi_1(q)g(p)\\
&-\Psi_2(p)g(p)-\Psi_1(p)\Psi_2(p)+\Psi_1(q)\Psi_2(q){}^s_kI^\lambda_H(I)(r)-\Psi_1(q)g(q){}^s_kI^\lambda_H(I)(r).
\end{aligned}\tag{80}$$

Multiplying by $\frac{1}{k\Gamma_k(\lambda)}[\log\frac{z}{q}]^{\frac{\lambda}{k}-1}[\frac{q}{z}]^s\frac{1}{q}$ and integrating w.r.t. q on $[1,\infty]$, and using equation (1), we can get (79). □

Theorem 2.11. *Let g,h be two positive functions on $[0,\infty]$, such that $\forall z>1$, $s\epsilon$, $\alpha, k\epsilon$, $q\geq 1$ and ${}^s_kI^\alpha_H g^p(z)<\infty$, ${}^s_kI^\alpha_H h^p(z)<\infty$. If $0<m\leq\frac{g(v)}{h(v)}\leq M$, m, M,ϵ, $\forall v\epsilon(0,z)$. Then, we have*

$$[{}^s_kI^\alpha_H g^q(z)]^{\frac{1}{q}}+[{}^s_kI^\alpha_H[h]^q(z)]^{\frac{1}{q}}\leq\frac{M(m+2)+1}{(M+1)(m+1)}[{}^s_kI^\alpha_H[g+h]^q(z)]^{\frac{1}{q}},\tag{81}$$

$$[{}^s_kI^\alpha_H g^q(z)]^{\frac{2}{q}}+[{}^s_kI^\alpha_H[h]^q(z)]^{\frac{2}{q}}\geq\frac{(M+1)(m+1)}{(M)}[{}^s_kI^\alpha_H[g]^q(z)]^{\frac{1}{q}}[{}^s_kI^\alpha_H[h]^q(z)]^{\frac{1}{q}}.\tag{82}$$

Proof. Using $\frac{g(v)}{h(v)}\leq M, \forall v\epsilon(0,z)$, we can get $\frac{[g+h](v)}{h(v)}\leq M+1$ and hence

$$(M+1)^q g^q\leq M^q[g+h]^q(v).\tag{83}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,\infty]$,

$$\frac{(M+1)}{k\Gamma_k(\alpha)}[\int_1^z\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}\leq\frac{(M)^q}{k\Gamma_k(\alpha)}[\int_1^z\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}.\tag{84}$$

Using the equation (1)

$$(M+1)^q{}^s_kI^\alpha_H g^q(z)\leq(M)^q{}^s_kI^\alpha_H[g+h]^q(z).\tag{85}$$

Which gives

$$[{}^s_kI^\alpha_H g^q(z)]^{\frac{1}{q}} \leq \frac{M}{M+1}[{}^s_kI^\alpha_H[g+h]^q(z)]^{\frac{1}{q}}. \tag{86}$$

Using $m \leq \frac{g(v)}{h(v)}$, $\forall v\epsilon(0,\infty)$, we get

$$(1+\frac{1}{m})^q h^q(v) \leq (\frac{1}{m})^q[g+h]^q(v), \tag{87}$$

Multiplying by $\frac{1}{k\Gamma_k(\alpha)}[\log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s\frac{1}{v}$ and integrating w.r.t. v on $[1,\infty]$,

$$\begin{aligned}&\frac{(1+\frac{1}{m})}{k\Gamma_k(\alpha)}[\int_1^z \log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s h^q\frac{dv}{v}\\ &\leq \frac{(\frac{1}{m})^q}{k\Gamma_k(\alpha)}[\int_1^z \log\frac{z}{v}]^{\frac{\alpha}{k}-1}[\frac{v}{z}]^s[g+h]^q(v)\frac{dv}{v}.\end{aligned} \tag{88}$$

Using the equation (1)

$$(1+\frac{1}{m})^q{}^s_kI^\alpha_H h^q(v) \leq (\frac{1}{m})^q{}^s_kI^\alpha_H[g+h]^q(v). \tag{89}$$

Which gives,

$$[{}^s_kI^\alpha_H h^q(v)]^{\frac{1}{q}} \leq (\frac{1}{m+1})[{}^s_kI^\alpha_H[g+h]^q(z)]^{\frac{1}{q}}. \tag{90}$$

Adding (42) and (43), we get (40).

Multiplying (42) and (43)

$$\frac{(M+1)(m+1)}{M}[{}^s_kI^\alpha_H g^q(z)]^{\frac{1}{q}}[{}^s_kI^\alpha_H h^q(z)]^{\frac{1}{q}} \leq [{}^s_kI^\alpha_H[g+h]^q(z)]^{\frac{1}{q}}. \tag{91}$$

Using Minkowski inequalities on R.H.S, we can get (41). □

Applications

There are many applications of the fractional integral inequalities. Some of them are as under. In boundary value problems, we can use fractional integral inequalities to establish uniqueness of the solutions. They are also used in finding the unique solutions in fractional partial differential equations.

Conclusions

Results of fractional integral inequalities are determined by using extended Hadamard k-fractional integral. With these inequalities the uniqueness and continuous dependence of the solution of the nonlinear fractional differential equations can also be established. Furthermore, we can also extend these inequalities for α by analytical continuation.

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Convergence of a modified PRP conjugate gradient method with a new formula of step-size

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Abstract. We present in this paper the global convergence of a modified PRP (Polak-Ribière-Polyak) conjugate gradient method suggested by Min and Jing [11], by using a new formula of step-size that combination by Wu [14], and by Sun and colleagues [3, 12]. Some numerical results are also presented.
Keywords: conjugate gradient methods, global convergence, PRP method, step-size, line search.

1. Introduction

Let us consider the following unconstrained minimization problem: $f(x), x \in R^n$, where f is a differentiable objective function, has the following form

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1.1}$$

$$\text{where } \; d_k = \begin{cases} -g_k, & \text{for } k = 1, \\ -g_k + \beta_k d_{k-1}, & \text{for } \; k \geq 2, \end{cases} \tag{1.2}$$

where $g_k = \nabla f(x_k)$ is the gradient of f at x_k.

Motivated by the ideas of Wei and al. [14] and Dai and Wen [5], which spured Min and Jing [11] construct two modified PRP methods, in which the parameter β_k is specified as follows:

$$\beta_k^{MPRP} = \frac{g_k^T y_{k-1}}{\mu |g_k^T d_{k-1}| + \|g_{k-1}\|^2}, \tag{1.3}$$

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where $\|.\|$ means the Euclidean norm, $y_{k-1} = g_k - g_{k-1}$, and $\mu \geq 0$ is a constant. Let us remark that the descent direction d_k is defined by

$$g_k^T d_k = -c \|g_k\|^2 , \tag{1.4}$$

where $0 < c < 1$.

The global convergence properties of conjugate gradient method have been studied by many researchers [2-9].

In the implementation of any conjugate gradient (CG) method, the step-size is often determined by certain line search conditions such as the Wolfe conditions [13]. These types of line search involve extensive computation of function values and gradients, which often becomes a significant burden for large-scale problems, which spured Sun [12], and Wu [14] to pursue the conjugate gradient method where they calculated the step-size instead of the line search. The new formula for step-size α_k in the form

$$\alpha_k = \frac{-\delta g_k^T d_k}{(\bar{g}_{k+1} - g_k)^T d_k + \gamma \|d_k\|^2}, \tag{1.5}$$

where

$$\delta \in (0, (\kappa + \gamma)/\tau), \gamma \geq 0, \tag{1.6}$$

τ and κ confirm the Assumption 2.1 below, $\bar{g}_{k+1}$ denote $\bigtriangledown f(x_k + d_k)$.

In this paper, our goal is to employ the step-formula (1.5) to prove the convergence of a modified PRP conjugate gradient method.

This paper is organized as follows. Some preliminary results on the family of CG methods with the new-form step-size formula (1.5) are given in Section 2. Section 3 includes the main convergence properties of the modified PRP conjugate gradient method.

2. Properties of the new step-size

The present section gathers technical results concerning the step-size α_k generated by (1.5).

Assumption 2.1. *The function f is LC^1 and strongly convex in R^n, i.e, there exists constants $\tau > 0$ and $\kappa \geq 0$ such that*

$$\|\bigtriangledown f(u) - \bigtriangledown f(v)\| \leq \tau \|u - v\| , \forall u, v \in R^n, \tag{2.1}$$

and

$$[\bigtriangledown f(u) - \bigtriangledown f(v)]^T (u - v) \geq \kappa \|u - v\|^2 , \forall u, v \in R^n. \tag{2.2}$$

Note that Assumption 2.1 implies that the level set $L = \{x \in R^n | f(x) \leq f(x_1)\}$ is bounded.

Lemma 2.2 *Suppose that Assumption 2.1 holds. Then the following inequalities*

$$\kappa \|s_k\|^2 \le y_k^T s_k \le \tau \|s_k\|^2 , \tag{2.3}$$

where $s_k = x_{k+1} - x_k$, $y_k = g_{k+1} - g_k$ *and*

$$(\kappa + \gamma) \|d_k\|^2 \le (\bar{g}_{k+1} - g_k)^T d_k + \gamma \|d_k\|^2 \le (\tau + \gamma) \|d_k\|^2 , \tag{2.4}$$

hold for all k.

Proof. It is straightforward from (2.1) and (2.2) that (2.3) holds. Now, we prove (2.4) is true

$$\begin{aligned}(\bar{g}_{k+1} - g_k)^T d_k + \gamma \|d_k\|^2 &\le \|\bar{g}_{k+1} - g_k\| \|d_k\| + \gamma \|d_k\|^2 \\ &\le (\tau + \gamma) \|d_k\|^2 .\end{aligned} \tag{2.5}$$

Then, by (2.2), we have

$$(\bar{g}_{k+1} - g_k)^T d_k + \gamma \|d_k\|^2 \ge \kappa \|d_k\|^2 + \gamma \|d_k\|^2 \ge (\kappa + \gamma) \|d_k\|^2 . \tag{2.6}$$

Hence, it follows from (2.5) and (2.6) that (2.4) hold for all k. □

Lemma 2.3. *Suppose that* x_k *is given by* (1.1), (1.2) *and* (1.5). *Then*

$$g_{k+1}^T d_k = \rho_k g_k^T d_k, \tag{2.7}$$

holds for all k*, where* $0 < \rho_k = 1 - \delta\Phi_k \|d_k\|^2 / [(\bar{g}_{k+1} - g_k)^T d_k + \gamma \|d_k\|^2]$, *and*

$$\Phi_k = \begin{cases} 0, & \text{for } \alpha_k = 0, \\ (g_{k+1} - g_k)^T (x_{k+1} - x_k) / \|x_{k+1} - x_k\|^2 , & \text{for } \alpha_k \ne 0. \end{cases} \tag{2.8}$$

Proof. If $\alpha_k = 0$, then $\rho_k = 1$ and $x_{k+1} = x_k$. Thus, (2.7) is true.

Now, we suppose that $\alpha_k \ne 0$. From (2.8) and (2.6), we have

$$\begin{aligned} g_{k+1}^T d_k &= g_k^T d_k + (g_{k+1} - g_k)^T d_k \\ &= g_k^T d_k + \alpha_k^{-1} (g_{k+1} - g_k)^T (x_{k+1} - x_k) \\ &= g_k^T d_k + \alpha_k^{-1} \Phi_k \|x_{k+1} - x_k\|^2 \\ &= g_k^T d_k + \alpha_k \Phi_k \|d_k\|^2 \\ &= g_k^T d_k - \{\delta g_k^T d_k / [(\bar{g}_{k+1} - g_k)^T d_k + \gamma \|d_k\|^2]\} \Phi_k \|d_k\|^2 \\ &= \{1 - \delta \Phi_k \|d_k\|^2 / [(\bar{g}_{k+1} - g_k)^T d_k + \gamma \|d_k\|^2]\} g_k^T d_k \\ &= \rho_k g_k^T d_k. \end{aligned}$$

The proof is complete. □

Corollary 2.4. *Suppose that Assumption 2.1 holds. Then*

$$\frac{\delta\kappa}{\tau+\gamma} \le 1-\rho_k \le \frac{\delta\tau}{\kappa+\gamma}, \tag{2.9}$$

holds for all k.

Proof. It follows From (2.3) and (2.4), we obtain (2.9).

Lemma 2.5. *Suppose that Assumption 2.1 holds and $\{x_k\}$ is generated by* (1.1), (1.2) *and* (1.5). *Then*

$$\sum_{d_k\neq 0} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty. \tag{2.10}$$

Proof. By the mean-value theorem, we have

$$f(x_{k+1}) - f(x_k) = \bar{g}^T(x_{k+1}-x_k), \tag{2.11}$$

where $\bar{g} = \bigtriangledown f(\bar{x})$ for some $\bar{x} \in [x_{k,}x_{k+1}]$. Now, by the Cauchy-Schwartz inequality, (1.5), and Assumption 2.1 we obtain

$$\begin{aligned}
\bar{g}^T(x_{k+1}-x_k) &= g_k^T(x_{k+1}-x_k) + (\bar{g}-g_k)^T(x_{k+1}-x_k)\\
&\le g_k^T(x_{k+1}-x_k) + \|\bar{g}-g_k\|\,\|x_{k+1}-x_k\|\\
&\le g_k^T(x_{k+1}-x_k) + \tau\,\|x_{k+1}-x_k\|^2\\
&= \alpha_k g_k^T d_k + \tau\alpha_k^2\,\|d_k\|^2\\
&= \alpha_k g_k^T d_k - \tau\alpha_k\delta g_k^T d_k\,\|d_k\|^2/[(\bar{g}_{k+1}-g_k)^T d_k + \gamma\,\|d_k\|^2]\\
&= \alpha_k g_k^T d_k(1 - \frac{\tau\delta\,\|d_k\|^2}{(\bar{g}_{k+1}-g_k)^T d_k + \gamma\,\|d_k\|^2}).
\end{aligned} \tag{2.12}$$

By from (2.4) and (2.12), we obtain

$$\begin{aligned}
\alpha_k g_k^T d_k &= -\frac{\delta}{(\bar{g}_{k+1}-g_k)^T d_k + \gamma\,\|d_k\|^2}(g_k^T d_k)^2\\
&\le -\frac{\delta}{(\tau+\gamma)}\frac{(g_k^T d_k)^2}{\|d_k\|^2},
\end{aligned} \tag{2.13}$$

by (2.12) and (1.6), we have

$$1 - \frac{\tau\delta\,\|d_k\|^2}{(\bar{g}_{k+1}-g_k)^T d_k + \gamma\,\|d_k\|^2} \ge 1 - \frac{\tau\delta}{\kappa+\gamma} > 0. \tag{2.14}$$

From (2.13) and (2.14), it follows that

$$\varOmega = \frac{\delta}{\tau+\gamma}(1 - \frac{\tau\delta}{\kappa+\gamma}) > 0. \tag{2.15}$$

From (2.11) we have,

$$f(x_{k+1}) - f(x_k) \le -\Omega \frac{(g_k^T d_k)^2}{\|d_k\|^2} \le 0, \tag{2.16}$$

which implies $f(x_{k+1}) \le f(x_k)$. Hence, it follows from (2.16) that (2.10) is true. The proof is complete. □

Lemma 2.6. *Suppose that Assumption* 2.1 *holds, then we have*

$$\sum_k \alpha_k^2 \|d_k\|^2 < \infty. \tag{2.17}$$

Proof. By (1.5) and (2.4) we have

$$\begin{aligned}\sum_k \alpha_k^2 \|d_k\|^2 &= \sum_k \frac{(\delta g_k^T d_k)^2}{[(\bar{g}_{k+1} - g_k)^T d_k + \|d_k\|^2]^2} \|d_k\|^2 \\ &\le (\frac{\delta}{\kappa + \gamma})^2 \sum_{d_k \neq 0} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty.\end{aligned} \tag{2.18}$$

The proof is complete. □

3. Global convergence of the modified PRP method

In this section, we discuss the convergence properties of a modified PRP method conjugate gradient method, in which β_k^{MPRP} is given by (1.3).

We give the following algorithm firstly.

Algorithm 3.1
Step 0: Given $x_1 \in R^n$, set $d_1 = -g_1$, $k = 1$.
Step 1: If $\|g_k\| = 0$ then stop else go to Step 2.
Step 2: Set $x_{k+1} = x_k + \alpha_k d_k$ where d_k is defined by (1.2), and α_k is defined by (1.5).
Step 3: Compute β_{k+1}^{MPRP} using formula (1.3).
Step 4: Set $k := k + 1$, go to Step 1.

In 1992, Gilbert and Nocedal introduced the property (*) which plays an important role in the studies of CG methods. This property means that the next research direction approaches the steepest direction automatically when a small step-size is generated, and the step-sizes are not produced successively [15].

Property (*). Consider a CG method of the form (1.1) and (1.2). Suppose that, for all k,

$$0 < r \le \|g_k\| \le \bar{r}, \tag{3.1}$$

where r and $\bar{r}$ are two constants. If there exist $b > 1$ and $\lambda > 0$ such that for all k,

$$|\beta_k^{MPRP}| \leq b, \tag{3.2}$$

and

$$\|s_k\| \leq \lambda \Longrightarrow |\beta_k^{MPRP}| \leq \frac{1}{2b}, \tag{3.3}$$

where $s_{k-1} = \alpha_{k-1} d_{k-1}$.

The following Lemma shows that the MPRP method has Property (*).

Lemma 3.2. *Consider the method of form* (1.1) *and* (1.2). *Suppose that Assumption 2.1 hold, then, the method β_k^{MPRP} has Property* $(*)$.

Proof. Consider any constant r and $\bar{r}$ which satisfy (3.1).

Let $b = \dfrac{2\bar{r}^2}{r^2} > 1$, $\lambda = \dfrac{r^4}{4\tau\bar{r}^3}$. By (1.3) we have

$$|\beta_k^{MPRP}| \leq |\frac{g_k^T y_{k-1}}{\mu|g_k^T d_{k-1}| + \|g_{k-1}\|^2}| \leq \frac{\|g_k\|^2 + \|g_k\| \, \|g_{k-1}\|}{\|g_{k-1}\|^2} \leq \frac{2\bar{r}^2}{r^2} = b. \tag{3.4}$$

From (2.1), holds. If then

$$|\beta_k^{MPRP}| \leq \frac{\|g_k\| \, \|g_k - g_{k-1}\|}{\|g_{k-1}\|^2} \leq \frac{\tau \, \|s_{k-1}\| \, \|g_k\|}{\|g_{k-1}\|} \leq \frac{\tau\lambda\bar{r}}{r^2} = \frac{1}{2b}. \tag{3.5}$$

The proof is finished. □

Theorem 3.3. *Under Assumption 2.1, the method defined by* (1.1), (1.2), (1.5) *and* (1.3) *will generate a sequence $\{x_k\}$ such that* $\lim_{k\longrightarrow\infty} \inf \|g_k\| = 0$.

Proof. Suppose on the contrary that $\|g_k\| \geq \psi$, for all k.

Since L is bounded, both $\{x_k\}$ and $\{g_k\}$ are bounded. By using

$$\|d_k\| \leq \|g_k\| + |\beta_k^{MPRP}| \, \|d_{k-1}\| \,, \tag{3.6}$$

one can show that $\{\|d_k\|\}$ is uniformly bounded. Definition (1.2) implies the following relation

$$|g_k^T d_k| = |g_k^T(-g_k + \beta_k^{MPRP} d_{k-1})| \tag{3.7}$$

$$\geq \|g_k\|^2 - |\beta_k^{MPRP}| \, \|g_k\| \, \|d_{k-1}\| \, . \tag{3.8}$$

From (1.3) and using the Cauchy-Schwarz inequality, we have

$$|\beta_k^{MPRP}| = |\frac{g_k^T(g_k - g_{k-1})}{\mu|g_k^T d_{k-1}| + \|g_{k-1}\|^2}|. \tag{3.9}$$

From (2.1) and (2.18) we have

$$\begin{aligned}\|g_k - g_{k-1}\| &\leq \tau\alpha_{k-1}\|d_{k-1}\| \\ &\leq (\frac{\tau\delta}{\kappa+\gamma})\frac{|g_{k-1}^T d_{k-1}|}{\|d_{k-1}\|} \leq \frac{|g_{k-1}^T d_{k-1}|}{\|d_{k-1}\|}.\end{aligned} \tag{3.10}$$

From (1.4), (2.7) we have

$$\mu|g_k^T d_{k-1}| + \|g_{k-1}\|^2 = (\mu\rho_{k-1} + \frac{1}{c})|g_{k-1}^T d_{k-1}| = m|g_{k-1}^T d_{k-1}|, (m > 1). \tag{3.11}$$

By (3.9), (3.10), and (3.11) we have

$$|\beta_k^{MPRP}|\,\|d_{k-1}\| \leq \frac{\|g_k\|}{m}. \tag{3.12}$$

Hence by substituting (3.12) in (3.8), we have

$$|g_k^T d_k| \geq A\,\|g_k\|^2, A = \frac{m-1}{m}, \tag{3.13}$$

for large k. Thus we have

$$\frac{(g_k^T d_k)^2}{\|d_k\|^2\,\|g_k\|^2} \geq A^2\frac{\|g_k\|^2}{\|d_k\|^2}. \tag{3.14}$$

Since $\|g_k\| \geq \psi$ and $\|d_k\|$ is bounded above, we conclude that there is $\varepsilon > 0$ such that $\dfrac{(g_k^T d_k)^2}{\|d_k\|^2\,\|g_k\|^2} \geq \varepsilon$, which implies $\sum_{d_k\neq 0}\frac{(g_k^T d_k)^2}{\|d_k\|^2} = \infty$.

This is a contradiction to Lemma 2.5. □

4. Numerical experiments and discussions

In this part, we present the numerical experiments of the new formula (1.5) and apply it using (1.3), computer
(Processor: Intel(R)core(TM)i3-3110M cpu@2.40GHZ, Ram 4.00 GB) through the Matlab programme.
10 testing problems have been taken from [1].

This will lead us to test for the global convergence properties of our method. Stopping criteria is set to $\|g_k\| \leq \varepsilon$ where $\varepsilon = 10^{-6}$. Taking into consideration the following parameters: $\gamma = 1.5$ and $\mu = 0.5$.

Table 1 list numerical results. The meaning of each column is as follows: "Problem "the name of the test problem, "δ", "Xzero", "k "the number of iterations, "Time", "Xoptimal".

The following results showed the effectiveness of the proposed method.

Table 1

	Problem	δ	Xzero	k	Time	Xopimal
1	Booth	1	(1 1)	46	0.118	(1.0 3.0)
2	Branin	1.5	(1 1)	54	0.113	(3.1416 2.275)
3	Sphere	1	(-1 1)	64	0.015	(-0.230 -0.230)
4	Exponential	1	(-1 1)	59	0.082	(-0.6406 -0.6406)
5	Himmelblau	2	(1 1)	258	0.084	(0.6403 -0.6403)
6	Matyas	1	(-1 1)	34	0.047	(0.6403 -0.6403)
7	McCormick	1	(-1 1)	36	0.048	(-0.5472 -1.5472)
8	Rosenbrock	0.4	(1 1)	4999	0.735	(0.4198 1.9116)
9	SIX-HUMP CAMEL	2	(1 1)	15	0.031	(-0.0898 0.7127)
10	THREE-HUMP CAMEL	1.5	(1 1)	46	0.1180	(0.2665 -0.2935)

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Nonexistence of global solutions of some nonlinear ultra-parabolic equations on the Heisenberg group

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Abstract. This article provides sufficient conditions for non existence Global weak solutions for non-local and non-linear equivalent equations on $\mathbb{H}^{\mathbb{N}} \times (0,\infty) \times (0,\infty)$, where $\mathbb{H}^{\mathbb{N}}$ is the Heisenberg group. Our method of proof relies on a suitable choice of a test function and the weak formulation approach of the sought for solutions.
Keywords: Timoshenko system, second sound well-posedness, exponential stability, distributed delay.

1. Introduction

In this article we are concerned with the nonexistence of global solutions of nonlocal nonlinear ultra-parabolic two-times equation posed on the Heisenberg group.

We start with the equation:

$$\mathbf{D}^{\alpha_1}_{0|t_1}(u) + \mathbf{D}^{\alpha_2}_{0|t_2}(u) + (-\Delta_{\mathbb{H}})^{\alpha/2}(|u|^m) = |u|^p \tag{1.1}$$

posed for $\omega = (\eta, t_1, t_2) \in Q = \mathbb{H}^N \times \mathbb{R}^+ \times \mathbb{R}^+$, $N \in \mathbb{N}$ and supplemented with the initial conditions

$$u(\eta, t_1, 0) = u_1(\eta, t_1)\ \ , u(\eta, 0, t_2) = u_2(\eta, t_2)$$

Here, $p > 1$ are real number, $m \in \mathbb{N}$ and where for $0 < \alpha_1 < \alpha_2 < 1$ and $\mathbf{D}^{\alpha_1}_{0|t_1}$, $\mathbf{D}^{\alpha_2}_{0|t_2}$ is the fractional derivative in the sense of the so-called Caputo's. Then, we extend our results to the system of two equations

$$\begin{cases} \mathbf{D}^{\alpha_1}_{0|t_1}(u) + \mathbf{D}^{\alpha_2}_{0|t_2}(u) + (-\Delta_{\mathbb{H}})^{\alpha/2}(|u|^m) = |v|^p, \\ \mathbf{D}^{\beta_1}_{0|t_1}(v) + \mathbf{D}^{\beta_2}_{0|t_2}(v) + (-\Delta_{\mathbb{H}})^{\beta/2}(|v|^n) = |u|^q \end{cases} \tag{1.2}$$

posed for $\omega = (\eta, t_1, t_2) \in Q = \mathbb{H}^N \times \mathbb{R}^+ \times \mathbb{R}^+$, $N \in \mathbb{N}$ and supplemented with the initial conditions

$$\begin{aligned} &u(\eta, t_1, 0) = u_1(\eta, t_1)\ u(\eta, 0, t_2) = u_2(\eta, t_2), \\ &v(\eta, t_1, 0) = v_1(\eta, t_1)\ v(\eta, 0, t_2) = v_2(\eta, t_2). \end{aligned}$$

Here p, q are positive real numbers and $0 < \alpha_1 < \alpha_2 < 1$, $0 < \beta_1 < \beta_2 < 1, 0 < \alpha, \beta \leq 2$.

Heisenberg group. The Heisenberg group $\mathbb{H}^N$, whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = (x + \tilde{x},\ y + \tilde{y},\ \tau + \tilde{\tau} + 2\left(\langle x, \tilde{y}\rangle - \langle \tilde{x}, y\rangle\right)),\ \ \eta^{-1} = (-x,\ -y,\ \tau),$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^{N}$[1].

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau} \qquad \text{and} \qquad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$$

as

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(X_i^2 + Y_i^2\right), \tag{1.3}$$

An explicit calculation gives us the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4\left(x_i^2 + y_i^2\right) \frac{\partial^2}{\partial \tau^2} \right). \tag{1.4}$$

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator verifying the so-called Hormander condition of order 1. It is invariant by left multiplication in the group since

$$\Delta_{\mathbb{H}}\left(u\left(\eta \circ \tilde{\eta}\right)\right) = \left(\Delta_{\mathbb{H}} u\right)\left(\eta \circ \tilde{\eta}\right),\ \forall \left(\eta, \tilde{\eta}\right) \in \mathbb{H}^N \times \mathbb{H}^N.$$

The homogeneous norm of the space is

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} \left(x_i^2 + y_i^2\right)\right)^2\right)^{1/4} \tag{1.5}$$

and the natural distance is accordingly defined $d\left(\eta, \tilde{\eta}\right) = |\tilde{\eta}^{-1} \circ \eta|_{\mathbb{H}}$. It is also important to observe that, $\eta \to |\eta|_{\mathbb{H}}$ is homogeneous of degree one compared to the natural group of dilatations

$$\delta_\lambda\left(\eta\right) = \left(\lambda x, \lambda y, \lambda^2 \tau\right), \tag{1.6}$$

whose Jacobian determinant is λ^Λ, where $\Lambda = 2N + 2$, is the homogeneous dimension of $\mathbb{H}$. Note also, that the $\Delta_{\mathbb{H}}$ operator is homogeneous of degree 2 with respect to the dilatation δ_λ defined in (1.6), namely

$$\Delta_{\mathbb{H}} = \lambda^2 \delta_\lambda\left(\Delta_{\mathbb{H}}\right).$$

1. here $\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i$

See [3], [4], [9], [11], [13].

Now, we call *sub-elliptic gradient*

$$\nabla_{\mathbb{H}} = (X, Y) = (X_1, \ldots, X_N, Y_1, \ldots, Y_N)\,. \tag{1.7}$$

A remarkable property of the Kohn Laplacian is that a fundamental solution of $-\Delta_{\mathbb{H}}$ with pole at zero is given by

$$\Gamma(\eta) = \frac{C_\Lambda}{|\eta|_{\mathbb{H}}^{\Lambda-2}}, \tag{1.8}$$

where C_Λ is a suitable positive constant.

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality

$$\|v\|_{\Lambda^*}^2 \leq c\|\nabla_{\mathbb{H}} v\|_2^2, \forall v \in \mathrm{C}_0^\infty\left(\mathbb{H}^N\right), \tag{1.9}$$

where $\Lambda^* = \frac{2\Lambda}{\Lambda-2}$ and c is a positive constant.

This inequality ensures in particular that for every domain Ω the function

$$\|v\| \leq \|\nabla_{\mathbb{H}} v\|_2$$

is a norm on $\mathrm{C}_0^\infty(\Omega)$. We denote by $\mathrm{S}_0^1(\Omega)$ the closure of $\mathrm{C}_0^\infty(\Omega)$ with respect to this norm; $\mathrm{S}_0^1(\Omega)$ becomes a Hilbert space with the inner product

$$< u, v >_{\mathrm{S}_0^1} = \int_\Omega < \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} v > .$$

Fractional powers of sub-elliptic Laplacians. Here, we recall a result on fractional powers of sub-Laplacian in the Heisenberg group. Let $\mathcal{N}(t, x)$ be the fundamental solution of $\Delta_{\mathbb{H}} + \frac{\partial}{\partial t}$. For all $0 < \beta < 4$, the integral

$$R_\beta(x) = \frac{1}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t, x)\, dt$$

converges absolutely for $x \neq 0$. If $\beta < 0, \beta \notin \{0, -2, -4, \ldots\}$, then

$$\tilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} \mathcal{N}(t, x)\, dt$$

defines a smooth function in $\mathbb{H}\backslash\{0\}$, since $t \mapsto \mathcal{N}(t, x)$, vanishes of infinite order as $t \to 0$ if $x \neq 0$. In addition, $\tilde{R}_\beta$ is positive and $\mathbb{H}$-homogeneous of degree $\beta - 4$.

1.1 Theorem

For every $v \in \mathcal{S}(\mathbb{H})^2$ $(-\Delta_{\mathbb{H}})^s v \in L^2(\mathbb{H})$ and

$$(-\Delta_{\mathbb{H}})^s v(x) = \int_{\mathbb{H}} (v(x \circ y) - v(x) - \chi(y) < \nabla_{\mathbb{H}} v(x), y >) \tilde{R}_{-2s}(y)\, dy,$$

where χ is the characteristic function of the unit ball $B_\rho(0,1)$, $(\rho(x) = R_{2-\alpha}^{\frac{-1}{2-\alpha}}(x)$, $0 < \alpha < 2$, ρ is an $\mathbb{H}$-homogeneous norm in $\mathbb{H}$ smooth outside the origin).

1.2 Note

Proof of the Theorem 1.1 (see [2]).

1.3 Note

For $\alpha = 2$ in equation (1.1) (see, ([1], [6], [11]).

2. Preliminaries

The nonlocal operator (the left-sided Riemann-Liouville) $D^\alpha_{0|t}$ is defined, for a an absolutely continuous function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$\left(D^\alpha_{0|t}\right) g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{g(\tau)}{(t-\tau)^\alpha} d\tau$$

and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler gamma function. And the right-sided Riemann-Liouville derivatives of order $0 < \alpha < 1$. are defined, by:

$$\left(D^\alpha_{t|T}\right) g(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{g(\tau)}{(\tau - t)^\alpha} d\tau.$$

Note that for a differentiable function g, we have the left-sided Caputo derivatives of order α are defined as:

$$\mathbf{D}^\alpha_{0|t}(g)(t) = D^\alpha_{0|t}(g - g(0))(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(\tau)}{(\tau - t)^\alpha} d\tau.$$

Finally, taking into account the following integration by parts formula:

$$\int_0^T f(t)\, D^\alpha_{0|t} g(t)\, dt = \int_0^T D^\alpha_{t|T} f(t)\, g(t)\, dt.$$

Now, we define the regular function $\psi : \psi \in C_0^2(\mathbb{R}^+)^3$ by

$$\psi(\xi) = \begin{cases} 1, & \text{if } 0 \le \xi \le 1, \\ \searrow, & \text{if } 1 \le \xi \le 2, \\ 0, & \text{if } \xi \ge 2, \end{cases} \tag{2.1}$$

2. Schwartz's class
3. Space defined and continuous functions and differentiable twice and compact support on $\mathbb{R}^+$

which will be used hereafter.

3. Résults

3.1 Definition

A locally integrable function $u \in L^m_{loc}(Q_T) \cap L^p_{loc}(Q_T)$ is called a local weak solution of (1.1) in Q_T $\left(Q_T = \mathbb{H}^N \times [0,T] \times [0,T]\right)$ subject to the initial data $u_1, u_2 \in L^1_{loc}\left(\mathbb{H}^N \times [0,T]\right)$ if the equality

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi d\omega + \int_{Q_T} u_2 D^{\alpha_1}_{t_1|T} \varphi d\omega + \int_{Q_T} u_1 D^{\alpha_2}_{t_2|T} \varphi d\omega \\ &= \int_{Q_T} u D^{\alpha_1}_{t_1|T} \varphi d\omega + \int_{Q_T} u D^{\alpha_2}_{t_2|T} \varphi d\omega + \int_{Q_T} |u|^m \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2} \varphi d\omega \end{aligned} \tag{3.1}$$

is satisfied for any φ be a smooth test function ($\varphi \in \mathrm{C}^\infty_0(Q_T)$) with

$$\varphi\left(., T, .\right) = \varphi\left(., ., T\right) = 0, \;\; \varphi \geq 0, \;\; d\omega = d\eta dt_1 dt_2$$

and the solution is called global if $T = +\infty$.

3.2 Theorem

Let $1 < m < p < p_c$, where

$$p_c = m + \frac{m\alpha - (m-1)(\frac{\alpha}{\alpha_1} + \frac{\alpha}{\alpha_2})}{2N + 2 - \alpha + (\frac{\alpha}{\alpha_1} + \frac{\alpha}{\alpha_2})}, \;\; \text{(c for critical)}$$

and

$$\int_Q u_2 D^{\alpha_1}_{t_1|T} \varphi d\omega > 0, \quad \int_Q u_1 D^{\alpha_2}_{t_2|T} \varphi d\omega > 0.$$

Then, (1.1) does not have a nontrivial global weak solution. For the proof, we need to recall the following proposition from Proposition 2.3.

3.3 Proposition

Consider a convex function $F \in C^2(\mathbb{R})$. Assume that $\varphi \in C^\infty_0(\mathbb{R}^{2N+1})$, then

$$F'(\varphi)(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi \geq (-\Delta_{\mathbb{H}})^{\alpha/2} F(\varphi). \tag{3.2}$$

In particular, if $F(0) = 0$ and $\varphi \in C^\infty_0(\mathbb{R}^{2N+1})$, then

$$\int_{\mathbb{R}^{2N+1}} F'(\varphi)(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi d\eta \geq 0. \tag{3.3}$$

Let us mention that hereafter we will use inequality (2.1) for $F(\varphi) = \varphi^\sigma$, $\sigma \gg 1$[4], $\varphi \geq 0$; in this case it reads

$$\sigma\varphi^{\sigma-1}(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi \geq (-\Delta_{\mathbb{H}})^{\alpha/2}\varphi^\sigma. \tag{3.4}$$

We need the following Lemma taken from [32].

4. σ is much larger than 1

3.4 Lemma

Let $f \in L^1\left(\mathbb{R}^{2N+1}\right)$ and $\int_{\mathbb{R}^{2N+1}} f d\eta \geq 0$. Then there exists a test function $0 \leq \varphi \leq 1$ such that

$$\int_{\mathbb{R}^{2N+1}} f\varphi d\eta \geq 0. \tag{3.5}$$

3.5 Note

Let us set

$$\int_{Q_T} = \int_0^T \int_0^T \int_{\mathbb{R}^{2N+1}}, \quad \int_Q = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{2N+1}}.$$

Proof of Theorem 3.2. The proof is by contradiction. For that, let u be a solution and φ be a smooth nonnegative test function such that

$$\begin{aligned}
\mathcal{A}(\varphi) &= \int_Q |D^{\alpha_1}_{t_1|T}\varphi^\sigma|^{\frac{p}{p-1}}\varphi^{\frac{-\sigma}{p-1}} d\omega < \infty, \\
\mathcal{B}(\varphi) &= \int_Q |D^{\alpha_2}_{t_2|T}\varphi^\sigma|^{\frac{p}{p-1}}\varphi^{\frac{-\sigma}{p-1}} d\omega < \infty, \\
\mathcal{K}(\varphi) &= \int_Q |(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi|^{\frac{p}{p-m}}\varphi^{(\sigma-\frac{p}{p-m})} d\omega < \infty.
\end{aligned} \tag{3.6}$$

Then, taking φ^σ, $\sigma \gg 1$ instead of φ in (3.1) and using inequality (3.4), we obtain

$$\begin{aligned}
&\int_Q |u|^p\varphi^\sigma d\omega + \int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega + \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega \\
&= \int_Q u D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega + \int_Q u D^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega + \int_Q |u|^m \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi^\sigma d\omega \\
&\leq \int_Q u D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega + \int_Q u D^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega + \int_Q |u|^m \sigma\varphi^{\sigma-1} \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi d\omega.
\end{aligned}$$

- For $\int_Q u D^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega$ by means of the ε-Young's inequality $ab \leq \varepsilon a^p + C(\varepsilon)b^{\frac{p}{p-1}}$, $\frac{1}{p} + \frac{p-1}{p} = 1, a \geq 0, b \geq 0$, we obtain

$$u D^{\alpha_1}_{t_1|T}\varphi^\sigma \leq |u||D^{\alpha_1}_{t_1|T}\varphi^\sigma| = \varphi^{\frac{\sigma}{p}}\varphi^{\frac{-\sigma}{p}}|u||D^{\alpha_1}_{t_1|T}\varphi^\sigma| = |u|\varphi^{\frac{\sigma}{p}}|D^{\alpha_1}_{t_1|T}\varphi^\sigma|\varphi^{\frac{-\sigma}{p}}$$

because $\varphi^{\frac{\sigma}{p}}\varphi^{\frac{-\sigma}{p}} = \varphi^{\frac{\sigma}{p}-\frac{\sigma}{p}} = \varphi^0 = 1$ if we pose $a = |u|\varphi^{\frac{\sigma}{p}}$ and $b = |D^{\alpha_1}_{t_1|T}\varphi^\sigma|\varphi^{\frac{-\sigma}{p}}$, then

$$u D^{\alpha_1}_{t_1|T}\varphi^\sigma \leq ab \leq \varepsilon a^p + C(\varepsilon)b^{\frac{p}{p-1}} = \varepsilon\left(|u|\varphi^{\frac{\sigma}{p}}\right)^p + C(\varepsilon)\left(|D^{\alpha_1}_{t_1|T}\varphi^\sigma|\varphi^{\frac{-\sigma}{p}}\right)^{\frac{p}{p-1}}$$

$$\Updownarrow$$

$$u D^{\alpha_1}_{t_1|T}\varphi^\sigma \leq \varepsilon|u|^p\varphi^\sigma + C(\varepsilon)|D^{\alpha_1}_{t_1|T}\varphi^\sigma|^{\frac{p}{p-1}}\varphi^{\frac{-\sigma}{p-1}}$$

$$\Downarrow$$

$$\int_Q uD^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega \le \varepsilon \int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\int_Q |D^{\alpha_1}_{t_1|T}\varphi^\sigma|^{\frac{p}{p-1}}\varphi^{\frac{-\sigma}{p-1}}d\omega$$

$$\Updownarrow$$

$$(I)\qquad \int_Q uD^{\alpha_1}_{t_1|T}\varphi^\sigma d\omega \le \varepsilon \int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\mathcal{A}(\varphi)$$

• For $\int_Q uD^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega$ we take the previous method with placing t_2 place t_1 and α_2 place α_1 to get the

$$(II)\qquad \int_Q uD^{\alpha_2}_{t_2|T}\varphi^\sigma d\omega \le \varepsilon \int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\mathcal{B}(\varphi)$$

• For $\int_Q |u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi d\omega$ by means of the ε-Young's inequality $ab \le \varepsilon a^{\frac{p}{m}} + C(\varepsilon)b^{\frac{p}{p-m}}$, $\frac{m}{p}+\frac{p-m}{p}=1, a\ge 0, b\ge 0$, we obtain $|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi = 1.|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi = \varphi^{\frac{m\sigma}{p}}\varphi^{\frac{-m\sigma}{p}}|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi$ because $\varphi^{\frac{m\sigma}{p}}\varphi^{\frac{-m\sigma}{p}} = \varphi^{\frac{m\sigma}{p}-\frac{m\sigma}{p}} = \varphi^0 = 1$, if we pose $a = |u|^m\varphi^{\frac{m\sigma}{p}}$ and $b = |\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|\sigma\varphi^{\sigma-1-\frac{m\sigma}{p}}$, then

$$|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi \le ab \le \varepsilon a^{\frac{p}{m}} + C(\varepsilon)b^{\frac{p}{p-m}}$$

$$\varepsilon a^{\frac{p}{m}} + C(\varepsilon)b^{\frac{p}{p-m}} = \varepsilon\left(|u|^m\varphi^{\frac{m\sigma}{p}}\right)^{\frac{p}{m}} + C(\varepsilon)\left(|\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|\sigma\varphi^{\sigma-1-\frac{m\sigma}{p}}\right)^{\frac{p}{p-m}}$$

$$\Updownarrow$$

$$|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi \le \varepsilon|u|^p\varphi^\sigma + C(\varepsilon)|\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|^{\frac{p}{p-m}}\sigma^{\frac{p}{p-m}}\varphi^{(\sigma-1-\frac{m\sigma}{p})\frac{p}{p-m}}$$

$$\Updownarrow$$

$$|u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi \le \varepsilon|u|^p\varphi^\sigma + C(\varepsilon)|\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|^{\frac{p}{p-m}}\sigma^{\frac{p}{p-m}}\varphi^{(\sigma-\frac{p}{p-m})}$$

$$\Downarrow$$

$$\begin{aligned}&\int_Q |u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi d\omega\\ &\le \varepsilon\int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\sigma^{\frac{p}{p-m}}\int_Q |\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi|^{\frac{p}{p-m}}\varphi^{(\sigma-\frac{p}{p-m})}d\omega\end{aligned}$$

$$\Updownarrow$$

$$(III)\qquad \int_Q |u|^m\sigma\varphi^{\sigma-1}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi d\omega \le \varepsilon\int_Q |u|^p\varphi^\sigma d\omega + C(\varepsilon)\sigma^{\frac{p}{p-m}}\mathcal{K}(\varphi).$$

Now, we choose $\varepsilon = \frac{1}{6}$ and $C = \max\left\{C(\varepsilon), C(\varepsilon)\sigma^{\frac{p}{p-m}}\right\}$ and the (I), (II), (III) we obtain

$$\begin{aligned}
&\int_Q |u|^p \varphi^\sigma d\omega + \int_Q u_2 D^{\alpha_1}_{t_1|T} \varphi^\sigma d\omega + \int_Q u_1 D^{\alpha_2}_{t_2|T} \varphi^\sigma d\omega \\
&\leq \frac{1}{2}\int_Q |u|^p \varphi^\sigma d\omega + C\left(\mathcal{A}(\varphi) + \mathcal{B}(\varphi) + \mathcal{K}(\varphi)\right).
\end{aligned} \tag{3.7}$$

We choose the test function $\varphi(\eta, t_1, t_2)$, in the form

$$\varphi(\eta, t_1, t_2) = \varphi_1(\eta)\varphi_2(t_1)\varphi_3(t_2), \tag{3.8}$$

where $\varphi_1(\eta) = \psi(\frac{\tau^2+|x|^4+|y|^4}{R^4})$, and $\varphi_2(t_1) = \psi(\frac{t_1}{R^{\rho_1}})$, and $\varphi_3(t_2) = \psi(\frac{t_2}{R^{\rho_2}})$, and $\rho_1 = \frac{\alpha(p-1)}{\alpha_1(p-m)}$, and $\rho_2 = \frac{\alpha(p-1)}{\alpha_2(p-m)}$. Set

$$\begin{aligned}
\Omega_1 &= \left\{\tilde{\eta} \in \mathbb{H};\ 0 < \tilde{\tau}^2 + |\tilde{x}|^4 + |\tilde{y}|^4 \leq 2\right\},\\
\Omega_2 &= \left\{\tilde{t_1};\ 0 \leq \tilde{t_1} \leq 2\right\},\\
\Omega_3 &= \left\{\tilde{t_2};\ 0 \leq \tilde{t_2} \leq 2\right\},
\end{aligned}$$

we apply the change of next variables $\tilde{\tau} = R^{-2}\tau$, $\tilde{x} = R^{-1}x$, $\tilde{y} = R^{-1}y$, $\tilde{t}_1 = R^{-\rho_1}t_1$, $\tilde{t}_2 = R^{-\rho_2}t_2$, we obtain the estimates,

$$\mathcal{A}(\varphi) \leq \mathbf{A}R^{\mathbf{a}},\ \ \mathcal{B}(\varphi) \leq \mathbf{B}R^{\mathbf{b}},\ \ \mathcal{K}(\varphi) \leq \mathbf{K}R^{\mathbf{k}} \tag{3.9}$$

with

$$\begin{aligned}
\mathbf{a} &= -\frac{\alpha_1\rho_1 p}{p-1} + 2N + 2 + \rho_1 + \rho_2 = -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2,\\
\mathbf{b} &= -\frac{\alpha_2\rho_2 p}{p-1} + 2N + 2 + \rho_1 + \rho_2 = -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2,\\
\mathbf{k} &= -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2,
\end{aligned}$$

we put $\mathbf{v} = -\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2$. Then

$$\mathcal{A}(\varphi) \leq \mathbf{A}R^{\mathbf{v}},\ \ \mathcal{B}(\varphi) \leq \mathbf{B}R^{\mathbf{v}},\ \ \mathcal{K}(\varphi) \leq \mathbf{K}R^{\mathbf{v}} \tag{3.10}$$

the constants $\mathbf{A}$; $\mathbf{B}$; $\mathbf{K}$ are $\mathcal{A}(\varphi)$ and $\mathcal{B}(\varphi)$ and $\mathcal{K}(\varphi)$ evaluated on $\Omega_1 \times \Omega_2 \times \Omega_3$. Now, if

$$-\frac{\alpha p}{p-m} + 2N + 2 + \rho_1 + \rho_2 < 0 \ \Leftrightarrow\ p < p_c$$

by letting $R \to \infty$ in (2.6), we obtain

$$\int_Q |u|^p d\omega = 0 \quad \Rightarrow \quad u \equiv 0$$

this is a contradiction. □

4. System of fractional equations

We consider

$$\begin{cases} \mathbf{D}_{0|t_1}^{\alpha_1}(u) + \mathbf{D}_{0|t_2}^{\alpha_2}(u) + (-\Delta_{\mathbb{H}})^{\alpha/2}(|u|^m) = |v|^p \\ \mathbf{D}_{0|t_1}^{\beta_1}(v) + \mathbf{D}_{0|t_2}^{\beta_2}(v) + (-\Delta_{\mathbb{H}})^{\beta/2}(|v|^n) = |u|^q \end{cases}$$

posed for $\omega = (\eta, t_1, t_2) \in Q = \mathbb{H}^N \times \mathbb{R}^+ \times \mathbb{R}^+$, $N \in \mathbb{N}$ and supplemented with the initial conditions $u(\eta, t_1, 0) = u_1(\eta, t_1)$, $u(\eta, 0, t_2) = u_2(\eta, t_2)$, $v(\eta, t_1, 0) = v_1(\eta, t_1)$, $v(\eta, 0, t_2) = v_2(\eta, t_2)$, Here, p, q are positive real numbers and $0 < \alpha_1 < \alpha_2 < 1$, $0 < \beta_1 < \beta_2 < 1$,$0 < \alpha, \beta \leq 2$.

Let us set

$$I_0 = \int_Q u_2 D_{t_1|T}^{\alpha_1}\varphi d\omega + \int_Q u_1 D_{t_2|T}^{\alpha_2}\varphi d\omega,$$
$$J_0 = \int_Q v_2 D_{t_1|T}^{\beta_1}\varphi d\omega + \int_Q v_1 D_{t_2|T}^{\beta_2}\varphi d\omega$$

where $d\omega = d\eta dt_1 dt_2$.

4.1 Definition

We say that $(u, v) \in \left(L_{loc}^q(Q) \cap L_{loc}^m(Q)\right) \times \left(L_{loc}^p(Q) \cap L_{loc}^n(Q)\right)$ is a weak formulation to system (1.2) if

$$\begin{cases} \displaystyle\int_Q |v|^p \varphi d\omega + I_0 = \int_Q u D_{t_1|T}^{\alpha_1}\varphi d\omega + \int_Q u D_{t_2|T}^{\alpha_2}\varphi d\omega + \int_Q |u|^m (-\Delta_{\mathbb{H}})^{\alpha/2}\varphi d\omega, \\ \displaystyle\int_Q |u|^q \varphi d\omega + J_0 = \int_Q v D_{t_1|T}^{\beta_1}\varphi d\omega + \int_Q v D_{t_2|T}^{\beta_2}\varphi d\omega + \int_Q |v|^n (-\Delta_{\mathbb{H}})^{\beta/2}\varphi d\omega \end{cases}$$

for any test function φ (see the equality (3.1)). Now, set

$$\sigma_1 = -\frac{1}{pq-1}\left[pq\alpha_1 + p\beta_1 - 2(pq-1) - \left(\frac{(pq-p)\alpha_1}{\alpha} + \frac{(p-1)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_2 = -\frac{1}{pq-1}\left[pq\alpha_1 + p\beta_2 - 2(pq-p) - \left(\frac{(pq-p)\alpha_1}{\alpha} + \frac{(p-1)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_3 = -\frac{1}{pq-n}\left[pq\alpha_1 + p\beta_1 - 2(2pq-nq-p)\right.$$
$$\left. - \left(\frac{(pq-p)\alpha_1}{\alpha} + \frac{(pq-nq)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_4 = -\frac{1}{pq-1}\left[pq\alpha_2 + p\beta_1 - 2(pq-1) - \left(\frac{(pq-p)\alpha_1}{\alpha} + \frac{(p-1)\beta_1}{\beta}\right)(2N+2)\right],$$
$$\sigma_5 = -\frac{1}{pq-1}\left[pq\alpha_2 + p\beta_2 - 2(pq-1) - \left(\frac{(p-1)\alpha_1}{\alpha} + \frac{(pq-p)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_6 = -\frac{1}{pq-n}\left[pq\alpha_2 + p\beta_1 - 2\left(pq-n\right) - \left(\frac{(pq-p)\alpha_1}{\alpha} + \frac{(p-n)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_7 = -\frac{1}{pq-m}\left[pq\alpha_1 + pm\beta_1 - 2\left(pq-m\right)\right.$$
$$\left. - \left(\frac{(pq-pm)\alpha_1}{\alpha} + \frac{(mp-m)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_8 = -\frac{1}{pq-m}\left[pq\alpha_1 + pm\beta_2 - 2\left(pq-m\right)\right.$$
$$\left. - \left(\frac{(pq-pm)\alpha_1}{\alpha} + \frac{(mp-m)\beta_1}{\beta}\right)(2N+2)\right],$$

$$\sigma_9 = -\frac{1}{pq-nm}\left[pq\alpha_1 + pm\beta_1 - 2\left(pq-nm\right)\right.$$
$$\left. - \left(\frac{(pq-pm)\alpha_1}{\alpha} + \frac{(mp-nm)\beta_1}{\beta}\right)(2N+2)\right].$$

Note. The way we calculate $\{\sigma_1, \ldots, \sigma_9\}$ is the same as the way we calculate $\{\delta_1, \ldots, \delta_9\}$.

4.2 Theorem

Let $q > 1$, $p > 1$, $q > m$, $p > n$ and suppose that

$$\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\mu d\omega > 0, \ \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\mu d\omega > 0,$$
$$\int_Q v_2 D^{\beta_1}_{t_1|T}\varphi^\mu d\omega > 0 \ \ \int_Q v_1 D^{\beta_2}_{t_2|T}\varphi^\mu d\omega > 0.$$

If $\max\{\sigma_1, \ldots, \sigma_9, \delta_1, \ldots, \delta_9\} \leq 0$.

Then, the system (1.2) does not admit local nontrivial weak solution[5].

Proof. As in the proof of Theorem 1, we reason by the absurd. Suppose (u; v) is a weak non-trivial solution that exists globally in time. Next,replacing φ by φ^μ in (4.1). Since the initial conditions u_0 and v_0 are positive, the variational formulation (4.1) leads to

$$\begin{cases} \int_Q |v|^p \varphi^\mu d\omega \leq \int_Q u D^{\alpha_1}_{t_1|T}\varphi^\mu d\omega + \int_Q u D^{\alpha_2}_{t_2|T}\varphi^\mu d\omega \\ + \int_Q |u|^m \left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\varphi^\mu d\omega, \\ \int_Q |u|^q \varphi^\mu d\omega \leq \int_Q v D^{\beta_1}_{t_1|T}\varphi^\mu d\omega + \int_Q v D^{\beta_2}_{t_2|T}\varphi^\mu d\omega \\ + \int_Q |v|^n \left(-\Delta_{\mathbb{H}}\right)^{\beta/2}\varphi^\mu d\omega. \end{cases} \tag{4.1}$$

5. Then solutions to system (1.2) blow-up whenever $\max\{\sigma_1, \ldots, \sigma_9, \delta_1, \ldots, \delta_9\} \leq 0$

Applying Hölder's inequality, we obtain the following estimates:

• For $q > m$

$$\int_Q |u|^m \left|(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi^\mu\right| d\omega \leq \mu\left(\int_Q |u|^q \varphi^\mu d\omega\right)^{\frac{m}{q}} \times \left(\int_Q \varphi^{\mu-\frac{q}{q-m}} \left|(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi\right|^{\frac{q}{q-m}} d\omega\right)^{\frac{q-m}{q}}. \tag{4.2}$$

• For $q > 1$:

$$\int_Q u \left|D^{\alpha_1}_{t_1|T}\varphi^\mu\right| d\omega \leq \left(\int_Q |u|^q \varphi^\mu d\omega\right)^{\frac{1}{q}} \times \left(\int_Q \left|D^{\alpha_1}_{t_1|T}\varphi^\mu\right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega\right)^{\frac{q-1}{q}} \tag{4.3}$$

and

$$\int_Q u \left|D^{\alpha_2}_{t_2|T}\varphi^\mu\right| d\omega \leq \left(\int_Q |u|^q \varphi^\mu d\omega\right)^{\frac{1}{q}} \times \left(\int_Q \left|D^{\alpha_2}_{t_2|T}\varphi^\mu\right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega\right)^{\frac{q-1}{q}}. \tag{4.4}$$

Similarly, we have:

• For $p > n$:

$$\int_Q |v|^n \left|(-\Delta_{\mathbb{H}})^{\beta/2}\varphi^\mu\right| d\omega \leq \mu\left(\int_Q |v|^p \varphi^\mu d\omega\right)^{\frac{n}{p}} \times \left(\int_Q \varphi^{\mu-\frac{p}{p-n}} \left|(-\Delta_{\mathbb{H}})^{\beta/2}\varphi\right|^{\frac{p}{p-n}} d\omega\right)^{\frac{p-n}{p}} \tag{4.5}$$

• For $p > 1$:

$$\int_Q v \left|D^{\beta_1}_{t_1|T}\varphi^\mu\right| d\omega \leq \left(\int_Q |v|^p \varphi^\mu d\omega\right)^{\frac{1}{p}} \times \left(\int_Q \left|D^{\beta_1}_{t_1|T}\varphi^\mu\right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega\right)^{\frac{p-1}{p}} \tag{4.6}$$

and

$$\int_Q v \left|D^{\beta_2}_{t_2|T}\varphi^\mu\right| d\omega \leq \left(\int_Q |v|^p \varphi^\mu d\omega\right)^{\frac{1}{p}} \times \left(\int_Q \left|D^{\beta_2}_{t_2|T}\varphi^\mu\right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega\right)^{\frac{p-1}{p}}. \tag{4.7}$$

If, we set

$$I_u = \int_Q |u|^q \varphi^\mu d\omega, \quad I_v = \int_Q |v|^p \varphi^\mu d\omega,$$

$$A(q,m) = \mu\left(\int_Q \varphi^{\mu-\frac{q}{q-m}} \left|(-\Delta_{\mathbb{H}})^{\alpha/2}\varphi\right|^{\frac{q}{q-m}} d\omega\right)^{\frac{q-m}{q}},$$

$$A(p,n) = \mu\left(\int_Q \varphi^{\mu-\frac{p}{p-n}} \left|(-\Delta_{\mathbb{H}})^{\beta/2}\varphi\right|^{\frac{p}{p-n}} d\omega\right)^{\frac{p-n}{p}},$$

$$
\begin{aligned}
B(q) &= \left(\int_Q \left|D^{\alpha_1}_{t_1|T}\varphi^{\mu}\right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega\right)^{\frac{q-1}{q}}, \\
B(p) &= \left(\int_Q \left|D^{\beta_1}_{t_1|T}\varphi^{\mu}\right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega\right)^{\frac{p-1}{p}}, \\
C(q) &= \left(\int_Q \left|D^{\alpha_2}_{t_2|T}\varphi^{\mu}\right|^{\frac{q}{q-1}} \varphi^{\frac{-\mu}{q-1}} d\omega\right)^{\frac{q-1}{q}}, \\
C(p) &= \left(\int_Q \left|D^{\beta_2}_{t_2|T}\varphi^{\mu}\right|^{\frac{p}{p-1}} \varphi^{\frac{-\mu}{p-1}} d\omega\right)^{\frac{p-1}{p}}, \\
I_0^{\mu} &= \int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^{\mu} d\omega + \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^{\mu} d\omega, \\
J_0^{\mu} &= \int_Q v_2 D^{\beta_1}_{t_1|T}\varphi^{\mu} d\omega + \int_Q v_1 D^{\beta_2}_{t_2|T}\varphi^{\mu} d\omega
\end{aligned}
$$

then, using estimates (4.3)-(4.4)-(4.5), we can write (4.1) as

$$
\begin{aligned}
I_v + I_0^{\mu} &\leq I_u^{\frac{1}{q}} B(q) + I_u^{\frac{1}{q}} C(q) + I_u^{\frac{m}{q}} A(q,m), \\
I_u + J_0^{\mu} &\leq I_v^{\frac{1}{p}} B(p) + I_v^{\frac{1}{p}} C(p) + I_v^{\frac{n}{p}} A(p,n).
\end{aligned}
$$

Since $I_0^{\mu}, J_0^{\mu} > 0$, we have

$$
I_v \leq I_u^{\frac{1}{q}} B(q) + I_u^{\frac{1}{q}} C(q) + I_u^{\frac{m}{q}} A(q,m), \tag{4.8}
$$

$$
I_u \leq I_v^{\frac{1}{p}} B(p) + I_v^{\frac{1}{p}} C(p) + I_v^{\frac{n}{p}} A(p,n). \tag{4.9}
$$

Now, from (4.8) and (4.9), we have

$$
\begin{aligned}
I_v + I_0^{\mu} \leq &\left(I_v^{\frac{1}{pq}} B^{\frac{1}{q}}(p) + I_v^{\frac{1}{pq}} C^{\frac{1}{q}}(p) + I_v^{\frac{n}{pq}} A^{\frac{1}{q}}(p,n)\right) B(q), \\
&+ \left(I_v^{\frac{1}{pq}} B^{\frac{1}{q}}(p) + I_v^{\frac{1}{pq}} C^{\frac{1}{q}}(p) + I_v^{\frac{n}{pq}} A^{\frac{1}{q}}(p,n)\right) C(q) \\
&+ \left(I_v^{\frac{m}{pq}} B^{\frac{m}{q}}(p) + I_v^{\frac{m}{pq}} C^{\frac{m}{q}}(p) + I_v^{\frac{nm}{pq}} A^{\frac{m}{q}}(p,n)\right) A(q,m).
\end{aligned}
$$

Then, Young's inequality implies

$$
\begin{aligned}
I_v + I_0^{\mu} \leq K \Bigg[&\left(B^{\frac{1}{q}}(p)B(q)\right)^{\frac{pq}{pq-1}} + \left(C^{\frac{1}{q}}(p)B(q)\right)^{\frac{pq}{pq-1}} + \left(A^{\frac{1}{q}}(p,n)B(q)\right)^{\frac{pq}{pq-n}} \\
&+ \left(B^{\frac{1}{q}}(p)C(q)\right)^{\frac{pq}{pq-1}} + \left(C^{\frac{1}{q}}(p)C(q)\right)^{\frac{pq}{pq-1}} + \left(A^{\frac{1}{q}}(p,n)C(q)\right)^{\frac{pq}{pq-n}} \\
&+ \left(B^{\frac{m}{q}}(p)A(q,m)\right)^{\frac{pq}{pq-m}} + \left(C^{\frac{m}{q}}(p)A(q,m)\right)^{\frac{pq}{pq-m}} + \left(A^{\frac{m}{q}}(p,n)A(q,m)\right)^{\frac{pq}{pq-nm}} \Bigg].
\end{aligned}
$$

Let's take now the test function $\varphi(\eta, t_1, t_2)$ in the form

$$\varphi(\eta, t_1, t_2) = \psi\left(\frac{\tau^{2\theta_j} + |x|^{4\theta_j} + |y|^{4\theta_j}}{R^4}\right)\psi\left(\frac{t_1}{R}\right)\psi\left(\frac{t_2}{R}\right), \quad j = 1, 2,$$

and θ_j will be determined further. Then

$$\Delta_{\mathbb{H}}\varphi(\eta) = \psi\left(\frac{t_1}{R}\right)\psi\left(\frac{t_2}{R}\right)\Delta_{\mathbb{H}}\psi(\rho),$$

where

$$\rho = \frac{\tau^{2\theta_j} + |x|^{4\theta_j} + |y|^{4\theta_j}}{R^4}$$

and

$$\Delta_{\mathbb{H}}\psi(\rho) = \sum_{i=1}^{N}\left(\frac{\partial^2\psi(\rho)}{\partial x_i^2} + \frac{\partial^2\psi(\rho)}{\partial y_i^2} + 4y_i\frac{\partial^2\psi(\rho)}{\partial x_i\partial\tau}\right.$$
$$\left. -4x_i\frac{\partial^2\psi(\rho)}{\partial y_i\partial\tau} + 4\left(x_i^2 + y_i^2\right)\frac{\partial^2\psi(\rho)}{\partial\tau^2}\right)$$

we have

$$\frac{\partial^2\psi(\rho)}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial\rho}{\partial x_i}\frac{\partial\psi(\rho)}{\partial\rho}\right) = \frac{\partial^2\rho}{\partial x_i^2}\psi'(\rho) + \left(\frac{\partial\rho}{\partial x_i}\right)^2\psi''(\rho),$$
$$= \frac{4\theta_j}{R^4}\left(|x|^{4\theta_j-2} + (4\theta_j - 2)\,x_i^2|x|^{4\theta_j-4}\right)\psi'(\rho) + \frac{16\theta_j^2}{R^8}x_i^2|x|^{8\theta_j-4}\psi''(\rho)$$

and

$$\frac{\partial^2\psi(\rho)}{\partial y_i^2} = \frac{\partial}{\partial y_i}\left(\frac{\partial\rho}{\partial y_i}\frac{\partial\psi(\rho)}{\partial\rho}\right) = \frac{\partial^2\rho}{\partial y_i^2}\psi'(\rho) + \left(\frac{\partial\rho}{\partial y_i}\right)^2\psi''(\rho),$$
$$= \frac{4\theta_j}{R^4}\left(|y|^{4\theta_j-2} + (4\theta_j - 2)\,y_i^2|y|^{4\theta_j-4}\right)\psi'(\rho) + \frac{16\theta_j^2}{R^8}y_i^2|y|^{8\theta_j-4}\psi''(\rho)$$

and

$$4y_i\frac{\partial^2\psi(\rho)}{\partial x_i\partial\tau} = 4y_i\frac{\partial}{\partial x_i}\left(\frac{\partial\rho}{\partial\tau}\frac{\partial\psi(\rho)}{\partial\rho}\right) = 4y_i\left[\frac{\partial^2\rho}{\partial x_i\partial\tau}\psi'(\rho) + \left(\frac{\partial\rho}{\partial\tau}\right)\left(\frac{\partial\rho}{\partial x_i}\right)\psi''(\rho)\right],$$
$$= 4y_i\left[\frac{\partial}{\partial x_i}\left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\psi'(\rho) + \left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\left(\frac{4\theta_j}{R^4}|x|^{4\theta_j-2}x_i\right)\psi''(\rho)\right]$$
$$= \frac{8\theta_j^2}{R^8}\tau^{2\theta_j-1}|x|^{4\theta_j-2}x_iy_i\psi''(\rho)$$

and

$$-4x_i\frac{\partial^2\psi(\rho)}{\partial y_i\partial\tau} = -4x_i\frac{\partial}{\partial y_i}\left(\frac{\partial\rho}{\partial\tau}\frac{\partial\psi(\rho)}{\partial\rho}\right)$$

$$
\begin{aligned}
&= -4x_i\left[\frac{\partial^2\rho}{\partial y_i\partial\tau}\psi'(\rho) + \left(\frac{\partial\rho}{\partial\tau}\right)\left(\frac{\partial\rho}{\partial y_i}\right)\psi''(\rho)\right] \\
&= -4x_i\left[\frac{\partial}{\partial y_i}\left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\psi'(\rho) + \left(\frac{2\theta_j}{R^4}\tau^{2\theta_j-1}\right)\left(\frac{4\theta_j}{R^4}|y|^{4\theta_j-2}y_i\right)\psi''(\rho)\right] \\
&= -\frac{8\theta_j^2}{R^8}\tau^{2\theta_j-1}|y|^{4\theta_j-2}x_iy_i\psi''(\rho)
\end{aligned}
$$

and

$$
\begin{aligned}
&4\left(x_i^2+y_i^2\right)\frac{\partial^2\psi(\rho)}{\partial\tau^2} = 4\left(x_i^2+y_i^2\right)\frac{\partial}{\partial\tau}\left(\frac{\partial\rho}{\partial\tau}\frac{\partial\psi(\rho)}{\partial\rho}\right) \\
&= 4\left(x_i^2+y_i^2\right)\left[\left(\frac{\partial^2\rho}{\partial\tau^2}\right)\psi'(\rho) + \left(\frac{\partial\rho}{\partial\tau}\right)^2\psi''(\rho)\right] \\
&= 4\left(x_i^2+y_i^2\right)\left[\left(\frac{2\theta_j(2\theta_j-1)}{R^4}\tau^{2\theta_j-2}\right)\psi'(\rho) + \left(\frac{4\theta_j^2}{R^8}\tau^{4\theta_j-2}\right)\psi''(\rho)\right]
\end{aligned}
$$

finally

$$
\begin{aligned}
\Delta_{\mathbb{H}}\psi(\rho) = {} & \frac{4\theta_j}{R^4}[(N+(4\theta_j-2))(|x|^{4\theta_j-2}+|y|^{4\theta_j-2}) \\
& + (4\theta_j-2)\tau^{2\theta_j-2}(|x|^2+|y|^2)]\psi'(\rho) \\
& + \frac{16\theta_j^2}{R^8}[|x|^{8\theta_j-2}+|y|^{8\theta_j-2}+\frac{1}{2}\tau^{2\theta_j-1}\langle x,y\rangle(|x|^{4\theta_j-2}-|y|^{4\theta_j-2}) \\
& + \tau^{4\theta_j-2}(|x|^2+|y|^2)]\psi''(\rho)
\end{aligned}
$$

and we apply the change of next variables in the form

$$\eta = (x,y,\tau) \longrightarrow \tilde{\eta} = (\tilde{x},\tilde{y},\tilde{\tau}),$$

where

$$\tilde{x} = R^{\frac{-1}{\theta_j}}x \;\; \tilde{y} = R^{\frac{-1}{\theta_j}}y \;\; \tilde{\tau} = R^{\frac{-2}{\theta_j}}\tau, \quad \tilde{t}_1 = R^{-1}t_1, \quad \tilde{t}_2 = R^{-1}t_2$$

we put

$$\Omega_1^j = \left\{\tilde{\eta}\in\mathbb{H}: \; \tilde{\rho} = \tilde{\tau}^{2\theta_j} + |\tilde{x}|^{4\theta_j} + |\tilde{y}|^{4\theta_j} \le 2\right\},$$

for Ω_2 and Ω_3, see the equality (3.7). Then

$$
\begin{aligned}
\Delta_{\mathbb{H}}\psi(\rho) = {} & \frac{4\theta_j}{R^{\frac{2}{\theta_j}}}[(N+(4\theta_j-2))(|\tilde{x}|^{4\theta_j-2}+|\tilde{y}|^{4\theta_j-2}) \\
& + (4\theta_j-2)\tilde{\tau}^{2\theta_j-2}(|\tilde{x}|^2+|\tilde{y}|^2)]\psi'(\tilde{\rho}) \\
& + \frac{16\theta_j^2}{R^{\frac{2}{\theta_j}}}[|\tilde{x}|^{8\theta_j-2}+|\tilde{y}|^{8\theta_j-2}+\frac{1}{2}\tilde{\tau}^{2\theta_j-1}\langle\tilde{x},\tilde{y}\rangle(|\tilde{x}|^{4\theta_j-2}-|\tilde{y}|^{4\theta_j-2}) \\
& + \tilde{\tau}^{4\theta_j-2}(|\tilde{x}|^2+|\tilde{y}|^2)]\psi''(\tilde{\rho})
\end{aligned}
$$

this means

$$\Delta_{\mathbb{H}}\psi\left(\rho\right)=\frac{1}{R^{\frac{2}{\theta_j}}}\Delta_{\mathbb{H}}\psi\left(\tilde{\rho}\right),\ \forall\tilde{\eta}\in\Omega_1^j,$$
$$\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\psi\left(\rho\right)=R^{\frac{-\alpha}{\theta_j}}\left(-\Delta_{\mathbb{H}}\right)^{\alpha/2}\psi\left(\tilde{\rho}\right),$$
$$\left(-\Delta_{\mathbb{H}}\right)^{\beta/2}\psi\left(\rho\right)=R^{\frac{-\beta}{\theta_j}}\left(-\Delta_{\mathbb{H}}\right)^{\beta/2}\psi\left(\tilde{\rho}\right)$$

as

$$d\eta=R^{\frac{N}{\theta_j}+\frac{N}{\theta_j}+\frac{2}{\theta_j}}d\tilde{\eta}=R^{\frac{2N+2}{\theta_j}}d\tilde{\eta},$$

we make the following estimates:

- For $j=1$, we choose $\theta_1=\frac{\alpha}{\alpha_1}$ and as $\alpha_1<\alpha_2$ we obtain

$$A\left(q,m\right)=\mathrm{C}_1R^{-\alpha_1+\frac{(q-m)}{q}\left(\frac{(2N+2)\alpha_1}{\alpha}+2\right)}$$

where

$$\mathrm{C}_1=\mu\left(\int_{\Omega_1^1}\left|(-\Delta_{\mathbb{H}})^{\alpha/2}\psi(\tilde{\rho})\right|^{\frac{q}{q-m}}\psi^{\mu-\frac{q}{q-m}}(\tilde{\rho})d\tilde{\eta}\int_{\Omega_2}\psi^{\mu}(\tilde{t}_1)d\tilde{t}_1\int_{\Omega_3}\psi^{\mu}(\tilde{t}_2)d\tilde{t}_2\right)^{\frac{q-m}{q}}$$

and

$$B\left(q\right)=\mathrm{C}_2R^{-\alpha_1+\frac{(q-1)}{q}\left(\frac{(2N+2)\alpha_1}{\alpha}+2\right)}$$

where

$$\mathrm{C}_2=\left(\int_{\Omega_1^1}\psi^{\mu}\left(\tilde{\eta}\right)d\tilde{\eta}\int_{\Omega_2}\left|D^{\alpha_1}_{\tilde{t}_1|R^{-1}T}\psi^{\mu}\left(\tilde{t}_1\right)\right|^{\frac{q}{q-1}}\psi^{\frac{-\mu}{q-1}}\left(\tilde{t}_1\right)d\tilde{t}_1\int_{\Omega_3}\psi^{\mu}\left(\tilde{t}_2\right)d\tilde{t}_2\right)^{\frac{q-1}{q}}$$

and

$$C\left(q\right)=\mathrm{C}_3R^{-\alpha_2+\frac{(q-1)}{q}\left(\frac{(2N+2)\alpha_1}{\alpha}+2\right)}$$

where

$$\mathrm{C}_3=\left(\int_{\Omega_1^1}\psi^{\mu}\left(\tilde{\eta}\right)d\tilde{\eta}\int_{\Omega_2}\psi^{\mu}\left(\tilde{t}_1\right)d\tilde{t}_1\int_{\Omega_3}\left|D^{\alpha_2}_{\tilde{t}_2|R^{-1}T}\psi^{\mu}\left(\tilde{t}_2\right)\right|^{\frac{q}{q-1}}\psi^{\frac{-\mu}{q-1}}\left(\tilde{t}_2\right)d\tilde{t}_2\right)^{\frac{q-1}{q}}.$$

- For $j=2$, we choose $\theta_2=\frac{\beta}{\beta_1}$ and as $\beta_1<\beta_2$ we obtain

$$A\left(p,n\right)=\mathrm{C}_4R^{\frac{-\beta}{\theta_2}+\frac{(p-n)}{p}\left(\frac{(2N+2)\beta_1}{\beta}+2\right)}$$

where

$$\mathrm{C}_4=\mu\left(\int_{\Omega_1^2}\left|(-\Delta_{\mathbb{H}})^{\beta/2}\psi(\tilde{\rho})\right|^{\frac{p}{p-n}}\psi^{\mu-\frac{p}{p-n}}(\tilde{\rho})d\tilde{\eta}\int_{\Omega_2}\psi^{\mu}(\tilde{t}_1)d\tilde{t}_1\int_{\Omega_3}\psi^{\mu}(\tilde{t}_2)d\tilde{t}_2\right)^{\frac{p-n}{p}}$$

and
$$B\left(p\right)=\mathrm{C}_5R^{-\beta_1+\frac{(p-1)}{p}\left(\frac{(2N+2)\beta_1}{\beta}+2\right)},$$
where
$$\mathrm{C}_5=\left(\int_{\Omega_1^1}\psi^{\mu}\left(\tilde{\eta}\right)d\tilde{\eta}\int_{\Omega_2}\left|D^{\beta_1}_{\tilde{t}_1|R^{-1}T}\psi^{\mu}\left(\tilde{t}_1\right)\right|^{\frac{p}{p-1}}\psi^{\frac{-\mu}{p-1}}\left(\tilde{t}_1\right)d\tilde{t}_1\int_{\Omega_3}\psi^{\mu}\left(\tilde{t}_2\right)d\tilde{t}_2\right)^{\frac{p-1}{p}}$$
and
$$C\left(p\right)=\mathrm{C}_6R^{-\beta_2+\frac{(p-1)}{p}\left(\frac{(2N+2)\beta_1}{\beta}+2\right)}$$
where
$$\mathrm{C}_6=\left(\int_{\Omega_1^1}\psi^{\mu}\left(\tilde{\eta}\right)d\tilde{\eta}\int_{\Omega_2}\psi^{\mu}\left(\tilde{t}_1\right)d\tilde{t}_1\int_{\Omega_3}\left|D^{\beta_2}_{\tilde{t}_2|R^{-1}T}\psi^{\mu}\left(\tilde{t}_2\right)\right|^{\frac{p}{p-1}}\psi^{\frac{-\mu}{p-1}}\left(\tilde{t}_2\right)d\tilde{t}_2\right)^{\frac{p-1}{p}},$$
for some positive constant $K\widehat{C}$, where
$$\widehat{C}=\max\left\{(C_5^{\frac{1}{q}}C_2)^{\frac{pq}{pq-1}},(C_6^{\frac{1}{q}}C_2)^{\frac{pq}{pq-1}},(C_4^{\frac{1}{q}}C_2)^{\frac{pq}{pq-n}},(C_5^{\frac{1}{q}}C_3)^{\frac{pq}{pq-1}}\right.,$$
$$\left.(C_6^{\frac{1}{q}}C_3)^{\frac{pq}{pq-1}},(C_4^{\frac{1}{q}}C_3)^{\frac{pq}{pq-n}},(C_5^{\frac{m}{q}}C_1)^{\frac{pq}{pq-m}},(C_6^{\frac{m}{q}}C_1)^{\frac{pq}{pq-m}},(C_4^{\frac{m}{q}}C_1)^{\frac{pq}{pq-nm}}\right\}.$$
Hence, we obtain
$$I_v+I_0^{\mu}\leq K\widehat{C}\left\{R^{\sigma_1}+R^{\sigma_2}+\ldots+R^{\sigma_9}\right\}. \tag{4.10}$$
Similarly, we obtain for I_u the estimate
$$I_u+J_0^{\mu}\leq K\widehat{\widehat{C}}\left\{R^{\delta_1}+R^{\delta_2}+\ldots+R^{\delta_9}\right\}, \tag{4.11}$$
where the value $\widehat{\widehat{C}}$ is set as the value setting $\widehat{C}$. Finally, by tending $R\to\infty$, we observe that:

Either $\max\{\sigma_1,\ldots,\sigma_9,\delta_1,\ldots,\delta_9\}<0$ and in this case, the right hand side tends to zero while the left hand side is strictly positive. Hence, we obtain a contradiction. Or, $\max\{\sigma_1,\ldots,\sigma_9,\delta_1,\ldots,\delta_9\}=0$ and in this case, following the analysis similar as in one equation, we prove a contradiction. □

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Neural dynamic optimization algorithm based on event triggered algorithm and its application

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Abstract. Wireless sensor networks consist of microprocessor controlled sensors that communicate with each other over multi-hop communication networks. In WSN, the energy consumption of sensor networks for communication can be obviously bigger than the energy required to operate computation that would bring us unimaginable benefits if communication and computation between each node can be somehow isolated. In this paper, a neurodynamic optimization approach is proposed based on the event-triggered algorithm for handling standard NUM problem in WSN. We first confirm that the equilibrium point set of the designed neural network model based on event-triggered algorithm corresponds to the optimal solution of the NUM problem. Then, it is proved that the proposed neural network model is stable in the sense of Lyapunov and is convergent to the optimal solution. Finally, a numerical example is provided to illustrate the performance of the proposed neural network.

Keywords: wireless sensor networks, neural network, distributed optimization, event triggered, network utility maximization.

1. Introduction

In recent years, the distributed optimization algorithm, which is more powerful than traditional optimization in large-scale problems, has attracted attention from more and more researchers. Various optimization problems in sensor networks, smart grids, computation, etc. [1, 2, 3, 4], have been studied by using distributed algorithms.

The *wireless sensor network* (WSN) consists of nodes with limited energy and memory and helps to monitor the far located areas out of the human reach. The sensor nodes gather and send information to the public mobile communication base station. The nodes can only communicate with the nearby nodes. The human administrators control the sensor network convey orders in time and get reactions throughout the public mobile communication base station. The sensor nodes and the battery are both small in size only to provide limited energy and power to the nodes.

Generally speaking, it is impractical and sometimes impossible to replace the battery to maintain a longer network lifetime. The sensor nodes utilize high amount of energy in sensing the environmental activities and communicating with other nodes in the network, sensation and communication with affects the network lifetime. The lifetime of the network can be increased by using various protocols that conserve the residual energy of the sensor nodes [5]. For instance, an artificial bee colony algorithm can be applied to extend the network lifetime [6]. The reference [7] proposed distributed algorithms to calculate and compute the best routing scheme that maximizes the time where the initial node in the network runs out of energy.

One way of conserving energy of the sensor nodes is reducing the complexity of messaging by means of applying a novel distributed algorithms called the *event-triggered algorithm*. Under the event-triggered algorithm, each agent sends information from itself to its neighbours when a local "error" signal is bigger than a state dependent threshold. The activation of the event-triggered system is due to the occurrence of a major event. In a time-triggered system, the activities are initiated periodically at a preset point in real-time [8]. In order to reduce the demand of communication in smart grid, Li et al. [9] presented a distributed optimization approach based on an event-triggered communication under the economic dispatch problem. In order to reduce the demand of information and communication in WSN, Pu and Lemmon [10] presented a distributed optimization approach based on the event-triggered communication. Similar approaches of resource allocation were used in [11] and [12]. However, these approaches are traditional optimization algorithms and lower in efficiency in large-scale computing problems. Because of the inborn large-scale parallelism, the neural network method can solve optimization problems in calculating time at the order of magnitude. It is much faster than those optimization algorithms implemented on general-purpose digital computers [13].

Since the mid-1980s, the neurodynamic optimization approach based on continuous-time *recurrent neural networks* (RNNs) has been extensively studied. Hopfield and Tank [14] proposed a neural network approach to solve linear programming problems. In [15], an RNN to handle a class of nonlinear optimization problem was discovered by Kennedy and Chua. Since then, many neural network models were proposed and studied. Yan, Wang and Li [16] presented a neurodynamic approach for bound-constrained global optimization problem. Qin et al. [17] presented a neurodynamic approach for solving a class of convex optimization problems with equality and inequality constraints. In [18], a complex-valued neural network was presented to handle a class of complex-valued nonlinear convex optimization problem. In [19], a one-layer RNN was proposed for solving a class of non-linear non-smooth pseudoconvex optimization problem with linear equality constraints. Recently, a collaborative neurodynamic approach to multiobjective optimization was presented to attain both goals of pareto optimality and solution diversity[20]. The collaborative neurodynamic approach demonstrates higher efficiency in seeking for the global optimal solution [21]. A collaborative neurodynamic approach based on the distributed constrained optimization was proposed in [22]. A collaborative neurodynamic optimization approach for solving global and combinatorial optimization was designed in [23]. [16] presented a new collaborative neurodynamic optimization approach for solving a class of nonconvex optimization problems with bound constraints.

Because of the inherent massive parallelism, the neurodynamic optimization approach can solve optimization problems in running time much faster than those of the most popular optimization algorithms executed on general-purpose digital computers [25]. However, in WSN, there is a dearth of literature on the neurodynamic optimization algorithms. In this paper, we use a neurodynamic optimization approach to solve the *network utility maximization* (NUM) problems in WSN.

The rest of this paper is organized as follows: Some basic concepts of the problem are introduced in Section 2. Descriptions of the neurodynamic approach are provided in Section 3. In Section 4, the simulation examples are given to show the performance and effectiveness of the proposed neural network model. In Section 5, we give the conclusion of this paper.

2. Problem formulation

In this section, we introduce some background information. The data gathering problem in WSN studied in [10] is formulated as

$$\begin{array}{ll} \max & U(x) = \sum_{i \in V} U_i(x_i) \\ \text{s.t.} & Ax \leq \bar{c} \\ & (e_{tx} + e_{rx})Ax - e_{rx}x \leq b(t), \end{array}$$

where $x = [x_1, x_2, \cdots, x_N]^T$, $x_i \in \mathbb{R}$ and $x_i \geq 0$ stands for the data rate of the node i. $\bar{c} \in \mathbb{R}^M$ is the vector of node capacitie. $A \in \mathbb{R}^{N \times N}$ is the routing matrix of the relaying relationship between nodes. A_{ji}, the ji-th component, is 1 if node i communicates with j and is 0 if node i does not communicate with j. The j-th row of Ax means the total data rates node j requires to send, which is not higher than its capacity $\bar{c}_j$. e_{tx} represents the energy used in transmitting and e_{rx} represents the energy used in receiving one unit of data. In the last inequality constraint, $b(t) = [b_1(t), b_2(t), \cdots, b_N(t)]^T$ represents the expected energy rate of reduction on node i at instant t. The cost function U is the sum of the node utility functions $U_i(x_i)$. By simplifying the notations and descriptions, we obtain that

$$\begin{array}{ll} \max & U(x) = \sum_{i \in V} U_i\left(x_i\right) \\ \text{s.t.} & Ax \leqslant c, \quad x \geqslant 0, \end{array} \tag{1}$$

where

$$c = \min\left\{\bar{c}, \frac{b(t)}{e_{tx} + e_{rt}}\right\}.$$

Equation (1) is a NUM problem and c is a constant.

In NUM problem, we have the following equation by applying the augmented Lagrangian method,

$$\begin{aligned} \bar{L}(x, s; \lambda, w) = & -\sum_{i \in V} U_i\left(x_i\right) + \sum_{j \in V} \lambda_j \left(a_j^T x - c_j + s_j\right) \\ & + \tfrac{1}{2} \sum_{j \in V} \tfrac{1}{w_j} \left(a_j^T x - c_j + s_j\right), \end{aligned} \tag{2}$$

where $s \in \mathbb{R}^n$ represents the slack variable and $s_j \geq 0, j \in \mathcal{V}$. The vector $a_j^T = [A_{j1}, A_{j2}, \cdots, A_{jN}]$ is the j-th row of the routing matrix A. The penalty parameter w_j is related to all constraints, and $w = diag\{w_1, w_2 \cdots, w_N\}$ is the diagonal matrix and the elements in the matrix are made up of penalty parameters. Under the Karush-Kuhn-Tucker condition, λ_j is the Lagrange multiplier as related to node constraint $c_j - a_j^T x \geq 0$.

Equation (2) can be rewritten as

$$L(x;\lambda,w)=\sum_{j\in V}\psi_j(x;\lambda,w)\ -\sum_{i\in V}U_i\left(x_i\right),\tag{3}$$

where

$$\psi_j(x;\lambda,w)=\begin{cases}-\frac{1}{2}w_j\lambda_j^2, & \text{if } c_j-a_j^Tx-w_j\lambda_j\geq 0,\\ \lambda_j\left(a^Tx-c_j\right)+\frac{1}{2w_j}\left(a^Tx-c_j\right)^2, & \text{otherwise.}\end{cases}$$

The minimizer of $L(x;\lambda,w)$ is sufficiently approximated to the solution of problem (1) when $\lambda_j=0$ and w_j is sufficiently small. The algorithm is as follows [24]:
Step 1. Select initial data rate $x^0>0$, let $\lambda_j=0$ and sufficiently small $w_j>0,\ j\in V$.
Step 2. Minimize $L(x;\lambda,w)$, where γ is a sufficiently small step size,

$$x=\max\left\{0,x^0-\gamma\nabla_xL\left(x^0;\lambda,w\right)\right\},\quad x^0=x.$$

In paper [10], we know that ρ is a constant, $\bar{L}$ is the maximum number of relay nodes and $\bar{S}$ is the maximum number of nodes. For all $i\in\mathcal{V}$, $\ell\in\mathbb{R}^+$ and any $t\in\left[T_j^L[\ell],T_j^L[\ell+1]\right)$,

$$\begin{aligned}z_i(t)=\dot{x}_i(t)&=\left(\nabla U_i\left(x_i(t)\right)-\sum_{j\in\mathcal{L}_i}\hat{\mu}_j(t)\right)^+_{x_i(t)},\\ \mu_j(t)&=\frac{1}{w_j}\left(a_j^Tx(t)-c_j\right)^+,\\ \hat{z}_i(t)&=z_i\left(T_i^S[\ell]\right).\end{aligned}$$

$T_i^S[\ell]$ is the ℓ- th time instant when node i sends information of its user state to all nodes $j\in\mathcal{L}_i$.

$$\hat{\mu}_j(t)=\mu_j\left(T_j^L[\ell]\right).$$

The sequence $T_j^L[\ell]$ represents time instants when node j transmits its link state to the relay nodes. Then, we have the following lemma.

Lemma 1 ([10])**.** *The data rate $x(t)$ asymptotically converges to the unique minimizer of $L(x;\lambda,w)$, while*

$$z_i^2(t)-\rho\hat{z}_i^2(t)\geq 0,$$

for $t\in\left[T_i^S[\ell],T_i^S[\ell+1]\right)$,

$$\rho\sum_{i\in\mathcal{S}_j}\frac{1}{\bar{L}}\hat{z}_i^2(t)-\bar{L}\bar{S}\left(\mu_j(t)-\hat{\mu}_j(t)\right)^2\geq 0,$$

for $t\in\left[T_i^L[\ell],T_i^L[\ell+1]\right)$.

3. Neural network model

Lemma 1 supplies the infrastructure for constructing a neurodynamic optimization approach based on the event-triggered message-passing protocol. Accordingly, we propose the following neural network based on the event-triggered algorithm adopted from [10]

$$\begin{cases} \dot{x}_i = x_i - \left(x_i + \gamma\left(z_i - \left(\mu_j - \hat{\mu}_j\right) A_{ij}\right)\right)^+ \\ \hat{z}_i(t) = \begin{cases} z_i(T^+),\ z_i \geq \sqrt{\rho}\hat{z}_i \\ z_i(t),\ z_i < \sqrt{\rho}\hat{z}_i \end{cases} \\ \hat{\mu}_j(t) = \begin{cases} \mu_j(T^+),\ \sqrt{\rho}\frac{1}{\sqrt{\bar{L}}}\hat{z}_i \geq \sqrt{\bar{L}\bar{S}}(\mu_j - \hat{\mu}_j) \\ \mu_j(t),\ \sqrt{\rho}\frac{1}{\sqrt{\bar{L}}}\hat{z}_i < \sqrt{\bar{L}\bar{S}}(\mu_j - \hat{\mu}_j), \end{cases} \end{cases} \tag{4}$$

where, $\bar{L}\sqrt{\bar{S}} \leq A_{ji}$, T^+ is the time instant when $z_i \leqslant \sqrt{\rho}\hat{z}_i$ and $\sqrt{\rho}\frac{1}{\sqrt{\bar{L}}}\hat{z}_i \leqslant \sqrt{\bar{L}\bar{S}}(\mu_j - \hat{\mu}_j)$.

Lemma 2. *System* (4) *is convergent to the unique optimal solution of problem* (1).

Proof. Let

$$M(x) = -\left[z - (\mu - \hat{\mu})A\right] = M,$$

then, we have

$$\dot{x} = x - (x - \gamma M)^+.$$

The dynamic equation of the proposed continuous-time projection neural network model is

$$\begin{aligned} \frac{dy}{dt} &= g(y) - \gamma M(g(y)) - y \\ \textit{output equation} \quad x &= g(y). \end{aligned} \tag{5}$$

Assume that x^* is the solution of (4). According to

$$x^* = \left[-\gamma M\left(x^*\right) + x^*\right]^+,$$

we obtain that

$$x^* = g\left(-\gamma M\left(x^*\right) + x^*\right),$$

where $g(x)$ is the projection operator.

Let $y^* = -\gamma M\left(x^*\right) + x^*$, then $x^* = g\left(y^*\right)$. It follows that

$$\begin{aligned} y^* &= -\gamma M\left(x^*\right) + g\left(y^*\right), \\ y^* &= -\gamma M\left(g\left(y^*\right)\right) + g\left(y^*\right). \end{aligned}$$

Then $0 = g(y^*) - \gamma M(x^*) - y^*$, thus y^* is an equilibrium point of the system

$$\frac{dy}{dt} = g(y) - \gamma M(g(y)) - y.$$

Assume $\bar{y}$ is an equilibrium point of (5), they satisfies

$$\bar{x} - \gamma M(\bar{x}) - \bar{y} = 0,$$

where $\bar{x} = g(\bar{y})$.

Let

$$\bar{M} = M(\bar{x}) = M(g(\bar{y})) = \left[\bar{z}_i - (\bar{\mu}_j - \bar{\hat{\mu}}_j)A_{ji}\right],$$

where $\bar{z}_i = \bar{z}_i(x) = z_i(\bar{x})$ and $\bar{\hat{\mu}}_j = \bar{\hat{\mu}}_j(x) = \hat{\mu}_j(\bar{x})$.

According to

$$-\bar{y} + \bar{x} - \gamma M(\bar{x}) = 0$$

and

$$\frac{dy}{dt} = -y + g(y) - \gamma M(g(y)),$$

we put the equilibrium point to the origin

$$\frac{dy}{dt} = -\gamma(M(x) - M(\bar{x})) - (y - \bar{y}) + (x - \bar{x}).$$

Consider the following Lyapunov function $V(y) = \|y - \bar{y}\|^2$. According to the chain rule, we have

$$\begin{aligned}
\dot{V}(t) &= \dot{y}(t)(\nabla V(y))^T \\
&= 2[-(y - \bar{y}) + (x - \bar{x}) - \gamma(M - \bar{M}))](y - \bar{y})^T \\
&= -2(y - \bar{y})^T(y - \bar{y}) + 2(y - \bar{y})^T(x - \bar{x}) \\
&\quad - 2\gamma(y - \bar{y})^T(M - \bar{M})) \\
&= 2\gamma\left\{(x - \bar{x})^T\left(M - \bar{M}\right) - \left(M - \bar{M}\right)\right\}.
\end{aligned}$$

According to Lemma of variational inequality [26], we have

$$\begin{aligned}
(x - \bar{x})^T\left(M - \bar{M}\right) &= (x - \bar{x})^T\left(M - \bar{M} + \bar{x} - \bar{x}\right) \\
&= -(x - \bar{x})^T\left(\bar{x} - (M - \bar{M} + \bar{x})\right) \leq 0,
\end{aligned}$$

when $\left(M - \bar{M}\right) \geq 0$, we have $\dot{V} \leq 0$.

According to [10], we have $\bar{\mu}_j = \bar{\hat{\mu}}_j, \bar{z}_i = 0$ then

$$\begin{aligned}
\left(M - \bar{M}\right) &= -[(z - \bar{z}) - A_{ji}\left((\mu - \hat{\mu}_j) - (\bar{\mu} - \bar{\hat{\mu}}_j)\right)] \\
&= -[z - A_{ji}(\mu - \hat{\mu}_j)] \\
&= -[z - \sqrt{\rho}\hat{z} + \sqrt{\rho}\hat{z} - A_{ji}(\mu - \hat{\mu}_j)].
\end{aligned}$$

In the conditions of the proposed model, $z_i \leqslant \sqrt{\rho}\hat{z}_i$, $\sqrt{\rho}\frac{1}{\sqrt{\bar{L}}}\hat{z}_i \leqslant \sqrt{\bar{L}\bar{S}}(\mu_j - \hat{\mu}_j)$, $\bar{L}\sqrt{\bar{S}} \leq A_{ji}$, we have $z_i \leqslant \sqrt{\rho}\hat{z}_i$, $\sqrt{\rho}\hat{z}_i \leqslant A_{ji}(\mu_j - \hat{\mu}_j)$, then $\left(M - \bar{M}\right) \geq 0$, thus, we have $\dot{V} \leq 0$ then, system (4) is convergent to the unique optimal solution of problem (1). □

4. Illustrative example

In this part, the effectiveness of the neural dynamic optimization approach based on event-triggered algorithm is demonstrated by a simulation example.

Consider the following NUM problem

$$\begin{array}{rl} \min & U(x) \\ \text{s.t.} & Ax \leq C \end{array} \tag{6}$$

As a special case, the number of communication nodes we consider is 2, and the node utility function $U(x) = x_1^2 + x_2^2$.

Let $C = [2,3]^T$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\rho = 0.25$, $\lambda = 0$, $\gamma = 0.09$, $\omega = 0.05$, and $x \in [0, 10]$. We apply the neurodynamic optimization approach based on event-triggered algorithm (4) to solve this example. By the above assumptions, as shown in Figure 1 and Figure 2, it can be seen that all trajectories converge to the solutions 2.223 and 3.337.

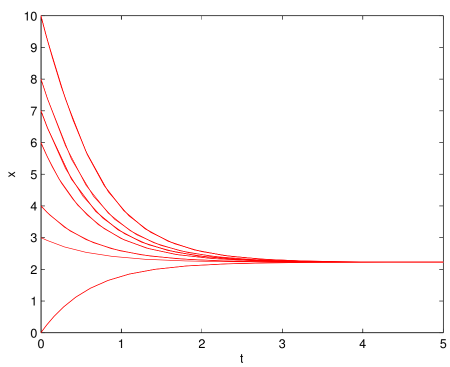

Figure 1: Transient behaviors of x_1

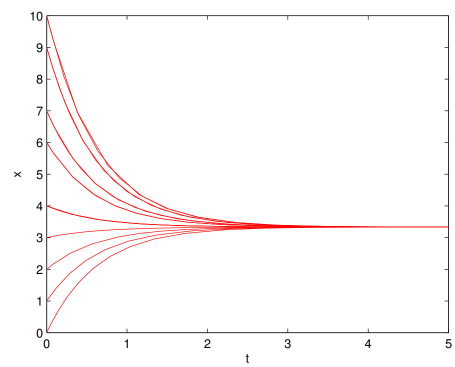

Figure 2: Transient behaviors of x_2

5. Conclusions

In this paper, we propose a neurodynamic optimization approach based on event-triggered algorithm for solving the NUM problems in WSN. The paper shows that the proposed neural network model based on event-triggered algorithm is stable in the sense of Lyapunov and converges to the optimal solution under event-triggered mechanism. Moreover, in traditional optimization algorithms, their efficiency is lower than the neurodynamic optimization approach. Finally, the effectiveness of the neural dynamic optimization method based on event-triggered algorithm is demonstrated by a simulation example. In the future study, we will look more into the neurodynamic optimization approach based on time-triggered algorithm.

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A note on extended Hurwitz-Lerch Zeta function

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Abstract. In the present research note, we introduce another extension of Hurwitz-Lerch Zeta function (HLZF) by using the generalized extended Beta function defined by Parmar [7]. We investigate its integral representations, Mellin transform, generating relations and differential formula. In view of diverse applications of the Hurwitz-Lerch Zeta functions, the results presented here are potentially useful in some other related research areas.

Keywords: Generalized Hurwitz-Lerch Zeta function, extended beta function, extended hypergeometric function, Mellin transform.

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1. Introduction

The well known Hurwitz-Lerch Zeta function (HLZF) is defined by (see, [2], [10], [11]):

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \tag{1.1}$$

$(a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C}, \text{ when } \mid z\mid<1; \Re(s)>1, \text{when } \mid z\mid=1)$.

Due to diverse applications of Hurwitz-Lerch Zeta function (HLZF), several extensions of $\Phi(z,s,a)$ have been introduced and studied (see, for example [1], [3], [4], [5], [8], [10], [11], [12] etc).

Very recently, Parmar et al. [9] defined the following extended Hurwitz-Lerch Zeta function (HLZF):

$$\Phi^{(\rho,\sigma)}_{\lambda,\mu;\gamma}(z,s,a;p)=\sum_{n=0}^{\infty}\frac{(\lambda)_n B^{(\rho,\sigma)}_p(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}, \tag{1.2}$$

$(p\geq 0, \Re(\rho)>0, \Re(\sigma)>0; \lambda,\mu\in\mathbb{C}; \gamma,a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C}, \text{ when } \mid z\mid< 1; \Re(s+\gamma-\lambda-\mu)>1, \text{ when } \mid z\mid=1)$, where $B^{(\rho,\sigma)}_p(x,y)$ is the extended Beta function (see [6]):

$$B^{(\rho,\sigma)}_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\rho;\sigma;\frac{-p}{t(1-t)})dt. \tag{1.3}$$

They also presented the integral representation of (1.2)

$$\Phi^{(\rho,\sigma)}_{\lambda,\mu;\gamma}(z,s,a;p)=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-at}F^{(\rho,\sigma)}_p(\lambda,\mu;\gamma;ze^{-t})dt. \tag{1.4}$$

For $\rho=\sigma$, (1.2) reduces to the Hurwitz-Lerch Zeta function (HLZF) defined by Parmar and Raina [8], which, further for $p=0$, gives the known extension of (1.1) (see [3]).

In a sequel of above-mentioned works, we introduce a further extension of $\Phi^{(\rho,\sigma)}_{\lambda,\mu;\gamma}(z,s,a;p)$ by using the generalized Beta function defined by Parmar [7].

2. Extended Hurwitz-Lerch Zeta function (EHLZF)

Here we define the following extension of (1.2):

$$\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)=\sum_{n=0}^{\infty}\frac{(\lambda)_n B^{(\rho,\sigma;m)}_p(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}, \tag{2.1}$$

$(p\geq 0, \Re(\rho)>0, \Re(\sigma)>0, \Re(m)>0; \lambda,\mu\in\mathbb{C}; \gamma,a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C}, \text{ when } \mid z\mid< 1; \Re(s+\gamma-\lambda-\mu)>1, \text{ when } \mid z\mid=1)$, where $B^{(\rho,\sigma;m)}_p(a,b)$ is the extended Beta function (see, Parmar [7])

$$B^{(\rho,\sigma;m)}_p(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}{}_1F_1(\rho;\sigma;\frac{-p}{x^m(1-x)^m})\ dx. \tag{2.2}$$

He [7] also gave the following extension of Gauass hypergeometric function:

$$F_p^{(\rho,\sigma;m)}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n B_p^{(\rho,\sigma;m)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}. \tag{2.3}$$

If we set $m = 1$ in (2.1), we get the known extended Hurwitz-Lerch Zeta function given by (1.2).

Remark 2.1. The extended Hurwitz-Lerch Zeta function (EHLZF)

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$$

has the following limiting case:

$$\begin{aligned} \Phi_{\mu;\gamma}^{*(\rho,\sigma;m)}(z,s,a;p) &= \lim_{|\lambda|\longrightarrow\infty} \left\{ \Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}\left(\frac{z}{\lambda},s,a;p\right)\right\} \\ &= \sum_{n=0}^{\infty} \frac{B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)} \frac{z^n}{(n+a)^s}. \end{aligned} \tag{2.4}$$

$(p \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(m) > 0; \mu \in \mathbb{C};\ \gamma, a \in \mathbb{C} \setminus \mathbb{Z}_0; s \in \mathbb{C},$ when $\mid z \mid< 1; \Re(s+\gamma-\mu) > 1,$ when $\mid z \mid= 1).$

3. Integral representations of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$

In this section we deal with some integral representations of (2.1):

Theorem 3.1. *The following integral representation of EHLZF* $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ *holds true:*

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} F_p^{(\rho,\sigma;m)}(\lambda,\mu;\gamma;ze^{-t})dt, \tag{3.1}$$

$(\Re(p) \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(m) > 0;\ p = 0, \Re(a) > 0; \Re(s) > 0,$ when $\mid z \mid\leq 1 (z \neq 1); \Re(s) > 1,$ when $z = 1).$

Proof. We have

$$\frac{1}{(n+a)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt.$$

By using the above result in (2.1) and after changing the order of summation and integration, which is guaranteed under the conditions, we get

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)=\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \left(\sum_{n=0}^{\infty} \frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+\gamma,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{(ze^{-t})^n}{n!} \right) dt,$$

which upon using (2.3), yields the required result.

Theorem 3.2. *The following integral representation of EHLZF* $\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)$ *holds true:*

$$\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p) = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\Phi^{*(\rho,\sigma;m)}_{\mu,\gamma}(zt,s,a;p)dt. \tag{3.2}$$

$(\Re(p) \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(m) > 0; p = 0, \Re(\lambda) > 0;\ \Re(a) > 0; \Re(s) > 0,$ when $\mid z \mid \leq 1 (z \neq 1); \Re(s) > 1$, when $z = 1)$.

Proof. We have

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda+n-1}e^{-t}dt.$$

By using the above result in (2.1) and after changing the order of summation and integration, which is guaranteed under the conditions, we get

$$\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-t}\left(\sum_{n=0}^\infty \frac{B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{(zt)^n}{n!(n+a)^s}\right)dt,$$

which upon using (2.4), we arrive at our required result.

4. Mellin transform

The Mellin transform of the function $f(u)$ is given by

$$M\{f(u);s\} = \phi(s) = \int_0^\infty u^{s-1}f(u)du. \tag{4.1}$$

Theorem 4. *For the new extended Hurwitz-Lerch Zeta function EHLZF* $\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)$*, we have the following Mellin transform representation:*

$$\begin{aligned} &M\left\{\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)\right\} \\ &= \frac{\Gamma^{(\rho,\sigma)}(s)\Gamma(m\alpha+\mu)}{B(\mu,\gamma-\mu)\Gamma(2m\alpha+\gamma)}\Phi_{\lambda,m\alpha+\mu;2m\alpha+\gamma}(z,s,a). \end{aligned} \tag{4.2}$$

$(\Re(s) > 0, \Re(m\alpha+\mu) > 0, \Re(2m\alpha+\gamma) > 0, 0 < \Re(\mu) < \Re(\gamma))$, where $\Gamma^{(\rho,\sigma)}(s)$ and $\Phi_{\lambda,\mu;\gamma}(z,s,a)$ are the extended Gamma function and Hurwitz-Lerch Zeta function, respectively ([7] and [3, p.313]).

Proof. Using the definition (4.1) on the L.H.S of (4.2) and then expanding $\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)$ with the help of (2.1), we get

$$\begin{aligned} &M\left\{\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)\right\} \\ &= \int_0^\infty p^{\alpha-1}\left(\sum_{n=0}^\infty \frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}\right)dp. \end{aligned}$$

Now, changing the order of integration and summation, we get

$$\begin{aligned}&M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right\}\\&=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{n!(n+a)^s B(\mu,\gamma-\mu)}\int_0^{\infty}p^{\alpha-1}B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)dp\\&=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{n!(n+a)^s}\frac{B(m\alpha+\mu+n,\gamma-\mu+m\alpha)}{B(\mu,\gamma-\mu)}\Gamma^{(\rho,\sigma)}(s),\end{aligned}$$

where $\Gamma^{(\rho,\sigma)}(s)$ is the generalized Gamma function given in Parmar [7].

Now, expanding $B(m\alpha+\mu+n,\gamma-\mu+m\alpha)$ in terms of Gamma function and then by using the result $\Gamma(\lambda+n)=\Gamma(\lambda)(\lambda)_n$, we get

$$\begin{aligned}&M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right\}\\&=\frac{\Gamma^{(\rho,\sigma)}(s)\Gamma(m\alpha+\mu)}{B(\mu,\gamma-\mu)\Gamma(2m\alpha+\gamma)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(m\alpha+\mu)_n}{n!(2m\alpha+\gamma)_n}\frac{z^n}{(n+a)^s}.\end{aligned}$$

Finally, using the definition of Hurwitz-Lerch Zeta function (HLZF) given in [3, p.313], we arrive at our required result.

5. Generating relations

Theorem 5.1. *For $p\geq 0$, $\lambda\in\mathbb{C}$ and $\mid t\mid<1$, the following generating function of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ holds true:*

$$\sum_{n=0}^{\infty}(\lambda)_n\Phi_{\lambda+n,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\frac{t^n}{n!}=(1-t)^{-\lambda}\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}\left(\frac{z}{1-t},s,a;p\right).\tag{5.1}$$

Proof. Let us denote the left hand side of (5.1) by L_1. By using the series expression given in (2.1) into L_1, we find that

$$L_1=\sum_{n=0}^{\infty}(\lambda)_n\left\{\sum_{k=0}^{\infty}\frac{(\lambda+n)_k B_p^{(\rho,\sigma;m)}(\mu+k,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{z^k}{k!(k+a)^s}\right\}\frac{t^n}{n!},\tag{5.2}$$

which by changing the order of summation, gives

$$L_1=\sum_{k=0}^{\infty}\frac{(\lambda)_k B_p^{(\rho,\sigma;m)}(\mu+k,\gamma-\mu)}{B(\mu,\gamma-\mu)}\left\{\sum_{n=0}^{\infty}(\lambda+k)_n\frac{t^n}{n!}\right\}\frac{z^k}{k!(k+a)^s}.\tag{5.3}$$

Now, applying the following binomial expansion:

$$(1-\lambda)^{-(\lambda+k)}=\sum_{n=0}^{\infty}(\lambda+k)_n\frac{t^n}{n!},(\mid t\mid<1),$$

for evaluating the inner sum in (5.3) and then by using (2.1), we get our desired result.

Theorem 5.2. *For $p \geq 0$, $\lambda \in \mathbb{C}$ and $|\,t\,|<|\,a\,|; s \neq 1$, the following generating function of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ holds true:*

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a-t;p)=\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}\left(z,s+n,a;p\right)t^n. \tag{5.4}$$

Proof. Let us denote the left hand side of (5.4) by L_2. Then by using (2.1), we get

$$\begin{aligned}
L_2 &= \sum_{l=0}^{\infty}\frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{z^l}{l!(l+a-t)^s}\\
&= \sum_{l=0}^{\infty}\frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{z^l}{l!(l+a)^s}(1-\frac{t}{l+a})^{-s}\\
&= \sum_{l=0}^{\infty}\frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{z^l}{l!(l+a)^s}\left\{\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\left(\frac{t}{l+a}\right)^n\right\}\\
&= \sum_{n=0}^{\infty}\frac{(s)_n}{n!}\left(\sum_{l=0}^{\infty}\frac{(\lambda)_l B_p^{(\rho,\sigma;m)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{z^l}{l!(l+a)^{s+n}}\right)t^n.
\end{aligned}$$

Finally, by making use of (2.1), we get the desired assertion (5.4).

Remark 5.1. For $m=1$, the generating function (5.1) and (5.4) asserted by Theorem 5.1 and Theorem 5.2, respectively, were derived earlier by Parmar et al. [9].

6. Derivation of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$

In this section we provide a differential formula of our new extended Hurwitz-Lerch Zeta function (EHLZF) $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$.

Theorem 6. *The following differential formula holds true:*

$$\frac{d^r}{dz^r}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right]=\frac{(\mu)_r(\lambda)_r}{(\gamma)_r}\Phi_{\lambda+r,\mu+r;\gamma+r}^{(\rho,\sigma;m)}(z,s,a+r;p), \tag{6.1}$$

where $r \in \mathbb{N}=\{1,2,3,\cdots\}$.

Proof. Taking the derivative of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ with respect to z, we get

$$\begin{aligned}
\frac{d}{dz}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)\right] &= \frac{d}{dz}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}\right]\\
&= \sum_{n=1}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma;m)}(\mu+n,\gamma-\mu)}{(n-1)!B(\mu,\gamma-\mu)}\frac{z^{n-1}}{(n+a)^s}.
\end{aligned}$$

Replacing n by $n+1$, we get

$$\begin{aligned}
&\frac{d}{dz}\left[\Phi^{(\rho,\sigma;m)}_{\lambda,\mu;\gamma}(z,s,a;p)\right]\\
&=\frac{\mu\lambda}{\gamma}\sum_{n=0}^{\infty}\frac{(\lambda+1)_n B_p^{(\rho,\sigma;m)}(\mu+n+1,\gamma-\mu)}{n!B(\mu+1,\gamma-\mu)}\frac{z^n}{((n+1+a)^s}\\
&=\frac{\mu\lambda}{\gamma}\Phi^{(\rho,\sigma;m)}_{\lambda+1,\mu+1;\gamma+1}(z,s,a+1;p).
\end{aligned}$$

Recursive of this procedure yields us the desired result (6.1).

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Global stability analysis and persistence for an ecological food web-model

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Abstract. The ecological food web problems and their impact on the environment play vital role for balancing of some environments in our daily life. In the present work, the analytic results of an ecological food web-model are rigorously examined and analyzed. The model includes interactions and natural variables occur in different organisms of the species that influence by the competition and refuge as two basic conditions. The persistence of variant species for the resources competition is also analyzed. The global asymptotic stability of the positive equilibrium points is investigated numerically based on the Runge-Kutta predictor-corrector algorithm. Finally, the effects of the variation of each parameter on the proposed model are inspected numerically.

Keywords: food web-model, global stability, persistence, period dynamic, stage-structure.

1. Introduction

Our external environment suffers from many problems, including environmental, economic, social, ..., etc, as well as, the spread of epidemics and infectious diseases of all kinds. However, with the current technological universe and the increase of population, many scientists are motivating to orient their interest

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for studying such natural phenomena, through mathematical modeling to be analyzed deeply [1, 2, 3, 4, 5, 6, 7, 8].

With the time, and the arrival of the age of technology accompanied by the increase in the population, these problems have become more complex and more difficult. Therefore, it has become necessary to use modern technologies to help us in diagnosing and analyzing the scientific results that obtained in theory. A comprehensive number of studies have been developed to solve such difficulty and complexity. Marcus R. [9] presented two finite-difference algorithms for studying the dynamics of spatially extended predator-prey interactions with the Holling type II functional response and logistic growth of the prey. Naji,and Hussien, [10] proposed an epidemic model that describes the dynamics of the spread of two different types of infectious diseases that spread through both horizontal and vertical transmission in the host population. Whereas, Li, Hongli, et al. [11] investigated a three-species food chain model in a patchy environment, where the prey species, mid-level predator species, and top predator species can disperse among n different patches ($n \geq 0$).

The environmental-model that deals with endangered species (lemur animals) and two types of hunters (the black panthers and hyenas animals) that are link together by a food web is studied and analyzed theoretically by Al-Jubouri, et al. in [12].

The essential contribution of this study lies in demonstrating the theoretical aspect of model (2) given in [12]. New criteria are introduced to study the global stability of its unique equilibrium points, as well as, their existence. The simulations results substantiate the feasibility of the analytical findings.

2. Mathematical formulation

The idea of the proposed ecological-model is based on three-types of different species link together by a food web model. A high dimensional prey- predator model proposed in [12] is shown in Figure 1, and expressed mathematically in equation (1).

This model will be represented by the following nonlinear autonomous differential equations,

$$\begin{aligned}
\frac{dI_1}{dT} &= rI_2 - I_1\left(\frac{\rho_1(1-m)}{b_1+(1-m)I_1}P_1 + a_1(1-m)P_2 + \beta + d_1 + c_1I_1\right),\\
\frac{dI_2}{dT} &= \beta I_1 - I_2\left(\frac{\rho_2(1-m)}{b_2+(1-m)I_2}P_1 + a_2(1-m)P_2 + d_2 + c_2I_2\right),\\
\frac{dP_1}{dT} &= P_1\left(\frac{e_1\rho_1(1-m)I_1}{b_1+(1-m)I_1} + \frac{e_2\rho_2(1-m)I_2}{b_2+(1-m)I_2} - d_3 - c_3P_2\right),\\
\frac{dP_2}{dT} &= P_2\left(e_3a_1(1-m)I_1 + e_4a_2(1-m)I_2 - d_4 - c_4P_1\right).
\end{aligned} \tag{2.1}$$

This model consists of a two stage-structure of prey species (Lemur animals), which is an immature $I_1(t)$, and a mature $I_2(t)$, with a mid-level predator

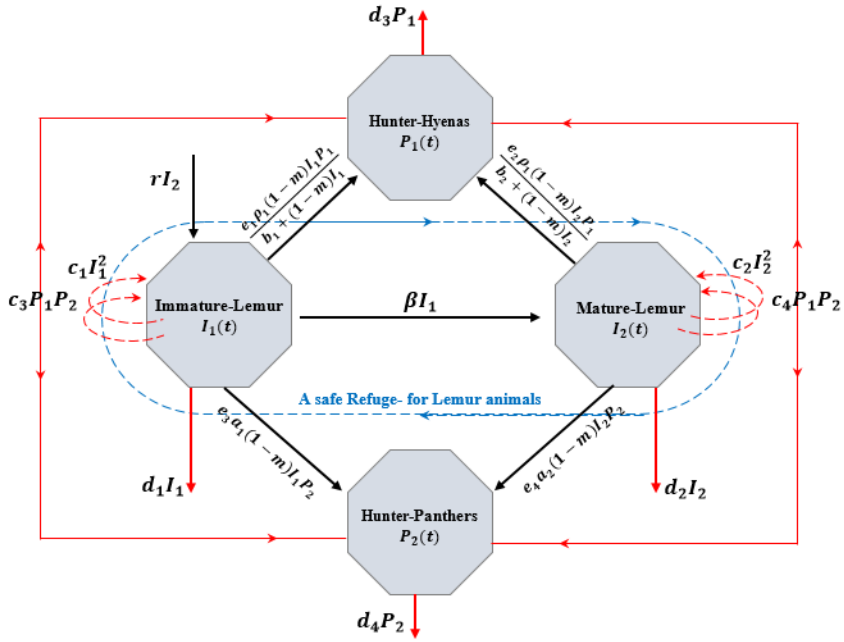

Figure 1: *Sketch, showing the idea of mathematical simulation of an Ecological-model.*

(Hyenas) $P_1(t)$, and a top-level predator (the black panthers) $P_2(t)$. Each of, I_1, I_2, P_1, and P_2 are representing the densities of populations at time (t). Furthermore, all the parameters used are positive and will be described biologically through Table 1.

Table 1: The inputs of the mathematical model(1).

Parameters Code in Model (1)	Biological Description
r	Actual increase average of a mature prey
β	Actual increase average of an immature prey
$c_{1,2}$	Competition average for an immature and mature prey
$d_{1,2}$	Natural death average for an immature and mature prey
$\rho_{1,2}$	Predation average for the prey- by a mid- level predator
$b_{1,2}$	Semi saturation average for a mid- level predator
$a_{1,2}$	Predation average for the prey- by a top- level predator
$c_{3,4}$	Competition average between a predators species
$d_{3,4}$	Death average for a predators after loss prey species
m	Refuge average
$(1-m)$	The number of prey exposed to predation by a predators
$e_{1,\ldots,4}$	Conversion average of a sustenance

Using the dimensionless variables technique, we have,

$$t = rT, \quad x = \frac{c_1}{r} I_1, \quad y = \frac{c_2}{r} I_2, \quad z = \frac{\rho_1 c_1}{r^2} P_1 \quad ,and \quad w = \frac{a_1(1-m)}{r} P_2.$$

A coordination to these assumptions, the model becomes as,

$$\begin{aligned}
\frac{dx}{dt} &= \upsilon_1 y - x\left(\frac{z}{\upsilon_2 + x} + w + (\upsilon_3 + \upsilon_4) + x\right) = f_1(x,y,z,w); x(0) \geq 0,\\
\frac{dy}{dt} &= \upsilon_5 x - y\left(\frac{\upsilon_6 z}{\upsilon_7 + y} + \upsilon_8 w + \upsilon_9 + y\right) = f_2(x,y,z,w); y(0) \geq 0,\\
\frac{dz}{dt} &= z\left(\frac{\upsilon_{10} x}{\upsilon_2 + x} + \frac{\upsilon_{11} y}{\upsilon_7 + y} - \upsilon_{12} - \upsilon_{13} w\right) = f_3(x,y,z,w); z(0) \geq 0,\\
\frac{dw}{dt} &= w\left(\upsilon_{14} x + \upsilon_{15} y - \upsilon_{16} - \upsilon_{17} z\right) = f_4(x,y,z,w); w(0) \geq 0.
\end{aligned} \tag{2.2}$$

Here:

$\upsilon_1 = \frac{c_1}{c_2}$; $\upsilon_2 = \frac{b_1 c_1}{r(1-m)}$; $\upsilon_3 = \frac{\beta}{r}$; $\upsilon_4 = \frac{d_1}{r}$; $\upsilon_5 = \frac{\beta c_2}{r c_1}$; $\upsilon_6 = \frac{\rho_2 c_2}{\rho_1 c_1}$; $\upsilon_7 = \frac{b_2 c_2}{r(1-m)}$; $\upsilon_8 = \frac{a_2}{a_1}$; $\upsilon_9 = \dfrac{d_2}{r}$; $\upsilon_{10} = \frac{e_1 \rho_1}{r}$; $\upsilon_{11} = \frac{e_2 \rho_2}{r}$; $\upsilon_{12} = \frac{d_3}{r}$; $\upsilon_{13} = \frac{c_2}{a_1(1-m)}$; $\upsilon_{14} = \frac{e_3 a_1(1-m)}{c_1}$; $\upsilon_{15} = \frac{e_4 a_2(1-m)}{c_2}$; $\upsilon_{16} = \frac{d_4}{r}$; $\upsilon_{17} = \frac{r c_4}{\rho_1 c_1}$.

Since these functions are Lipschitzian on $\mathbb{R}^4_+ = \{(x,y,z,w) \in \mathbb{R}^4_+ : x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$ and $w(0) \geq 0\}$, then the solution of the model (2) exists and unique.

3. Boundedness

Theorem 1. *All the trajectories of model (2), with the initial points in $\mathbb{R}^4_+$ are uniformly bounded. For the proof, we refer the reader to see [12].*

4. Existence and stability analysis

The model (2) have at most five- biologically reasonable equilibrium points $H_i = (x,y,z,w),\ i = 0,...,4$, which are exist under the conditions established in [12].

In the following, the stability of model (2) near proper equilibrium points $H_i, i = 0,...,4$ is discussed in [12].

1. The trivial point $H_0 = (0,0,0,0)$, if the following condition hold

$$u_5 < \frac{u_9(u_3 + u_4)}{u_1}. \tag{4.1}$$

Then, the trajectories of model (2) tending to the asymptotically stable point H_0.

2. The predators-free point $H_1 = (\bar{x}, \bar{y}, 0, 0)$, if the following conditions hold

$$u_{12} + u_{16} > n_3 + n_4, \tag{4.2}$$

$$n_3 n_4 + u_{12} u_{16} > u_{16} n_3 + u_{12} n_4. \tag{4.3}$$

Then, the trajectories of model (2) tending to the asymptotically stable point H_1.

3. The mid-level predator-free point $H_2 = (\bar{\bar{x}}, \bar{\bar{y}}, 0, \bar{\bar{w}})$, if the following conditions hold

$$\bar{\bar{w}} > \max\{\Gamma_1, \Gamma_2\}, \tag{4.4}$$

$$\Gamma_3 > \Gamma_4, \tag{4.5}$$

$$\Gamma_5 > \Gamma_6. \tag{4.6}$$

Then, the trajectories of model(2)tending to the asymptotically stable point H_2. For more details see [12].

4. The top-level predator-free point $H_3 = (\bar{\bar{\bar{x}}}, \bar{\bar{\bar{y}}}, \bar{\bar{\bar{z}}}, 0)$, if the following conditions hold

$$\bar{\bar{\bar{z}}} > \frac{u_{14}\bar{\bar{\bar{x}}} + u_{15}\bar{\bar{\bar{y}}}}{u_{17}}, \tag{4.7}$$

$$u_{12} < c_3 < u_{12} + c_1 + c_2, \tag{4.8}$$

$$((u_{12} + c_1 + c_2) - c_3)\psi_1 > (c_3 - (u_{12} + c_1 + c_2))\psi_2 + Q_3. \tag{4.9}$$

Then, the trajectories of model(2) tending to the asymptotically stable point H_2. For more details see [12].

5. Finally, the coexistence equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$, if the following conditions hold

$$p_{12}^2 < \frac{4}{9}\, p_{11}p_{22}, \tag{4.10}$$

$$p_{13}^2 < \frac{4}{9}\, p_{11}p_{33}, \tag{4.11}$$

$$p_{14}^2 < \frac{4}{9}\, p_{11}p_{44}, \tag{4.12}$$

$$p_{23}^2 < \frac{4}{9}\, p_{22}p_{33}, \tag{4.13}$$

$$p_{24}^2 < \frac{4}{9}\, p_{22}p_{44}, \tag{4.14}$$

$$p_{34}^2 > \frac{4}{9}\, p_{33}p_{44}. \tag{4.15}$$

Then, the trajectories of model(2) tending to the asymptotically stable point H_2. For more details see [12].

5. Numerical simulations

In this section, the quantitative behavior of model (2) is determined based on Runge-Kutta predictor-corrector method using MATLAB. These simulations demonstrate the previously obtained theoretical results of stability and equilibrium of the proposed model given in [12]. Also, the global dynamics and persistence have been proven and materialized numerically. As in the Figures 2-9. Furthermore, the effects of changing parameter values of model (2) were investigated. The proposed system was simulated numerically for the following parameter values:

$$\begin{aligned} &\upsilon_1 = 0.9, \upsilon_2 = 0.4, \upsilon_3 = 0.3, \upsilon_4 = 0.4, \upsilon_5 = 2.5, \upsilon_6 = 0.4, \\ &\upsilon_7 = 0.6, \upsilon_8 = 0.2, \upsilon_9 = 0.7, \upsilon_{10} = 0.9, \upsilon_{11} = 0.9, \\ &\upsilon_{12} = 0.25, \upsilon_{13} = 0.09, \upsilon_{14} = 0.9, \upsilon_{15} = 1.1, \upsilon_{16} = 0.04, \upsilon_{17} = 0.33. \end{aligned} \tag{5.1}$$

Taking the above data into consideration, the time series of the trajectories of model (2) are shown in Figure 2.

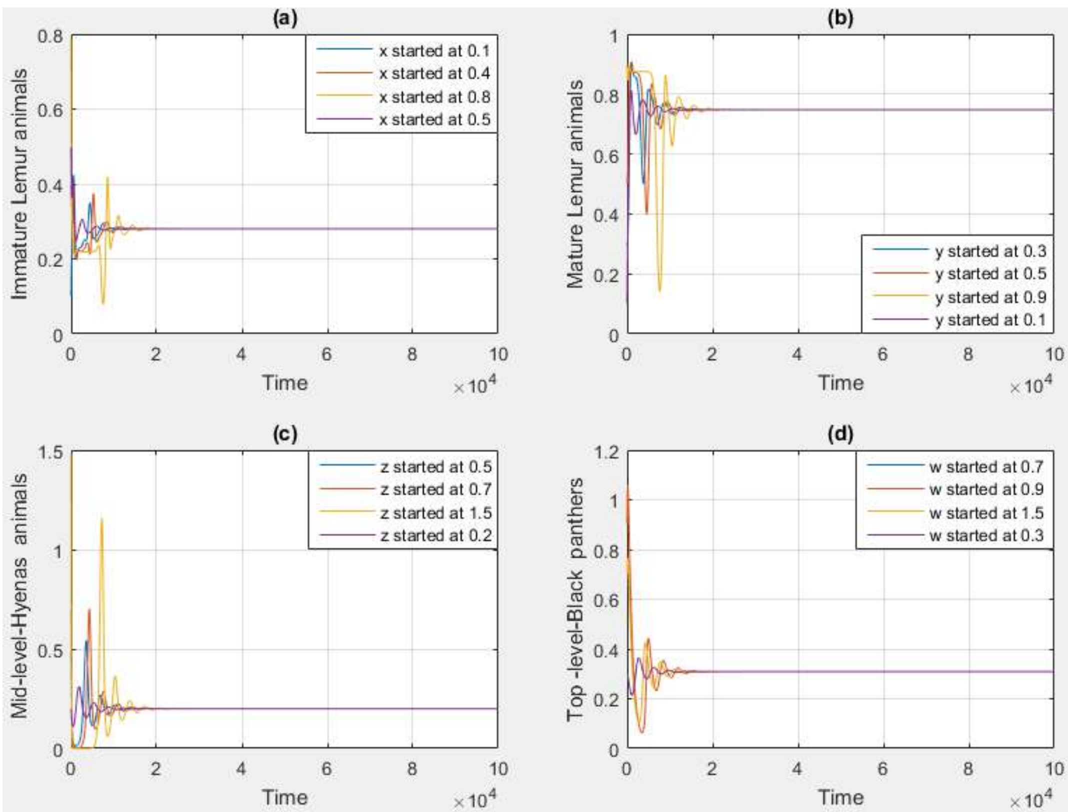

Figure 2: The time series of system (2), starting with four different initial points $(0.1, 0.3, 0.5, 0.7)$, $(0.4, 0.5, 0.7, 0.9)$, $(0.8, 0.9, 1.5, 1.5)$ and $(0.5, 0.7, 0.2, 0.4)$.

It illustrates that model (2) has globally asymptotically stable as the solution of model (2) approaches asymptotically to the positive equilibrium point $H_4 = (0.254, 0.714, 0.2, 0.309)$, which confirmed the obtained analytical results.

Next, we need to analyze the results of the asymptotic stability of points $H_i, i = 0, 1, 2, 3$. Some parameter values affect the dynamical behavior of model (2). At each time, the effect of varying of one parameter while the others are fixed is discussed. The results are summarized in Table 1.

It can be seen that varying the value of parameters $\upsilon_i, i = 3, 4, 5$ does not affect the dynamic of model(2). Therefore, the solution of the model (2) still converges to the coexisting (positive) equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$. Figures (3-6) show the time series of model (2) according to different parameters, which converge to the equilibrium points H_i ; $for\ i = 0, 1, 2, 3$.

Table 2: The numerical behaviors and persistence of model (2) by changing of a specific parameter and fixing the other.

Variable parameters in model(2)	Numerical behavior of model(2)	Persistence of model(2)
$0.01 \leqslant \upsilon_1 < 0.9$	Converge to stable point $H_0 = (0, 0, 0, 0)$	Not Persist
$0.9 \leqslant \upsilon_1 \leqslant 1$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$\upsilon_1 > 1$	Converge to stable point in $xy - plane$	Persist
$0.3 < \upsilon_2 \leqslant 1.1$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.01 \leqslant \upsilon_5 < 0.6$	Converge to stable point $H_0 = (0, 0, 0, 0)$	Not Persist
$0.6 \leqslant \upsilon_5 < 2.5$	Converge to stable point in $xyw - space$	Persist
$2.5 \leqslant \upsilon_5 < 3$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$\upsilon_5 \geqslant 3$	Converge to periodic dynamics in Int.$\mathbb{R}^4_+$	Persist
$0.13 < \upsilon_7 < 0.6$	Converge to periodic dynamics in Int.$\mathbb{R}^4_+$	Persist
$0.6 \leqslant \upsilon_7 < 1$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.01 \leqslant \upsilon_8 \leqslant 0.35$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.3 < \upsilon_9 < 0.7$	Converge to stable point in $xyz - space$	Persist
$0.7 \leqslant \upsilon_9 \leqslant 0.95$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.8 \leqslant \upsilon_{10} < 1$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$1 \leqslant \upsilon_{10} < 2$	Converge to stable point in $xyz - space$	Persist
$0.9 \leqslant \upsilon_{11} < 2$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$\upsilon_{11} \geqslant 2$	Converge to periodic dynamics in Int.$\mathbb{R}^4_+$	Persist
$0.2 < \upsilon_{12} \leqslant 0.25$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.01 \leqslant \upsilon_{13} \leqslant 0.1$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.1 < \upsilon_{13} \leqslant 0.7$	Converge to stable point in $xyw - space$	Persist
$0.01 \leqslant \upsilon_{14} \leqslant 0.9$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.3 < \upsilon_{15} \leqslant 1.1$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.035 < \upsilon_{16} \leqslant 0.5$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.5 < \upsilon_{16} \leqslant 2$	Converge to stable point in $xyz - space$	Persist
$0.33 \leqslant \upsilon_{17} < 0.8$	Converge to stable point in Int.$\mathbb{R}^4_+$	Persist
$0.8 \leqslant \upsilon_{17} \leqslant 2$	Converge to stable point in $xy - plane$	Persist

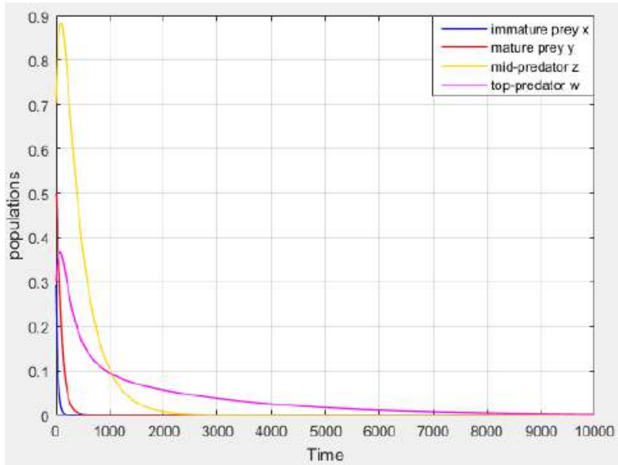

Figure 3: Time series of the trajectories for the data given in equation (18), with $\upsilon_1 = 0.01$, which shows that the trajectories converge asymptotically to the vanishing equilibrium point $H_0 = (0, 0, 0, 0)$.

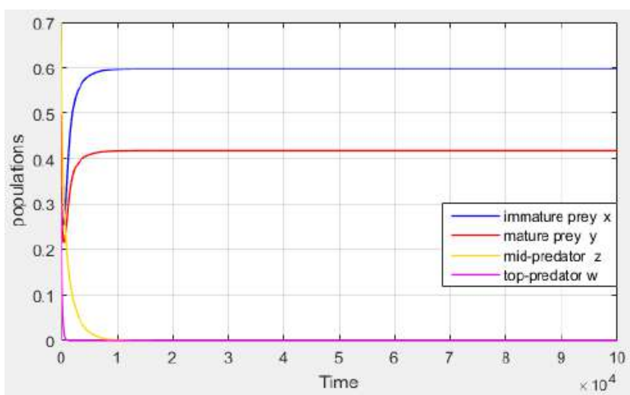

Figure 4: Time series of the trajectories for the data given in equation (18), with $\upsilon_{17} = 0.8$, which shows that the trajectories converge asymptotically to the predators-free equilibrium point $H_1 = (0.596, 0.416, 0, 0)$.

Figure 3 confirms the obtained analytic results regarding the existence of a locally asymptotically stable trivial equilibrium point $H_0 = (0, 0, 0, 0)$, when decreasing the intra-specific competition rate between the prey species (immature and mature prey) relative to the food and a refuge within limits $(0.01 \leqslant \upsilon_1 < 0.9)$. Increasing υ_1 in the range $(0.9 \leqslant \upsilon_1 \leqslant 1)$ and keeping other parameters constant as equation (18) shows that the solution of model(2) converges asymptotically to the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$ in Int.$\mathbb{R}^4_+$, see Figure 2. In addition, model (2) converges asymptotically to the predators-free

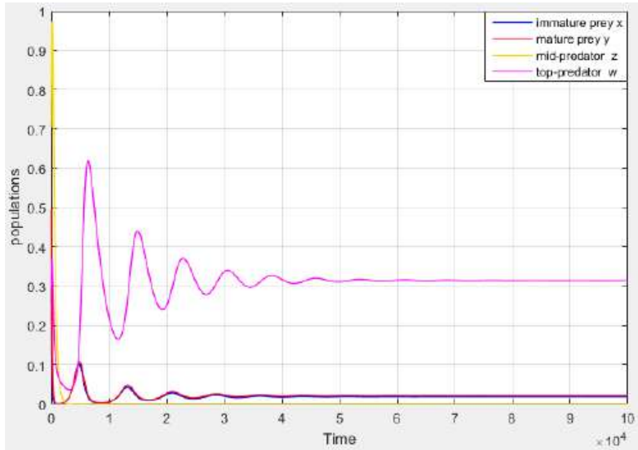

Figure 5: Time series of the trajectories for the data given in equation (18), with $\upsilon_{13} = 0.7$, which shows that the trajectories converge asymptotically to the mid-level predator-free equilibrium point $H_2 = (0.018, 0.212, 0, 0.314)$.

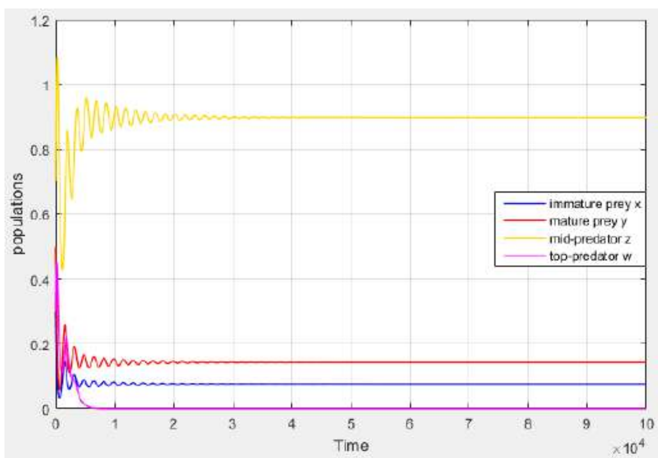

Figure 6: Time series of the trajectories for the data given in equation (18), with $\upsilon_{16} = 0.95$, which shows that the trajectories converge asymptotically to the top-level predator-free equilibrium point $H_3 = (0.075, 0.142, 0.898, 0)$.

point $H_1 = (\bar{x}, \bar{y}, 0, 0)$ when $\upsilon_1 > 1$. In Figure 4, $(0.33 \leqslant \upsilon_{17} < 0.8)$ represents the growth rate of the prey populations (immature and mature prey). To compete the predators for the predation of prey, it will expand, so the solution of model (2) converges asymptotically to the predators-free equilibrium point

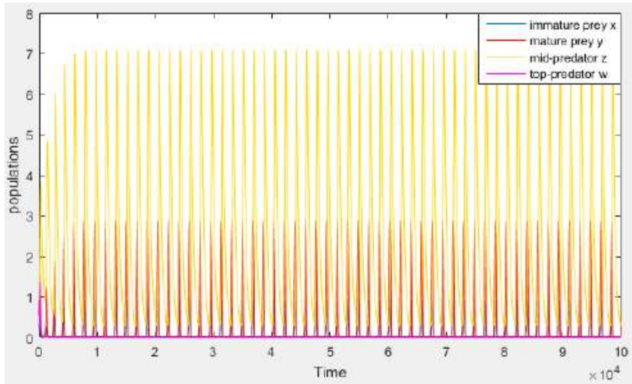

Figure 7: Time series of the trajectories for the data given in equation (18), with $\upsilon_5 = 3$, which shows that the trajectories approach the period dynamics in Int.$\mathbb{R}^4_+$.

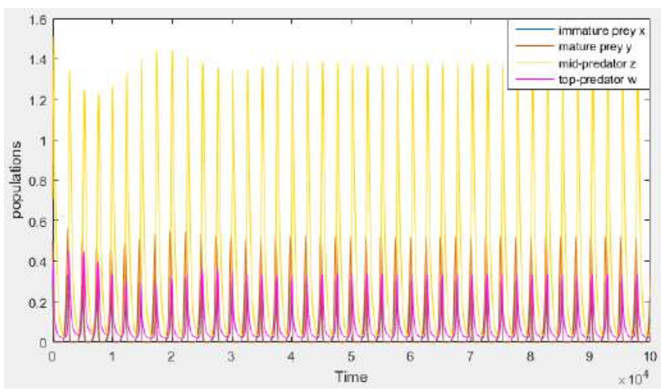

Figure 8: Time series of the trajectories for the data given in equation (18), with $\upsilon_7 = 0.1$, which shows that the trajectories approach the period dynamics in Int.$\mathbb{R}^4_+$.

$H_1 = (0.596, 0.416, 0, 0)$. Decreasing the range $(0.33 \leqslant \upsilon_{17} < 0.8)$, lead to approaching the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$ in Int.$\mathbb{R}^4_+$. For the constant parameters of equation (18), with $(0.1 < \upsilon_{13} \leqslant 0.7)$, which represents the inter-specific competition rate between the predators species (Top and mid-level predators) relative to the food and existence, the solution of model (2) converges asymptotically to the mid-level predator-free equilibrium point $H_2 = (0.018, 0.212, 0, 0.314)$ as shown in Figure 5, while it approaches asymptotically to the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$ when $(0.01 \leqslant \upsilon_{13} \leqslant 0.1)$ as shown in Figure 2. When the natural death rate for the top-level predator population relative to their growth rate is in the range

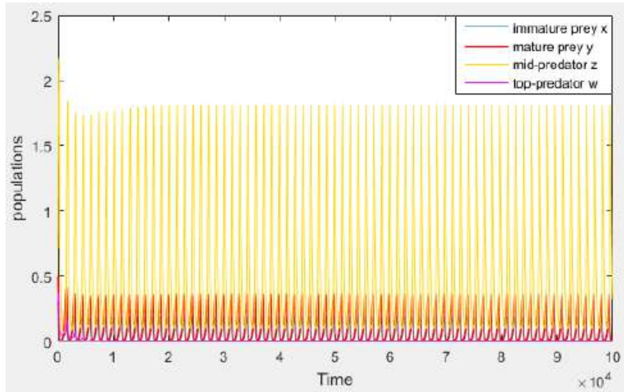

Figure 9: Time series of the trajectories for the data given in equation (18), with $\upsilon_{11} = 2$, which shows that the trajectories approach the period dynamics in Int.$\mathbb{R}^4_+$.

$(0.5 < \upsilon_{16} \leqslant 2)$, the solution of model (2) converges asymptotically to the top-level hunter-free equilibrium point $H_3 = (0.075, 0.142, 0.898, 0)$ as shown in Figure 6. But, it still approaches to positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$ in $(0.035 < u_{16} \leqslant 0.5)$.

Numerical simulations of model(2) shows that the model has periodic dynamics, as presented in Figures (7-9). For constant parameters of equation (18), Figure 7 shows that the solution curves of model (2) approach to periodic dynamics in Int.$\mathbb{R}^4_+$, when $(\upsilon_5 \geqslant 3)$, which represents the growing rate for immature prey relative to compete the prey population for existence. Reducing the half-saturation constant for mid-level predator relative to compete the mature prey population to refuge within the limits $(0.13 < \upsilon_7 < 0.6)$ which causes an approaching to periodic dynamics in Int.$\mathbb{R}^4_+$, see Figure 8. Expansion the predation rate, the mid-level predator for the mature prey population relative to their growth rate within limits $(\upsilon_{11} \geqslant 2)$ leads to approaching the periodic dynamics in Int.$\mathbb{R}^4_+$, as shown in Figure 9. Otherwise, model (2) still has a globally asymptotically stable positive equilibrium point.

6. Discussion and conclusions

This study aims to analyze a mathematical model (2) describing a food web-model with ecological reactions that occur between different species. We used computational algorithms, through which, the nature of the relationship of these organisms to the external environment and its direct impact on maintaining the balance of nature has been known to them through computer simulation. Initially, the effects of the variation of each parameter on the proposed model are studied and analyzed numerically. This can be summarize as follows:

1. Consider that the parameters' values in equation (18) are fixed, then the time series of the trajectories of model (2) converges to a globally asymptotically stable positive equilibrium point $H_4 = (0.254, 0.714, 0.2, 0.309)$, this can be seen obviously in Figure 2.

2. The trajectories of model (2) again converges to the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$, when changing the parameters values $\upsilon_i, i = 3, 4, 5$, because it does not affect the nature of the dynamic behavior of model(2).

3. The trajectories of model (2) converges to the trivial equilibrium point $H_0 = (0, 0, 0, 0)$, when decreasing the intra-specific competition rate between the prey species (immature and mature prey) are relative to the food and refuge within the limits $(0.01 \leqslant \upsilon_1 < 0.9)$, as shown in Figure 3. Otherwise, it converges to the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$.

4. The trajectories of model (2) converges to the predators-free equilibrium point $H_1 = (0.596, 0.416, 0, 0)$, when expanding the growth rate of the prey species (immature and mature prey) within the limits $(0.33 \leqslant \upsilon_{17} < 0.8)$, as shown in Figure 4. Otherwise, it converges to the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$.

5. The trajectories of model (2) converges to the mid-level predator-free equilibrium point $H_2 = (0.018, 0.212, 0, 0.314)$, when the inter-specific competition rate between the predators species (Top and mid-level predators) are relative to the food and existence within the limits $(0.1 < \upsilon_{13} \leqslant 0.7)$, as shown in Figure 5. Otherwise, it converges to the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$.

6. The trajectories of model (2) converges to the top-level predator-free equilibrium point $H_3 = (0.075, 0.142, 0.898, 0)$, when the natural death rate for the top-level predator population is relative to their growth rate in the range $(0.5 < \upsilon_{16} \leqslant 2)$, as shown in Figure 6. Otherwise, it converges to the positive equilibrium point $H_4 = (x^*, y^*, z^*, w^*)$.

   Moreover, the computer simulations of food web-model (2) showed us that model (2) possesses periodic dynamic behavior, this can be seen obviously in Figures (7-9). For more details see Table 2.

However, from these numerical analyses and results discussion, we conclude that the global stability of such complex ecological model that includes interactions and occur in different organisms is demonstrated. Moreover, such environmental, which has many conflicting can coexist within a common environment. Besides, the numerical experiments give a guarantee that a balance can be reached and the organisms can overcomes the danger of extinction.

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On bi-univalent functions involving Srivastava-Attiya operator

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Abstract. Two subclasses $L\sum_{\Sigma}^{b,\delta}(\mu,\alpha)$ and $L\sum_{\Sigma}^{b,\delta}(\mu,\beta)$ of the class $\sum$ of Bi-univalent functions have been introduced by making use of the Srivastava-Attiya operator. The estimates of the coefficients $|a_2|$ and $|a_3|$ of functions have been found for these subclasses. The results obtained are quit interesting and new.

Keywords: univalent function, bi-univalent function, coefficients bounds, and Srivastava-Attiya operator.

1. Introduction

Let Σ denotes the class of functions f of the form [1]

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which satisfy the following two continuous:

i. Holomorphic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$,

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ii. Normalized by $f(0) = f'(0) - 1 = 0$.

In addition, with $z \in U$, the general form of Hurwitz-Lerch Zeta function $\varphi(\delta, b, z)$ which are used with the convolution of holomorphic function can be defined by:

$$\varphi(\delta, b, z) = \sum_{K=0}^{\infty} \frac{z^k}{(k+b)^\delta} = b^{-\delta} + \frac{z}{(1+b)^\delta} + \sum_{k=2}^{\infty} \frac{z^k}{(k+b)^\delta}, \tag{2}$$

such that b is a complex number with $b \neq 0, -1, -2, \ldots, \mu \in \mathbb{C}$, and $\mathrm{Re}(\delta) > 1$.

Also, Srivastava and Attiya [2] defined the linear operators $Q_{\delta,b} : \Sigma \longrightarrow \Sigma$ by means of:

$$Q_{\delta,b} f(z) = G_{\delta,b} * f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^\delta a_k z^k, \tag{3}$$

where $G_{\delta,b} \in \Sigma$ to be:

$$G_{\delta,b} = (1+b)^\delta \left[\varphi(\delta, b, z) - b^{-\delta}\right] = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^\delta z^k. \tag{4}$$

Remark 1.1. $Q_{0,b}$ and $Q_{-\delta,b}$ denotes the identity and inverse operator of $Q_{\delta,b}$ respectively.

Koebe one-quarter theorem includes the image of U under every univalent functions $f \in A$ with an open disk centered at origin and radius $\frac{1}{4}$. Therefore, the inverse of every univalent function $f \in A$ can be written as $f^{-1} : f(U) \longrightarrow U$ satisfying:

$$f^{-1}(f(z)) = z, \ z \in U,$$
$$f\left(f^{-1}(\omega)\right) = \omega, \ |\omega| < r_0(f), r_0(f) \geq \frac{1}{4}.$$

Furthermore, we notice that the inverse function has the series expansion which can be written in the form:

$$f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \ldots \tag{5}$$

In addition, a function $f \in A$ is bi-univalent if both f and the inverse $g = f^{-1}$ are univalent in U.

Recently, several authors are concentrating on these functions, which are defined to the class Σ composed with various other features of the bi-univalent function class Σ, considering the most two important subclasses of univalent functions $S^*(\beta)$ and $C(\beta)$ of order β (see [2-7]). Consequently via definition, the classes $S^*(\beta)$ and $C(\beta)$ can be written as:

$$S^*(\beta) = \left\{ f \in S : Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \, z \in U \text{ and } 0 \leq \beta < 1 \right\}$$

and

$$C(\beta) = \left\{ f \in S : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in U \text{ and } 0 \leq \beta < 1 \right\}.$$

For $0 \leq \beta < 1$, if both f and its inverse f^{-1} are starlike and convex function of order β, then a function $f \in \sum d$ is in the class $S^*_\Sigma(\beta)$, or $C_\Sigma(\beta)$. These classes are introduced and investigated by Brannan and Taha [3]. Moreover, the coefficients $|a_2|$ and $|a_3|$ for functions in the classes $S^*_\Sigma(\beta)$ and $C_\Sigma(\beta)$ have been found.

The main objective of this study is to present new two subclasses of the class Σ related to the Srivastava-Attiya operator [2,7,10] and accordingly to find the estimation of the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses [11-13].

In order to prove our main results, we have to remembrance the following lemma.

Lemma 1.1. *If a function $h(z) \in P$ ([6])*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots,\ z \in U. \tag{6}$$

Then

$$|c_k| \leq 2,\ k \in N. \tag{7}$$

P is the family of all functions p, that an analytic in U for which $h(0) = 1$ and $Re(h(z)) > 0$.

2. Coefficient estimates of $L\sum_\Sigma^{b,\delta}(\mu, \alpha)$

Definition 2.1. *Let $f(z)$ related by* (1). *Then, it be in the class $L\sum_\Sigma^{b,\delta}(\mu, \alpha)$, if the following are fulfilled* [4]*:*

$$f \in \sum,\ \arg\left|\left\{(1-\mu)\frac{Q_{\delta,b}f(z)}{z} + \mu\left(Q_{\delta,b}f(z)\right)'\right\}\right| < \frac{\alpha\pi}{2} \tag{8}$$

with $0 < \alpha \leq 1, \mu \geq 1, z \in U$, and

$$arg\left|\left\{(1-\mu)\frac{Q_{\delta,b}g(\omega)}{\omega} + \mu\left(Q_{\delta,b}g(\omega)\right)'\right\}\right| < \frac{\alpha\pi}{2}, \tag{9}$$

where $0 < \alpha \leq 1$, $\mu \geq 1$, $z \in U$, and the function g is extended by $g = f^{-1}$ and given by:

$$f^{-1}(\omega) = \omega - a_2\omega^2 + \left(2a_2^2 - a_3\right)\omega^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \dots \tag{10}$$

For the functions in the class $L\sum_\Sigma^{b,\delta}(\mu, \alpha)$, we find the coefficients $|a_2|$ and $|a_3|$.

Theorem 2.1. *Let $f(z)$ which is given by (1) supposed to be in the class $L\sum_{\Sigma}^{b,\delta}(\mu,\alpha)$, $0<\alpha\leq 1$ and $\mu\geq 1$, $\delta\geq 1$. Then:*

$$|a_2| \leq \frac{2a}{\sqrt{\alpha\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right)+(1-\alpha)\left(\frac{1+b}{2+b}\right)^{2\delta}(1+\mu)^2}} \tag{11}$$

and

$$|a_3| \leq \frac{2\alpha}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{12}$$

Proof. The inequalities (11) and (12) are equivalent to:

$$(1-\mu)\frac{Q_{\delta,b}f(z)}{z}+\mu\left(Q_{\delta,b}f(z)\right)'=(p(z))^{\alpha} \tag{13}$$

and

$$(1-\mu)\frac{Q_{\delta,b}f(\omega)}{\omega}+\mu\left(Q_{\delta,b}f(\omega)\right)'=(q(\omega))^{\alpha}, \tag{14}$$

where $p(z)$ and $q(w)$ satisfies the inequalities:

$$\mathrm{Re}(p(z))>0,\ z\in U \text{ and } \mathrm{Re}(q(w))>0,\ w\in U.$$

Moreover, the functions $p(z)$ and $q(w)$ can be written as:

$$p(z)=1+p_1z+p_2z^2+\dots \tag{15}$$

and

$$q(\omega)=1+q_1\omega+q_2\omega^2+\dots \tag{16}$$

As well, $g(w)$ is given as in (3).

Now, by equating the coefficients in equations (11) and (12), we get:

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=p_1\alpha, \tag{17}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3=p_2\alpha+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{18}$$

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=-q_1\alpha, \tag{19}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)\left(2a_2^2-a_3\right)=q_2\alpha+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{20}$$

From equations (18) and (19), we obtained:

$$p_1 = -q_1 \tag{21}$$

and

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(2\mu+1)^2 a_2^2 = \alpha\left(p_1+q_1\right) + \frac{\alpha(\alpha-1)}{2}\left(q_1^2+p_1^2\right). \tag{22}$$

From (20), (21) and (22), we got:

$$a_2^2 = \frac{\alpha^2\left(q_2^2+p_2^2\right)}{\alpha\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right) + (1-\alpha)\left(\frac{1+b}{2+b}\right)^{2\delta}(1+\mu)^2}. \tag{23}$$

By applying lemma (1) on the coefficients p_2 and q_2, we obtained:

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right) + (1-\alpha)\left(\frac{1+b}{2+b}\right)^{2\delta}(1+\mu)^2}}. \tag{24}$$

Now, to find the bound on $|a_3|$, subtract (20) from (18) to get:

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3 - 2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_2^2 = \alpha\left(p_2-q_2^2\right). \tag{25}$$

From (23) and with the help of $p_1^2 = q_1^2$, substitute the value of a_2^2 to get:

$$a_3 = \frac{\alpha\left[\left(4\alpha(1+2\mu)\left(\frac{1+b}{3+b}\right)^{\delta} + (1-\alpha)(1+\mu)^2\left(\frac{1+b}{3+b}\right)^{\delta}\right)p_2 - (1-\alpha)(1+\mu)^2\left(\frac{1+b}{2+b}\right)^{2\delta}q_2^2\right]}{2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)\left[2\alpha(1+2\mu)\left(\frac{1+b}{3+b}\right)^{\delta} + (1-\alpha)(1+\mu)^2\left(\frac{1+b}{2+b}\right)^{2\delta}\right]}. \tag{26}$$

Now, considering Lemma 1.1 again and using the substations of coefficients p_1, p_2, q_1 and q_2, to get:

$$|a_3| \leq \frac{2\alpha}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{27}$$

Hence, the proof of the Theorem 2.2 is completed. □

Now, assuming $\mu = 1$ and $b = 1$ in above theorem, then we have:

Corollary 2.1. *If $f(z)$ given by* (1) *is in $L_{\Sigma}^{1,\delta}(1,\alpha)$, $0<\alpha\leq 1$ and $\mu\geq 1$, $\delta\geq 1$, then we have:*

$$|a_2| \leq \frac{2\alpha}{\sqrt{6\alpha\left(\frac{1}{2}\right)^{\delta}+4(1-\alpha)\left(\frac{1}{2}\right)^{2\delta}}} \tag{28}$$

and

$$|a_3| \leq \frac{2\alpha}{\left(\frac{1}{2}\right)^{\delta}(1+2\mu)}. \tag{29}$$

Assuming $\alpha=1$ in theorem 2.2, we have the following corollary:

Corollary 2.2. *Let $f(z)$ which is given by* (1) *belonged to the class $L\sum_{\Sigma}^{b,\delta}(\mu,1)$, $0<\alpha\leq 1$ and $\mu\geq 1$, $\delta\geq 1$. Then:*

$$|a_2| \leq \frac{2}{\sqrt{\left(\left(\frac{1+b}{3+b}\right)^{\delta}(2+4\mu)\right)}} \tag{30}$$

and

$$|a_3| \leq \frac{2}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{31}$$

3. Coefficient estimates of $L\sum_{\Sigma}^{b,\delta}(\beta,\mu)$

Definition 3.1. *Let $f(z)$ related by* (1)*. Then it be in the class $L\sum_{\Sigma}^{b,\delta}(\beta,\mu)$ if the following conditions are fulfilled:*

$$f\in\sum \text{ and } \operatorname{Re}\left\{(1-\mu)\frac{Q_{\delta,b}f(z)}{z}+\mu\left(Q_{\delta,b}f(z)\right)'\right\}>\beta, \tag{32}$$

where $0<\beta\leq 1$, $\mu\geq 1$, $z\in U$ and

$$\operatorname{Re}\left\{(1-\mu)\frac{Q_{\delta,b}g(\omega)}{\omega}+\mu\left(Q_{\delta,b}g(\omega)\right)\right\}>\beta, \tag{33}$$

such that $0<\beta\leq 1$, $\mu\geq 1$, $z\in U$. Thus, the function g is introduced that the inverse of f given as in (5)*.*

Theorem 3.1. *If $f(z)$ which is given by* (1) *supposed to be in $L\sum_{\Sigma}^{b,\delta}(\beta,\mu)$, $0\leq\beta<1$ and $\mu\geq 0$, then:*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}} \tag{34}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{35}$$

Proof. The inequalities (32) and (33) are equivalent to:

$$(1-\mu)\frac{Q_{\delta,b}f(z)}{z}+\mu\left(Q_{\delta,b}f(z)\right)'=\beta+(1-\beta)p(z) \tag{36}$$

and

$$(1-\mu)\frac{Q_{\delta,b}g(\omega)}{\omega}+\mu\left(Q_{\delta,b}g(\omega)\right)'=\beta+(1-\beta)q(z). \tag{37}$$

By equating coefficients in equations (36) and (37) produces:

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=p_1(1-\beta), \tag{38}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3=p_2(1-\beta) \tag{39}$$

and

$$\left(\frac{1+b}{2+b}\right)^{\delta}(1+\mu)a_2=-q_1(1-\beta), \tag{40}$$

$$\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)\left(2a_2^2-a_3\right)=q_2(1-\beta). \tag{41}$$

From equations (39) and (40), we obtained:

$$p_1=-q_1 \tag{42}$$

and also from (39) and (41), we obtain:

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_2^2=(1-\beta)\left(p_2+q_2\right). \tag{43}$$

So, we get:

$$a_2^2=\frac{\left(p_2+q_2\right)(1-\beta)}{2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{44}$$

By applying lemma 1.1 for the coefficients p_2 and q_2, we obtained:

$$\left|a_2^2\right|\le\sqrt{\frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}} \tag{45}$$

which is the looked-for inequality as given in the (34).

Now, by subtracting (41) from (39), we have:

$$2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_3^2=(1-\beta)\left(p_2-q_2\right)+2\left(\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)a_2^2\right. \tag{46}$$

Then, substitute the value of a_2^2 from (43), to obtain:

$$a_3^2 = \frac{(p_2 - q_2)(1-\beta)}{2\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{47}$$

Now, with the help of the Lemma 1.1, we got:

$$\left|a_3^2\right| \leq \frac{2(1-\beta)}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}, \tag{48}$$

which is the bound on $|a_3^2|$ as stated in (35). $\square$

Assuming $\beta = 0$ in theorem 3.2, we have the following corollary:

Corollary 3.1. *Let $f(z)$ given by (1) supposed to be in the class $L_{\Sigma}^{b,\delta}(0,\mu)$, $0 \leq \beta < 1$ and $\mu \geq 0$, $z \in U$. Then:*

$$\left|a_2^2\right| \leq \sqrt{\frac{2}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}}, \tag{49}$$

$$\left|a_3^2\right| \leq \frac{2}{\left(\frac{1+b}{3+b}\right)^{\delta}(1+2\mu)}. \tag{50}$$

Conclusion

We have been shown the existence of novel two subclasses types paly an interested roll to the Srivastava-Attiya operator with their original results. Consequently, the obtained outcomes have demonstrated the estimation of the coefficients $|a2|$ and $|a3|$ for associated complex functions in new subclasses. Many problem still opened for example extend the obtained results to the case of differential operator in Hebert space as in [14-15] or with another operator types (see [16-17]).

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On a complex Matsumoto space

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Abstract. In this paper we studied complex Matsumoto space. The expressions for fundamental metric tensor, angular metric tensor, Chern-Finsler connection coefficients and the formula of holomorphic curvature are obtained.
Keywords: complex Finsler space, complex Matsumoto space, Chern-Finsler connection coefficients, holomorphic curvature.

1. Introduction

The Finsler geometry has its origins in the famous dissertation of Germann mathematician Finsler [1] in 1918. A Finsler manifold is a manifold M where each tangent space is equipped with a Minkowski norm. J. H. Taylor and J. L. Synge introduced a special parallelism and the concept of connection in the theory of Finsler space was introduced by L. Berwald. Later on, E. Cartan, H. Rund, M. Matsumoto, D. Bao, Z. Shen etc, made effective contributions in this field. Finsler geometry has many applications in theories of physics, biology and mechanics. Especially, quantum physics has stimulated the study of complex structures.

After, that as compared to the real case in complex Finsler geometry are not known so many classes of complex Finsler metrics. Besides the significant Kobayashi and Caratheodary metrics [1], which quickened the study of such Finsler geometry and we know rather trivial classes of complex Finsler metrics, one is to Hermitian metrics on the base manifold [5] and second is to the locally Minkowski complex metrics. Therefore, any new class of complex Finsler spaces

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with some presence in both theory and applications is welcomed (see more details in ([2, 6, 9, 11, 13, 16]).

The aim of the present paper is to study the complex Matsumoto metric. In the third section to determine the fundamental metric tensor and angular metric tensor of the complex Matsumoto space. In the last section is to characterize the Chern-Finsler connection coefficients, Cartan tensor and the formula for holomorphic curvature of complex Matsumoto metric.

2. Priliminaries

Let M be a complex manifold of dimension n and $(z^k)_{k=1,2,3,..,,}$ complex coordinates in a local chart. Its complexified tangent bundle $T_{\mathbb{C}}M$ splits in to holomorphic tangent bundle $T'M$ and antiholomorphic tangent bundle $T''M$, i.e $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is itself a complex manifold with local coordinates (z^k, η^k) in a chart, which changes by the following rules.

$$z'^k = z'^k(z), \quad \eta^k = \frac{\partial z'^k}{\partial z^j}\eta^j \tag{1}$$

Further, $T_{\mathbb{C}}(T'M)$ decomposes into holomorphic and antiholomorphic tangent bundles $T'(T'M)$ and $T''(T'M)$ respectively.

A natural local frame $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$ for $T'(T'M)$ change according to the rules from Jacobi matrix of (1). Since the changing rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives, the concept of complex non-linear connection (c.n.c.) was introduced.

Let $V(T'M) = ker\pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. The complex non-linear connection (c.n.c.), determines a supplementary complex sub-bundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. It determines an adapted frame $\frac{\delta}{\delta z^k}$=$\frac{\partial}{\partial z^k}$-$N^j_k \frac{\partial}{\partial \eta^j}$, where $N^j_k(z, \eta)$ are the coefficients of the complex nonlinear connections:[1, 11]

A complex Finsler metric F on complex manifold M is a continuous function $F : T'M \to \mathbb{R}$ satisfying following conditions [11, 14]

(i) $L := F^2$ is smooth on $T'\tilde{M} := T'M\backslash\{0\}$;

(ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

(iii) $F(z,\ \lambda\eta)$=$|\lambda|F(z, \eta)$; for $\lambda \in \mathbb{C}$;

(iv) the Hermitian matrix $(g_{i\bar{j}}(z, \eta))$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, called the fundamental metric tensor, is positive definite on $T'M \setminus \{0\}$.

Let us write $L = F^2$. Then, the pair (M, F) is called a complex Finsler space. The strongly pseudoconvexity of the Finsler metric F on complex indicatrix, $I_{F,z} = \{\eta \in T'_zM \mid F(z, \eta) < 1\}$ is implied by assumption (iv). A Hermitian connection of $(1, 0)$ type named as the Chern-Finsler Connection [1] has a special meaning in a complex Finsler space. Notationally, it is $D\Gamma N = (L^i_{jk}, 0, C^i_{jk}, 0)$,

where

$$\overset{CF}{N^i_j} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l, \quad L^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k} = \frac{\partial N^i_k}{\partial \eta^j}, \quad C^i_{jk} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k}, \tag{2}$$

where $(g^{\bar{m}i})$ is the inverse of the metric tensor $(g_{\bar{m}i})$. The horizontal lift of the Liouville complex field (or the vertical radial vector field) $\eta^k\dot{\partial}_k$ is $\chi = \eta^k\dot{\partial}_k$, where $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and $\delta_k = \frac{\delta}{\delta z^k}$. The holomorphic curvature [11] of the complex Finsler space (M, F) in the direction η is

$$K_F(z,\eta) = \frac{2}{L^2(z,\eta)}G(R(\chi,\tilde{\chi})\chi,\tilde{\chi}), \tag{3}$$

where G is the $N-$lift of the complex Finsler metric tensor $g_{i\bar{j}}$ defined by $G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\partial\eta^i \otimes \partial\bar{\eta}^j$ and R is the curvature of Chern Finsler connection. Locally, it has the following expression [7],

$$K_F(z,\eta) = \frac{2}{L^2}R_{\bar{j}k}\bar{\eta}^j\eta^k \tag{4}$$

where

$$R_{\bar{j}k} = -g_{l\bar{j}}\delta_{\bar{h}}(\overset{CF}{N^l_k})\bar{\eta}^h. \tag{5}$$

3. Notion of complex Matsumoto space

In this section, we find the fundamental metric tensor $g_{i\bar{j}}$ and its inverse and determinant of the complex Matsumoto metric.

Let M be an n-dimensional complex manifold and $(z,\eta) \in T'M, \eta = \eta^k\frac{\partial}{\partial z^k}$. Let a purely Hermitian positive metric a, and a differential $(1,0)$ from b, be defined on M as $a = a_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ and $b = b_i(z)dz^i$. Aldea and munteanu [5] defined the complex Finsler metric on $T'M$ by

$$F(z,\eta) = F(\alpha(z,\eta),\, |\beta(z,\eta)|), \tag{6}$$

where

(a) $\alpha(z,\eta) = \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$;

(b) $|\beta(z,\eta)| = \sqrt{\beta(z,\eta)\ \overline{\beta(z,\eta)}}$ *with* $\beta(z,\eta) = b_i(z)\eta^i$.

We introduce a metric function F on the complex manifold M by

$$F = \frac{\alpha^2}{\alpha - |\beta|} \quad (|\beta| \neq 0). \tag{7}$$

We call the metric F defined by (7) as a complex Matsumoto metric and the manifold together with this complex Matsumoto metric as a complex Matsumoto space. The above complex Matsumoto metric is positive and smooth on $T'M\backslash\{0\}$. This metric is purely Hermitian if and only if β vanishes identically. The function $L = F^2$ depends on z and η because of $\alpha = \alpha(z,\eta)$ and $|\beta| = |\beta(z,\eta)|$. Also, α and β are homogeneous with respect to η, i.e. $\alpha(z,\ \lambda\eta)$=$|\lambda|\alpha(z,\eta)$ and $\beta(z,\ \lambda\eta) = \lambda\beta(z,\eta)$ for $\forall\lambda \in \mathbb{C}$. Therefore, $L(z,\lambda\eta) = \lambda\bar{\lambda}L(z,\eta)$ for any $\lambda \in \mathbb{C}$. From the homogeneity property, we have

$$\frac{\partial\alpha}{\partial\eta^i}\eta^i = \frac{1}{2}\alpha, \quad \frac{\partial|\beta|}{\partial\eta^i}\eta^i = \frac{1}{2}|\beta|. \tag{8}$$

Differentiating $\alpha(z,\eta)$ and $|\beta(z,\eta)|$ partially with respect to η^i and $\overline{\eta}^j$, we obtain the following

$$\frac{\partial\alpha}{\partial\eta^i} = \frac{l_i}{2\alpha}, \quad \frac{\partial|\beta|}{\partial\eta^i} = \frac{\bar{\beta}b_i}{2|\beta|}, \quad \frac{\partial\alpha}{\partial\bar{\eta}^j} = \frac{l_{\bar{j}}}{2\alpha}, \quad \frac{\partial|\beta|}{\partial\bar{\eta}^j} = \frac{\beta b_{\bar{j}}}{2|\beta|}, \tag{9}$$

$$\frac{\partial^2\alpha}{\partial\eta^i\partial\bar{\eta}^j} = \frac{a_{i\bar{j}}}{2\alpha} - \frac{l_i l_{\bar{j}}}{4\alpha^3}, \quad \frac{\partial^2|\beta|}{\partial\eta^i\partial\bar{\eta}^j} = \frac{b_i b_{\bar{j}}}{4|\beta|}, \tag{10}$$

where

$$l_i = a_{i\bar{j}}\overline{\eta}^j, \quad l_{\bar{j}} = a_{k\bar{j}}\eta^k,$$

now

$$\eta_i = \frac{\partial L}{\partial\eta^i} = L_\alpha\frac{\partial\alpha}{\partial\eta^i} + L_{|\beta|}\frac{\partial|\beta|}{\partial\eta^i}.$$

Using (9) and (10) we get

$$\eta_i = \frac{(\alpha - 2|\beta|)}{(\alpha-|\beta|)^2}Fl_i + \frac{1}{(\alpha-|\beta|)}\frac{F^2\bar{\beta}b_i}{|\beta|}, \tag{11}$$

$$\bar{\eta}_j = \frac{(\alpha - |\beta|)}{(\alpha-|\beta|)^2}Fl_{\bar{j}} + \frac{1}{(\alpha-|\beta|)}\frac{F^2\beta b_{\bar{j}}}{|\beta|}. \tag{12}$$

The fundamental metric tensor $g_{i\bar{j}}$ of the complex Matsumoto space (M,F) is given by

$$\begin{aligned} g_{i\bar{j}} &= \frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j} \\ &= L_{\alpha\alpha}\frac{\partial\alpha}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j} + L_{\alpha|\beta|}\left(\frac{\partial\alpha}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j} + \frac{\partial|\beta|}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j}\right) + L_{|\beta||\beta|}\left(\frac{\partial|\beta|}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j}\right) \\ &+ L_\alpha\frac{\partial^2\alpha}{\partial\eta^i\partial\bar{\eta}^j} + L_{|\beta|}\frac{\partial^2|\beta|}{\partial\eta^i\partial\bar{\eta}^j}, \end{aligned}$$

where

$$L_\alpha = \frac{\partial L}{\partial\alpha}, \quad L_{|\beta|} = \frac{\partial L}{\partial|\beta|}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, \quad L_{|\beta||\beta|} = \frac{\partial^2 L}{\partial|\beta|^2}, \quad L_{\alpha|\beta|} = \frac{\partial^2 L}{\partial\alpha\partial|\beta|}.$$

Using (9), (10) and (11), (12) we have

$$\begin{aligned} g_{i\bar{j}} &= \frac{F(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}\, a_{i\bar{j}} + \frac{-F|\beta|}{2\alpha^2}\left(\frac{\alpha-4|\beta|}{(\alpha-|\beta|)^3}\right) l_i l_{\bar{j}} + \frac{F^2}{2|\beta|}\left(\frac{\alpha+2|\beta|}{(\alpha-|\beta|)^2}\right) b_i b_{\bar{j}} \\ &+ \frac{F(\alpha-4|\beta|)}{2|\beta|(\alpha-|\beta|)^3}\left(l_i \beta b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}\right), \end{aligned}$$

which may be written as

$$g_{i\bar{j}} = p_0 a_{i\bar{j}} + p_{-2} l_i l_{\bar{j}} + q_0' b_i b_{\bar{j}} + q_{-2}\left(l_i \beta b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}\right), \tag{13}$$

where

$$\begin{aligned} p_0 &= \frac{F(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}, \quad p_{-2} = \frac{-F|\beta|}{2\alpha^2}\left(\frac{\alpha-4|\beta|}{(\alpha-|\beta|)^3}\right), \\ q_{-2} &= \frac{F(\alpha-4|\beta|)}{2|\beta|(\alpha-|\beta|)^3}, \quad q_0' = \frac{F^2}{2|\beta|}\left(\frac{\alpha+2|\beta|}{(\alpha-|\beta|)^2}\right). \end{aligned}$$

A simple calculation shows that

$$q_{-2}\left(l_i \beta b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}\right) = \frac{q_{-2}}{q_0 p_0}\eta_i \eta_{\bar{j}} - \frac{q_0 q_{-2}}{p_0}|\beta|^2 b_i b_{\bar{j}} - \frac{p_0 q_{-2}}{q_0} l_i l_{\bar{j}}, \tag{14}$$

substituting (14) in (13) we obtain a new expression for $g_{i\bar{j}}$ as

$$g_{i\bar{j}} = p_0 a_{i\bar{j}} + p'_{-2} l_i l_{\bar{j}} + q''_0 b_i b_{\bar{j}} + q''_{-2}\eta_i \eta_{\bar{j}}, \tag{15}$$

where

$$\begin{aligned} p'_{-2} &= p_{-2} - \frac{p_0 q_{-2}}{q_0} = \frac{-F}{2(\alpha-|\beta|)^3}\left(\frac{|\beta|}{\alpha^2}(\alpha-4|\beta|) + \alpha^2 - 4|\beta|^2\right), \\ q''_0 &= q'_0 - \frac{q_0 q_{-2}}{p_0}|\beta|^2 = \frac{F}{2(\alpha-|\beta|)}\left(\frac{\alpha-4|\beta|}{|\beta|(\alpha-|\beta|)^2} - \frac{F^2(\alpha+2|\beta|)}{(\alpha-2|\beta|)}\right), \\ q''_{-2} &= \frac{q_{-2}}{q_0 p_0} = \frac{(\alpha-|\beta|)}{2F(\alpha-2|\beta|)}(\alpha+2|\beta|). \end{aligned}$$

Theorem 3.1. *Let* $F = \frac{\alpha^2}{\alpha-|\beta|}$, $(|\beta| \neq 0)$ *be the complex Matsumoto metric, then the fundamental metric tensor* $g_{i\bar{j}}$ *is given by equation (15).*

Proof. Here the expression (15) is the fundamental metric tensor of the complex Matsumoto space. Now our next aim is to find the formulas for the inverse as well as for the determinant of the fundamental metric tensor $g_{i\bar{j}}$. For this purpose we use the following proposition given below [5]. □

Proposition 3.1. *Suppose:*
- $(Q_{i\bar{j}})$ *is a non-singular* $n \times n$ *complex matrix with inverse* $(Q^{\bar{j}i})$.
- C_i *and* $C_{\bar{i}} = \bar{C}_i, i = 1, 2, 3, \ldots, n$ *are complex numbers.*
- $C^i := Q^{\bar{j}i} C_{\bar{j}}$ *and its conjugates;* $C^2 := C^i C_i = \bar{C}^i C_{\bar{i}}$; $H_{i\bar{j}} := Q_{i\bar{j}} \pm C_i C_{\bar{j}}$.

Then:

1. *$det(H_{i\bar{j}}) = (1 \pm C^2) det(Q_{i\bar{j}})$,*
2. *whenever $(1 \pm C^2) \neq 0$, the matrix $(H_{i\bar{j}})$ is invertible and in this case its inverse is $H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \mp C^2} C^i \bar{C}^j$.*

From (15) we may re written as,

$$g_{i\bar{j}} = p_0 \left(a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}} + \frac{q''_{-2}}{p_0} \eta_i \eta_{\bar{j}} \right).$$

Assuming $Q_{i\bar{j}} = a_{i\bar{j}}$ and $C_i = \sqrt{\frac{p'_{-2}}{p_0}} l_i$ and applying Proposition (3.1), we find

$$Q^{\bar{j}i} = a^{\bar{j}i} \text{ and } C^i = Q^{\bar{j}i} C_{\bar{j}}, \quad C^2 = \alpha^2 \frac{p'_{-2}}{p_0},$$

where $(a^{\bar{j}i})$ is the Hermitian inverse of $(a_{i\bar{j}})$ since $1 \pm C^2 \neq 0$, and $1 - C^2 = \frac{p_0 - \alpha^2 p'_{-2}}{p_0}$.

The matrix $H_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}}$ is invertible with the inverse as,

$$H^{\bar{j}i} = a^{\bar{j}i} + R \eta^i \bar{\eta}^j \text{ and } det(H_{i\bar{j}}) = \left(1 + \frac{\alpha^2 p'_2}{p_0} \right) det(a_{i\bar{j}}) = \frac{p'_{-2}}{Rp_0} det(a_{i\bar{j}})$$

where $R = \frac{p'_2}{p_0 - \alpha^2 p'_2}$.

Taking $Q_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}}$ and $C_i = \sqrt{\frac{q''_0}{p_0}} b_i$ and again applying Proposition (3.1), we obtain $Q^{\bar{j}i} = a^{\bar{j}i} + R \eta^i \bar{\eta}^j$ and $C^i = \sqrt{\frac{q''_0}{p_0}} (b_i + R \bar{\beta} \eta^i)$, since $1 \pm C^2 \neq 0$, the inverse of $H_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}$ exists and is given by

$$H^{\bar{j}i} = a^{\bar{j}i} + R \eta^i \bar{\eta}^j + \frac{q''_0}{p_0} (b^i + R \bar{\beta} \eta^i)(b^{\bar{j}} + R \beta \bar{\eta}^j),$$

and also

$$det(a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}) = \frac{\gamma p'_{-2}}{Rp_0} det(a_{i\bar{j}}),$$

where $\gamma = 1 + \frac{q_0^2}{p_0} (\|b\|^2 + R|\beta|^2)$.

We set $Q_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}$ and $C_i = \sqrt{\frac{q''_{-2}}{p_0}} \eta_i$. Then, we have

$$\begin{aligned} Q^{\bar{j}i} &= a^{\bar{j}i} + R \eta^i \bar{\eta}^j + \frac{q''_0}{\gamma p_0} (b^i + R \bar{\beta} \eta^i)(b^{\bar{j}} + R \beta \bar{\eta}^j), \\ C^2 &= \frac{q''_{-2}}{p_0} \left[a^{\bar{j}i} \eta_i \eta_{\bar{j}} + RF^4 + \frac{q''_0}{\gamma p_0} \eta_i \eta_{\bar{j}} (b^i + R \bar{\beta} \eta^i)(b^{\bar{j}} + R \beta \bar{\eta}^j) \right]. \end{aligned}$$

Since $1 \pm C^2 \neq 0$, $H_{i\bar{j}} = a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}} + \frac{q''_{-2}}{p_0} \eta_i \eta_{\bar{j}}$, is invertible with the inverse.

$$(16) \quad H^{\bar{j}i} = \frac{a^{\bar{j}i} + R\eta^i \bar{\eta}^j + \frac{q''_0}{\gamma p_0}(b^i + R\bar{\beta}\eta^i)(b^{\bar{j}} + R\beta\bar{\eta}^j)}{1 + \frac{q''_{-2}}{p_0}\eta_m \bar{\eta}_n \left[a^{\bar{n}m} + R\eta^m \bar{\eta}^n + \frac{q''_0}{\gamma p_0}(b^m + R\bar{\beta}\eta^m)(b^{\bar{n}} + R\beta\bar{\eta}^n)\right]}.$$

In view of (11), we have

$$\begin{aligned} a^{\bar{n}m}\eta_i \bar{\eta}_{\bar{j}} &= p_0^2\alpha^2 + 2q_0p_0|\beta|^2 + q_0^2\|b\|^2|\beta|^2, \\ b^m \eta_m &= (p_0\bar{\beta} + q_0\|b\|^2\bar{\beta}), \\ b^{\bar{n}}\bar{\eta}_n &= (p_0\beta + q_0\|b\|^2\beta). \end{aligned}$$

Therefore,

$$\begin{aligned} (17) \quad (b^m\eta_m + R\bar{\beta}F^2)(b^{\bar{n}}\bar{\eta}_n + R\beta F^2) &= p_0^2|\beta|^2 + 2p_0q_0|\beta|^2\|b\|^4 \\ &\quad + q_0^2|\beta|^2\|b\|^4 + 2p_0RF^2|\beta|^2 \\ &\quad + 2q_0RF^2|\beta|^2\|b\|^2 + R^2F^4|\beta|^2, \end{aligned}$$

substitute (17), in (16), we get

$$(18) \qquad H^{\bar{j}i} = \frac{1}{M}\left[a^{\bar{j}i} + R\eta^i\bar{\eta}^j + \frac{q''_0}{\gamma p_0}(b^i + R\bar{\beta}\eta^i)(b^{\bar{j}} + R\beta\bar{\eta}^j)\right],$$

where

$$\begin{aligned} M &= 1 + \frac{q''_{-2}}{p_0}\Bigg\{p_0^2\alpha^2 + 2q_0p_0|\beta|^2 + q_0^2\|b\|^2|\beta|^2 \\ (19) \qquad &+ RF^4 + \frac{q''_0}{\gamma p_0}\Bigg(p_0^2|\beta|^2 + 2p_0q_0|\beta|^2\|b\|^4 \\ &+ q_0^2|\beta|^2\|b\|^4 + 2p_0RF^2|\beta|^2 + 2q_0RF^2|\beta|^2\|b\|^2 + R^2F^4|\beta|^2\Bigg)\Bigg\}, \end{aligned}$$

and

$$det(H_{i\bar{j}}) = (1 + c^2)det(Q_{i\bar{j}}) = M det(a_{i\bar{j}} + \frac{p'_{-2}}{p_0} l_i l_{\bar{j}} + \frac{q''_0}{p_0} b_i b_{\bar{j}}).$$

On simplifying, above we have

$$(20) \qquad det(H_{i\bar{j}}) = M\gamma\left(1 + \frac{\alpha^2 p'_{-2}}{p_0}\right)det(a_{i\bar{j}}) = \frac{M\gamma}{R}\frac{p'_{-2}}{p_0}det(a_{i\bar{j}}).$$

Since $g_{i\bar{j}} = p_0 H_{i\bar{j}}$, the inverse of the fundamental metric tensor is given by

$$(21) \qquad g^{\bar{j}i} = \frac{1}{p_0}H^{\bar{j}i},$$

where $H^{\bar{j}i}$ is given by (18). Also, the determinant of the fundamental metric tensor is given by

$$det(g_{i\bar{j}}) = p_0^n det(H_{i\bar{j}}) = p_0^n \frac{M\gamma}{R}\frac{p'_{-2}}{p_0} det(a_{i\bar{j}}). \tag{22}$$

The angular metric tensors of the complex Matsumoto space (M, F) is given by

$$\begin{aligned} K_{i\bar{j}} = \frac{\partial^2 F}{\partial\eta^i\partial\bar{\eta}^j} &= F_{\alpha\alpha}\frac{\partial\alpha}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j} + F_{\alpha|\beta|}\left(\frac{\partial\alpha}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j} + \frac{\partial|\beta|}{\partial\eta^i}\frac{\partial\alpha}{\partial\bar{\eta}^j}\right) \\ &+F_{|\beta||\beta|}\left(\frac{\partial|\beta|}{\partial\eta^i}\frac{\partial|\beta|}{\partial\bar{\eta}^j}\right) + F_\alpha\frac{\partial^2\alpha}{\partial\eta^i\partial\bar{\eta}^j} + F_{|\beta|}\frac{\partial^2|\beta|}{\partial\eta^i\partial\bar{\eta}^j}, \end{aligned} \tag{23}$$

where

$$F_\alpha = \frac{\partial F}{\partial\alpha},\quad F_{|\beta|} = \frac{\partial F}{\partial|\beta|},\quad F_{\alpha\alpha} = \frac{\partial^2 F}{\partial\alpha^2},\quad F_{|\beta||\beta|} = \frac{\partial^2 F}{\partial|\beta^2|}\ and\ F_{\alpha|\beta|} = \frac{\partial^2 F}{\partial\alpha\partial|\beta|}.$$

On differentiating (6) with respect to α and β, we obtain

$$\begin{aligned} &F_\alpha = \frac{\alpha^2 - 2\alpha|\beta|}{(\alpha - |\beta|)^2},\quad F_{|\beta|} = \frac{\alpha^2}{(\alpha-|\beta|)^2}, \\ &F_{\alpha|\beta|} = \frac{-2\alpha|\beta|}{(\alpha-|\beta|)^3},\quad F_{\alpha\alpha} = \frac{2|\beta|^2}{(\alpha-|\beta|)^3}, \\ &F_{|\beta||\beta|} = \frac{2\alpha^2}{(\alpha-|\beta|)^3}. \end{aligned} \tag{24}$$

On substituting (24) and (9) in (23), we obtain

$$K_{i\bar{j}} = \xi_0 a_{i\bar{j}} + \xi_{-2} l_i l_{\bar{j}} + \chi_{-2}(\beta l_i b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}) + \chi'_0 b_i b_{\bar{j}}, \tag{25}$$

where,

$$\xi_0 = \frac{\alpha - 2|\beta|}{2\,(\alpha-|\beta|)^2},\ \xi_{-2} = \frac{-(\alpha - 3|\beta|)}{4\alpha\,(\alpha-|\beta|)^3},\ \chi'_0 = \frac{F}{4|\beta|}\frac{(\alpha+3|\beta|)}{(\alpha-|\beta|)^2},\ \chi_{-2} = \frac{-1}{2(\alpha-|\beta|)^3}.$$

Again differentiatting (7) partially with respect to η^i and $\bar{\eta}^j$, respectively we have,

$$\begin{aligned} \frac{\partial F}{\partial\eta^i} &= \left(\frac{\alpha^2 - 2\alpha|\beta|}{(\alpha-|\beta|)^2}\right)\frac{l_i}{2\alpha} + \left(\frac{\alpha^2}{(\alpha-|\beta|)^2}\right)\frac{\bar{\beta} b_i}{2|\beta|}, \\ \frac{\partial F}{\partial\bar{\eta}^j} &= \left(\frac{\alpha^2 - 2\alpha|\beta|}{(\alpha-|\beta|)^2}\right)\frac{l_{\bar{j}}}{2\alpha} + \left(\frac{\alpha^2}{(\alpha-|\beta|)^2}\right)\frac{\beta b_{\bar{j}}}{2|\beta|}. \end{aligned} \tag{26}$$

From (24) and (26), we obtain

$$\begin{aligned} &\frac{\frac{\partial F}{\partial\eta^i}\frac{\partial F}{\partial\bar{\eta}^j}}{\frac{F_\alpha}{2\alpha}\frac{F_{|\beta|}}{2|\beta|}} - \frac{F_{|\beta|}/2|\beta|}{F_\alpha/2\alpha}|\beta|^2 b_i b_{\bar{j}} - \frac{F_\alpha/2\alpha}{F_{|\beta|}/2|\beta|} l_i l_{\bar{j}} = (l_i\beta b_{\bar{j}} + l_{\bar{j}}\bar{\beta} b_i), \\ &\left\{\frac{4|\beta|(\alpha-|\beta|)^4\frac{\partial F}{\partial\eta^i}\frac{\partial F}{\partial\bar{\eta}^j}}{\alpha^3(\alpha-2|\beta|)} - \frac{|\beta|(\alpha-2|\beta|)}{\alpha^2} l_i l_{\bar{j}} - \frac{\alpha^2|\beta|}{(\alpha-2|\beta|)} b_i b_{\bar{j}}\right\} \\ &\qquad = (l_i\beta b_{\bar{j}} + l_{\bar{j}}\bar{\beta} b_i). \end{aligned} \tag{27}$$

Substituting the value of $(l_i\beta b_{\bar{j}} + l_{\bar{j}}\bar{\beta}b_i)$ from (27) in (25), then we have

$$
\begin{aligned}
(28)\ K_{i\bar{j}} &= \frac{\alpha-2|\beta|}{2\left(\alpha-|\beta|\right)^2}a_{i\bar{j}} + \frac{-(\alpha-3|\beta|)}{4\alpha\left(\alpha-|\beta|\right)^3}l_il_{\bar{j}} \\
&+\frac{F}{4|\beta|}\frac{(\alpha+3|\beta|)}{\left(\alpha-|\beta|\right)^2}b_ib_{\bar{j}} + \frac{-1}{2(\alpha-|\beta|)^3} \\
&\left\{\frac{4|\beta|(\alpha-|\beta|)^4\frac{\partial F}{\partial\eta^i}\frac{\partial F}{\partial\bar{\eta}^j}}{\alpha^3(\alpha-2|\beta|)} - \frac{|\beta|(\alpha-2|\beta|)}{\alpha^2}l_il_{\bar{j}} - \frac{\alpha^2|\beta|}{(\alpha-2|\beta|)}b_ib_{\bar{j}}\right\}.
\end{aligned}
$$

Since $\eta_i = \frac{\partial L}{\partial\eta^i} = 2F\frac{\partial F}{\partial\eta^i}$ and $\bar{\eta}_j = \frac{\partial L}{\partial\bar{\eta}^j} = 2F\frac{\partial F}{\partial\bar{\eta}^j}$, we have $\eta_i\bar{\eta}_j = 4L\frac{\partial F}{\partial\eta^i}\frac{\partial F}{\partial\bar{\eta}^j}$.

Substituting these values in (28) we get

$$
(29) \qquad K_{i\bar{j}} = \xi_0 a_{i\bar{j}} + \xi'_{-2}l_il_{\bar{j}} + \frac{\chi''_{-2}}{2L}\eta_i\eta_{\bar{j}} + \chi''_0 b_ib_{\bar{j}},
$$

where

$$
\begin{aligned}
\xi'_{-2} &= \frac{1}{2(\alpha-|\beta|)^3}\left(\frac{3|\beta|-\alpha}{2\alpha} + \frac{|\beta|(\alpha-2|\beta|)}{F(\alpha-|\beta|)}\right), \quad \chi''_{-2} = \frac{-2|\beta|(\alpha-|\beta|)}{F(\alpha-2|\beta|)}, \\
\chi''_0 &= \frac{F}{2|\beta|(\alpha-|\beta|)^2}\left(\frac{\alpha+3|\beta|}{2} + \frac{1}{\alpha-2|\beta|}\right),
\end{aligned}
$$

or, in the equivalent form:

$$
(30) \qquad K_{i\bar{j}} = \xi_0\left(a_{i\bar{j}} + \frac{\xi'_{-2}}{\xi_0}l_il_{\bar{j}} + \frac{\chi''_0}{\xi_0}b_ib_{\bar{j}} + \frac{\chi''_{-2}}{2L\xi_0}\eta_i\eta_{\bar{j}}\right).
$$

Notice that (30) we obtain following lemma;

Lemma 3.1. *Let (M, F) be a complex Matsumoto space then the angular metric tensor is given by* (30).

Remark 3.1. Apply the same procedure of Proposition (3.1) then we obtain the inverse and determinant value of angular metric tensor $K_{i\bar{j}}$ as in Proposition (3.2).

Proposition 3.2. *Let $F = \frac{\alpha^2}{\alpha-|\beta|}$ be a complex Matsumoto metric with $|\beta| \neq 0$. Then, they have the following:*

(i) The inverse tensor $K^{\bar{j}i}$ of the angular tensor fields $K_{i\bar{j}}$ is.

$$
(31) \quad K^{\bar{j}i} = \frac{1}{\xi_0 M_1}\left\{a^{\bar{j}i} + R_1\eta^i\bar{\eta}^j + \frac{\chi''_0}{\gamma_1\xi_0}(b^i + R_1\bar{\beta}\eta^i)(b^{\bar{j}} + R_1\beta\bar{\eta}^j)\right\}
$$

where $R_1 = \frac{\xi'_{-2}}{\xi_0+\alpha^2\xi'_{-2}}$, $\gamma_1 = 1 + \frac{\chi''_0}{\xi_0}(\|b\|^2 + R_1|\beta|^2)$, *and*

$$\begin{aligned} M_1 = 1 + \frac{\chi''_{-2}}{2L\xi_0}\Bigg\{&\xi_0^2\alpha^2 + 2\chi_0\xi_0|\beta|^2 + \chi_0^2\|b\|^2|\beta|^2 \\ &+ R_1F^4 + \frac{\chi''_0}{\gamma_1\xi_0}\Big[\xi_0^2|\beta|^2 + 2\xi_0\chi_0|\beta|^2\|b\|^4 \\ &+ \chi_0^2|\beta|^2\|b\|^4 + 2\xi_0R_1F^2|\beta|^2 + 2\chi_0R_1F^2|\beta|^2\|b\|^2 + R_1^2F^4|\beta|^2\Big]\Bigg\}. \end{aligned} \tag{32}$$

(*ii*)

$$det(K_{i\bar{j}}) = (\xi_0)^n det(H_{i\bar{j}}) = (\xi_0)^n \frac{M_1\xi'_{-2}}{\gamma_1\xi_0} det(a_{i\bar{j}}). \tag{33}$$

4. Holomorphic curvature of complex Matsumoto metric

In this section, we study the Chern-Finsler connection, complex Cartan tensor and holomorphic curvature of complex Matsumoto metric.

Now, the Chern–Finsler connection coefficients (c.n.c.) and the horizontal and vertical coefficients are computed. By definition,

$$\overset{CF}{N^i_j} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l = g^{\bar{m}i}\frac{\partial \bar{\eta}_m}{\partial z^j}. \tag{34}$$

From (9) and (10), we compute the following

$$\begin{aligned} \bar{\eta}_m &= (p_0 l_{\bar{m}} + q_0 b_{\bar{m}}\beta), \\ \bar{\eta}_m &= \frac{(\alpha - 2|\beta|)}{(\alpha-|\beta|)^2}Fl_{\bar{m}} + \frac{1}{(\alpha-|\beta|)}\frac{F^2\beta b_{\bar{m}}}{|\beta|}. \end{aligned} \tag{35}$$

Differentiating (35) with respect to z^j we have

$$\begin{aligned} \frac{\partial\bar{\eta}_m}{\partial z^j} = \frac{(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}\Bigg\{& F\frac{\partial a_{i\bar{m}}}{\partial z^j}\eta^i + \frac{1}{(\alpha-|\beta|)^2}\Big[\frac{\alpha}{2}\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s \\ &- \frac{\alpha^2}{2|\beta|}(\beta\frac{\partial b_{\bar{s}}}{\partial z^j}\bar{\eta}^s + \bar{\beta}\frac{\partial b_s}{\partial z^j}\eta^s) - (\alpha-|\beta|)\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s\Big]l_{\bar{m}} \\ &+ \frac{F}{(\alpha-|\beta|)}\Big[\frac{1}{|\beta|}(\beta\frac{\partial b_{\bar{m}}}{\partial z^j}\bar{\eta}^m + \bar{\beta}\frac{\partial b_m}{\partial z^j}\eta^m) - \frac{1}{\alpha}\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s\Big]l_{\bar{m}}\Bigg\} \\ &+ \frac{F}{(\alpha-|\beta|)^2}\Big[\frac{\partial a_{i\bar{s}}}{\partial z^j}\eta^i\bar{\eta}^s - \frac{1}{|\beta|}(\beta\frac{\partial b_{\bar{m}}}{\partial z^j}\bar{\eta}^m + \bar{\beta}\frac{\partial b_m}{\partial z^j}\eta^m)\Big]l_{\bar{m}} \end{aligned} \tag{36}$$

$$+\frac{F^2}{|\beta|(\alpha-|\beta|)}\left[\beta\frac{\partial b_{\bar m}}{\partial z^j}+\frac{b_{\bar m}}{2\beta}\frac{\partial b_i}{\partial z^j}\eta^i-\frac{\beta b_{\bar m}}{2|\beta|^2}(\beta\frac{\partial b_{\bar m}}{\partial z^j}\bar\eta^m+\bar\beta\frac{\partial b_m}{\partial z^j}\eta^m)\right]$$
$$+\frac{\beta b_{\bar m}}{|\beta|}\left\{\frac{2F}{(\alpha-|\beta|)^3}\left[\frac{\alpha}{2}\frac{\partial a_{i\bar s}}{\partial z^j}\eta^i\bar\eta^s-\frac{\alpha^2}{2|\beta|}(\beta\frac{\partial b_{\bar s}}{\partial z^j}\bar\eta^s+\bar\beta\frac{\partial b_s}{\partial z^j}\eta^s)-(\alpha-|\beta|)\frac{\partial a_{i\bar s}}{\partial z^j}\eta^i\bar\eta^s\right]\right.$$
$$\left.+F^2\left[\frac{\partial a_{i\bar s}}{\partial z^j}\eta^i\bar\eta^s-\frac{1}{2|\beta|}(\beta\frac{\partial b_{\bar m}}{\partial z^j}\bar\eta^m+\bar\beta\frac{\partial b_m}{\partial z^j}\eta^m)\right]\right\}.$$

Using (21) and (36) in (34), we have

$$\begin{aligned}\overset{CF}{N^i_j} &= \overset{a}{N^i_j}+\frac{1}{p_0}a^{\bar m i}\left(\frac{\partial\bar\eta_m}{\partial z^j}-p_0\frac{\partial a_{l\bar m}}{\partial z^j}\eta^l\right)\\ &- \frac{1}{p_0}\left\{R\eta^i\bar\eta^m+\frac{q''_0}{\gamma p_0}(b^i+R\bar\beta\eta^i)(b^{\bar m}+R\beta\bar\eta^m)+\frac{q''_{-2}}{p_0M}\right.\\ &\left.\left[a^{\bar m i}+R\eta^i\bar\eta^m+\frac{q''_0}{\gamma p_0}(b^i+R\bar\beta\eta^i)(b^{\bar m}+R\beta\bar\eta^m)\right]^2\right\}\frac{\partial\bar\eta_n}{\partial z^j},\end{aligned}\tag{37}$$

where $\frac{\partial\bar\eta_n}{\partial z^j}$ is same as in (36) and $\overset{a}{N^i_j}=a^{\bar m i}\frac{\partial a_{l\bar m}}{\partial z^j}\eta^l$.

Next, we have to find the expression for the vertical and horizontal coefficients of Chern–Finsler connection. Consider the following complex Cartan tensor [2]

$$C_{j\bar h k}=\frac{\partial g_{j\bar h}}{\partial\eta^k}=\frac{\partial g_{j\bar h}}{\partial\alpha}\frac{\partial\alpha}{\partial\eta^k}+\frac{\partial g_{j\bar h}}{\partial|\beta|}\frac{\partial|\beta|}{\partial\eta^k}.\tag{38}$$

On calculating values of $\frac{\partial g_{j\bar h}}{\partial\alpha}$ and $\frac{\partial g_{j\bar h}}{\partial|\beta|}$ and substituing in (38), we get

$$\begin{aligned}C_{j\bar h k} &= \left\{\left(\frac{\alpha^2-\alpha^3+4\alpha|\beta|^2}{(\alpha-|\beta|)^4}\right)\frac{l_k}{2\alpha}+\left(\frac{3\alpha^3-2\alpha^2-6\alpha^2|\beta|}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}a_{j\bar h}\\ &+\left\{\left(\frac{4|\beta|^2-\alpha|\beta|}{2(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}+\left(\frac{12|\beta|^2-\alpha^2+4\alpha|\beta|}{2(\alpha-|\beta|)^6}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}l_jl_{\bar h}\\ &+\left\{\left(\frac{\alpha^4-4\alpha^3|\beta|}{|\beta|(\alpha-|\beta|)^4}-\frac{12\alpha^3}{(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}+\left(\frac{(3\alpha+9|\beta|)}{2(\alpha-|\beta|)^3}\right.\right.\\ &\left.\left.+\frac{(\alpha+2|\beta|)}{(\alpha-|\beta|)^2}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}b_jb_{\bar h}+\left\{\left(\frac{-\alpha^3+5\alpha^2|\beta|+8\alpha|\beta|^2}{2|\beta|(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}\right.\\ &\left.+\left(\frac{5F^2}{2\alpha(\alpha-|\beta|)^3}-\frac{F^2}{2|\beta|^2(\alpha-|\beta|)^3}-\frac{8F}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar k}}{2|\beta|}\right\}\\ &(l_j\beta b_{\bar h}+\bar\beta b_jl_{\bar h}).\end{aligned}\tag{39}$$

Also, the vertical coefficients of Chern–Finsler connections are defined as

$$C^i_{jk} = g^{\bar{m}i}\frac{\partial g_{k\bar{m}}}{\partial \eta^j} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k} = g^{\bar{m}i}C_{j\bar{m}k}. \tag{40}$$

Using (22) and (39) in (40), we have

$$\begin{aligned} C^i_{jk} \;=\; & \frac{1}{M}\Bigg[a^{\bar{m}i} + \left(1+\frac{q_0'' R|\beta|^2}{p_0\gamma}\right)R\eta^i\bar{\eta}^m + \frac{q_0''}{p_0\gamma}b^i b^{\bar{m}} \\ & + \frac{q_0'' R}{p_0\gamma}(\beta b^i\bar{\eta}^m + \bar{\beta}\eta^i b^{\bar{m}})\Bigg] \\ & \Bigg[\left\{\left(\frac{\alpha^2-\alpha^3+4\alpha|\beta|^2}{(\alpha-|\beta|)^4}\right)\frac{l_k}{2\alpha} + \left(\frac{3\alpha^3-2\alpha^2-6\alpha^2|\beta|}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\right\}a_{j\bar{h}} \\ & + \left\{\left(\frac{4|\beta|^2-\alpha|\beta|}{2(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \left(\frac{12|\beta|^2-\alpha^2+4\alpha|\beta|}{2(\alpha-|\beta|)^6}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\right\}l_j l_{\bar{h}} \\ & + \left\{\left(\frac{\alpha^4-4\alpha^3|\beta|}{|\beta|(\alpha-|\beta|)^4} - \frac{12\alpha^3}{(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \left(\frac{(3\alpha+9|\beta|)}{2(\alpha-|\beta|)^3}\right.\right. \\ & \left.\left. + \frac{(\alpha+2|\beta|)}{(\alpha-|\beta|)^2}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\right\}b_j b_{\bar{h}} + \left\{\left(\frac{-\alpha^3+5\alpha^2|\beta|+8\alpha|\beta|^2}{2|\beta|(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}\right. \\ & \left. + \left(\frac{5F^2}{2\alpha(\alpha-|\beta|)^3} - \frac{F^2}{2|\beta|^2(\alpha-|\beta|)^3} - \frac{8F}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\right\} \\ & (l_j\beta b_{\bar{h}} + \bar{\beta}b_j l_{\bar{h}})\Bigg]. \end{aligned} \tag{41}$$

Also,

$$C_k = C_{k\bar{h}j}g^{\bar{h}j}. \tag{42}$$

Plugging (22) and (40) in (42) gives us

$$\begin{aligned} C_k \;=\; & \Bigg[\left\{\left(\frac{\alpha^2-\alpha^3+4\alpha|\beta|^2}{(\alpha-|\beta|)^4}\right)\frac{l_k}{2\alpha} + \left(\frac{3\alpha^3-2\alpha^2-6\alpha^2|\beta|}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\right\}a_{j\bar{h}} \\ & + \left\{\left(\frac{4|\beta|^2-\alpha|\beta|}{2(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \left(\frac{12|\beta|^2-\alpha^2+4\alpha|\beta|}{2(\alpha-|\beta|)^6}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\right\}l_j l_{\bar{h}} \\ & + \left\{\left(\frac{\alpha^4-4\alpha^3|\beta|}{|\beta|(\alpha-|\beta|)^4} - \frac{12\alpha^3}{(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha} + \left(\frac{(3\alpha+9|\beta|)}{2(\alpha-|\beta|)^3}\right.\right. \\ & \left.\left. + \frac{(\alpha+2|\beta|)}{(\alpha-|\beta|)^2}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\right\}b_j b_{\bar{h}} + \left\{\left(\frac{-\alpha^3+5\alpha^2|\beta|+8\alpha|\beta|^2}{2|\beta|(\alpha-|\beta|)^5}\right)\frac{l_k}{2\alpha}\right. \end{aligned} \tag{43}$$

$$+\left(\frac{5F^2}{2\alpha(\alpha-|\beta|)^3}-\frac{F^2}{2|\beta|^2(\alpha-|\beta|)^3}-\frac{8F}{(\alpha-|\beta|)^4}\right)\frac{\beta b_{\bar{k}}}{2|\beta|}\Bigg\}$$
$$(l_j\beta b_{\bar{h}}+\bar{\beta}b_j l_{\bar{h}})\Bigg]\Bigg[a^{\bar{h}i}+\left(1+\frac{q_0''R|\beta|^2}{p_0\gamma}\right)R\eta^j\bar{\eta}^h+\frac{q_0''}{p_0\gamma}b^j b^{\bar{h}}$$
$$+\frac{q_0''R}{p_0\gamma}(\beta b^j\bar{\eta}^h+\bar{\beta}\eta^j b^{\bar{h}})\Bigg].$$

Now, we find the Holomorphic curvature, first we compute Ricci curvature $R_{\bar{j}k}$ in (4).

Substituting the values of $g_{i\bar{j}}$ and $\overset{CF}{N}{}^i_k$ in (5) then, we have

$$\begin{aligned}
R_{\bar{j}k} \;=\; & -\Bigg[\frac{F(\alpha-2|\beta|)}{(\alpha-|\beta|)^2}\,a_{l\bar{j}}+\frac{-F|\beta|}{2\alpha^2}\left(\frac{\alpha-4|\beta|}{(\alpha-|\beta|)^3}\right)l_l l_{\bar{j}}\\
& +\frac{1}{2|\beta|}\left(\frac{3F^2}{(\alpha-|\beta|)^2}+\frac{2F^2}{(\alpha-|\beta|)}\right)b_l b_{\bar{j}}+\frac{F(\alpha-4|\beta|)}{2|\beta|(\alpha-|\beta|)^3}\\
& \left(l_l\beta b_{\bar{j}}+\bar{\beta}b_l l_{\bar{j}}\right)\Bigg]\;\delta_{\bar{h}}\Bigg\{\overset{a}{N}{}^l_k+\frac{1}{p_0}a^{\bar{m}l}\left(\frac{\partial\bar{\eta}_m}{\partial z^k}-p_0\frac{\partial a_{q\bar{m}}}{\partial z^j}\eta^q\right)\\
& -\frac{1}{p_0}\Bigg[R\eta^l\bar{\eta}^m+\frac{q_0''}{\gamma p_0}(b^l+R\bar{\beta}\eta^l)(b^{\bar{m}}+R\beta\bar{\eta}^m)+\frac{q_{-2}''}{p_0 M}\\
& \left(a^{\bar{m}l}+R\eta^l\bar{\eta}^m+\frac{q_0''}{\gamma p_0}(b^l+R\bar{\beta}\eta^l)(b^{\bar{m}}+R\beta\bar{\eta}^m)\right)^2\Bigg]\frac{\partial\bar{\eta}_m}{\partial z^k}\Bigg\}\bar{\eta}^h.
\end{aligned}\tag{44}$$

Theorem 4.1. *Let (M,F) be a complex Matsumoto space with the metric (7). Then, Ricci curvature is given by equation* (44) *and holomorphic curvature is given by* (4).

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On closedness of rectangular bands and left[right] normal bands

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Abstract. In this paper, first we have shown that the variety of rectangular bands is closed in the variety of all left[right] semiregular bands. Further, we have shown that the variety of left[right] normal bands are closed in some containing varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$, where ($\mathbf{n} \in \mathbb{N}$).

Keywords: zigzag equations, varieties, identity, closed and bands.

1. Introduction

Let U be a subsemigroup of a semigroup S. Following, Isbell [9], we say that U dominates an element d of S if for every semigroup P and for all homomorphisms $\alpha, \delta : S \longrightarrow P$ and $u\alpha = u\delta$ for every u in U implies $d\alpha = d\delta$. The set of all elements of S dominated by U is called the *dominion* of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of semigroup S is called closed if $Dom(U, S) = U$. A semigroup is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{D}$ be a class of semigroups. A semigroup U is said to be $\mathcal{D}$-closed if $Dom(U, S) = U$, for all $S \in \mathcal{D}$ such that $U \subseteq S$. Let $\mathcal{A}$ and $\mathcal{D}$ be

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classes of semigroups such that $\mathcal{A}$ is a subclass of $\mathcal{D}$. We say that $\mathcal{A}$ is $\mathcal{D}$-closed if every member of $\mathcal{A}$ is $\mathcal{D}$-closed. A class $\mathcal{D}$ of semigroups is said to be closed if $Dom(U, S) = U$, for all $U, S \in \mathcal{D}$ with U as a subsemigroup of S. Let $\mathcal{B}$ and $\mathcal{C}$ be two categories of semigroups with $\mathcal{B}$ as a subcategory of $\mathcal{C}$. It can be easily verified that a semigroup U is $\mathcal{B}$-closed if it is $\mathcal{C}$-closed.

A (semigroup)amalgam $\mathcal{A} = [S_i\colon\ i \in I;\ U;\ \phi_i\colon\ i \in I]$ consists of a semigroup U (called the core of the amalgam), a family $S_i\colon\ i \in I$ of semigroups disjoint from each other and from U, and a family $\phi_i\colon\ U \to S_i(i \in I)$ of monomorphisms. We shall simplify the notation to $\mathcal{U} = [S_i;\ U;\ \phi_i]$ or to $\mathcal{U} = [S_i;\ U\ ]$ when the context allows. We shall say that the amalgam $\mathcal{A}$ is embedded in a semigroup T if there exist a monomorphism $\lambda\colon\ U \to T$ and, for each $i \in I$, a monomorphism $\lambda_i\colon\ S_i \to T$ such that:

(a) $\phi_i\lambda_i{=}\lambda$ for each $i \in I$;

(b) $S_i\lambda_i \cap S_j\lambda_j = U\lambda$, for all $i,\ j \in I$ such that $i\ \neq j$.

A semigroup amalgam $\mathcal{U} = [\ S,\ S';\ U;\ i,\ \alpha \mid U\ ]$ consisting of a semigroup S, a subsemigroup U of S, an isomorphic copy S' of S, where $\alpha\colon\ S \to\ S'$ is an isomorphism and i is the inclusion mapping of U into S, is called a special semigroup amalgam. A class $\mathcal{C}$ of semigroups is said to have the special amalgamation property if every special semigroup amalgam in $\mathcal{C}$ is embeddable in $\mathcal{C}$.

Theorem 1.1 ([8], Theorem VII.2.3)**.** *et U be a subsemigroup of a semigroup S, S' be a semigroup disjoint from S and let α: $S \to S'$ be an isomorphism. Let $P = S{*_U}S'$, be the free product of the amalgam*

$$\mathcal{U} = [S, S'; U; i, \alpha \mid U],$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P . Then

$$(S\mu\ \cap\ S'\mu')\mu^{-1} = Dom(U, S).$$

From the above result, it follows that a special semigroup amalgam $[S,\ S';\ U;\ i,\ \alpha \mid U]$ is embeddable in a semigroup if and only if $Dom(U, S) = U$. Therefore, the above amalgam with core U is embeddable in a semigroup if and only if U is closed in S.

The following theorem provided by Isbell [9], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.2 ([9], Theorem 2.3)**.** *Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

(1) $\quad d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m},$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, *and*

$$a_0 = y_1 a_1, \qquad a_{2m-1} t_m = a_{2m},$$
$$a_{2i-1} t_i = a_{2i} t_{i+1}, \qquad y_i a_{2i} = y_{i+1} a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorizations is called a zigzag in S *over* U *with value* d, *length* m *and spine* $a_0, a_1, \ldots, a_{2m}$.

The following result is from Khan [10] and is also necessary for our investigations.

Theorem 1.3 ([10], Result 3)**.** *Let* U *and* S *be semigroups with* U *as a subsemigroup of* S. *Take any* $d \in S \backslash U$ *such that* $d \in Dom(U, S)$. *Let* (1) *be a zigzag of minimal length* m *over* U *with value* d. *Then,* $t_j, y_j \in S \backslash U$, *for all* $j = 1, 2, \ldots, m$.

Definition 1.1. *A semigroup* S *is said to be a band if* S *satisfies the identity* $a^2 = a$, *for all* $a \in S$.

Definition 1.2. *A band* S *is said to be a rectangular band if* S *satisfies the identity* $a = axa$, *for all* $a, x \in S$.

Definition 1.3. *A band* S *is said to be a left[right] normal band if* S *satisfies the identity* $axy = ayx[axy = xay]$, *for all* $a, x, y \in S$.

Definition 1.4. *A band* S *is said to be a left[right] semiregular band if* S *satisfies the identity* $axy = axyayxy[axy = axayaxy]$, *for all* $a, x, y \in S$.

The reader is referred to Petrich [11] for a complete description of all varieties of bands. The semigroup theoretic notations and conventions of Clifford and Preston [6] and Howie [8] will be used throughout without explicit mention.

2. Closedness of rectangular bands

In general, varieties of bands containing the varieties of rectangular bands are not absolutely closed as Higgins [7, Chapter 4] gave an example to show that the variety of all rectangular bands is not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which rectangular bands is closed. In this direction, we show that the variety of rectangular bands is closed in the variety of all left[right] semiregular bands.

Proposition 2.1. *Let* U *be any rectangular band and* S *be any semiregular band containing* U. *Assume that* $d \in Dom(U, S) \backslash U$. *If* (1) *is a zigzag in* S *over* U *with value* d *of minimal length* m, *then:*

(*a*) $y_i a_{2i-1} t_i = y_i a_{2i-1} a_{2i} y_{i+1} a_{2i+1} t_{i+1}$;

(b) $(y_i a_{2i-1} a_{2i})(y_{i+1} a_{2i+1} a_{2i+2}) = y_i a_{2i-1} a_{2i} a_{2i+2}$,

for all $i = 1, 2, ..., m-1$.

Proof.

$$
\begin{aligned}
(a)\quad y_i a_{2i-1} t_i &= y_i a_{2i-1} a_{2i-1} t_i \text{ (since } U \text{ is a band)}\\
&= y_i a_{2i-1} a_{2i} t_{i+1} \text{ (by zigzag equations)}\\
&= (y_i a_{2i-1} a_{2i}) t_{i+1}\\
&= (y_i a_{2i-1} a_{2i} y_i a_{2i} a_{2i-1} a_{2i}) t_{i+1} \text{ (since } S \text{ is a left semi-regular band)}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i} a_{2i-1} a_{2i}) t_{i+1}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i}) t_{i+1} \text{ (since } U \text{ is a rectangular band)}\\
&= y_i a_{2i-1} a_{2i} y_{i+1} a_{2i+1} t_{i+1} \text{ (by zigzag equations)},
\end{aligned}
$$

as required.

$$
\begin{aligned}
(b)\quad (y_i a_{2i-1} a_{2i})(y_{i+1} a_{2i+1} a_{2i+2}) &= y_i a_{2i-1} a_{2i} (y_i a_{2i}) a_{2i+2} \text{ (by zigzag equations)}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i}) a_{2i+2}\\
&= y_i a_{2i-1} a_{2i} y_i (a_{2i} a_{2i-1} a_{2i}) a_{2i+2}\\
&\qquad \text{(since } U \text{ is a rectangular band)}\\
&= (y_i a_{2i-1} a_{2i} y_i a_{2i} a_{2i-1} a_{2i}) a_{2i+2}\\
&= (y_i a_{2i-1} a_{2i}) a_{2i+2}\\
&\qquad \text{(since } S \text{ is a left semi-regular band)},
\end{aligned}
$$

as required. □

Theorem 2.1. *Rectangular bands are closed in left semiregular bands.*

Proof. Let U be a rectangular band and S be any left semiregular band containing U as a subband. We have to show that $Dom(U, S) = U$. Take any $d \in Dom(U, S) \backslash U$. Then d has zigzag of type (1) in S over U with value d of minimal length m. Now

$$
\begin{aligned}
d &= a_0 t_1 \text{ (by zigzag equations)}\\
&= y_1 a_1 t_1 \text{ (by zigzag equations)}\\
&= y_1 a_1 a_2 y_2 a_3 t_2 \text{ (by Proposition 2.1 (a))}\\
&= y_1 a_1 a_2 (y_2 a_3 t_2)\\
&= y_1 a_1 a_2 (y_2 a_3 a_4 y_3 a_5 t_3) \text{ (by Proposition 2.1 (a))}\\
&\vdots\\
&= y_1 a_1 a_2 y_2 a_3 a_4 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_m (a_{2m-1}) t_m
\end{aligned}
$$

$= y_1a_1a_2y_2a_3a_4\cdots y_{m-1}a_{2m-3}a_{2m-2}y_m(a_{2m-1}a_{2m-1})t_m$ (since U is a band)
$= y_1a_1a_2y_2a_3a_4\cdots y_{m-1}a_{2m-3}a_{2m-2}y_m a_{2m-1}a_{2m}$ (by zigzag equations)
$= y_1a_1a_2y_2a_3a_4\cdots y_{m-2}a_{2m-5}a_{2m-4}((y_{m-1}a_{2m-3}a_{2m-2})(y_m a_{2m-1}a_{2m}))$
$= y_1a_1a_2y_2a_3a_4\cdots y_{m-2}a_{2m-5}a_{2m-4}(y_{m-1}a_{2m-3}a_{2m-2}a_{2m})$
(by Proposition 2.1 (b))
$= y_1a_1a_2y_2a_3a_4\cdots y_{m-3}a_{2m-7}a_{2m-6}((y_{m-2}a_{2m-5}a_{2m-4})(y_{m-1}a_{2m-3}a_{2m-2}))a_{2m}$
$= y_1a_1a_2y_2a_3a_4\cdots y_{m-3}a_{2m-7}a_{2m-6}(y_{m-2}a_{2m-5}a_{2m-4}a_{2m-2})a_{2m}$
(by Proposition 2.1 (b))
$\vdots$
$= y_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}$
$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}$ (by zigzag equations)
$\in U$

$\Rightarrow Dom(U,S) = U$.

Hence, rectangular bands are closed in left semiregular bands. Thus, the proof of the theorem is complete. $\square$

Dually, we can prove the following Theorem

Theorem 2.2. *Rectangular bands are closed in right semiregular bands.*

Corollary 2.1. *The variety of all rectangular bands is closed in the variety of all left[right] semiregular bands.*

Corollary 2.2. *The variety of all rectangular bands is closed in the following varieties of bands:*

(*i*) *The variety of all regular bands.*

(*ii*) *The variety of all left[right] seminormal bands.*

(*iii*) *The variety of all left[right] quasinormal bands.*

(*iv*) *The variety of all normal bands.*

3. Closedness of left[right] normal bands

In general, varieties of bands containing the variety of normal bands are not absolutely closed as Higgins [7, Chapter 4] had shown that variety of right [left] normal bands is not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which the variety of right [left] normal bands is closed. As a first step in this direction, one attempts to find those varieties of semigroups that are closed in itself. Encouraged by the fact that Scheiblich [12] had shown that the variety of all normal bands was closed, Alam and Khan in [3, 4, 5] had shown that the variety of left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [2], Ahanger and Shah had proved a stronger fact that the variety of left [right] regular bands was closed in the variety of all bands and, recently, Abbas and Ashraf [1] had shown that a variety of left [right] normal bands was closed in some containing homotypical varieties (varieties admitting an identity containing same variables on both sides) of semigroups.

To this end, we first note that Petrich [11, Theorem II.5.1] has classified an identity on bands in atmost three variables. Therefore, on the class of bands, varieties of semigroups defined by the identities $axy = a^n yax$ $[axy = xyay^n]$ and $axy = aya^n x$ $[axy = xy^n ay]$ are equivalent to left[right] normal bands. In this section, we have shown that varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$, where ($\mathbf{n} \in \mathbb{N}$), are closed in itself and, as an application and consequence of these results, we conclude that the varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$ have special amalgamation property and left[right] normal bands are closed in $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$ and, thus a modest, but an important step towards the solution of the above problem. However the problem of finding out largest varieties of semigroups in which the varieties of semigroups defined by the identities $axy = a^n yax[axy = xyay^n]$ and $axy = aya^n x[axy = xy^n ay]$ are closed still remains open.

Lemma 3.1. *Let U be a subsemigroup of semigroup S such that S satisfies an identity $axy = a^n yax$ $[axy = aya^n x]$ and let $d \in Dom(U,S)\backslash U$ has a zigzag of type* (1) *in S over U with value d of shortest possible length m. Then*

$$d = y_k a_{2k-1} t_k (\prod_{i=1}^{k} a_{2k-(2i-1)}^n),$$

for each $k = 1, 2, \ldots, m$.

Proof. Let $\mathcal{V}_1 = [axy = a^n yax]$ and $\mathcal{V}_2 = [axy = aya^n x]$ be the varieties of semigroups.

First, we show that in both cases whether $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$, S satisfies $xyz = xyzy^n$.

Case (i). When $S \in \mathcal{V}_1$, then for any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= (x^n(zx)y) \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= (x^n)^n(yx^n z)x \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= ((x^n)^n(y^n zy)x^n x) \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= ((x^n x)(y^n z)y) \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= ((x^n x)^n y(x^n x)y^n)z \text{ (as } S \in \mathcal{V}_1 \text{)} \end{aligned}$$

$$
\begin{aligned}
&= x^n x y^n y z \text{ (as } S \in \mathcal{V}_1 \text{)}\\
&= (x^{n-1}(xx)y^n)yz \text{ (for } n=1, \text{ we treat } x^{n-1}x \text{ as } x)\\
&= ((x^{n-1})^n y^n x^{n-1} x)xyz \text{ (as } S \in \mathcal{V}_1 \text{)}\\
&= x^{n-1} x y^n x y z \text{ (as } S \in \mathcal{V}_1 \text{)}\\
&= (x^n y^n x(yz))\\
&= xyzy^n \text{ (as } S \in \mathcal{V}_1 \text{).} \qquad (2)
\end{aligned}
$$

Case (ii): When $S \in \mathcal{V}_2$, then for any $x, y, z \in S$, we have

$$
\begin{aligned}
xyz &= (x(zx^n)y) \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= x(y(x^n z)x^n) \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= (xyx^n y^n)x^n z \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= xy^n y x^n z \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= (x(yy^n)(x^n z))\\
&= (x(x^n z)x^n y)y^n \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= (xyx^n)zy^n \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= x(x^n(x^n y)z)y^n \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= x(x^n z(x^n)^n x^n)yy^n \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= (xx^n x^n z)yy^n \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= (xzx^n y)y^n \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= xyzy^n \text{ (as } S \in \mathcal{V}_2 \text{).}
\end{aligned}
$$

Thus, the claim is proved.

Now, we shall prove the lemma by using induction on k. Let U be a subsemigroup of semigroup S such that S belongs to either $\mathcal{V}_1$ or $\mathcal{V}_2$ and let $d \in Dom(U, S) \backslash U$ has a zigzag of type (1) in S over U with value d of shortest possible length m.

Now, for $k = 1$, we have

$$
\begin{aligned}
d &= y_1 a_1 t_1 \text{ (by zigzag equations)}\\
&= y_1 a_1 t_1 a_1^n \text{ (by equation (2)).}
\end{aligned}
$$

Thus, the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. Then, we shall show that it also holds for $k = j + 1$. Now,

$$
\begin{aligned}
d &= y_j a_{2j-1} t_j (\prod_{i=1}^{j} a_{2j-(2i-1)}^n) \text{ (by inductive hypothesis)}\\
&= y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j} a_{2j-(2i-1)}^n) \text{ (by zigzag equations)}
\end{aligned}
$$

$$= y_{j+1}a_{2j+1}t_{j+1}a_{2j+1}^n(\prod_{i=1}^{j} a_{2j-(2i-1)}^n) \text{ (by equation (2))}$$
$$= y_{j+1}a_{2j+1}t_{j+1}(\prod_{i=1}^{j+1} a_{2(j+1)-(2i-1)}^n),$$

as required and, by induction, the lemma is established. □

Theorem 3.1. *The variety $\mathcal{V} = [axy = a^n yax]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = a^n yax$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\backslash U$. Let d has zigzag of type (1) in S over U of shortest possible length m. Now,

$$d = y_m a_{2m-1} t_m (\prod_{i=1}^{m} a_{2m-(2i-1)}^n) \text{ (by Lemma 3.1)}$$
$$= (y_m(a_{2m-1}t_m)a_{2m-1})a_{2m-1}^{n-1}(\prod_{i=2}^{m} a_{2m-(2i-1)}^n) \text{ (for } n = 1,$$
$$\text{we treat } a_{2m-1}a_{2m-1}^{n-1} \text{ as } a_{2m-1})$$
$$= (y_m^n a_{2m-1} y_m a_{2m-1}) t_m a_{2m-1}^{n-1}(\prod_{i=2}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$
$$= (y_m a_{2m-1})(a_{2m-1}t_m)a_{2m-1}^{n-1}(\prod_{i=2}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$
$$= y_{m-1}a_{2m-2}a_{2m}a_{2m-1}^{n-1}(\prod_{i=2}^{m} a_{2m-(2i-1)}^n) \text{ (by zigzag equations)}$$
$$= (y_{m-1}(a_{2m-2}a_{2m})a_{2m-1}^{n-1})a_{2m-3}a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^n)$$
$$= y_{m-1}^n(a_{2m-1}^{n-1}(y_{m-1}a_{2m-2}a_{2m})a_{2m-3})a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$
$$= y_{m-1}^n((a_{2m-1}^{n-1})^n a_{2m-3}a_{2m-1}^{n-1}y_{m-1})a_{2m-2}a_{2m}a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$
$$= (y_{m-1}^n a_{2m-1}^{n-1} y_{m-1}(a_{2m-3}a_{2m-2}a_{2m}))a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$
$$= (y_{m-1}a_{2m-3})a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^n) \text{ (as } S \in \mathcal{V})$$

$$
\begin{aligned}
&= y_{m-2}a_{2m-4}a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^{n}) \text{ (by zigzag equations)}\\
&\vdots\\
&= y_1a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n}\\
&= (y_1(a_2a_4\cdots a_{2m-2}a_{2m})(a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}))a_1a_1^{n-1}\\
&= y_1^{n}((a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})(y_1a_2a_4\cdots a_{2m-2}a_{2m})a_1)a_1^{n-1} \text{ (as } S\in\mathcal{V}\text{)}\\
&= y_1^{n}((a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})^{n}a_1(a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})y_1)a_2a_4\cdots a_{2m-2}a_{2m}a_1^{n-1}\\
&\quad \text{(as } S\in\mathcal{V}\text{)}\\
&= (y_1^{n}(a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})y_1(a_1a_2a_4\cdots a_{2m-2}a_{2m}))a_1^{n-1} \text{ (as } S\in\mathcal{V})\\
&= (y_1a_1)a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1} \text{ (as } S\in\mathcal{V})\\
&= a_0a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1} \text{ (by zigzag equations)}\\
&\in U
\end{aligned}
$$

$\Rightarrow Dom(U,S)=U$.

Thus, the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 3.1:

Corollary 3.1. *The variety of all left normal bands is closed in the variety* $\mathcal{V}=[axy=a^{n}yax]$ *of semigroups.*

Dually, we may prove the following results.

Theorem 3.2. *The variety* $\mathcal{V}=[axy=xyay^{n}]$ *of semigroups, i.e. the class of all semigroups satisfying the identity* $axy=xyay^{n}$*, is closed.*

Corollary 3.2. *The variety of all right normal bands is closed in the variety* $\mathcal{V}=[axy=xyay^{n}]$ *of semigroups.*

Theorem 3.3. *The variety* $\mathcal{V}=[axy=aya^{n}x]$ *of semigroups, i.e. the class of all semigroups satisfying the identity* $axy=aya^{n}x$*, is closed.*

Proof. Take any U, $S\in\mathcal{V}$ with U as a subsemigroup of S such that $d\in Dom(U,S)\backslash U$. Let d has zigzag of type (1) in S over U of shortest possible length m. Now,

$$
\begin{aligned}
d &= y_m a_{2m-1}t_m(\prod_{i=1}^{m} a_{2m-(2i-1)}^{n}) \text{ (by Lemma 3.1)}\\
&= (y_m(a_{2m-1}t_m)a_{2m-1})a_{2m-1}^{n-1}(\prod_{i=2}^{m} a_{2m-(2i-1)}^{n}) \text{ (for } n\text{=1, we treat } a_{2m-1}a_{2m-1}^{n-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } a_{2m-1})
\end{aligned}
$$

$$
\begin{aligned}
&= (y_m a_{2m-1} y_m^n a_{2m-1}) t_m a_{2m-1}^{n-1} (\prod_{i=2}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V} \text{)}\\
&= (y_m a_{2m-1})(a_{2m-1} t_m) a_{2m-1}^{n-1} (\prod_{i=2}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V} \text{)}\\
&= y_{m-1} a_{2m-2} a_{2m} a_{2m-1}^{n-1} (\prod_{i=2}^{m} a^n_{2m-(2i-1)}) \text{ (by zigzag equations)}\\
&= (y_{m-1}(a_{2m-2}a_{2m}) a_{2m-1}^{n-1}) a_{2m-3} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a^n_{2m-(2i-1)})\\
&= y_{m-1}(a_{2m-1}^{n-1}(y_{m-1}^n a_{2m-2}a_{2m}) a_{2m-3}) a_{2m-3}^{n-1} (\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})\\
&= y_{m-1}(a_{2m-1}^{n-1} a_{2m-3} (a_{2m-1}^{n-1})^n y_{m-1}^n) a_{2m-2} a_{2m} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})\\
&= (y_{m-1} a_{2m-1}^{n-1} y_{m-1}^n (a_{2m-3} a_{2m-2} a_{2m})) a_{2m-3}^{n-1} (\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})\\
&= (y_{m-1} a_{2m-3}) a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V})\\
&= y_{m-2} a_{2m-4} a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a^n_{2m-(2i-1)}) \text{ (by zigzag equations)}\\
&\vdots\\
&= y_1 a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^n\\
&= (y_1 (a_2 a_4 \cdots a_{2m-2} a_{2m})(a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1})) a_1 a_1^{n-1}\\
&= (y_1 (a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} y_1^n a_2 a_4 \cdots a_{2m-2} a_{2m}) a_1) a_1^{n-1} \text{ (as } S \in \mathcal{V} \text{)}\\
&= (y_1 a_1 y_1^n (a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} y_1^n)) a_2 a_4 \cdots a_{2m-2} a_{2m} a_1^{n-1} \text{ (as } S \in \mathcal{V})\\
&= (y_1 (a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1}) y_1^n (a_1 a_2 a_4 \cdots a_{2m-2} a_{2m})) a_1^{n-1} \text{ (as } S \in \mathcal{V})\\
&= (y_1 a_1) a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^{n-1} \text{ (as } S \in \mathcal{V})\\
&= a_0 a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^{n-1} \text{ (by zigzag equations)}\\
&\in U
\end{aligned}
$$

$\Rightarrow Dom(U, S) = U$.

Thus, the proof of the theorem is complete. □

The following corollary is an immediate consequence of Theorem 3.3:

Corollary 3.3. *The variety of all left normal bands is closed in the variety* $\mathcal{V} = [axy = aya^n x]$ *of semigroups.*

Dually, we may prove the following results.

Theorem 3.4. *The variety* $\mathcal{V} = [axy = xy^n ay]$ *of semigroups, i.e. the class of all semigroups satisfying the identity* $axy = xy^n ay$*, is closed.*

Corollary 3.4. *The variety of all right normal bands is closed in the variety* $\mathcal{V} = [axy = xy^n ay]$ *of semigroups.*

In the view of Section 2, we propose an important open problem.

Problem 1. Is the variety of normal bands closed in the variety of left[right] semiregular bands?

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Junction interface conditions for asymptotic gradient full-observer in Hilbert space

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Abstract. The fundamentals concept of boundary asymptotic gradient observer of full order type $\partial\Omega AGFO$-observer via internal case in link with the strategic sensors in different system domains have been presented. The results so obtained for linear dynamical systems which is created by a strongly continuous semi-group (SCS-group) in Hilbert space $H^{1/2}(\partial\Omega)$ have been analyzed. Consequently, the existence of sufficient conditions for $\partial\Omega AGFO$-estimator in parabolic infinite dimensional systems have been studied and scrutinized. In addition to that, we have observed at the junction interface that the interior solution is harmonized with the exterior solution for asymptotic gradient full observation.
Keywords: $\partial\Omega AGFO$-observer, $\partial\Omega AG$-detectability, $\partial\Omega G$-strategic sensor, junction conditions.

1. Introduction

In literature, a distributed parameter systems and observability concepts on a special domain Ω have been widely developed and tackled by several authors [1-2]. The determination of Luenberger observer is to offer an asymptotic for-

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mal approximation for the current state of deliberated system [3-4]. Recently, Al-Saphory and El Jai et al. have explored a new direction of regional analysis for distributed parameter systems in finite time interval and infinite, with regional or regional boundary cases associated with strategic sensors and actuators as in [5-15]. In this paper, we familiarize and explore the notion of $\partial\Omega AGFO$-observer connected to extended internal region approach of the considered system domain [7, 12]. Therefore the usual boundary case have been developed through an extension to previous works as in [13-14]. In addition to that, boundary detectability and boundary strategic sensors have been deliberated and analyzed.

The incentive of studying this notion is there exist several problem in the real world needs to be studied as in [4, 16]. Indeed, the authors have obtained a more general mathematical model of the $BAGFO$-observer which characterized by internal gradient strategic (zone, pointwise or filament) sensors (Figure 1).

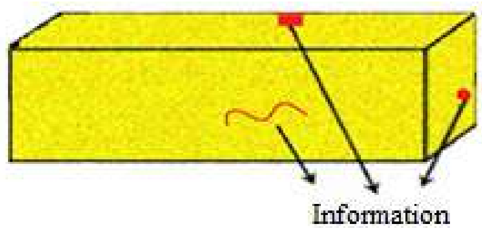

Figure 1: Mathematical modeling with positions of sensors.

The rest of the paper is prearranged as follows. Section 2 is enthusiastic to the considered system and preliminaries. In Section 3, we study $\partial\Omega G$-observability and $\partial\Omega AG$-detectability and extant some original results. In section 4 we familiarize $\partial\Omega AGFO$-observer concepts in terms of $\partial\Omega AG$-detectability and $\partial\Omega G$-strategic sensors. Also, the matching of inside to the outside solution at a junction interface has been studied in the sense of Banerjee et al. [16]. Finally, some applications for distributed diffusion systems with the devoted different domains and strategic sensors have been demonstrated.

2. Preliminaries of system inceptions

Assume Ω be an open set in $\mathbb{R}^n$, through smooth boundary $\partial\Omega$ with the following sets

$$\Pi = \Omega \times (0, \infty);\ \Xi = \partial\Omega \times (0, \infty).$$

Suppose that the following spaces specify as separable Hilbert type given by

$$\mathbb{W} = H^1(\Omega);\ \mathbb{U} = L^2(0, \infty, R^p);\ \mathbb{Y} = L^2(0, \infty, R^q).$$

So, these spaces represent respectively as state space; input space and measurement space such that p with q the numbers of controls and information [17].

Thus, the system can be written as:

$$(1)\qquad \begin{cases} \dfrac{\partial w}{\partial t}(\xi,t) = \mathcal{A}w(\xi,t) + Bu(t) & \Pi \\ w(\mu,t) = 0 & \Xi \\ w(\zeta,0) = w(\xi) & \Omega \end{cases}$$

augmented with the output function

$$(2)\qquad y(.,t) = Cw(.,t) \qquad \Pi$$

where Ω grips for the closure of Ω and $w_0(\xi)$ which is made-up to be unidentified in the state space $\mathbb{W} = H^1(\Omega)$. Therefore, $\mathcal{A}$ is a linear self-adjoint transformation of 2^{nd} differential case, with compact resolvent. Now, operators $B \in L(\mathbb{R}^p, \mathbb{W})$ and $C \in L(H^1(\Omega), \mathbb{R}^q)$ be contingent on the structures of control and information [18], which means, in various situations [12]. Consequently, we obtain $B \notin L(\mathbb{R}^p, \mathbb{W})$ and $C \notin L(H^{1/2}(\partial\Omega), \mathbb{R}^q)$. Accordingly, the system (1) has a unique solution [17-18] which is assumed as

$$(3)\qquad w(\zeta,t) = S_{\mathcal{A}}(t)w_0(\zeta) + \int_0^t S_{\mathcal{A}}(t-\tau)Bu(\tau)\,d\tau \qquad \Pi$$

. K an operator is defined by following

$$K : \mathbb{W} \longrightarrow \mathbb{Y}, \qquad w \longrightarrow CS_{\mathcal{A}}(\cdot)w$$

and,

$$y(\cdot,t) = K(t)w(\cdot,0),$$

where K is bounded linear operator [14-15].

. $K^* : \mathbb{Y} \longrightarrow \mathbb{W}$ is the adjoint operator of K, is defined by

$$K^*y^* = \int_0^t S^*_{\mathcal{A}}(\tau)C^*y^*(\cdot,\tau)\,d\tau.$$

. The operator ∇ signifies the gradient, which is assumed to have the form

$$(4)\qquad \begin{cases} \nabla : H^1(\Omega) \longrightarrow (H^1(\Omega))^n \\ w \longrightarrow \nabla_w = \left(\dfrac{\partial w}{\partial \zeta_1}, \cdots, \dfrac{\partial w}{\partial \zeta_n}\right) \end{cases}$$

∇^* is the adjoint of ∇ and specified by

$$\begin{cases} \nabla^* : (H^1(\Omega))^n \longrightarrow H^1(\Omega) \\ w \longrightarrow \nabla^*_w = v \end{cases}$$

Such that v is characterized a solution of the Dirichlet problem

$$\begin{cases} \Delta_v = -\text{div}(w) & \Omega \\ v = 0 & \partial\Omega \end{cases}$$

Thus, an extension of the trace operator [19] which is denoted by γ defined as

$$\gamma : (H^1(\Omega))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$$

and the adjoints is correspondingly given by γ^*.

. Systems (1)-(2) are assumed to be exactly observable (or $\mathbb{E}\Omega$-observable) and weakly observable (or $\mathbb{W}\Omega$- observable) on $[0,T]$ if $\operatorname{Im} K^* = H^1(\Omega)$ and $\overline{\operatorname{Im} K^*} = H^1(\Omega)$ respectively.

. The semi-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ is asymptotically stable in $H^1(\Omega)$ (or ΩA-stable), if, for +ve constants M_Ω and α_Ω, then

$$\|S_{\mathcal{A}}(\cdot)\|_{L(H^1(\Omega),\mathbb{W})} \leq M_\Omega e^{-\alpha_\Omega t},\ t \geq 0.$$

. System (1) is called ΩA-stable if the transformation $\mathcal{A}$ produces SCS-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ which is ΩA-stable.

. Systems (1)-(2) are assumed to be asymptotically detectable (ΩA-detectable) if the transformation $H_\Omega : \mathbb{Y} \longrightarrow H^1(\Omega)$ such that the operator $(\mathcal{A} - H_\Omega C)$ creates a SCS-group $(S_{H_\Omega}(t))_{t\geq 0}$, which is ΩA-stable.

3. $\partial\Omega G$-observability and $\partial\Omega AG$-detectability

The observability definitions to boundary case for parabolic, hyperbolic linear, semi-linear and nonlinear have been extended [20-23] with duel concept [24]. Though, we come across to some definitions and theorems to elucidate the concept of $\partial\Omega AG$-detectability and $\partial\Omega G$-observability in the state space $H^{1/2}(\partial\Omega)$ in it's an extension from [5,14].

Definition 3.1. *System* (1) *together with information (2) is assumed to be exactly gradient observable (or $\mathbb{E}\Omega G$-observable) on* $[0,T]$ *if:*

$$\operatorname{Im}\nabla K^* = \left(H^1(\Omega)\right)^n.$$

Definition 3.2. *System* (1) *together with information (2) is assumed to be weakly gradient observable (or $\mathbb{W}\Omega G$-observable) on* $[0,T]$ *if:*

$$\overline{\operatorname{Im}\nabla K^*} = \left(H^1(\Omega)\right)^n.$$

Definition 3.3. *System* (1) *together with information (2) is assumed to be exactly boundary gradient observable ($\mathbb{E}\partial\Omega G$-observable) on* $[0,T]$, *if:*

$$\operatorname{Im}\gamma\nabla K^* = \left(H^{1/2}(\partial\Omega)\right)^n.$$

Definition 3.4. *System* (1) *together with information (2) is assumed to be weakly gradient observable* $\mathbb{W}\partial\Omega G$*-observable on [0,T], if:*

$$\overline{\mathrm{Im}\gamma\nabla K^*} = \left(H^{1/2}(\partial\Omega)\right)^n.$$

Remark 3.1. We can deduced that, the equation:

$$\overline{\mathrm{Im}\gamma\nabla K^*} = \left(H^{1/2}(\partial\Omega)\right)^n$$

is equivalent to:

$$\ker \nabla K^*\gamma^* = \{0\}.$$

From previous results, we present the characterization of exactly boundary gradient observable system in Ω ($EG\partial\Omega$-observable) in the following result.

Proposition 3.1. *System* (1) *together with information (2) is said to be* $E\partial\Omega G$*-observable on* $[0,T]$ *if and only if* $\exists\ \alpha_{E\partial\Omega G} \geq 0$*, such that:*

$$\|\gamma\nabla w\|_{L\left(H^1(\Omega),\left(H^{1/2}(\partial\Omega)\right)^n\right)} \leq \alpha_{E\partial\Omega G}\,\|Kw_0\|_Y\,,\ for\ all\ w_0 \in \mathbb{W} \tag{5}$$

Now, we give the concept of boundary gradient strategic sensor ($\partial\Omega G$-strategic sensor).

Definition 3.5. *Sensor* (D, f) *is* $\partial\Omega G$*-strategic, if the corresponding system is* $\mathbb{W}\partial\Omega G$*-observable.*

Definition 3.6. *The semi-group* $(S_{\mathcal{A}}(t))_{t\geq 0}$ *is supposed to be boundary asymptotic gradient stable on* $(H^{1/2}(\partial\Omega))^n$ *(*$\partial\Omega AG$*-stable), if for some positive constants* $M_{\partial\Omega AG}, \alpha_{\partial\Omega AG} > 0$*, then:*

$$\|\gamma\nabla S_A(t)\|_{\left(H^{1/2}(\partial\Omega)\right)^n} \leq M_{\partial\Omega AG}e^{-\alpha_{\partial\Omega AG}t},\ for\ all\ t \geq 0.$$

Remark 3.2. If the semi-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ is $\partial\Omega AG$-stable, then for all $w_0 \in (H^{1/2}(\partial\Omega))^n$ the solutions associated to the autonomous system of (1) satisfies:

$$\lim_{t\to\infty}\|\gamma\nabla S_{\mathcal{A}}w(\cdot,t)\|_{\left(H^{1/2}(\partial\Omega)\right)^n} = \lim_{t\to\infty}\|\gamma\nabla S_{\mathcal{A}}w(\cdot)\|_{\left(H^{1/2}(\partial\Omega)\right)^n} = 0 \tag{6}$$

Definition 3.7. *System* (1) *is assumed to be* $\partial\Omega AG$*-stable, if the transformation* $\mathcal{A}$ *produces* SCS*-group* $(S_{\mathcal{A}}(t))_{t\geq 0}$ *which is* $\partial\Omega AG$*-stable.*

Definition 3.8. *System* (1) *together with the information (2) is assumed to be* $\partial\Omega AG$*-detectable if there is transformation such that* $(\mathcal{A} - H_{\partial\Omega AG}C)$ *, produces a* SCS*-group* $(S_{H_{\partial\Omega AG}}(t))_{t\geq 0}$ *which is* $\partial\Omega AG$*-stable.*

Though, one can assume the following results. Consequently, the notion of $\partial\Omega AG$-detectability is a weaker property than the exact $E\partial\Omega G$-observability [1,14].

4. Boundary asymptotic gradient full-order observer

A methodology that permits a construction and reconstruct asymptotically gradient in full order estimator ($\partial\Omega AGFO$-estimator) of $\hat{T}w(\xi,t)$ has been presented in this section. This technique evades the evaluation inverse problem, and related to calculate the unknown initial state [3,10], which permits to guess a current state in $\partial\Omega$ with no needs to the outcome of the initial state of the main system.

4.1 Modernization of $\partial\Omega AGFO$-estimator

Assume the following system:

$$(7)\qquad \begin{cases} \dfrac{\partial w}{\partial t}(\zeta,t) = Aw(\zeta,t) + Bu(t) & \Pi \\ w(\mu,t) = 0 & \Xi \\ w(\zeta,0) = w_0(\zeta) & \Omega \\ y(\cdot,t) = Cw(\cdot,t) & \Pi \end{cases}$$

For a region $\partial\Omega$, assume that for $\hat{T} \in L((H^{1/2}(\Omega))^n, (H^{1/2}(\partial\Omega))^n)$ and $\hat{T} = \gamma T$, $\exists\ \mathcal{V}(\cdot,t)$, such that:

$$(8)\qquad \mathcal{V}(\zeta,t) = \hat{T}w(\zeta,t) \qquad \Pi$$

where $\mathcal{V}(\cdot,t)$ is a state system. Therefore, if we can form a system which is an asymptotic approach for $\mathcal{V}(\cdot,t)$, then it will be give an asymptotic estimation for $\hat{T}w(\zeta,t)$ (i.e. it structure an asymptotic observer to the restriction of $Tw(\zeta,t)$ on $\partial\Omega$).

Equations (2)-(8) provides:

$$(9)\qquad \begin{bmatrix} y \\ \mathcal{V} \end{bmatrix} = \begin{bmatrix} C \\ \hat{T} \end{bmatrix} w$$

Suppose there exists two linear bounded operators $\mathbb{R}$ and $\mathbb{S}$, where $\mathbb{R} : \mathcal{R} \longrightarrow (H^{1/2}(\partial\Omega))^n$ and $\mathbb{S} : (H^{1/2}(\partial\Omega))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$, such that $\mathbb{R}C + \mathbb{S}\hat{T} = I$, then by deriving $\mathcal{V}(\zeta,t)$, we have:

$$\begin{aligned} \tfrac{\partial \mathcal{V}}{\partial t} &= \hat{T}\tfrac{\partial w}{\partial t}(\zeta,t) = \hat{T}\mathcal{A}w(\zeta,t) + \hat{T}Bu(t) \\ &= \hat{T}\mathcal{A}\mathbb{S}\mathcal{V}(\zeta,t) + \hat{T}\mathcal{A}\mathbb{S}\mathbb{R}y(\zeta,t) + \hat{T}Bu(t) \qquad \Pi \end{aligned}$$

Consider ($\partial\Omega AGFO$-estimator for x) as:

$$(10)\qquad \begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(\zeta,t) = F_{\partial\Omega AG}\mathcal{V}(\zeta,t) + G_{\partial\Omega AG}u(t) + H_{\partial\Omega AG}y(\cdot,t) & \Pi \\ \mathcal{V}(\zeta,0) = \mathcal{V}_0(\zeta) & \Omega \\ \mathcal{V}(\mu,t) = 0 & \Xi \end{cases}$$

with $F_{\partial\Omega\Omega AG}$ generates SCS-group $(S_{F_{\partial\Omega}}(t))_{t\geq 0}$, that is $\partial\Omega AG$-stable on $\mathbb{W} = H^{1/2}(\partial\Omega)$, that means $\exists\ M_{F_{\partial\Omega}}, \alpha_{F_{\partial\Omega}} > 0$, such that:

$$\|\chi_{\partial\Omega} S_{F_{\partial\Omega}}(.)\|_{L\left(\left(H^{1/2}(\partial\Omega)\right)^n, \left(H^{1/2}(\partial\Omega)\right)^n\right)} \leq M_{F_{\partial\Omega}} e^{-\alpha_{F_{\partial\Omega}} t},\ \ t \geq 0 \qquad \Pi \tag{11}$$

and $G_{\partial\Omega} \in L(\mathbb{R}^p, (H^{1/2}(\partial\Omega))^n)$ and $H_{\partial\Omega} \in L(\mathbb{R}^p, (H^{1/2}(\partial\Omega))^n)$. The solution of (10) is given by:

$$\mathcal{V}(.,t) = S_{F_{\partial\Omega}}(t)\mathcal{V}_0(\cdot) + \int_0^t S_{F_{\partial\Omega}}(t-\tau)\left[G_{\partial\Omega} u(\tau) + H_{\partial\Omega} y(\cdot,\tau)\right] d\tau \qquad \Pi \tag{12}$$

Now, in the case when $\hat{T} = I$ and $\mathbb{W} = \mathbb{V}$ in equation (8), the operator equation [4]:

$$\hat{T}\mathcal{A} - F_{\partial\Omega AG}\hat{T} = H_{\partial\Omega}\mathcal{A}GC$$

of the $\partial\Omega EFO$-observer becomes to:

$$F_{\partial\Omega AG}\mathcal{A}G = \mathcal{A} - H_{\partial\Omega AG}C,$$

where $\mathcal{A}$ and C are identified. Hence, the operator $H_{\partial\Omega AG}$ has to be known such that the operator $F_{\partial\Omega AG}$ is $\partial\Omega AG$-stable.

Also, for the equation (7), the dynamical system can be deliberated as:

$$\begin{cases} \dfrac{\partial\mathcal{V}}{\partial t}(\zeta,t) = \mathcal{A}\mathcal{V}(\zeta,t) + Bu(t) + H_{\partial\Omega AG}\left(y(.,t) - C\mathcal{V}(\zeta,t)\right) & \Pi \\ \mathcal{V}(\mu,t) = 0 & \Xi \\ \mathcal{V}(\zeta,0) = 0 & \Omega \end{cases} \tag{13}$$

which is named $\partial\Omega EFO$-observer.

4.2 Junction interface conditions

We examine the three regions E, E_1 and E_2 as in (Figure 2) of junction conditions [16] used to generalize an approach may be called asymptotic observer to build the gradient of current state on the $\partial\Omega$. Thus, the boundary observer on

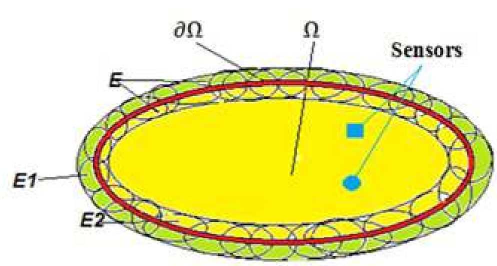

Figure 2: Ω, $\partial\Omega$ and regions junction conditions.

$\partial\Omega$ might be gotten as an observer of internal regional type in E_2. If we have the following mapping $\mathfrak{R}$ holds an extension of continuous linear operator [19], $\mathfrak{R} : (H^{1/2}(\partial\Omega))^n \longrightarrow (H^1(\Omega))^n$, such that:

$$\gamma \nabla \mathfrak{R} h(\mu, t) = h(\mu, t), \quad for\ all\ h \in \left(H^{1/2}(\partial\Omega)\right)^n \qquad \Xi \tag{14}$$

Let $\forall\, w_0 \in \partial\Omega$ there exists $r > 0$ is an random and appropriately small real with the following sets:

$$E = \bigcup_{w_0 \in \partial\Omega} B(w_0, r) = \{w \in \Omega \ or\ w \in E_1 : \|w - w_0\| < r,\ w_0 \in \partial\Omega\},$$

where:

$$E_1 = \bigcup_{w_0 \in \partial\Omega} B(w_0, r) = \{w \in E \ or\ w \notin \Omega : \|w - w_0\| < r,\ w_0 \in \partial\Omega\}$$

and

$$E_2 = \bigcup_{w_0 \in \partial\Omega} B(w_0, r) = \{w \in E \ or\ w \notin E_1 : \|w - w_0\| < r,\ w_0 \in \partial\Omega\} \subset \Omega$$

and then we have:

$$E = E_1 \cup E_2,\ \partial\Omega = E_1 \cap \bar{E}_2 \ and\ E_2 = E \cap \Omega,$$

where $B(w_0, r)$ represents a ball of radius r centered in $w_0(\mu, t)$ and $\partial\Omega$ is boundary of the domain Ω.

For the a region E_2 of the domain Ω and let χ_{E_2} be a function assumed as:

$$\chi_{E_2} : \left(H^1(\Omega)\right)^n \to \left(H^1(E_2)\right)^n,$$
$$w : \chi_{E_2} w = w|_{E_2},$$

where $w|_{E_2}$ is the restriction of w to E_2 with adjoint operator $\chi^*_{E_2}$ (for more details see references [7-9]).

Definition 4.1. *System* (7) *is E_2AG-stable, then the autonomous system solution linked to* (7)*, asymptotically converges to 0 when t approaches to ∞.*

Definition 4.2. *System* (7) *is called E_2AG-detectable, if there exists an operator $H_{E_2AG} : \mathcal{O} \longrightarrow (H^1(E_2))^n$, such that the operator $A - H_{E_2AG}C$ produces a SCS-group $(S_{E_2AG}(t))_{t>0}$, which is E_2AG-stable.*

So, the process of junction conditions from interior to exterior of E_2AG-detectability might be assumed as follows [23-26]:

Proposition 4.1. *If the system* (7) *is $\bar{E}_2AG$-detectable, then it is $\partial\Omega AG$-detactable.*

Proof. Suppose that $w(\zeta,t) \in H^{1/2}(\partial\Omega)$ and $\bar{w}(\zeta,t)$ be an extension to $H^{1/2}(\bar{E}_2)$ with $\partial\Omega \subset \bar{E}_2$.

Trace theorem [19] with equation (14) tells, there exist $\mathfrak{R}\bar{w}(\zeta,t) \in (H^1(\Omega))^n$ with a bounded support such that:

$$\gamma\left(\mathfrak{R}\mathfrak{R}_{E_2}\bar{w}(\zeta,t)\right) = w(\zeta,t) \qquad \Pi \tag{15}$$

where $\mathfrak{R}_{E_2} : (H^1(E_2))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$. Since the system (7) is $\bar{E}_2$-detectable, then it is E_2-detectable [18, 25]. Accordingly, there exists an operator $\chi_{E_2}\nabla K^* : \mathcal{O} \longrightarrow (H^1(E_2))^n$ specified by:

$$H_{\partial\Omega AG}w(\cdot,t) = \gamma\nabla K^* y(\zeta,t) \qquad \Pi \tag{16}$$

such that the operator $A - H_{\partial\Omega AG}C$ produces a SCS-group $(S_{\partial\Omega}(t))_{t>0}$ which is $\partial\Omega AG$-stable. For every $\in \mathcal{O}$, then we get:

$$\chi_{E_2}\nabla K^* y(\xi,t) = \chi_{E_2}\mathfrak{R}\mathfrak{R}_{E_2}\bar{w}(\xi,t)$$

and hence:

$$H_{\partial\Omega AG} = \left(\gamma\chi^*_{E_2}\nabla K^* y\right) : \mathbb{Y} \longrightarrow \left(H^{1/2}(\partial\Omega)\right)^n$$

such that $A - H_{\partial\Omega AG}C$produces a semi-group $(S_{\partial\Omega}(t))_{t>0}$, which is $\partial\Omega AG$-stable. To conclude, the system (7) is $\partial\Omega AG$-detectable. □

Proposition 4.2. *If the dynamical system (13) is $\bar{E}_2AGFO$-observer for the systems* (7) *then, its $\partial\Omega AGFO$-observer.*

Proof. In view of assumptions as in Proposition 4.3 with equations (15) and (16) and since the dynamical systems (13) $\bar{E}_2AGFO$-observer, so we can assume that:

I- The systems (13) is E_2AGFO-observer [25-26], thus there exists a dynamical system with $w(\xi,t) \in \mathbb{W}$, such that:

$$\chi_{E_2}\hat{T}w(\zeta,t) = \chi_{E_2}\mathfrak{R}\mathfrak{R}_{E_2}\bar{w}(\zeta,t).$$

Then, we have:

$$\left(\gamma\chi^*_{E_2}\chi_{E_2}\mathfrak{R}\hat{T}w\right)(\zeta,t) = w(\zeta,t) \qquad \Pi \tag{17}$$

II- The equations (2) and (16) allow:

$$\begin{bmatrix} y \\ \mathcal{V} \end{bmatrix}(\zeta,t) = \begin{bmatrix} C \\ \left(\gamma\chi^*_{E_2}\chi_{E_2}\mathfrak{R}\hat{T}\right) \end{bmatrix} w(\zeta,t) \qquad \Pi$$

and there exists two linear bounded operator $\bar{R}$ and C satisfy the relation:

$$\bar{R}C\left(\gamma\chi^*_{E_2}\chi_{E_2}\mathfrak{R}\hat{T}\right) + \gamma\chi^*_{E_2}\chi_{E_2}\mathfrak{R}\hat{T} = I_{\partial\Omega AG}.$$

III- There exist an operator $F_{\bar{E}_2}$ is $\bar{E}_2AGFO$-observer, such that $\partial\Omega AG$-stable (see [28]). To end with the dynamical system (13) is $\partial\Omega AGFO$-observer for the system (7).

□

4.3 Sensors and $\partial\Omega AG$ -detectability

The boundary asymptotic gradient detectability concept with the spatial structure of sensors can be linked [6]. Now, for that determination assume that J has unstable modes to have a clear picture of this concept with respect to sensor structures [5].

Proposition 4.3. *Assume that there is q zone sensors $(D_i, f_i)_{1\leq i\leq q}$ and $\rho(\mathcal{A})$ is the spectrum of A holds for finite J eigenvalues of $Re\lambda_J \geq 0$. Then,* (7) *is $\partial\Omega AG$-detectable if and only if:*

I- $q \geq m$,

II- $rank\, G_i = m_i, i = 1, 2, \cdots, J$, *where* $sup m_i = m < \infty$ *and* $j = 1, 2, \cdots, m_i$,

$$G = (G)_{ij} = \begin{cases} <\varphi_{nj}, f_i(.)>_{L^2(D_i)}, & \textit{zone case} \\ \varphi_{nj}(b_i), & \textit{poitwise case} \\ <\Phi_{nj}, f_i(.)>_{L^2(\Gamma_i)}, & \textit{boundary zone case} \end{cases}$$

Proof. The proof can be established as in [25] with case of state gradient $w(\xi, t)$ belong to sub region Γ, such that $\Gamma = \partial\Omega$. □

Remark 4.1. If the system (7) is $\partial\Omega AG$-detectable, then it is possible to construct an $\partial\Omega AGFO$-observer for the original system [5, 25].

Proposition 4.4. *If the systems* (7) *is $\partial\Omega AG$-detectable, then the dynamical system* (13) *is $\partial\Omega AGFO$-observer of the systems* (7) *that means:*

$$\lim_{t\to\infty} [w(\zeta, t) - \mathcal{V}(\zeta, t)] = 0 \qquad \Pi \tag{18}$$

Proof. Assume $\varphi(\zeta, t) = w(\zeta, t) - \mathcal{V}(\zeta, t)$, where $\mathcal{V}(\zeta, t)$ is the solution of (13). Differentiate equation (18) and use of equations (7) and (13), we attain:

$$\begin{aligned} \frac{\partial\varphi}{\partial t}(\zeta, t) &= \frac{\partial w}{\partial t}(\zeta, t) - \frac{\partial V}{\partial t}(\zeta, t) \\ &= (A - H_{\partial\Omega AG}C)\,\varphi(\zeta, t) \qquad \Pi. \end{aligned}$$

The system (7) is $\partial\Omega AG$-detectable. Hence, there exists an operator $H_{\partial\Omega AG} \in L(\mathcal{O}, (H^{1/2}(\partial\Omega))^n)$, such that $A - H_{\partial\Omega AG}C$ produces a SCS-group $(S_{\partial\Omega AG}(t))_{t\geq 0}$m which is $\partial\Omega AG$-stable on $(H^{1/2}(\partial\Omega))^n$, and there exists $M_{\partial\Omega AG}$, $\omega_{\partial\Omega AG} > 0$, such that:

$$\|\varphi\|_{(H^{1/2}(\partial\Omega))^n} \leq \|\gamma\nabla S_{\partial\Omega AG}(t)\|_{(H^{1/2}(\partial\Omega))^n}\|\varphi_0\| \leq M_{\partial\Omega AG}e^{-\omega_{\partial\Omega AG}t}\|\varphi_0\|$$

with

$$\varphi_0(\zeta) = w_0(\zeta) - \mathcal{V}_0(\zeta)$$

and hereafter, we got the following:

$$\lim_{t\to\infty} [w(\zeta, t) - \mathcal{V}(\zeta, t)] = 0 \qquad \Pi.$$

□

5. Applications to $\partial\Omega AGFO$-Observer

The distributed diffusion systems defined in the domain Ω have been considered as an application to $\partial\Omega AGFO$-observer [12, 27]. Several applications in real life problems associated with different types of sensor have been prolonged. For two-dimensional system, the domain:

$$\Omega =]0, a_1[\times]0, a_2[$$

with the boundary is given by the following form:

$$\partial\Omega = [0, a_1] \times \{a_2\} \cup [0, a_1] \times \{0\} \cup \{0\} \times [0, a_2] \cup a_1 \times [0, a_2]$$

is a region of $\bar{\Omega}$.

The eigenfunctions of (16) are defined by:

$$\varphi_{nm}(\zeta_1, \zeta_2) = \left(\frac{4}{a_1 a_2}\right)^{1/2} \cos n\pi \left(\frac{\zeta_1}{a_1}\right) \cos n\pi \left(\frac{\zeta_2}{a_2}\right) \tag{19}$$

associated with eigenvalues:

$$\lambda_{nm} = -\left(\frac{n^2}{a_1^2} + \frac{m^2}{a_2^2}\right)\pi^2, \; n, m \geq 1 \tag{20}$$

If we assume that $\frac{a_1^2}{a_2^2} \notin Q$ [28-30], then the multiplicity of the eigenvalues λ_{nm} is $r_{nm} = 1$ for every $n, m = 1, 2, \cdots, J$, then one sensor (D, f) may be sufficient for $\partial\Omega AGFO$-observer [26-30].

5.1 Rectangular domain

A sufficient conditions which is characterized some cases of the $\partial\Omega AGFO$-observer in the rectangular domain of system (21) with various sensor locations cases have been provided in this section.

5.1.1 Internal zone sensors case

Assume the following two dimensional system that is defined by parabolic equation:

$$\begin{cases} \dfrac{\partial w}{\partial t}(\zeta_1, \zeta_2, t) = \dfrac{\partial^2 w}{\partial w \partial \zeta_1^2}(\zeta_1, \zeta_2, t) + \dfrac{\partial^2 w}{\partial \zeta_1^2}(\zeta_1, \zeta_2, t) & \Pi \\ w(\mu_1, \mu_2, t) = 0 & \Xi \\ w(\zeta_1, \zeta_2, 0) = w_0(\zeta_1, \zeta_2) & \Omega \end{cases} \tag{21}$$

together with the information is represented via internal pointwise or zone sensors

$$y(\cdot, t) = \int_D w(\zeta_1, \zeta_2, t) f(\zeta_1, \zeta_2)\, d\zeta_1 d\zeta_2 \qquad \Pi \tag{22}$$

where the zone sensor is situated interior to the domain Ω (Figure 3), with support of:

$$D = [\zeta_{1_0} - l_1, \zeta_{1_0} + l_1] \times [\zeta_{2_0} - l_2, \zeta_{2_0} + l_2] \subset \Omega \text{ and } L^2(D)$$

In this case the system (21) together with the information (22) have an associ-

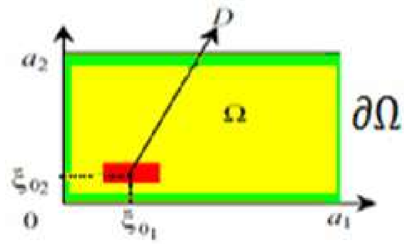

Figure 3: Ω, $\partial\Omega$ with sensor position D of internal zone type.

ated dynamical system, that is specified by the following formula:

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(\zeta_1, \zeta_2, t) = \dfrac{\partial^2 \mathcal{V}}{\partial w \partial \zeta_1^2}(\zeta_1, \zeta_2, t) + \dfrac{\partial^2 \mathcal{V}}{\partial \zeta_1^2}(\zeta_1, \zeta_2, t) & \\ \qquad\qquad -H_{\partial\Omega GA}(C\mathcal{V}(\zeta_1, \zeta_2, t) - y(t)) & \Pi \\ \mathcal{V}(\mu_1, \mu_2, t) = 0 & \Xi \\ \mathcal{V}(\zeta_1, \zeta_2, 0) = z_0(\zeta_1, \zeta_2) & \Omega \end{cases} \tag{23}$$

Hence, the following important result is obtained.

Proposition 5.1. *Assume that f_1 and f_2 are symmetric about $\zeta = \zeta_{01}$ and $\zeta = \zeta_{02}$ respectively, then the process* (23) *is $\partial\Omega AGFO$-observer for systems* (21)-(22) *if $n\zeta_{01}/a_1$ and $m\zeta_{02}/a_2 \notin N$, for every $n, m = 1, 2, \cdots, J$.*

5.1.2 Pointwise sensors case

Assume the system (21) together with information (24) which is measured by internal pointwise sensors. Then, the output function can be formulated as:

$$y(t) = \int_\Omega w(\zeta_1, \zeta_2, t)\delta(\zeta_1 - b_1, \zeta_2 - b_2)\, d\zeta_1 d\zeta_2 \qquad \Pi \tag{24}$$

So, the following result is prophesied.

Proposition 5.2. *Let $b = (b_1, b_2)$ is the sensor positioned in Ω, then the dynamic system* (23) *is $\partial\Omega AGFO$-observer for the system* (21)-(24), *if $(nb_1)/(a_1)$ and $(mb_2)/(a_2 \notin N)$, for every $n, m = 1, 2, \cdots, J$.*

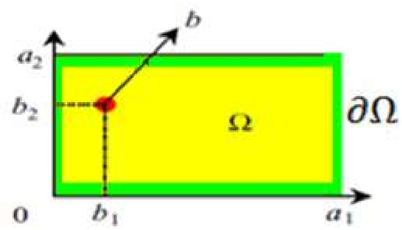

Figure 4: Ω, $\partial\Omega$ with sensor position b of pointwise zone type.

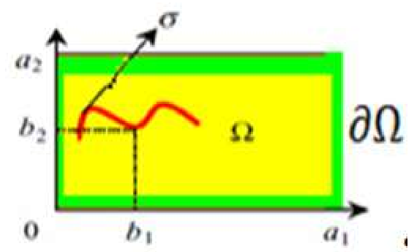

Figure 5: Ω, $\partial\Omega$ with sensor position σ of filament zone type.

5.1.3 Filament pointwise sensors case

Assume that the filament sensor positioned in Ω, where $\sigma = Im(\gamma) \subset \Omega$ is symmetric with respect to the line $b = (b_1, b_2)$ (Figure 5). More precisely, the sensor is line of pointwise positioned in Ω, then the output function still given by equation (21).

Proposition 5.3. *Let the sensor is located in* $b = (b_1, b_2)$*, then the process* (23) *is* $\partial\Omega AGFO$*-observer to* (21)-(24)*, if* $(nb_1)/(a_1)$ *and* $(mb_2)/(a_2 \notin N)$*, for every* $n, m = 1, 2, \cdots, J$.

5.2 Circular domain

Remark 5.1. The results in 5.1 can be extended to the case of circular domain with the internal zone and pointwise sensor as in [27-28].

6. Conclusion

The crossing problem from interior to exterior of asymptotic gradient full order observer have been explored and achieved in rigorous results. Thus, the characterizations of this approach are presented in connection with corresponding notions as stability, detectability, strategic sensor and considered domain. Then, the boundary asymptotic gradient reconstruction state via full-order observer in parabolic distributed parameter systems is examined and proved. Many interesting results concerning the choice of sensors structure are given and illustrated in specific situations to diffusion systems. Moreover, many problem still opened for instance, hyperbolic distributed parameter systems and it's development of the sense of these results as in [22] with another operators (see [31-32]).

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Fixed point theorems for monotone mappings on partial M^*-metric spaces

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Abstract. In this paper, we introduce the concept of partial M^*-metric on a nonempty set X, and we give some properties supported by some examples to illustrate

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our results. Furthermore, we establish some fixed points results for partial M^*-metric. Also, we extend our result for monotone mappings on partial M^*-metric spaces.

Keywords: M^*-metric spaces, fixed point, partial metric.

1. Introduction

Bakhtin [2] and Czerwik [3] are defined a $b-$metric space and the idea of a $b-$metric space the triangle inequality axiom is weaker than for metric space. Also, many authors gives many fixed point theorems in a b-metric space (see [6−15]), Aydi et al. [8] gave some interesting theories for fixed point for set-valid quasi contraction in b-metric space.

In 2021 [37], Malkawi et al. introduced the notion of MR-metric space and MR-metric space is a generalization of a b-metric space [2, 3] and the tetrahedral inequality axiom is weaker than for a $D-metric$ space [1]. Also, there are many fixed point theorems in different type spaces for more information. I Refer to the reader to look at $[4-36]$.

Definition 1 ([37])**.** *Let X be a non empty set and $R \geq 1$ be a real number. $M : X \times X \times X \to [0,\infty)$ a function which is called an MR-metric, if it satisfies the following axioms for each $x, y, z \in X$.*

$(M1) : M(x, y, z) \geq 0.$

$(M2) : M(x, y, z) = 0$ *iff* $x = y = z.$

$(M3) : M(x, y, z) = M(p(x, y, z))$; *for any permutation* $p(x, y, z)$ *of* $x, y, z.$

$(M4) : M(x, y, z) \leq R\left[M(x, y, \ell) + M(x, \ell, z) + M(\ell, y, z)\right].$

A pair (X, M) is called an MR-metric space.

Also, Gharib et al. [38] introduced the concept of M^*-metric spaces, the importance of which lies in this property $M^*(x, x, y) = M^*(x, y, y)$. It is worth noting that these characteristics need not be satisfied in MR-metric space [37].

Definition 2 ([38])**.** *Let X be a non empty set and $R \geq 1$ be a real number. A function $M^* : X \times X \times X \to [0,\infty)$ is called M^*-metric, if the following properties are satisfied for each $x, y, z \in X$.*

$(M^*1) : M^*(x, y, z) \geq 0.$

$(M^*2) : M^*(x, y, z) = 0$ *iff* $x = y = z.$

$(M^*3) : M^*(x, y, z) = M^*(p(x, y, z))$; *for any permutation* $p(x, y, z)$ *of* $x, y, z.$

$(M^*4) : M^*(x, y, z) \leq RM^*(x, y, u) + M^*(u, z, z).$

A pair (X, M^*) is called an M^*-metric space.

The following are examples of M^*-metric space.

Example 1. a) Let (X, d) be a metric space then $M^*(x, y, z) = \frac{1}{R}\max\{d(x, y), d(y, z), d(z, x)\}$ and $M^*(x, y, z) = \frac{1}{R}[d(x, y), d(y, z), d(z, x)]$ are M^*-metric on X.

b) If $X = \mathbb{R}^n$, then

$$M^*(x, y, z) = \frac{1}{R}[\|x + y - 2z\| + \|y + z - 2x\| + \|z + x - 2y\|],$$

for every $x, y, z \in \mathbb{R}^n$ is an M^*-metric on X.

Example 2. Let $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be a mapping defined as the following:

$$\psi(x,y) = 0 \text{ if } x = y, \psi(x,y) = \frac{1}{2} \text{ if } x > y, \psi(x,y) = \frac{1}{3} \text{ if } x < y.$$

Then, clearly ψ is not a metric, since $\psi(1,2) \neq \psi(2,1)$. Define $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by

$$G(x,y,z) = \frac{1}{R} \max\{\psi(x,y), \psi(y,z), \psi(z,x)\}.$$

Then, G is an M^*-metric.

Example 3. Let $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping defined as the following:

$\psi(x,y) = \max\{x,y\}$. Clearly it is not a metric. Define $G : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(x,y) = \frac{1}{R}[\max\{x,y\} + \max\{y,x\} + \max\{z,x\}] - x - y - z,$$

for every $x, y, z \in \mathbb{R}^+$. Then G is an M^*-metric.

2. Partial M^*-metric space

The Authors defined $b-$metric space by replacing the triangular inequality axiom with a weaker one. Also, for some work on b-metric, we refer the reader to [40, 41, 42, 43, 44, 45, 46].

Now, we present the concept of a partial M^*-metric space and prove its properties.

Definition 3. *A partial M^*-metric on a nonempty set X is a function $M_p^* : X \times X \times X \to \mathbb{R}^+$ such that for all $x, y, z, a \in X$:*

(M_p^*1) $x = y = z \Leftrightarrow M_p^*(x,x,x) = M_p^*(x,y,z) = M_p^*(y,y,y) = M_p^*(z,z,z)$,

(M_p^*2) $M_p^*(x,x,x) \leq M_p^*(x,y,z)$,

(M_p^*3) $M_p^*(x,y,z) = M_p^*(p\{x,y,z\})$, *where p is a permutation function,*

(M_p^*4) $M_p^*(x,y,z) \leq RM_p^*(x,y,a) + M_p^*(a,z,z) - M_p^*(a,a,a)$.

(X, M_p^*) is a partial M^*-metric space on a nonempty set X and M_p^* is a partial M^*-metric on X. It is clear that, if $M_p^*(x,y,z) = 0$, then from (M_p^*1) and (M_p^*2) $x = y = z$. But if $x = y = z$, $M_p^*(x,y,z)$ may not be 0. The basic example of a partial M^*-metric space $(\mathbb{R}^+, M_p^*)$ is $M_p^*(x,y,z) = \frac{1}{R}\max\{x,y,z\}$ for all $x, y, z \in \mathbb{R}^+$.

It is obvious that every M^*-metric is a partial M^*-metric, but the converse need not be true. We will explain this in the following example.

Example 4. Let $M_p^* :: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a nonempty defined as follows:

$$M_p^*(x,y,z) = \frac{1}{R}[|x-y| + |y-z| + |x-z|] + \max\{x,y,z\},$$

such that $R \geq 1$. Then clearly it is a partial M^*-metric, but it is not an M^*-metric.

Example 5. Let (X, p) be a partial b-metric space and $M_p^* :: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a nonempty defined as:

$$M_p^*(x, y, z) = \frac{1}{R}[p(x, y) + p(x, z) + p(y, z)] - p(x, x) - p(y, y) - p(z, z).$$

Then, clearly M_p^* is a partial M^*-metric, but it is not an M^*-metric.

Remark 1. $M_p^*(x, x, y) = M_p^*(x, y, y)$

Proof.

$$\begin{aligned} M_p^*(x, x, y) &\leq RM_p^*(x, x, x) + M_p^*(x, y, y) - M_p^*(x, x, x) \\ &\leq RM_p^*(x, x, x) + M_p^*(x, y, y) - RM_p^*(x, x, x) \\ &\leq M_p^*(x, y, y). \end{aligned} \tag{2.1}$$

$$\begin{aligned} M_p^*(x, y, y) &\leq RM_p^*(y, y, y) + M_p^*(y, x, x) - M_p^*(y, y, y) \\ &\leq RM_p^*(y, y, y) + M_p^*(y, x, x) - RM_p^*(y, y, y) \\ &\leq M_p^*(y, x, x). \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we get $M_p^*(x, x, y) = M_p^*(x, y, y)$. □

Lemma 1. *Let (X, M_p^*) be a partial M^*-metric space. If we define $p(x, y) = M_p^*(x, y, y)$, then (X, p) is a partial b-metric space*

Proof. (M_p^*1) $x = y \Leftrightarrow M_p^*(x, x, x) = M_p^*(x, y, y) = p(y, y, y) \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(M_p^*2) $M_p^*(x, x, x) \leq M_p^*(x, y, y)$ implies that $p(x, x) \leq p(x, y)$,

(M_p^*3) $M_p^*(x, y, y) = M_p^*(y, x, x)$ implies that $p(x, y) = p(y, x)$,

(M_p^*4) $M_p^*(y, y, x) \leq RM_p^*(y, y, z) + M_p^*(z, x, x) - M_p^*(z, z, z)$ implies that

$$p(x, y) \leq R[p(y, z) + p(z, x)] - p(z, z).$$ □

Let (X, M_p^*) be a partial M^*-metric space. For $r > 0$ define

$$B_{M_p^*}(x, r) = \{y \in X : M_p^*(x, y, y) < M_p^*(x, x, x) + r\}.$$

Definition 4. *Let (X, M_p^*) be a partial M^*-metric space and $A \subset X$.*

(1) *If, for every $x \in A$ there exists $r > 0$ such that $B_{M_p^*}(x, r) \subset A$, then the subset A is called an open subset of X.*

(2) *$\{x_n\}$ is a sequence in a partial M^*-metric space (X, M_p^*) converges to x if and only if $M_p^*(x, x, x) = \lim_{n\to\infty} M_p^*(x_n, x_n, x)$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$M_p^*(x, x, x_n) < M_p^*(x, x, x) + \epsilon \ \forall n \geq n_0, \tag{1}$$

or equivalently, for each $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$M_p^*(x, x_n, x_m) < M_p^*(x, x, x) + \epsilon \ \forall n, m \geq n_0. \tag{2}$$

Indeed, if (1) *holds then*

$$\begin{aligned} M_p^*(x, x_n, x_m) &= M_p^*(x_n, x, x_m) \\ &\leq RM_p^*(x_n, x, x) + M_p^*(x, x_m, x_m) - M_p^*(x, x, x) \\ &< R\epsilon + \epsilon + M_p^*(x, x, x). \end{aligned}$$

Conversely, set $m = n$ *in* (2) *we have* $M_p^*(x_n, x_n, x) < M_p^*(x, x, x) + \epsilon$.

(3) $\{x_n\}$ *is a sequence in a partial* M^*-*metric space* (X, M_p^*) *is called a Cauchy if* $\lim_{n\to\infty} M_p^*(x_n, x_m, x_m)$ *exists.*

Let $\tau_{M_p^*}$ *be the set of all open subsets* X*, then* $\tau_{M_p^*}$ *is a topolpgy on* X *(induced by the partial* M^*-*metric* M_p^**).*

A partial M^*-*metric space* (X, M_p^*) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *with respect to* $\tau_{M_p^*}$*.*

If a sequence $\{x_n\}$ in a partial M^*-metric space (X, M_p^*) converges to x, then we have

$$\begin{aligned} M_p^*(x_n, x_n, x_m) &\leq RM_p^*(x_n, x_n, x) + M_p^*(x, x_m, x_m) - M_p^*(x, x, x) \\ &< R\epsilon + \epsilon + M_p^*(x, x, x). \end{aligned}$$

Lemma 2. *Let* (X, M_p^*) *be a partial* M^*-*metric space. If* $r > 0$*, then the ball* $B_{M_p^*}(x, r)$ *with center* $x \in X$ *and radius* r *is an open ball.*

Proof. Let $y \in B_{M_p^*}(x, r)$, then $M_p^*(x, y, y) < M_p^*(x, x, x)+r$. Let $RM_p^*(x, y, y)-M_p^*(x, x, x) = \delta$. Let $z \in B_{M_p^*}(y, r - \delta)$, by triangular inequality, we have

$$\begin{aligned} M_p^*(x, x, z) &\leq RM_p^*(x, y, y) + M_p^*(y, z, z) + M_p^*(y, y, y) \\ &= RM_p^*(x, y, y) - M_p^*(x, x, x) + M_p^*(z, z, y) \\ &\quad -M_p^*(y, y, y) + M_p^*(x, x, x) \\ &< \delta + r - \delta + M_p^*(x, x, x) \\ &= M_p^*(x, x, x) + r. \end{aligned}$$

Thus, $z \in B_{M_p^*}(x, r)$. Hence $B_{M_p^*}(y, r - \delta) \subseteq B_{M_p^*}(x, r)$. Therefore, the ball $B_{M_p^*}(x, r)$ is an open ball. □

Each partial M^*-metric M_p^* on X generates a topology $\tau_{M_p^*}$ on X which has as a base the family of open M_p^*-balls $\{B_{M_p^*}(x, \epsilon) : x \in X, \epsilon > 0\}$.

The following example shows that a convergent sequence $\{x_n\}$ in a partial M^*-metric space (X, M_p^*) need not be a Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not necessarily unique, to explain that see the following example

Example 6. Let $X = [0, \infty)$ and $M_p^*(x,y,z) = \frac{1}{R}\max\{x,y,z\}$. Then, it is clear that (X, M_p^*) is a complete partial M^*-metric space. Let

$$x_n = \begin{cases} 1, & n = 2k \\ 2, & n = 2k+1. \end{cases}$$

Then, clearly it is convergent sequence and for every $x \geq 2$ we have

$$\lim_{n\to\infty} M_p^*(x_n, x_n, x) = M_p^*(x, x, x),$$

therefore

$$L(x_n) = \{x : x_n \to x\} = [2, \infty).$$

But, $\lim_{n\to\infty} M_p^*(x_n, x_m, x_m)$ does not exist. Hence $\{x_n\}$ is not a Cauchy sequence.

The following Lemma plays an important role in this paper.

Lemma 3. *Let (X, p) be a partial $b-$metric space then there exists a partial M^*-metric M_p^* on X such that*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the partial M^*-metric space (X, M_p^*),

(b) the partial $b-$metric space (X, p) is complete if and only if the partial M^*-metric space (X, M_p^*) is complete. Furthermore, $M_p^*(x,x,y) = p(x,y)$, for every $x, y \in X$.

Proof. Define

$$M_p^*(x,y,z) = \frac{1}{R}\max\{p(x,y), p(x,z), p(y,z)\}, \ \ \forall x,y,z \in X.$$

Then, it is easy to see that M_p^* is a partial M^*-metric and $M_p^*(x,x,y) = p(x,y)$, for every $x, y \in X$. □

The following Lemma shows that under certain conditions the limit is unique.

Lemma 4. *Let $\{x_n\}$ be a convergent sequence in a partial M^*-metric space (X, M_p^*) such that $x_n \to x$ and $x_n \to y$. If*

$$\lim_{n\to\infty} M_p^*(x_n, x_n, x_n) = M_p^*(x,x,x) = M_p^*(y,y,y),$$

then $x = y$.

Proof. As

$$M_p^*(x,y,y) = M_p^*(x,x,y) \leq RM_p^*(x,x,x_n) + M_p^*(x_n,y,y) - M_p^*(x_n,x_n,x_n),$$

therefore

$$M_p^*(x_n,x_n,x_n) \leq RM_p^*(x,x,x_n) + M_p^*(x_n,y,y) - M_p^*(x,y,y).$$

By given assumptions, we have

$$\begin{aligned}\lim_{n\to\infty} M_p^*(x_n, x_n, x) &= M_p^*(x, x, x),\\ \lim_{n\to\infty} M_p^*(x_n, x_n, y) &= M_p^*(y, y, y),\\ \lim_{n\to\infty} M_p^*(x_n, x_n, x_n) &= M_p^*(x, x, x).\end{aligned}$$

Therefore

$$M_p^*(x, x, x) \leq RM_p^*(x, x, x) + M_p^*(y, y, y) - M_p^*(x, y, y),$$

which shows that $M_p^*(y, y, y) \leq (1 - R)M_p^*(x, x, x) + M_p^*(x, y, y) \leq M_p^*(y, y, y)$. So,

$$M_p^*(y, y, y) \leq M_p^*(x, y, y) \leq M_p^*(y, y, y).$$

Also,

$$M_p^*(x, y, y) = M_p^*(y, y, x) \leq RM_p^*(y, y, x_n) + M_p^*(x_n, x, x) - M_p^*(x_n, x_n, x_n),$$

implies that

$$M_p^*(x_n, x_n, x_n) \leq RM_p^*(y, y, x_n) + M_p^*(x_n, x, x) - M_p^*(x, y, y),$$

by on taking limit as $n \to \infty$ gives

$$M_p^*(y, y, y) \leq RM_p^*(y, y, y) + M_p^*(x, x, x) - M_p^*(x, y, y),$$

which shows that

$$M_p^*(x, x, x) \leq (1 - R)M_p^*(y, y, y) + M_p^*(x, y, y) \leq M_p^*(x, x, x).$$

So,

$$M_p^*(x, x, x) \leq M_p^*(x, y, y) \leq M_p^*(x, x, x).$$

Thus, $M_p^*(x, x, x) = M_p^*(x, y, y) = M_p^*(y, y, y)$. Therefore, $x = y$. □

Lemma 5. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial M^*-metric space (X, M_p^*) such that*

$$\lim_{n\to\infty} M_p^*(x_n, x, x) = \lim_{n\to\infty} M_p^*(x_n, x_n, x_n) = M_p^*(x, x, x),$$

and

$$\lim_{n\to\infty} M_p^*(y_n, y, y) = \lim_{n\to\infty} M_p^*(y_n, y_n, y_n) = M_p^*(y, y, y).$$

Then $\lim_{n\to\infty} M_p^*(x_n, y_n, y_n) = M_p^*(x, y, y)$. In particular, $\lim_{n\to\infty} M_p^*(x_n, y_n, z)$ $= M_p^*(x, y, z)$, for every $z \in X$.

Proof. As $\{x_n\}$ and $\{y_n\}$ converges to a $x \in X$ and $y \in X$ respectively, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} M_p^*(x,x,x_n) &< M_p^*(x,x,x) + \frac{\epsilon}{2R}, \\ M_p^*(y,y,y_n) &< M_p^*(y,y,y) + \frac{\epsilon}{2R}, \\ M_p^*(x,x,x_n) &< M_p^*(x_n,x_n,x_n) + \frac{\epsilon}{2R} \end{aligned}$$

and

$$M_p^*(y,y,y_n) < M_p^*(y_n,y_n,y_n) + \frac{\epsilon}{2R}$$

for $n \geq n_0$. Now,

$$\begin{aligned} M_p^*(x_n,x_n,y_n) &\leq RM_p^*(x_n,x_n,x) + M_p^*(x,y_n,y_n) - M_p^*(x,x,x) \\ &\leq RM_p^*(x_n,x_n,x) + RM_p^*(y,y_n,y_n) + M_p^*(x,x,y) \\ &\quad -M_p^*(y,y,y) - M_p^*(x,x,x) \\ &< M_p^*(x,y,y) + \frac{R\epsilon}{2R} + \frac{R\epsilon}{2R} \\ &= M_p^*(x,y,y) + \epsilon, \end{aligned} \tag{1}$$

and so we have

$$M_p^*(x_n,y_n,y_n) - M_p^*(x,y,y) < \epsilon.$$

Also,

$$\begin{aligned} M_p^*(x,y,y) &\leq RM_p^*(x_n,y,y) + M_p^*(x,x,x_n) - M_p^*(x_n,x_n,x_n) \\ &\leq RM_p^*(x,x,x) + RM_p^*(x_n,x_n,y_n) + M_p^*(y_n,y,y) \\ &\quad -M_p^*(y_n,y_n,y_n) - M_p^*(x_n,x_n,x) \\ &< M_p^*(x_n,x_n,y) + \frac{R\epsilon}{2R} + \frac{R\epsilon}{2R} \\ &= M_p^*(x,y,y) + \epsilon. \end{aligned} \tag{2}$$

Thus,

$$M_p^*(x,x,y) - M_p^*(x_n,x_n,y_n) < \epsilon.$$

Hence, for all $n \geq n_0$, we have $\left|M_p^*(x_n,x_n,y_n) - M_p^*(x,x,y)\right| < \epsilon$. Hence, the result follows. □

Lemma 6. *If M_p^* is a partial M^*-metric on X, then the functions $M_{p^s}^*, M_{p^m}^* : X \times X \times X \to \mathbb{R}^+$ are given by:*

$$\begin{aligned} M_{p^s}^*(x,y,z) = RM_p^*(x,x,y) + RM_p^*(y,y,z) + M_p^*(z,z,x) \\ - M_p^*(x,x,x) - M_p^*(y,y,y) - M_p^*(z,z,z) \end{aligned}$$

and

$$M^*_{p^m}(x,y,z) = \max\left\{\begin{array}{c} 2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y), \\ 2RM^*_p(y,y,z) - M^*_p(y,y,y) - M^*_p(z,z,z), \\ 2RM^*_p(z,z,x) - M^*_p(z,z,z) - M^*_p(x,x,x) \end{array}\right\},$$

for every $x,y,z \in X$ are equivalent M^*-metrics on X.

Proof. It is easy to see that $M^*_{p^s}$ and $M^*_{p^m}$ are M^*-metrics on X. Let $x,y,z \in X$. It is obvious that

$$M^*_{p^m}(x,y,z) \leq 2M^*_{p^s}(x,y,z).$$

On the other hand, since $a+b+c \leq 3\max\{a,b,c\}$, it provides that

$$\begin{aligned} M^*_{p^s}(x,y,z) &= RM^*_p(x,x,y) + RM^*_p(y,y,z) + M^*_p(z,z,x) \\ &- M^*_p(x,x,x) - M^*_p(y,y,y) - M^*_p(z,z,z) \\ &\leq \frac{1}{2}\left[2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y)\right] \\ &+ \frac{1}{2}\left[2RM^*_p(y,y,z) - M^*_p(y,y,y) - M^*_p(z,z,z)\right] \\ &+ \frac{1}{2}\left[2RM^*_p(z,z,x) - M^*_p(z,z,z) - M^*_p(x,x,x)\right] \\ &\leq \frac{3}{2}\max\left\{\begin{array}{c} 2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y), \\ 2RM^*_p(y,y,z) - M^*_p(y,y,y) - M^*_p(z,z,z), \\ 2RM^*_p(z,z,x) - M^*_p(z,z,z) - M^*_p(x,x,x) \end{array}\right\} \\ &= \frac{3}{2}M^*_{p^m}(x,y,z). \end{aligned}$$

Thus, we have

$$\frac{1}{2}M^*_{p^m}(x,y,z) \leq M^*_{p^s}(x,y,z) \leq \frac{3}{2}M^*_{p^m}(x,y,z).$$

These inequalities implies that $M^*_{p^s}$ and $M^*_{p^m}$ are equivalent. □

Remark 2. Note that:

$$M^*_{p^s}(x,x,y) = 2RM^*_p(x,x,y) - M^*_p(x,x,x) - M^*_p(y,y,y) = M^*_{p^m}(x,x,y).$$

A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_{M^*_p}(x_0,\delta)) \subseteq B_{M^*_p}(Fx_0,\epsilon)$.

The following lemma plays an important role to prove fixed point results on a partial M^*-metric space.

Lemma 7. *Let (X, M^*_p) be a partial M^*-metric space.*

(a) $\{x_n\}$ is a Cauchy sequence in (X, M^*_p) if and only if it is a Cauchy sequence in the M^*-metric space $(X, M^*_{p^s})$

(*b*) *A partial M^*-metric space (X, M_p^*) is complete if and only if the M^*-metric space $(X, M_{p^s}^*)$ is complete. Furthermore,*

$$\lim_{n\to\infty} M_{p^s}^*(x_n, x_n, x) = 0$$

if and only if

$$M_p^*(x,x,x) = \lim_{n\to\infty} M_p^*(x_n, x_n, x) = \lim_{n,m\to\infty} M_p^*(x_n, x_n, x_m).$$

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, M_p^*), we want to prove $\{x_n\}$ is a Cauchy sequence in the M^*-metric space $(X, M_{p^s}^*)$.

Since $\{x_n\}$ be a Cauchy sequence in (X, M_p^*), then there exists $\alpha \in \mathbb{R}$ and for every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $\left|M_p^*(x_n, x_n, x_m) - \alpha\right| < \frac{\epsilon}{4R}$ for all $n, m \geq n_\epsilon$. Hence

$$\begin{aligned}
M_{p^s}^*(x_n, x_n, x_m) &\leq \big|2RM_p^*(x_n, x_n, x_m) - M_p^*(x_n, x_n, x_n) \\
&\quad -M_p^*(x_m, x_m, x_m) + 2\alpha - 2\alpha\big| \\
&\leq \big|2RM_p^*(x_n, x_n, x_m) - 2\alpha\big| + \big|M_p^*(x_n, x_n, x_n) - \alpha\big| \\
&\quad + \big|M_p^*(x_m, x_m, x_m) - \alpha\big| \leq 4R\frac{\epsilon}{4R} = \epsilon,
\end{aligned}$$

for all $n, m \geq n_\epsilon$. Thus $\{x_n\}$ is a Cauchy sequence in $(X, M_{p^s}^*)$.

Now, we prove that completeness of $(X, M_{p^s}^*)$ implies completeness of (X, M_p^*).

Indeed, if $\{x_n\}$ be a Cauchy sequence in (X, M_p^*) then it is $\{x_n\}$ be a Cauchy sequence in $(X, M_{p^s}^*)$. Since the M^*-metric space $(X, M_{p^s}^*)$ is complete we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} M_{p^s}^*(x_n, x_n, y) = 0$. Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \sup \big|M_p^*(x_n, x_n, y) - M_p^*(y, y, y)\big| \\
&\leq \lim_{n\to\infty} \big|2RM_p^*(x_n, x_n, y) - M_p^*(x_n, x_n, x_n) - M_p^*(y, y, y)\big| = 0.
\end{aligned}$$

Hence, we follow that $\{x_n\}$ is a convergent sequence in (X, M_p^*). That is meaning

$$\lim_{n\to\infty} M_p^*(x_n, x_n, y) = M_p^*(y, y, y).$$

Now, we prove that every Cauchy sequence $\{x_n\}$ in $(X, M_{p^s}^*)$ is a Cauchy sequence in (X, M_p^*). Let $\epsilon = \frac{1}{2R}$, then there exists $n_0 \in \mathbb{N}$ such that $M_{p^s}^*(x_n, x_n, x_m) < \frac{1}{2R}$ for all $n, m \geq n_0$. Since

$$\begin{aligned}
M_p^*(x_n, x_n, x_n) &\leq 4RM_p^*(x_{n_0}, x_{n_0}, x_n) - 3M_p^*(x_n, x_n, x_n) \\
&\quad - M_p^*(x_{n_0}, x_{n_0}, x_{n_0}) + M_p^*(x_n, x_n, x_n) \\
&\leq 2RM_{p^s}^*(x_n, x_n, x_{n_0}) + M_p^*(x_{n_0}, x_{n_0}, x_{n_0}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
M_p^*(x_n, x_n, x_n) &\leq 2RM_{p^s}^*(x_n, x_n, x_{n_0}) + M_p^*(x_{n_0}, x_{n_0}, x_{n_0}) \\
&\leq 1 + M_p^*(x_{n_0}, x_{n_0}, x_{n_0}).
\end{aligned}$$

Consequently the sequence $\{M_p^*(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$ and so there exists an $a \in \mathbb{R}$ such that a sub sequence $\{M_p^*(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to a, i.e. $\lim_{k\to\infty} M_p^*(x_{n_k}, x_{n_k}, x_{n_k}) = 0$.

It remains to prove that $\{M_p^*(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in $(X, M_{p^s}^*)$, for $\epsilon > 0$, there exists n_ϵ such that $M_{p^s}^*(x_n, x_n, x_m) < \frac{\epsilon}{2R}$ for all $n, m \geq n_\epsilon$. Hence, for all $n, m \geq n_\epsilon$,

$$\begin{aligned}&\left|M_p^*(x_n, x_n, x_n) - M_p^*(x_m, x_m, x_m)\right| \leq 4RM_p^*(x_n, x_n, x_m) - 3M_p^*(x_n, x_n, x_n)\\&- M_p^*(x_m, x_m, x_m) + M_p^*(x_n, x_n, x_n) - M_p^*(x_m, x_m, x_m)\\&\leq 2RM_{p^s}^*(x_n, x_n, x_m) < \epsilon.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|M_p^*(x_n, x_n, x_n) - a\right| &\leq \left|M_p^*(x_n, x_n, x_m) - M_p^*(x_{n_k}, x_{n_k}, x_{n_k})\right|\\&\quad + \left|M_p^*(x_{n_k}, x_{n_k}, x_{n_k}) - a\right| < \epsilon + \epsilon = 2\epsilon,\end{aligned}$$

for all $n, n_k \geq n_\epsilon$. Hence $\lim_{n\to\infty} M_p^*(x_n, x_n, x_n) = a$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X, M_p^*). We have

$$\begin{aligned}&\left|2RM_p^*(x_n, x_n, x_m) - 2a\right|\\&= \left|RM_{p^s}^*(x_n, x_n, x_m) + M_p^*(x_n, x_n, x_n) - a + M_p^*(x_m, x_m, x_m) - a\right|\\&\leq RM_{p^s}^*(x_n, x_n, x_m) + \left|M_p^*(x_n, x_n, x_n) - a\right| + \left|M_p^*(x_m, x_m, x_m) - a\right|\\&< \frac{\epsilon}{2R} + 2\epsilon + 2\epsilon = (\frac{1}{2R} + 4)\epsilon.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X, M_p^*).

We shall have established the lemma if we prove that $(X, M_{p^s}^*)$ is complete if so is (X, M_p^*). Let $\{x_n\}$ be a Cauchy sequence in $(X, M_{p^s}^*)$. Then $\{x_n\}$ is a Cauchy sequence in (X, M_p^*) and so it is convergent to point $y \in X$ with

$$\lim_{n,m\to\infty} M_p^*(x_n, x_n, x_m) = \lim_{m\to\infty} M_p^*(y, y, x_m) = M_p^*(y, y, y).$$

Thus, for $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$\left|M_p^*(y, y, x_n) - M_p^*(y, y, y)\right| < \frac{\epsilon}{2R}$$

and

$$\left|M_p^*(y, y, y) - M_p^*(x_n, x_n, x_n)\right| < \frac{\epsilon}{2R}$$

whenever $n \geq n_\epsilon$. As a consequence, we have

$$\begin{aligned}M_{p^s}^*(y, y, x_n) &= 2RM_p^*(y, y, x_n) - M_p^*(x_n, x_n, x_n) - M_p^*(y, y, y)\\&\leq \left|RM_p^*(y, y, x_n) - M_p^*(y, y, y)\right| + \left|RM_p^*(y, y, x_n) - M_p^*(x_n, x_n, x_n)\right|\\&< R\frac{\epsilon}{2R} + R\frac{\epsilon}{2R} = \epsilon,\end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore $(X, M^*_{p^s})$ is complete.

Finally, it is easy to check that $\lim_{n\to\infty} M^*_{p^s}(a, a, x_n) = 0$ if and only if

$$M^*_p(a, a, a) = \lim_{n\to\infty} M^*_p(a, a, x_n) = \lim_{n,m\to\infty} M^*_p(x_n, x_n, x_m). \qquad \square$$

Definition 5. *Let (X, M^*_p) be a partial M^*-metric space, then M^*_p is said to first type if*

$$M^*_p(x, x, y) \leq M^*_p(x, y, z),$$

for all $x, y, z \in X$.

3. Fixed point result

We begin this section giving the concept of weakly increasing mappings.

Definition 6 ([39]). *Let $(X, \preceq)$ be a partially ordered set. Two mappings $S, T : X \to X$ are said to be $S - T$ weakly increasing if $Sx \preceq TSx$ for all $x \in X$.*

Remark 3 ([39]). (i) Two weakly increasing mappings need not be nondecreasing. for examples see [4].

(ii) $\mathcal{F}$ denote the set of all functions $F : [0, \infty) \to [0, \infty)$ such that F is nondecreasing and continuous, $F(0) = 0 < F(t)$, for every $t > 0$ and $F(x+y) \leq F(x) + F(y)$ for all $x, y \in [0, +\infty)$.

(iii) Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ where ψ is continuous, nondecreasing function such that $\sum_{n=0}^{\infty} \psi^n(t)$ is convergent for each $t > 0$. From the conditions on ψ, it is clear that $\lim_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$, for every $t > 0$.

Now, we begin the our main results is as follows:

Theorem 8. *Let $(X, \preceq)$ be a partially ordered set and suppose that the partial M^*-metric space M^*_p is a first type on X and (X, M^*_p) is a complete partial M^*-metric space. Let $S, T, G : X \to X$ be three self-mappings such that $S - T$, $T - G$ and $G - S$ are weakly increasing mappings such that*

$$F(M^*_p(Sx, Ty, Gz)) \leq \frac{1}{R}\psi(RF(\varphi(x, y, z)) \tag{3.1}$$

for all $x, y, z \in X$ with x, y, z are comparable with respect to partially order $\preceq$, where $F \in \mathcal{F}$, $\psi \in \Psi$ and

$$\varphi(x, y, z) = \max \left\{ \begin{array}{c} M^*_p(x, y, z), M^*_p(x, x, Sx), \\ M^*_p(y, y, Ty), M^*_p(z, z, Gz) \end{array} \right\}. \tag{3.2}$$

Further assume that if, for every increasing sequence $\{x_n\}$ convergent to $x \in X$, we have $x_n \preceq x$. Then S, T and G have a common fixed point.

Proof. Let x_0 be arbitrary point of X. We can define a sequence in X as follows $x_{3n+1} = Sx_{3n}$, $x_{3n+2} = Tx_{3n+1}$ and $x_{3n+3} = Gx_{3n+2}$ for $n = 0, 1, 2, ...$

Since $S - T$, $T - G$ and $G - S$ are weakly increasing mappings, we have $x_1 = Sx_0 \preceq TSx_0 = x_2 = Tx_1 \preceq GTx_1 = x_3 = Gx_2 \preceq SGx_2 = x_4$ and continuing this process, we have $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$

Case 1. Suppose there exists $n_0 \in \mathbb{N}$ such that $M_p^*(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}) = 0$. Now, we show that $M_p^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3}) = 0$. Otherwise, from (3.1), we get

$$\begin{aligned}
F(M_p^*(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3})) &\leq F(M_p^*(x_{3n_0+1}, x_{3n_0+2}, x_{3n_0+3})) \\
&= F(M_p^*(Sx_{3n_0}, Tx_{3n_0+1}, Gx_{3n_0+2})) \\
&\leq \frac{1}{R}\psi(RF(\varphi(x_{3n_0}, x_{3n_0+1}, x_{3n_0+2}))) \\
&= \frac{1}{R}\psi(RF(\varphi(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3}))) \\
&< F(x_{3n_0+2}, x_{3n_0+2}, x_{3n_0+3}),
\end{aligned}$$

which is a contradiction. Hence $M_p^*(x_{3n_0}, x_{3n_0+1}, x_{3n_0+1}) = 0$. Therefore, $x_{3n_0} = x_{3n_0+1} = x_{3n_0+2} = x_{3n_0+3}$. Thus, $Sx_{3n_0} = Tx_{3n_0} = Gx_{3n_0} = x_{3n_0}$. That is x_{3n_0} is a common fixed point of S, T and G.

Case 2: Assume $M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}) > 0$ for all $n \in \mathbb{N}$. Now, we want to prove

$$F(M_p^*(x_{n-1}, x_n, x_{n+1})) \leq \psi(F(M_p^*(x_{n-2}, x_{n-1}, x_n))). \tag{3.3}$$

Setting $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$ in (3.2), we have

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) = \max\left\{\begin{array}{c} M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}), \\ M_p^*(x_{3n}, x_{3n}, x_{3n+1}), \\ M_p^*(x_{3n}, x_{3n}, x_{3n+2}), \\ M_p^*(x_{3n+2}, x_{3n+2}, x_{3n+3}) \end{array}\right\}.$$

Since M_p^* is of the first type, we get

$$\varphi(x_{3n}, x_{3n+1}, x_{3n+2}) \leq \max\left\{M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}), M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})\right\}.$$

If $M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})$ is maximum in the R.H.S. of the above inequuality, we have from (3.1) that

$$\begin{aligned}
F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) &\leq F(M_p^*(Sx_{3n}, Tx_{3n+1}, Gx_{3n+2})) \\
&< \frac{1}{R}\psi(RF(\varphi(x_{3n}, x_{3n+1}, x_{3n+2}))) \\
&\leq \frac{1}{R}\psi(RF(\max\left\{M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}),\right. \\
&\quad \left. M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})\right\})) \\
&= \frac{1}{R}\psi(RF(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))) \\
&< F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})),
\end{aligned}$$

which is a contradiction. Thus,

$$F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3})) \leq \psi(F(M_p^*(x_{3n}, x_{3n+1}, x_{3n+2}))).$$

Similarly, we have

$$F(M_p^*(x_{3n+2}, x_{3n+3}, x_{3n+4})) \leq \psi(F(M_p^*(x_{3n+1}, x_{3n+2}, x_{3n+3}))),$$

and

$$F(M_p^*(x_{3n}, x_{3n+1}, x_{3n+2})) \leq \psi(F(M_p^*(x_{3n-1}, x_{3n}, x_{3n+1}))).$$

Therefore, for every $n \in \mathbb{N}$, we have

$$F(M_p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi(F(M_p^*(x_{n-1}, x_n, x_{n+1}))).$$

Now, we have $F(M_p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi(F(M_p^*(x_{n-1}, x_n, x_{n+1}))) \leq \cdots \leq \psi^n(F(M_p^*(x_0, x_1, x_2)))$.

Hence

$$\lim_{n\to\infty} F(M_p^*(x_n, x_{n+1}, x_{n+2})) = 0,$$

so that

$$\lim_{n\to\infty} M_p^*(x_n, x_{n+1}, x_{n+2}) = 0. \tag{3.4}$$

Since M_p^* is first type and F is non-decreasing, we have

$$F(M_p^*(x_n, x_n, x_{n+1})) \leq F(M_p^*(x_n, x_{n+1}, x_{n+2})) \leq \psi^n(F(M_p^*(x_0, x_1, x_2))).$$

Since $F(x, y) \leq F(x) + F(y)$ and $M_{p^s}^*(x_n, x_n, x_{n+1}) \leq 2RM_p^*(x_n, x_n, x_{n+1})$, we have

$$F(M_{p^s}^*(x_n, x_n, x_{n+1})) \leq 2RF(M_p^*(x_n, x_n, x_{n+1})) \leq 2R\psi^n(F(M_p^*(x_0, x_1, x_2))).$$

Now, from

$$\begin{aligned} M_{p^s}^*(x_{n+k}, x_n, x_n) \leq{}& RM_{p^s}^*(x_{n+k}, x_{n+k-1}, x_{n+k-1}) \\ &+ RM_{p^s}^*(x_{n+k-1}, x_{n+k-2}, x_{n+k-2}) + \cdots + M_{p^s}^*(x_{n+1}, x_n, x_n), \end{aligned}$$

we have

$$\begin{aligned} F(M_{p^s}^*(x_{n+k}, x_n, x_n)) &\leq F(RM_{p^s}^*(x_{n+k}, x_{n+k-1}, x_{n+k-1})) \\ &\quad + \cdots + F(M_{p^s}^*(x_{n+1}, x_n, x_n)) \\ &\leq 2R^2\psi^{n+k-1}(M_p^*(x_0, x_1, x_2)) \\ &+ \cdots + 2R^2\psi^n(M_p^*(x_0, x_1, x_2)) \\ &\leq 2R^2\sum_{i=n}^{\infty}\psi^i(M_p^*(x_0, x_1, x_2)). \end{aligned}$$

Since $\sum_{n=0}^{\infty}\psi^n(t)$ is convergent for each $t > 0$ it follows that $\{x_n\}$ is a Cauchy sequence in the M^*-metric space $(X, M^*_{p^s})$. Since (X, M^*_p) is complete, then from Lemma 2.7 follows that the sequence $\{x_n\}$ converges to some x in the M^*-metric space $(X, M^*_{p^s})$. Hence $\lim_{n\to\infty} M^*_{p^s}(x_n, x, x) = 0$. Again, from Lemma 2.7, we have

$$M^*_p(x, x, x) = \lim_{n\to\infty} M^*_p(x, x, x_n) = \lim_{n,m\to\infty} M^*_p(x_n, x_m, x_m). \tag{3.5}$$

Since $\{x_n\}$ is a Cauchy sequence in the M^*-metric space $(X, M^*_{p^s})$ and

$$M^*_{p^s}(x_n, x_m, x_m) = 2RM^*_p(x_n, x_m, x_m) - M^*_p(x_n, x_n, x_n) - M^*_p(x_m, x_m, x_m),$$

we have

$$\lim_{n,m\to\infty} M^*_{p^s}(x_n, x_m, x_m) = 0$$

and by (3.4), we have

$$\lim_{n\to\infty} M^*_p(x_n, x_n, x_n) = 0,$$

thus by definition $M^*_{p^s}$, we have

$$\lim_{n,m\to\infty} M^*_p(x_n, x_m, x_m) = 0.$$

Therefore, by (3.5), we have

$$\begin{aligned} M^*_p(x, x, x) &= \lim_{n\to\infty} M^*_p(x_n, x, x) \\ &= \lim_{n,m\to\infty} M^*_p(x_n, x_m, x_m) = 0. \end{aligned}$$

Now, by the inequality (3.1) for $x = x$, $y = x_{3n+1}$ and $z = x_{3n+2}$, then we have

$$F(M^*_p(Sx, x_{3n+2}, x_{3n+3})) \leq \frac{1}{R}\psi(RF(\varphi(x, x_{3n+1}, x_{3n+2}))),$$

and by letting $n \to \infty$ and using Lemma 2.5, we obtain

$$F(M^*_p(Sx, x, x)) \leq \frac{1}{R}\psi(RF(M^*_p(Sx, x, x))) < F(M^*_p(Sx, x, x)),$$

which is a contradiction. Hence, $M^*_p(Sx, x, x) = 0$. Thus $Sx = x$. Similarly, by using the inequality (3.1) for $y = x$, $x = x_{3n}$ and $z = x_{3n+2}$, then we have

$$F(M^*_p(x_{3n}, Tx, x_{3n+3})) \leq \frac{1}{R}\psi(RF(\varphi(x_{3n}, x, x_{3n+2}))),$$

and letting $n \to \infty$ and using Lemma 2.5, we obtain

$$F(M^*_p(x, Tx, x)) \leq \frac{1}{R}\psi(RF(M^*_p(x, Tx, x)) < F(M^*_p(x, Tx, x)),$$

which is a contradiction. Hence, $Tx = x$. Similarly, by using the inequality (3.1) for $z = x$, $x = x_{3n}$ and $y = x_{3n+1}$, we can show that $Gx = x$. □

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Certain classes of meromorphic functions by using the linear operator

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Abstract. In this paper, we introduce a new certain differential operator $A_\lambda^n f(z)$ with subclass $S_p^*(\alpha, \lambda, n, \beta)$ for functions of the form $f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^k$. For functions in $S_p^*(\alpha, \lambda, n, \beta)$, we give coefficient inequalities, distortion theorem, radii of starlikeness and convexity.

Keywords: analytic functions, meromorphic functions, starlike, convex.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. As usual, we denote by S the subclass of $\mathcal{A}$, consisting of functions which are also univalent in $\mathbb{U}$. We recall here the definitions of the well-known classes of starlike functions and convex functions:

$$S^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\} \quad (z \in \mathbb{U}),$$

$$S^c = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\} \quad (z \in \mathbb{U})$$

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Later Acu and Owa [2] studied the classes extensively. The class S_w^* is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class S_w^c is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by Amourah in [3] and [4] for starlike and convex functions.

Let S denoted the subclass of $\mathcal{A}(p)$ consisting of the function of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p-1} z^{k+p-1}, \tag{2}$$
$$(a_{k+p-1} > 0,\ z \in \mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}).$$

The function $f(z)$ in S is said to be starlike functions of order α if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}^*), \tag{3}$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Similarly, a function f in S is said to be convex of order α if and only if

$$\operatorname{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}^*), \tag{4}$$

for some $\alpha (0 \leq \alpha < 1)$. We denote by $VR(\alpha)$ the class of all convex functions of order α. We note that the class $S^*(\alpha)$ and various other subclasses have been studied rather extensively by Nehari and Netanyahu [5], Acu and Owa [2], Amourah ([6],[7],[10],[11],[13]), Aouf [12], Miller [8] and Royster [9].

For the function $f \in \mathcal{A}(p)$, the definition of linear operator $A_\lambda^n f(z)$ introduced by [1] to define the linear operator $A_\lambda^n f(z)$ as the following:

$$A_\lambda^0 f(z) = f(z),$$
$$A_\lambda^1 f(z) = (1 + p\lambda)\, A_\lambda^0 f(z) + \lambda z \left(A_\lambda^0 f(z)\right)',$$

and for $n = 1, 2, 3, \cdots$

$$A_\lambda^n f(z) = A(A_\lambda^{n-1} f(z)), \tag{5}$$
$$= \frac{1}{z^p} + \sum_{k=1}^{\infty} [1 + 2p\lambda + k\lambda - \lambda]^n\, a_{k+p-1} z^{k+p-1}, \tag{6}$$

for $\lambda \geq 0$, $z \in \mathbb{U}^*$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Then, we can observe easily that for

$$\lambda z \left(A_\lambda^n f(z)\right)' = A_\lambda^{n+1} f(z) - (1 + p\lambda) A_\lambda^n f(z), \quad (p \in \mathbb{N},\ n \in \mathbb{N}_0)$$

Definition 1.1. *A function* $f(z) \in S$ *is said to be in* $S_p(\alpha, \lambda, n, \beta)$ *if and only if*

$$\left|\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)} + \alpha + \alpha\beta\right| \leq \operatorname{Re}\left\{-\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}\right\} + \alpha - \alpha\beta, \tag{7}$$

for some $0 \leq \beta < 1$, $\alpha \geq \frac{1}{2+\beta}$, $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$ *and for all* $z \in \mathbb{U}^*$.

Let $\mathcal{A}^*(p)$ denote the subclass of $\mathcal{A}(p)$ consisting of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^k, \qquad (a_k \geq 0). \tag{8}$$

Further, we define the class $S_p(\alpha, \lambda, n, \beta)$ by

$$S_p^*(\alpha, \lambda, n, \beta) = S_p(\alpha, \lambda, n, \beta) \cap \mathcal{A}^*(p). \tag{9}$$

In this paper, coefficient inequalities, growth and distortion theorem, radii of starlikeness and convexity.

2. Coefficient inequalities

In this section, the result provides a sufficient condition for a function, regular in $\mathbb{U}^*$, to be in $S_p^*(\alpha, \lambda, n, \beta)$.

Theorem 2.1. *Let the function* $f(z)$ *be given by* (8). *If*

$$\sum_{k=1}^{\infty} [p(\alpha\beta + 1) + k - 1] \gamma_n a_{k+p-1} \leq p(1 - \alpha\beta), \quad (z \in \mathbb{U}^*) \tag{10}$$

where $\gamma_n = (1 + 2p\lambda + k\lambda - \lambda)^n$, $0 \leq \beta < 1, \alpha \geq \frac{1}{2+\beta}$, $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$.

Proof. Suppose that $f \in S_p^*(\alpha, \lambda, n, \beta)$. Then, by the inequality (7), we get that

$$\left|\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)} + \alpha + \alpha\beta\right| \leq \operatorname{Re}\left\{-\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}\right\} + \alpha - \alpha\beta.$$

That is,

$$\operatorname{Re}\left\{\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)} + \alpha + \alpha\beta\right\} \leq \left|\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)} + \alpha + \alpha\beta\right|$$
$$\leq \operatorname{Re}\left\{-\frac{z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)}\right\} + \alpha - \alpha\beta.$$

That is,

$$\operatorname{Re}\left\{\frac{2z(A_\lambda^n f(z))'}{pA_\lambda^n f(z)} 2\alpha\beta\right\} \leq 0.$$

Hence, by the inequalities (7) and (8)

$$\operatorname{Re}\left\{\frac{-2p(1-\alpha\beta)+\sum_{k=1}^{\infty}2\left[p(\alpha\beta+1)+k-1\right]\gamma_n a_{k+p-1}z^{k+2p-1}}{p+\sum_{k=1}^{\infty}p\gamma_n a_{k+p-1}z^{k+2p-1}}\right\}\leq 0. \tag{11}$$

Taking z to be real and putting $z\to 1^{-}$ through real values, then the inequality (11) yields

$$\frac{-2p(1-\alpha\beta)+\sum_{k=1}^{\infty}2\left[p(\alpha\beta+1)+k-1\right]\gamma_n a_{k+p-1}}{p+\sum_{k=1}^{\infty}p\gamma_n a_{k+p-1}}\leq 0$$

Hence,

$$\sum_{k=1}^{\infty}\left[p(\alpha\beta+1)+k-1\right]\gamma_n a_{k+p-1}\leq p(1-\alpha\beta)$$

This completes the proof of Theorem 2.1. □

Corollary 2.1. *Let the function* $f(z)$ *be defined by* (8). *If* $f\in S_p^*(\alpha,\lambda,n,\beta)$, *then*

$$a_{k+p-1}\leq\frac{p(1-\alpha\beta)}{\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n},\quad (k\in\mathbb{N}). \tag{12}$$

The result (12) *is sharp for functions of the form:*

$$f(z)=\frac{1}{z^p}+\frac{p(1-\alpha\beta)}{\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n}z^{k+p-1},\ (k\in\mathbb{N}). \tag{13}$$

where $0\leq\beta<1,\alpha\geq\frac{1}{2+\beta}$, $p\in\mathbb{N}$ *and* $n\in\mathbb{N}_0$.

Proof. Since $f\in S_p^*(\alpha,\lambda,n,\beta)$, then from Theorem 2.1 above, we get that

$$\sum_{k=1}^{\infty}\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n a_{k+p-1}\leq p(1-\alpha\beta).$$

Next, note that

$$\begin{aligned}&\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n a_{k+p-1}\\ &\leq\sum_{k=1}^{\infty}\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n a_{k+p-1}\leq p(1-\alpha\beta).\end{aligned}$$

Hence,

$$a_{k+p-1}\leq\frac{p(1-\alpha\beta)}{\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n}$$

Thus, the equality (12) is attained for the function f given by

$$f(z)=\frac{1}{z^p}+\frac{p(1-\alpha\beta)}{\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n}z^{k+p-1}.$$ □

3. Growth and distortion theorem

In this section we will prove the following growth and distortion theorems for the class $S_p^*(\alpha,\lambda,n,\beta)$.

Theorem 3.1. *Let the function $f(z)$ given by (8) be in the class $S_p^*(\alpha,\lambda,n,\beta)$, where $0 \leq \beta < 1, \alpha \geq \frac{1}{2+\beta}$, $p \in \mathbb{N}$, $p > m$, $0 < |z| = r < 1$ and $n \in \mathbb{N}_0$. Then, we have*

$$\begin{aligned}
&\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}.\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)} \\
&\leq \left|f^{(m)}(z)\right| \\
&\leq \left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}.\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)}.
\end{aligned} \tag{14}$$

The result is sharp for the function f given by

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n} z^{k+p-1}. \tag{15}$$

Proof. Since $f \in S_p^*(\alpha,\lambda,n,\beta)$, from Theorem 2.1 readily yields the inequality

$$\frac{(1+\alpha\beta)(1+2p\lambda)^n}{(p-1)!}\sum_{k=1}^{\infty}(k+p-1)!a_{k+p-1}, \tag{16}$$

$$\leq [p(\alpha\beta+1)+k-1](1+2p\lambda+k\lambda-\lambda)^n a_{k+p-1} \leq p(1-\alpha\beta), \tag{17}$$

that is,

$$\begin{aligned}
\sum_{k=1}^{\infty}(k+p-1)!a_{k+p-1} &\leq \frac{p(1-\alpha\beta)(p-1)!}{(1+\alpha\beta)(1+2p\lambda)^n} \\
&= \frac{(1-\alpha\beta)p!}{(1+\alpha\beta)(1+2p\lambda)^n}.
\end{aligned} \tag{18}$$

By differentiating the function f in the form m times with respect to z, we get that

$$\begin{aligned}
f^{(m)}(z) &= (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} \\
&+ \sum_{k=1}^{\infty} \frac{(k+p-1)!}{(k+p-m-1)!} a_{k+p-1} z^{k+p-m-1}.
\end{aligned} \tag{19}$$

From (18) and (19), we get that

$$\begin{aligned}
\left|f^{(m)}(z)\right| &\leq \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)}+\sum_{k=1}^{\infty}\frac{(k+p-1)!}{(k+p-m-1)!}a_{k+p-1}r^{k+p-m-1} \\
&\leq \left\{\frac{(p+m-1)!}{(p-1)!}+\sum_{k=1}^{\infty}\frac{(k+p-1)!}{(2p-m-1)!}a_{k+p-1}r^{3p-1}\right\}r^{-(p+m)} \qquad (20)\\
&\leq \left\{\frac{(p+m-1)!}{(p-1)!}+\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)}, \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}
\left|f^{(m)}(z)\right| &\geq \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)} \\
&\quad -\sum_{k=1}^{\infty}\frac{(k+p-1)!}{(k+p-m-1)!}a_{k+p-1}r^{k+p-m-1} \qquad (22)\\
&\geq \left\{\frac{(p+m-1)!}{(p-1)!}-\sum_{k=1}^{\infty}\frac{(k+p-1)!}{(2p-m-1)!}a_{k+p-1}r^{3p-1}\right\}r^{-(p+m)} \\
&\geq \left\{\frac{(p+m-1)!}{(p-1)!}-\frac{(1-\alpha\beta)}{(1+\alpha\beta)(1+2p\lambda)^n}\frac{p!}{(2p-m-1)!}r^{3p-1}\right\}r^{-(p+m)}.
\end{aligned}$$

We can easily prove that the bounds of (14) are attained for the function f given by the form (15).

This completes the proof of Theorem 3.1. □

4. Radii of starlikeness and convexity

The radii of starlikeness and convexity for the class $S_p^*(\alpha,\lambda,n,\beta)$ is given by the following theorems.

Theorem 4.1. *If the function $f(z)$ given by (8) is in the class $S_p^*(\alpha,\lambda,n,\beta)$, where $0<\beta\leq 1$ and $n\in\mathbb{N}_0$, then $f(z)$ is starlike of order $\mu(\ 0\leq\mu<p)$ in $|z|<r_1$, that is*

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}>\mu, \qquad (23)$$

where

$$r_1=\inf_{k\geq 1}\left\{\frac{(p-\mu)\left[p(\alpha\beta+1)+k-1\right](1+2p\lambda+k\lambda-\lambda)^n}{p(k+2\mu-1)(1-\alpha\beta)}\right\}^{\frac{1}{k+2p-1}}. \qquad (24)$$

Proof. It suffices to prove that

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right| = \left|\frac{\sum_{k=1}^{\infty}\left(k+2p-1\right)a_{k+p-1}z^{k+2p-1}}{2(p-\mu)-\sum_{k=1}^{\infty}\left(k+2\mu-1\right)a_{k+p-1}z^{k+2p-1}}\right| \tag{25}$$

$$\leq \frac{\sum_{k=1}^{\infty}\left(k+2p-1\right)a_{k+p-1}\left|z\right|^{k+2p-1}}{2(p-\mu)-\sum_{k=1}^{\infty}\left(k+2\mu-1\right)a_{k+p-1}\left|z\right|^{k+2p-1}}.$$

Then, the following

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\mu}\right| \leq 1, \ \ (0\leq\mu<p, \ p\in\mathbb{N}) \tag{26}$$

will hold if

$$\sum_{k=1}^{\infty}\frac{k+2\mu-1}{p-\mu}a_{k+p-1}\left|z\right|^{k+2p-1}\leq 1. \tag{27}$$

Then, by Corollary 2.1 the inequality (27) will be true if

$$\frac{k+2\mu-1}{(p-\mu)}\left|z\right|^{k+2p-1}\leq\frac{\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^{n}}{p\left(1-\alpha\beta\right)}$$

that is,

$$\left|z\right|^{k+2p-1}\leq\frac{(p-\mu)\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^{n}}{p\left(k+2\mu-1\right)\left(1-\alpha\beta\right)}. \tag{28}$$

This completes the proof of Theorem 4.1. □

Theorem 4.2. *If the function $f(z)$ given by (8) is in the class $S_p^*\left(\alpha,\lambda,n,\beta\right)$, where $0<\beta\leq 1$ and $n\in\mathbb{N}_0$, then $f(z)$ is convex of order μ($0\leq\mu<p$) in $|z|<r_2$, that is,*

$$\operatorname{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\}>\mu,$$

where

$$r_2=\inf_{k\geq 1}\left\{\frac{(p-\mu)\left[p\left(\alpha\beta+1\right)+k-1\right]\left(1+2p\lambda+k\lambda-\lambda\right)^{n}}{(k+\mu-1)(k+2\mu-1)\left(1-\alpha\beta\right)}\right\}^{\frac{1}{k+2p-1}},$$

(29) $\quad (k\geq 1)$.

Proof. By using the same technique employed in the proof of Theorem 4.1, we can show that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{\frac{zf''(z)}{f'(z)}-p+2\mu}\right| = \left|\frac{\sum_{k=1}^{\infty}(k+p-1)(k+2p-1)a_{k+p-1}z^{k+2p-1}}{2p(p-\mu)z^{-p}-\sum_{k=1}^{\infty}(k+p-1)(k+2\mu-1)a_{k+p-1}z^{k+2p-1}}\right|$$

$$\leq\frac{\sum_{k=1}^{\infty}(k+p-1)(k+2p-1)a_{k+p-1}\left|z\right|^{k+2p-1}}{2p(p-\mu)-\sum_{k=1}^{\infty}(k+p-1)(k+2\mu-1)a_{k+p-1}\left|z\right|^{k+2p-1}}.$$

Then, the following

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} + p}{\frac{zf''(z)}{f'(z)} - p + 2\mu} \right| \leq 1 \tag{30}$$

will hold if

$$\sum_{k=1}^{\infty} \frac{(k+\mu-1)(k+2\mu-1)}{p(p-\mu)} a_{k+p-1} |z|^{k+2p-1} \leq 1. \tag{31}$$

Then, by Corollary 2.1 the inequality (31) will be true if

$$\frac{(k+\mu-1)(k+2\mu-1)}{p(p-\mu)} |z|^{k+2p-1} \leq \frac{[p(\alpha\beta+1)+k-1](1+2p\lambda+k\lambda-\lambda)^n}{p(1-\alpha\beta)}$$

that is,

$$|z|^{k+2p-1} \leq \frac{(p-\mu)[p(\alpha\beta+1)+k-1](1+2p\lambda+k\lambda-\lambda)^n}{(k+\mu-1)(k+2\mu-1)(1-\alpha\beta)}. \tag{32}$$

Therefore, the inequality (32) leads us to the disk $|z| < r_2$, where r_2 is given by the form (29). □

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The total graph of a commutative ring with respect to multiplication

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Abstract. Let R be a commutative ring with $1 \neq 0$, $Z(R)$ be the set of zero-divisors of R, and $Reg(R)$ be the set of regular elements of R. In this paper, we introduce and investigate the *dot total graph of* R and denote by $T_{Z(R)}(\Gamma(R))$. It is the (undirected) simple graph with all elements of R as vertices, and any two distinct vertices $x, y \in R$ are adjacent if and only if $xy \in Z(R)$. The graph $T_{Z(R)}(\Gamma(R))$ is shown to be connected and has a small diameter of at most two. Furthermore, $T_{Z(R)}(\Gamma(R))$ divides into two distinct subsets of R, i.e., $Z(R)$ and $Reg(R)$. Following that, the connectivity, clique number, and girth of the graph $T_{Z(R)}(\Gamma(R))$ were investigated. Finally, the traversability of the graph $T_{Z(R)}(\Gamma(R))$ is investigated.
Keywords: commutative rings, zero-divisor graph, regular elements, zero-divisors.

1. Introduction

Throughout this paper, let R be a commutative ring with unity $1 \neq 0$. In 1988, Beck [10] considered $\Gamma(R)$ as a simple graph, whose vertices are the elements of R and any two different elements x and y are adjacent if and only if $xy = 0$, but he was mainly interested in colorings. In 1993, Anderson and Naseer [6] continued this study by giving a counterexample, where R is a finite local ring.

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In 1999, Anderson and Livingston [3], associated a (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$ and they were interested to study the interplay of ring-theoretic properties of R with graph-theoretic properties of $\Gamma(R)$. In 2008, Anderson and Badawi [4] introduced the total graph of R, denoted by $T(\Gamma(R))$, as the (undirected) graph with all elements of R as vertices and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. In 2012, Abbasi and Habibi [2] introduced and studied the total graph of a commutative ring R with respect to proper ideal I, denoted by $T(\Gamma_I(R))$. In addition, some fundamental graphs with vector spaces can be identified in [7, 8].

Let G be a graph. We say that G is connected if there is a path between any two distinct vertices of G. For distinct vertices x and y of G, we define $d(x, y)$ to be the length of the shortest path from x to y ($d(x, y) = \infty$ if there is no such path). The diameter of G is $diam(G) = sup\{d(x, y) \mid x$ and y are distinct vertices of $G\}$. The girth of G, denoted by $gr(G)$, is defined as the length of the shortest cycle in G ($gr(G) = \infty$ if G contains no cycle). Note that if G contains a cycle, then $gr(G) \leq 2\ diam(G) + 1$. The complement $\overline{G}$ of a graph G is that graph whose vertex set is $V(G)$ and such that for each pair u, v of distinct vertices of G, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G. The degree of vertex v, written $deg_G(v)$ or $deg(v)$, is the number of edges incident to v, (or the degree of the vertex v is the number of vertices adjacent to v). In a connected graph G, a vertex v is said to be a cut-vertex of G if and only if $G \setminus \{v\}$ is disconnected. Let $V(G)$ be a vertex set of G. Then the subset $U \subseteq V(G)$ is called as vertex-cut if $G \setminus U$ is disconnected. The connectivity of a graph G denoted by $k(G)$ and is defined as the cardinality of a minimum vertex-cut of G, also the same concepts we have for the edges. In a connected graph G, an edge e is said to be a bridge of G if and only if $G \setminus \{e\}$ is disconnected. Let $E(G)$ be an edge set of G. Then the subset $X \subseteq E(G)$ is called an edge-cut if $G \setminus X$ is disconnected. The edge-connectivity of a graph G denoted by $\lambda(G)$ and is defined as the cardinality of a minimum edge-cut of G. A complete subgraph of a graph G is called a clique. The clique number denoted by $\omega(G)$, is the greatest integer $n \geqslant 1$ such that $K_n \subseteq G$, and $\omega(G) = \infty$ if $K_n \subseteq G$ for all $n \geqslant 1$. A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree. Also, G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. In addition, let G be a graph of order $n \geq 3$. If $deg(u) + deg(v) \geq n$ for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian. The present paper is organise as follows:

In Section 2, we introduce the definition of the total graph of R with respect to multiplication. We give some examples, and show that $T_{Z(R)}(\Gamma(R))$ is always connected with $diam(T_{Z(R)}(\Gamma(R))) \leqslant 2$ and $gr(T_{Z(R)}(\Gamma(R))) \leqslant 5$, and we establish if the graph $T_{Z(R)}(\Gamma(R))$ is a complete graph or a star graph based on the type of ring and we observe that if R is not trivial then $T_{Z(R)}(\Gamma(R))$ is not null graph. Also, we find the degree of each vertex of $T_{Z(R)}(\Gamma(R))$. Further, in Section 3, we study the connectivity of $\overline{K}_n \vee K_m$ when $T_{Z(R)}(\Gamma(R))$ has no cut-vertex and $T_{Z(R)}(\Gamma(R))$ has a bridge. We also, find the $k(T_{Z(R)}(\Gamma(R)))$. Furthermore, in Section 4, we study the clique number of the graph $\overline{K}_n \vee K_m$. Also, we find the girth of $T_{Z(R)}(\Gamma(R))$ i.e., $gr(\overline{K}_n \vee K_m)$. Finally, in Section 5, we study the traversability of the graph $T_{Z(R)}(\Gamma(R))$ when the graph $T_{Z(R)}(\Gamma(R))$ have an Eulerian trail and $T_{Z(R)}(\Gamma(R))$ is Hamiltonian. Further, we generalized the definition of the graph $T_{Z(R)}(\Gamma(R))$ and denoted by $T_A(\Gamma(B))$. Also, we investigate some properties viz complement graph, spaning subgraph, induced subgraph of $T_A(\Gamma(B))$.

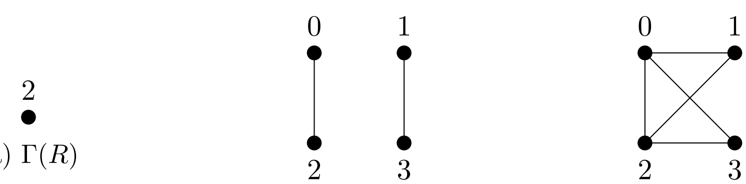
2. Definition and properties of $T_{Z(R)}(\Gamma(R))$

We begin this section by define dot total graph of a commutative ring and denoted by $T_{Z(R)}(\Gamma(R))$. We demonstrate that $T_{Z(R)}(\Gamma(R))$ is always connected and has small diameter which is less than or equal to two and girth which is less than or equal to five. We start with some examples which motivate later results and we associate some examples from zero-divisor graph of a commutative ring, total graph and compare them with $T_{Z(R)}(\Gamma(R))$.

Definition 2.1. *Let R be a commutative ring with $1 \neq 0$ and $Z(R)$ be the set of zero-divisors of R, and $Reg(R)$ be the set of regular elements of R. We define an undirected simple graph $T_{Z(R)}(\Gamma(R))$, whose vertices are all the elements of R and any two distinct vertices x and y of $T_{Z(R)}(\Gamma(R))$ are adjacent if and only if $xy \in Z(R)$.*

Example 2.1. We have several rings with its set of zero-divisor $Z(R)$ and its set of regular elements $Reg(R)$ and comparisons $\Gamma(R)$, $T(\Gamma(R))$ and $T_{Z(R)}(\Gamma(R))$:

(i) $R = \mathbb{Z}_4, Z(R) = \{0, 2\}$ and $Reg(R) = \{1, 3\}$ (see Fig.1)

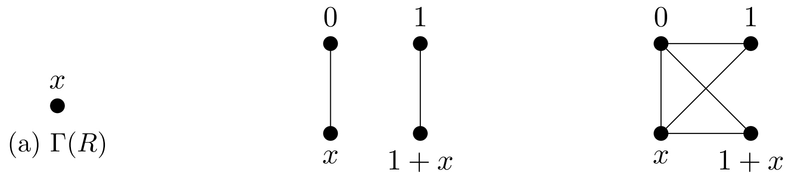


(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 1: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_4$

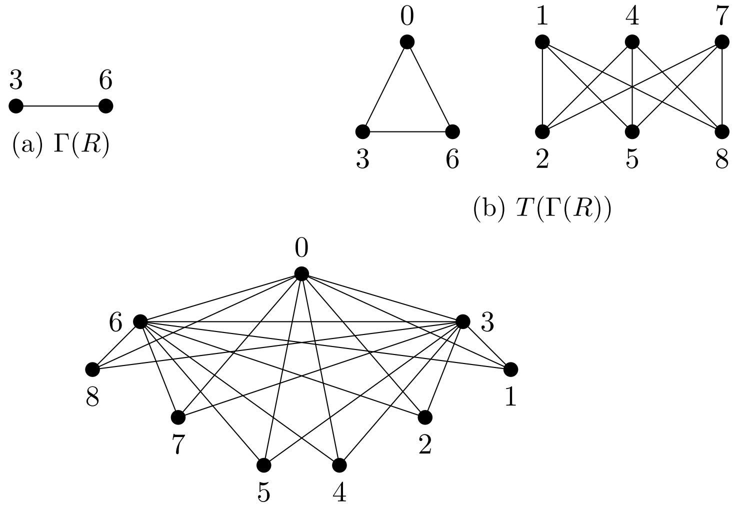
(ii) $R = \mathbb{Z}_2[x]/(x^2) = \{0, 1, x, 1 + x\}, Z(R) = \{0, x\}$ and $Reg(R) = \{1, 1 + x\}$ (see Fig. 2)

(iii) $R = \mathbb{Z}_9, Z(R) = \{0, 3, 6\}$ and $Reg(R) = \{1, 2, 4, 5, 7, 8\}$ (see Fig. 3)



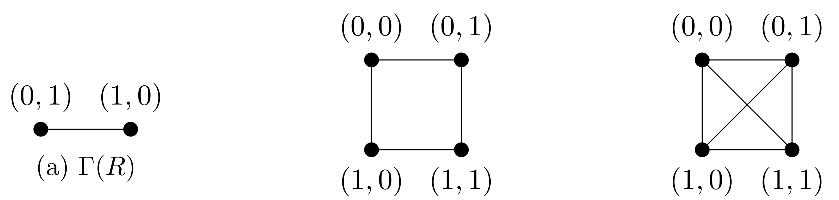
(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 2: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_2[x]/(x^2)$



(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 3: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_9$



(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 4: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_2 \times \mathbb{Z}_2$

(iv) $R = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}, Z(R) = \{(0,0),(0,1),(1,0)\}$ and $Reg(R) = \{(1,1)\}$ (see Fig. 4)

(v) $R = \mathbb{Z}_3[x]/(x^2) = \{0,1,2,x,2x,1+x,2+x,1+2x,2+2x\}$, $Z(R) = \{0,x,2x\}$ and $Reg(R) = \{1,2,1+x,2+x,1+2x,2+2x\}$ (see Fig. 5)

(vi) $R = \mathbb{Z}_6, Z(R) = \{0,2,3,4\}$ and $Reg(R) = \{1,5\}$ (see Fig. 6)

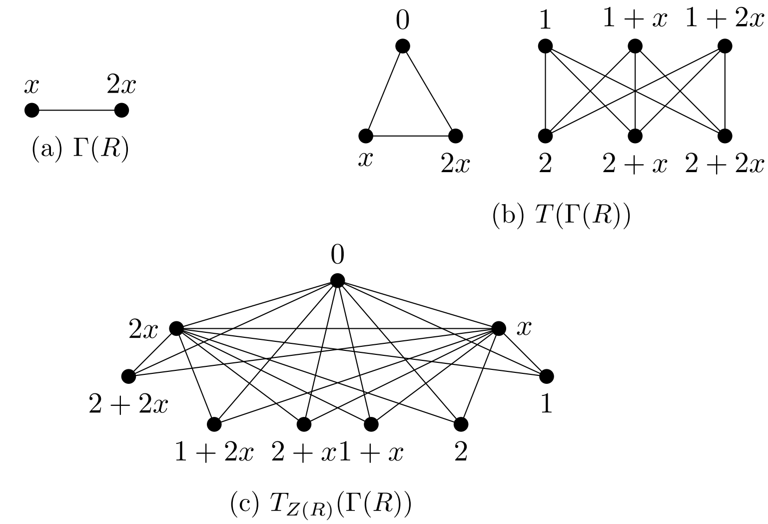


Figure 5: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_3[x]/(x^2)$

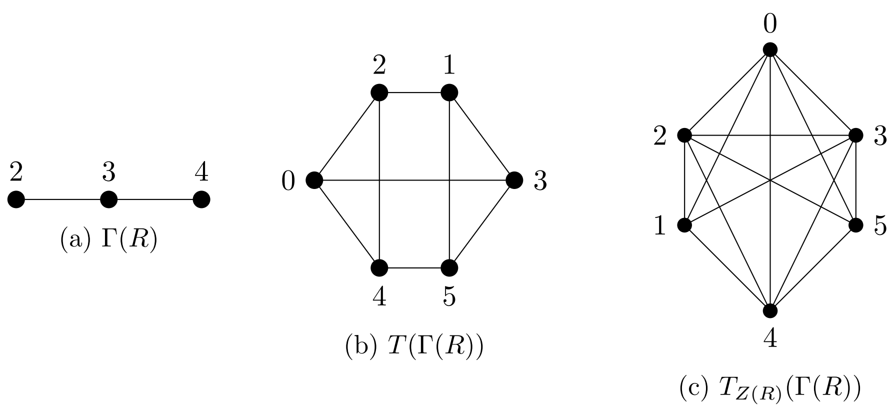


Figure 6: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_6$

(vii) $R = \mathbb{Z}_8, Z(R) = \{0, 2, 4, 6\}$ and $Reg(R) = \{1, 3, 5, 7\}$ (see Fig. 7)

$(viii)$ $R = \mathbb{Z}_7, Z(R) = \{0\}$ and $Reg(R) = \{1, 2, 3, 4, 5, 6\}$ (see Fig 8)

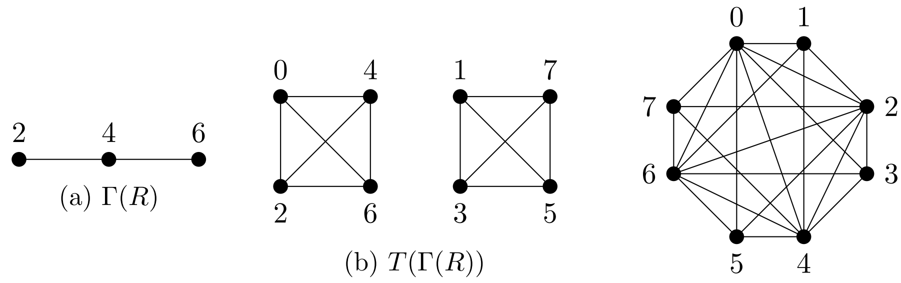
Remark 2.1.

(1) Note that these examples show that non isomorphic rings may have the same zero-divisor graph, but in dot total graph the non isomorphic rings R_1 and R_2 have the following:

   (a) If $|R_1| \neq |R_2|$, then $T_{Z(R_1)}(\Gamma(R_1)) \ncong T_{Z(R_2)}(\Gamma(R_2))$.

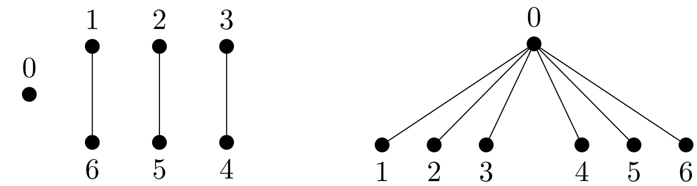
   (b) If $|R_1| = |R_2|$, then they may have the same dot total graph.

(2) For any integral domain R, we know that $\Gamma(R) = \emptyset$ (null graph), but here $T_{Z(R)}(\Gamma(R))$ is complete bipartite graph of the form $K_{1,n}$ is called



(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 7: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_8$



(a) $\Gamma(R)$ (b) $T(\Gamma(R))$ (c) $T_{Z(R)}(\Gamma(R))$

Figure 8: (a)$\Gamma(R)$, (b)$T(\Gamma(R))$ and (c)$T_{Z(R)}(\Gamma(R))$, when $R = \mathbb{Z}_7$

a star graph and $n = |R| - 1$, if R is finite (previous Example $(viii)$) otherwise $n = \infty$, if R is infinite.

(3) Let R be a commutative ring. Then the following statements hold:

- (i) If $x \in Z(R)$, then x is adjacent to each vertex $y \in R$.
- (ii) If $x \in Reg(R)$, then x is adjacent to $y \in Z(R)$, only.
- (iii) Any two distinct verties of $Reg(R)$ are not adjacent in $T_{Z(R)}(\Gamma(R))$.

(4) $T(\Gamma(R))$ may be connected and may not. That is, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected [4], but $T_{Z(R)}(\Gamma(R))$ is connected as we prove in next theorem.

We next show that the all dot total graphs of R are connected and study the diameter and girth.

Theorem 2.1. *$T_{Z(R)}(\Gamma(R))$ is connected and $diam(T_{Z(R)}(\Gamma(R))) \leq 2$. Moreover, if $T_{Z(R)}(\Gamma(R))$ contains a cycle, then $gr(T_{Z(R)}(\Gamma(R))) \leq 5$.*

Proof. Let x and y be distinct vertices of $T_{Z(R)}(\Gamma(R))$.

$Case(i)$ If $x, y \in Z(R)$, then $x - y$ is a path in $T_{Z(R)}(\Gamma(R))$.

$Case(ii)$ If $x, y \in Reg(R)$, then there is some $z \in Z(R)$ such that $xz \in Z(R)$ and $yz \in Z(R)$. Thus $x - z - y$ is a path.

Case(iii) If $x \in Z(R)$ and $y \in Reg(R)$, then $x - y$ is a path.

Thus $T_{Z(R)}(\Gamma(R))$ is connected and $diam(T_{Z(R)}(\Gamma(R))) \leq 2$. Since for any undirected graph H, $gr(H) \leq 2\, diam(H) + 1$, H contains a cycle (for reference see [12]). Thus $gr(T_{Z(R)}(\Gamma(R))) \leq 5$. □

Remark 2.2. For any commutative ring R with $1 \neq 0$, we know that $\Gamma(R)$ is connected and has $diam(\Gamma(R)) \leq 3$ and if $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 7$ (for reference see [3]). Also, the same results hold for $\Gamma_I(R)$ (for reference see [15]). In addition, if $T(\Gamma(R))$ is connected, then $diam(T(\Gamma(R))) = d(0,1)$ (for reference see [4]). But for $T_{Z(R)}(\Gamma(R))$, we get a connected graph which has $diam(T_{Z(R)}(\Gamma(R))) \leq 2$ and if $T_{Z(R)}(\Gamma(R))$ contains a cycle, then $gr(T_{Z(R)}(\Gamma(R))) \leq 5$.

The graph $T_{Z(R)}(\Gamma(R))$ has a very special form. In fact, if $|Z(R)| = m$ and $|Reg(R)| = n$ then $T_{Z(R)}(\Gamma(R)) \cong \overline{K}_n \vee K_m$, where $\vee$ is used for the join of two graphs.

Theorem 2.2. *The graph* $\overline{K}_n \vee K_m$ *is complete iff* $n = 1$*.*

Proof. Suppose $\overline{K}_n \vee K_m$ is complete. Then each distinct vertices in R are adjacent. If $n > 1$, then there is at least two vertices x and y in $\overline{K}_n$ which are non adjacent, which is a contradiction. Hence $n = 1$.

Conversely, suppose that $n = 1$. Then it is clear that $\overline{K}_n \vee K_m$ is complete graph. □

Corollary 2.1. $T_{Z(R)}(\Gamma(R))$ *is not complete if and only if* $|Reg(R)| \geqslant 2$*.*

Corollary 2.2. $\overline{T_{Z(R)}(\Gamma(R))}$ *is* K_n *with vertices of regular elements of* R*, where* $n = |Reg(R)|$ *and other vertices are isolated (elements of* $Z(R)$*).*

Remark 2.3. Let R be a finite commutative ring. Then the following statements hold:

(i) If $Z(R)$ is an ideal, then $T(\Gamma(R))$ is not connected [11, 13] and for any element $x \in R$, there are two possibilities:

- (a) If $2 \in Z(R)$, then $deg(x) = |Z(R)| - 1$ for each $x \in R$.
- (b) If $2 \notin Z(R)$, then $deg(x) = |Z(R)| - 1$ for each $x \in Z(R)$ and $deg(x) = |Z(R)|$ for each $x \in Reg(R)$.

(ii) If $Z(R)$ is not an ideal, then $T(\Gamma(R))$ is connected and $deg(x) = |Z(R)| - 1$ for each $x \in R$.

In the next theorem, we find the degree of each vertex of $T_{Z(R)}(\Gamma(R)) \cong \overline{K}_n \vee K_m$.

Theorem 2.3. *The degree of vertices in the graph* $\overline{K}_n \vee K_m$ *are* m *or* $m+n-1$*.*

Proof. Since vertices in the graph $\overline{K}_n \vee K_m$ are belong to either K_m or $\overline{K}_n$, we have the following two cases:

Case(i) If $x \in K_m$, then x is adjacent to each vertex in $\overline{K}_n \vee K_m$ except x, that is, x is adjacent to $m+n-1$ vertices and hence degree of x is $m+n-1$.

Case(ii) If $x \in \overline{K}_n$, then x is adjacent to the vertices, which belongs to K_m, that is, x is adjacent to m vertices and hence $deg(x) = m$. □

Corollary 2.3. *The graph $\overline{K}_n \vee K_m$ is regular graph iff $n = 1$.*

Remark 2.4. For any graph G, $\delta(G)$ is the minimum degree of G and $\Delta(G)$ is the maximum degree of G. Here for $G = T_{Z(R)}(\Gamma(R))$, $\delta(G) = |Z(R)|$ and $\Delta(G) = |R| - 1$.

3. Connectivity of $\overline{K}_n \vee K_m$

In this section, we study the connectivity of $\overline{K}_n \vee K_m$.

Theorem 3.1. *The graph $\overline{K}_n \vee K_m$ has a cut vertex iff $m = 1$. i.e., R is an integral domain.*

Proof. Assume that the vertex x of $\overline{K}_n \vee K_m$ is a cut-vertex. Then there exist $u, w \in \overline{K}_n \vee K_m$ such that x lies on every path from u to w. Thus we have the following two cases:

Case(i) If u is adjacent to w, then we get a contradiction.

Case(ii) If u is not adjacent to w, then $u, w \in \overline{K}_n$ and $x \in K_m$. Now, if $m > 1$, then K_m have more than one vertices. i.e., $x \neq y \in K_m$. Therefore, there is at least one path from u to w and x does not lie on it, which is a contradiction. Hence $m = 1$.

Conversely, assume that $m = 1$. Then it is clear that $\overline{K}_n \vee K_m$ has a cut vertex. □

Theorem 3.2. *The graph $\overline{K}_n \vee K_m$ has a bridge iff $m = 1$. i.e., R is an integral domain.*

Proof. Suppose that $\overline{K}_n \vee K_m$ has a bridge. Now we have the following cases:

Case(i) If $|R| = 2$, then it is clear that $m = 1$.

Case(ii) If $|R| \geqslant 3$, then either $V(\overline{K}_n \vee K_m) \subseteq V(\overline{K}_n)$ or $V(\overline{K}_n \vee K_m) \subseteq V(K_m)$ and we know that there is no edge between any two elements of $\overline{K}_n$, and we have an edge either between each $x, y \in V(K_m)$ or each $x \in V(K_m)$ with all $y \in R$. Therefor we have the following subcases:

Subcase(a) If $x, y \in V(K_m)$ and $|R| \geqslant 3$, then there exists $z \in R$ such that x and y are adjacent to z. We note that $x - y - z - x$ is a cycle, and there is no bridge between them, we get a contradiction.

Subcase(b) If $x \in V(K_m)$, $y \in V(\overline{K}_n)$ and $|R| \geqslant 3$, then there exists at least one element $z \in R \setminus \{x, y\}$. There are two possibilities:

If $z \in V(K_m)$, then z is adjacent to x and y. Thus $x - z - y - x$ is a cycle and there is no bridge between them. This is a contradiction.

If $z \in V(\overline{K}_n)$ and only $x \in V(K_m)$ (here $x = 0$, additive identity), then x is adjacent to each $z \in V(\overline{K}_n)$ and there is no adjacency between any two elements of $\overline{K}_n$. Thus there are more than one vertex adjacent to x and $0 = x \in V(K_m)$ only, otherwise, there is a cycle. Thus all edges are bridge. Hence $m = 1$.

Converse of the proof is trivial. □

Remark 3.1. If the ring $R \cong \mathbb{Z}_2$ or R is an integral domain, then $T_{Z(R)}(\Gamma(R))$ has a bridge and vice versa.

Theorem 3.3. *$k(\overline{K}_n \vee K_m) = m$.*

Proof. We know that, for any graph G, $k(G) \leqslant \lambda(G) \leqslant \delta(G)$ and by Remark 2.4, $\delta(\overline{K}_n \vee K_m) = |Z(R)| = m$. Therefore,

$$k(\overline{K}_n \vee K_m) \leqslant m.$$

Now if $x \in V(K_m)$, then x is adjacent to each vertex $y \in R$. Hence the minimum vertex-cut is the set of all those vertices in $V(K_m)$, otherwise, $\overline{K}_n \vee K_m$ is connected. Hence $k(\overline{K}_n \vee K_m)) = m$. □

Remark 3.2. For any commutative ring R with $1 \neq 0$, $Z(R)$ is the minimum vertex-cut of $T_{Z(R)}(\Gamma(R))$.

4. Clique number of $\overline{K}_n \vee K_m$

In this section, we study the clique number of $\overline{K}_n \vee K_m$.

Theorem 4.1. *$\omega(\overline{K}_n \vee K_m) = m + 1$.*

Proof. We know that each pair of elements in K_m are adjacent. In general, they are adjacent to all elements of $\overline{K}_n \vee K_m$. Thus each element is adjacent at least to one element in $\overline{K}_n$. Since $|K_m| = m$, we find that $m + 1$ elements are adjacent. This completes the proof. □

Corollary 4.1. *If $m \geqslant 2$, then $gr(\overline{K}_n \vee K_m) = 3$. If $m = 1$, i.e., R is an integral domain, then $gr(\overline{K}_n \vee K_m) = \infty$.*

Proof. Suppose that $m \geqslant 2$. Then by the same arguments as used in the above theorem, and given that $|K_m| = m \geqslant 2$, we find that at least two elements are in K_m. Let $u, v \in K_m$. Also, $\overline{K}_n \vee K_m$ has at least one element $w \in \overline{K}_n$. Then $u - w - v - u$ is a cycle of length 3, which is the smallest cycle in $\overline{K}_n \vee K_m$. Hence $gr(\overline{K}_n \vee K_m) = 3$.

Suppose that $m = 1$. Then there is no cycle in $\overline{K}_n \vee K_m$. Hence $gr(\overline{K}_n \vee K_m) = \infty$. □

5. Traversability of $T_{Z(R)}(\Gamma(R))$

In this section, we show that $T_{Z(R)}(\Gamma(R))$ can not be an Eulerian graph. Also, we discover the types of rings that make the graph $T_{Z(R)}(\Gamma(R))$ have an Eulerian trail. Further, we find out when the graph $T_{Z(R)}(\Gamma(R))$ is Hamiltonian graph.

Theorem 5.1. *$T_{Z(R)}(\Gamma(R))$ can not be an Eulerian graph.*

Proof. First of all, we prove that $T_{Z(R)}(\Gamma(R))$ is an Eulerian if and only if $|R|$ is odd and $|Z(R)|$ is even. Moreover, $|Reg(R)|$ is odd. Suppose that $T_{Z(R)}(\Gamma(R))$ is an Eulerian. Then every vertex of $T_{Z(R)}(\Gamma(R))$ has even degree. Since the degree of each vertex of $T_{Z(R)}(\Gamma(R))$ either $(|R| - 1)$ or $|Z(R)|$ (Theorem 2.3), we have the following cases:

$Case(i)$ If $x \in Z(R)$, then $deg(x) = |R| - 1$, which is even, and we get $|R|$ is odd.

$Case(ii)$ If $x \in Reg(R)$, then $deg(x) = |Z(R)|$, which is even. Thus $|Z(R)|$ is even.

Hence $|R|$ is odd and $|Z(R)|$ is even. Moreover, $|Reg(R)|$ is odd.

Conversely, suppose that $|R|$ is odd and $|Z(R)|$ is even. Then $|R| - 1$ is even and $|Z(R)|$ is also even. Since the degree of each vertex of $T_{Z(R)}(\Gamma(R))$ is either $|R| - 1$ or $|Z(R)|$, degree of each vertex of $T_{Z(R)}(\Gamma(R))$ is even. Hence $T_{Z(R)}(\Gamma(R))$ is an Eulerian.

Second, we show that there is no ring R such that $T_{Z(R)}(\Gamma(R))$ be an Eulerian graph. If $u \in Reg(R) = U(R)$ then $u^n = 1$ where $n = |U(R)|$. So, if n is an odd number, then $-1 = (-1)^n = 1$. Hence $2 = 0$ and $Char(R) = 2$. Thus $|R| = 2^k$. So, there is no ring R such that $T_{Z(R)}(\Gamma(R))$ be an Eulerian graph. □

Theorem 5.2. *$T_{Z(R)}(\Gamma(R))$ has an Eulerian trail iff $R \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}$.*

Proof. Suppose that $|Z(R)| = |Reg(R)| = 1$. Then $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail and $R \cong \mathbb{Z}_2$. Now suppose that $|Z(R)| > 1$ or $|Reg(R)| > 1$. Then we prove that $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail if and only if either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Suppose that $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail. Then exactly two vertices of $T_{Z(R)}(\Gamma(R))$ have odd degree. Let u and v be the two vertices of odd degree and let $x_1, x_2, ..., x_n$ be the vertices of even degree. Then we have the following cases:

Case(i) If $u, v \in Z(R)$ and $x_i \in Reg(R)$ for all $1 \leq i \leq n$, then $deg(u) = deg(v)$ is odd and $deg(x_i)$ for all $1 \leq i \leq n$ is even, therefor $|R| - 1$ is odd and $|Z(R)| = 2$ is even, thus $|R|$ is even and $|Z(R)| = 2$. Hence $|Z(R)| = 2$ and $|Reg(R)|$ is even. Moreover, $|R|$ is even.

Case(ii) If $u, v \in Z(R)$ and there exists at least one $x_j \in Z(R)$, then $deg(u) = deg(v) = deg(x_j)$ is odd. Hence there are more than two odd vertices in $T_{Z(R)}(\Gamma(R))$, we get a contradiction.

Case(iii) If $u, v \in Reg(R)$ and $x_i \in Z(R)$ for all $1 \leq i \leq n$, then $deg(u) = deg(v)$ is odd and $deg(x_i)$ for all $1 \leq i \leq n$ is even. Note that $|Z(R)|$ is odd and $|R| - 1$ is even. We get $|Z(R)|$ is odd and $|R|$ is odd. Since $u, v \in Reg(R)$ only, we have $|Reg(R)| = 2$. Hence $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Moreover, $|R|$ is odd.

Case(iv) If $u, v \in Reg(R)$ and there exists at least one $x_j \in Reg(R)$, then $deg(u) = deg(v) = deg(x_j)$ is odd. Thus there are more than two odd vertices in $T_{Z(R)}(\Gamma(R))$, we get a contradiction.

Case(v) If $u \in Z(R)$ and $v \in Reg(R)$, then $deg(u) = deg(v) = deg(x_i)$ for all $1 \leq i \leq n$ is odd. Thus all the vertices of $T_{Z(R)}(\Gamma(R))$ have odd degree, we get a contradiction.

Therefore in all the cases, we get that either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd.

Conversely, suppose that either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Now we assume that $|Z(R)| = 2$ and $|Reg(R)|$ is even, let x be any vertex of $T_{Z(R)}(\Gamma(R))$, then we have the following cases:

Case(i) If $x \in Z(R)$, then $deg(x) = |R| - 1$, which is odd. Since $|Z(R)| = 2$ and $|Reg(R)|$ is even, there are only two vertices in $Z(R)$ have odd degree and each other vertices in $Reg(R)$ have even degree. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

Case(ii) If $x \in Reg(R)$, then $deg(x) = |Z(R)| = 2$, which is even, by the same argument, there are only two vertices $x_1, x_2 \in Z(R)$ such that x_1 and x_2 are adjacent to each vertices in $Reg(R)$ and x_1 adjacent to x_2 and $deg(x_1) = deg(x_2) = |Reg(R)| + 1$ which is odd. Therefor, there are only two vertices in $Z(R)$ have odd degree and each other vertices in $Reg(R)$ have even degree. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

After that, we assume that $|Reg(R)| = 2$ and $|Z(R)|$ is odd. Then $|R|$ is odd, and let x be any vertex of $T_{Z(R)}(\Gamma(R))$. Then we have the following cases:

Case(i) If $x \in Z(R)$, then $deg(x) = |R| - 1$, which is even. Since $|Reg(R)| = 2$ and $|Z(R)|$ is odd, there are only two vertices in $Reg(R)$ have odd degree and each other vertices in $Z(R)$ have even degree. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

Case(ii) If $x \in Reg(R)$, then $deg(x) = |Z(R)|$, which is odd, thus $|R|$ is odd. By the same argument, there are only two vertices in $Reg(R)$ have odd degree and each other vertices in $Z(R)$ have degree $|R|-1$, which is even. Hence $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

From all the above cases we conclude that $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail. Hence if either $|Z(R)| = 2$ and $|Reg(R)|$ is even or $|Reg(R)| = 2$ and $|Z(R)|$ is odd, then $T_{Z(R)}(\Gamma(R))$ contains an Eulerian trail.

Second, we show that $T_{Z(R)}(\Gamma(R))$ has an Eulerian trail iff $R \cong \mathbb{Z}_3, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}$.

(i) Assume $|Z(R)| = 2$. Let $0 \neq x \in Z(R)$. Since $ann(x), Rx \subseteq Z(R)$ we conclude $ann(x) = Rx = Z(R)$. So, the isomorphism $\frac{R}{ann(x)} \cong Rx$ implies $|R| = 4$. Hence $R \cong \mathbb{Z}_4$ or $R \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$.

(ii) It is well known that every commutative artinian ring is isomorphic to direct product of finitely many local rings. If R is a finite local ring with the unique maximal ideal M, then $|R| = p^r, m = |Z(R)| = |M| = p^s$ and $n = |Reg(R)| = |U(R)| = p^r - p^s$. In particular, $m|n$. So, if $n = 1$, then $R \cong \mathbb{Z}_2$. If $n = 2$, then $m = 1$ or $m = 2$. If $m = 1$, then $R \cong \mathbb{Z}_3$. If $m = 2$, then $|R| = 4$ and $R \cong \mathbb{Z}_4$. So, the only odd order ring with $|Reg(R)| = 2$ is $\mathbb{Z}_3$. □

Remark 5.1. In the above theorem, if $|Z(R)| = 2$, then Eulerian trail of $T_{Z(R)}(\Gamma(R))$ begins at one of these two elements of $Z(R)$ and ends at other. Also, if $|Reg(R)| = 2$, then Eulerian trail of $T_{Z(R)}(\Gamma(R))$ begins at one of these two elements of $Reg(R)$ and ends at other.

Theorem 5.3. *Let R has a maximal ideal of index 2 and $|R| > 2$, then $T_{Z(R)}(\Gamma(R))$ is Hamiltonian.*

Proof. The graph $\overline{K}_n \vee K_m$ is Hamiltonian iff $m \geq max\{n, 2\}$. So, $T_{Z(R)}(\Gamma(R))$ is Hamiltonian iff $|R| > 2$ and $|Z(R)| \geq \frac{|R|}{2}$ iff $|R| > 2$ and $\frac{|U(R)|}{|R|} \leq \frac{1}{2}$. Since $\frac{|U(R)|}{|R|} = \frac{|U(\frac{R}{J(R)})|}{|\frac{R}{J(R)}|}$ ($J(R)$ is the Jacobson radial of R). So, $T_{Z(R)}(\Gamma(R))$ is Hamiltonian if $T_{Z(\frac{R}{J(R)})}(\Gamma(\frac{R}{J(R)}))$ is Hamiltonian. Also If $T_{Z(R)}(\Gamma(R))$ is Hamiltonian and $|R/J(R)| > 2$, then $T_{Z(\frac{R}{J(R)})}(\Gamma(\frac{R}{J(R)}))$ is Hamiltonian. Since $\frac{R}{J(R)} \cong \prod_{M_i \in Max(R)} \dfrac{R}{M_i} \cong \prod F_{q_i}$ ($\frac{R}{M_i} \cong F_{q_i}$ is a field). So, $\frac{|U(\frac{R}{J(R)})|}{|\frac{R}{J(R)}|} = \prod \frac{q_i - 1}{q_i}$. In particular, if R has a maximal ideal of index 2 and $|R| > 2$, then $T_{Z(R)}(\Gamma(R))$ is Hamiltonian. Also $T_{Z(R)}(\Gamma(R))$ is Hamiltonian for a local ring (R, M) iff $|R/M| = 2$ and $|R| > 2$. □

Corollary 5.1. *Let R be a local ring and has k maximal ideal. If $T_{Z(R)}(\Gamma(R))$ is Hamiltonian, then $R/J(R) \cong F$, i.e., $k = 1$ and $J(R)$ is maximal ideal of R.*

Corollary 5.2. *Let R be a finite commutative ring with $1 \neq 0$, such that $|R| = n \geq 3$. If $|Z(R)| \geq \frac{n}{2}$ for each pair u, v of $Reg(R)$, then $T_{Z(R)}(\Gamma(R)) + uv$ is Hamiltonian if and only if $T_{Z(R)}(\Gamma(R))$ is Hamiltonian.*

Let $A, B \subseteq R$. Define $T_A(\Gamma(B))$ be a graph whose vertex set is B and two distinct vertices x, y are adjacent if $xy \in A$.

Theorem 5.4. *The graph $T_A(\Gamma(B))$ is the complement graph of $T_{A^c}(\Gamma(B))$ where $A^c = R \setminus A$.*

Proof. Let u and v be two distinct vertices of B. Then $T_A(\Gamma(B))$ and $T_{A^c}(\Gamma(B))$ have the same set of vertices. Since $uv \in A$ if and only if $uv \notin A^c$, we get that uv is an edge of $T_A(\Gamma(B))$ if and only if uv is not an edge of $T_{A^c}(\Gamma(B))$. $\square$

Theorem 5.5. *The graph $T_A(\Gamma(B))$ is a spaning subgraph of $T_C(\Gamma(B))$ if $A \subseteq C$.*

Proof. Let $A \subseteq C$. Since $T_A(\Gamma(B))$ and $T_C(\Gamma(B))$ have the same set of vertices depending on B, we have to prove that the edge set of $T_A(\Gamma(B))$ contains in the edge set of $T_C(\Gamma(B))$. To complete the prove, assume, on contrary, that the edge set of $T_A(\Gamma(B))$ contains the edge set of $T_C(\Gamma(B))$. Then for every two distinct vertices $u, v \in B$ that adjacent in $T_C(\Gamma(B))$ should be adjacent in $T_A(\Gamma(B))$. By definitions of $T_C(\Gamma(B))$ and $T_A(\Gamma(B))$, we get that $C \subseteq A$, which is a contradiction. Hence $T_A(\Gamma(B))$ is the spanning subgraph of $T_C(\Gamma(B))$. $\square$

Corollary 5.3. *The graph $T_A(\Gamma(B))$ is an induced subgraph of $T_A(\Gamma(C))$ if $B \subseteq C$.*

Corollary 5.4. *If A is multiplicatively closed subset of R and $B \subseteq A$, then $T_A(\Gamma(B))$ is a complete graph.*

Corollary 5.5. *If A and B are two disjoint multiplicatively closed subsets of R, then $T_A(\Gamma(B))$ is the empty graph.*

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On nonsolvability of exponential Diophantine equations via transformation to elliptic curves

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Abstract. Exponential Diophantine equations of the form $p^X + q^Y = Z^2$, with unknowns (X, Y, Z) in the set of positive integers, are of interest to many number theorists. Many of these equations are solved using congruence techniques and the quadratic reciprocity. The goal of this paper is to show unsolvability of some Diophantine equations of this type using the concept of elliptic curves. Similar types of exponential Diophantine equations are also considered in this study. To illustrate the results, examples are provided.

Keywords: exponential Diophantine equation, elliptic curve, congruences, factorization.

1. Introduction

Solving Diophantine equations is one of the oldest problems in Number Theory but is one of the hot topics of research in this field of mathematics in the past few years. Recently, several papers have been devoted in finding the non-negative integer solutions of Diophantine equations of the form

$$p^X + q^Y = Z^2 \tag{1}$$

with unknowns (X, Y, Z). Such equations are called exponential Diophantine equations as they require solutions in the exponents. In 2007, Acu [1] found the complete set of solutions of the Diophantine equation $2^X + 5^Y = Z^2$. In 2012 and 2013, Sroysang ([8],[9]) worked on the equations $3^X + 5^Y = Z^2$ and $8^X +$

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$19^Y = Z^2$. On the other hand, Rabago [4] looked into Diophantine equations $3^X+19^Y = Z^2$ and $3^X+91^Y = Z^2$. Many other similar problems were considered in the references [11], [10], [5], [6], [7], [15] and [2].

Most of the tools used in the said studies were the congruence and factorization techniques. In 2019, Mina and Bacani [3] were able to provide some criteria for showing non-existence of solutions over the set of positive integers for such equations by using the values of the Legendre and Jacobi symbols.

In this paper, we will present several ways on determining whether equation (1) has no solutions in the set of positive integers. These are done by transforming such equations to another family of equations whose rational points form an abelian group structure. These equations are no other than equations that describe elliptic curves. The use of elliptic curves in solving Diophantine equations is not new and has already been done in the past. A classical example would be the Fermat equation

$$a^4 + b^4 = c^4,\ a \neq 0. \tag{2}$$

Using the transformation

$$x = 2\frac{b^2+c^2}{a^2} \quad \text{and} \quad y = 4\frac{b(b^2+c^2)}{a^3},$$

we get a corresponding elliptic curve

$$y^2 = x^3 - 4x$$

which has only the following rational solutions: $(x, y) = (0, 0), (2, 0), (-2, 0)$. These all correspond to $b = 0$, so there are no nontrivial solutions to (2).

We will also be dealing with equations of the form

$$p^X + q^Y = Z^n \tag{3}$$

where $n = 3$ or 6. This type of equation is generally not possible to study when using only congruence techniques. Most of the cases we will be dealing with require one of the exponents to be even. There are some theorems that guarantee non-existence of solutions to (1), such as those presented in [3] which deal with the case where one of the exponents is odd. Note that since computation of ranks of elliptic curves is generally a hard problem, most of the results will focus on the case where the bases p and q are fixed. We use a free mathematical software *SAGE* [13] for the computation of ranks and torsion subgroups of elliptic curves.

Throughout the paper, we will denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{Q}$ the sets of positive integers, non-negative integers, integers and rational numbers, respectively.

2. Basic concepts about elliptic curves

An elliptic curve defined over $\mathbb{Q}$ is a curve that is described by the following general Weierstrass equation:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_1, a_2, a_3, a_4, a_6 \in \mathbb{Q}$. By completing the square, we get

$$\left(y + \frac{a_1 x}{2} + \frac{a_2}{2}\right)^2 = x^3 + \left(a_2 + \frac{a_1^2}{4}\right) x^2 + \left(a_4 + \frac{a_1 a_3}{2}\right) x + \left(\frac{a_3^2}{4} + a_6\right),$$

which can be written as

$$y_1^2 = x^3 + a_2' x^2 + a_4' x + a_6',$$

where $y_1 = y + a_1 x/2 + a_3/2$ and $a_2' = a_2 + a_1^2/4$. Furthermore, if we let $x_1 = x + a_2'/3$, then we get the simpler Weierstrass equation

$$y_1^2 = x_1^3 + Ax_1 + B, \quad \text{for some } A, B \in \mathbb{Q}.$$

In other words, an elliptic curve defined over the rationals is given by the following equation:

$$E : y^2 = x^3 + Ax + B,$$

where A and B are rational numbers. In addition, the discriminant $\Delta := 4A^3 + 27B^2$ must be nonzero. It is well-known that the rational points on the elliptic curve E over $\mathbb{Q}$ forms an abelian group called the Mordell-Weil group with the point at infinity $\mathcal{O}$ acting as the identity. The group is isomorphic to $E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$, where $E(\mathbb{Q})_{\text{tors}}$ is the group of elements of finite order, called the torsion subgroup, and $r \geq 0$ is called the rank of E. There are ways of solving the torsion subgroup, such as using the well-known Nagell-Lutz Theorem, but the computation of rank r is generally a hard problem.

One of the results in the theory of elliptic curves is the transformation of a quartic equation to the Weierstrass equation of elliptic curve, and vice-versa. The proof of this theorem can be seen in [12].

Theorem 2.1. *Consider the following equation*

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2,$$

with coefficients $a, b, c, d, q \in \mathbb{Q}$. *Let*

$$x = \frac{2q(v+q) + du}{u^2}, \qquad y = \frac{4q^2(v+q) + 2q(du + cu^2) - (d^2u^2/2q)}{u^3}.$$

Define $a_1 = d/q,\ a_2 = c - (d^2/4q^2),\ a_3 = 2qb,\ a_4 = -4q^2 a,\ a_6 = a_2 a_4$. *Then,*

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

The inverse transformation is given by

$$u = \frac{2q(x+c) - (d^2/2q)}{y}, \quad v = -q + \frac{u(ux - d)}{2q}.$$

The point $(u, v) = (0, q)$ *corresponds to the point* $(x, y) = \mathcal{O}$, *and* $(u, v) = (0, -q)$ *corresponds to* $(x, y) = (-a_2, a_1 a_2 - a_3)$.

The next theorem is a well-known result regarding the torsion subgroup of the elliptic curve $y^2 = x^3 + B$.

Theorem 2.2. *Let $E : y^2 = x^3 + B$ be an elliptic curve for some sixth power-free integer B. Then, the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ of $E(\mathbb{Q})$ is isomorphic to one of the following groups:*

1. *$\mathbb{Z}/6\mathbb{Z}$ if $B = 1$,*
2. *$\mathbb{Z}/3\mathbb{Z}$ if $B \neq 1$ is a square or $B = -432$,*
3. *$\mathbb{Z}/2\mathbb{Z}$ if $B \neq 1$ is a cube,*
4. *$\{\mathcal{O}\}$, otherwise.*

3. Main results

For the first two theorems, we present some results about the transformation of the exponential Diophantine equation (1) into the Weierstrass equation of elliptic curve.

Theorem 3.1. *Let p be prime and q be an odd number such that* $\gcd(p, q) = 1$. *Then, the exponential Diophantine equation* (1) *has no solutions* (X, Y, Z) *in* $\mathbb{N}$ *if* $X \equiv 0 \pmod 3$ *and* $Y \equiv 0 \pmod 4$.

Proof. Suppose (X, Y, Z) is a solution of (1) such that $X \equiv 0 \pmod 3$ and $Y \equiv 0 \pmod 4$. This implies that $X = 3X_1$ and $Y = 4Y_1$, for some X_1, $Y_1 \in \mathbb{N}$. By factoring $p^X + q^Y = Z^2$, we get

$$(p^{X_1})^3 = (Z + (q^{Y_1})^2)(Z - (q^{Y_1})^2).$$

Since p is prime, there exist two non-negative integers α and β, $\alpha < \beta$ such that $\alpha + \beta = 3X_1$ and

$$p^\alpha(p^{\beta-\alpha} - 1) = p^\beta - p^\alpha = (Z + (q^{Y_1})^2) - (Z - (q^{Y_1})^2) = 2(q^{Y_1})^2.$$

Since $\gcd(p, q) = 1$, we find that $\alpha = 0$ and we get the equation

$$(p^{X_1})^3 - 1 = 2(q^{Y_1})^2.$$

Multiplying both sides by 8 yields $(4q^{Y_1})^2 = (2p^{X_1})^3 - 8$. By substituting $x = 2p^{X_1}$ and $y = 4q^{y_1}$, we obtain the elliptic curve $E_1 : y^2 = x^3 - 8$. Using *SAGE*, we find that its rank is $r = 0$ and its torsion subgroup $E_1(\mathbb{Q})_{\text{tors}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We see that the rational points on $E_1(\mathbb{Q})$ are $(2, 0)$ and the point at infinity $\mathcal{O}$, all of which are of finite order. This yields $(p^{X_1}, q) = (1, 0)$, which is a contradiction to the assumption that q is positive. Therefore, (1) has no solutions in $\mathbb{N}$. $\square$

Theorem 3.2. *Let p be prime and q be an odd number such that* $\gcd(p,q)=1$. *Then, the Diophantine equation* (1) *can be transformed to the elliptic curve* $E_2 : y^2 = x^3 - 8q^3$ *if* $X \equiv 0 \pmod 3$ *and* $Y \equiv 2 \pmod 4$. *Moreover, if the rank of* E_2 *is zero, then* (1) *has no solutions in* $\mathbb{N}$.

Proof. Let (X,Y,Z) be a solution such that $X = 3X_1$ and $Y = 4Y_1 + 2$, for some $X_1 \in \mathbb{N}$ and $Y_1 \in \mathbb{N}_0$. By factoring $p^X + q^Y = Z^2$, we have

$$(p^{X_1})^3 = (Z + q^{2Y_1+1})(Z - q^{2Y_1+1}).$$

Since p is prime, there exist two non-negative integers α and β, $\alpha < \beta$ such that $\alpha + \beta = 3X_1$, and

$$p^{\alpha}(p^{\beta-\alpha} - 1) = p^{\beta} - p^{\alpha} = (Z + q^{2Y_1+1}) - (Z - q^{2Y_1+1}) = 2q^{2Y_1+1}.$$

Since $\gcd(p,q)=1$, we find that $\alpha = 0$ and we get the equation

$$(p^{X_1})^3 - 1 = 2q^{2Y_1+1}, \ \text{ or } \ (p^{X_1})^3 - 1 = 2q \cdot (q_1^Y)^2.$$

Multiplying both sides by $8q^3$ yields $(4q^{Y_1+2})^2 = (2q \cdot p^{X_1})^3 - 8q^3$. By substituting $x = 2q \cdot p^{X_1}$ and $y = 4q^{Y_1+2}$, we obtain the elliptic curve $E_2 : y^2 = x^3 - 8q^3$. Using Theorem 2.2, we find that $E_2(\mathbb{Q})_{\text{tors}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The torsion points on $E_2(\mathbb{Q})$ are $(2q,0)$ and the point at infinity $\mathcal{O}$. This yields $(p^{X_1}, q) = (1,0)$ which gives no solutions to the original equation since q is assumed to be positive. Moreover, since the rank of E_2 is assumed to be zero, this means that there are no other rational points on E_2, and consequently on (1). Hence, (1) has no solutions in $\mathbb{N}$. □

Remark 3.1. In Theorems 3.1 and 3.2, the elliptic curves $y^2 = x^3 - 8$ and $y^2 = x^3 - 8q^3$ are also called Mordell curves [14]. The determination of values of q for which the second curve has rank zero is a difficult problem.

Let us now apply these two theorems to a specific exponential Diophantine equation of the form (1).

Example 3.1. The Diophantine equation $19^X + 27^Y = Z^2$ has no solutions (X,Y,Z) in $\mathbb{N}$.

Proof. Taking the equation in modulo 4 gives us $3^X + 3^Y \equiv Z^2 \pmod 4$. Since Z^2 is even, $Z^2 \equiv 0 \pmod 4$. Thus, $3^X + 3^Y \equiv 0 \pmod 4$. This implies that X and Y are of different parity. For the sake of our purpose, we will only deal with the case where X is odd and Y is even. The other cases yield no solution via congruence considerations. By letting $Y = 2Y_1$, for some $Y_1 \in \mathbb{N}$ and factoring, we get

$$19^X = (Z + 27^{Y_1})(Z - 27^{Y_1}).$$

Using the same reasoning as done in the proof of Theorem 3.1, we get the equation

$$19^X - 1 = 2 \cdot 27^{Y_1}.$$

Factoring this equation, we get $(19-1)(19^{X-1}+19^{X-2}+\cdots+1) = 18\cdot 3\cdot 27^{Y_1-1}$. This implies that $19^{X-1}+19^{X-2}+\cdots+1 = 3\cdot 27^{Y_1-1}$. Taking modulo 3 yields $X \equiv 0 \pmod 3$, i.e. $X = 3X_1$, for some $X_1 \in \mathbb{N}$.

Now, if Y_1 is odd, then our equation becomes $(19^{X_1})^3 + (27^{2Y_2+1})^2 = Z^2$, where $Y_1 = 2Y_2+1$. Now, we can transform the equation into the elliptic curve $y^2 = x^3 - 157464$. This has rank zero with torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Using Theorem 3.2, the equation has no solutions in $\mathbb{N}$.

For the second part of the proof, if Y_1 is even, i.e., $Y_1 = 2Y_2$, for some $Y_2 \in \mathbb{N}$, then the original equation becomes $(19^{X_1})^3 + (27^{Y_1})^4 = Z^2$. This resulting equation has no solutions in $\mathbb{N}$ using Theorem 3.1. $\square$

For the next results, we will consider another family of exponential Diophantine equations of the form (3).

Theorem 3.3. *Let p be prime and q be an odd number such that* $\gcd(p,q) = 1$. *Then, the Diophantine equation $p^X+q^Y = Z^6$ can be transformed into the elliptic curve $E_3 : y^2 = x^3 - 4p^3$ if $X \equiv 1 \pmod 2$ and $Y \equiv 0 \pmod 2$. Moreover, if the rank of E_3 is zero, then $p^X + q^Y = Z^6$ has no solutions in* $\mathbb{N}$.

Proof. Let $X = 2X_1+1$ and $Y = 2Y_1$, for some $X_1 \in \mathbb{N}_0$, $Y_1 \in \mathbb{N}$. By factoring, we have the following,

$$p^{2X_1+1} = (Z^3 + (q^{Y_1})^2)(Z^3 - (q^{Y_1})^2).$$

Since p is prime, there exist two non-negative integers α and β, $\alpha < \beta$ such that $\alpha + \beta = 2X_1 + 1$ and

$$p^{\alpha}(p^{\beta-\alpha} + 1) = p^{\beta} + p^{\alpha} = (Z^3 + (q^{Y_1})^2) + (Z^3 - (q^{Y_1})^2) = 2Z^3.$$

Note that $p \not| Z$, otherwise $p \mid q$ which is not possible since $\gcd(p,q) = 1$. Hence, $\alpha = 0$ and we get the equation

$$p \cdot (p^{X_1})^2 - 1 = 2Z^3.$$

Multiplying both sides by $4p^3$ yields $(2p^{X_1+2})^2 = (2pZ)^3 - 4p^3$. By substituting $x = 2pZ$ and $y = 2p^{X_1+2}$, we obtain the elliptic curve $E_3 : y^2 = x^3 - 4p^3$. Using Theorem 2.2 the torsion subgroup $E_3(\mathbb{Q})_{\text{tors}}$ is isomorphic to $\{\mathcal{O}\}$. This implies that if the rank of E_3 is zero, then there are no solutions in $\mathbb{N}$ to the original equation. $\square$

Next, we are going to consider a larger family of equations of the form $p^X + q^Y = Z^3$, but this time both p and q are primes. In this case, we have the following two results.

Theorem 3.4. *Let p and q be distinct odd primes. Then, the Diophantine equation $p^X + q^Y = Z^3$ has no solutions (X,Y,Z) in $\mathbb{N}$ with $X \equiv 0 \pmod 2$ and $Y \equiv 0 \pmod 6$.*

Proof. Let $X = 2X_1$ and $Y = 6Y_1$, for some $X_1, Y_1 \in \mathbb{N}$. By factoring, we have the following:

$$p^{2X_1} = (Z - q^{2Y_1})(Z^2 + Zq^{2Y_1} + q^{4Y_1}).$$

Since p is prime, there exist two non-negative integers α and β, $\alpha < \beta$ such that $\alpha + \beta = 2X_1$ and

$$p^{\beta-\alpha} = \frac{p^\beta}{p^\alpha} = \frac{Z^2 + Zq^{2Y_1} + q^{4Y_1}}{Z - q^{2Y_1}} = Z^2 + 2q^{2Y_1} + \frac{3q^{4Y_1}}{Z - q^{2Y_1}}.$$

This means that $Z - q^{2Y_1}$ divides $3q^{4Y_1}$. Since q is prime, $Z - q^{2Y_1} = 3q^j$ or $Z - q^{2Y_1} = q^j$, for some $0 \leq j \leq 4Y_1$. If $j > 0$, then $q \mid Z$ which is a contradiction. Hence $j = 0$ and we have either $Z - q^{2Y_1} = 3$ or $Z - q^{2Y_1} = 1$. For the first case, we have

$$p^X + q^{6Y_1} = (q^{2Y_1} + 3)^3 = q^{6Y_1} + 9q^{4Y_1} + 27q^{2Y_1} + 27,$$

which implies that $p^X = 9q^{4Y_1} + 27q^{2Y_1} + 27$. This means that $9 \mid p^X$ or $p = 3$. This gives us $3^{X-2} = q^{4Y_1} + 3q^{2Y_1} + 3$. Since $\gcd(p, q) = \gcd(3, q) = 1$, we have $X = 2$ and consequently, $1 = q^{4Y_1} + 3q^{2Y_1} + 3$, a contradiction.

For the second case, we have

$$p^X + q^{6Y_1} = (q^{2Y_1} + 1)^3 = q^{6Y_1} + 3q^{4Y_1} + 3q^{2Y_1} + 1.$$

This implies that $p^{2X_1} = 3q^{4Y_1} + 3q^{2Y_1} + 1$. Let $u = q^{Y_1}$ and $v = p^{X_1}$ so that $v^2 = 3u^4 + 3u^2 + 1$. Using Theorem 2.1, if we let $\widehat{x} = \frac{2v+2}{u^2}$ and $\widehat{y} = \frac{4v+4+6u^2}{u^3}$ and define $a_1 = 0$, $a_2 = 3$, $a_3 = 0$, $a_4 = -12$ and $a_6 = -36$, then we get the elliptic curve

$$\widehat{y}^2 = \widehat{x}^3 + 3\widehat{x}^2 - 12\widehat{x} - 36.$$

By letting $x = \widehat{x} + 1$ and $y = \widehat{y}$, we obtain the elliptic curve

$$E_4 : y^2 = x^3 - 15x - 22.$$

We have computed its rank to be $r = 0$ and the torsion subgroup $E_4(\mathbb{Q})_{\text{tors}}$ of E_4 to be $\{\mathcal{O}, (-2, 0)\} \cong \mathbb{Z}/2\mathbb{Z}$. This means that $(x, y) = (-2, 0)$ is the only $\mathbb{Q}$-rational point on E_4. We retrieve $(\widehat{x}, \widehat{y}) = (-3, -1)$, which corresponds to no integer point in the original equation. $\square$

Theorem 3.5. *Let $\widehat{p}$ and q be distinct odd primes, and $p = \widehat{p}^k$, for some $k \in \mathbb{N}$. Then, the Diophantine equation $p^X + q^Y = Z^3$ can be transformed into the elliptic curve $E_4 : y^2 = x^3 - 15p^2x - 22p^3$ if $X \equiv 1 \pmod 2$, $Y \equiv 0 \pmod 6$ and k is even. Moreover, if the Mordell-Weil group of E_4 is trivial, then $p^X + q^Y = Z^3$ has no solutions in $\mathbb{N}$.*

Proof. Let $k = 2k_1$, $X = 2X_1 + 1$ and $Y = 6Y_1$, for some $X_1, \in \mathbb{N}_0, k_1, Y_1 \in \mathbb{N}$. By factoring, we have the following:

$$p^{2X_1+1} = (Z - q^{2Y_1})(Z^2 + Zq^{2Y_1} + q^{4Y_1}).$$

Since $p = \widehat{p}^k$, where $\widehat{p}$ is prime, there exist two non-negative integers α and β, $\alpha < \beta$ such that $\alpha + \beta = k(2X_1 + 1)$ and

$$\widehat{p}^{\beta-\alpha} = \frac{\widehat{p}^\beta}{\widehat{p}^\alpha} = \frac{Z^2 + Zq^{2Y_1} + q^{4Y_1}}{Z - q^{2Y_1}} = Z^2 + 2q^{2Y_1} + \frac{3q^{4Y_1}}{Z - q^{2Y_1}}.$$

This means that $Z - q^{2Y_1}$ divides $3q^{4Y_1}$. Since q is prime, $Z - q^{2Y_1} = 3q^j$ or $Z - q^{2Y_1} = q^j$, for some $0 \leq j \leq 4Y_1$. If $j > 0$, then $q \mid Z$ which is a contradiction. Hence, $j = 0$ and we have either $Z - q^{2Y_1} = 3$ or $Z - q^{2Y_1} = 1$. For the first case, we have

$$p^X + q^{6Y_1} = (q^{2Y_1} + 3)^3 = q^{6Y_1} + 9q^{4Y_1} + 27q^{2Y_1} + 27,$$

which implies that $p^X = 9q^{4Y_1} + 27q^{2Y_1} + 27$. This means that $9 \mid p^X$ or that is, $\widehat{p} = 3$. This gives us $3^{k(X-2)} = q^{4Y_1} + 3q^{2y_1} + 3$. Since $\gcd(p, q) = \gcd(3^k, q) = 1$, we have $X = 2$ which gives $1 = q^{4Y_1} + 3q^{2Y_1} + 3$, a contradiction. For the second case, we have

$$p^X + q^{6Y_1} = (q^{2Y_1} + 1)^3 = q^{6Y_1} + 3q^{4Y_1} + 3q^{2Y_1} + 1.$$

This implies that $p \cdot p^{2X_1+1} = 3pq^{4Y_1} + 3pq^{2Y_1} + p$. Let $u = q^{Y_1}$ and $v = p^{X_1+1}$ so that $v^2 = 3pu^4 + 3pu^2 + 1$. Using Theorem 2.1, if we let $\widehat{x} = \frac{2\widehat{p}^{k_1}v+2p}{u^2}$ and $\widehat{y} = \frac{4p^2v+4p\widehat{p}^{k_1}+6p\widehat{p}^{k_1}u^2}{u^3}$ and define $a_1 = 0$, $a_2 = 3p$, $a_3 = 0$, $a_4 = -12p^2$ and $a_6 = -36p^3$, then we get the elliptic curve

$$\widehat{y}^2 = \widehat{x}^3 + 3p\widehat{x}^2 - 12p^2\widehat{x} - 36p^3.$$

By letting $x = \widehat{x} + 1$ and $y = \widehat{y}$, we obtain the elliptic curve

$$E_5 : y^2 = x^3 - 15p^2x - 22p^3.$$

One can use the SAGE to determine the torsion and the rank of this elliptic curve for a specific value of p. Moreover, if the Mordell-Weil group of E_4 is trivial, then $p^X + q^Y = Z^3$ has no solutions in $\mathbb{N}$. $\square$

We have the following example demonstrating the use of Theorem 3.4 and Theorem 3.5.

Example 3.2. Consider the Diophantine equation $7^X + 11^Y = Z^3$ over the set of positive integers. Taking modulo 7, we get $Z^3 \equiv 4^Y \pmod 7$. Now, since $Z^3 \equiv 0, 1, 6 \pmod 7$, we get $4^Y \equiv 1 \pmod 7$ or that is $Y \equiv 0 \pmod 3$. Using Theorem 3.4, the equation has no solutions in $\mathbb{N}$ if X and Y are even. On the other hand, consider the Diophantine equation $49^X + 11^Y = Z^3$ where X is odd and Y is even. Using Theorem 3.5, this can be transformed into the elliptic curve $E : y^2 = x^3 - 36015x - 2588278$ which has rank 0. Furthermore, its torsion subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We can easily see that $(-98, 0)$ is the only non-trivial torsion point of E which does not correspond to an integer solution in the original equation.

4. Summary

In this paper, we presented a way of determining nonsolvability of exponential Diophantine equations of type $p^X + q^Y = Z^n$, where n is either $2, 3$ or 6, via transformation to a Weierstrass equation of elliptic curves. We did this because the rational points on an elliptic curve form an abelian group, and so are easier to determine. Theorems 3.1 and 3.2 are dedicated for the case when $n = 2$, and Theorems 3.3, 3.4 and 3.5 for the case when $n = 3$ and 6. These theorems do not cover all possible scenarios when solving a certain Diophantine equation but are effective in reducing the number of cases to be considered when solving for its solutions. For future works, some of the results can be extended to a more general family of exponential Diophantine equations or to any similar types of equation.

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A nonmonotone damped Gauss-Newton method for nonlinear complementarity problems

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Abstract. The damped Gauss-Newton methods have been successfully applied to solve the nonlinear complementarity problem (NCP). This class of methods is usually designed based on a monotone Armijo line search. In this paper, we propose a damped Gauss-Newton method with a nonmonotone line search to solve the NCP. Without requiring any problem assumptions, we prove that the proposed method is well defined and it is globally convergent. Moreover, under the nonsingularity assumption, we show that the proposed method is locally superlinearly/quadratically convergent. Some numerical results are reported.
Keywords: nonlinear complementarity problem, Gauss-Newton method, nonmonotone line search, quadratic convergence.

1. Introduction

The nonlinear complementarity problem (NCP) is to find $x \in \mathcal{R}^n$ such that

$$x \geq 0,\ F(x) \geq 0,\ x^T F(x) = 0, \tag{1}$$

where $F : \mathcal{R}^n \to \mathcal{R}^n$ is a continuously differentiable function. The NCP has been studied extensively due to its various applications in operations research, economic equilibrium and engineering design.

There has been developed a number of numerical algorithms for solving the NCP. Among them, the Newton-type algorithm is one kind of the most effective algorithms which is designed based on some equation reformulation of the NCP. One class of well-known Newton-type algorithms is the smoothing Newton methods (e.g., [2, 3, 5, 6, 11, 13]). This class of algorithms usually reformulates the NCP as a smooth nonlinear equation and then solves it by Newton method. It is worth pointing out that, in these smoothing Newton methods, to ensure Newton step be feasible, one usually requires that the function F has Cartesian P_0-property, that is, for every x and y in $\mathcal{R}^n$ with $x \neq y$, there is an index $i_0 \in \{1, ..., n\}$ such that $x_{i_0} \neq y_{i_0}$ and $(x_{i_0} - y_{i_0})(F_{i_0}(x) - F_{i_0}(y)) \geq 0$.

Another class of Newton-type algorithms is the damped Gauss-Newton methods (e.g, [4, 9, 10]). Different from smoothing Newton methods, the damped

Gauss-Newton methods usually reformulate the NCP as a nonsmooth nonlinear equation and then solve it. Since the Gauss-Newton equation is always solvable, the damped Gauss-Newton method is well defined without requiring that the function F has Cartesian P_0-property. It is worth pointing out that, in many damped Gauss-Newton methods (e.g., [8, 10]), the nonmonotone line search technique has been used to improve numerical results when the methods are implemented. However, the theoretical analyses are based on the methods with some monotone line search. As is well known, the nonmonotone line search technique can improve the likelihood of finding a global optimal solution and convergence speed in cases where the involving function is highly nonconvex and has a valley in a small neighbourhood of some point (e.g., [1, 14]). Encouraging numerical results have been reported when smoothing Newton methods with nonmonotone line search schemes were applied to solve NCPs (e.g., [2, 7, 11]).

In this paper, we propose a damped Gauss-Newton method to solve the NCP which is designed based on a nonmonotone line search scheme. We prove that the proposed method is well defined and it is globally convergent without requiring any problem assumptions. Moreover, we show that the convergence rate of the proposed method is local superlinear/quadratic under the nonsingularity assumption. We also report some numerical results which indicate that our method is very effective for solving NCPs even though these problems have no Cartesian P_0-property.

2. A nonmonotone damped Gauss-Newton method

2.1 The reformulation of the NCP

In this paper, we consider the following Fischer-Burmeister function:

$$\phi(a,b) := \sqrt{a^2+b^2} - (a+b), \quad \forall (a,b) \in \mathcal{R}^2, \tag{2}$$

which satisfies

$$\phi(a,b) = 0 \Longleftrightarrow a \geq 0,\ b \geq 0,\ ab = 0. \tag{3}$$

By using ϕ, we can reformulate the NCP as the following nonsmooth equation:

$$H(x) := \begin{pmatrix} \phi(x_1, F_1(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix} = 0. \tag{4}$$

Obviously, x is a solution of the NCP if and only if $H(x) = 0$.

Define the merit function $\psi(x) : \mathcal{R}^n \to \mathcal{R}$ as

$$\psi(x) := \frac{1}{2}\|H(x)\|^2 = \frac{1}{2}\sum_{i=1}^{n}(\phi(x_i, F_i(x)))^2. \tag{5}$$

The following lemma gives some useful properties which can be found in [4].

Lemma 2.1. (a) *$H(x)$ defined in (4) is semismooth on $\mathcal{R}^n$ and it is strongly semismooth on $\mathcal{R}^n$ if $F'(x)$ is Lipschitz continuous on $\mathcal{R}^n$.*

(b) *For any $x \in \mathcal{R}^n$ and $V \in \partial H(x)$, V can be represented as follows*

$$V = \mathrm{diag}(a_i)\nabla F(x)^T + \mathrm{diag}(b_i),$$

where $\mathrm{diag}(\alpha_i)$ denotes a diagonal matrix with the diagonal elements $\alpha_1, ..., \alpha_n$ and $(a_i+1)^2 + (b_i+1)^2 \leq 1,\ i = 1, ..., n$.

(c) *$\psi(x)$ defined by (5) is continuously differentiable on $\mathcal{R}^n$ and its gradient $\nabla\psi(x)$ can be represented as $\nabla\psi(x) = V^T H(x)$ for any $V \in \partial H(x)$.*

2.2 The algorithm

We now describe our nonmonotone damped Gauss-Newton method (NDGNM) as follows.

Algorithm NDGNM

Step 1. Choose $\gamma \in (0, 1/2), \eta \in (0, 1)$ and $x^0 \in \mathcal{R}^n$. Choose a sequence $\{\mu_k\}$ such that $\mu_k > 0$ for all $k \geq 0$. Choose a sequence $\{\tau_k\}$ such that $\tau_k \in (\tau, 1]$ where $\tau > 0$ is a constant. Set $R_0 := \psi(x^0)$. Set $k := 0$.

Step 2. Choose $V_k \in \partial H(x^k)$ and compute $\nabla\psi(x^k) = V_k^T H(x^k)$. If $\nabla\psi(x^k) = 0$, then stop.

Step 3. Let d_k be the solution of the following linear system

$$(V_k^T V_k + \mu_k I)d = -\nabla\psi(x^k). \tag{6}$$

Step 4. Find a step-size $\lambda_k := \eta^{m_k}$, where m_k is the smallest nonnegative integer m satisfying

$$\psi(x^k + \eta^m d_k) \leq R_k + \gamma\eta^m \nabla\psi(x^k)^T d_k. \tag{7}$$

Step 5. Set $x^{k+1} := x^k + \lambda_k d_k$ and

$$R_{k+1} := (1 - \tau_k)R_k + \tau_k \psi(x^{k+1}). \tag{8}$$

Set $k := k + 1$. Go to Step 2.

Theorem 2.1. *Algorithm NDGNM is well defined and its generated sequence $\{x^k\}$ satisfies $\psi(x^k) \leq R_k$ for all $k \geq 0$.*

Proof. Suppose that $\psi(x^k) \leq R_k$ holds for some k. If $\nabla\psi(x^k) = 0$, then Algorithm NDGNM terminates. Now, we suppose that $\nabla\psi(x^k) \neq 0$. Since $\mu_k > 0$, $V_k^T V_k + \mu_k I$ is positive definite and the search direction d_k in Step 3 is well defined. Moreover, since $\nabla\psi(x^k) \neq 0$, we have $d_k \neq 0$ and hence

$$\nabla\psi(x^k)^T d_k = -d_k^T (V_k^T V_k + \mu_k I) d_k < 0. \tag{9}$$

Next we show that there exists at least a nonnegative integer m satisfying (7). On the contrary, we suppose that for any nonnegative integer m,

$$\psi(x^k + \eta^m d_k) > R_k + \gamma\eta^m \nabla\psi(x^k)^T d_k. \tag{10}$$

Since $\psi(x^k) \leq R_k$, by (10), we have

$$\frac{\psi(x^k + \eta^m d_k) - \psi(x^k)}{\eta^m} > \gamma \nabla \psi(x^k)^T d_k.$$

Since ψ is continuously differentiable at x^k, by letting $m \to \infty$ in the above inequality, we have $\nabla \psi(x^k)^T d_k \geq \gamma \nabla \psi(x^k)^T d_k$. This contradicts (9) and $\gamma \in (0, 1/2)$. Hence, we can find a step-size λ_k in Step 4 and get the $(k+1)$-th iteration $x^{k+1} = x^k + \lambda_k d_k$. Moreover, from (7) and (9) we have

$$\psi(x^{k+1}) \leq R_k + \gamma \lambda_k \nabla \psi(x^k)^T d_k \leq R_k. \tag{11}$$

Using this fact, we obtain from (8) that

$$\psi(x^{k+1}) = (1-\tau_k)\psi(x^{k+1}) + \tau_k \psi(x^{k+1}) \leq (1-\tau_k)R_k + \tau_k \psi(x^{k+1}) = R_{k+1}.$$

Hence, we can conclude that if $\psi(x^k) \leq R_k$, then x^{k+1} can be generated by Algorithm NDGNM and it satisfies $\psi(x^{k+1}) \leq R_{k+1}$. Since $\psi(x^0) = R_0$, by the mathematical induction, we prove the theorem. The proof is completed. □

3. Convergence analysis

In this section, we assume that Algorithm NDGNM does not terminate in finitely many steps, i.e., $\nabla \psi(x^k) \neq 0$ for all $k \geq 0$. To establish the global convergence of Algorithm NDGNM, we need the following result.

Lemma 3.1 ([12], Corollary 1)**.** *Let $\{x^k\} \subset \mathcal{R}^n$ be a sequence converging to x. Let $\{V_k\}$ be a sequence such that $V_k \in \partial H(x^k)$ for all $k \geq 0$. Then $\{V_k\}$ is bounded. Moreover, if $\{V_k\}$ converges to V, then $V \in \partial H(x)$.*

Theorem 3.1 (Global convergence)**.** *Assume that x^* is an accumulation point of $\{x^k\}$ generated by Algorithm NDGNM. Then x^* is a stationary point of the merit function $\psi(x)$ if any one of the following conditions holds:*

(i) both $\{\mu_k\}$ and $\{d_k\}$ are bounded.

(ii) $\tilde{\mu} < \mu_k < \bar{\mu}$ for some $\bar{\mu} > \tilde{\mu} > 0$.

(iii) $\mu_k = \alpha \|H(x)\|^\beta$ for any $\alpha, \beta > 0$.

Moreover, x^* is a solution of the NCP if there exists a nonsingular element in $\partial H(x^*)$.

Proof. By (11), we have $\psi(x^{k+1}) \leq R_k$ for all $k \geq 0$. Then, it follows from (8) that for all $k \geq 0$

$$R_{k+1} = (1-\tau_k)R_k + \tau_k \psi(x^{k+1}) \leq (1-\tau_k)R_k + \tau_k R_k = R_k. \tag{12}$$

Thus, there exists a constant $R^* \geq 0$ such that $\lim_{k\to\infty} R_k = R^*$. By (8), we have for all $k \geq 1$

$$\psi(x^k) = R_{k-1} + \frac{R_k - R_{k-1}}{\tau_{k-1}}.$$

Since $\tau_k \geq \tau > 0$, we have $\lim_{k\to\infty} \psi(x^k) = R^*$. Now, without loss of generality, we assume that $\lim_{(K\ni)k\to\infty} x^k = x^*$ where K is a subsequence of $\{0, 1, ...\}$.

First, we consider the condition (i). Since $\{V_k\}_{k\in K}$ is bounded by Lemma 3.1, and $\{\mu_k\}_{k\in K}$ and $\{d^k\}_{k\in K}$ are bounded by the condition (i), by passing to the subsequence, we may assume that

$$\lim_{(K\ni)k\to\infty} V_k = V^*, \quad \lim_{(K\ni)k\to\infty} \mu_k = \mu^*, \quad \lim_{(K\ni)k\to\infty} d^k = d^*.$$

Moreover, by Lemma 3.1 we have $V^* \in \partial H(x^*)$. Thus, from Lemma 2.1 (c) it follows that $\nabla\psi(x^*) = (V^*)^T H(x^*)$ and

$$\lim_{(K\ni)k\to\infty} \nabla\psi(x^k) = \lim_{(K\ni)k\to\infty} V_k^T H(x^k) = (V^*)^T H(x^*) = \nabla\psi(x^*). \tag{13}$$

Now, we prove that $\nabla\psi(x^*)^T d^* = 0$. We divide the proof into the following two parts:

Part 1. $\lambda_k \geq c > 0$ for all $k \in K$ where c is a fixed constant. In this case, it follows from (7) and (9) that for all $k \in K$,

$$0 \leq -\gamma c \nabla\psi(x^k)^T d_k \leq -\gamma\lambda_k \nabla\psi(x^k)^T d_k \leq R_k - \psi(x^{k+1}). \tag{14}$$

Since $\lim_{k\to\infty} R_k = \lim_{k\to\infty} \psi(x^k) = R^*$, by letting $k \to \infty$ with $k \in K$ in (14), we have $\nabla\psi(x^*)^T d^* = 0$.

Part 2. $\{\lambda_k\}_{k\in K}$ has a subsequence converging to zero. We may pass to the subsequence and assume that $\lim_{(K\ni)k\to\infty} \lambda_k = 0$. From the line search (7), we get that for all sufficiently large $k \in K$,

$$\psi(x^k + \eta^{-1}\lambda_k d_k) > R_k + \gamma\eta^{-1}\lambda_k \nabla\psi(x^k)^T d_k.$$

Since $\psi(x^k) \leq R_k$ for all $k \geq 0$, it follows that

$$\psi(x^k + \eta^{-1}\lambda_k d_k) - \psi(x^k) \geq \gamma\eta^{-1}\lambda_k \nabla\psi(x^k)^T d_k,$$

i.e.,

$$\frac{\psi(x^k + \eta^{-1}\lambda_k d_k) - \psi(x^k)}{\eta^{-1}\lambda_k} \geq \gamma \nabla\psi(x^k)^T d_k. \tag{15}$$

Since ψ is continuously differentiable at x^*, by letting $k \to \infty$ with $k \in K$ in (15), we have

$$\nabla\psi(x^*)^T d^* \geq \gamma \nabla\psi(x^*)^T d^*. \tag{16}$$

On the other hand, since $\nabla\psi(x^k)^T d_k < 0$ for all $k \geq 0$ by (9), we have

$$\nabla\psi(x^*)^T d^* \leq 0. \tag{17}$$

Since $\gamma \in (0,1)$, we obtain from (16) and (17) that $\nabla\psi(x^*)^T d^* = 0$.

By Part 1 and Part 2, we can conclude that $\nabla\psi(x^*)^T d^* = 0$. Moreover, from (6) we have

$$\nabla\psi(x^*)^T d^* + (d^*)^T((V^*)^T V^* + \mu^* I)d^* = 0,$$

which gives

$$(d^*)^T((V^*)^T V^* + \mu^* I)d^* = 0. \tag{18}$$

If $\mu^* > 0$, then the matrix $(V^*)^T V^* + \mu^* I$ is positive definite. By (18), we have $d^* = 0$ which together with (6) gives $\nabla\psi(x^*) = -((V^*)^T V^* + \mu^* I)d^* = 0$. If $\mu^* = 0$, then by (18) we have $V^* d^* = 0$. Using (6) again, we have $\nabla\psi(x^*) = -(V^*)^T V^* d^* = 0$. This proves that x^* is a stationary point of ψ.

Next, we consider the condition (ii). Since $0 < \tilde{\mu} < \mu_k < \bar{\mu}$, the matrices $\{V_k^T V_k + \mu_k I\}$ are uniformly positive definite for all k. It follows from (6) that

$$\begin{aligned}
\|d_k\| &= \|(V_k^T V_k + \mu_k I)^{-1}\nabla\psi(x^k)\| \\
&\leq \|(V_k^T V_k + \mu_k I)^{-1}\|\|\nabla\psi(x^k)\| \\
&\leq \frac{1}{\mu_k}\|\nabla\psi(x^k)\| \\
&\leq \frac{1}{\tilde{\mu}}\|\nabla\psi(x^k)\|.
\end{aligned}$$

Since $\{\|\nabla\psi(x^k)\|\}_{k\in K}$ is bounded, $\{d_k\}_{k\in K}$ is bounded. So, by following from (i), we obtain the desired result.

At last, we consider the condition (iii). For all $k \geq 0$, since $\nabla\psi(x^k) \neq 0$, we have $H(x^k) \neq 0$ and hence $\mu_k = \alpha\|H(x)\|^\beta > 0$. Suppose that $\nabla\psi(x^*) \neq 0$. Then $\|H(x^*)\| > 0$. Since $\lim_{k\to\infty}\psi(x^k) = R^*$, by (5) and the continuity of H, we have

$$\lim_{k\to\infty}\mu_k = \lim_{k\to\infty}\alpha\left(\sqrt{2\psi(x^k)}\right)^\beta = \alpha(\sqrt{2R^*})^\beta = \alpha\|H(x^*)\|^\beta > 0.$$

So, there exists $\bar{\mu} > \tilde{\mu} > 0$ such that $\tilde{\mu} < \mu_k < \bar{\mu}$. By (ii), x^* must be a stationary point of $\psi(x)$. It is a contradiction. Thus, x^* is a stationary point of $\psi(x)$.

The second part of the theorem follows from Lemma 2.1 (c).

We complete the proof. $\square$

In a similar way as those in [10, Theorem 7.2], we can obtain the local superlinear/quadratic convergence of Algorithm NDGNM as follows.

Theorem 3.2 (Local superlinear/quadratic convergence)**.** *Assume that x^* is an accumulation point of $\{x^k\}$ generated by Algorithm NDGNM. Let $\mu_k = \alpha\|H(x)\|^\beta$ for some $\alpha, \beta > 0$. If all $V \in \partial H(x^*)$ are nonsingular, then the whole sequence $\{x^k\}$ converges to x^* superlinearly. Furthermore, if F' is Lipschitz continuous around x^* and $\beta \geq 1$, then the convergence rate is quadratic.*

In Theorem 3.1 and Theorem 3.2, we assume that the sequence $\{x^k\}$ generated by Algorithm NDGNM has one accumulation point x^* and all $V \in \partial H(x^*)$ are nonsingular. In the following, we show that this assumption is satisfied when F in the NCP is a uniform P-function. For this purpose, we need the following lemma.

Lemma 3.2 ([4], Lemma 4.1)**.** *Suppose that F is a uniform P-function, i.e., there exists a positive scalar $c > 0$ such that*

$$\max_{1\leq i\leq n} (x_i - y_i)(F_i(x) - F_i(y)) \geq c\|x - y\|^2, \quad \forall\ x, y \in \mathcal{R}^n.$$

Then, the following results hold:

- (i) *The NCP has a unique solution.*
- (ii) *For any $x \in \mathcal{R}^n$ and any $V \in \partial H(x)$, V is nonsingular.*
- (iii) *The level set $L(x^0) := \{x \in \mathcal{R}^n : \psi(x) \leq \psi(x^0)\}$ is bounded for any $x^0 \in \mathcal{R}^n$.*

Theorem 3.3. *If F is a uniform P-function, then the sequence $\{x^k\}$ generated by Algorithm NDGNM has at least one accumulation point x^* and all $V \in \partial H(x^*)$ are nonsingular.*

Proof. By Theorem 2.1 and (12), we have $\psi(x^k) \leq R_k \leq R_0 = \psi(x^0)$ for all $k \geq 0$. This together with Lemma 3.2 (iii) implies that $\{x^k\}$ is bounded and it has at least one accumulation point x^*. The second result holds by Lemma 3.2 (ii). $\square$

By Theorems 3.1–3.3 and Lemma 3.2 (i), we can directly have the following result

Theorem 3.4. *If F is a uniform P-function, then the sequence $\{x^k\}$ generated by Algorithm NDGNM converges to the unique solution of the NCP locally superlinearly/quadratically.*

4. Numerical results

In this section, we report some numerical results of Algorithm NDGNM. All experiments are carried on a PC with CPU of Inter(R) Core(TM)i7-7700 CPU @ 3.60 GHz and RAM of 8.00GB. The codes are written in MATLAB and run in MATLAB R2018a environment. The parameters used in Algorithm NDGNM are chosen as $\gamma = 0.1,\ \eta = 0.8, \tau_k = \frac{2^k+1}{2^{k+1}}$.

We consider the following linear complementarity problem (LCP):

$$x \geq 0,\ y \geq 0,\ y = Mx + q,\ x^T y = 0,$$

in which $M \in \mathcal{R}^{n\times n}$ and $q \in \mathcal{R}^n$. By using the Fischer-Burmeister function ϕ, we have a nonsmooth equation reformulation of the LCP:

$$H(x,y) := \begin{pmatrix} y - Mx - q \\ \phi(x_1, y_1) \\ \vdots \\ \phi(x_n, y_n) \end{pmatrix} = 0,$$

namely, (x, y) is a solution of the LCP if and only if $H(x, y) = 0$.

We apply Algorithm NDGNM to solve $H(x, y) = 0$ and use $\|H(x^k, y^k)\| \leq 10^{-6}$ as the stopping criterion. In our experiments, we investigate the following two LCPs:

(I) Let M be the block diagonal matrix with $M_1, ..., M_4$ as block diagonals, i.e., $M = \mathrm{diag}(M_1, ..., M_4)$, in which $M_i = \frac{N_i^T N_i}{\|N_i^T N_i\|}$ with $N_i = \mathrm{rand}(\frac{n}{4}, \frac{n}{4})$ for $i = 1, ..., 4$. Take $q = \mathrm{rand}(n, 1)$. In this case, the function $F(x) = Mx + q$ has the Cartesian P_0-property.

(II) Let $M = \mathrm{diag}(M_1, ..., M_4)$, in which $M_i = \frac{N_i}{\|N_i\|} - \mathrm{eye}(n/4)$ with $N_i = \mathrm{rand}(\frac{n}{4}, \frac{n}{4})$ for $i = 1, ..., 4$. Take $q = \mathrm{rand}(n, 1)$. In this case, the function $F(x) = Mx + q$ may have no Cartesian P_0-property.

In the experiments, we generate 10 problem instances for each size of n. We use the following two starting points: (1) $x^0 = (1, 0, ..., 0)^T$, $y^0 = (1, 1, ..., 1)^T$; (2) $x^0 = (1, 0, ..., 0)^T$, $y^0 = Mx^0 + q$. Numerical results are listed in Table 1 where **SP** denotes the starting point, **aIT** denotes the average value of the iteration numbers, **aCPU** denotes the average value of the CPU time in seconds and **aHK** denotes the average value of $\|H(x^k, y^k)\|$ when Algorithm NDGNM terminates among the 10 testing. From Table 1, we can see that Algorithm NDGNM is very effective for solving LCPs even though these problems have no Cartesian P_0-property.

Table 1 Numerical results of Algorithm NDGNM

LCP	SP	n	aIT	aCPU	aHK
(I)	(1)	1000	4.9	1.83	2.1706e-07
		1500	5.5	5.68	8.2870e-08
		2000	5.5	14.50	2.6248e-07
		2500	5.4	22.64	1.9886e-07
		3000	6.2	53.75	2.2750e-07
	(2)	1000	4.2	1.54	1.2565e-07
		1500	4.5	4.67	2.2360e-07
		2000	4.8	10.69	1.1695e-07
		2500	4.2	18.02	1.2727e-07
		3000	4.7	34.96	2.3348e-07
(II)	(1)	1000	4.1	1.47	1.2654e-07
		1500	4.0	9.07	2.3691e-07
		2000	4.2	17.62	1.1274e-07
		2500	4.2	30.95	3.7991e-07
		3000	4.4	41.55	1.1590e-07
	(2)	1000	3.2	1.13	1.1403e-07
		1500	3.4	3.47	8.1548e-08
		2000	3.6	8.18	2.1214e-08
		2500	3.2	13.17	4.4488e-08
		3000	3.5	24.83	1.4597e-07

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e-semicommutative modules

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Abstract. e-semicommutative modules and their related properties. Also, we investigate some extensions of rings and modules in terms of e-semicommutativity.
Keywords: reduced ring, symmetric ring, e-reduced ring, e-symmetric ring.

1. Introduction

Throughout this paper, all rings are associative with unity. R denotes an associative ring with unity, M_R is a unitary right R-module, $\mathrm{Id}\,(R)$ denotes the set of all idempotent elements of R, $\mathrm{N}\,(R)$ denotes the set of all nilpotent elements of R, $\mathrm{C}(R)$ denotes the center of R, $\mathrm{S_r}(R) = \{e \in \mathrm{Id}\,(R) : e\,R\,e = e\,R\}$ denotes the set of all right semicentral idempotent elements of R, $\mathrm{S}_\ell(R) = \{e \in \mathrm{Id}\,(R) : e\,R\,e = R\,e\}$ denotes the set of all left semicentral idempotent elements of R, and $\mathrm{r}_R(M) = \{a \in R : Ma = 0\}$ denotes the right annihilator of M in R.

A ring R is said to be abelian if $\mathrm{Id}\,(R) \subseteq \mathrm{C}(R)$. A ring R is called reduced if $\mathrm{N}\,(R) = 0$. This concept of reduced rings was extended to modules [9] as follows: a right R-module M_R is reduced if, for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Recall from [10], R is a right e-reduced ring, where $e \in \mathrm{Id}(R)$, if $\mathrm{N}\,(R)\,e = 0$. A ring R is called symmetric [8] if whenever $a, b, c \in R$ such that $abc = 0$, we have $acb = 0$. Recall from Refs. [8] and [11], a right R-module M_R is called *symmetric* if whenever $a, b \in R$ and $m \in M$ such that $mab = 0$ implies $mba = 0$. Following [10], a ring R is called *e-symmetric*, for $e \in \mathrm{Id}(R)$, if whenever $a, b, c \in R$ such that $abc = 0$, we have $acbe = 0$.

Introduce of these properties via idempotents, inspires us to extend the notions of e-reduced and e-symmetric to modules as follows:

Definition 1.1 ([1]). *A right R-module M_R is called e-reduced, where $e \in \mathrm{Id}(R)$, if whenever $a \in R$ and $m \in M$ such that $ma = 0$ implies $mR \cap Mae = 0$.*

Definition 1.2 ([1]). *A right R-module M_R is called e-symmetric, where $e \in \mathrm{Id}(R)$, if whenever $a, b \in R$ and $m \in M$ such that $mab = 0$ implies $mbae = 0$.*

A ring R is an e-reduced (e-symmetric) ring if and only if R_R is an e-reduced (e-symmetric) module.

According to [3] a ring R is called semicommutative, if whenever $a, b \in R$ satisfy $ab = 0$, then $aRb = 0$. A right R-module M_R is called semicommutative [5], if whenever $a \in R$ and $m \in M$ satisfy $ma = 0$, then $mRa = 0$. Recall from [7], a ring R is called e-semicommutative, for $e \in \mathrm{Id}(R)$, if whenever $a, b \in R$ such that $ab = 0$, we have $aRbe = 0$.

So it is natural to motivate us to extend the condition of e-semicommutativity to Module Theory.

2. Modules with e-semicommutative condition

In this section, we extend the notion of e-semicommutative rings to modules as follows:

Definition 2.1. *A right R-module M_R is called e-semicommutative, where $e \in \mathrm{Id}(R)$, if whenever $a \in R$ and $m \in M$ such that $ma = 0$ implies $mRae = 0$.*

Obviously, R is an e-semicommutative ring if and only if R_R is an e-semicommutative module.

Clearly, any semicommutative module is an e-semicommutative module, for any $e \in \mathrm{Id}(R)$, and every an e-reduced (e-symmetric) module is e-semicommutative. The following examples demonstrate rather strikingly that the class of e-semicommutative modules is properly contains the class of semicommutative modules.

Example 2.1. Let S be a semicommutative ring and $R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$. Consider a right R-module $M_R = R[x]_R$. Assume that $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$. We see that $(Ax + A)B = 0$ but $(Ax + A)CB \neq 0$. Then, M_R is not semicommutative. Now for the idempotent $E = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we can show that M_R is E-semicommutative. Let $f(x) = \sum_{i=0}^{n} A_i x^i \in M$, where $A_i = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} \in R$ for every $i = 0, 1, ..., n$, and $B = \begin{pmatrix} w & u \\ 0 & v \end{pmatrix} \in R$ such that $f(x)B = 0$. Then, $0 = A_i B = \begin{pmatrix} a_i w & a_i u + b_i v \\ 0 & c_i v \end{pmatrix}$ for every $i = 0, 1, ..., n$. Hence, $a_i w = 0$, $c_i v = 0$ and $a_i u + b_i v = 0$. For any element $C = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$, we have $f(x)CBE = \sum_{i=0}^{n} (A_i CBE)\, x^i = 0$. Therefore, M_R is E-semicommutative.

Example 2.2. Let S be a semicommutative ring and $R = \begin{pmatrix} S & 0 & 0 \\ S & S & S \\ 0 & 0 & S \end{pmatrix}$. Consider R_R as a right R-module. Assume that $m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R$. We see that $ma = 0$ but $mba \neq 0$. Then, R_R is not semicommutative. Now for the idempotent $e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we can show that R_R is e-semicommutative. Let $m = \begin{pmatrix} x_1 & 0 & 0 \\ y_1 & z_1 & w_1 \\ 0 & 0 & v_1 \end{pmatrix}$, $a = \begin{pmatrix} x_2 & 0 & 0 \\ y_2 & z_2 & w_2 \\ 0 & 0 & v_2 \end{pmatrix} \in R$ such that $ma = 0$. Hence, $x_1x_2 = z_1z_2 = v_1v_2 = z_1w_2 = w_1v_2 = 0$ and $y_1x_2 + z_1y_2 = 0$. For any element $r = \begin{pmatrix} x & 0 & 0 \\ y & z & w \\ 0 & 0 & v \end{pmatrix} \in R$, we have $mrae = \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_1zz_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, since S_S is semicommutative. Therefore, R_R is e-semicommutative.

Proposition 2.1. *The class of e-semicommutative modules is closed under submodules, direct products and so direct sums.*

Proof. The proof is immediate from the definitions and algebraic structures. □

Proposition 2.2. *Let R be a ring, $e \in \mathrm{Id}(R)$ and M_R a right R-module. M_R is e-semicommutative if and only if every cyclic submodule of M_R is e-semicommutative.*

Proof. Assume that every cyclic submodule of M_R is e-semicommutative. Let $a \in R$ and $m \in M$ such that $ma = 0$ in M. Consider the cyclic submodule mR, we have $ma = 0$ in mR. Since mR is e-semicommutative, we get $mRae = 0$. Hence, M_R is e-semicommutative. □

Proposition 2.3. *Let R be a ring, $e \in \mathrm{Id}(R)$ and M_R a right R-module. Then, the following two conditions are equivalent:*

1) M_R is an e-semicommutative module.

2) $NA = 0$ implies $NRAe = 0$ for any nonempty subset N in M and A in R.

Proof. "(1) $\Longrightarrow$ (2)" Assume that M_R is e-semicommutative and N is a subset of M and A is a subset of R such that $NA = 0$. Then, for any $n \in N$ and $a \in A$, we have $na = 0$. Thus, $nRae = 0$. Then, $\sum_{n\in N, a\in A} nRae = 0$. Hence, $NRAe = 0$.

"(2) $\Longrightarrow$ (1)" Assume that $a \in R$ and $m \in M$ such that $ma = 0$. Then, M_R is e-semicommutative follows directly if we set $N = \{m\}$ and $A = \{a\}$. $\square$

Proposition 2.4. *Let R be a ring with every right ideal is two sided and $e \in \mathrm{Id}(R)$. Then, every right R-module is e-semicommutative.*

Proof. Suppose that M_R is a right R-module. Let $a \in R$ and $m \in M$ such that $ma = 0$. From our assumption, the right ideal $ae\,R$ is two sided. Then, we have $R\,ae \subseteq ae\,R$. So, we get $m\,R\,ae \subseteq mae\,R = 0$. Therefore, M_R is e-semicommutative. $\square$

Proposition 2.5. *Let R, S be rings, $e \in \mathrm{Id}(R)$ and $\varphi : R \to S$ be a ring homomorphism. If M_S is a right S-module, then M is a right R-module via $mr = m\varphi(r)$ for all $r \in R$ and $m \in M$. Then, we get:*

(1) If M_S is a $\varphi(e)$-semicommutative module, then M_R is an e-semicommutative module.

(2) If φ is onto and M_R is an e-semicommutative module, then M_S is a $\varphi(e)$-semicommutative module.

Proof. (1) Suppose that M_S is a $\varphi(e)$-semicommutative module. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, $m\varphi(a) = 0$. Since M_S is $\varphi(e)$-semicommutative, we have $ms\varphi(a)\varphi(e) = 0$ for all $s \in S$. Hence, for any $r \in R$, we have $mrae = m\varphi(rae) = m\varphi(r)\varphi(a)\varphi(e) = 0$. Therefore, M_R is an e-semicommutative module.

(2) Suppose that φ is onto and M_R is an e-semicommutative module. Let $x \in S$ and $m \in M$ such that $mx = 0$. Since φ is onto, there exists $a \in R$ such that $x = \varphi(a)$. Then, $0 = mx = m\varphi(a) = ma$. Since M_R is e-semicommutative, implies $mRae = 0$. Hence, $0 = m\varphi(R)\varphi(a)\varphi(e) = mSx\varphi(e)$. Thus M_S is a $\varphi(e)$-semicommutative module. $\square$

Corollary 2.1. *Let R be a ring, $e \in \mathrm{Id}(R)$, M_R a right R-module and $\overline{R} = R/\mathrm{r}_R(M)$. M_R is an e-semicommutative module if and only if $M_{\overline{R}}$ is an $\overline{e}$-semicommutative module.*

Proof. This is a consequence of Proposition 2.5, if we consider the canonical epimorphism $\varphi : R \to \overline{R}$ defined by $\varphi(r) = \overline{r} = r + \mathrm{r}_R(M)$, for all $r \in R$. $\square$

Proposition 2.6. *Let R be a ring, $e \in \mathrm{C}(R)$ and M_R a right R-module. Then, M_R is an e-semicommutative module if and only if M_{Re} is a semicommutative module.*

Proof. "$\Longrightarrow$" Assume that M_R is an e-semicommutative module. Let $a \in R\,e \subseteq R$ and $m \in M$ such that $ma = 0$. Then, we get $mRae = 0$. Since $e \in \mathrm{C}(R)$, we have $m\,R\,ea = 0$. Hence, M_{Re} is a semicommutative module.

"$\Longleftarrow$" Assume that M_{Re} is a semicommutative module. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, we get $m\,R\,ea = 0$. Since $e \in \mathrm{C}(R)$, we have $m\,R\,ae = 0$. Thus M_R is an e-semicommutative module. $\square$

Corollary 2.2. *Let R be a ring, $e \in \mathrm{C}(R)$ and M_R a right R-module. If M_{Re} and $M_{R(1-e)}$ are semicommutative modules, then M_R is a semicommutative module.*

Proof. We can easily check that $e \in \mathrm{C}(R)$ if and only if $(1-e) \in \mathrm{C}(R)$. From Proposition 2.6, we conclude that M_R is both e-semicommutative and $(1-e)$-semicommutative. Now let $a \in R$ and $m \in M$ such that $ma = 0$. Thus $mRae = 0$ and $mRa(1-e) = 0$, which implies that $mRa = 0$. Therefore, M_R is a semicommutative module. □

Proposition 2.7. *Let R be a ring, $e \in \mathrm{S}_\ell(R)$ and M_R a right R-module. Then, M_R is an e-semicommutative module if and only if M_{eRe} is a semicommutative module.*

Proof. "$\Longrightarrow$" Assume that M_R is an e-semicommutative module. Let $ere \in e\,R\,e$ and $m \in M$ such that $m\,(ere) = 0$. Then, we get $m\,R\,(ere) = 0$. Since $e \in \mathrm{S}_\ell(R)$, we have $0 = m\,(R\,e)\,(ere) = m\,(e\,R\,e)\,(ere)$. Hence, M_{eRe} is a semicommutative module.

"$\Longleftarrow$" Assume that M_{eRe} is a semicommutative module. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, we get $mae = 0$. Since $e \in \mathrm{S}_\ell(R)$, we have $meae = 0$. Hence, $0 = m\,(e\,R\,e)\,(eae) = m\,(R\,e)\,(eae) = mR\,(eae) = mRae$. Thus M_R is an e-semicommutative module. □

Recall from [4], that a right R-module M_R is called principally quasi-Baer (p.q.-Baer for short) if for any $m \in M$, $\mathrm{r}_R(mR) = gR$, where $g \in \mathrm{Id}(R)$.

Proposition 2.8. *Let R be an abelian ring, $e \in \mathrm{Id}(R)$ and M_R a p.q.-Baer right R-module. If M_R is e-semicommutative, then M_R is e-reduced.*

Proof. Assume that M_R is e-semicommutative. Let $a \in R$ and $m \in M$ such that $ma = 0$. Then, we get $mRae = 0$. Let $x \in mR \cap Mae$, so there exist $r \in R$ and $n \in M$ such that $x = mr$ and $x = nae$. Since $ae \in \mathrm{r}_R(mR) = gR$, where $g \in \mathrm{Id}(R)$, we get $ae = gae$. Thus $x = ngae = naeg = xg = mrg = mgr = 0$. Hence, $mR \cap Mae = 0$. Therefore, M_R is an e-reduced module. □

3. Matrix extensions

This section is devoted to characterize right e-semicommutative 2-by-2 generalized upper triangular matrix rings. Moreover, as a corollary we obtain that a ring R is a right e-semicommutative ring if and only if $T_n(R)$ is right E-semicommutative for all positive integers n.

Theorem 3.1. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ where R and S are rings, and ${}_RM_S$ an (R,S)-bimodule. If T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in \mathrm{Id}(T)$, then:*

(1) R is a right e-semicommutative ring;
(2) S is a right g-semicommutative ring;
(3) M_S is a right g-semicommutative S-module.

Proof of Theorem 3.1. Assume that T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in \mathrm{Id}(T)$. Then, by easy computations, we can check that $e \in \mathrm{Id}(R), g \in \mathrm{Id}(S)$ and $ek + kg = k$.

(1) Assume that $ab = 0$, for $a, b \in R$. Consider the following elements $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in T$. We have $0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$. Since T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, we get for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T$,

$$0 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}.$$

Hence, $axbe = 0$ in R, for any $x \in R$. Therefore, R is a right e-semicommutative ring.

(2) Assume that $\alpha\beta = 0$, for $\alpha, \beta \in S$. Consider the following elements $\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \in T$. We have $0 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}$. Since T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, we get for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T$,

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}.$$

Hence, $\alpha z \beta g = 0$ in S, for any $z \in S$. Therefore, S is a right g-semicommutative ring.

(3) Let $a \in S$ and $m \in M$ such that $ma = 0$. Consider the following elements $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in T$. We have $0 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$. Since T is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, we get for any $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T$,

$$0 = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}.$$

Hence, $mzag = 0$ in M_S, for any $z \in S$. Therefore, M_S is a right g-semicommutative S-module.

Theorem 3.2. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are rings, and ${}_RM_S$ an (R, S)-bimodule. If T is a left $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}$-semicommutative ring, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in$ $\mathrm{Id}(T)$, then:*

(1) R is a left e-semicommutative ring,
(2) S is a left g-semicommutative ring, and
(3) $_RM$ is a left e-semicommutative R-module.

Proof of Theorem 3.2. The proof is similar to the proof of Theorem 3.1.

Theorem 3.3. *Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ where R and S are rings, and $_RM_S$ an (R, S)-bimodule. If R is a right e-semicommutative ring, where $e \in \mathrm{Id}(R)$, then T is a right $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$-semicommutative ring.*

Proof of Theorem 3.3. Assume that R is a right e-semicommutative ring, where $e \in \mathrm{Id}(R)$. Let $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}, \begin{pmatrix} q & n \\ 0 & p \end{pmatrix} \in T$ such that

$$0 = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} q & n \\ 0 & p \end{pmatrix} = \begin{pmatrix} aq & an + mp \\ 0 & bp \end{pmatrix}.$$

Hence, $aq = 0$ in R. Since R is a right e-semicommutative ring, we have $auqe = 0$, for any $u \in R$. Thus, for any $\begin{pmatrix} u & t \\ 0 & v \end{pmatrix} \in T$, we have

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} u & t \\ 0 & v \end{pmatrix} \begin{pmatrix} q & n \\ 0 & p \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Therefore, T is a right $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$-semicommutative ring.

Corollary 3.1. *Let $T_n(R)$ be the n-by-n upper triangular matrix ring over a ring R, where $n \geq 1$. Then, the following are equivalent:*
(1) *R is a right e-semicommutative ring, where $e \in \mathrm{Id}(R)$.*
(2) *$T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is a right $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$-semicommutative ring.*
(3) *$T_n(R)$ is a right $\begin{pmatrix} e & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$-semicommutative ring for every positive integer n.*

Proof. "(3) $\Longrightarrow$ (1)" follows directly from the fact that $T_1(R) \cong R$.
"(1) $\Longrightarrow$ (2)" is clear from Theorem 3.3.
"(2) $\Longrightarrow$ (3)" Note that $T_{n+1}(R) \cong \begin{pmatrix} R & M \\ 0 & T_n(R) \end{pmatrix}$ where M is the 1-by-n row matrix with R in every entry and 0 is the n-by-1 column zero matrix. So, this implication is proved by using induction on n. □

4. Polynomial extensions

This section is intended to motivate our investigation of the behavior of right e-semicommutative modules with respect to polynomial extensions.

Recall the following extensions of a right R-module M_R :

$$M[x] = \left\{ \varphi(x) = \sum_{i=0}^{n} m_i x^i : m_i \in M \right\}.$$

$M[x]$ is a right $R[x]$-module and $M[x]_{R[x]}$ is called *the usual polynomial extension of* M_R.

$$M[x, x^{-1}] = \left\{ \varphi(x) = \sum_{i=-k}^{n} m_i x^i : m_i \in M \right\}.$$

$M[x, x^{-1}]$ is a right $R[x, x^{-1}]$-module and $M[x, x^{-1}]_{R[x,x^{-1}]}$ is called *the usual Laurent polynomial extension of* M_R.

We mean by a regular element of a ring R, a nonzero element which is not a zero divisor.

Theorem 4.1. *Let R be a ring, Δ be a multiplicatively closed subset of R consisting of central regular elements, $1 \in \Delta$ and $e \in \mathrm{Id}(R)$. Then, M_R is e-semicommutative if and only if $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$ is $\left(1^{-1}e\right)$-semicommutative.*

Proof of Theorem 4.1. Suppose that M_R is e-semicommutative. Let $a \in R$, $m \in M$ and $u, w \in \Delta$ such that $\left(w^{-1}m\right)\left(u^{-1}a\right) = 0$ in $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$. Since Δ is contained in the center of R, we have $0 = \left(w^{-1}u^{-1}\right)(ma) = (wu)^{-1}(ma)$, and so $ma = 0$. Hence, for any $r \in R$, we have $mrae = 0$. So, in $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$, we have for any $v \in \Delta$, $0 = (wvu)^{-1}(mrae) = \left(w^{-1}v^{-1}u^{-1}1^{-1}\right)(mrae)$. Thus

$$\left(w^{-1}m\right)\left(v^{-1}r\right)\left(u^{-1}a\right)\left(1^{-1}e\right) = 0.$$

Hence, $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$ is $\left(1^{-1}e\right)$-semicommutative.

It is clear that if $\left(\Delta^{-1}M\right)_{(\Delta^{-1}R)}$ is $\left(1^{-1}e\right)$-semicommutative, then M_R is e-semicommutative.

Corollary 4.1. *Let R be a ring and $e \in \mathrm{Id}(R)$. Then, $M[x]_{R[x]}$ is e-semicommutative if and only if $M[x, x^{-1}]_{R[x,x^{-1}]}$ is e-semicommutative.*

Proof. Consider the multiplicatively closed set $\Delta = \left\{1, x, x^2, x^3, ...\right\}$ which is clearly a subset of $R[x]$ consisting of central regular elements. Since $\Delta^{-1}R[x] = R[x, x^{-1}]$ and $\Delta^{-1}M[x] = M[x, x^{-1}]$, the result follows directly from Theorem 4.1. $\square$

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The quasi frame and equations of non-lightlike curves in Minkowski $\mathbb{E}_1^3$ and $\mathbb{E}_1^4$

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Abstract. The quasi frame is an alternate frame to the Frenet-Serret frame but it is defined when the second derivative of the curve vanishes. It has the same behavior as a parallel transport frame but is easier in computation and has the same accuracy. In this paper, we investigate the quasi frame and equations of non-lightlike curves in 3-dimensional Minkowski space $\mathbb{E}_1^3$ and in 4-dimensional Minkowski space-time $\mathbb{E}_1^4$. Furthermore, we show the quasi frame can be considered as a generalization of Bishop frame in $\mathbb{E}_1^3$ and $\mathbb{E}_1^4$.

Keywords: Minkowski space, spacelike curve, timelike curve, Bishop frame.

1. Introduction

The Frenet frame was created to study the behavior of curves. The two curvatures $\{\kappa_i(s) \mid i = 1, 2\}$ in $\mathbb{E}^3$ (the three curvatures $\{\kappa_i(s) \mid i = 1, 2, 3\}$ in $\mathbb{E}^4$) play an effective role to identify the shape and size of the curve. The main disadvantage that appeared on this frame is when the second derivative in $\mathbb{E}^3$ (one of the curvatures $\{\kappa_i(s) \mid i = 1, 2, 3\}$ in $\mathbb{E}^4$) of a curve vanishes i.e. if the curve was

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a straight line or at an inflection point, the Frenet frame in these cases becomes undefined [1].

In 1975, R. Bishop created a frame that called an alternative frame or parallel transport frame that is well defined when the second derivative in $\mathbb{E}^3$ (one of the curvatures $\{\kappa_i(s) \mid i = 1,2,3\}$ in $\mathbb{E}^4$) is zero. This frame was known as Bishop frame [1]. The idea of Bishop in $\mathbb{E}^3$ ($\mathbb{E}^4$) based on the observation of a tangent vector field takes place in the same direction and the other vector fields take place in a plane perpendicular to the tangent vector field so, their derivatives take the same direction of the tangent vector field.

In 1983, Bishop and Hanson gave the advantages of a parallel transport frame [7] and regarded it as a developed frame of the Frenet frame. Many researchers have been using Bishop's concepts. In Euclidean space, see [3, 6]; in Minkowski space, see [2, 12]; In dual space, see [9] and this frame is developed to study of canal and tubular surfaces, see [8].

In 2015, C. Ekici and H. Tozak [4] defined a framing alternative to the Frenet-Serret frame called the quasi frame. The behavior of the quasi frame is similar to Bishop Frame but it is easier in computing, although both frames have similar accuracy. In 2020, M. Khalifa and R. A. Abdel-Baky used the quasi frame to study the skew ruled surfaces in Euclidean space[10].

In this paper, we investigate the quasi frame and equations of non-lightlike curves in 3-dimensional Minkowski space $\mathbb{E}^3_1$ and in 4-dimensional Minkowski space-time $\mathbb{E}^4_1$. Furthermore, we show the quasi frame can be considered as a generalization of Bishop frame in $\mathbb{E}^3_1$ and $\mathbb{E}^4_1$. This paper is organized as follows: In section 2, some basic definitions of the frame and equations of Frenet are presented in 3-dimensional Minkowski space $\mathbb{E}^3_1$ and 4-dimensional Minkowski space-time $\mathbb{E}^4_1$. In section 3, we investigate the quasi equations in 3-dimensional Minkowski space $\mathbb{E}^3_1$ in the three different cases of a non-lightlike curve by using the transformation matrix between the quasi and Frenet frames. In section 4, we investigate the quasi equations in 4-dimensional Minkowski space $\mathbb{E}^4_1$ in the four different cases of a non-lightlike curve by using the transformation matrices between the quasi and Frenet frames.

2. Preliminaries

The Minkowski space $\mathbb{E}^3_1$ is the space $\mathbb{R}^3$ with a metric g, where g is defined by $g = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a coordinate system of $\mathbb{E}^3_1$. If $v \in \mathbb{E}^3_1$ then, the vector v is called a spacelike, a timelike or a lightlike(null), if $g(v,v) > 0$, $g(v,v) < 0$ or $g(v,v) = 0$ and $v \neq 0$, respectively. In particular, the vector $v = 0$ is a spacelike.

Let $\alpha(s)$ be any curve in Minkowski $\mathbb{E}^3_1$, then frenet equations are given by

$$(1) \qquad \begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ \epsilon_1\kappa_1 & 0 & \epsilon_2\kappa_2 \\ 0 & \epsilon_3\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

• If $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $(i = 2, 3)$ then, the curve is spacelike with spacelike principal normal.

• If $\epsilon_i = 1$ for $(i = 1, 2, 3)$ then, the curve is spacelike with spacelike binormal.

• If $\epsilon_3 = -1$ and $\epsilon_i = 1$ for $(i = 1, 2)$ then, the curve is timelike.

The Minkowski space $\mathbb{E}_1^4$ is the space $\mathbb{R}^4$ with a metric g, where g is defined by
$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, where (x_1, x_2, x_3, x_4) is a coordinate system of $\mathbb{E}_1^4$. If $v \in \mathbb{E}_1^4$ then, the vector v is called a spacelike, a timelike or a lightlike(null), if $g(v, v) > 0$, $g(v, v) < 0$ or $g(v, v) = 0$ and $v \neq 0$, respectively. In particular, the vector $v = 0$ is a spacelike.

Let $\alpha(s)$ be any curve in Minkowski $\mathbb{E}_1^4$ then, Frenet equations are given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \epsilon_1\kappa_1 & 0 & \epsilon_2\kappa_2 & 0 \\ 0 & \epsilon_3\kappa_2 & 0 & \epsilon_4\kappa_3 \\ 0 & 0 & \epsilon_5\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{2}$$

• If $\epsilon_1 = \epsilon_3 = -1$ *and* $\epsilon_i = 1$ for $(i = 2, 4, 5)$ then, the curve is spacelike with spacelike principal normal with spacelike principal first binormal.

• If $\epsilon_1 = \epsilon_5 = -1$ and $\epsilon_i = 1$ for $(i = 2, 3, 4)$ then, the curve is spacelike with spacelike principal normal with spacelike principal second binormal.

• If $\epsilon_5 = -1$ and $\epsilon_i = 1$ for $(i = 1, 2, 3, 4)$ then, the curve is spacelike with spacelike principal first and second binormals.

• If $\epsilon_3 = \epsilon_5 = -1$ and $\epsilon_i = 1$ for $(i = 1, 2, 4)$ then, the curve is timelike.

In $\mathbb{E}^3$, let $\alpha(s)$ be a curve, quasi frame depends on three orthonormal vectors, $\mathbf{T}(s)$ is the tangent vector, $\mathbf{N}_q(s)$ is the quasi normal and $\mathbf{B}_q(s)$ is the quasi binormal vector. The quasi frame $\{\mathbf{T}(s), \mathbf{N}_q(s), \mathbf{B}_q(s)\}$ is given by

$$\mathbf{T} = \frac{\alpha'}{\| \alpha' \|}, \quad \mathbf{N}_q = \frac{\mathbf{T} \times \mathbf{k}}{\| \mathbf{T} \times \mathbf{k} \|}, \quad \mathbf{B}_q = \mathbf{T} \times \mathbf{N}_q, \tag{3}$$

where $\mathbf{k}$ is the projection vector.

For our calculations, we have chosen $\mathbf{k} = (0, 0, 1)$ in this paper. In all cases that $\mathbf{T}$ and $\mathbf{k}$ are parallel then, the quasi frame is singular. Thus, in those cases $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

Let $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the Frenet-Serret frame vectors and $\theta(s)$ is an Euclidean angle between principal normal $\mathbf{N}(s)$ and quasi normal $\mathbf{N}_q(s)$ then, we obtain

$$\begin{aligned} \mathbf{N}_q &= cos\theta \ \mathbf{N} + sin\theta \ \mathbf{B}, \\ \mathbf{B}_q &= -sin\theta \ \mathbf{N} + cos\theta \ \mathbf{B}. \end{aligned} \tag{4}$$

• Let us consider a line curve parametrized by

$$\alpha(t) = (t, t, 0).$$

Easily, we see the Frenet frame is not suitable for this curve, while the quasi frame is given by

$$\begin{aligned}\mathbf{T}(t) &= (1/\sqrt{2}, 1/\sqrt{2}, 0),\\ \mathbf{N_q}(t) &= (1/\sqrt{2}, -1/\sqrt{2}, 0),\\ \mathbf{B_q}(t) &= (0, 0, 1).\end{aligned}$$

Which indicates that the quasi frame is better than Frenet frame.

• Let us consider a curve parametrized by

$$\alpha(t) = (t, t, t^9).$$

Easily, we get

$$\kappa(t) = \frac{72\sqrt{2}}{(2 + 82t^{16})^{3/2}}, \qquad \tau(t) = 0.$$

Since $\tau \equiv 0$ then, the angle between the Bishop frame and the Frenet frame is constant, therefore the Bishop frame is also not suitable for this curve, while the quasi frame is given by

$$\begin{aligned}\mathbf{T}(t) &= \frac{(1, 1, 9t^2)}{\sqrt{2 + 81t^{16}}},\\ \mathbf{N_q}(t) &= \frac{1}{2}(\sqrt{2}, -\sqrt{2}, 0),\\ \mathbf{B_q}(t) &= \frac{(9\sqrt{2}t^8, 9\sqrt{2}t^8, -2\sqrt{2})}{2\sqrt{2 + 81t^{16}}}.\end{aligned}$$

• Let us consider the curve parametrized by

$$\alpha(t) = (2t, t^2, t^3/3).$$

The quasi and Bishop frames of the curve have the same behavior but, the computing of the Bishop frame along the curve is difficult, although both of the frames have similar accuracy.

In $\mathbb{E}^4$, let $\alpha(s)$ be a curve, quasi frame depends on four orthonormal vectors, $\mathbf{T}(s)$ is the tangent vector, $\mathbf{N}_q(s)$ is the quasi normal, $\mathbf{B}_{1q}(s)$ is the quasi first binormal vector and $\mathbf{B}_{2q}(s)$ is the second binormal. The quasi frame $\{\mathbf{T}(s), \mathbf{N}_q(s), \mathbf{B}_{1q}(s), \mathbf{B}_{2q}(s)\}$ is given by

$$\begin{aligned}&\mathbf{T} = \frac{\alpha'}{\| \alpha' \|}, \quad \mathbf{N}_q = \frac{\mathbf{T} \times \mathbf{k}_1 \times \mathbf{k}_2}{\| \mathbf{T} \times \mathbf{k}_1 \times \mathbf{k}_2 \|},\\ &\mathbf{B}_{2q} = \zeta \frac{\mathbf{T} \times \mathbf{N}_q \times \alpha'''}{\| \mathbf{T} \times \mathbf{N}_q \times \alpha''' \|}, \quad \mathbf{B}_{1q} = \zeta \mathbf{B}_{2q} \times \mathbf{T} \times \mathbf{N}_q,\end{aligned} \tag{5}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the projection vectors and ζ is ± 1 where the determinant of matrix is equal to 1.

For simplicity, we choose $\mathbf{k}_1 = (0,0,0,1)$ and $\mathbf{k}_2 = (0,0,1,0)$ in our calculations. However, the quasi frame is singular when $\mathbf{T}$ and $\mathbf{k}_1$ or $\mathbf{T}$ and $\mathbf{k}_2$ or $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel and in those cases we may change our projection vectors.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ are the Frenet-Serret frame vectors, where $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}_1(s)$ and $\mathbf{B}_2(s)$ are tangent, principal normal, first and second binormal vector fields, respectively and $\theta(s)$ is an Euclidean angle between principal normal $\mathbf{N}(s)$ and quasi normal $\mathbf{N}_q(s)$ then, we obtain

$$\begin{aligned}
\mathbf{N}_q &= \cos\theta\ \cos\psi\ \mathbf{N} + (-\cos\phi\ \sin\psi + \sin\theta\ \sin\phi\ \cos\psi)\mathbf{B}_1 \\
&\quad + (\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)\mathbf{B}_2 \\
\mathbf{B}_{1q} &= \cos\theta\ \sin\psi\ \mathbf{N} + (\cos\phi\ \cos\psi + \sin\theta\ \sin\phi\ \sin\psi)\mathbf{B}_1 \\
&\quad + (-\sin\phi\ \cos\psi + \cos\phi\ \sin\theta\ \sin\psi)\mathbf{B}_2 \\
\mathbf{B}_{2q} &= \sin\theta\ N + \cos\theta\ \sin\phi\ \mathbf{B}_1 + \cos\theta\ \cos\phi\ \mathbf{B}_2.
\end{aligned} \tag{6}$$

3. Quasi equations in $\mathbb{E}_1^3$

In this section, we investigate quasi equations in 3-dimensional Minkowski space $\mathbb{E}_1^3$ in the three different cases of a non-lightlike curve by using the transformation matrix between quasi and Frenet-Serret frames. Furthermore, we introduce the quasi curvatures in Minkowski 3-space.

Theorem 3.1. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi timelike binormal vector field $\boldsymbol{B}_q(s)$ then, the quasi equations are given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & -K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{bmatrix}, \tag{7}$$

where $K_1 = \kappa_1 \cosh\theta$, $K_2 = \kappa_1 \sinh\theta$ and $K_3 = \kappa_2 + \theta'$.

Proof 3.1. Let the transformation matrix is given by

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{8}$$

By using Equation (1), we obtain

$$\mathbf{T}' = \kappa_1\ \mathbf{N} = \kappa_1\ \cosh\theta\ \mathbf{N}_q - \kappa_1\ \sinh\theta\ \mathbf{B}_q, \tag{9}$$

Since $\mathbf{N}_q = \cosh\theta\ \mathbf{N} + \sinh\theta\ \mathbf{B}$, $\mathbf{B}_q = \sinh\theta\ \mathbf{T} + \cosh\theta\ \mathbf{B}$ and by differentiating with respect to arc length s, we get

$$\begin{aligned}
\mathbf{N}_q' &= -\kappa_1\ \cosh\theta\ \mathbf{T} + (\kappa_2 + \theta')\ \mathbf{B}_q, \\
\mathbf{B}_q' &= -\kappa_1\ \sinh\theta\ \mathbf{T} + (\kappa_2 + \theta')\ \mathbf{N}_q.
\end{aligned} \tag{10}$$

Therefore, the proof is completed.

Corollary 3.1. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $N_q(s)$ and a quasi timelike binormal vector field $B_q(s)$ then, quasi curvatures $\{K_i|i=1,2,3\}$ can be determined by*

$$\begin{aligned} K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}'_q, \boldsymbol{T}),\\ K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_q) = -g(\boldsymbol{B}'_q, \boldsymbol{T}),\\ K_3 &= -g(\boldsymbol{N}'_q, \boldsymbol{B}_q) = g(\boldsymbol{B}'_q, \boldsymbol{N}_q). \end{aligned} \tag{11}$$

Corollary 3.2. *If we put $(\kappa_2 = -\acute{\theta})$ in Equation (7), we get the same results as Bishop frame.*

The next two theorems can be proved analogously so, we omit their proofs.

Theorem 3.2. *If $\alpha(s)$ be a curve is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector field $\boldsymbol{B}_q(s)$ then, quasi equations is given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}'_q \\ \boldsymbol{B}'_q \end{bmatrix} = \begin{bmatrix} 0 & K_1 & -K_2 \\ K_1 & 0 & K_3 \\ K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{bmatrix}, \tag{12}$$

where $K_1 = \kappa_1 \ \cosh\theta$, $K_2 = \kappa_1 \ \sinh\theta$ and $K_3 = \kappa_2 + \theta'$.

Corollary 3.3. *If $\alpha(s)$ be a curve is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector field $\boldsymbol{B}_q(s)$ then, quasi curvatures $\{K_i|i=1,2,3\}$ can be determined by*

$$\begin{aligned} K_1 &= -g(\boldsymbol{T}', \boldsymbol{N}_q) = g(\boldsymbol{N}'_q, \boldsymbol{T}),\\ K_2 &= -g(\boldsymbol{T}', \boldsymbol{B}_q) = g(\boldsymbol{B}'_q, \boldsymbol{T}),\\ K_3 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_q) = -g(\boldsymbol{B}'_q, \boldsymbol{N}_q). \end{aligned} \tag{13}$$

Corollary 3.4. *If we put $(\kappa_2 = -\acute{\theta})$ in Equation (12), we get the same results as Bishop frame.*

Theorem 3.3. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector $\boldsymbol{B}_q(s)$ then, quasi equations is given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}'_q \\ \boldsymbol{B}'_q \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 \\ K_1 & 0 & K_3 \\ K_2 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{bmatrix}, \tag{14}$$

where $K_1 = \kappa_1 \cos\theta$, $K_2 = -\kappa_1 \sin\theta$ and $K_3 = \kappa_2 + \theta'$.

Note that: The transformation matrix is given by

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{15}$$

Corollary 3.5. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector $\boldsymbol{B}_q(s)$ then, quasi curvatures $\{K_i|i=1,2,3\}$ can be determined by*

$$
\begin{aligned}
K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}_q', \boldsymbol{T}),\\
K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_q) = -g(\boldsymbol{B}_q', \boldsymbol{T}),\\
K_3 &= g(\boldsymbol{N}_q', \boldsymbol{B}_q) = -g(\boldsymbol{B}_q', \boldsymbol{N}_q).
\end{aligned} \tag{16}
$$

Corollary 3.6. *If we put $(\kappa_2 = -\acute{\theta})$ in equations (14), we get the same results as Bishop frame.*

4. Quasi equations in $\mathbb{E}_1^4$

In this section, we investigate quasi equations in 4-dimensional Minkowski space $\mathbb{E}_1^4$ in the four different cases of a non-lightlike curve by using the transformation matrices between quasi and Frenet-Serret frames.

Theorem 4.1. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi equations is given by*

$$
\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_{1q}' \\ \boldsymbol{B}_{2q}' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ K_1 & 0 & K_4 & K_5 \\ K_2 & -K_4 & 0 & K_6 \\ K_3 & -K_5 & -K_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_{1q} \\ \boldsymbol{B}_{2q} \end{bmatrix}, \tag{17}
$$

where

$$
\begin{aligned}
K_1 &= \kappa_1 \ \cos\phi \ cos\psi,\\
K_2 &= \kappa_1 \ (-\cos\psi \ \sin\theta \ \sin\phi + \cos\theta \ \sin\psi),\\
K_3 &= \kappa_1 \ (\cos\theta \ \cos\psi \ \sin\phi + \sin\theta \ sin\psi),\\
K_4 &= \sin\theta \ (\kappa_3 \ \sin\psi + \phi') + \cos\theta \ \Big(\kappa_3 \ \cos\psi \ \sin\phi + \cos\phi \ (\kappa_2 - \psi')\Big),\\
K_5 &= -\cos\theta \ \cos^2\phi \ (\kappa_3 \ \sin\psi + \phi') + \cos\phi \ \sin\theta \ (\kappa_2 - \psi')\\
&\quad + \sin\phi \ \Big(\kappa_3 \ \cos\psi \ \sin\theta - \cos\theta \ \sin\phi \ (\kappa_3 \ \sin\psi + \phi')\Big),\\
K_6 &= -\sin^2\theta \ \Big[-\kappa_3 \ \cos\phi \ \cos\psi + \cos^2\psi\Big(\theta' + \sin\phi \ (\kappa_2 - \psi')\Big)\\
&\quad + \sin^2\psi\Big(\theta' + \sin\phi(\kappa_2 - \psi')\Big)\Big] - \cos^2\theta\Big[-\kappa_3 \ \cos\phi \ \cos\psi + \cos^2\phi \ \theta'\\
&\quad + \cos^2\psi \ \sin\phi\Big(\kappa_2 + \sin\phi \ \theta' - \psi'\Big) + \sin\phi \ \sin^2\psi \ \Big(\kappa_2 + \sin\phi \ \theta' - \psi'\Big)\Big].
\end{aligned}
$$

Proof 4.1. We have three possible simple rotations. The first rotation exists on the spacelike plane spanned by the spacelike Frenet first binormal $\mathbf{B}_1$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

so,

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_{1q} \\ \mathbf{B}_{2q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi\cos\psi & -\cos\phi\sin\psi & -\sin\phi \\ 0 & -\cos\psi\sin\theta\sin\phi + \cos\theta\sin\psi & \cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\phi\sin\theta \\ 0 & \cos\theta\cos\psi\sin\phi + \sin\theta\sin\psi & \cos\psi\sin\theta - \cos\theta\sin\phi\sin\psi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{19}$$

By using Equations (2), we obtain

$$\begin{aligned} \mathbf{N}'_q =& \Big[\kappa_1 \ \cos\phi \ \cos\psi\Big]\mathbf{T} \\ &+ \Big[\kappa_2 \ \cos\phi \ \sin\psi - \cos\psi \ \sin\phi \ \phi' - \cos\phi \ \sin\psi \ \psi'\Big]\mathbf{N} \\ &+ \Big[\kappa_2 \ \cos\phi \ \cos\psi + \kappa_3 \ \sin\phi + \sin\phi \ \sin\psi \ \phi' - \cos\phi \ \cos\psi \ \psi'\Big]\mathbf{B}_1 \\ &+ \Big[-\kappa_3 \ \cos\phi \ \sin\psi + (-\cos\phi \ \phi')\Big]\mathbf{B}_2, \\ \mathbf{B}'_{1q} =& \Big[\kappa_1(-\cos\psi\sin\theta\sin\phi + \cos\theta\sin\psi)\Big]\mathbf{T} \\ &+ \Big[-\kappa_2(\cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi) - \sin\theta\sin\psi\theta' \\ &- \cos\psi(\cos\theta\sin\phi\theta' + \cos\phi\sin\theta\phi') + (\cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi)\psi'\Big]\mathbf{N} \\ &+ \Big[\kappa_2(-\cos\psi\sin\theta\sin\phi + \cos\theta\sin\psi) + \kappa_3\cos\phi\sin\theta - \cos\psi\sin\theta\theta' \\ &+ \sin\psi(\cos\theta\sin\phi\theta' + \cos\phi\sin\theta\phi') + (\cos\psi\sin\theta\sin\phi - \cos\theta\sin\psi)\psi'\Big]\mathbf{B}_1 \end{aligned} \tag{20}$$

$$
\begin{aligned}
&+\Big[\kappa_3(\cos\theta\cos\psi+\sin\theta\sin\phi\sin\psi)-\cos\theta\cos\phi\theta'+\sin\theta\sin\phi\phi'\Big]\mathbf{B}_2,\\
\mathbf{B}'_{2q}=&\Big[\kappa_1(\cos\theta\cos\psi\sin\phi+\sin\theta\sin\psi)\Big]\mathbf{T}\\
&+\Big[-\kappa_2(\cos\psi\sin\theta-\cos\theta\sin\phi\sin\psi)+\cos\theta\sin\psi\theta'\\
&+\cos\psi(-\sin\theta\sin\phi\theta'+\cos\theta\cos\phi\phi')+\cos\psi\sin\theta-\cos\theta\sin\phi\sin\psi)\psi'\Big]\mathbf{N}\\
&+\Big[\kappa_2(\cos\theta\cos\psi\sin\phi+\sin\theta\sin\psi)-\kappa_3\cos\theta\cos\phi+\cos\theta\cos\psi\theta'\\
&-\cos\theta\cos\phi\sin\psi\phi'-\sin\theta\sin\psi\psi'-\sin\phi(-\sin\theta\sin\psi\theta'+\cos\theta\cos\psi\psi')\Big]\mathbf{B}_1\\
&+\Big[\kappa_3(\cos\psi\sin\theta-\cos\theta\sin\phi\sin\psi)-\cos\phi\sin\theta\theta'-\cos\theta\sin\phi\phi'\Big]\mathbf{B}_2.
\end{aligned}
$$

Therefore,

$$
\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'_q\\ \mathbf{B}'_{1q}\\ \mathbf{B}'_{2q}\end{bmatrix}=\begin{bmatrix}0 & K_1 & K_2 & K_3\\ K_1 & 0 & K_4 & K_5\\ K_2 & -K_4 & 0 & K_6\\ K_3 & -K_5 & -K_6 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}_q\\ \mathbf{B}_{1q}\\ \mathbf{B}_{2q}\end{bmatrix}.
$$

Corollary 4.1. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i|i=1,2,3,4,5,6\}$ can be determined by*

$$
\begin{aligned}
&K_1=g(\boldsymbol{T}',\boldsymbol{N}_q)=-g(\boldsymbol{N}'_q,\boldsymbol{T}),\\
&K_2=g(\boldsymbol{T}',\boldsymbol{B}_{1q})=-g(\boldsymbol{B}'_{1q},\boldsymbol{T}),\\
&K_3=g(\boldsymbol{T}',\boldsymbol{B}_{2q})=-g(\boldsymbol{B}'_{2q},\boldsymbol{T}),\\
&K_4=g(\boldsymbol{N}'_q,\boldsymbol{B}_{1q})=-g(\boldsymbol{B}'_{1q},\boldsymbol{N}_q)\\
,&K_5=g(\boldsymbol{N}'_q,\boldsymbol{B}_{2q})=-g(\boldsymbol{B}'_{2q},\boldsymbol{N}_q)\\
&K_6=g(\boldsymbol{B}'_{1q},\boldsymbol{B}_{2q})=-g(\boldsymbol{B}'_{2q},\boldsymbol{B}_{1q}).
\end{aligned}\tag{21}
$$

Corollary 4.2. *If we put $\kappa_2=\psi'+\phi'\ tan\phi\ cot\psi$ and $\kappa_3=-\frac{\phi'}{sin\psi}$ in equations (17), we can easily find ($K_4=0=K_5=K_6$) and hence, we have the same result as Bishop frame.*

The next three theorems can be proved analogously so, we omit their proofs.

Theorem 4.2. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi timelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi equations are given by*

$$
\begin{bmatrix}\boldsymbol{T}'\\ \boldsymbol{N}'_q\\ \boldsymbol{B}'_{1q}\\ \boldsymbol{B}'_{2q}\end{bmatrix}=\begin{bmatrix}0 & K_1 & K_2 & K_3\\ -K_1 & 0 & K_4 & K_5\\ -K_2 & -K_4 & 0 & K_6\\ K_3 & K_5 & K_6 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{N}_q\\ \boldsymbol{B}_{1q}\\ \boldsymbol{B}_{2q}\end{bmatrix},\tag{22}
$$

where

$$
\begin{aligned}
K_1 &= \kappa_1\ \cos\psi\ \cosh\phi,\\
K_2 &= \kappa_1\ (\cosh\theta\ \sin\psi + \cos\psi\ \sinh\theta\ \sinh\phi),\\
K_3 &= -\kappa_1\ (\sin\psi\ \sinh\theta + \cos\psi\ \cosh\theta\ \sinh\phi),\\
K_4 &= \kappa_3\ \cos\psi\ \cosh\theta\ \sinh\phi + \sinh\theta\ \Big[\cosh^2\phi\ (\kappa_3\ \sin\psi - \phi')\\
&\quad + \sin\psi\ \sinh^2\phi(-\kappa_3 + \sin\psi\ \phi')\Big] + \cos^2\psi\ \Big[\sinh\theta\ \sinh^2\phi\ \phi'\\
&\quad + \cosh\theta\ \cosh\phi\ (\kappa_2 - \psi')\Big] + \cosh\theta\ \cosh\phi\ \sin^2\psi\ (\kappa_2 - \psi'),\\
K_5 &= -\kappa_3\ \cos\psi\ \sinh\theta\ \sinh\phi + \cosh\theta\ \cosh^2\phi\ (-\kappa_3\ \sin\psi + \phi')\\
&\quad + \cosh\theta\ \sin\psi\ \sinh^2\phi\ (\kappa_3 - \sin\psi\ \phi') - \cos^2\psi\ \Big[\cosh\theta\ \sinh^2\phi\ \phi'\\
&\quad + \cosh\phi\ \sinh\theta\ (\kappa_2 - \psi')\Big] + \cosh\phi\ \sin^2\psi\ \sinh\theta\ (-\kappa_2 + \psi'),\\
K_6 &= \kappa_3\ \cos\psi\ \cosh\phi - \sin^2\psi\ \sinh^2\theta\ \Big(\theta' + \sinh\phi\ (\kappa_2 - \psi')\Big)\\
&\quad - \cos^2\psi\ \Big[\sinh^2\theta\ \Big(\theta' + \sinh\phi\ (\kappa_2 - \psi')\Big) + \cosh^2\theta\ \sinh\phi\ (-\kappa_2 + \sinh\phi\ \theta' + \psi')\Big]\\
&\quad + \cosh^2\theta\ \Big(\cosh^2\phi\ \theta' - \sin^2\psi\ \sinh\phi\ (-\kappa_2 + \sinh\phi\ \theta' + \psi')\Big).
\end{aligned}
$$

Note that: We have three possible simple rotations. The first rotation exists on the timelike plane spanned by the spacelike Frenet first binormal $\mathbf{B}_1$ and the timelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the timelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the timelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$
\begin{aligned}
R &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cosh\theta & \sinh\theta\\ 0 & 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \cosh\phi & 0 & \sinh\phi\\ 0 & 0 & 1 & 0\\ 0 & \sinh\phi & 0 & \cosh\phi \end{bmatrix}\\
&\quad \cdot \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\psi & -\sin\psi & 0\\ 0 & \sin\psi & \cos\psi & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{23}
$$

so,

$$
\begin{bmatrix} \mathbf{T}\\ \mathbf{N}_q\\ \mathbf{B}_{1q}\\ \mathbf{B}_{2q} \end{bmatrix}
$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi\cosh\phi & -\cosh\phi\sin\psi & \sinh\phi \\ 0 & \cosh\theta\sin\psi + cos\psi\sinh\theta\sinh\phi & \cos\psi\cosh\theta - \sin\psi\sinh\theta\sin\phi & \cosh\phi\sinh\theta \\ 0 & \sin\psi\sinh\theta + \cos\psi\cosh\theta\sinh\phi & \cos\psi\sinh\theta - \cosh\theta\sin\psi\sinh\phi & \cosh\theta\cosh\phi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

Corollary 4.3. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi timelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i | i = 1,2,3,4,5,6\}$ can be determined by*

$$\begin{aligned} K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}_q', \boldsymbol{T}), \\ K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}_{1q}', \boldsymbol{T}), \\ K_3 &= -g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = g(\boldsymbol{B}_{2q}', \boldsymbol{T}), \\ K_4 &= g(\boldsymbol{N}_q', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}_{1q}', \boldsymbol{N}_q), \\ K_5 &= -g(\boldsymbol{N}_q', \boldsymbol{B}_{2q}) = g(\boldsymbol{B}_{2q}', \boldsymbol{N}_q) \\ K_6 &= -g(\boldsymbol{B}_{1q}', \boldsymbol{B}_{2q}) = g(\boldsymbol{B}_{2q}', \boldsymbol{B}_{1q}). \end{aligned} \tag{24}$$

Corollary 4.4. *If we put $\kappa_2 = \psi' - \phi'\ tanh\phi\ cot\psi$ and $\kappa_3 = -\frac{\phi'}{sin\psi}$ in equations (22), we can easily find ($K_4 = 0 = K_5 = K_6$) and hence, we have the same result as Bishop frame.*

Theorem 4.3. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi timelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi equations are given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_{1q'} \\ \boldsymbol{B}_{2q}' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & K_4 & K_5 \\ K_2 & K_4 & 0 & K_6 \\ -K_3 & -K_5 & K_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_{1q} \\ \boldsymbol{B}_{2q} \end{bmatrix}, \tag{25}$$

where

$$\begin{aligned} K_1 &= \kappa_1\ \cos\phi\ \cosh\psi, \\ K_2 &= -\kappa_1\ (\cosh\psi\ \sin\phi\ \sinh\theta + \cosh\theta\ \sin\psi), \\ K_3 &= \kappa_1\ (\cosh\theta\ \cosh\psi\ \sin\phi + \sinh\theta\ \sinh\psi), \\ K_4 &= \sin\phi\ \Big[\kappa_3\ \cosh\theta\ \cosh\psi - sin\phi\ \sinh\theta\ (\kappa_3\ \sinh\psi - \phi')\Big] + cos\phi \cosh\theta\ (\kappa_2 + \psi') \\ &\quad + \cos^2\phi\ \sinh\theta\ (-\kappa_3\ \sinh\psi + \phi'), \\ K_5 &= \cos^2\phi\ \cosh\theta\ (\kappa_3\ \sinh\psi - \phi') + \sin\phi\ \Big((\cosh\psi\ \sinh\theta + \cosh\theta\ \sin\phi\ \sinh\psi)\ \kappa_3 \\ &\quad - \cosh\theta\ \sin\phi\ \phi'\Big) - \cos\phi\ \sinh\theta\ (\kappa_2 + \psi'), \\ K_6 &= \cosh^2\theta\ \Big[\kappa_3\ \cos\phi\ \cosh\psi + cos^2\phi\ \theta' + \cosh^2\psi\ sin\phi\ (\kappa_2 + \sin\phi\ \theta' + \psi') \\ &\quad - sin\phi\ \sinh^2\psi\ (\kappa_2 + \sin\phi\ \theta' + \psi')\Big] - \sinh^2\theta\ \Big[\kappa_3\ \cos\phi\ \cosh\psi \\ &\quad + \cosh^2\psi\ \Big(\theta' + sin\phi\ (\kappa_2 + \psi')\Big) - \sinh^2\psi\ \Big(\theta' + \sin\phi\ (\kappa_2 + \psi')\Big)\Big]. \end{aligned}$$

Note that: We have three possible simple rotations. The first rotation exists on the timelike plane spanned by the timelike Frenet first binormal $\mathbf{B}_1$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the timelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the timelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$
\begin{aligned}
R = &\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & \sinh\theta \\ 0 & 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{bmatrix} \\
&\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\psi & \sinh\psi & 0 \\ 0 & \sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{26}
$$

so,

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_{1q} \\ \mathbf{B}_{2q} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi\cosh\psi & \cos\phi\sinh\psi & -\sin\phi \\ 0 & \cosh\psi\sin\phi\sinh\theta + \cosh\theta\sinh\psi & \cosh\theta\cosh\psi + \sin\phi\sinh\theta\sinh\psi & \cos\phi\sinh\theta \\ 0 & \cosh\theta\cosh\psi\sin\phi + \sinh\theta\sinh\psi & \cosh\psi\sinh\theta + \cosh\theta sin\phi\sinh\psi & \cos\phi\cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.
\end{aligned} \tag{27}
$$

Corollary 4.5. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi timelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i | i = 1,2,3,4,5,6\}$ can be determined by*

$$
\begin{aligned}
K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}_q', \boldsymbol{T}), \\
K_2 &= -g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = g(\boldsymbol{B}_{1q}', \boldsymbol{T}), \\
K_3 &= g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}_{2q}', \boldsymbol{T}), \\
K_4 &= -g(\boldsymbol{N}_q', \boldsymbol{B}_{1q}) = g(\boldsymbol{B}_{1q}', \boldsymbol{N}_q), \\
K_5 &= g(\boldsymbol{N}_q', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}_{2q}', \boldsymbol{N}_q), \\
K_6 &= g(\boldsymbol{B}_{1q}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}_{2q}', \boldsymbol{B}_{1q}).
\end{aligned} \tag{28}
$$

Corollary 4.6. *If we put $\kappa_2 = \psi' - \phi'\ tan\phi\ coth\psi$ and $\kappa_3 = \frac{\phi'}{\sinh\psi}$ in equation (25), we can easily find ($K_4 = 0 = K_5 = K_6$) and hence, we have the same result as Bishop frame.*

Theorem 4.4. *If $\alpha(s)$ is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$, then quasi equations are given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_{1q}' \\ \boldsymbol{B}_{2q}' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ K_1 & 0 & K_4 & K_5 \\ -K_2 & K_4 & 0 & K_6 \\ -K_3 & K_5 & -K_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_{1q} \\ \boldsymbol{B}_{2q} \end{bmatrix}, \tag{29}$$

where

$$\begin{aligned}
K_1 &= \kappa_1 \ \cosh\phi \ \cosh\psi, \\
K_2 &= \kappa_1 \ (\cosh\psi \ \sin\theta \ \sinh\phi - \cos\theta \ \sinh\psi), \\
K_3 &= -\kappa_1 \ (\cos\theta \ \cosh\psi \ \sinh\phi + \sin\theta \ \sinh\psi), \\
K_4 &= -\sin\theta \ (\sinh\psi \ \kappa_3 + \phi') + \cos\theta \ \Big(- \cosh\psi \ \sinh\phi \ \kappa_3 + \cosh\phi \ (\kappa_2 + \psi')\Big), \\
K_5 &= \cos\theta \ \cosh^2\phi \ (\sinh\psi \ \kappa_3 + \phi') - \sinh\phi \ \Big(\cosh\psi \ \sin\theta \ \kappa_3 \\
&\quad + \cos\theta \ \sinh\phi \ (\sinh\psi \ \kappa_3 + \phi')\Big) + \cosh\phi \ \sin\theta(\kappa_2 + \psi'), \\
K_6 &= \cos^2\theta \ \Big[\cosh\phi \ \cosh\psi \ \kappa_3 - \cosh^2\phi \ \theta' + \cosh^2\psi \ \sinh\phi \ (-\kappa_2 + \sinh\phi \ \theta' \\
&\quad + -\psi')\sinh\phi \ \sinh^2\psi \ (\kappa_2 - \sinh\phi \ \theta' + \psi')\Big] + \sin^2\theta \ \Big[\cosh\phi \ \cosh\psi \ \kappa_3 \\
&\quad - \cosh^2\psi \ \Big(\theta' + \sinh\phi \ (\kappa_2 + \psi')\Big) + \sinh^2\psi \ \Big(\theta' + \sinh\psi \ (\kappa_2 + \psi')\Big)\Big].
\end{aligned}$$

Note that: We have three possible simple rotations. The first rotation exists on the spacelike plane spanned by the spacelike Frenet first binormal $\mathbf{B}_1$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the timelike plane spanned by the timelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the timelike plane spanned by the timelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$\begin{aligned}
R = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\phi & 0 & \sinh\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sinh\phi & 0 & \cosh\phi \end{bmatrix} \\
& \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\psi & \sinh\psi & 0 \\ 0 & \sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{31}$$

so,

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_{1q} \\ \mathbf{B}_{2q} \end{bmatrix}$$

(32)

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\phi\cosh\psi & \cosh\phi\sinh\psi & \sinh\phi \\ 0 & \cos\theta\sinh\psi - \cosh\psi\sin\theta\sinh\phi & \cos\theta\cosh\psi - \sin\theta\sinh\phi\sinh\psi & -\cosh\phi sin\theta \\ 0 & \cos\theta\cosh\psi\sinh\phi + \sin\theta\sinh\psi & \cosh\psi\sin\theta + \cos\theta\sinh\phi\sinh\psi & \cos\theta\cosh\phi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

Corollary 4.7. *If $\alpha(s)$ is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i | i = 1, 2, 3, 4, 5, 6\}$ can be determined by*

$$\begin{aligned} K_1 &= -g(\boldsymbol{T}', \boldsymbol{N}_q) = g(\boldsymbol{N}'_q, \boldsymbol{T}), \\ K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{T}), \\ K_3 &= g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{T}), \\ K_4 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{N}_q), \\ K_5 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{N}_q), \\ K_6 &= g(\boldsymbol{B}'_{1q}, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{B}_{1q}). \end{aligned}$$

Corollary 4.8. *If we put $\kappa_2 = -\psi' - \phi' tanh\phi coth\psi)$ and $\kappa_3 = -\frac{\phi'}{\sinh\psi}$ in equations* (29)*, we can easily find $(K_4 = 0 = K_5 = K_6)$ and hence, we have the same row result as Bishop frame.*

5. Conclusion

In this paper, we investigated the frame and equations of quasi for non-lightlike curves in 3-dimensional Minkowski space $\mathbb{E}_1^3$ and in 4-dimensional Minkowski space-time $\mathbb{E}_1^4$. Furthermore, we showed the quasi frame can be considered as a generalization of Bishop frame in $\mathbb{E}_1^3$ and $\mathbb{E}_1^4$.

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Upper and lower estimates for products of two hyperbolic p-convex functions

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Abstract. In this study, we obtain upper and lower estimates for product of two hyperbolic p-convex functions, which is analogous to Hermite-Hadamard type inequalities for product of two hyperbolic p-convex functions.

Keywords: convex functions, hyperbolic p-convex functions, Hadamard's inequality.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see [7], [8]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Over the last twenty

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years, the numerous studies have focused on to establish generalization of the Inequality (1) and to obtain new bounds for left hand side and right hand side of the Inequality (1) (see [12], [17], [6], [22], [16] and [13]).

In [20], Pachpatte established some new integral inequalities analogous to that of Hadamard's inequality given in (1) involving two convex functions.

Theorem 1.1 ([20]). *Let f and g be real-valued, non-negative and convex functions on $[a,b]$. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{2}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{3}M(a,b)-\frac{1}{6}N(a,b)\le \frac{1}{b-a}\int_a^b f(x)g(x)dx, \tag{3}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Over the year, the generalized versions of Inequalities (2) and (3) for several convexity has been proved. For some of them please refer to ([16], [4], [14], [15], [21] and [23]). In [3], on the other hand, F. Chen proved Hermite-Hadamard type inequalities for product of two convex functions via Riemann-Liouville fractional integrals. For the other results of this topic, please refer to ([2], [18] and [19]). In [1], Mohamed S. S. Ali introduced the definition of hyperbolic p-convex functions as follows:

Definition 1.1 ([1]). *A function $f : I \to \mathbb{R}$ is said to be sub H-function on I or hyperbolic p-convex function, if for any arbitrary closed subinterval $[a,b]$ of I the graph of $f(x)$ for $x\in[a,b]$ lies nowhere above the graph of function, determined by the equation:*

$$H(x)=H(x,a,b,f)=A\cosh px+B\sinh px,\quad p\in\mathbb{R}\setminus\{0\},$$

where A and B are chosen such that $H(a)=f(a)$, and $H(b)=f(b)$.

Equivalently, for all $x\in[a,b]$

$$f(x)\le H(x)=\frac{f(a)\sinh p(b-x)+f(b)\sinh p(x-a)}{\sinh p(b-a)}. \tag{4}$$

For $x=(1-t)a+tb$, $t\in[a,b]$, the condition (4) becomes

$$f((1-t)a+tb)\le \frac{\sinh[p(1-t)(b-a)}{\sinh[p(b-a)])}f(a)+\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}f(b). \tag{5}$$

If the inequality (4) holds with $''\ge''$, then the function is called hyperbolic p-concave on I.

For some properties and results concerning the class of hyperbolic p-convex functions (see [1], [6], [10], [12], [11], [9] and [5]).

In 2016, Mohamed S. S. Ali introduced the following Hermite-Hadamard inequality for hyperbolic p-convex functions. The analysis of this proof is depending on the geometric observation which shows that the graph of the function $f(x)$ lies between the chord $H(x)$ and the support $S_u(x)$ (see [1]). In 2018, Dragomir proved Hermite-Hadamard Inequality (1) for hyperbolic p-convex functions with different method (see [6]).

Theorem 1.2 ([1]). *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic p-convex on I and $a, b \in I$ with $a < b$. Then, one has the inequality*

$$\frac{2}{p} f\left(\frac{a+b}{2}\right) \sinh p\left(\frac{b-a}{2}\right) \leq \int_a^b f(x)dx \leq \frac{1}{p}[f(a)+f(b)] \tanh p(\frac{b-a}{2}). \tag{6}$$

Theorem 1.3 ([6]). *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic p-convex on I and $a, b \in I$ with $a < b$. Then, one has the inequality*

$$f(\frac{a+b}{2}) \leq \int_a^b f(x) sech\left[p\left(x - \frac{a+b}{2}\right)\right] dx \leq \frac{f(a)+f(b)}{2}. \tag{7}$$

The aim of this paper is to establish new integral inequalities for product of two hyperbolic p-convex functions.

2. Results

Theorem 2.1. *If f, $g : I \to \mathbb{R}$ are two real-valued, non-negative and hyperbolic p-convex functions on I, then for any $a, b \in I$, we have*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)dx \quad &\leq \quad \frac{M(a,b)}{2}\left[\frac{\coth[p(b-a)]}{p(b-a)} - csch^2[p(b-a)]\right] \\ &+ \quad \frac{N(a,b)}{2\sinh[p(b-a)]}\left[\coth[p(b-a) - \frac{1}{p(b-a)}\right], \end{aligned}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b), \quad N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. Since f and g are hyperbolic p-convex functions on $[a, b]$, then from (5), we have

$$f((1-t)a+tb) \leq \frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]} f(a) + \frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]} f(b), \tag{8}$$

and

$$g((1-t)a+tb) \leq \frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]} g(a) + \frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]} g(b). \tag{9}$$

By using (8) and (9), we have

$$
\begin{aligned}
& f((1-t)a+tb)g((1-t)a+tb) \le \frac{\sinh^2[p(1-t)(b-a)]}{\sinh^2[p(b-a)]} f(a)g(a) \\
& + \frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]} \frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]} [f(a)g(b)+f(b)g(a)] \\
& + \frac{\sinh^2[pt(b-a)]}{\sinh^2[p(b-a)]} f(b)g(b). \qquad (10)
\end{aligned}
$$

Integrating the both sides of (10) with respect to t from 0 to 1, then we obtain

$$
\begin{aligned}
& \int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt \le \qquad (11) \\
& \le \frac{f(a)g(a)}{\sinh^2[p(b-a)]} \int_0^1 \sinh^2[p(1-t)(b-a)]dt \\
& + \frac{f(a)g(b)+f(b)g(a)}{\sinh^2[p(b-a)]} \int_0^1 \sinh[pt(b-a)] \sinh[p(1-t)(b-a)]dt \\
& + \frac{f(b)g(b)}{\sinh^2[p(b-a)]} \int_0^1 \sinh^2[pt(b-a)]dt. \qquad (12)
\end{aligned}
$$

By changing of variable $x = (1-t)a + tb$, we get

$$
\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt = \frac{1}{b-a} \int_a^b f(x)g(x)dx. \qquad (13)
$$

Moreover, it is easy observe that

$$
\begin{aligned}
\int_0^1 \sinh^2[p(1-t)(b-a)]dt &= \int_0^1 \sinh^2[pt(b-a)]dt \qquad (14) \\
&= \frac{1}{4p(b-a)} \sinh[2p(b-a)] - \frac{1}{2} \qquad (15)
\end{aligned}
$$

and

$$
\int_0^1 \sinh[pt(b-a)] \sinh[p(1-t)(b-a)]dt = \frac{1}{2}\left[\cosh p(b-a) - \frac{\sinh p(b-a)}{p(b-a)}\right]. \qquad (16)
$$

By substituting by the Equalities (13), (14) and (16) in (11), then we have

$$
\begin{aligned}
& \frac{1}{b-a} \int_a^b f(x)g(x)dx \le \frac{f(a)g(a)+f(b)g(b)}{2\sinh^2[p(b-a)]} \left[\frac{\sinh[2p(b-a)]}{2p(b-a)} - 1\right] \\
& + \frac{f(a)g(b)+f(b)g(a)}{2\sinh^2[p(b-a)]} \left[\cosh[p(b-a)] - \frac{\sinh[p(b-a)]}{p(b-a)}\right].
\end{aligned}
$$

Hence

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{M(a,b)}{2}\left[\frac{\coth[p(b-a)]}{p(b-a)} - csch^2[p(b-a)]\right] \\ &+ \frac{N(a,b)}{2\sinh[p(b-a)]}\left[\coth[p(b-a) - \frac{1}{p(b-a)}\right].\end{aligned}$$

This completes the proofs. □

Remark 2.1. For $p \to 0$, we observe that

$$\lim_{p\to 0}\frac{1}{2}\left[\frac{\coth[p(b-a)]}{p(b-a)} - csch^2[p(b-a)]\right] = \frac{1}{3}, \tag{17}$$

and

$$\lim_{p\to 0}\frac{1}{2\sinh[p(b-a)]}\left[\coth[p(b-a) - \frac{1}{p(b-a)}\right] = \frac{1}{6}. \tag{18}$$

Corollary 2.1. *With the notations in Theorem 2.1, if $p \to 0$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b) + \frac{1}{2}N(a,b), \tag{19}$$

which is analogous to inequality (2) *in case of convex functions.*

Theorem 2.2. *If f, $g : I \to \mathbb{R}$ are two real-valued, non-negative and hyperbolic p-convex functions on I, then for any $a, b \in I$, we have*

$$\begin{aligned}&2\cosh^2\left[\frac{p(b-a)}{2}\right]f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ &-\frac{M(a,b)}{2\sinh p(b-a)}\left[\coth p(b-a) - \frac{1}{p(b-a)}\right] \\ &-\frac{N(a,b)}{2}\left[\frac{\coth p(b-a)}{p(b-a)} - csch^2 p(b-a)\right] \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx,\end{aligned}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b), \quad N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. For $t \in [a,b]$, we can write

$$\frac{a+b}{2} = \frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}.$$

Using the hyperbolic p-convexity of f and g, we have

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)=f\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\\
&\times g\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\\
&\le\left[\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}f((1-t)a+tb)+\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}f(ta+(1-t)b)\right]\\
&\times\left[\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}g((1-t)a+tb)+\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}g(ta+(1-t)b)\right]\\
&=\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}[f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g(ta+(1-t)b)]\\
&+\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}[f((1-t)a+tb)g(ta+(1-t)b)+f(ta+(1-t)b)g((1-t)a+tb)].
\end{aligned}
$$

For the second expression in the last equality, by using again the hyperbolic p-convexity of f and g, we obtain

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\Bigg[f((1-t)a+tb)g((1-t)a+tb)\\
&+f(ta+(1-t)b)g(ta+(1-t)b)\Bigg]\\
&+\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\Bigg\{\left[\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}f(a)+\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}f(b)\right]\\
&\times\left[\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}g(a)+\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}g(b)\right]\\
&+\left[\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}f(a)+\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}f(b)\right]\\
&\times\left[\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}g(a)+\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}g(b)\right]\Bigg\}.
\end{aligned}
\tag{20}
$$

Hence,

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\Bigg[f((1-t)a+tb)g((1-t)a+tb)\\
&+f(ta+(1-t)b)g(ta+(1-t)b)\Bigg]\\
&+\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\Bigg\{\frac{2M(a,b)}{\sinh^2[p(b-a)]}\sinh[pt(b-a)]\sinh[p(1-t)(b-a)]\\
&+\frac{N(a,b)}{\sinh^2[p(b-a)]}[\sinh^2[pt(b-a)]+\sinh^2[p(1-t)(b-a)]]\Bigg\}.
\end{aligned}
\tag{21}
$$

Integrating the both sides of (21) with respect to t from 0 to 1, and by using the equalities (13), (14) and (16), we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\frac{2}{b-a}\int_a^b f(x)g(x)dx \\
&+ \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\frac{M(a,b)}{\sinh^2[p(b-a)]}\left[\cosh p(b-a) - \frac{\sinh[p(b-a)]}{p(b-a)}\right] \\
&+ \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\frac{N(a,b)}{\sinh^2[p(b-a)]}\left[\frac{\sinh[2p(b-a)]}{2p(b-a)} - 1\right]. \qquad (22)
\end{aligned}
$$

By multiplying the both sides of (22) by $\frac{\sinh^2[p(b-a)]}{2\sinh^2[\frac{p(b-a)}{2}]}$, then, we obtain the desired inequality

$$
\begin{aligned}
&2\cosh^2\left[\frac{p(b-a)}{2}\right]f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&- \frac{M(a,b)}{2\sinh p(b-a)}\left[\coth p(b-a) - \frac{1}{p(b-a)}\right] \\
&- \frac{N(a,b)}{2}\left[\frac{\coth p(b-a)}{p(b-a)} - csch^2 p(b-a)\right] \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx. \qquad \square
\end{aligned}
$$

Corollary 2.2. *With the notations in Theorem 2.2, if $p \to 0$, then*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{6}M(a,b) - \frac{1}{3}N(a,b) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx,$$

which is analogous to inequality (3) *in case of convex functions.*

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Chaotic dynamics of the Duffing-Holms model with external excitation

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Abstract. The dynamics of the Duffing-Holms model are researched, and the critical conditions for chaos of the model with external excitation are obtained using Melnikov method. The expression of Melnikov function is given. The results show that the criteria obtained for chaos motion in the sense of the Smale horseshoe is consistent with that obtained by the numerical simulation. Research shows the Melnikov function is an effective analytical method to judge the occurrence of chaotic motion.
Keywords: financial market, Duffing Holms model, chaos, Melnikov method.

1. Introduction

Financial market is a complex economic system. Financial risk can be divided into endogenous financial risk and exogenous financial risk. Endogenous financial risk refers to the financial risk generated in the process of commercial financial transactions. It mainly includes market risk, credit risk, liquidity risk and operational risk, etc., which are generally manifested as non systematic risks. Exogenous financial risk refers to the financial risk generated in the process of non-commercial financial transactions. It mainly includes currency risk, legal risk, policy risk, social and economic environment change risk, and most of them show systemic financial risk. From the characteristics and properties of financial risk, financial risk has the characteristics of uncertainty, universality, diffusion, concealment and suddenness. All these indicate that the financial system and

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its evolution have obvious characteristics of chaos. Chaos is a harmful form of movement, which may lead to the system out of control and cause the system to collapse completely. Chaos theory can provide a new method and idea for solving financial crisis and related problems.

Considering the real financial system as a discrete dynamic system and describing it with a nonlinear chaotic dynamic model, it can accurately reflect the operation and law of the financial system. In 1988, Ramsay [1] used Duffing-Holmes equation to test the fluctuation of M2 money in the US financial market from January 1959 to November 1987, and the stock fluctuation in the US commodity market from July 1962 to August 1985, the sufficient evidence of the existence of chaos is obtained. Therefore, Duffing-Holmes model has become an effective tool to study financial chaos. G. H. Zhou [2] uses Duffing-Holmes model to discuss the conditions and mechanism of financial system risk and some specific measures of risk control, and discusses the application of chaos theory in financial system. H. X. Yao, T. L. Shi [3] obtains the sufficient conditions of system hope bifurcation by using harmonic balance method and bifurcation theory. Using periodic excitation method and constant external excitation method, the chaotic behavior of the Duffing-Holms system can be effectively controlled to a stable periodic orbit or equilibrium point. Combined with the financial market, the relationship between the occurrence of chaos and financial crisis is explained. New order parameters are explored of the Duffing-Holms model in [4]. Xu, et al. studied the interaction effect among several financial factors in a financial system using mathematical models [5]. Their investigation showed that such model displays rich dynamical behaviours including chaos. Synchronization of the model were considered in their work. Zhang et al. studied a 4D chaos financial system [6]. Liao, et al. used a system of differential equations to model the evolution of a financial system and study its complexity [7].

Melnikov function is an effective analytical method for theoretical prediction of chaotic motions in nonlinear systems [8, 9]. This method can be used to analyze and judge whether Smale chaos occurs. Compared with the traditional numerical simulation methods of chaotic motion [10-15], Melnikov method can give analytical conditions for existence of chaotic motions.

In this paper, the dynamics of the Duffing-Holms model is investigated, and the critical conditions for chaos of the model with external excitation are obtained by Melnikov method. The expression of Melnikov function is given. Research shows the Melnikov function method is an effective analytical method to judge the occurrence of chaotic motion. The research results can better guide the government departments to make some strategic adjustments and avoid financial problems.

2. Duffing-Holms model and the equilibrium analysis

Duffing-Holms model is a nonlinear equation with oscillation introduced by Duffing in 1918. It is considered as a typical example of chaotic phenomena, and has been widely used in financial markets. Duffing-Holms model can accurately describe the various states of a complex system under different conditions [16],

$$\ddot{x} + \varepsilon\delta\dot{x} + ax + bx^3 = \varepsilon f \cos\omega t. \tag{1}$$

Where x is the state of the financial market, $\dot{x}$ is the rate at which the state of the financial market changes, $\ddot{x}$ is the acceleration of changes in the state of the financial market, δ represents the government's ability to prevent financial risks, ε is the control parameter of the policy, f is the speculative disturbance parameter, ω is the self regulating ability of financial market order, a, b are the coefficients of a cubic function, $a < 0, b > 0$.

Take the state variable of Eq. (1), let $(u, v) = (x, \dot{x})$ and reduce Eq. (1) to the first order differential equations

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -au - bu^3 - \varepsilon\delta v + \varepsilon f \cos(\omega\tau). \end{cases} \tag{2}$$

Because the exciting force f is small, $-\varepsilon\delta v + \varepsilon f \cos(\omega\tau)$ is considered as disturbance terms. When $\varepsilon = 0$, Eq. (2) is called a conservative undisturbed system,

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -au - bu^3. \end{cases} \tag{3}$$

When $a > 0$, $b > 0$, Eq. (3) has a unique equilibrium point $P_0(0,0)$. The characteristic roots of the linearized system are a pair of pure imaginary roots $\lambda_1 = \sqrt{a}\mathrm{i}$, , $\lambda_2 = -\sqrt{a}\mathrm{i}$ and the equilibrium point is a center. When $b < 0$, there are three equilibrium points $P_0\,(0,0)$, $P_1(\sqrt{-a/b}, 0)$, $P_2(-\sqrt{-a/b}, 0)$. $P_0(0,0)$ is a center, the characteristic roots of the linearized system corresponding to $P_0(0,0)$ are a pair of pure imaginary roots $\lambda_1 = \sqrt{a}\mathrm{i}$, $\lambda_2 = -\sqrt{a}\mathrm{i}$. For the other two non-zero equilibrium points $P_1(\sqrt{-a/b}, 0)$, $P_2(-\sqrt{-a/b}, 0)$, the characteristic roots of the linearized system are a pair of real roots, so the two non-zero equilibrium points are saddle points. When $a > 0$, the time process diagram $(t - u, t - v)$ and the phase diagram $(u - v)$ of the conservative undisturbed Eq. (3) are shown in Fig. 1-3. The system appears a period-1 motion. When $a < 0$, the time process diagram $(t - u, t - v)$ and the phase diagram $(u - v)$ of the conservative undisturbed Eq. (3) are shown in Fig. 4-6. We find that when the speculative disturbance parameter f is zero, the financial system is in a stable state.

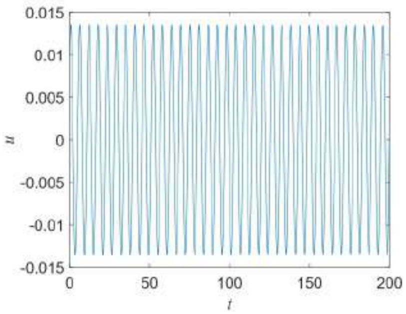


Figure 1: The time process diagram $(t-u)$.

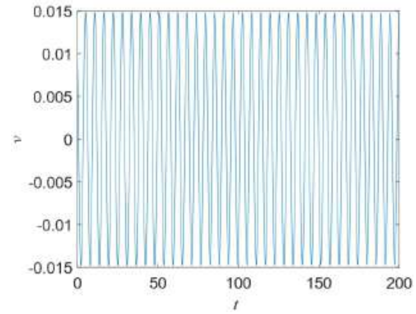


Figure 2: The time process diagram $(t-v)$.

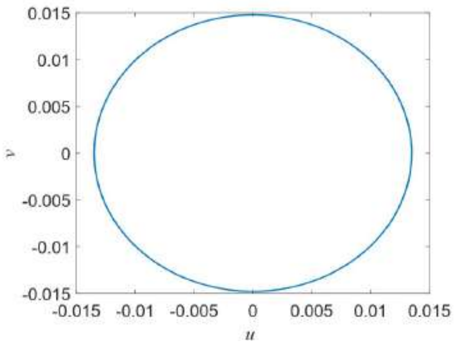


Figure 3: Phase diagram $(u-v)$ of the unperturbed system.

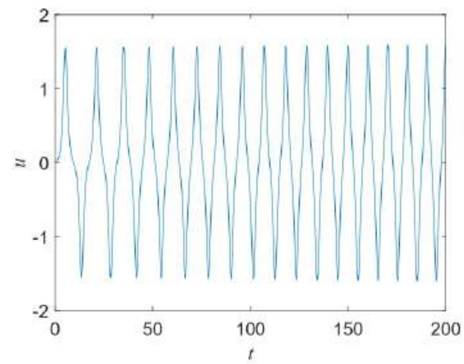


Figure 4: The time process diagram $(t-u)$.

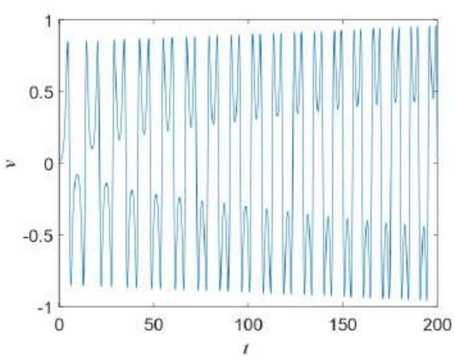


Figure 5: The time process diagram $(t-v)$.

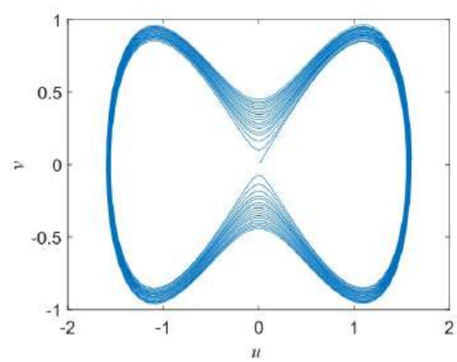


Figure 6: Phase diagram $(u-v)$ of the unperturbed system.

When $a < 0$, the homoclinic orbit of Hamilton system corresponding to Eq. (3) is

$$2\left(v^2 + au^2\right) + bu^4 = 4h. \tag{4}$$

According Eq. (4), the homoclinic orbits of the hyperbolic saddle points $P_1(\sqrt{-a/b}, 0)$, $P_2(-\sqrt{-a/b}, 0)$ of Eq. (3) are obtained,

$$q_0^{\pm}(\tau) : \begin{cases} u_0(\tau) = \pm\sqrt{-2a/b}\,\mathrm{sech}(\sqrt{-a}\tau), \\ v_0(\tau) = \pm a\sqrt{2/b}\,\mathrm{sech}(\sqrt{-a}\tau)\mathrm{th}(\sqrt{-a}\tau). \end{cases} \tag{5}$$

Where $(+)$ represents the positive axis part of the homoclinic orbits and $(-)$ represents the negative axis part of the homoclinic orbits.

3. Melnikov method for chaotic motion analysis of the system

When $\varepsilon = 0$, the homoclinic orbits of Eq. (2) appear under some range of parameters a, b. For the perturbed system, as $\varepsilon \neq 0$, the cross section homoclinic may appear. The quasi Hamiltonian system of Eq. (2) can be written

$$\dot{x} = f(x) + \varepsilon g(x, \tau). \tag{6}$$

Where $x = \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$, $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $g = \begin{pmatrix} g_1(x,\tau) \\ g_2(x,\tau) \end{pmatrix}$, and $g(x, \tau)$ is a periodic function. Melnikov integral is constructed to determine the distance between stable and unstable manifolds, which is defined as

$$M_{\pm}(\tau_0) = \int_{-\infty}^{+\infty} f(q_0^{\pm}(\tau)) \wedge g(q_0^{\pm}(\tau), \tau + \tau_0) \exp[-\operatorname{trace} Df(q_0^{\pm}(s))ds]d\tau. \tag{7}$$

Where $\operatorname{trace} Df(q_0^{\pm}(s)$ is the trace of the matrix $Df(q_0^{\pm}(s))$ and "$\wedge$" denotes the Possion symbol, which is defined as

$$f \wedge g = f_1 g_2 - f_2 g_1. \tag{8}$$

Since the unperturbed system is a Hamiltonian system, the trace $Df(q_0^{\pm}(s) \equiv 0$, Eq. (7) can be reduced,

$$M_{\pm}(\tau_0) = \int_{-\infty}^{+\infty} f(q_0^{\pm}(\tau)) \wedge g(q_0^{\pm}(\tau), \tau + \tau_0) d\tau. \tag{9}$$

Let

$$\begin{aligned} &f_1 = v,\ g_1 = 0, \\ &f_2 = -au - bu^3, \\ &g_2 = -\delta v + f\cos(\omega\tau). \end{aligned} \tag{10}$$

The Melnikov function corresponding to Eq. (2) is

$$
\begin{aligned}
M_{\pm}(\tau_0) &= \int_{-\infty}^{+\infty} v_0^{\pm}(\tau)\left(-\delta v_0^{\pm}(\tau) + f\cos\left(\omega\left(\tau+\tau_0\right)\right)\right)d\tau \\
&= -\delta\int_{-\infty}^{+\infty}\left[v_0^{\pm}(\tau)\right]^2 d\tau + f\int_{-\infty}^{+\infty} v_0^{\pm}(\tau)\sin\left(\omega\tau_0\right)\sin(\omega\tau)d\tau \\
&= -\delta B + f\sin\left(\omega\tau_0\right)A.
\end{aligned}
\tag{11}
$$

The Melnikov integral can be calculated

$$
A = \int_{-\infty}^{+\infty} v_0^{\pm}(\tau)\sin(\omega\tau)d\tau, \quad B = \int_{-\infty}^{+\infty}\left[v_0^{\pm}(\tau)\right]^2 d\tau. \tag{12}
$$

According to Melnikov's theory [9, 16], if Melnikov's function has a simple zero point which does not depend on ε, there exists t_0 such that $M_{\pm}(t_0) = 0$. For a sufficiently small ε, on the Poincaré mapping of Eq. (2), there is a chaotic motion in the sense of Smale horseshoe. For a certain frequency ω, if

$$
f/\delta > B/A. \tag{13}
$$

then Eq. (2) appears a Smale horseshoe type chaotic motion.

4. Numerical integration method for calculating Melnikov function

From the above, in order to get the threshold value $|B/A|$, the calculation of the Melnikov integral is very important. Although the analytical expression of the non perturbed homoclinic orbit can be obtained, it is difficult to get the Melnikov integral analytically, so it is necessary to use the numerical integration method to calculate it. Now we calculate the Melnikov integral by the methods in [17, 18]. The idea of this method is that the time variable t is a function of the state variable u of the homoclinic orbit. The Melnikov integral of the time variable t can be transferred to the integral of the state variable u, and then it can be solved by the numerical calculation.

If $\tau > 0$, according to Eq. (4), we obtain,

$$
\frac{du}{d\tau} = \mp\sqrt{h - au^2 - \frac{b}{2}u^4}. \tag{14}
$$

On the homoclinic orbit $q_0^{\pm}$, for Eq. (10), we separate variables, and integrate both sides, then we obtain,

$$
\tau = \mp\int_{u_{1,2}}^{u}\frac{d\xi}{\sqrt{h - a\xi^2 - \frac{b}{2}\xi^4}}. \tag{15}
$$

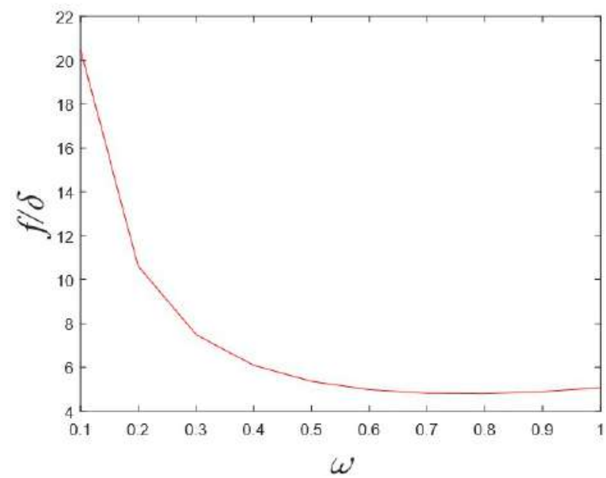


Figure 7: The time process diagram $(t - v)$.

It follows from Eq. (14, 15), and Eq. (12),

$$A = 2\int_0^{+\infty} v_0^{\pm}(\tau)\sin(\omega\tau)d\tau = 2\int_{u_{1,2}}^{u} \sin(\omega\tau)d\tau$$
$$= \mp\int_{u_{1,2}}^{u} \sin\left(\omega\frac{d\xi}{\sqrt{h - a\xi^2 - \frac{b}{2}\xi^4}}\right) du. \tag{16}$$

$$B{=}2\int_0^{+\infty} \left[v_0^{\pm}(\tau)\right]^2 d\tau{=}2\int_{u_{1,2}}^{u} v_0^{\pm}(\tau)du{=}\mp 2\int_{u_{1,2}}^{u} \left(h{-}au^2{-}\frac{b}{2}u^4\right)du. \tag{17}$$

When the frequency ω of the applied force is in the interval $[0, 1]$, the complex Simpson formula is used to integrate u in 1000 steps and u in 500 steps, then the A and B values corresponding to each value ω can be obtained. We can get the Melnikov threshold value with ω. When the ratio of the speculative disturbance parameter f and the government's ability to prevent financial risks δ is greater than the value of $|B/A|$, the chaotic motion of the system appears in the sense of Smale horseshoe. The financial system will change from stable equilibrium to chaos, which will cause financial system shock or economic crisis.

$$f/\delta > 4/(\sqrt[3]{2}\pi\omega \sec h(\pi\omega/2)). \tag{18}$$

5. Numerical simulation of the system

When $\omega \neq 0$, when the market is disturbed by changing external force, the system will gradually lose its stability, generate bifurcation and finally generate chaos. The bifurcation diagrams of u and v with the change of external force f are shown in Fig. 8-9, as $\omega = 1.5$. It shows the complex dynamic behavior of Eq. (2). With the change of the ratio of the speculative disturbance parameter, the stability of the state of the financial market and the rate at which the state of the financial market changes will change.

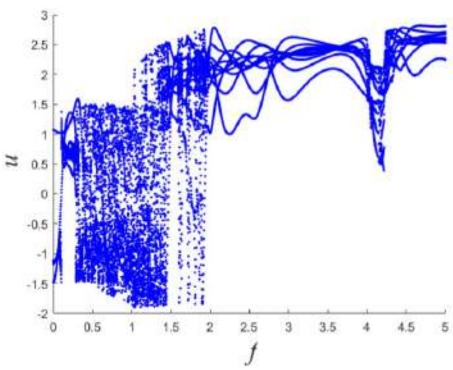


Figure 8: The bifurcation diagram $(f - u)$ of Eq. (2).

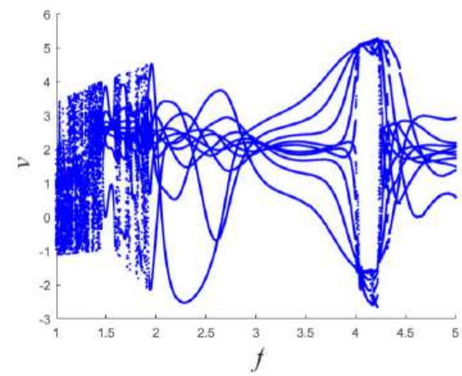


Figure 9: The bifurcation diagram $(f - v)$ of Eq. (2).

Let $a = -1.2, \varepsilon = 0.8, \delta = 0.2, b = 1, f = 1.5$, when $\omega = 1.5$, the Eq. (2) appears chaotic motion. The time process diagram $(t-u)$ and the phase diagram $(u - v)$ of the conservative undisturbed Eq. (2) are shown in Fig. 10-11.

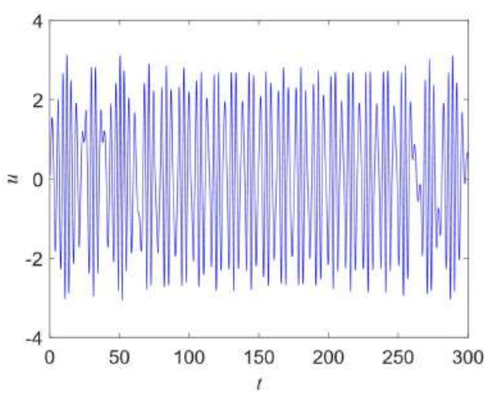


Figure 10: The time process diagram $(t - u)$ of Eq. (2).

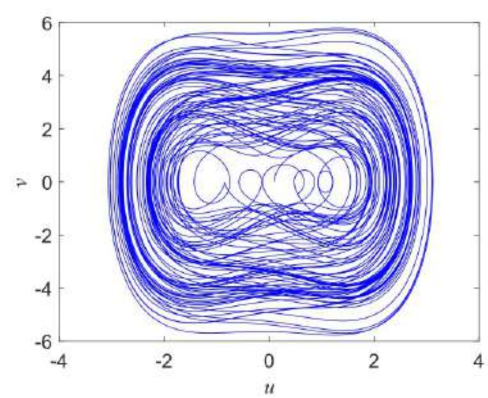


Figure 11: The phase diagram $(u - v)$ of Eq. (2).

When $\omega = 1.85$, the time process diagram $(t - u)$ and the phase diagram $(u - v)$ of the conservative undisturbed Eq. (2) are shown in Fig. 12-13. The chaotic attractor of the system on Poincaré section is shown in Fig. 14. It can be seen from the calculation results that the critical value of the parameters obtained from the numerical simulation results is consistent with the critical value determined by Melnikov method.

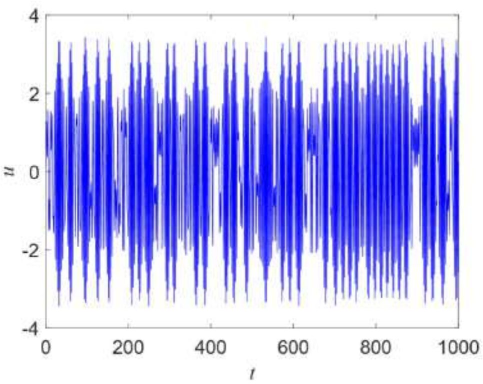


Figure 12: The time process diagram $(t-u)$ of Eq. (2).

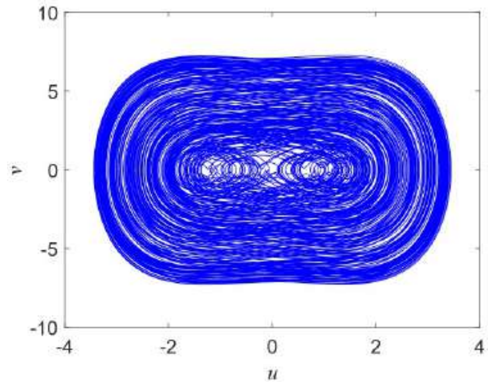
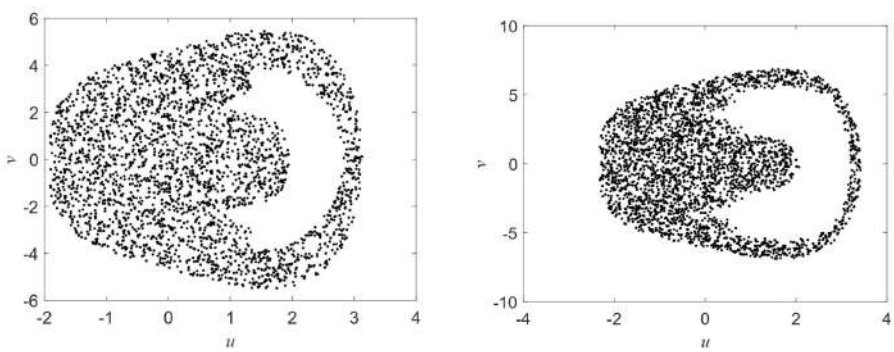


Figure 13: The phase diagram $(u-v)$ of Eq. (2).



(a) $\omega = 1.5$

(b) $\omega = 1.85$

Figure 14: Chaotic attractor on Poincaré section of Eq. (2)

When the dynamic property of chaos appears in the Duffing-Holms model, that is, the financial system appears chaos or economic crisis, the government's ability to prevent financial risks and the control parameter of the policy need to be adjusted and made to solve the current crisis.

6. Conclusion

The dynamics of the Duffing-Holms model are researched, and the critical conditions for chaos of the model with external excitation are obtained using Melnikov method. The results show that the criteria obtained for chaos motion by the Smale horseshoe mapping is consistent with that obtained by the numerical simulation. Research shows the Melnikov function is an effective analytical method to judge the occurrence of chaotic motion.

The research results can better guide the government departments to make strategic adjustments and avoid financial crisis. First of all, it is necessary to enhance the government's ability δ to prevent financial crisis, that is, to increase the government's foreign exchange reserves, reduce foreign debts, and improve the capital structure, so as to prevent the international financial giants from speculating on foreign exchange and stocks in a country or a region's foreign exchange market and stock market. Secondly, when δ becomes great under the interference of international speculators, the financial market will break away from the undisturbed track and enter into an unstable state. Thirdly, the financial system reform should be gradual, the policy parameter ε should be adjusted, and the psychological preparation for medium and long-term governance of the financial market should be made. When a country's financial market is in an unstable or chaotic state, the government should not make sudden and significant changes to the financial system (such as the exchange rate system), but should take a small range of policy fine-tuning measures, and actively strive for a large amount of financial assistance from international financial organizations.

Acknowledgement

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Strongly m-system and strongly primary ideals in posets

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Abstract. In this paper, we study and establish some interesting results of strongly prime ideal and strongly m-system in posets. Also, we study the notion of strongly primary ideals in posets and show some properties of the set $\sqrt{I} = \{x : L(x)^* \cap I \neq \phi\}$ for ideal I of P.

Keywords: Posets, ideals, strongly prime ideal, strongly m-system, strongly primary ideal, minimal strongly prime ideal.

1. Introduction

Throughout this paper $(P, \leq)$ denotes a poset with smallest element 0. For basic terminology and notation for posets, we refer [8] and [9]. For $M \subseteq P$, let $L(M) = \{x \in P : x \leq m, \text{ for all } m \in M\}$ denote the lower cone of M in P and $U(M) = \{x \in P : m \leq x, \text{ for all } m \in M\}$ be the upper cone of M in P. Let $A, B \subseteq P$, we write $L(A, B)$ instead of $L(A \cup B)$ and dually for the upper cones. If $M = \{x_1, x_2, ..., x_n\}$ is finite, then we use the notation $L(x_1, x_2, ..., x_n)$ instead of $L(\{x_1, x_2, ..., x_n\})$(and dually). It is clear that for any subset A of P, we have $A \subseteq L(U(A))$ and $A \subseteq U(L(A))$. If $A \subseteq B$, then $L(B) \subseteq L(A)$ and $U(B) \subseteq U(A)$. Moreover, $LUL(A) = L(A)$ and $ULU(A) = U(A)$. Following [12], a non-empty subset I of P is called semi-ideal if $b \in I$ and $a \leq b$, then $a \in I$. A subset I of P is called ideal if $a, b \in I$ implies $L(U((a, b)) \subseteq I$ (see [8]). Following [7], for any subset X of P, $[X]$ is the smallest ideal of P containing X and $X^* = X \backslash \{0\}$. If $X = \{b\}$, then $L(b)$ is called the principle ideal of P generated by b. A proper semi-ideal (ideal) I of P is called prime if $L(a, b) \subseteq I$ implies that either $a \in I$ or $b \in I$ (see [9]). An ideal I of P is called semi-prime if $L(a, b) \subseteq I$ and $L(a, c) \subseteq I$ together imply $L(a, U(b, c))) \subseteq I$ (see [8]). Following

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[4], an ideal I of P is called strongly prime if $L(A^*, B^*) \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for different proper ideals A, B of P. A non-empty subset M of P is called m-system if for any $x_1, x_2 \in M$, there exists $t \in L(x_1, x_2)$ such that $t \in M$. Following [6], a non-empty subset M of P is called strongly m-system if $A \cap M \neq \phi$ and $B \cap M \neq \phi$ imply $L(A^*, B^*) \cap M \neq \phi$ for any different proper ideals A, B of P. It is clear that an ideal I of P is strongly prime if and only if $P \backslash I$ is a strongly m- system of P and every strongly m-system of P is m-system. Following [4], an ideal I of P is called strongly semi-prime if $L(A^*, B^*) \subseteq I$ and $L(A^*, C^*) \subseteq I$ together imply $L(A^*, U(B^*, C^*)) \subseteq I$ for any different proper ideals A, B and C of P. For any semi-ideal I of P and a subset A of P, we define $\langle A, I \rangle = \{z \in P : L(a, z) \subseteq I, \text{ for all } a \in A\} = \bigcap_{a \in A} \langle a, I \rangle$ (see [4]). If $A = \{x\}$, then we write $\langle x, I \rangle$ instead of $\langle \{x\}, I \rangle$. For any ideal I of P, a strongly prime ideal Q of P is said to be a minimal strongly prime ideal of I if $I \subseteq Q$ and there is no strongly prime ideal R of P such that $I \subset R \subset Q$. The set of all strongly prime ideals of P is denoted by $Sspec(P)$ and the set of minimal strongly prime ideals of P is denoted by $Smin(P)$. For any ideal I of P, $P(I)$ and $SP(I)$ denotes the intersection of all prime semi-ideals and strongly prime ideals of P containing I respectively. It is clear from Theorem 6 of [9] and Example 1.1 of [6] that $P(I) = I$ and $SP(I) \neq I$ for any ideal I of P. Following [2], let I be a semi-ideal of P. Then, I is said to have (*) condition if whenever $L(A, B) \subseteq I$, we have $A \subseteq \langle B, I \rangle$ for any subsets A and B of P. From [8], a non empty subset F of a poset P is called semi-filter if $x \leq y$ and $x \in F$, then $y \in F$. It is clear that for any subset I of P, I is a semi-ideal of P if and only if $P \backslash I$ is a semi-filter of P. A subset F of P is called filter if for $x, y \in F$ implies $U(L(x, y)) \subseteq F$. A filter F is called prime, whenever $U(x, y) \subseteq F$ implies $x \in F$ or $y \in F$.

2. Minimal strongly prime ideals

Lemma 2.1. *Let M be a strongly m-system of P. Then, the following statements hold:*

- *(i) $P \backslash M$ satisfies the condition that $L(A^*, B^*) \subseteq P \backslash M$ implies $A \subseteq P \backslash M$ or $B \subseteq P \backslash M$ for any different proper ideals A, B of P.*
- *(ii) If $P \backslash M$ is a semi-ideal of P, then M is a prime filter of P.*
- *(iii) If $P \backslash M$ is an ideal of P, then $P \backslash M$ is a strongly prime ideal of P.*

Proof. (i) Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq P \backslash M$. If $A \nsubseteq P \backslash M$ and $B \nsubseteq P \backslash M$, then $A \cap M \neq \phi$ and $B \cap M \neq \phi$ imply that $L(A^*, B^*) \cap M \neq \phi$, a contradiction.

(ii) Let $x, y \in M$. Then, $L(x) \cap M \neq \phi$ and $L(y) \cap M \neq \phi$, there exists $t \in L(x, y) \cap M$ with $U(L(x, y)) \subseteq U(t) \subseteq M$. So, M is a filter.

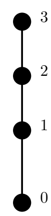
Let $U(a, b) \subseteq M$ for some $a, b \in P$. Then, $U(a) \cap M \neq \phi$ and $U(b) \cap M \neq \phi$ which imply there exists $a_1 \in U(a) \cap M$ and $b_1 \in U(b) \cap M$ such that

$L(L(a_1)^*, L(b_1)^*) \cap M \neq \phi$, so $L(L(a)^*, L(b)^*) \cap M \neq \phi$. Thus, $L(L(a)^*, L(b)^*) \nsubseteq P \backslash M$. By (i), we have $a \in M$ and $b \in M$. So, M is a prime filter.

(iii) It is trivial from (i). □

The following example shows the condition "$P \backslash M$ is an ideal of P "is not superficial in Lemma 2.1 (iii).

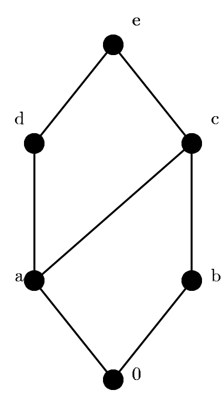
Example 2.2. Consider $P = \{0, 1, 2, 3\}$ and define a relation $\leq$ on P as follows.



Then, $(P, \leq)$ is a poset and $M = \{1, 2\}$ is a strongly m-system of P, but $P \backslash M$ is not an ideal of P. □

The below example shows that every prime filter of P need not to be strongly m-system of P in general.

Example 2.3. Consider $P = \{0, a, b, c, d, e\}$ and define a relation $\leq$ on P as follows.



Then, $(P, \leq)$ is a poset and $F = \{b, c, e\}$ is a prime filter of P, but not strongly m-system as $A = \{0, b\}$ and $B = \{0, a, b, c\}$ are the ideals of P with $A \cap F \neq \phi$ and $B \cap F \neq \phi$, but $L(A^*, B^*) \cap F = \phi$. □

In the papers [10], [11] and [13], authors related the concept of minimal prime ideal over an ideal I and the maximal multiplicative system disjoint from I in rings, semigroups and lattices. Following the above papers, we have some interesting results in posets.

Theorem 2.4. *Let I be an ideal of P. If $P \backslash I$ is a maximal strongly m-system of P, then I is a minimal strongly prime of P.*

Proof. Let I be an ideal of P such that $P\backslash I$ is a maximal strongly m-system of P. Then, I is strongly prime ideal. If J is a strongly prime ideal of P such that $J \subset I$, then $P\backslash I \subset P\backslash J$, a contradiction to the maximality of $P\backslash I$. □

Example 2.5. Let $n \in Z^{+}\backslash\{0,1\}$ and ρ be the "less than or equal "relation on set of integers. Then, $P_n = \{a : a$ is an integer and $a\rho n\}$ is a poset and $I_n = \{a : a\rho(n-1)\}$ is a minimal strongly prime ideal of P_n. Here $P_n\backslash I_n$ is not a maximal strongly m-system of P_n as $P_n\backslash I_n$ is contained in a strongly m-system $P_n\backslash\{0\}$ of P_n.

The above example shows that the converse of Theorem 2.4 is not true in general, but we have the following.

Theorem 2.6. *Let I be an ideal of P. If the complement of every strongly m-system of P is a semi-ideal of P and I is minimal strongly prime ideal, then $P\backslash I$ is a maximal strongly m-system of P.*

Proof. Let I be a minimal strongly prime ideal of P. Then, $P\backslash I$ is a strongly m-system of P. If there exists a strongly m-system M of P such that $P\backslash I \subset M$. Then, $P\backslash M \subset I$. We now prove $P\backslash M$ is an ideal of P. Let $x, y \in P\backslash M$ and $L(U(x,y)) \nsubseteq P\backslash M$. Then, there exists $t \in L(U(x,y)) \cap M$ with $U(x,y) \subseteq U(t) \subseteq M$ which implies that $U(x) \cap M \neq \phi$ and $U(y) \cap M \neq \phi$, there exists $t_1 \in U(x) \cap M$ and $t_2 \in U(y) \cap M$ such that $t_1, t_2 \in M$. Since M is strongly m-system, we have $L(L(t_1)^*, L(t_2)^*) \cap M \neq \phi$ which implies $L(L(x)^*, L(y)^*) \cap M \neq \phi$. Thus $L(L(x)^*, L(y)^*) \nsubseteq P\backslash M$. By Lemma 2.1(i), we have $x \in M$ and $y \in M$, a contradiction. So, $P\backslash M$ is an ideal of P. By Lemma 2.1(iii), we have $P\backslash M$ is a strongly prime ideal of P, a contradiction to the minimality of I. □

As a consequence of above theorem, we have the following.

Corollary 2.7. *Let M be a strongly m-system of P. If M is a semi-filter of P, then $P\backslash M$ is an ideal of P.*

Theorem 2.8. *Let $I \neq 0$ be an ideal of P satisfies $(*)$ condition and M be a strongly m-system of P. If M is semi-filter, then the following are equivalent:*

(*i*) *M is a maximal strongly m-system of P with respect to $M \cap I = \phi$.*

(*ii*) *$P\backslash M$ is a minimal strongly prime ideal of P containing I.*

(*iii*) *For a strongly prime ideal $P\backslash M$ containing I, for each $x \in P\backslash M$, there exists $t \in U(x)$ and $y \in M$ such that $L(L(t)^*, L(y)^*) \subseteq I$.*

Proof. (i)⇒(ii) It follows from Corollary 2.7 and Theorem 2.4, $P\backslash M$ is a minimal strongly prime ideal of P containing I.

(ii)⇒(iii) It is trivial from Theorem 2.2 of [3].

(iii)⇒(i) From (iii), we have M is a strongly m-system of P with $M \cap I = \phi$.

Suppose N is a strongly m-system of P such that $N \cap I = \phi$ and $M \subset N$. Then, there exists $a \in N \backslash M$, $y \in M$ and $t \in U(a)$ such that $L(L(t)^*, L(y)^*) \subseteq I$ which implies $L(y)^* \subseteq \langle L(t)^*, I\rangle \subseteq \langle L(a)^*, I\rangle$. So, $L(L(a)^*, L(y)^*) \subseteq I$. Since $y, a \in N$ and N is strongly m-system, we have $L(L(a)^*, L(y)^*) \cap N \neq \phi$ which implies $I \cap N \neq \phi$, a contradiction. □

Theorem 2.9. *Let I be an ideal of P and M be a strongly m-system of P such that $M \cap I = \phi$. Then, there exists a maximal strongly m-system N containing M with $N \cap I = \phi$.*

Proof. It follows from Theorem 2.1 of [3]. □

Lemma 2.10. *Let P be a poset and $r \in P$. If $P\backslash U(r)$ satisfies $(*)$ condition, then $U(r)$ is a strongly m-system of P.*

Proof. Let A and B be different proper ideals of P such that $A \cap U(r) \neq \phi$ and $B \cap U(r) \neq \phi$. Suppose $L(A^*, B^*) \cap U(r) = \phi$. Then, $L(A^*, B^*) \subseteq P\backslash U(r)$ and $B^* \subseteq \langle A^*, P\backslash U(r)\rangle = \bigcap_{a \in A^*} \langle a, P\backslash U(r)\rangle \subseteq \langle q, P\backslash U(r)\rangle \subseteq \langle r, P\backslash U(r)\rangle$ for some $q \in A \cap U(r)$. Since $U(r)$ is a m-system of P, then $P\backslash U(r)$ is a prime semi-ideal of P. By Theorem 20 of [8], we have $B^* \subseteq \langle r, P\backslash U(r)\rangle = P\backslash U(r)$, a contradiction. □

For any subset X of P, we define $V^{'}(X) = \{Q \in Smin(P) : X \subseteq Q\}$ and $D^{'}(X) = Smin(P)\backslash V^{'}(X)$.

Theorem 2.11. *Let A be a non empty subset of P and $J \neq \{0\}$ be an ideal of P. If every semi-ideal of P satisfies $(*)$ condition and every m-system of P is a semi-filter of P, then $\langle A, J\rangle = \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$.*

Proof. Let $x \in \langle A, J\rangle$. Then, $L(a, x) \subseteq J$, for all $a \in A$. For $Q \in V^{'}(J) \cap D^{'}(A)$, there exists $a_1 \in A\backslash Q$ such that $L(L(x)^*, L(a_1)^*) \subseteq J \subseteq Q$ which implies $x \in Q$. Hence, $x \in \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$.

Conversely, let $x \in \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$ and $x \notin \langle A, J\rangle$. Then, $L(x, t) \nsubseteq J$ for some $t \in A$, so there exists $r \in L(x, t)\backslash J$ with $U(r) \cap J = \phi$. By Lemma 2.10, we have $U(r)$ is a strongly m-system such that $U(r) \cap J = \phi$. Then, by Theorem 2.9, there exists a maximal strongly m-system K of P containing $U(r)$ such that $K \cap J = \phi$ and, by Theorem 2.8, $P\backslash K \in V^{'}(J)$. Since $r \leq x$ and $r \in K$, we have $U(x) \subseteq U(r) \subseteq K$ which implies $x \notin \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$, a contradiction. □

Theorem 2.12. *Let $J \neq \{0\}$ be an ideal of P. If every maximal m-system is a semi-filter of P and every semi-ideal satisfies $(*)$ condition, then J is a strongly semi-prime ideal of P.*

Proof. Let J be an ideal of P such that $L(A^*, B^*) \subseteq J$ and $L(A^*, C^*) \subseteq J$ for different proper ideals A, B, C of P. If $L(A^*, \cup(B^*, C^*)) \nsubseteq J$, then there exists $t \in L(A^*, U(B^*, C^*))\backslash J$ with $U(t) \cap J = \phi$. By Lemma 2.10 and Theorem

2.9, there exists a maximal strongly m-system K of P containing $U(t)$ of P such that $K \cap J = \phi$. Then, by Theorem 2.8, $P \backslash K \in V^{'}(J)$ which implies $L(A^*, B^*) \subseteq P \backslash K$ and $L(A^*, C^*) \subseteq P \backslash K$. Since $P \backslash K$ is strongly prime ideal, we have $A \subseteq P \backslash K$ or $B, C \subseteq P \backslash K$ which imply $L(L(a)^*, L(t)^*)) \subseteq P \backslash K$, for all $a \in A^*$ and $t \in L(U(B^*, C^*)$. Since $t \in U(t) \subseteq K$ with $t \leq a$ and K is strongly m-system, we have $L(L(a)^*, L(t)^*)) \cap K \neq \phi$, a contradiction. □

Following [6], for an ideal I and a strongly prime ideal Q of P, $I_Q = \{x \in P : L(x, y) \subseteq I \text{ for some } y \notin Q\}$.

Theorem 2.13. *Let I be a strongly prime ideal of P and $J \neq \{0\}$ be an ideal of P with $(*)$ condition. Then, the following statements are equivalent:*

(i) $I \in V^{'}(J)$.

(ii) *I contains precisely one of x or $\langle x, J \rangle$, for any $x \in P$.*

(iii) $\langle x, J \rangle \backslash I \neq \phi$, *for any* $x \in I$.

(iv) $J_I = I$.

Proof. (i)⇒(ii) Assume on the contrary that $\langle x, J \rangle \subseteq I$ for $x \in I$. Since $I \in V^{'}(J)$, we have by Theorem 2.2 of [3], for each $x \notin P \backslash I$, there exists $t \in U(x)$ and $y \in P \backslash I$ such that $L(L(t)^*, L(y)^*) \subseteq J$ which implies $L(y) \subseteq \langle L(t)^*, J \rangle \subseteq \langle L(x)^*, J \rangle \subseteq \langle x, J \rangle$. So, $y \in I$, a contradiction. If $x \notin I$, let $t \in \langle x, J \rangle$. Then, $L(L(t)^*, L(x)^*) \subseteq L(x, t) \subseteq J \subseteq I$. Since I is strongly prime ideal and $x \notin I$, we have $t \in I$.

(ii)⇒(iii) It is trivial.

(iii)⇒(iv) By the definition of J_I, we have $J_I \subseteq I$. Let $x \in I$. Then, $\langle x, J \rangle \nsubseteq I$ which implies there exists $t \in \langle x, J \rangle \backslash I$. Hence, $L(t, x) \subseteq J$ for some $t \notin I$. So, $x \in J_I$.

(iv)⇒(i) It is follows from Theorem 2.10 of [6]. □

Theorem 2.14. *Let $J \neq \{0\}$ be an ideal of P with $(*)$ condition and $I \in V^{'}(J)$. Then, $\langle \langle x, J \rangle, J \rangle \subseteq I$.*

Proof. Let $I \in V^{'}(J)$ and $x \in I$. Then, by Theorem 2.2 of [3], there exists $t \in U(x)$ and $y \in P \backslash I$ such that $L(L(t)^*, L(y)^*) \subseteq J$, so $y \in \langle L(t)^*, J \rangle \subseteq \langle L(x)^*, J \rangle \subseteq \langle x, J \rangle$. Suppose $\langle \langle x, J \rangle, J \rangle \nsubseteq I$. Then, there exists $z \in \langle \langle x, J \rangle, J \rangle \backslash I$. Now, for $y, z \in P \backslash I$, we have $L(L(z)^*, L(y)^*) \cap P \backslash I \neq \phi$ which implies $L(z, y) \cap P \backslash I \neq \phi$. Then, there exists $t \in L(y, z)$ and $t \in P \backslash I$. Since $z \in \langle \langle x, J \rangle, J \rangle$, we have $L(z, r) \subseteq J$, for all $r \in \langle x, J \rangle$ which imply $L(z, y) \subseteq J \subseteq I$, a contradiction. □

Theorem 2.15. *Let I be an ideal of P with $(*)$ condition and $M = \{x : \langle x, I \rangle = I\}$. Then, M is a strongly m-system of P.*

Proof. Let A and B be different proper ideals of P such that $A \cap M \neq \phi$ and $B \cap M \neq \phi$. Then, there exists $x \in A$ and $y \in B$ such that $x, y \in M$. Suppose $L(A^*, B^*) \cap M = \phi$. Then, for all $t \in L(A^*, B^*)$ there exists $r \in P \backslash I$ and $L(r, t) \subseteq I$ which implies $t \in \langle r, I \rangle$. So, $L(A^*, B^*) \subseteq \langle r, I \rangle$ which implies $L(A^*, B^*, r) \subseteq I$. Since I satisfies (*) condition, we have $L(B^*, r) \subseteq \langle A^*, I \rangle \subseteq \langle x, I \rangle = I$ which implies $r \in \langle B^*, I \rangle \subseteq \langle y, I \rangle = I$, a contradiction. □

Lemma 2.16. *Let I be an ideal of P. Then, $SP(I) = \{c \in P$: every strongly m-system in P which contains c has a non empty intersection with $I\}$.*

Proof. Let $H = \{c \in P$: every strongly m-system in P which contains c has a non empty intersection with $I\}$ and $c \notin H$. Then, there is a strongly m-system M of P which contains c and $M \cap I = \phi$. By Theorem 2.1 of [3], there exists a strongly prime ideal Q of P with $I \subseteq Q$ and $Q \cap M = \phi$ which implies $c \notin \cap Q_i$. So, $\cap Q_i \subseteq H$.

Conversely, let $c \notin \cap Q_i$. Then, there is a strongly prime ideal Q_i of P for some i such that $c \notin Q_i$ which implies $c \in P \backslash Q_i$ and $P \backslash Q_i$ is a strongly m-system of P. Since $P \backslash Q_i \cap I = \phi$, we have $c \notin H$. Hence, $H \subseteq \cap Q_i$. □

Theorem 2.17. *Let A and B be ideals of P. Then, the following statements hold:*

(*i*) $A \subseteq B$ *implies* $SP(A) \subseteq SP(B)$.

(*ii*) $SP(L(A^*, B^*)) = SP(A \cap B) = SP(A) \cap SP(B)$.

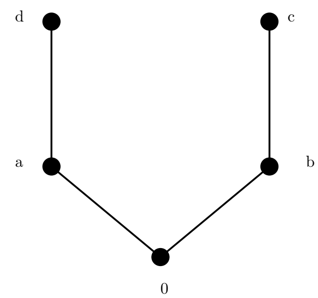
Proof. (i) It is trivial.

(ii) We have $L(A^*, B^*) \subseteq A \cap B \subseteq A$. Then, by (i), $SP(L(A^*, B^*)) \subseteq SP(A \cap B) \subseteq SP(A)$ which imply $SP(L(A^*, B^*)) \subseteq SP(A \cap B) \subseteq SP(A) \cap SP(B)$. Let $x \in SP(A) \cap SP(B)$ and K be a strongly m-system containing x. Then, by Lemma 2.16, $K \cap A \neq \phi$ and $K \cap B \neq \phi$. Since K is strongly m-system, we have $L(A^*, B^*) \cap K \neq \phi$ which implies $x \in SP(L(A^*, B^*))$. □

3. Strongly primary ideals

Theory of primary ideals played an important role in commutative ring theory. Because every ideal can be written as the intersection of finitely many primary ideals. In [1], A. Anjaneyulu developed the theory of primary ideals in arbitrary semigroup. Primary ideals in semigroup. In this section we study the notion of primary in poset. Following [1], we define $\sqrt{I} = \{x : L(x)^* \cap I \neq \phi\}$ for ideal I of P. An ideal I of P is called *primary* if $L(a, b) \subseteq I$ implies $a \in I$ or $b \in \sqrt{I}$. An ideal I of P is called *strongly primary* if $L(A^*, B^*) \subseteq I$ implies $A \subseteq I$ or $B \subseteq \sqrt{I} \cup \{0\}$ for different proper ideals A, B of P. Every strongly primary ideal of P is a primary ideal of P, and every strongly prime ideal of P is a strongly primary ideal of P. But the converse need not be true in each case in general.

Example 3.1. Consider $P = \{0, a, b, c, d, e\}$ and define a relation $\leq$ on P as follows.



Then, $(P, \leq)$ is a poset and $I = \{0, a\}$, $A = \{0, b\}$ and $B = \{0, a, d\}$ are ideals of P. Here I is a strongly primary ideal of P, but not a strongly prime as $L(A^*, B^*) \subseteq I$ with $A \nsubseteq I$ and $B \nsubseteq I$. $\square$

Lemma 3.2. *Let A and B be ideals of P. Then, the following statements hold:*

(i) $A \subseteq \sqrt{A} \cup \{0\}$.

(ii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

(iii) *If* $A \subseteq B$, *then* $\sqrt{A} \subseteq \sqrt{B}$.

(iv) $\sqrt{L(A^*, B^*)} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

The following theorem relates the strongly primary and strongly primness between I and $\sqrt{I}$.

Theorem 3.3. *Let I be a strongly primary ideal of P and $\sqrt{I} \cup \{0\}$ be an ideal of P. Then, $\sqrt{I} \cup \{0\}$ is a strongly prime ideal of P.*

Proof. Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq \sqrt{I} \cup \{0\}$ and $A \nsubseteq \sqrt{I} \cup \{0\}$. Then, for all $t \in L(A^*, B^*)$, we have $t \in \sqrt{I} \cup \{0\}$ which imply $L(t)^* \cap I \neq \phi$. There exists $s \in I$ and $s \in A^*, B^*$ which imply $L(A^*, B^*) \subseteq L(s) \subseteq I$. Since I is strongly primary ideal and $A \nsubseteq I$, we have $B \subseteq \sqrt{I} \cup \{0\}$. $\square$

The condition "$\sqrt{I} \cup \{0\}$ is an ideal of P "is not superficial in Theorem 3.3. In Example 2.3, if $I = \{0, b\}$, then $\sqrt{I} \cup \{0\} = \{0, b, c, e\}$ is not an ideal of P.

Definition 3.4. *Let Q be a strongly prime ideal of P. A strongly primary ideal I of P is said to be Q- strongly primary if $\sqrt{I} \cup \{0\} = Q$.*

Theorem 3.5. *Let $I_1, I_2, ..., I_n$ be Q-strongly primary ideals of P. Then, $\bigcap_{i=1}^{n} I_i$ is a Q-strongly primary ideal of P.*

Proof. Let $J = \bigcap_{i=1}^{n} I_i$. Then, $\sqrt{J} \cup \{0\} \subseteq \bigcap_{i=1}^{n} \sqrt{I_i} \cup \{0\}$ and $\bigcap_{i=1}^{n} \sqrt{I_i} \cup \{0\} \subseteq \sqrt{J} \cup \{0\}$ as $J \subseteq I_i \subseteq \sqrt{I_i}$. Since $I_i's$ are Q-strongly primary ideals, we have $\sqrt{J} \cup \{0\} = \bigcap_{i=1}^{n} \sqrt{I_i} \cup \{0\} = Q$. We now prove that J is a strongly primary ideal of P. Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq J$ and $A \nsubseteq J$. Then, there is an ideal I_j of P such that $A \nsubseteq I_j$. Since $L(A^*, B^*) \subseteq J \subseteq I_j$ and I_j is strongly primary, we have $B \subseteq \sqrt{I_j} \cup \{0\} = Q = \sqrt{J} \cup \{0\}$. $\square$

Theorem 3.6. *Let I be a strongly primary ideal of P. If I is a semi-prime ideal of P, then $\langle x, I\rangle$ is a strongly primary ideal of P for any $x \in P$.*

Proof. Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq \langle x, I\rangle$ for any $x \in P \backslash I$. Then, $L(A^*, B^*, L(x)^*) \subseteq I$. If $L(A^*, B^*) \subseteq I$, then $A \subseteq I \subseteq \langle x, I\rangle$ or $B \subseteq \sqrt{I} \cup \{0\} \subseteq \sqrt{\langle x, I\rangle} \cup \{0\}$. If $L(A^*, B^*) \nsubseteq I$, then by Theorem 2.4 of [5], $L(A^*, L(x)^*) \subseteq I$ and $L(L(x)^*, B^*) \subseteq I$. Since I is primary and $x \notin I$, we have $A \subseteq \langle x, I\rangle \cup \{0\}$ and $B \subseteq \langle x, I\rangle \cup \{0\}$. $\square$

Lemma 3.7. *Let I be an ideal of P and $I \subseteq Q$ for some strongly prime ideal Q of P. Then, $SP(I) \subseteq \sqrt{I} \cup \{0\}$.*

Proof. Let $x \in SP(I)$. Then, $x \in \bigcap_{I \subseteq Q_i} Q_i$, where Q_i's are strongly prime ideals of P which implies $L(Q_i) \cap I \neq \phi$ and $L(x) \cap I \neq \phi$, so $x \in \sqrt{I} \cup \{0\}$. Hence, $SP(I) \subseteq \sqrt{I} \cup \{0\}$. $\square$

Theorem 3.8. *Let I be an ideal of P and $I \subseteq Q$ for some strongly prime ideal Q of P. Then, I is a strongly primary ideal of P.*

Theorem 3.9. *Let I be an ideal of P with $(*)$ condition and Q be a strongly prime ideal of P. If $I_Q \subseteq \sqrt{I} \cup \{0\}$, then I is strongly primary.*

Proof. Let $I_Q \subseteq \sqrt{I} \cup \{0\}$ and $L(A^*, B^*) \subseteq I$ with $A \nsubseteq I$ for different proper ideals A, B of P.
Case (i). If $I \subseteq Q$, then by Theorem 3.8, I is a strongly primary ideal of P.
Case (ii). Let $I \nsubseteq Q$. Then, there is $x \in I \backslash Q$. We now prove $B \subseteq \sqrt{I} \cup \{0\}$. Suppose not, $B \nsubseteq \sqrt{I} \cup \{0\}$. Since $I_Q \subseteq \sqrt{I} \cup \{0\}$, we have $B \nsubseteq I_Q$. Then, there exists $y \in B \backslash I_Q$ which implies $L(y, t) \nsubseteq I$, for all $t \notin Q$. In particular $L(x, y) \nsubseteq I$ which implies $B^* \nsubseteq \langle x, I\rangle = P$, a contradiction. $\square$

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Bivariate extension of λ-hybrid type operators

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Abstract. In this manuscript, we develop a bi-variate extension of hybrid type operators. We discuss the order of approximation via modulus of continuity, Peetre's K-functional,the rate of convergence, Lipschitz maximal functions and Voronovskaja type result. In addition to this, we investigate global approximation results. In the last section, we study the approximation properties of the operators in Bögel-spaces in terms of mixed-modulus of continuity.

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Keywords: hybrid type operators, modulus of continuity, Peetre's K-functional, Voronovskaja type result, mixed-modulus of continuity, Bögel functions.

1. Introduction

Approximation theory is an important part of mathematical analysis where the main purpose of investigation is to approximate a delicate, difficult and sophisticated function with the help of simple and smooth function. Karl Weierstrass (1885) developed an elegant theorem called as Weierstrass approximation theorem [25] which is widely known and accepted that all types of algebraic polynomial in the category of continuous real valued function on closed interval are dense. Among these, S. N. Bernstein (1912)[4] introduced the polynomials via binomial distribution to give the simplest and easiest proof of this celebrated theorem as follows:

$$B_n(f;x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right), n \in \mathbb{N}, \tag{1}$$

where $p_{n,\nu}(x) = \binom{n}{k} x^\nu (1-x)^{n-\nu}$ and $f \in [0,1]$. He established that $B_n(f;x) \rightrightarrows f$ for each $f \in C[0,1]$ where $\rightrightarrows$ holds for uniform convergence. Szász [1] generalized the operators defined by (1) on unbounded interval, i.e. on $[0,\infty)$ as

$$S_n(f;x) = e^{-nx} \sum_{\nu=0}^{\infty} \frac{(nx)^\nu}{\nu!} f\left(\frac{\nu}{n}\right), \ n \in \mathbb{N}. \tag{2}$$

Several generalizations studied for (2) to yield the convergence properties by these sequences on $[0,\infty)$. Operators (1) and (2) are limited for continuous functions only. Durrmeyer [2] suggested for an integral modification of Bernstein operators (1) on an interval [0,1] to study the approximation properties for Lebsgue integrable functions given by

$$D_n(f;x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) \int_0^1 p_{n,\nu}(t) f(t) dt. \tag{3}$$

With the help of the Bézier bases and shape parameter $\lambda \in [1,-1]$, Cai et. al. [14] obtained a generalization of classical Bernstein operators. In the sequence, Cai [10], Srivastava et al. [23] and Ozger ([18], [19]) constructed Stancu, Shurer and Kantorovich variants of λ-Bernstein operators. Motivated with the idea of λ-Bernstein polynomials, Acu et al. [26] introduced a new family of modified U_m^ρ operators and the operator is denoted by $U_{m,\lambda}^\rho$. Recently, Rao et al [6], introduced a new sequence of Hybrid type operators as:

$$A^*_{n,\alpha}(f;x) \quad = \quad \sum_{k=0}^{\infty} P^{\alpha}_{n,k}(x) \frac{n^{k+\lambda+1}}{\Gamma(k+\lambda+1)} \int_0^\infty t^{k+\lambda+1} e^{-nt} dt \tag{4}$$

where

$$\begin{aligned} P_{n,k}^{\alpha}(x) &= \frac{x^{k-1}}{(1+x)^{n+k-1}}\left\{\frac{\alpha x}{1+x}\binom{n+k-1}{k} - (1-\alpha)(1+x)\binom{n+k-3}{k-2}\right. \\ &+ \left.(1-\alpha)x\binom{n+k-1}{k}\right\} \end{aligned}$$

with $\binom{n-3}{-2} = \binom{n-2}{-1} = 0$, and the gamma function

$$\Gamma n = \int_0^\infty x^{n-1}e^{-x}dx, \quad \Gamma z = (z-1)\Gamma(z-1) = (z-1)!.$$

2. Construction of bivariate extension of λ-hybrid type operators and their basic estimates

Let $\mathcal{I}^2 = \{(u_1,u_2) : 0 \leq u_1 < \infty,\ 0 \leq u_2 < \infty\}$ and $C\left(\mathcal{I}^2\right)$ be the class of all continuous functions on $\mathcal{I}^2$ equipped with the norm

$$\|g\|_{C(\mathcal{I}^2)} = \sup_{(u_1,u_2)\in\mathcal{I}^2} |g(u_1,u_2)|.$$

Then, for all $f \in C\left(\mathcal{I}^2\right)$ and $n_1, n_2 \in \mathbb{N}$, we construct a new sequences of bi-variate extension of λ-Hybrid type operators as follow:

$$\begin{aligned} B_{n_1,n_2}^{\alpha}(f;u_1,u_2) = &\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P_{1,n_1,k_1}(u_1)P_{2,n_2,k_2}(u_2) \\ &\cdot\int_0^\infty\int_0^\infty G_1^*(u_1)G_2^*(u_2)f(t_1,t_2)dt_1dt_2. \end{aligned} \tag{5}$$

where

$$\begin{aligned} P_{i,n,k}^{*}(u_i) = &\frac{{u_i}^{k_i-1}}{(1+u_i)^{n_i+k_\nu-1}}\left\{\frac{\alpha u_i}{1+u_i}\binom{n_i+k_i-1}{k_i}\right. \\ &\left. - (1-\alpha)(1+u_i)\binom{n_i+k_i-3}{k_i-2} + (1-\alpha)u_1\binom{n_i+k_i}{k_i}\right\} \end{aligned}$$

and $\mathcal{G}_i^*(u_i) = \frac{{n_i}^{k_i+\lambda_i+1}}{\Gamma(k_i+\lambda_i+1)}\int_0^\infty {t_i}^{k_i+\lambda_i+1}e^{-n_it_i}dt_i$ for $i = 1, 2$.

Lemma 2.1 ([6]). *For the operators defined by* (4) *and* $e_i(x) = x^i, i \in \{0,1,2\}$, *test function, we have the following identities:*

$$\begin{aligned} A_{n,\alpha}^*(e_0;x) &= 1, \\ A_{n,\alpha}^*(e_1;x) &= x + \frac{2}{n}(\alpha-1)x + \frac{\lambda+1}{n}, \\ A_{n,\alpha}^*(e_2;x) &= x^2\left(1+\frac{4\alpha-3}{n}\right) + x\left(\frac{2\lambda+3}{n} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n^2}\right) \\ &+ \frac{\lambda^2+3\lambda+2}{n^2}, \end{aligned}$$

where $n \in \mathbb{N}$, *and* $\alpha \in [-1, 1]$.

Lemma 2.2 ([6])**.** *Let* $\eta_j(x) = (t-x)^j$, $j \in \{0, 1, 2\}$ *be the central moments. Then for the operator* $A^*_{n,\alpha}(.;.)$, *given by* (4), *we have the following equalities:*

$$\begin{aligned} A^*_{n,\alpha}(\eta_0; x) &= 1, \\ A^*_{n,\alpha}(\eta_1; x) &= \frac{2(\lambda-1)x}{n} + \frac{\lambda+1}{n}, \\ A^*_{n,\alpha}(\eta_2; x) &= O\left(\frac{1}{n}\right)\left(x^2 + x + 1\right). \end{aligned}$$

Lemma 2.3. *Let* $e_{i,j} = {u_1}^i {u_2}^j$. *Then, for the operator* $B^{\alpha}_{n_1,n_2}(.;.)$, *we have*

$$\begin{aligned} &B^{\alpha}_{n_1,n_2}(e_{0,0}; u_1, u_2) = 1, \\ &B^{\alpha}_{n_1,n_2}(e_{1,0}; u_1, u_2) = u_1 + \frac{2}{n_1}(\alpha-1)u_1 + \frac{\lambda+1}{n_1}, \\ &B^{\alpha}_{n_1,n_2}(e_{0,1}; u_1, u_2) = u_2 + \frac{2}{n_2}(\alpha-1)u_2 + \frac{\lambda+1}{n_2}, \\ &B^{\alpha}_{n_1,n_2}(e_{1,1}; u_1, u_2) = \left(u_1 \frac{2}{n_1}(\alpha-1)u_1 + \frac{\lambda+1}{n_1}\right)\left(u_2 + \frac{2}{n_2}(\alpha-1)u_2 + \frac{\lambda+1}{n_2}\right), \\ &B^{\alpha}_{n_1,n_2}(e_{2,0}; u_1, u_2) = u_1^2\left(1+\frac{4\alpha-3}{n_1}\right)+u_1\left(\frac{2\lambda+3}{n_1}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_1^2}\right) \\ &\qquad\qquad + \frac{\lambda^2+3\lambda+2}{n_1^2}, \\ &B^{\alpha}_{n_1,n_2}(e_{0,2}; u_1, u_2) = u_2^2\left(1+\frac{4\alpha-3}{n_2}\right)+u_2\left(\frac{2\lambda+3}{n_2}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_2^2}\right) \\ &\qquad\qquad + \frac{\lambda^2+3\lambda+2}{n_2^2}. \end{aligned}$$

Proof. In the light of lemma (2.1) and linearly property, we have

$$\begin{aligned} B^{\alpha}_{n_1,n_2}(e_{0,0}; u_1, u_2) &= B^{\alpha}_{n_1,n_2}(e_0; u_1, u_2) B^{\alpha}_{n_1,n_2}(e_0; u_1, u_2), \\ B^{\alpha}_{n_1,n_2}(e_{1,0}; u_1, u_2) &= B^{\alpha}_{n_1,n_2}(e_1; u_1, u_2) B^{\alpha}_{n_1,n_2}(e_0; u_1, u_2), \\ B^{\alpha}_{n_1,n_2}(e_{0,1}; u_1, u_2) &= B^{\alpha}_{n_1,n_2}(e_0; u_1, u_2) B^{\alpha}_{n_1,n_2}(e_1; u_1, u_2), \\ B^{\alpha}_{n_1,n_2}(e_{1,1}; u_1, u_2) &= B^{\alpha}_{n_1,n_2}(e_1; u_1, u_2) B^{\alpha}_{n_1,n_2}(e_1; u_1, u_2), \\ B^{\alpha}_{n_1,n_2}(e_{2,0}; u_1, u_2) &= B^{\alpha}_{n_1,n_2}(e_2; u_1, u_2) B^{\alpha}_{n_1,n_2}(e_0; u_1, u_2), \\ B^{\alpha}_{n_1,n_2}(e_{0,2}; u_1, u_2) &= B^{\alpha}_{n_1,n_2}(e_0; u_1, u_2) B^{\alpha}_{n_1,n_2}(e_2; u_1, u_2), \end{aligned}$$

which proves Lemma (2.3). □

Lemma 2.4. *Let* $\Psi^{u_1,u_2}_{i,j}(t,s) = \eta_{i,j}(t,s) = (t-u_1)^i(s-u_2)^j$, $i, j \in \{0, 1, 2\}$ *be the central moments. Then from the operators* $B^{\alpha}_{n_1,n_2}(.;.)$ *defined by* (5) *satisfies* *the following identities*

$$\begin{aligned}
B^{\alpha}_{n_1,n_2}(\eta_{0,0};u_1,u_2) &= 1,\\
B^{\alpha}_{n_1,n_2}(\eta_{1,0};u_1,u_2) &= \frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1},\\
B^{\alpha}_{n_1,n_2}(\eta_{0,1};u_1,u_2) &= \frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2},\\
B^{\alpha}_{n_1,n_2}(\eta_{1,1};u_1,u_2) &= \frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2},\\
B^{\alpha}_{n_1,n_2}(\eta_{2,0};u_1,u_2) &= O\left(\frac{1}{n}\right)\left(u_1^2+u_1+1\right),\\
B^{\alpha}_{n_1,n_2}(\eta_{0,2};u_1,u_2) &= O\left(\frac{1}{n}\right)\left(u_2^2+u_2+1\right).
\end{aligned}$$

Proof. Using lemma (2.2) and linearly property, we have

$$\begin{aligned}
B^{\alpha}_{n_1,n_2}(\eta_{0,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_{0};u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_{0};u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{1,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_{1};u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_{0};u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{0,1};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_{0};u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_{1};u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{1,1};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_{1};u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_{1};u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{2,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_{2};u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_{0};u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{0,2};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_{0};u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_{2};u_1,u_2),
\end{aligned}$$

which proves Lemma 2.4. □

Lemma 2.5. *For all $u_1, u_2 \in \mathcal{I}^2$ and sufficiently large $n_1, n_2 \in \mathbb{N}$ the operators $H^{*}_{n_1,n_2}(.\ ;\ .)$ satisfy following*

(1) $B^{\alpha}_{n_1,n_2}(\Psi^{2,0}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_1}\right)(u_1+1)^2 \le C_1(u_1+1)^2$ *as* $n_1, n_2 \to \infty$,

(2) $B^{\alpha}_{n_1,n_2}(\Psi^{0,2}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_2}\right)(u_2+1)^2 \le C_2(u_2+1)^2$ *as* $n_1, n_2 \to \infty$,

(3) $B^{\alpha}_{n_1,n_2}(\Psi^{4,0}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_1^2}\right)(u_1+1)^4 \le C_3(u_1+1)^4$ *as* $n_1, n_2 \to \infty$,

(4) $B^{\alpha}_{n_1,n_2}(\Psi^{0,4}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_2^2}\right)(u_2+1)^4 \le C_4(u_2+1)^4$ *as* $n_1, n_2 \to \infty$.

3. Some approximation results in weighted space and their degree of convergence

Let φ be weight function such that $\varphi(u_1,u_2) = 1 + u_1^2 + u_2^2$ and satisfying $B_{\varphi}\left(\mathcal{I}^2\right) = \{g :\mid g(u_1,u_2) \mid \le C_g\varphi(u_1,u_2), \quad C_g > 0\}$, where $B_{\varphi}\left(\mathcal{I}^2\right)$ is the set of all bounded function on $\mathcal{I}^2 = [0,\infty)\times[0,\infty)$. Suppose $C^{(m)}\left(\mathcal{I}^2\right)$ be the m-times continuously differentiable functions defined on $\mathcal{I}^2 = \{(u_1,u_2) \in \mathcal{I}^2 : u_1, u_2 \in [0,\infty)\}$. The equipped norm on B_φ defined by $\| g \|_\varphi = \sup_{u_1,u_2\in\mathcal{I}^2} \frac{|g(u_1,u_2)|}{\varphi(u_1,u_2)}$. Moreover, we have classified here some classes of function as follows:

$$C_\varphi^m\left(\mathcal{I}^2\right) = \left\{ g : g \in C_\varphi\left(\mathcal{I}^2\right); \quad \text{such that } \lim_{(u_1,u_2)\to\infty} \frac{g(u_1,u_2)}{\varphi(u_1,u_2)} = k_g < \infty \right\},$$
$$C_\varphi^0\left(\mathcal{I}^2\right) = \left\{ f : f \in C_\varphi^m\left(\mathcal{I}^2\right); \quad \text{such that } \lim_{(u_1,u_2)\to\infty} \frac{g(u_1,u_2)}{\varphi(u_1,u_2)} = 0 \right\},$$
$$C_\varphi\left(\mathcal{I}^2\right) = \{g : g \in B_\varphi \cap C_\varphi\left(\mathcal{I}^2\right)\}.$$

Suppose $\omega_\varphi(g;\delta_1,\delta_2)$ is the weighted modulus of continuity for all $g \in C_\varphi^0\left(\mathcal{I}^2\right)$ and $\delta_1, \delta_2 > 0$, defined by

$$\omega_\varphi(g;\delta_1,\delta_2) = \sup_{(u_1,u_2)\in[0,1]} \sup_{0\le|\theta_1|\le\delta_1, 0\le|\theta_2|\le\delta_2} \frac{| g(u_1+\theta_1,u_2+\theta_2) - g(u_1,u_2) |}{\varphi(u_1,u_2)\varphi(\theta_1,\theta_2)}. \tag{6}$$

For any $\eta_1, \eta_2 > 0$ one has

$$\omega_\varphi(g;\eta_1\delta_1,\eta_2\delta_2) \le 4(1+\eta_1)(1+\eta_2)(1+\delta_1^2)(1+\delta_2^2)\omega_\varphi(g;\delta_1,\delta_2),$$
$$| g(t,s) - g(u_1,u_2) | \le \varphi(u_1,u_2)\varphi\left(| t-u_1 |, | s-u_2 |\right)\omega_\varphi\left(g; | t-u_1 |, | s-u_2 |\right)$$
$$\le (1+u_1^2+u_2^2)(1+(t-u_1)^2)(1+(s-u_2)^2)\omega_\varphi\left(g; | t-u_1 |, | s-u_2 |\right).$$

Theorem 3.1. *Let $g \in C_\varphi^0\left(\mathcal{I}^2\right)$, then for sufficiently large $n_1, n_2 \in \mathbb{N}$ operator $B_{n_1,n_2}^\alpha(.;.)$ satisfying the inequality*

$$\frac{| B_{n_1,n_2}^\alpha(g;u_1,u_2) - g(u_1,u_2) |}{(1+u_1^2+u_2^2)}$$
$$\le \Psi_{u_1,u_2}\left(1 + O\left(n_1^{-1}\right)\right)\left(1 + O\left(n_2^{-1}\right)\right)\omega_\varphi\left(g; O\left(n_1^{-\frac{1}{2}}\right), \left(n_2^{-\frac{1}{2}}\right)\right),$$

where $\Psi_{u_1,u_2} = (1+(u_1+1)+C_1(u_1+1)^2+\sqrt{C_3}(u_1+1)^3)(1+(u_2+1)+C_2(u_2+1)^2+\sqrt{C_4}(u_2+1)^3)$ and $C_1, C_2, C_3, C_4 > 0$.

Proof. For all $\delta_{n_1}, \delta_{n_2} > 0$ we have

$$| g(t,s) - g(u_1,u_2) | \le 4(1+u_1^2+u_2^2)\left(1+(t-u_1)^2\right)\left(1+(s-u_2)^2\right)$$
$$\times \left(1 + \frac{| t-u_1 |}{\delta_{n_1}}\right)\left(1 + \frac{| s-u_2 |}{\delta_{n_2}}\right)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right)$$
$$= 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)$$
$$\times \left(1 + \frac{| t-u_1 |}{\delta_{n_1}} + (t-u_1)^2 + \frac{1}{\delta_{n_1}} | t-u_1 | (t-u_1)^2\right)$$
$$\times \left(1 + \frac{| s-u_2 |}{\delta_{n_2}} + (s-u_2)^2 + \frac{| s-u_2 |}{\delta_{n_2}}(s-u_2)^2\right)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right).$$

Applying $B^{\alpha}_{n_1,n_2}(.;.)$ both the sides and then using Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid \leq B^{\alpha}_{n_1,n_2}\left(\mid g(.,.)-g(u_1,u_2)\mid ;u_1,u_2\right)4(1+u_1^2+u_2^2)\\
&\times B^{\alpha}_{n_1,n_2}\left(1+\frac{\mid t-u_1\mid}{\delta_{n_1}}+(t-u_1)^2+\frac{1}{\delta_{n_1}}\mid t-u_1\mid (t-u_1)^2;u_1,u_2\right)\\
&\times B^{\alpha}_{n_1,n_2}\left(1+\frac{\mid s-u_2\mid}{\delta_{n_2}}+(s-u_2)^2+\frac{\mid s-u_2\mid}{\delta_{n_2}}(s-u_2)^2;u_1,u_2\right)\\
&\times (1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&=4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&\times\Bigg(1+\frac{1}{\delta_{n_1}}B^{\alpha}_{n_1,n_2}(\mid t-u_1\mid ;u_1,u_2)+B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_1}}B^{\alpha}_{n_1,n_2}(\mid t-u_1\mid (t-u_1)^2;u_1,u_2\Bigg)\\
&\times\Bigg(1+\frac{1}{\delta_{n_2}}B^{\alpha}_{n_1,n_2}(\mid s-u_2\mid ;u_1,u_2)+B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_2}}B^{\alpha}_{n_1,n_2}(\mid s-u_2\mid (s-u_2)^2;u_1,u_2\Bigg);
\end{aligned}
$$

$$
\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid \leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&\times\Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)}+B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_1}}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^4;u_1,u_2)}\Bigg]\\
&\times\Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)}+B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_2}}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^4;u_1,u_2)}\Bigg].
\end{aligned}
$$

In view of Lemma 2.5 and choose $\delta_{n_1}=O(n_1^{-\frac{1}{2}})$ and $\delta_{n_2}=O(n_2^{-\frac{1}{2}})$, then

$$
\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid \leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}\left(g;\delta_{n_1},\delta_{n_2}\right)\\
&\times\Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^2}+O\left(\frac{1}{n_1}\right)(u_1+1)^2\\
&+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^2}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^4}\Bigg]\\
&\times\Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^2}+O\left(\frac{1}{n_2}\right)(u_2+1)^2\\
&+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^2}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^4}\Bigg]
\end{aligned}
$$

$$\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\leq 4(1+u_1^2+u_2^2)(1+\delta^2_{n_1})(1+\delta^2_{n_2})\omega_{\varphi}\left(g;\delta_{n_1},\delta_{n_2}\right)$$
$$\times\left[1+(u_1+1)+C_1(u_1+1)^2+\sqrt{C_2}(u_1+1)^3\right]\Big[1+(u_2+1)$$
$$+C_3(u_2+1)^2+\sqrt{C_4}(u_2+1)^3\Big].$$

Which completes the proof. □

Lemma 3.1 ([38, 39])**.** *For the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$, which acting $C_\varphi\to B_\varphi$ defined earlier then there exists some positive K such that*

$$\parallel L_{n_1,n_2}(\varphi;u_1,u_2)\parallel_{\varphi}\leq K.$$

Theorem 3.2 ([38, 39])**.** *For the positive sequence of operators $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$ acting $C_\varphi\to B_\varphi$ defined earlier satisfying the following conditions*

$$\begin{array}{rrcl}
(1) & \lim\limits_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}(1;u_1,u_2)-1\parallel_{\varphi} &=& 0,\\
(2) & \lim\limits_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}(t;u_1,u_2)-u_1\parallel_{\varphi} &=& 0,\\
(3) & \lim\limits_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}(s;u_1,u_2)-u_2\parallel_{\varphi} &=& 0,\\
(4) & \lim\limits_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}((t^2+s^2);u_1,u_2)-(u_1^2+u_2^2)\parallel_{\varphi} &=& 0.
\end{array}$$

Then, for all $g\in C^0_\varphi$,

$$\lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}g-g\parallel_{\varphi}=0$$

and there exists another function $f\in C_\varphi\setminus C^0_\varphi$, such that

$$\lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}f-f\parallel_{\varphi}\geq 1.$$

Theorem 3.3. *If $g\in C^0_\varphi\left(\mathcal{I}^2\right)$ then, we have*

$$\lim_{n_1,n_2\to\infty}\parallel B^{\alpha}_{n_1,n_2}(g)-g\parallel_{\varphi}=0.$$

Proof.

$$\parallel B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2)\parallel_{\varphi}=\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{\mid B^{\alpha}_{n_1,n_2}(1+u_1^2+u_2^2;u_1,u_2)\mid}{1+u_1^2+u_2^2}$$
$$=1+\sup_{(u_1,u_2)\in\mathcal{I}^2}\left[\frac{1}{1+u_1^2+u_2^2}\left|\left(1+B^{\alpha}_{n_1,n_2}(u_1^2;u_1,u_2)+B^{\alpha}_{n_1,n_2}(u_2^2;u_1,u_2)\right)\right|\right]$$

$$\begin{aligned}
\| B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2) \|_{\varphi} &= 1 + \left|1+\frac{4\alpha-3}{n_1}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_1^2}{1+u_1^2+u_2^2} \\
&+ \left|\frac{2\lambda+3}{n_1} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_1^2}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_1}{1+u_1^2+u_2^2} \\
&+ \left|\frac{\lambda^2+3\lambda+2}{n_1^2}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{1}{1+u_1^2+u_2^2} \\
&+ \left|1+\frac{4\alpha-3}{n_2}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_2^2}{1+u_1^2+u_2^2} \\
&+ \left|\frac{2\lambda+3}{n_2} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_2^2}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{u_2}{1+u_1^2+u_2^2} \\
&+ \left|\frac{\lambda^2+3\lambda+2}{n_2^2}\right| \sup_{(u_1,u_2)\in\mathcal{I}^2} \frac{1}{1+u_1^2+u_2^2},
\end{aligned}$$

$$\begin{aligned}
\| B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2) \|_{\varphi} &\leq 1 + \left|1+\frac{4\alpha-3}{n_1}\right| + \left|\frac{2\lambda+3}{n_1} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_1^2}\right| \\
&+ \left|\frac{\lambda^2+3\lambda+2}{n_1^2}\right| + \left|1+\frac{4\alpha-3}{n_2}\right| \\
&+ \left|\frac{2\lambda+3}{n_2} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_2^2}\right| + \left|\frac{\lambda^2+3\lambda+2}{n_2^2}\right|.
\end{aligned}$$

Now, for all $n_1, n_2 \in \mathbb{N}\setminus\{1,2\}$, there exists a positive constant K such that

$$\| B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2) \|_{\varphi} \leq K.$$

Therefore, in order to prove Theorem 3.3 it is sufficient from the Lemma 3.1 and Theorem 3.2. Thus we arrive at the prove of Theorem 3.3. □

For any $g \in C(\mathcal{I}^2)$ and $\delta > 0$ modulus of continuity of order second is given by

$$\omega\left(g;\delta_{n_1},\delta_{n_2}\right) = \sup\{\mid g(t,s)-g(u_1,u_2)\mid : (t,s),(u_1,u_2)\in\mathcal{I}^2\}$$

with $\mid t-u_1 \mid \leq \delta_{n_1}$, $\mid s-u_2 \mid \leq \delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1\left(g;\delta\right) = \sup_{0\leq u_2\leq 1}\ \sup_{|x_1-x_2|\leq\delta} \{\mid g(x_1,u_2)-g(x_2,u_2)\mid\},$$

$$\omega_2\left(g;\delta\right) = \sup_{0\leq u_1\leq 1}\ \sup_{|u_1-u_2|\leq\delta} \{\mid g(u_1,u_1)-g(u_1,u_2)\mid\}.$$

Theorem 3.4. *For any $g \in C(\mathcal{I}^2)$, we have*

$$\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid \leq 2\bigg(\omega_1(g;\delta_{u_1,n_1}) + \omega_2(g;\delta_{n_2,u_2})\bigg).$$

Proof. In order to give the prove of Theorem 3.4, in general we use well-known Cauchy-Schwarz inequality. Thus, we see that

$$\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid \leq B^{\alpha}_{n_1,n_2}\left(\mid g(t,s)-g(u_1,u_2)\mid ;u_1,u_2\right)\\
&\leq B^{\alpha}_{n_1,n_2}\left(\mid g(t,s)-g(u_1,s)\mid ;u_1,u_2\right)\\
&+B^{\alpha}_{n_1,n_2}\left(\mid g(u_1,s)-g(u_1,u_2)\mid ;u_1,u_2\right)\\
&\leq B^{\alpha}_{n_1,n_2}\left(\omega_1(g;\mid t-u_1\mid);u_1,u_2\right)+B^{\alpha}_{n_1,n_2}\left(\omega_2(g;\mid s-u_2\mid);u_1,u_2\right)\\
&\leq \omega_1(g;\delta_{n_1})\left(1+\delta_{n_1}^{-1}B^{\alpha}_{n_1,n_2}(\mid t-u_1\mid ;u_1,u_2)\right)\\
&+\omega_2(g;\delta_{n_2})\left(1+\delta_{n_2}^{-1}B^{\alpha}_{n_1,n_2}(\mid s-u_2\mid ;u_1,u_2)\right)\\
&\leq \omega_1(g;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)}\right)\\
&+\omega_2(g;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)}\right).
\end{aligned}$$

If we choose $\delta_{n_1}^2=\delta_{n_1,u_1}^2=B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)$ and $\delta_{n_2}^2=\delta_{n_2,u_2}^2=B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)$, then we easily to reach our desired results. $\square$

Here, we find convergence in terms of the Lipschitz class for bivariate function. For $M>0$ and $\rho_1,\rho_2\in[0,1]$, Lipschitz maximal function space on $E\times E\subset\mathcal{I}^2$ defined by

$$\begin{aligned}
\mathcal{L}_{\rho_1,\rho_2}(E\times E) &= \Big\{g:\sup(1+t)^{\rho_1}(1+s)^{\rho_2}\left(g_{\rho_1,\rho_2}(t,s)-g_{\rho_1,\rho_2}(u_1,u_2)\right)\\
&\leq M\frac{1}{(1+u_1)^{\rho_1}}\frac{1}{(1+u_2)^{\rho_2}}\Big\},
\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$g_{\rho_1,\rho_2}(t,s)-g_{\rho_1,\rho_2}(u_1,u_2)=\frac{\mid g(t,s)-g(u_1,u_2)\mid}{\mid t-u_1\mid^{\rho_1}\mid s-u_2\mid^{\rho_2}};\quad (t,s),(u_1,u_2)\in\mathcal{I}^2. \tag{7}$$

Theorem 3.5. *Let $g\in\mathcal{L}_{\rho_1,\rho_2}(E\times E)$, then for any $\rho_1,\rho_2\in[0,1]$, there exists $M>0$ such that*

$$\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\\
&\leq M\Bigg\{\left((d(u_1,E))^{\rho_1}+\left(\delta^2_{n_1,u_1}\right)^{\frac{\rho_1}{2}}\right)\left((d(u_2,E))^{\rho_2}+\left(\delta^2_{n_2,u_2}\right)^{\frac{\rho_2}{2}}\right)\\
&+(d(u_1,E))^{\rho_1}\,(d(u_2,E))^{\rho_2}\Bigg\},
\end{aligned}$$

where δ_{n_1,u_1} and δ_{n_2,u_2} defined by Theorem 3.4.

Proof. Take $\mid u_1-x_0\mid=d(u_1,E)$ and $\mid u_2-y_0\mid=d(u_2,E)$. For any $(u_1,u_2)\in\mathcal{I}^2$, and $(x_0,y_0)\in E\times E$ we let $d(u_1,E)=\inf\{\mid u_1-u_2\mid:u_2\in E\}$. Thus, we can write here

$$\mid g(t,s)-g(u_1,u_2)\mid\leq M\mid g(t,s)-g(x_0,y_0)\mid+\mid g(x_0,y_0)-g(u_1,u_2)\mid. \tag{8}$$

Apply $B^{\alpha}_{n_1,n_2}$, we obtain

$$\begin{aligned}
&| B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2) |\leq B^{\alpha}_{n_1,n_2}(| g(u_1,u_2)-g(x_0,y_0) | \\
&+| g(x_0,y_0)-g(u_1,u_2) |) \\
&\leq MB^{\alpha}_{n_1,n_2}(| t-x_0 |^{\rho_1}| s-y_0 |^{\rho_2};u_1,u_2) \\
&+M | u_1-x_0 |^{\rho_1}| u_2-y_0 |^{\rho_2} .
\end{aligned}$$

For all $A,B\geq 0$ and $\rho\in[0,1]$ we know inequality $(A+B)^{\rho}\leq A^{\rho}+B^{\rho}$, thus

$$| t-x_0 |^{\rho_1}\leq| t-u_1 |^{\rho_1}+| u_1-x_0 |^{\rho_1},$$

$$| s-y_0 |^{\rho_1}\leq| s-u_2 |^{\rho_2}+| u_2-y_0 |^{\rho_2} .$$

Therefore,

$$\begin{aligned}
| B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2) | &\leq MB^{\alpha}_{n_1,n_2}(| t-u_1 |^{\rho_1}| s-u_2 |^{\rho_2};u_1,u_2) \\
&+M | u_1-x_0 |^{\rho_1} B^{\alpha}_{n_1,n_2}(| s-u_2 |^{\rho_2};u_1,u_2) \\
&+M | u_2-y_0 |^{\rho_2} B^{\alpha}_{n_1,n_2}(| t-u_1 |^{\rho_1};u_1,u_2) \\
&+2M | u_1-x_0 |^{\rho_1}| u_2-y_0 |^{\rho_2} B^{\alpha}_{n_1,n_2}(\mu_{0,0};u_1,u_2) .
\end{aligned}$$

On apply Hölder inequality on $B^{\alpha}_{n_1,n_2}$, we get

$$\begin{aligned}
&B^{\alpha}_{n_1,n_2}(| t-u_1 |^{\rho_1}| s-u_2 |^{\rho_2};u_1,u_2) \\
&=\mathcal{U}^{\alpha_1}_{n_1,k}(| t-u_1 |^{\rho_1};u_1,u_2)\,\mathcal{V}^{\alpha_2}_{n_2,l}(| s-u_2 |^{\rho_2};u_1,u_2) \\
&\leq \left(B^{\alpha}_{n_1,n_2}(| t-u_1 |^2;u_1,u_2)\right)^{\frac{\rho_1}{2}}\left(B^{\alpha}_{n_1,n_2}(\mu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_1}{2}} \\
&\times\left(B^{\alpha}_{n_1,n_2}(| s-u_2 |^2;u_1,u_2)\right)^{\frac{\rho_2}{2}}\left(B^{\alpha}_{n_1,n_2}(\mu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_2}{2}} .
\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}
&| B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2) | \\
&\leq M\left(\delta^2_{n_1,u_1}\right)^{\frac{\rho_1}{2}}\left(\delta^2_{n_2,u_2}\right)^{\frac{\rho_2}{2}}+2M\,(d(u_1,E))^{\rho_1}\,(d(u_2,E))^{\rho_2} \\
&+M\,(d(u_1,E))^{\rho_1}\left(\delta^2_{n_2,u_2}\right)^{\frac{\rho_2}{2}}+L\,(d(u_2,E))^{\rho_2}\left(\delta^2_{n_1,u_1}\right)^{\frac{\rho_1}{2}} .
\end{aligned}$$

We have complete the proof. $\square$

Theorem 3.6. *If $g\in C'(\mathcal{I}^2)$, then for all $(u_1,u_2)\in\mathcal{I}^2$, operator $B^{\alpha}_{n_1,n_2}$ satisfying*

$$| B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2) |\leq\| g_{u_1} \|_{C(\mathcal{I}^2)}\left(\delta^2_{n_1,u_1}\right)^{\frac{1}{2}}+\| g_{u_2} \|_{C(\mathcal{I}^2)}\left(\delta^2_{n_2,u_2}\right)^{\frac{1}{2}},$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.4.

Proof. Let $g \in C'(\mathcal{I}^2)$, and for any fixed $(u_1, u_2) \in \mathcal{I}^2$ we have

$$g(t,s) - g(u_1,u_2) \quad = \quad \int_{u_1}^{t} g_\varrho(\varrho, s)\mathrm{d}\varrho + \int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu.$$

On apply $B^{\alpha}_{n_1,n_2}$

$$\begin{aligned} & B^{\alpha}_{n_1,n_2}\left(g(t,s); u_1, u_2\right) - g(u_1,u_2) \\ (9) \qquad & = B^{\alpha}_{n_1,n_2}\left(\int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho; u_1, u_2\right) + B^{\alpha}_{n_1,n_2}\left(\int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu; u_1, u_2\right). \end{aligned}$$

From the sup-norm on $\mathcal{I}^2$ we can see that

$$(10) \qquad \mid \int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho \mid \leq \int_{u_1}^{t} \mid g_\varrho(\varrho,s)\mathrm{d}\varrho \mid \leq \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} \mid t - u_1 \mid,$$

$$(11) \qquad \mid \int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu \mid \leq \int_{u_2}^{s} \mid g_\mu(u_1,\mu)\mathrm{d}\mu \mid \leq \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} \mid s - u_2 \mid.$$

In the view of (9), (10) and (11) we can obtain

$$\begin{aligned} & \mid B^{\alpha}_{n_1,n_2}\left(g(u_1,u_2); u_1, u_2\right) - g(u_1,u_2) \mid \\ & \leq B^{\alpha}_{n_1,n_2}\left(\left| \int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho \right|; u_1, u_2\right) \\ & + B^{\alpha}_{n_1,n_2}\left(\left| \int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu \right|; u_1, u_2\right) \\ & \leq \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} B^{\alpha}_{n_1,n_2}\left(\mid t - u_1 \mid; u_1, u_2\right) \\ & + \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} B^{\alpha}_{n_1,n_2}\left(\mid s - u_2 \mid; u_1, u_2\right) \\ & \leq \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} \left(B^{\alpha}_{n_1,n_2}((t-u_1)^2; u_1,u_2) B^{\alpha}_{n_1,n_2}(1; u_1,u_2)\right)^{\frac{1}{2}} \\ & + \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} \left(B^{\alpha}_{n_1,n_2}((s-u_2)^2; u_1,u_2) B^{\alpha}_{n_1,n_2}(1; u_1,u_2)\right)^{\frac{1}{2}} \\ & = \parallel g_{u_1} \parallel_{C(\mathcal{I}^2)} \left(\delta^2_{n_1,u_1}\right)^{\frac{1}{2}} + \parallel g_{u_2} \parallel_{C(\mathcal{I}^2)} \left(\delta^2_{n_2,u_2}\right)^{\frac{1}{2}}. \end{aligned}$$

□

Theorem 3.7. *For any $f \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that*

$$\begin{aligned} & R^{\alpha_1,\alpha_2}_{n_1,n_2}(f; u_1, u_2) = B^{\alpha}_{n_1,n_2}(g; u_1, u_2) + f(u_1,u_2) \\ & - f\left(\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0}; u_1, u_2), \mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1}; u_1, u_2)\right), \end{aligned}$$

where, from Lemma 2.4, $\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0}; u_1, u_2) = \frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1}, n_1 > 1$ *and*

$$\mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1}; u_1, u_2) = \frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2}, n_2 > 1.$$

Then, for all $g \in C'(\mathcal{I}^2)$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying

$$\begin{aligned} R^{\alpha_1,\alpha_2}_{n_1,n_2}(g;t,s)-g(u_1,u_2) &\le \left\{\delta^2_{n_1,u_1}+\delta^2_{n_2,u_2}+\left(\frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}-u_1\right)^2\right.\\ &+\left.\left(\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2}-u_2\right)^2\right\}\parallel g\parallel_{C^2(\mathcal{I}^2)}. \end{aligned}$$

Proof. In the light of operator $R^{\alpha_1,\alpha_2}_{n_1,n_2}(f;u_1,u_2)$ and Lemma 2.4, we obtain $R^{\alpha_1,\alpha_2}_{n_1,n_2}(1;u_1,u_2)=1$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}(t-u_1;u_1,u_2)=0$ and $R^{\alpha_1,\alpha_2}_{n_1,n_2}(s-u_2;u_1,u_2)=0$. For any $g \in C'(\mathcal{I}^2)$ the Taylor series give us:

$$\begin{aligned} g(t,s)-g(u_1,u_2) &= \frac{\partial g(u_1,u_2)}{\partial u_1}(t-u_1)+\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\\ &+ \frac{\partial g(u_1,u_2)}{\partial u_2}(s-u_2)+\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi. \end{aligned}$$

On apply $R^{\alpha_1,\alpha_2}_{n_1,n_2}$, we see that

$$\begin{aligned} &R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(t,s);u_1,u_2\right)-R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(u_1,u_2)\right)\\ &=R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda;u_1,u_2\right)\\ &+R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;u_1,u_2\right)\\ &=B^{\alpha}_{n_1,n_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda;u_1,u_2\right)\\ &+B^{\alpha}_{n_1,n_2}\left(\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi;u_1,u_2\right)\\ &-\int_{u_1}^{\frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}}\left(\frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}-\lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\\ &-\int_{u_2}^{\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2}}\left(\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2}-\psi\right)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi. \end{aligned}$$

From hypothesis we easily obtain

$$\begin{aligned} &\left|\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\right|\le\int_{u_1}^{t}\left|(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\right|\le\parallel g\parallel_{C^2(\mathcal{I}^2)}(t-u_1)^2,\\ &\left|\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right|\le\int_{u_2}^{s}\left|(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial\psi^2}\mathrm{d}\psi\right|\le\parallel g\parallel_{C^2(\mathcal{I}^2)}(s-u_2)^2,\\ &\left|\int_{u_1}^{\frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}}\left(\frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}-\lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial\lambda^2}\mathrm{d}\lambda\right| \end{aligned}$$

$$\leq \| g \|_{C^2(\mathcal{I}^2)} \left(\frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1} - u_1\right)^2$$

$$\left| \int_{u_2}^{\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2}} \left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - \psi\right) \frac{\partial^2 g(u_1,\psi)}{\partial \psi^2} \mathrm{d}\psi \right|$$

$$\leq \| g \|_{C^2(\mathcal{I}^2)} \left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - u_2\right)^2.$$

Thus,

$$\begin{aligned}
&| R^{\alpha_1,\alpha_2}_{n_1,n_2}(g;t,s) - g(u_1,u_2) | \\
&\leq \Bigg\{ B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2) + B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2) \\
&+ \left(\frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1} - u_1\right)^2 \\
&+ \left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - u_2\right)^2 \Bigg\} \| g \|_{C^2(\mathcal{I}^2)}.
\end{aligned}$$

We complete the proof of desired Theorem 3.7. □

4. Some approximation results in Bögel space

Take any function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ for a real compact intervals of $\mathcal{I}_1 \times \mathcal{I}_2$. For all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ suppose $\Delta^*_{(t,s)} g(u_1,u_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta^*_{(t,s)} g(u_1,u_2) = g(t,s) - g(t,u_2) - g(u_1,s) + g(u_1,u_2).$$

If at any point $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ the function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ defined on $\mathcal{I}_1 \times \mathcal{I}_2$, then $\lim_{(t,s)\to(u_1,u_2)} \Delta^*_{(t,s)} g(u_1,u_2) = 0$. If set of all the space of all Bögel-continuous(B-continuous) denoted by $C_B(\mathcal{I}_1 \times \mathcal{I}_2)$ on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and be defined such that $C_B(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is $g,\ B-$bounded on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Next, the Bögel-differentiable function on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ and limit exists finite defined by

$$\lim_{(t,s)\to(u_1,u_2),\, t\neq u_1,\, s\neq u_2} \frac{1}{(t-u_1)(s-u_2)} \left(\Delta^*_{(t,s)}\right) = D_B g(u_1,u_2) < \infty.$$

Let the classes of all Bögel-differentiable function denoted by $D_\varphi g(u_1,u_2)$ and be $D_\varphi(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is g, B$-$differentiable on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Suppose the function g is B-bounded on D and be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$, then for all $(t,s),\ (u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ there exists positive constant M such that

$\mid \Delta^*_{(t,s)} g(u_1,u_2) \mid \leq M$. The classes of all B-continuous function is called a B-bounded on $\mathcal{I}_1 \times \mathcal{I}_2$, whene $\mathcal{I}_1 \times \mathcal{I}_2$ is compact subset. Let $B_\varphi(\mathcal{I}_1 \times \mathcal{I}_2)$ denote the classes of all B-bounded function defined on $\mathcal{I}_1 \times \mathcal{I}_2$ which equipped the norm on B as $\parallel g \parallel_B = \sup_{(t,s),(u_1,u_2)\in \mathcal{I}_1\times\mathcal{I}_2} \mid \Delta^*_{(t,s)} g(u_1,u_2) \mid$. As we know to approximate the degree for a set of all B-continuous function on positive linear operators, it is essential to use the properties of mixed-modulus of continuity. So we let for all (t,s), $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and $g \in B_\varphi(\mathcal{I}_{\alpha_n})$, the mixed-modulus of continuity of function g bt $\omega_B : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and be defined such as:

$$\omega_B(g;\delta_1,\delta_2) = \sup\{\Delta^*_{(t,s)} g(u_1,u_2) :\mid t-u_1 \mid \leq \delta_1,\ \mid s-u_2 \mid \leq \delta_2\}.$$

For any $\mathcal{I}^2 = [0,1] \times [0,1]$, we suppose the classes of all B-continuous function defined on $\mathcal{I}^2$ denoted by $C_\varphi(\mathcal{I}^2)$. Moreover, let set of all ordinary continuous function defined on $\mathcal{I}^2$ be $C(\mathcal{I}^2)$. For further details on space of Bögel functions related to this paper we propose the article [35, 36].

Let $(u_1,u_2) \in \mathcal{I}^2$ and $n_1,n_2 \in \mathbb{N}$ then for all $g \in C(\mathcal{I}^2)$ we define the GBS type operators for the positive linear operators $B^\alpha_{n_1,n_2}$. Thus we suppose

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(t,s);u_1,u_2\right) = B^\alpha_{n_1,n_2}\bigg(g(t,u_2)+g(u_1,s)-g(t,s);u_1,u_2\bigg). \tag{12}$$

More precisely, the generalized GBS operator for bivariate function is defined as follows:

$$\begin{aligned} & K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(t,s);u_1,u_2\right) \\ & = \sum_{k,l=0}^{\infty} \mathcal{Q}^*_1(n_1,u_1)\mathcal{Q}^*_2(n_2,u_2)\int_0^\infty\int_0^\infty \mathcal{P}^*_1(n_1,u_1)P^*_2(n_2,u_2), \end{aligned} \tag{13}$$

where $P_{u_1,u_2}(t,s) = \big(g(t,u_2)+g(u_1,s)-g(t,s)\big)$.

Theorem 4.1. *For all $g \in C_\varphi(\mathcal{I}^2)$, it follows that*

$$\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(t,s);u_1,u_2\right) - g(u_1,u_2) \mid \leq 4\omega_B\left(g;\delta_{n_1,u_1},\delta_{n_2,u_2}\right),$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.4.

Proof. Let $(t,s),(u_1,u_2) \in \mathcal{I}^2$. For all $n_1,n_2 \in \mathbb{N}$ and $\delta_{n_1},\delta_{n_2} > 0$, it follows that

$$\begin{aligned} \mid \Delta^*_{(u_1,u_2)} g(t,s) \mid &\leq \omega_B\left(g;\mid t-u_1\mid \mid s-u_2 \mid\right) \\ &\leq \left(1+\frac{t-u_1}{\delta_{n_1}}\right)\left(1+\frac{s-u_2}{\delta_{n_2}}\right)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right). \end{aligned}$$

From (12) and well-known Cauchy-Schwarz inequality, we easily conclude that

$$\begin{aligned}
&\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(t,s);u_1,u_2\right)-g(u_1,u_2)\mid\leq B^{\alpha}_{n_1,n_2}\Big(\mid\Delta^{*}_{(u_1,u_2)}g(t,s)\mid;u_1,u_2\Big)\\
&\leq\Bigg(B^{\alpha}_{n_1,n_2}(\phi_{0,0};u_1,u_2)+\frac{1}{\delta_{n_1}}\left(B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}}\\
&+\frac{1}{\delta_{n_2}}\left(B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}}\\
&+\frac{1}{\delta_{n_1}}\left(B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}}\\
&\times\frac{1}{\delta_{n_2}}\left(B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}}\Bigg)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}$$

In the view of Theorem 3.4 we easily get our results. □

If we let $x=(t,s),\ y=(u_1,u_2)\in\mathcal{I}^2$, then the Lipschitz function in terms of B-continuous functions defined by

$$Lip^{\xi}_{M}=\left\{g\in C(\mathcal{I}^2):\mid\Delta^{*}_{(u_1,u_2)}g(x,y)\mid\leq M\parallel x-y\parallel^{\xi}\right\},$$

where M is a positive constant, $0<\xi\leq 1$, and Euclidean norm $\parallel x-y\parallel=\sqrt{(t-u_1)^2+(s-u_2)^2}$.

Theorem 4.2. *For all $g\in Lip^{\xi}_{M}$ operator $K^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying*

$$\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(x,y);u_1,u_2\right)-g(u_1,u_2)\mid\leq M\{\delta^2_{n_1,u_1}+\delta^2_{n_2,u_2}\}^{\frac{\xi}{2}},$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.4.

Proof. We easily see that

$$\begin{aligned}
K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(x,y);u_1,u_2\right) &= B^{\alpha}_{n_1,n_2}\left(g(u_1,y)+g(x,u_2)-g(x,s);u_1,u_2\right)\\
&= B^{\alpha}_{n_1,n_2}\Big(g(u_1,u_2)-\Delta^{*}_{(u_1,u_2)}g(x,s);u_1,u_2\Big)\\
&= g(u_1,u_2)-B^{\alpha}_{n_1,n_2}\Big(\Delta^{*}_{(u_1,u_2)}g(x,s);u_1,u_2\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(x,y);u_1,u_2\right)-g(u_1,u_2)\mid\leq B^{\alpha}_{n_1,n_2}\Big(\mid\Delta^{*}_{(u_1,u_2)}g(x,y)\mid;u_1,u_2\Big)\\
&\leq MB^{\alpha}_{n_1,n_2}\Big(\parallel x-y\parallel^{\xi};u_1,u_2\Big)\\
&\leq M\{B^{\alpha}_{n_1,n_2}\left(\parallel x-y\parallel^{2};u_1,u_2\right)\}^{\frac{\xi}{2}}\\
&\leq M\{B^{\alpha}_{n_1,n_2}\left((t-u_1)^2;u_1,u_2\right)+B^{\alpha}_{n_1,n_2}\left((s-u_2)^2;u_1,u_2\right)\}^{\frac{\xi}{2}}.
\end{aligned}$$

□

Theorem 4.3. *If $g \in D_{\varphi}(\mathcal{I}^2)$ and $D_B g \in B(\mathcal{I}^2)$, then*

$$\begin{aligned}
&\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid \\
&\leq C\Big\{3 \parallel D_B g \parallel_\infty + \varpi_{mixed}(D_B g;\delta_{n_1},\delta_{n_2})\Big\}(u_1+1)(u_2+1) \\
&+ \Big\{1+\sqrt{C_2}(u_1+1)+\sqrt{C_1}(u_2+1)\Big\} \\
&\times \varpi_{mixed}(D_B g;\delta_{n_1},\delta_{n_2})(u_1+1)(u_2+1),
\end{aligned}$$

where δ_{n_1}, δ_{n_2} defined by Theorem 3.4 and C is any positive constant.

Proof. Suppose $\rho \in (u_1,t)$, $\xi \in (u_2,s)$ and

$$\Delta^*_{(u_1,u_2)} g(t,s) = (t-u_1)(s-u_2) D_B g(\rho,\xi),$$

$$D_B g(\rho,\xi) = \Delta^*_{(u_1,u_2)} D_B g(\rho,\xi) + D_B g(\rho,y) + D_B g(x,\xi) - D_B g(u_1,u_2).$$

For all $D_B g \in B(\mathcal{I}^2)$, it follows that

$$\begin{aligned}
&\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Delta^*_{(u_1,u_2)} g(t,s);u_1,u_2\right) \mid = \mid K^{\alpha_1,\alpha_2}_{n_1,n_2}((t-u_1)(s-u_2)D_B g(\rho,\xi);u_1,u_2) \mid \\
&\leq K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\mid t-u_1 \mid\mid s-u_2 \mid\mid \Delta^*_{(u_1,u_2)} D_B g(\rho,\xi) \mid;u_1,u_2\right) \\
&+ K^{\alpha_1,\alpha_2}_{n_1,n_2}\Bigg(\mid t-u_1 \mid\mid s-u_2 \mid (\mid D_B g(\rho,u_2) \mid \\
&+ \mid D_B g(u_1,\xi) \mid + \mid D_B g(u_1,u_2) \mid);u_1,u_2\Bigg) \\
&\leq K^{\alpha_1,\alpha_2}_{n_1,n_2}\Bigg(\mid t-u_1 \mid\mid s-u_2 \mid \\
&\times \varpi_{mixed}(D_B g; \mid \rho-u_1 \mid, \mid \xi-u_2 \mid);u_1,u_2\Bigg) \\
&+ 3 \parallel D_B g \parallel_\infty K^{\alpha_1,\alpha_2}_{n_1,n_2}(\mid t-u_1 \mid\mid s-u_2 \mid;u_1,u_2).
\end{aligned}$$

Here, ϖ_{mixed}, is mixed-modulus of continuity and it follows that

$$\begin{aligned}
&\varpi_{mixed}(D_B g; \mid \rho-u_1 \mid, \mid \xi-u_2 \mid) \\
&\leq \varpi_{mixed}(D_B g; \mid t-u_1 \mid, \mid s-u_2 \mid) \\
&\leq \left(1+\delta_{n_1}^{-1} \mid t-u_1 \mid\right)\left(1+\delta_{n_2}^{-1} \mid s-u_2 \mid\right) \varpi_{mixed}(D_B g; \mid \delta_{n_1},\delta_{n_2}).
\end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned}
&\mid K^*_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid = \mid \Delta^*_{(u_1,u_2)} g(t,s);u_1,u_2 \mid \\
&\leq 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left((t-u_1)^2(s-u_2)^2;u_1,u_2\right)\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+ \Bigg(K^{\alpha_1,\alpha_2}_{n_1,n_2} \left(\mid t-u_1 \mid\mid s-u_2 \mid; u_1,u_2\right) \\
&+ \delta_{n_1}^{-1} K^{\alpha_1,\alpha_2}_{n_1,n_2} \left((t-u_1)^2 \mid s-u_2 \mid; u_1,u_2\right) \Bigg) \\
&+ \delta_{n_2}^{-1} K^{\alpha_1,\alpha_2}_{n_1,n_2} \left(\mid t-u_1 \mid (s-u_2)^2; u_1,u_2\right) \\
&+ \delta_{n_1}^{-1}\delta_{n_2}^{-1} K^{\alpha_1,\alpha_2}_{n_1,n_2} \left((t-u_1)^2(s-u_2)^2; u_1,u_2\right) \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right);
\end{aligned}
$$

$$
\begin{aligned}
&\mid K^{*}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid = \mid \Delta^{*}_{(u_1,u_2)} g(t,s); u_1,u_2 \mid \\
&\leq 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \Bigg\{ \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \delta_{n_1}^{-1} \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{4,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} + \delta_{n_2}^{-1} \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,4}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \delta_{n_1}^{-1}\delta_{n_2}^{-1} K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,2}_{u_1,u_2}; u_1,u_2\right) \Bigg\} \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).
\end{aligned}
$$

Which follows that

$$
\begin{aligned}
&\mid K^{*}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid \\
&= 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \Bigg\{ \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \delta_{n_1}^{-1} \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{4,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \delta_{n_2}^{-1} \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,4}_{u_1,u_2}; u_1,u_2\right)\right)^{\frac{1}{2}} \\
&+ \delta_{n_1}^{-1}\delta_{n_2}^{-1} K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{2,0}_{u_1,u_2}; u_1,u_2\right) K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Psi^{0,2}_{u_1,u_2}; u_1,u_2\right) \Bigg\} \\
&\times \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).
\end{aligned}
$$

From Lemma (2.5), we demonstrate

$$
\begin{aligned}
&\mid K^{*}_{n_1,n_2}(g;u_1,u_2) - g(u_1,u_2) \mid \leq 3 \parallel D_B g \parallel_\infty \left(\sqrt{C_1C_2}(u_1+1)(u_2+1)\right) \\
&+ \Bigg\{ \left(\sqrt{C_1C_2}(u_1+1)(u_2+1)\right)
\end{aligned}
$$

$$+ \delta_{n_1}^{-1}\left(\sqrt{C_2}\sqrt{O\left(\frac{1}{n_1}\right)}(u_1+1)^2(u_2+1)\right)$$
$$+ \delta_{n_2}^{-1}\left(\sqrt{C_1}\sqrt{O\left(\frac{1}{n_2}\right)}(u_2+1)^2(u_1+1)\right)$$
$$+ \delta_{n_1}^{-1}\delta_{n_2}^{-1}\left(\sqrt{O\left(\frac{1}{n_1}\right)}\sqrt{O\left(\frac{1}{n_2}\right)}(u_1+1)(u_2+1)\right)\Bigg\}$$
$$\times\, \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).$$

Which complete the proof of Theorem 4.3. □

5. Conclusion and remarks

These types of generalization, that is, Bivariate Szász operators is a new generalization. In this, manuscript our investigation is to generalize the Szász Durrmeyer operators based on Dunkl analogue [41] by introducing the bivariate functions. We study the bivariate properties of Szász Durrmeyer operators with the help of modulus of continuity, mixed-modulus of continuity and then find the approximation results in Peetre's K-functional, Voronovskaja type theorem and Lipschitz maximal functions for these bivariate operators. Next, we also construct the GBS type operator of these generalized operators and study approximation in Bögel continuous functions by use of mixed-modulus of continuity.

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Topological approaches for generalized multi-granulation rough sets with applications

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Abstract. Methods of data classifications are considered as a major preprocessing step for pattern recognition, machine learning, and data mining. In this paper, we give two topological approaches to generalize multi-granular rough sets using families of binary relations. In the first approach, we define a family of topological spaces using families of relations to maximize the interiors and minimize the closures. In the second approach we define minimal neighborhoods to classify multi-data of information systems and generate a multi-granular knowledge base. Moreover, we present some important algorithms to reduce all topological reductions of the information system using topological bases. We round off by studying real life applications of this work using medical data.

Keywords: multi-granulation, rough sets, data classifications, information systems, interior operators, closure operators, approximation spaces.

1. Introduction

According to the very rapid growth of data and the high incidence of Internet broadcasting it becomes a seriously urgent issue to extract useful information to make decisions. In order to do this accurately, quickly and cost less, researchers need to work together in this field to unify their research frameworks.

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Many researchers have solved some of the problems of data sharing, but without a general conceptual framework governing their techniques. Some of them have used old mathematical techniques, some have used modern statistical methods, and others have developed hybrid methods between mathematics, statistics, and computer science.

In 1982 Z.Pawlak, introduced the theory of rough sets, [1], which may considered as the first mathematical tool to deal with uncertainty, incomplete and imprecise knowledge. The approximation space in the sense of Pawlak is an ordered pair $AS = (U, R)$, where U is a universal set and R is an equivalence relation on U. The equivalence classes of U are called the knowledge base. The lower approximation of a subset A of U is the union of all equivalence classes contained in A, while the upper approximation is the intersection of all equivalence classes which intersect A non-trivially. A rough set is a pair of two exact sets the lower approximation of A and the upper approximation of A.

Since equivalence relations are too restrictive for many real life applications, the classical rough set theory of Pawlak needed to be generalized. The generalization process is twofold; The first, is to replace the equivalence relation by tolerance relation [2],[3], similarity relation [4] , characteristic relation [5],[6] and arbitrary binary relation [7]. The second, is to replace the partition induced by the equivalence relation by a covering and use it to approximate any subset of the universe [8].

These frameworks are called granular computing, which are models providing solutions to problems in data mining, machine learning, pattern recognition and cognitive science. But, still there are problems that require more extensions. In 2006, Y. Qian introduced the multi-granular computing using rough set instead of a single granular. Multi-granular computing approach is replacing the single relation used in a single granular by a set of relations on the same universe (see [9], [10], [11]).

One of the important branches in mathematics is topology. Topology is the best implementation of relationship between objects or features so when we deal with complicated relationships topology becomes a very satisfactory tool. Pawlak has pointed out that topology is closely linked to rough set theory and on the full conviction that the topological structure of the rough sets is one of the key issues of rough set theory. This convenient relationship has prompted researchers to study this relationship, it's properties and its applications in real life (see [12], [13], [14], [15], [16], [17], [18], [19], [20]]. In 2013, Y. Qian has investigated a new theory on multi-granulation rough sets from the topological point of view, by inducing n-topological spaces on the universe set U from n-equivalence relations on U. He also has studied the multi-granulation topological rough space and its topological properties (see [21]).

An improvement of rough sets' accuracy measure using containment neighborhoods with a medical application and a comparison of two types of rough approximations based on neighborhoods, for new applications at the same research point can found in (see [22],[23],[24],[25],[26],[27],[28]).

In this paper, we offer a convenient hybrid method using topology and rough set theory to solve the problem of multi-source, variable, and large-scale data-sharing. We also develop algorithms based on the extraction of knowledge from such data.

This paper is arranged as follows:

In Section 2, we present the fundamental concepts and properties of the general topology and some concepts of information systems. In section 3 we present two topological approaches for generalized multi-granulation in two categories. The first approach used minimization in the boundary region, while the second approach used the idea of minimal neighborhoods. In Section 4 we apply our results to the problem of attribute reduction in medical information systems. Section 5 lists some important results and some directions for future studies.

2. Preliminaries

In this section, we provide the basic definitions and results on topological spaces and rough sets. In classical rough set theory the approximation space is defined as $(U,\ R)$ where U is non-empty finite set and R is an equivalence relation on U.

Definition 2.1 ([1]). *Let (U,R) be a classical approximation space, the lower and upper approximation of a given set $X \subseteq U$ are defined as follows:*

$$\underline{R}X = \{x : [x]_R \subseteq X\},$$
$$\overline{R}X = \{x : [x]_R \cap X \neq \phi\},$$

where $[x]_R$ is the equivalence class of $x \in U$ with respect to the equivalence relation R.

Remark 1. The boundary region of X is given by $\overline{R}X - \underline{R}X$, $\underline{R}X$ is called the positive region while $U - \overline{R}X$ is called the negative region.

Definition 2.2 ([29]). *A topological space is a pair (U,τ) consisting of a set U and a family τ of subsets of U satisfying the following conditions:*

(τ1) $\emptyset \in \tau$ and $U \in \tau$.

(τ2) τ is closed under arbitrary union.

(τ3) τ is closed under finite intersection.

the members of τ are called open sets and the complement of members of τ are called closed sets.

Definition 2.3 ([29]). *Let (U,τ) be topological space then the $\tau -$ closure of a subset $A \subset U$ is defined as follows:*

$$\tau - cl(A) = \cap\{F \subseteq U : A \subseteq F \text{ and } F \text{ is closed set}\}.$$

Definition 2.4 ([29]). *Let (U,τ) be topological space then the $\tau-interior$ of a subset $A \subset U$ is defined as follows:*

$$\tau - int(A) = \cup\{G \subseteq U : G \subseteq A \text{ and } G \text{ is open set}\}.$$

Z. Pawlak pointed out in [1]that lower approximations correspond to interiors and upper approximations correspond to closures. This idea has prompted the researchers to study the theory of rough set from the topological point of view to know more about rough sets.

Definition 2.5 ([29]). *If U is a finite universe and R is a binary relation on U, then we define, the right neighborhood of $x \in U$ as follows:*

$$xR = \{y : xRy\}.$$

Definition 2.6 ([30]). *Let U be non-empty set, a basis for a topology on U is a collection β of subsets of U such that*

1. *For each $x \in U$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

There are many ways to induce a topology from a given relation. One of them is achieved as follows: using Definition 2.5 we construct the collection $\{xR\}$ for all x in U, the family of all intersections of $\{xR\}_{x\in U}$ is a base β for a topology on U. If the union of all members of $\beta \neq U$ then we add U to β to be a base for a topology on U.

The classification of a rough set to three region as in Remark 1 can also be done by a membership function as follows:

Definition 2.7 ([29]). *Let τ be a topology on a finite set U, with base β, then the rough membership function is*

$$\mu_X^{\tau}(x) = \frac{|\{\cap B_x\} \cap X|}{|\cap B_x|},\ x \in U,$$

where B_x is any member of β containing x.

Theorem 2.1 ([30]). *Let (U,τ) be a topological space, $A \subseteq U$ then $x \in \tau - cl(A)$ if and only if $G \cap A \neq \emptyset$, for all $G \in \tau$ and $x \in G$.*

The idea of the multi-granulation is based on using multi-relation instead of a single relation to obtain better approximation. Thus, we start by giving the definition of multi-granular rough sets based on equivalence relations.

Definition 2.8 ([21]). *Let $(\Omega,\tau_1),(\Omega,\tau_2),\ldots,(\Omega,\tau_n)$ be n topological spaces induced by equivalence relations $R_1,R_2,\ldots,R_n$, respectively, and $X \subseteq \Omega$. Then, we define mint and mcl operators of X with respect to Γ, where $\Gamma = \{\tau_1,\tau_2,\ldots,\tau_n\}$, respectively, as follows:*

$$mint(X) = \bigcup\{A \in \tau_i | \vee (A \subseteq X), i \leq n\},$$
$$mcl(X) = \bigcup\{A \in \tau_i | \wedge (A \cap X \neq \emptyset), i \leq n\}.$$

3. Topological approaches for generalized multi-granulation

In this section we introduce a new theory on multi-granulation rough sets from the point of view of topological spaces. We generalize the equivalence relations to binary relations to be suitable in real life problems in other branches like artificial intelligence, knowledge discovery, machine learning and data mining. Also, our approach can be regarded as a generalization of Pawlak rough set, and we introduce a new algorithmic method for the reduction of attributes in information (decision) system.

3.1 First approach (maximization of interior and minimization of closure)

Definition 3.1. *Let U be a non-empty set, $X \subset U$, $R_1, R_2, \ldots, R_n$ be n binary relations on U and $\tau_1, \tau_2, \ldots, \tau_n$ be n topologies on U induced by the binary relations $R_1, R_2, \ldots, R_n$. We define the Gmint and Gmcl of X as follows*

$$Gmint(X) = \bigcup_{i=1}^{n} \tau_i - int(X), \tag{1}$$

$$Gmcl(X) = \bigcap_{i=1}^{n} \tau_i - cl(X). \tag{2}$$

Lemma 3.1. *Let U be a non-empty set and $X \subseteq U$ and $\tau_1, \tau_2, \ldots, \tau_n$ be n topologies on U.Then*

1. $\tau_i - int(Gmint(X)) = \tau_i - int(X)$,
2. $\tau_i - cl(Gmcl(X)) = \tau_i - cl(X)$.

Proof. (1) By Definition 3.1, we have

$$Gmint(X) = \bigcup_{i=1}^{n} \tau_i - int(X) = \bigcup_{i=1}^{n} G_i,$$

where G_i is the greatest $\tau_i - open$ contained in X. Now,

$$\tau_i - int(Gmint(X)) = \tau_i - int(\bigcup_{i=1}^{n} G_i). = G_i.$$

Since $G_i \subseteq \bigcup_{i=1}^{n} G_i$ and G_i is the greatest $\tau_i - open$ contained in X which contains $G_i \subseteq \bigcup_{i=1}^{n} G_i$, then the greatest $\tau_i - open$ contained in $G_i \subseteq \bigcup_{i=1}^{n} G_i$ is G_i.

(2) By Definition 3.1, we have

$$Gmcl(X) = \bigcap_{i=1}^{n} \tau_i - cl(X) = \bigcap_{i=1}^{n} F_i,$$

where F_i is the smallest $\tau_i - closed$ containing X. Now,

$$\tau_i - cl(Gmcl(X)) = \tau_i - cl(\bigcap_{i=1}^{n} F_i).$$

We claim that $\tau_i - cl(\bigcap_{i=1}^{n} F_i) = F_i$.

Suppose contrarily that there exists a τ_i-closed $F_i{}'$ such that $F_i{}' \subsetneq F_i$ and $\tau_i - cl(\bigcap_{i=1}^{n} F_i) = F_i{}'$. Then, $X \subseteq (\bigcap_{i=1}^{n} F_i) \subseteq F_i{}' \subsetneq F_i$. Therefore, there exists a $\tau_i - closed$ $F_i{}'$ smaller than F_i containing X, which contradicts the fact that $\tau_i - cl(X) = F_i$. Hence, $\tau_i - cl(\bigcap_{i=1}^{n} F_i) = F_i$, and then $\tau_i - cl(Gmcl(X)) = \tau_i - cl(X)$. □

Proposition 3.1. *Let $(U,\tau_1),(U,\tau_2),\ldots,(U,\tau_n)$ be n topological spaces induced by n binary relations $R_1, R_2, \ldots, R_n$ respectively, and $X, Y \subseteq U$. Then,*

1. *Gmint(U)=U,*
2. *Gmint($\emptyset$)=$\emptyset$,*
3. *Gmint(X)$\subseteq$ X,*
4. *$X \subseteq Y \Rightarrow$ Gmint(X)$\subseteq$ Gmint(Y),*
5. *Gmint(Gmint(X))=Gmint(X).*

Proof. The first three assertions are direct consequences of Definition 3.1. For (4), we have

$$\begin{aligned} x \in Gmint(X) &\Rightarrow x \in \bigcup_{i=1}^{n} \tau_i - int(X) \\ &\Rightarrow x \in \tau_{i_0} - int(X) \ for\ some\ i_0 \in \{1,\ldots,n\} \\ &\Rightarrow x \in \tau_{i_0} - open\ G\ such\ that\ x \in G \subseteq X \\ &\Rightarrow x \in \tau_{i_0} - open\ G\ such\ that\ x \in G \subseteq Y,\ since\ X \subseteq Y \\ &\Rightarrow x \in \tau_{i_0} - int(Y)\ for\ some\ i_0 \in \{1,\ldots,n\} \\ &\Rightarrow x \in \bigcup_{i=1}^{n} \tau_i - int(Y),\ i.e\ x \in Gmint(Y) \\ &\Rightarrow (X \subseteq Y \Rightarrow Gmint(X) \subseteq Gmint(Y)). \end{aligned}$$

For (5), we observe that

$$\begin{aligned} Gmint(Gmint(X)) &= \bigcup_{i=1}^{n} \tau_i - int(Gmint(X)) \\ &= \bigcup_{i=1}^{n} \tau_i - int(X) \qquad by \quad \text{Lemma } 3.1. \end{aligned}$$ □

Proposition 3.2. *Let $(U,\tau_1),(U,\tau_2),\dots,(U,\tau_n)$ be n topological spaces induced by n binary relations $R_1,R_2,\dots,R_n$ respectively, and $X,Y\subseteq U$. Then,*

1. *Gmcl(U)=U,*
2. *Gmcl(∅)=∅,*
3. *Gmcl(X)⊆ X,*
4. *$X\subseteq Y \Rightarrow$ Gmcl(X)⊆ Gmcl(Y),*
5. *Gmcl(Gmcl(X))=Gmcl(X).*

Proof. The first three assertions are direct consequences of Definition 3.1. For (4), let $x\in Gmcl(x)$ then $x\in\tau_i-cl(x)$ for all $i\in\{1,2,\dots,n\}$, by applying Theorem 2.1 we get $x\in\tau_i-cl(X)$ if and only if $G\cap X\neq\emptyset$ for all $G\in\tau_i$, $x\in G$, because $X\subseteq Y$. Then, $G\cap Y\neq\emptyset$ for all $G\in\tau_i$, $x\in G$. Therefore, $x\in\tau_i-cl(Y)$ for all $i\in\{1,2,\dots,n\}$. Hence, $x\in Gmcl(Y)$, and then $X\subseteq Y$ $\Rightarrow Gmcl(X)\subseteq Gmcl(Y)$. For (5), we observe that

$$\begin{aligned} Gmcl(Gmcl(X)) &= \bigcap_{i=1}^{n}\tau_i-cl(Gmcl(X))\\ &= \bigcap_{i=1}^{n}\tau_i-cl(X) \qquad \textit{by}\quad \text{Lemma 3.1.} \end{aligned}$$ □

Proposition 3.3. *Let $(U,\tau_1),(U,\tau_2),\dots,(U,\tau_n)$ be n topological spaces induced by n binary relations $R_1,R_2,\dots,R_n$ respectively. If $X,Y\subseteq U$, then*

$$Gmint(X\cap Y)=Gmint(X)\cap Gmint(Y).$$

Proof. Because $X\cap Y\subseteq X$ and $X\cap Y\subseteq Y$ then $Gmint(X\cap Y)$ is a subset of both $Gmint(X)$ and $Gmint(Y)$. Hence, $Gmint(X\cap Y)\subseteq Gmint(X)\cap Gmint(Y)$. Now, if

$$\begin{aligned} p\notin Gmint(X\cap Y) &\Rightarrow p\notin\tau_i-int(X\cap Y)\ for\ all\ i\in\{1,2,\dots,n\}\\ &\Rightarrow p\notin\tau_i-int(X)\cap\tau_i-int(Y)\ for\ all\ i\\ &\Rightarrow p\notin Gmint(X)\cap Gmint(Y). \end{aligned}$$

Therefore, $Gmint(X\cap Y)\supseteq Gmint(X)\cap Gmint(Y)$. Thus,

$$Gmint(X\cap Y)=Gmint(X)\cap Gmint(Y).$$ □

Proposition 3.4. *Let $(U,\tau_1),(U,\tau_2),\dots,(U,\tau_n)$ be n topological spaces induced by binary relations $R_1,R_2,\dots,R_n$, respectively. If $X,Y\subseteq U$. Then*

$$Gmcl(X\cup Y)=Gmcl(X)\cup Gmcl(Y).$$

Proof. Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$ then $Gmcl(X) \subseteq Gmcl(X \cup Y)$ and $Gmcl(Y) \subseteq Gmcl(X \cup Y)$. Hence, $Gmcl(X \cup Y) \supseteq Gmcl(X) \cup Gmcl(Y)$. Now, let

$$\begin{aligned} p \in Gmcl(X \cup Y) &\Rightarrow p \in \tau_i - cl(X \cup Y) \ for\ all\ i \in \{1, 2, \ldots, n\} \\ &\Rightarrow p \in \tau_i - cl(X) \cup \tau_i - cl(Y) \ for\ all\ i \\ &\Rightarrow p \in Gmcl(X) \cup Gmcl(Y). \end{aligned}$$

Therefore, $Gmcl(X \cup Y) \subseteq Gmcl(X) \cup Gmcl(Y)$. Thus,

$$Gmcl(X \cup Y) = Gmcl(x) \cup Gmcl(Y).$$ □

Theorem 3.1. *Let $(U, \tau_1), (U, \tau_2), \ldots, (U, \tau_n)$ be n topological spaces induced by n binary relations $R_1, R_2, \ldots, R_n$, respectively. If $X, Y \subseteq U$, then, Gmint and Gmcl are interior and closure operators, respectively.*

Proof. The proof follows directly by applying Propositions 3.1, 3.2, 3.3 and 3.4. □

Example 1. Let $U = \{1, 2, 3, 4, 5\}$, $X_1 = \{1, 2, 4\}$, $X_2 = \{3, 4, 5\}$. R_1, R_2 and R_3 be binary relations on U defined as follows

$$\begin{aligned} R_1 &= \{(1, 2), (1, 3), (2, 4), (2, 5), (5, 1)\} \\ R_2 &= \{(2, 2), (3, 4), (4, 5), (4, 1), (5, 3)\} \\ R_3 &= \{(1, 1), (5, 2), (5, 3), (3, 4), (3, 2), (4, 1)\} \end{aligned}$$

according to Definition 2.5 we have the following induced topologies

$\tau_1 = \{\emptyset, \{2, 3\}, \{4, 5\}, \{1\}, \{2, 3, 4, 5\}, \{1, 2, 3\}, \{1, 4, 5\}, U\}$,
$\tau_2 = \{\emptyset, \{2\}, \{4\}, \{1, 5\}, \{3\}, \{2, 4\}, \{1, 2, 5\}, \{1, 4, 5\}, \{2, 3\}, \{3, 4\}, \{1, 3, 5\}, \{1, 2, 4, 5\}, \{2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 3, 4, 5\}, U\}$,
$\tau_3 = \{\emptyset, \{1\}, \{2, 4\}, \{2, 3\}, \{2\}, \{1, 2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2\}, \{1, 2, 3, 4\}, U\}$.

In tables 1 and 2 we make a comparison between the accuracy in each topology alone and in our approach to X_1 and X_2.

Approximation Space	**Int**(X_1)	**Cl**(X_1)	**Accuracy**
(U, τ_1)	{1}	U	0.2
(U, τ_2)	{2,4}	{1,2,4,5}	0.5
(U, τ_3)	{1,2,4}	U	0.6
our approach	{1,2,4}	{1,2,4,5}	0.75

Table 1: Comparison among accuracy measures of category X_1

Depends on Definition 2.7 we define rough membership function in our approach as follows.

Approximation Space	**Int**(X_2)	**Cl**(X_2)	**Accuracy**
(U,τ_1)	{4,5}	{2,3,4,5}	0.5
(U,τ_2)	{3,4}	{1,3,4,5}	0.5
(U,τ_3)	$\emptyset$	{3,4,5}	0
our approach	{3,4,5}	{3,4,5}	1

Table 2: Comparison among accuracy measures of category X_2

Definition 3.2. *Let $(U,\tau_1),(U,\tau_2),\dots,(U,\tau_n)$ be n topological spaces induced by n binary relations $R_1,R_2,\dots,R_n$, respectively. Then a membership function is defined, for every $x\in U$, as follows:*

$$\mu_X^{\Gamma}(x)=\begin{cases}1, & if\ \max_{i=1}^{n}(\mu_X^{\tau_i}(x))=1,\\ 0, & elseif\ \min_{i=1}^{n}(\mu_X^{\tau_i}(x))=0,\\ \max_{i=1}^{n}(\mu_X^{\tau_i}(x)), & otherwise,\end{cases}$$

where $\Gamma=\{\tau_1,\tau_2,\dots,\tau_n\}$.

The following example illustrates Definition 3.2.

Example 2. Let U, R_1,R_2 and R_3 be as in Example 1 and

$$\beta_1=\{\{2,3\},\{4,5\},\{1\}\},$$
$$\beta_2=\{\{2\},\{4\},\{1,5\},\{3\}\},$$
$$\beta_3=\{\{1\},\{2,4\},\{2,3\},\{2\},U\}.$$

where β_1,β_2 and β_3 *are the basis of* τ_1,τ_2 and τ_3, respectively. For $X_1=\{1,2,4\}$ we have,

$$\mu_{X_1}^{\tau_1}(1)=1,\quad \mu_{X_1}^{\tau_1}(2)=\frac{1}{2},\quad \mu_{X_1}^{\tau_1}(3)=\frac{1}{2},\quad \mu_{X_1}^{\tau_1}(4)=\frac{1}{2},\quad \mu_{X_1}^{\tau_1}(5)=\frac{1}{2},$$
$$\mu_{X_1}^{\tau_2}(1)=\frac{1}{2},\quad \mu_{X_1}^{\tau_2}(2)=1,\quad \mu_{X_1}^{\tau_2}(3)=0,\quad \mu_{X_1}^{\tau_2}(4)=1,\quad \mu_{X_1}^{\tau_2}(5)=\frac{1}{2},$$
$$\mu_{X_1}^{\tau_3}(1)=1,\quad \mu_{X_1}^{\tau_3}(2)=1,\quad \mu_{X_1}^{\tau_3}(3)=\frac{1}{2},\quad \mu_{X_1}^{\tau_3}(4)=1,\quad \mu_{X_1}^{\tau_3}(5)=\frac{3}{5}.$$

Then, $Gmint(X_1)=\{1,2,4\}$ and $Gmcl(X_1)=\{1,2,4,5\}$ *which ensures the result in* Example 1, Table 1. For $X_2=\{3,4,5\}$

$$\mu_{X_2}^{\tau_1}(1)=0,\quad \mu_{X_2}^{\tau_1}(2)=\frac{1}{2},\quad \mu_{X_2}^{\tau_1}(3)=\frac{1}{2},\quad \mu_{X_2}^{\tau_1}(4)=1,\quad \mu_{X_2}^{\tau_1}(5)=1,$$
$$\mu_{X_2}^{\tau_2}(1)=\frac{1}{2},\quad \mu_{X_2}^{\tau_2}(2)=0,\quad \mu_{X_2}^{\tau_2}(3)=1,\quad \mu_{X_2}^{\tau_2}(4)=1,\quad \mu_{X_2}^{\tau_2}(5)=\frac{1}{2},$$
$$\mu_{X_2}^{\tau_3}(1)=0,\quad \mu_{X_2}^{\tau_3}(2)=0,\quad \mu_{X_2}^{\tau_3}(3)=\frac{1}{2},\quad \mu_{X_2}^{\tau_3}(4)=\frac{1}{2},\quad \mu_{X_2}^{\tau_3}(5)=\frac{3}{5},$$

also $Gmint(X_2)=\{3,4,5\}=Gmcl(X_2)$ *which insures the result in* Example 1, Table 2.

3.2 Second approach (minimal neighborhood approach)

Definition 3.3 (Neighborhood-map). *Let U be a non-empty set, R be a binary relation on U, τ is the topology on U induced by R and β is a base for τ. Then, we define the map $N : U \longmapsto \beta$ as follows, for $x \in U$, $N(x) = \cap B_x$ where B_x is any member of β containing x (i.e., the map N is mapping the element x to the minimal member of β containing x). Obviously, the map N is a well defined map.*

Definition 3.4 (Minimal Neighborhood-map). *Let U be a non-empty set, $R_1, R_2, \ldots, R_n$ be n binary relations on U, and $\beta_1, \beta_2, \ldots, \beta_n$ be n basis for the topologies $\tau_1, \tau_2, \ldots, \tau_n$ on U induced by R_1, R_2, $\ldots, R_n$, respectively. We define the map H $H : U \longmapsto P(U)$ as follows: for $x \in U$, $H(x) = \bigcap_{i=1}^{n} N_i(x)$ where N_i is the neighborhood-map corresponding to the base β_i.* Briefly the map H obtain the smallest neighborhood of an element x in all topologies τ_i for $i \in \{1, 2, \ldots, n\}$ *and the collection $\{H(x) \ : \ x \in U\}$ is called multi-granular knowledge base denoted by $\between_{i=1}^{n} \beta_i$.*

Theorem 3.2. *Let U be a non-empty set, R_1, R_2 be two binary relations on U, and β_1, β_2 are two basis for the topologies τ_1, τ_2 on U induced by R_1, R_2, respectively. Then $\beta_1 \between \beta_2 = \{H(x) \ : \ x \in U\}$ is also a topological base for U.*

Proof. Clearly, for each $x \in U$, $H(x)$ contains x. Finally let $B_1, B_2 \in \beta_1 \between \beta_2$ and $p \in B_1 \cap B_2$ since $B_1, B_2 \in \beta_1 \between \beta_2$, then there exist $x, x' \in U$ such that $B_1 = H(x)$ and $B_2 = H(x')$. Therefore, $p \in B_1 \cap B_2$ if and only if $p \in B_1 = H(x)$ and $p \in B_2 = H(x')$. Clearly $p \in H(p) \subseteq H(x)$ and $p \in H(p) \subseteq H(x')$. Therefore, there exist $B_3 = H(p) \in \beta_1 \between \beta_2$ such that $p \in B_3 \subseteq B_1 \cap B_2$. □

Corollary 3.1. *Let U be a non-empty set, $R_1, R_2, \ldots, R_n$ be n binary relations on U, and $\beta_1, \beta_2, \ldots, \beta_n$ are n basis for the topologies $\tau_1, \tau_2, \ldots, \tau_n$ on U induced by $R_1, R_2, \ldots, R_n$, respectively. Then, $\between_{i=1}^{n} \beta_i = \{H(x) \ : \ x \in U\}$ is also a topological base on U.*

Proof. Using mathematical induction, we find that this is an immediate consequence of Theorem 3.2. □

Definition 3.5. *Let U be a non-empty set, $R_1, R_2, \ldots, R_n$ be n binary relations on U and $\beta_1, \beta_2, \ldots, \beta_n$ be n topological basis on U induced by the binary relations $R_1, R_2, \ldots, R_n$. We define a generalize multi-granular topological rough space as follows $GMgTRS(\between_{i=1}^{n} \beta_i) = (U, M\tau)$, where $M\tau$ is the topology generated by the base $\between_{i=1}^{n} \beta_i$.*

Theorem 3.3. *Let U be a non-empty finite set of order m, $R_1, R_2, \ldots, R_n$ be n binary relations on U, and $\beta_1, \beta_2, \ldots, \beta_n$ be n basis for topologies on U induced by $R_1, R_2, \ldots, R_n$, respectively. If R_{i_0} for $i_0 \in \{1, 2, \ldots, n\}$ is the identity relation then $GMgTRS(\between_{i=1}^{n} \beta_i)$ is equal to (U, τ_{i0}) where τ_{i_0} is the topology induced by R_{i_0}.*

Proof. Let $U = \{x_1, x_2, \ldots, x_m\}$. Since R_{i_0} is identity relation on U then the induced base by R_{i_0} is $\beta_{i_o} = \{\{x_1\}, \{x_2\}, \ldots, \{x_m\}\}$ and by Definition 3.4 we have $H(x) = \{x\}$ for all x in U so $\between_{i=1}^{n} \beta_i = \{\{x_1\}, \{x_2\}, \ldots, \{x_m\}\}$ and hence $\between_{i=1}^{n} \beta_i = \beta_{i_o}$ therefore they generate the same topology on U. □

The following example illustrates the second approach.

Example 3. Let $U = \{1, 2, 3, 4, 5, 6\}$, $X = \{2, 4, 5\} \subseteq U$ and R_1, R_2 and R_3 be binary relations on U defined as follows

$$
\begin{aligned}
R_1 &= \{(1,1),(1,2),(3,3),(3,5),(4,6),(6,4)\} \\
R_2 &= \{(1,5),(1,6),(2,1),(2,2),(3,3),(3,4),(4,4)\} \\
R_3 &= \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3), \\
&\quad (4,4),(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}
\end{aligned}
$$

According to the paragraph below Definition 2.2 we induced the following bases:

$$
\begin{aligned}
\beta_1 &= \{\{1,2\},\{3,5\},\{4\},\{6\}\} \\
\beta_2 &= \{\{1,2\},\{3,4\},\{4\},\{5,6\}\} \\
\beta_3 &= \{\{1,2\},\{3,4\},\{3,4,6\},\{5,6\},\{4,5,6\},\{4\},\{6\},\{4,6\}\}
\end{aligned}
$$

So, the multi-granular knowledge base $\between_{i=1}^{3} \beta_i = \{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$. *In Table 3 we make a comparison between the accuracy in each basis alone and in our approach to the set* X. *We compute the interior and closure using the Definition 2.7 of membership function.*

Approximation Space	**Int**(X)	**Cl**(X)	**Accuracy**
using β_1	{4}	{1,2,3,4,5}	20%
using β_2	{4}	U	16.66%
using β_3	{4}	{1,2,3,4,5}	20%
using $\between_{i=1}^{3} \beta_i$	{4,5}	{1,2,4,5}	50%

Table 3: Interior and closure comparison by basis

The reduction process of data is very important since we express the whole data by a part of it with conservation of the structure of the whole data. So we introduce two algorithms for bases reduction, the first algorithm gets one bases reduct in polynomial time and the second algorithm gets all reducts but in exponential time.

Algorithm 1 (**Bases Reduct**).
Input: The non-empty set U and the basis $(\beta_1, \beta_2, \ldots, \beta_n)$ induced by the binary relations $R_1, R_2, \ldots, R_n$.
Output: One reduct
Steps are shown as follows:

I: $X \longleftarrow$ *Compute* $(\between_{i=1}^{n} \beta_i)$

$reduct = list\ of\ (\beta_1, \beta_2, \ldots, \beta_n)$

II: *For*$(i = 1; i \leq n; i++)$

remove the first element in reduct and store it in E.

if ($\between$ *reduct* $==$ X)

continue

else

add E in the last position of reduct

end

III: *return reduct.*

The following example illustrate the Algorithm 1 to get bases reduct.

Example 4. Let $U = \{1, 2, 3, 4, 5, 6\}$, and

$$\begin{aligned}
\beta_1 &= \{\{1, 2, 3\}, \{4, 5, 6\}\}, \\
\beta_2 &= \{\{1, 2, 3, 4, 6\}, \{5\}\}, \\
\beta_3 &= \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}.
\end{aligned}$$

Then

$$\begin{aligned}
\between_{i=1}^{3} \beta_i &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}; & reduct &= \{\beta_1, \beta_2, \beta_3\}; \\
\beta_2 \between \beta_3 &= \{\{1, 4\}, \{2\}, \{3, 6\}, \{5\}\}; & reduct &= \{\beta_2, \beta_3, \beta_1\}; \\
\beta_3 \between \beta_1 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}; & reduct &= \{\beta_3, \beta_1\}; \\
\beta_1 &= \{\{1, 2, 3\}, \{4, 5, 6\}\}; & reduct &= \{\beta_1, \beta_3\};
\end{aligned}$$

Hence, β_2 is redundant base which can be omit, and the basis reduct needed for classification are β_1 and β_3

The following algorithm computing all reducts but with exponential run time because we compute the power set of the set of bases.

Algorithm 2 (**All Basis reduct**).
Input: The non-empty set U and the basis $(\beta_1, \beta_2, \ldots, \beta_n)$ induced by the binary relations $R_1, R_2, \ldots, R_n$.
Output: List of all reducts
Steps are shown as follows:

I: $X \longleftarrow$ *Compute*($\between_{i=1}^{n} \beta_i$) ;

allsubsets = *powerset of* $(\{\beta_1, \beta_2, \ldots, \beta_n\})$;

allReducts = *null;*

II: *For*$(i = 1; i \leq 2^n; i++)$

if ($\between$ $allsubsets[i] == X$);

add i into allReducts;

end

III: *return allReducts.*

4. Real life applications

4.1 Clinical data description

Patients with digestive disease have become so many of these lesions due to the high number of fast foods which contain high calories as well as processed meat. As a direct result of this food many people suffer from excessive infusion and as a result of the subsequent diseases of the digestive system, the most serious of which are stomach cancers and colon. Because of eradication of the stomach up the food, directly go to the intestine, causing confusion in the absorption. The patients have some violent symptoms after the meal, such as dizziness, headache, colic and increasing the blood sugar. After a period, the patient is highest and most dangerous complications such as high cholesterol and clogged arteries leading to heart attacks.

The most general forms of innate stomach and colon cancer syndromes are:

- Hereditary nonpolyposis colorectal cancer (HNPCC). HNPCC, also called Lynch syndrome, increases the risk of stomach and colon cancer and other cancers. People with HNPCC tend to expand stomach and colon cancer before age 50.

- Familial adenomatous polyposis (FAP). FAP is a rare confusion that causes you to expand thousands of polyps in the inside layer of your stomach and colon and rectum. People with unprocessed FAP have a very much-increased risk of developing stomach and colon cancer before age 40.

4.2 Analysis of the problem

Our aim in this study to find the recommendations for patients show them appropriately greeted approach combines treatment and exercise to reach results explain the function of each presentation of the positive and negative impact on the patient. The decision of the Physician, according to the medical reports is the continuation of the medical tests are all for another or off the medical analyzes the patient's condition is stable loft insensitively to healthy style workout constantly.

4.3 Problem formulation

According to the medical reports requested by the doctor for patients in this case the following attributes:

1) Liver Functions: of the type S. GPT (Natural percent between 0 to 45 U/L) and of the type S. GOT (Natural percent between 0 to 37 U/L).

2) Kidney Functions: The measurements of uric acid in the blood (Uric Acid varies between 3 to 7 mg/dl).

3) Fats Percentage: Fats in the blood are divided into two types, the cholesterol level that has a natural range less than 200 mg/dl. The border range is between 200 to 240 mg/dl. The critical range of it that causes arteriosclerosis or heart is higher than 240 mg/dl. Second, the so-called triglycerides range that has reference up to 150 mg/dl.

4) Heart Efficiency: we measured the enzyme (Serum LDH) that has ranged reference between 0 to 480 U/L.

5) Signs of Tumors: we tested the digestive system through the scale (CEA) and normal Non-smoking rooms if less than 5 mg/ml. The other measure so-called CA 19.9 and extent of reference from 0 to 39 U/ml.

6) Viruses Hepatitis: Test the patient's immunity against of viruses of type B (HBC) and of type C (Highly infectious) furthermore is positive or negative.

7) Blood Sugar: The patient measurement of sugar of fasting for 6 hours, and an hour after eating, and then two hours after eating.

The results of the seven patients were collected from official files in the physician, which has been done after six months of surgery (see Table 4).

Patients ID	**Age**	**LF1**	**LF2**	**VH1**	**VH2**	**KF**	**FP1**	**FP2**	**HE**	**ST1**	**ST2**	**BS**	**D**
P1	12	63	45	N	N	11.2	180	210	526	36	44	N	C
P2	5	50	44	N	P	4.7	255	188	512	11	26	N	C
P3	18	34.5	23	N	N	5.6	177	112	430	16	36	P	S
P4	22	55	33	P	P	14.2	311	240	515	28	49	P	S
P5	8	36	22	N	N	6.3	166	99	310	11	23	N	C
P6	13	49	50	P	N	8.5	230	120	420	18	24	N	C
P7	15	57.5	41	N	P	7.6	206	144	460	17	25	P	S

Table 4: Medical Decision Information System

We define a suitable relation for each attribute and apply our approach on this data as follows.

$$\begin{aligned}
R_{age} &= \{(x,y) : |f_{age}(x) - f_{age}(y)| \leq 3\},\\
R_{LF1} &= \{(x,y) : f_{FL1}(x) \textit{ and } f_{FL1}(y) \leq \ 45 \textit{ or } f_{FL1}(x) \textit{ and } f_{FL1}(y) > 45\},\\
R_{LF2} &= \{(x,y) : f_{FL2}(x) \textit{ and } f_{FL2}(y) \leq \ 37 \textit{ or } f_{FL2}(x) \textit{ and } f_{FL2}(y) > 37\},\\
R_{VH1} &= \{(x,y) : f_{VH1}(x) = f_{VH1}(y)\},\\
R_{VH2} &= \{(x,y) : f_{VH2}(x) = f_{VH2}(y)\},\\
R_{KF} &= \{(x,y) : 3 \leq f_{KF}(x) \textit{ and } f_{KF}(y) \leq \ 7, \ f_{KF}(x) \textit{ and } f_{KF}(y) < 3 \textit{ or}\\
&\qquad f_{KF}(x) \textit{ and } f_{KF}(y) > 7\},\\
R_{FP1} &= \{(x,y) : 200 \leq f_{FP1}(x) \textit{ and } f_{FP1}(y) \leq 240, f_{FP1}(x) \textit{ and } f_{FP1}(y) < 200\\
&\qquad \textit{or } f_{FP1}(x) \textit{ and } f_{FP1}(y) > 240\},\\
R_{FP2} &= \{(x,y) : f_{FP2}(x) \textit{ and } f_{FP2}(y) \leq 150 \textit{ or } f_{FP2}(x) \textit{ and } f_{FP2}(y) > 150\},\\
R_{HE} &= \{(x,y) : f_{HE}(x) \textit{ and } f_{HE}(y) \leq 480 \textit{ or } f_{HE}(x) \textit{ and } f_{HE}(y) > 480\},\\
R_{ST1} &= \{(x,y) : f_{ST1}(x) \textit{ and } f_{ST1}(y) \leq 5 \textit{ or } f_{ST1}(x) \textit{ and } f_{ST1}(y) \leq 15 \textit{ or}\\
&\qquad f_{ST1}(x) \textit{ and } f_{ST1}(y) > 15\},\\
R_{ST2} &= \{(x,y) : f_{ST2}(x) \textit{ and } f_{ST2}(y) \leq 39 \textit{ or } f_{ST2}(x) \textit{ and } f_{ST2}(y) > 39\},\\
R_{BS} &= \{(x,y) : f_{BS}(x) = f_{BS}(y)\}.
\end{aligned}$$

Hence, we compute the basis of every relation as we did before in Example 2 to get the following bases.

$$\begin{aligned}
\beta_1 &= \{\{3,7\},\{1,3,6,7\},\{1,6,7\},\{2,5\},\{7\},\{4\}\},\\
\beta_2 &= \{\{3,5\},\{1,2,4,6,7\}\},\\
\beta_3 &= \{\{1,2,6,7\},\{3,4,5\}\},\\
\beta_4 &= \{\{1,2,3,5,7\},\{4,6\}\},\\
\beta_5 &= \{\{1,3,5,6\},\{2,4,7\}\},\\
\beta_6 &= \{\{2,3,5\},\{1,4,6,7\}\},\\
\beta_7 &= \{\{1,3,5\},\{6,7\},\{2,4\}\},\\
\beta_8 &= \{\{1,2,4\},\{3,5,6,7\}\},\\
\beta_9 &= \{\{1,2,4\},\{3,5,6,7\}\},\\
\beta_{10} &= \{\{2,5\},\{1,3,4,6,7\}\},\\
\beta_{11} &= \{\{1,4\},\{2,3,5,6,7\}\},\\
\beta_{12} &= \{\{1,2,5,6\},\{3,4,7\}\},
\end{aligned}$$

hence the multi-granular knowledge base will be as follows

$$\between_{i=1}^{12} \beta_i = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$$

and M_τ is the topology generated by the base $\between_{i=1}^{12} \beta_i$. Now we want to approximate the concept $X_C = \{p_1, p_2, p_5, p_6\}$ represent the set of patients with decision C (continue check up)

$$M_\tau - int(X_C) = \{p_1, p_2, p_5, p_6\} = M_\tau - cl(X_C)$$

so, the accuracy of approximating the concept X_C only with information in the data table is 100 % and when we apply the basis reduct algorithm we get the bases $\beta_7, \beta_{11}, \beta_{12}$ is a reduct of $\beta_1, \beta_2, \ldots, \beta_{12}$. This reduct represent the information of the whole table where $\bigcup \{\beta_7, \beta_{11}, \beta_{12}\} = \bigcup_{i=1}^{12} \beta_i$ so we use it instead of the 12^{th} bases.

After reduction the table of information reduced to be as in Table 5 and has the same structure of original data in Table 4 where $\{\beta_7, \beta_{11}, \beta_{12}\}$ represent the attributes {FP1, ST2, BS} respectively. From this reduct we get the decision rules to be used in the decision making in the future tests by a decision program.

Patients ID	**FP1**	**ST2**	**BS**	**D**
P1	180	44	N	C
P2	255	26	N	C
P3	177	36	P	S
P4	311	49	P	S
P5	166	23	N	C
P6	120	24	N	C
P7	206	25	P	S

Table 5: Reduct Information System

4.4 Results analysis

This method of dividing patient data from the results of the 12 medical examinations has been reduced to only three tests to being sufficient to make the right decision for these patients. There are other alternatives for decision-making where using the pathological method of data analysis and division, we have been able to find more than one reduction of medical examinations and each patient can choose the appropriate alternative in terms of financial capacity and likelihood.

5. Conclusions and future works

The amount of research papers available online on the topological application is growing and this growth has generated a need for a unifying theory to compare the results. Also, we need new techniques and tools that can intelligently and automatically extract implicit knowledge from these data.

These tools and technicality are the subjects of future research trends using general topological concepts. We deduce that the development of topology in the construction of some knowledge base concepts will help to get rich results that yield a lot of logical statements that discover hidden relationships among data and moreover, probably help in producing accurate rules.

In future papers, we hope to study more generalizations using topological concepts such as near open and near closed sets. And apply these generalized concepts to realistic medical data of large size. The topic of multivariate data reduction can also be studied using generalized topological concepts.

The following are some problems and lines for future study:

1. Developing a unifying theory of topological generalizations that using rough concepts.

2. Scaling up for design topological softwares to handle big dimensional classification problems.

3. Topological methods for mining complex knowledge from complex data.

Abbreviations

Gmint Generalized multi interior.

Gmcl Generalized multi closure.

GMgTRS Generalized multi-granular topological rough space.

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On the oscillatory behavior of a class of fourth order nonlinear damped delay differential equations with distributed deviating arguments

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Abstract. The present study concerns the oscillation of a class of fourth-order nonlinear damped delay differential equations with distributed deviating arguments. We offer a new description of oscillation of the fourth-order equations in terms of oscillation of a related well studied second-order linear differential equation without damping. Some new oscillatory criteria are obtained by using the generalized Riccati transformation, integral averaging technique and comparison principles. The effectiveness of the obtained criteria is illustrated via example.

Keywords: oscillation, fourth-order, damping term, Riccati transformation, comparison theorem, distributed deviating arguments.

1. Introduction

The purpose of this work, we are concerned with fourth-order nonlinear damped delay differential equations with distributed deviating arguments

$$(E_1) \qquad \begin{aligned} &\big(x_2(\mu)\big(x_1(\mu)\big(u''(\mu)\big)^{\alpha}\big)'\big)' + p(\mu)\Big(u''(\delta(\mu))\Big)^{\alpha} \\ &+ \int_c^d q(\mu,\varrho) f(\mu, u(g(\mu,\varrho)))d\varrho = 0, \end{aligned}$$

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where $\alpha \geq 1$ is a quotient of odd positive integers and $c < d$. Throughout this paper, we use the following assumptions:

$$\begin{cases} x_1, x_2, p, \delta \in C\big(I, [0, \infty)\big) \text{ and } x_1, x_2 > 0, \text{ where } I = [\mu_0, +\infty); \\ q, g \in C[I \times [c, d], [0, \infty)), \delta(\mu) \leq \mu, \\ \lim_{\mu \to +\infty} \delta(\mu) = \infty, \ g(\mu, \varrho) \text{ is a nondecreasing} \\ \text{function for } \varrho \in [c, d] \text{ satisfying } g(\mu, \varrho) \leq \mu \text{ and } \lim_{\mu \to +\infty} g(\mu, \varrho) = \infty; \\ f \in C(\mathbb{R}, \mathbb{R}), \text{ there exists a contacts } k_1 > 0 \text{ such that } f(\mu, u(\mu))/u^{\beta} \geq k_1. \end{cases}$$

We define the operators,

$$L^{[0]}u = u,\ L^{[1]}u = u',\ L^{[2]}u = x_1\big(\big(L^{[0]}u\big)''\big)^{\alpha},\ L^{[3]}u = x_2\big(L^{[2]}u\big)', L^{[4]}u = (L^{[3]}u)'.$$

By a solution to (E_1), we mean a function $u(\mu)$ in $C^2[T_u, \infty)$ for which $L^{[2]}u$, $L^{[4]}u$ is in $C^1[T_u, \infty)$ and (E_1) is satisfied on some interval $[T_u, \infty)$, where $T_u \geq \mu_0$. We consider only solutions $u(\mu)$ for which $\sup\{|u(\mu)| : \mu \geq T\} > 0$ for all $T \geq T_u$. A solution of (E_1) is called oscillatory if it is neither eventually positive nor eventually negative on $[T_u, \infty)$ and otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

We define

$$\begin{aligned} A_1(\mu_1, \mu) &= \int_{\mu_1}^{\mu} x_1^{-1/\alpha}(s) ds, \\ A_2(\mu_1, \mu) &= \int_{\mu_1}^{\mu} x_2^{-1}(s) ds, \\ A_3(\mu_1, \mu) &= \int_{\mu_1}^{\mu} \Big(\big(x_1(s)\big)^{-1} A_2(\mu_1, s)\Big)^{1/\alpha} ds, \\ A_4(\mu_1, \mu) &= \int_{\mu_1}^{\mu} \int_{\mu_1}^{u} \Big(\big(x_1(s)\big)^{-1} A_2(\mu_1, s)\Big)^{1/\alpha} ds\, du, \end{aligned}$$

for $\mu_0 \leq \mu_1 \leq \mu < \infty$ and assume that

$$A_1(\mu_1, \mu) \to \infty, \quad A_2(\mu_1, \mu) \to \infty \quad \text{as} \quad \mu \to \infty. \tag{1}$$

In mathematical representations of numerous physical, chemical phenomena and biological, fourth-order differential equations are very commonly encountered [1, 3]. Applications involve, for example, problems with elasticity, structural deformation or soil settlement. Questions related to the presence of oscillatory and nonoscillatory solutions play an important role in mechanical and engineering problems [5]. Many authors have extensively studied the problem of the oscillation of fourth (higher) order differential equations, including many techniques for obtaining oscillatory criteria for fourth (higher) order differential equations. Several studies have had very interesting results related to oscillatory properties of solutions of neutral differential equations and damped delay differential equations with/without distributed deviating arguments [4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Dzurina et al. [8] presented oscillation results for a fourth-order equation

$$\big(r_3(\mu)\big(r_2(\mu)(r_1(\mu)y'(\mu))\big)'\big)' + p(\mu)y'(\mu) + q(\mu)y(\tau(\mu)) = 0.$$

More precisely, the existing literature does not provide any criteria for the oscillation of Eq. (E_1). Inspired by the above papers, in this paper, using suitable Riccati type transformation, integral averaging condition, and comparison method, we present some sufficient conditions which insure that any solution of Eq. (E_1) oscillates when the associated second order equation

$$(x_2(\mu)z'(\mu))' + \frac{p(\mu)}{x_1(\delta(\mu))}z(\mu) = 0, \tag{E_2}$$

is oscillatory or nonoscillatory.

2. Basic lemmas

In this section, we state and prove some Lemmas that are frequently used in the remainder of this paper.

Lemma 2.1 ([9])**.** *Assume that (E_2) is nonoscillatory. If Eq. (E_1) has a nonoscillatory solution $u(\mu)$ on I, $\mu_1 \geq \mu_0$, then there exists a $\mu_2 \in I$ such that $u(\mu)L^{[2]}u(\mu) > 0$ or $u(\mu)L^{[2]}u(\mu) < 0$ for $\mu \geq \mu_2$.*

Lemma 2.2. *If Eq. (E_1) has a nonoscillatory solution $u(\mu)$ which satisfies $u(\mu)L^{[2]}u(\mu) > 0$ in Lemma 2.1 for $\mu \geq \mu_1 \geq \mu_0$. Then,*

$$L^{[2]}u(\mu) > A_2(\mu_1,\mu)\,L^{[3]}u(\mu), \quad \mu \geq \mu_1, \tag{2}$$

$$L^{[1]}u(\mu) > A_3(\mu_1,\mu)\left(L^{[3]}u(\mu)\right)^{1/\alpha}, \quad \mu \geq \mu_1, \tag{3}$$

and

$$u(\mu) > A_4(\mu_1,\mu)\left(L^{[3]}u(\mu)\right)^{1/\alpha}, \quad \mu \geq \mu_1. \tag{4}$$

Proof. If Eq. (E_1) has a non-oscillatory solution u. We assume that there exists a $\mu_1 \geq \mu_0$ such that $u(\mu) > 0$ and $u(g(\mu,\varrho)) > 0$ for $\mu \geq \mu_1$. From Eq. (E_1), we have

$$L^{[4]}u(\mu) = -\Big(\frac{p(\mu)}{x_1(\delta(\mu))}\Big)L^{[2]}u(\delta(\mu)) - k_1\int_c^d q(\mu,\varrho)u^\beta(g(\mu,\varrho))d\varrho \leq 0,$$

and $L^{[3]}u(\mu)$ is non increasing on I, we get

$$L^{[2]}u(\mu) \geq \int_{\mu_1}^{\mu}\left(L^{[2]}u(s)\right)'ds = \int_{\mu_1}^{\mu}(x_2(s))^{-1}L^{[3]}u(s)\,ds \geq A_2(\mu_1,\mu)\,L^{[3]}u(\mu),$$

this implies that

$$u''(\mu) \geq \Big(L^{[3]}u(\mu)\Big)^{1/\alpha}\Big((x_1(\mu))^{-1}A_2(\mu_1,\mu)\Big)^{1/\alpha}.$$

Now, twice integrating above from μ_1 to μ and using $L^{[3]}u(\mu) \leq 0$, we find

$$u'(\mu) \geq \Big(L^{[3]}u(\mu)\Big)^{1/\alpha}\int_{\mu_1}^{\mu}\Big((x_1(s))^{-1}A_2(\mu_1,s)\Big)^{1/\alpha}ds$$

and

$$u(\mu) \geq \Big(L^{[3]}u(\mu)\Big)^{1/\alpha}\int_{\mu_1}^{\mu}\int_{\mu_1}^{u}\Big((x_1(s))^{-1}A_2(\mu_1,s)\Big)^{1/\alpha}ds\,du \quad \text{for } \mu \leq \mu_1.$$

□

Lemma 2.3 ([11])**.** *Let $\xi \in C^1(I,\mathbb{R}^+)$, $\xi(\mu) \leq \mu$, $\xi'(\mu) \geq 0$ and $G(\mu) \in C(I,\mathbb{R}^+)$ for $\mu \geq \mu_0$. Assume that $y(\mu)$ is a bounded solution of second order delay differential equation*

$$(E_3) \qquad \big(x_2(\mu)\,y'(\mu)\big)' - \Theta(\mu)\,y(\xi(\mu)) = 0.$$

If

$$(5) \qquad \limsup_{\mu\to\infty}\int_{\xi(\mu)}^{\mu}\Theta(s)\,A_2(\xi(\mu),\,\xi(s))\,ds > 1$$

or

$$(6) \qquad \limsup_{\mu\to\infty}\int_{\xi(\mu)}^{\mu}\Big(\big(x_2(\mu)\big)^{-1}\int_{u}^{\mu}\Theta(s)\,ds\Big)du > 1,$$

where $x_2(\mu)$ is as in (E_1). Then the solutions of (E_3) are oscillatory.

3. Oscillation-comparison principle method

In this section, we shall establish some oscillation criteria for Eq. (E_1). For convenience, we denote

$$Q(\mu) = \Big(\frac{p(\mu)}{x_1(\delta(\mu))}\Big)A_2(\mu_1,\delta(\mu)), \quad \psi(\mu) = \exp\Big(\int_{\mu_1}^{\mu}Q(s)ds\Big),$$

$$\widetilde{q}(\mu,\varrho) = \int_{c}^{d}q(\mu,\varrho)\,d\varrho, \quad \Theta^*(\mu) = k_1\,\widetilde{q}(\mu,\varrho)\Big(A_4(\mu_1,g(\mu,d))\Big)^{\beta}.$$

Theorem 3.1. *Assume $\alpha \geq \beta$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists a $\xi \in C^1(I,\mathbb{R})$ such that*

$$g(\mu,\varrho) \leq \xi(\mu) \leq \delta(\mu) \leq \mu, \quad \xi'(\mu) \geq 0 \quad \textit{for } \mu \geq \mu_1,$$

and (5) *or* (6) *holds with*

$$\Theta(\mu) = \ell_* \, k_1 \widetilde{q}(\mu, \varrho) g^{\beta}(\mu, d)\big(A_1(\xi(\mu), g(\mu, d))\big)^{\beta} - \frac{p(\mu)}{x_1(\delta(\mu))} \geq 0, \quad \mu \geq \mu_1,$$

for constant $\ell_* > 0$. *Moreover, suppose that every solution of the first-order delay equation*

$$z'(\mu) + \psi^{1-\frac{\beta}{\alpha}}(g(\mu, d))\, \Theta^*(\mu)\, z^{\frac{\beta}{\alpha}}(g(\mu, d)) = 0. \tag{7}$$

Then every solution of Eq. (E_1) *is oscillatory.*

Proof. Let Eq. (E_1) has a nonoscillatory solution $u(\mu)$. Assume that, there exists a $\mu \geq \mu_1$ such that $u(\mu) > 0$ and $u(g(\mu, \varrho)) > 0$ for some $\mu \geq \mu_0$. From Lemma 2.1, $u(\mu)$ has the conditions either $L^{[2]}u(\mu) > 0$ or $L^{[2]}u(\mu) < 0$ for $\mu \geq \mu_1$.

Assume $u(\mu)$ has the condition $L^{[2]}u(\mu) > 0$, for $\mu \geq \mu_1$, then one can easily see that $L^{[3]}u(\mu) > 0$ for $\mu \geq \mu_1$. We can choose $\mu_2 \geq \mu_1$ such that $g(\mu, \varrho) \geq \mu_1$ for $\mu \geq \mu_2$, $g(\mu, \varrho) \to \infty$ as $\mu \to \infty$ and we have (4),

$$u(g(\mu, d)) > A_4(\mu_1, g(\mu, d))\left(L^{[3]}u(g(\mu, d))\right)^{1/\alpha}, \quad \mu \geq \mu_2. \tag{8}$$

By substituting (2), (8) in Eq. (E_1) and $L^{[3]}u(\mu)$ is decreasing, then

$$\begin{aligned} &\Big(L^{[3]}u(\mu)\Big)' + \Big(\frac{p(\mu)}{x_1(\delta(\mu))}\Big) L^{[3]}u(\mu) A_2(\mu_1, \delta(\mu)) \\ &\qquad + k_1\, \widetilde{q}(\mu, \varrho)\Big(A_4(\mu_1, g(\mu, d))\Big)^{\beta}\Big(L^{[3]}u(g(\mu, d))\Big)^{\beta/\alpha} \leq 0. \end{aligned} \tag{9}$$

Take $\phi = L^{[3]}u$, we have

$$\phi'(\mu) + Q(\mu)\phi(\mu) + \Theta^*(\mu)\phi^{\frac{\beta}{\alpha}}(g(\mu, d)) \leq 0 \tag{10}$$

or

$$\Big(\psi(\mu)\, \phi(\mu)\Big)' + \psi(\mu)\Theta^*(\mu)\phi^{\frac{\beta}{\alpha}}(g(\mu, d)) \leq 0, \quad \text{for } \mu \geq \mu_2. \tag{11}$$

Next, setting $z = \psi\, \phi > 0$ and $\psi(g(\mu, d)) \leq \phi(\mu)$, thus we have

$$z'(\mu) + \psi^{1-\frac{\beta}{\alpha}}(g(\mu, d))\Theta^*(\mu) z^{\frac{\beta}{\alpha}}(g(\mu, d)) \leq 0. \tag{12}$$

This means (12) is a positive for this inequality. Also, by [[2], Corollary 2.3.5], it can be seen that (3.1) has a positive solution, a contradiction.

Next, assume $u(\mu)$ has the condition $L^{[2]}u(\mu) < 0$, for $\mu \geq \mu_1$, then one can easily see that $L^{[1]}u(\mu) \geq 0$, $L^{[3]}u(\mu) > 0$ for $\mu \geq \mu_3(\geq \mu_2)$. Using monotonicity of $u'(\mu)$ and mean value property of differentiation there exists a $\theta \in (0, 1)$ such that

$$u(\mu) \geq \theta\, \mu\, u'(\mu), \quad \text{for } \mu \geq \mu_3. \tag{13}$$

Set $w(\mu) = L^{[1]}u(\mu)$, then $w'(\mu) = u''(\mu) < 0$. Using (13) in Eq. (E_1) we get

$$\big(x_2(\mu)\big(x_1(\mu)\big[w'(\mu)\big]^{\alpha}\big)'\big)' + p(\mu)(w'(\delta(\mu)))^{\alpha} + k_1\,(\mu\theta)^{\beta}\,\widetilde{q}(\mu,\varrho)w^{\beta}(g(\mu,d)) \leq 0,$$

and so $\big(x_1(\mu)\big[w'(\mu)\big]^{\alpha}\big) < 0$, we have $\big(x_1(\mu)\big[w'(\mu)\big]^{\alpha}\big)' > 0$ for $\mu \geq \mu_3$.

Now, for $v \geq u \geq \mu_3$, we get

$$\begin{aligned} w(u) > w(u) - w(v) &= -\int_u^v -x_1^{-1/\alpha}(\tau)(x_1(\tau)(w'(\tau))^{\alpha})^{1/\alpha}d\tau \\ &\geq x_1^{1/\alpha}(v)(-w'(v)))\left(\int_u^v x_1^{-1/\alpha}(\tau)d\tau\right) \\ &= x_1^{1/\alpha}(v)(-w'(v))A_1(u,v). \end{aligned}$$

Taking $u = \xi(\mu)$ and $v = g(\mu,d)$, we obtain

$$\begin{aligned} w(g(\mu,d)) &> A_1(g(\mu,d),\xi(\mu))\big(x_1^{1/\alpha}(\xi(\mu))(-w'(\xi(\mu)))\big) \\ &= A_1(g(\mu,d),\xi(\mu))\ y(\xi(\mu)), \end{aligned}$$

where $y(\mu) = x_1^{1/\alpha}(\xi(\mu))(-w'(\xi(\mu)))\big) > 0$ for $\mu \geq \mu_3$. From Eq. (E_1), we have that $y(\mu)$ is decreasing and $g(\mu,d) \leq \xi(\mu) \leq \delta(\mu) \leq \mu$, we get

$$\begin{aligned} &(x_2(\mu)z'(\mu))' + \frac{p(\mu)}{x_1(\delta(\mu))}z(\delta(\mu)) \\ &\quad \geq k_1\,(\theta g(\mu,d))^{\beta}\,\widetilde{q}(\mu,\varrho)A_1(g(\mu,d),\xi(\mu))z^{\frac{\beta}{\alpha}-1}(\xi(\mu))z(\xi(\mu)). \end{aligned}$$

Since z is decreasing and $\alpha \geq \beta$, there exists a constant ℓ such that $z^{\frac{\beta}{\alpha}-1}(\mu) \geq \ell$ for $\mu \geq \mu_3$. Thus, we obtain

$$(x_2(\mu)z'(\mu))' \geq \Big(\ell\,k_1\,(\theta\,g(\mu,d))^{\beta}\,\widetilde{q}(\mu,\varrho)A_1(g(\mu,d),\xi(\mu)) - \frac{p(\mu)}{x_1(\delta(\mu))}\Big)z(\xi(\mu)).$$

Proceeding the rest of the proof in Lemma (2.3), we arrive at the required conclusion, and so is omitted. □

4. Oscillation-Riccati method

This section deals with some oscillation criteria for Equation Eq. (E_1) by using Ricatti Method.

Theorem 4.1. *Assume $\alpha \geq \beta$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists η, $\xi \in C^1(I,\mathbb{R})$ such that $g(\mu,\varrho) \leq \xi(\mu) \leq \delta(\mu) \leq \mu$, $\xi'(\mu) \geq 0$ and $\eta > 0$ for $\mu \geq \mu_1$ with*

$$\limsup_{\mu\to\infty}\int_{\mu_5}^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho) - \frac{A^2(s)}{4B(s)}\right)ds = \infty \text{ for all } \mu_1 \in I, \tag{14}$$

where, for $\mu \geq \mu_1$,

$$A(\mu) = \frac{\eta'(\mu)}{\eta(\mu)} - \frac{p(\mu)}{x_1(\delta(\mu))} A_2(\mu_1, \delta(\mu)) \tag{15}$$

and

$$B(\mu) = \frac{\beta\, \ell_2^{\beta-\alpha}\, g'(\mu, d)}{\eta(\mu)} \Big(A_4(\mu_1, g(\mu, d))\Big)^{\beta-1} \Big(A_3(\mu_1, g(\mu, d))\Big)^{1/\alpha}, \tag{16}$$

also (5) *or* (6) *holds with* $\Theta(\mu)$ *as in Theorem 3.1. Then, every solution of Eq.* (E_1) *is oscillatory.*

Proof. Let Eq. (E_1) has a nonoscillatory solution $u(\mu)$. Assume that, there exists a $\mu \geq \mu_1$ such that $u(\mu) > 0$ and $u(g(\mu, \varrho)) > 0$ for some $\mu \geq \mu_0$. From Lemma 2.1, $u(\mu)$ has the conditions either $L^{[2]}u(\mu) > 0$ or $L^{[2]}u(\mu) < 0$ for $\mu \geq \mu_1$. If condition $L^{[2]}u(\mu) < 0$ holds, the proof is follows from Theorem 3.1. Next, if condition $L^{[2]}u(\mu) > 0$ holds. Define

$$\omega(\mu) = \eta(\mu) \frac{L^{[3]}u(\mu)}{u^{\beta}(g(\mu, d))}, \quad \mu \in I, \tag{17}$$

then $\omega(\mu) > 0$ for $\mu \geq \mu_1$. From (4) and $L^{[4]}u(\mu) < 0$, we have

$$\begin{aligned} \omega(\mu) &= \eta(\mu) \frac{L^{[3]}u(\mu)}{u^{\beta}(g(\mu, d))} \leq \eta(\mu) \frac{L^{[3]}u(g(\mu, d))}{u^{\beta}(g(\mu, d))} \\ &\leq \eta(\mu)(A_4(\mu_1, g(\mu, d)))^{-\alpha} u^{\alpha-\beta}(g(\mu, d)), \end{aligned} \tag{18}$$

for $\mu \geq \mu_1$. From (3) and definition $L^{[2]}u(\mu)$, we find

$$\begin{aligned} u'(g(\mu, d)) &= L^{[1]}u(g(\mu, d)) \geq A_3(\mu_1, g(\mu, d))(L^{[3]}u(\delta(\mu)))^{1/\alpha} \\ &\geq A_3(\mu_1, g(\mu, d))(L^{[3]}u(g(\mu, d)))^{1/\alpha}. \end{aligned}$$

Then

$$\begin{aligned} \frac{u'(g(\mu, d))}{u(g(\mu, d))} &\geq \left(\frac{A_3(\mu_1, g(\mu, d))}{\eta(\delta(\mu))}\right)^{1/\alpha} \frac{\eta^{1/\alpha}(\delta(\mu))(L^{[3]}u(\mu))^{1/\alpha}}{u^{\beta/\alpha}(g(\delta(\mu), d))} u^{\beta/\alpha-1}(g(\delta(\mu), d)) \\ &= \left(\frac{A_3(\mu_1, g(\mu, d))}{\eta(\mu)}\right)^{1/\alpha} \omega^{1/\alpha}(\mu) u^{\beta/\alpha-1}(g(\delta(\mu), d)). \end{aligned} \tag{19}$$

Also, since there exists a constant ℓ_1 and $\mu_2 \geq \mu_1$ such that for $L^{[3]}u(\mu) \leq L^{[3]}u(\mu_2) = \ell_1$. Therefore,

$$\begin{aligned} L^{[2]}u(\mu) &= L^{[2]}u(\mu_2) + \int_{\mu_2}^{\mu} (L^{[2]}u(s))' ds \leq L^{[2]}u(\mu_2) + \ell_1 \int_{\mu_2}^{\mu} \frac{ds}{x_2(s)} \\ &= L^{[2]}u(\mu_2) + \ell_1 A_2(\mu_2, \mu) = \left[\frac{L^{[2]}u(\mu_2)}{A_2(\mu_2, \mu)} + \ell_1\right] A_2(\mu_2, \mu) \\ &\leq \left[\frac{L^{[2]}u(\mu_2)}{A_2(\mu_2, \mu_3)} + \ell_1\right] A_2(\mu_2, \mu) = \ell_1^{*}\, A_2(\mu_2, \mu), \end{aligned} \tag{20}$$

holds for all $\mu \geq \mu_2$, where $\ell_1^* = \ell_1 + \frac{L^{[2]}u(\mu_1)}{A_2(\mu_2,\mu_3)}$, this implies that,

$$u'(\mu) = u'(\mu_3) + \int_{\mu_3}^{\mu} u''(s)ds \leq u'(\mu_3) + \int_{\mu_3}^{\mu} \left(\frac{\ell_1^* A_2(\mu_2, s)}{x_1(s)} \right)^{1/\alpha} ds$$
$$= u(\mu_3) + \left(\ell_1^*\right)^{1/\alpha} A_3(\mu_3, \mu) = \ell_2\, A_3(\mu_3, \mu),$$

holds for all $\mu \geq \mu_3 (\geq \mu_2)$, where $\ell_2 = \frac{u(\mu_2)}{A_3(\mu_3,\mu_4)} + (\ell_1^*)^{1/\alpha}$. Then

$$u(\mu) = u(\mu_4) + \int_{\mu_4}^{\mu} u'(s)ds \leq u(\mu_4) + \int_{\mu_4}^{\mu} \Big(\ell_2\, A_3(\mu_3, s) \Big) ds$$
$$= u(\mu_4) + \ell_2\, A_4(\mu_4, \mu) = \ell_2^*\, A_4(\mu_4, \mu), \tag{21}$$

holds for all $\mu \geq \mu_4 (\geq \mu_3)$,where $\ell_2^* = \frac{u(\mu_4)}{A_4(\mu_4,\mu_1)} + \ell_2$. Further

$$u^{\beta/\alpha-1}(g(\mu, d)) \geq \left(\ell_2^*\right)^{\beta/\alpha-1} \big(A_4(\mu_4, g(\mu, d))\big)^{\beta/\alpha-1}, \quad \mu \geq \mu_4. \tag{22}$$

By using (21) in (18), we obtain

$$\omega(\mu) \leq \left(\ell_2^*\right)^{\alpha-\beta} \eta(\mu) \left(A_4(\mu_1, g(\mu, d))\right)^{-\beta}, \tag{23}$$

and hence

$$\omega^{\frac{1}{\alpha}-1}(\mu) \leq \left(\ell_2^*\right)^{(\alpha-\beta)(\frac{1}{\alpha}-1)} \eta^{\frac{1}{\alpha}-1}(\mu) \left(A_4(\mu_1, g(\mu, d))\right)^{-\beta(\frac{1}{\alpha}-1)}. \tag{24}$$

Now differentiating (17), we get

$$\omega'(\mu) = \frac{\eta'(\mu)}{\eta(\mu)}\omega(\mu) + \frac{L^{[4]}u(\mu)}{L^{[3]}u(\mu)}\omega(\mu) - \beta g'(\mu, d)\frac{u'(g(\mu, d))}{u(g(\mu, d))}\omega(\mu). \tag{25}$$

Using Eq. (E_1), (2) in (25), we have

$$\begin{aligned} \omega'(\mu) &\leq \Big[\frac{\eta'(\mu)}{\eta(\mu)} - \frac{p(\mu)}{x_1(g(\mu, d))}\, A_2(\mu_4, g(\mu, d))\Big]\omega(\mu) \\ &\quad -k_1\eta(\mu)\widetilde{q}(\mu, \varrho) - \beta g'(\mu)\frac{u'(g(\mu, d))}{u(g(\mu, d))}\omega(\mu) \\ &\leq A(\mu)\omega(\mu) - k_1\eta(\mu)\widetilde{q}(\mu, \varrho) - \beta g'(\mu)\frac{u'(g(\mu, d))}{u(g(\mu, d))}\omega(\mu). \quad (26) \end{aligned}$$

By using (19), (22) and (25) in (26), we have

$$\begin{aligned} \omega'(\mu) &\leq A(\mu)\omega(\mu) - k_1\eta(\mu)\widetilde{q}(\mu, \varrho) \\ &\quad -\frac{\beta\, \ell_2^{\beta-\alpha}\, g'(\mu)}{\eta(\mu)}\Big(A_4(\mu_1, g(\mu, d))\Big)^{\beta-1}\Big(A_3(\mu_1, g(\mu, d))\Big)^{1/\alpha}\omega^2(\mu) \\ &= A(\mu)\omega(\mu) - k_1\eta(\mu)\widetilde{q}(\mu, \varrho) + B(\mu)\omega^2(\mu) \quad (27) \\ &= -k_1\eta(\mu)\widetilde{q}(\mu, \varrho) + \left[\sqrt{B(\mu)}\,\omega(\mu) - \frac{1}{2}\frac{A(\mu)}{\sqrt{B(\mu)}}\right]^2 + \frac{1}{4}\frac{A^2(\mu)}{B(\mu)} \\ &\leq -k_1\eta(\mu)\widetilde{q}(\mu, \varrho) + \frac{1}{4}\frac{A^2(\mu)}{B(\mu)}. \quad (28) \end{aligned}$$

Integrating (28) from $\mu_5(>\mu_4)$ to μ gives

$$\int_{\mu_5}^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho)-\frac{1}{4}\frac{A^2(s)}{B(s)}\right)ds\le\omega(\mu_5), \tag{29}$$

which contradicts (14). □

Corollary 4.1. *Assume $\alpha\ge\beta$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists η, $\xi\in C^1(I,\mathbb{R})$ such that $g(\mu,\varrho)\le\xi(\mu)\le\delta(\mu)\le\mu$, $\xi'(\mu)\ge 0$ and $\eta>0$ for $\mu\ge\mu_1$ such that the function $A(\mu)\le 0$,*

$$\limsup_{\mu\to\infty}\int_{\mu_5}^{\mu}\Big(\eta(s)\,\widetilde{q}(s,\varrho)\Big)ds=\infty\ \textit{for all}\ \mu_1\in I, \tag{30}$$

where $A(\mu)$ is defined in (15)*, also* (5) *or* (6) *holds with $\Theta(\mu)$ as in Theorem 3.1. Then every solution of Eq.* (E_1) *is oscillatory.*

Next, we examine the oscillation results of solutions of (E_1) by Philos-type. Let $\mathbb{D}_0=\{(\mu,s):a\le s<\mu<+\infty\}$, $\mathbb{D}=\{(\mu,s):a\le s\le\mu<+\infty\}$ the continuous function $H(\mu,s)$, $H:\mathbb{D}\to\mathbb{R}$ belongs to the class function $\mathbb{R}$

(i) $H(\mu,\mu)=0$ for $\mu\ge\mu_0$ and $H(\mu,s)>0$ for $(\mu,s)\in\mathbb{D}_0$,

(ii) H has a continuous and non-positive partial derivative on $\mathbb{D}_0$ with respect to the second variable such that

$$-\frac{\partial H(\mu,s)}{\partial s}=h(\mu,s)[H(\mu,s)]^{1/2},$$

for all $(\mu,s)\in\mathbb{D}_0$.

Theorem 4.2. *Assume $\alpha\ge 1$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists η, $\xi\in C^1(I,\mathbb{R})$ such that $g(\mu,\varrho)\le\xi(\mu)\le\delta(\mu)\le\mu$, $\xi'(\mu)\ge 0$, $\eta>0$ and $H(\mu,s)\in\mathbb{R}$ for $\mu\ge\mu_1$ with*

$$\limsup_{\mu\to\infty}\frac{1}{H(\mu,\mu_5)}\int_{\mu_5}^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho)H(\mu,s)-\frac{P^2(\mu,s)}{4B(s)}\right)ds=\infty, \tag{31}$$

for all $\mu_1\in I$, where $P(\mu,s)=h(\mu,s)-A(s)\sqrt{H(\mu,s)}$ and $A(\mu)$, $B(\mu)$ are defined in Theorem 4.1, also (5) *or* (6) *holds with $\Theta(\mu)$ as in Theorem 3.1. Then every solution of Eq.* (E_1) *is oscillatory.*

Proof. Let Eq. (E_1) has a nonoscillatory solution $u(\mu)$. Assume that, there exists a $\mu\ge\mu_1$ such that $u(\mu)>0$ and $u(g(\mu,\varrho))>0$ for some $\mu\ge\mu_0$. Proceeding as in the proof of Theorem 4.1, we obtain the inequality (27), i.e.,

$$\omega'(\mu)\ \le\ A(\mu)\omega(\mu)-k_1\eta(\mu)\widetilde{q}(\mu,\varrho)+B(\mu)\omega^2(\mu),$$

and so,

$$\begin{aligned}
\int_{\mu_5}^{\mu} H(\mu,s)\eta(s)\widetilde{q}(s,\varrho)ds &\leq \int_{\mu_5}^{\mu} H(\mu,s)[-\omega'(s)+A(s)\omega(s)-B(s)\omega^2(s)]ds \\
&= -H(\mu,s)\Big[\omega(s)\Big]_{\mu_5}^{\mu} + \int_{\mu_5}^{\mu}\Big[\frac{\partial H(\mu,s)}{\partial s}\omega(s) \\
&\quad +H(\mu,s)\Big[A(s)\omega(s)-B(s)\omega^2(s)\Big]\Big]ds \\
&= H(\mu,\mu_5)\omega(\mu_5) - \int_{\mu_5}^{\mu}\Big[\omega^2(s)B(s)H(\mu,s) \\
&\quad +\omega(s)\Big(h(\mu,s)\sqrt{H(\mu,s)}-H(\mu,s)A(s)\Big)\Big]ds \\
&\leq H(\mu,\mu_5)\omega(\mu_5) + \int_{\mu_5}^{\mu}\frac{P^2(\mu,s)}{4B(s)}ds,
\end{aligned}$$

which contradicts to (31). The rest of the proof is similar to that of Theorem 4.1 and hence is omitted. □

5. Examples

Below, we present a example to show application of the main results.

Example 5.1. For $\mu \geq 1$, consider fourth order differential equation

$$(32)\quad \big(1/2\mu\big(9e^{-\mu}(\mu)\big(u''(\mu)\big)\big)'\big)' + 36e^{-s/2}u^{(ii)}(\frac{\mu}{2}) + \int_1^2 \frac{\mu}{3}u(\varrho, 36e^{\mu/3})d\varrho = 0.$$

Here, $x_1 = 9e^{-\mu}$, $x_2 = 1/2\mu$, $\alpha = \beta = 1$, $p(\mu) = 36e^{-s/2}$, $q(\mu,\varrho) = \mu/3$ and $\delta(\mu) = \mu/2$, $g(\mu,\varrho) = \mu/3$. Now Pick $\eta(\mu) = 36e^{\mu/3}$, we obtain

$$\begin{aligned}
A_1(\mu_1,\mu) &= \int_1^{\mu}(9e^s)^{-1}ds = 9(e^{\mu}-e), \\
A_2(\mu_1,\mu) &= \int_1^{\mu} 2s\,ds = \mu^2-1 = (\mu+1)(\mu-1), \\
A_3(\mu_1,\mu/3) &= \int_1^{\mu/3}(9e^s)^{-1}(s^2-1)ds = e^{\mu/3}(\mu-3)^2, \\
\widetilde{q}(s,\varrho) &= \frac{s}{3}\int_1^2 d\varrho = s/3,
\end{aligned}$$

$A^2(s) = \frac{(3\mu^2-5)^2}{9}$ and $B(s) = \frac{(s-3)^2}{36}$. Now,

$$\begin{aligned}
&\limsup_{\mu\to\infty}\int_2^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho) - \frac{A^2(s)}{4B(s)}\right)ds \\
&= \limsup_{\mu\to\infty}\int_2^{\mu}\Big(12k_1\,s\,e^{s/3} - \Big(\frac{3s^2-5}{s-3}\Big)^2\Big)ds \to \infty \text{ as } \mu\to\infty,
\end{aligned}$$

and all hypotheses of Theorem 4.1 are satisfied, so every solution of (32) is oscillatory.

6. Conclusions

It is clear that the form of problem Eq. (E_1) is more general than all the problems considered in the study. In this paper, using the suitable Riccati type transformation, integral averaging condition, and comparison method, we offer some oscillatory properties which ensure that any solution of Eq. (E_1) oscillates under assumption of $A_1(\mu_1, \mu) \to \infty$, $A_2(\mu_1, \mu) \to \infty$ as $\mu \to \infty$. Also, it would be useful to extend oscillation criteria of Eq. (E_1) under the condition of $A_1(\mu_1, \mu) < \infty$, $A_2(\mu_1, \mu) < \infty$ as $\mu \to \infty$. In addition, we can consider the oscillation of Eq. (E_1) when equation Eq. (E_2) is oscillatory, and we can try to get some oscillation criteria of Eq. (E_1) if $p(t) < 0$ in the future work.

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The shortest path problem on chola period built temples with Dijkstra's algorithm in intuitionistic triangular neutrosophic fuzzy graph

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Abstract. In this article, Intuitionistic Triangular Neutrosophic Fuzzy Graph of Shortest Path Problem was Inaugurated, which is drew on triangular numbers and Intuitionistic Neutrosophic Fuzzy Graph. Real-world application is given as an illustrative model for Intuitionistic Triangular Neutrosophic Fuzzy Graph. Here we introduced famous chola period temples. These types of temples builted in various king of cholas. Here we assume only seven types of temples as vertices of Intuitionistic Triangular Neutrosophic Fuzzy Graph. Use of fuzzification method, edge weights of this Graph was calculated. Score function of Intuitionistic Triangular Neutrosophic Fuzzy Graph is inaugurated, with the help of this score function in the proposed algorithm, shortest way is determined.. This present Chola period temples Shortest Path Problem. Obtained shortest path is verified through Dijkstra's Algorithm with the help of Python Jupyter Notebook (adaptation) programming.

Keywords: Intuitionistic Fuzzy Number (IFN), Triangular Fuzzy Number (TFN), Shortest Path (SP), Intuitionistic Triangular Neutrosophic Fuzzy Graph(ITNFG).

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1. Introduction

The creators of, Ahuja [1] examined systematic execution of Dijkstra's calculation. Arsham [2] introduced another crucial arrangement calculation which permits affectability examination without utilizing any counterfeit, slack or surplus factors. Anusuya [3] apply positioning capacity for briefest way issue. Broumi [4] proposed for extend esteemed Neutrosophic Number. Broumi [5] presented Neutrosophic charts with most limited way issues. Broumi [6] proposed calculation gives Shortest way issue on single esteemed Neutrosophic charts. Broumi [7] proposed the Shortest way under Bipolar Neutrosophic setting. Broumi [8] gave the Shortest way issue under span esteemed Neutrosophic setting. Chiranjibe Jana [9] Presented Trapezoidal Neutrosophic aggregation operators and its application in multiple attribute decision making process. De [10] Computation of Shortest Path in a Fuzzy organization. De [11] Study on Ranking of Trapezoidal Intuitionistic Fuzzy Numbers. Enayattabar [12] introduced Dijkstra calculation for briefest way issue under Pythagorean Fuzzy climate. Jana [13] presented stretch esteemed Trapezoidal Neutrosophic Set. Jayagowri [14] discover Optimized Path in a Network utilizing Trapezoidal Intuitionistic Fuzzy Numbers. Kalaiarasi [15] determine fuzzy optimal total cost and fuzzy optimal order quantity obtained by Ranking function method and Kuhn-tucker method for the proposed Inventory model. Kalaiarasi [16] constructed Inventory parameters that are Fuzzy using Trapezoidal Fuzzy Numbers. Kumar [17] proposed to tackling briefest way issue with edge weight. Kumar [18] introduced Algorithm for most limited way issue in an organization with span esteemed Intuitionistic Trapezoidal Fuzzy Number. Kumar [19] presented the SPP from an underlying hub to an objective hub on Neutrosophic chart. Majumdar [20] introduced an Intuitionistic Fuzzy most brief way organization. Nagoor Gani [21] looking Intuitionistic Fuzzy most brief organization. Ojekudo Nathaniel akpofure [22] tended to the most brief way utilizing Dijkstra's calculation. Said broumi [23] processing the most brief way Neutrosophic Information. Smarandache [24] summed up the Fuzzy rationale and presented two Neutrosophic ideas. Victor christianto [25] gave a Neutrosophic approach to futurology. Wang [26] contributed Neutrosophic sets with their properties. Xu [27] introduced a strategies for amassing span esteemed Intuitionistic Fuzzy data, Yang [28] introduced rectangular hindrance subject to various improvement capacities regarding the quantity of curves. Ye [29] proposed a Trapezoidal Fuzzy Neural Computing and Applications. Ye [30] developed of the Multi models dynamic strategy utilizing shape liking measure, Ye [31] presented a Prioritized aggregation operators of Trapezoidal Intuitionistic Fuzzy Sets and their Application.

Fuzzy graph theory is finding an increasing number in developing real time applications in modeling systems with accuracy varying at different levels of infor mation. The fuzzy set theory can play a significant role in this kind of decision making environment to tackle the unknown or the vagueness about the time du ration of activities in a project network. To effectively deal with the ambiguities involved in the process of linguistic estimate times. In the applied field, the success of the use of fuzzy set theory depends on the choice of the membership function that we make. However, there are applications in which experts do not have precise knowledge of the function that should be taken. In these cases, it is appropriate to represent the membership degree of each element of the fuzzy set by means of an interval. From these considerations arises the extension of fuzzy sets called the theory of interval-valued fuzzy sets. That is, fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval [0,1]. Replacing the membership function of vertices and edges in fuzzy graphs by interval-valued fuzzy sets such that they satisfy some particular conditions, interval valued fuzzy graphs (IVFG) were defined. Thus IVFG provide a better description of vagueness and uncertainty within the specific interval than the traditional fuzzy graph.

Triangular intuitionistic fuzzy numbers

Triangular intuitionistic fuzzy numbers (TIFNS) are a special kind of intuitionistic fuzzy sets (IFSS) on a real number set. TIFNs are useful to deal with ill-known quantities in decision data and decision making problems themselves.

Dijkstra's Algorithm

The shortest path algorithm is given a weighted graph or digraph $G = (V, E, W)$ and two specified vertices V and W; the algorithm finds a shortest path from V to W. The distance from a vertex V to a vertex W (denote $d(V, W)$) is the weight of a shortest path from V to W. Dijkstra's shortest path algorithm will find the shortest paths from V to the other vertices in order of increasing distance from V. The algorithm stops when it reaches W.

Here, in this paper disclosed the briefest way to Chola period temples utilized the proposed calculation.

Intuitionistic fuzzy number gives more accuracy than fuzzy numbers. So that intuitionistic fuzzy numbers are used for finding shortest path of a graph. In this paper Dijkstra's algorithm is the only algorithm suitable for verifying our real world problem, because of the edge weight of fuzzy graph, rather than other algorithms.

2. Methodology

In this section, we explain important notions of Intuitionistic Fuzzy Sets.

Definition 2.1 ([9]). *Let* $\overline{n_1} = [(t_1, t_2, t_3), (t_4, t_5, t_6)], [(i_1, i_2, i_3), (i_4, i_5, i_6)],$ $[(f_1, f_2, f_3), (f_4, f_5, f_6)]$ *and* $\overline{n_2} = [(T_1, T_2, T_3), (T_4, T_5, T_6)], [(I_1, I_2, I_3), (I_4, I_5, I_6)],$ $[(F_1, F_2, F_3), (F_4, F_5, F_6)]$. *Therefore, the conditions are*

1. $\overline{n_1} \oplus \overline{n_2} = \langle [(t_1 + T_1 - t_1T_1, t_2 + T_2 - t_2T_2, t_3 + T_3 - t_3T_3),$
$(t_4 + T_4 - t_4T_4, t_5 + T_5 - t_5T_5, t_6 + T_6 - t_6T_6)],$
$[(i_1I_1, i_2I_2, i_3I_3), (i_4I_4, i_5I_5, i_6I_6)],$
$[(f_1F_1, f_2F_2, f_3F_3), (f_4F_4, f_5F_5, f_6F_6)]\rangle.$

2. $\overline{n_1} \otimes \overline{n_2} = \langle [(t_1T_1, t_2T_2, t_3T_3), (t_4T_4, t_5T_5, t_6T_6)],$
$[(i_1 + I_1 - i_1I_1, i_2 + I_2 - i_2I_2, i_3 + I_3 - i_3I_3),$
$(i_4 + I_4 - i_4I_4, i_5 + I_5 - i_5I_5, i_6 + I_6 - i_6I_6)],$
$[(f_1 + F_1 - f_1F_1, f_2 + F_2 - f_2F_2, f_3 + F_3 - f_3F_3),$
$(f_4 + F_4 - f_4F_4, f_5 + F_5 - f_5F_5, f_6 + F_6 - f_6F_6)]\rangle.$

3. $\lambda\overline{n_1} = \langle [(1-(1-t_1)^\lambda), (1-(1-t_2)^\lambda), (1-(1-t_3)^\lambda), (1-(1-t_4)^\lambda), (1-(1-t_5)^\lambda), (1-(1-t_6)^\lambda)],$
$[((i_1)^\lambda, (i_2)^\lambda, (i_3)^\lambda), ((i_4)^\lambda, (i_5)^\lambda, (i_6)^\lambda)],$
$[((f_1)^\lambda, (f_2)^\lambda, (f_3)^\lambda), ((f_4)^\lambda, (f_5)^\lambda, (f_6)^\lambda)]\rangle$, *for* $\lambda > 0$.

4. $n_1^\lambda = [((t_1)^\lambda, (t_2)^\lambda, (t_3)^\lambda), ((t_4)^\lambda, (t_5)^\lambda, (t_6)^\lambda)],$
$[1-(1-i_1)^\lambda, (1-(1-i_2)^\lambda), (1-(1-i_3)^\lambda), (1-(1-i_4)^\lambda), (1-(1-i_5)^\lambda), (1-(1-i_6)^\lambda)],$
$[(1-(1-f_1)^\lambda), (1-(1-f_2)^\lambda), (1-(1-f_3)^\lambda), (1-(1-f_4)^\lambda), (1-(1-f_5)^\lambda), (1-(1-f_6)^\lambda)]\rangle$, *for* $\lambda > 0$.

Definition 2.2 ([9])**.** *Let* $\overline{n} = [(t_1, t_2, t_3), (t_4, t_5, t_6)], [(i_1, i_2, i_3), (i_4, i_5, i_6)], [(f_1, f_2, f_3), (f_4, f_5, f_6)]$ *be an intuitionistic triangular neutrosophic number, then defined as their score functions*

$$\begin{aligned} S(\overline{n}) = \frac{1}{3}\Bigg\{ 2 + \left(\frac{t_4 + 2t_5 + t_6}{4} - \frac{t_1 + 2t_2 + t_3}{4}\right) \\ - \left(\frac{i_4 + 2i_5 + i_6}{4} - \frac{i_1 + 2i_2 + i_3}{4}\right) - \left(\frac{f_4 + 2f_5 + f_6}{4} - \frac{f_1 + 2f_2 + f_3}{4}\right)\Bigg\}, \\ S(\overline{n}) \in [-1, 1], \end{aligned} \tag{1}$$

where the higher value of $S(\overline{n})$ *larger the intuitionistic triangular number* $\overline{n}$*.*

3. Intuitionistic triangular neutrosophic fuzzy graph

Advantages of the proposed Algorithm

It is easy to understand a step wise representation and not dependent on any programming language. So we introduce Intuitionistic Triangular Neutrosophic Fuzzy Graph algorithm.

Merits of Proposed Algorithms:

1. It is a step-wise representation of a solution to a given problem, which makes it easy to understand.
2. An algorithm uses a definite procedure.
3. It is not dependent on any programming language, so it is easy to understand for anyone even without programming knowledge.
4. Every step in an algorithm has its own logical sequence so it is easy to debug.
5. By using algorithm, the problem is broken down into smaller pieces or steps hence, it is easier for programmer to convert it into an actual program.

Demerits of Proposed Algorithms:

1. Alogorithms is Time consuming.
2. Difficult to show Branching and Looping in Algorithms.
3. Big tasks are difficult to put in Algorithms.

Algorithm:
In this research, we using proposed algorithm for finding shortest path.

Step 1. Let $d_1 = \langle[(0,0,0),(0,0,0)],[(1,1,1),(1,1,1)],[(1,1,1),(1,1,1)]\rangle$ the source node as $d_1 = \langle[(0,0,0),(0,0,0)],[(1,1,1),(1,1,1)],[(1,1,1),(1,1,1)]\rangle$.
Step 2. Find $d_j = minimum\{d_i + d_{ij}\}$, $j = 2, 3, ..., n$.
Step 3. If the minimum value of i. i.e., $i = r$ then the label node j as $[d_j, r]$. If minimum arise related to more than one values of i. Their position we choose minimum value of i.
Step 4. Let the destination node be $[d_n, l]$. Here source node is d_n. We conclude a Score function and we finds minimum value of Intuitionistic Triangular Neutrosophic Number.
Step 5. We calculate Shortest Path Problem between source and destination node. Review the label of node 1. Let it be as $[d_n, A]$. Now review the label of node A and so on. Replicate the same procedure until node 1 is procured.
Step 6. The Shortest Path can be procured by combined all the nodes by the step 5.

4. Data analysis

To find Shortest Path on Chola period built temples using Intuitionistic Triangular Neutrosophic Fuzzy Graph.

In this chapter, AST denotes AmarasundreashwararTemple, GKCT denotes Gangai konda cholapuram Temple, TKT denotes Thiruvanai Kovil Temple, MKT denotes Moovar Kovil Temple, SST denotes Shri Suryanar Temple, BT denotes Brihadeeswarar Temple, and SAT denotes Shri Airavatesvara Temple.

Here, each node is converts as ITNFN.

Here, we consider source node is Amarasundreashwarar Temple and destination node is Airavatesvara temple. To find shortest path on Amarasundreashwarar Temple to Airavatesvara temple.

Node 1	=	AmarasundreashwararTemple
Node 2	=	Gangai konda cholapuram Temple
Node 3	=	Thiruvanai Kovil Temple
Node 4	=	Moovar Kovil Temple
Node 5	=	Shri Suryanar Temple
Node 6	=	Brihadeeswarar Temple
Node 7	=	Shri Airavatesvara Temple

The distance (km) between temples are considered as the edges of the graph. Considered distance are converted as Intuitionistic Triangular Neutrosophic Fuzzy Graph using the score function(fuzzification) of Intuitionistic Triangular Neutrosophic Fuzzy Graph.

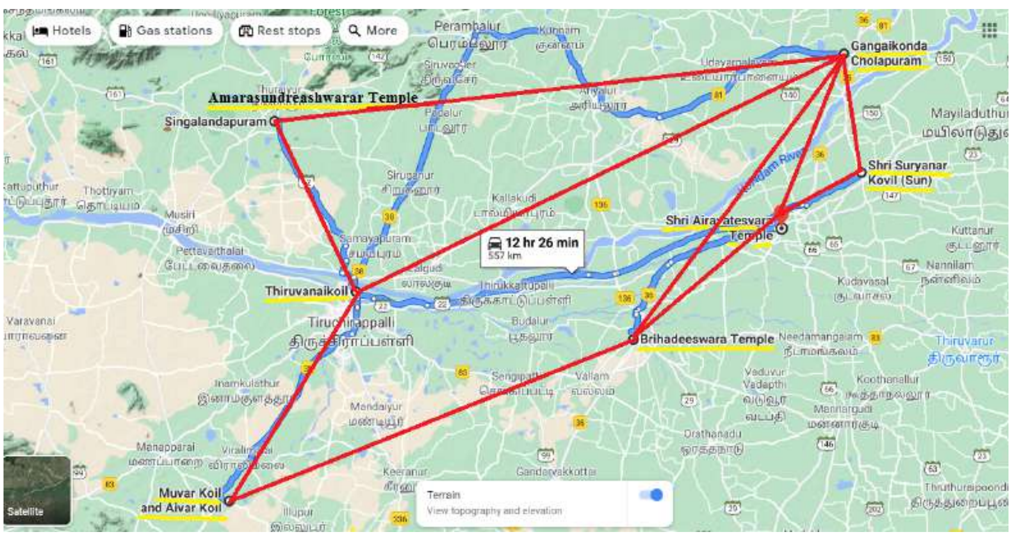


Figure 1: A graph of Chola period temples

Here, distance between one temple to another temple is calculated in kilometers. The numerical value of the distance is converted to Intuitionistic Triangular Neutrosophic Fuzzy Graphs with the help of through Neutrosopic Score function and trapezoidal signed distance.

The given distance (kilometer) converted to neutrosophic number $\frac{2+T-I-F}{3}$ (using score function). We converted neutrosophic number as (a_1, a_2, a_3) are membership function & $(a_1^*, a_2^{*\prime}, a_3^{*\prime})$ are non-membership function. These functions converted to fuzzy triangular numbers using triangular signed distance $\frac{a_1+2a_2+a_3}{4}$. Finally, converted Intuitionistic Triangular Neutrosophic Fuzzy Number.

Here, Apply the Intuitionistic Triangular Neutrosophic Fuzzy Number in our algorithm to find shortest path to Chola period temples. In this application, many paths have chola period temples. To calculate Shortest Path using score function(Definition 2.1 and 2.2). An algorithm is used to apply a definite procedure and the process has been expensive and time consuming. Here node 1-2 = 117 km

This km changed to neutrosophic number use neutrosophic score function, and each neutrosophic number converted to fuzzification method, so we get fuzzy number. Finally we convert membership and non-membership from fuzzy number because of Intuitionistic fuzzy number, and use triangular signed distance to membership and non-membership functions. At last we get Intuitionistic Triangular Neutrosophic Fuzzy Number.

820	340	131
0.82	0.34	0.131
(0.82,0.18)	(0.34,0.66)	(0.131,0.869)

$$\Big\langle [(0.65, 0.82, 0.99), (0.11, 0.18, 0.25)], [(0.16, 0.34, 0.52), (0.57, 0.66, 0.75)], [(0.07, 0.131, 0.192), (0.813, 0.869, 0.925)] \Big\rangle$$

1-3=35 km

350	142	105
0.85	0.142	0.105
(0.35,0.65)	(0.142,0.858)	(0.105,0.895)

$$\Big\langle [(0.21, 0.35, 0.49), (0.49, 0.65, 0.81)], [(0.088, 0.142, 0.196), (0.744, 0.858, 0.972)], [(0.018, 0.105, 0.192), (0.835, 0.895, 0.955)] \Big\rangle$$

2-3=125 km

$$\begin{array}{ccc} 955 & 425 & 157 \\ 0.955 & 0.425 & 0.157 \\ (0.955,0.045) & (0.425,0.575) & (0.157,0.843) \end{array}$$

$$\Big\langle [(0.911, 0.955, 0.999), (0.029, 0.045, 0.061)], [(0.229, 0.425, 0.621), (0.425, 0.575, 0.725)],$$
$$[(0.013, 0.157, 0.301), (0.722, 0.843, 0.964)] \Big\rangle$$

2-5=24 km

$$\begin{array}{ccc} 316 & 172 & 74 \\ 0.316 & 0.172 & 0.074 \\ (0.316,0.684) & (0.172,0.828) & (0.074,0.926) \end{array}$$

$$\Big\langle [(0.128, 0.316, 0.504), (0.556, 0.684, 0.812)], [(0.011, 0.172, 0.333), (0.721, 0.828, 0.935)],$$
$$[(0.049, 0.074, 0.099), (0.873, 0.926, 0.979)] \Big\rangle$$

2-6=71 km

$$\begin{array}{ccc} 650 & 330 & 109 \\ 0.65 & 0.33 & 0.109 \\ (0.65,0.35) & (0.33,0.67) & (0.109,0.891) \end{array}$$

$$\Big\langle [(0.59, 0.65, 0.71), (0.17, 0.35, 0.53)], [(0.17, 0.33, 0.49), (0.51, 0.67, 0.83)],$$
$$[(0.035, 0.109, 0.183), (0.826, 0.891, 0.956)] \Big\rangle$$

3-4=48 km

$$\begin{array}{ccc} 465 & 220 & 103 \\ 0.465 & 0.22 & 0.103 \\ (0.465,0.535) & (0.22,0.78) & (0.103,0.897) \end{array}$$

$$\Big\langle [(0.395, 0.465, 0.535), (0.455, 0.535, 0.615)], [(0.11, 0.22, 0.33), (0.71, 0.78, 0.85)],$$
$$[(0.011, 0.103, 0.195), (0.821, 0.897, 0.973)] \Big\rangle$$

4-6=95 km

950	435	232
0.95	0.435	0.232
(0.95,0.05)	(0.435,0.565)	(0.232,0.768)

$$\Big\langle [(0.91, 0.95, 0.99), (0.02, 0.05, 0.08)], [(0.333, 0.435, 0.537), (0.505, 0.565, 0.625)],$$
$$[(0.149, 0.232, 0.315), (0.733, 0.768, 0.803)] \Big\rangle$$

5-6=54 km

650	320	170
0.65	0.32	0.17
(0.65,0.35)	(0.32,0.68)	(0.17,0.83)

$$\Big\langle [(0.51, 0.65, 0.79), (0.24, 0.35, 0.46)], [(0.17, 0.32, 0.47), (0.6, 0.68, 0.76)],$$
$$[(0.09, 0.17, 0.25), (0.69, 0.83, 0.97)] \Big\rangle$$

5-7=20 km

180	72	50
0.18	0.072	0.05
(0.18,0.82)	(0.072,0.928)	(0.05,0.95)

$$\Big\langle [(0.09, 0.18, 0.27), (0.71, 0.82, 0.93)], [(0.045, 0.072, 0.099), (0.869, 0.928, 0.987)],$$
$$[(0.03, 0.05, 0.07), (0.93, 0.95, 0.97)] \Big\rangle$$

6-7=20 km

640	330	201
0.64	0.33	0.201
(0.64,0.36)	(0.33,0.67)	(0.201,0.799)

$$\Big\langle [(0.56, 0.64, 0.72), (0.28, 0.36, 0.44)], [(0.2, 0.33, 0.46), (0.59, 0.67, 0.75)],$$
$$[(0.069, 0.201, 0.333), (0.737, 0.799, 0.861)] \Big\rangle$$

In this iteration SPP was calculated through the proposed algorithm, the concept of the Chola period temples shortest path calculated from Amarasundreashwarar Temple to Shri Airavatesvara Temple.

Let $n = 7$ is the destination node, since there are totally 7 nodes.

Iteration 1. Assume the source node is Amarasundreashwarar Temple. Here we assume $d_1 = \langle[(0,0,0), (0,0,0)], [(1,1,1), (1,1,1)], [(1,1,1), (1,1,1)]\rangle$ and label of source node is $\{\langle[(0,0,0),(0,0,0)],[(1,1,1),(1,1,1)],[(1,1,1),(1,1,1)]\rangle, --\}$ the value of $d_j, j = 2,3,4,5,6$ is succeeding. Here we assume d_1 is the Amarasundreashwarar Temple.

Iteration 2. The node Gangai konda cholapuram Temple has only node Amarasundreashwarar Temple as the predecessor. Intuitionistic Triangular Fuzzy Neutrosophic Shortest Path is calculated from Gangai konda cholapuram Temple to Amarasundreashwarar Temple. Since node 2 has only node 1 as the predecessor. So fix $i = 1$ and $j = 2$ we apply step 2 at proposed algorithm

$$
\begin{aligned}
d_2 &= minimum\{d_1 \oplus d_{12}\}\\
&= minimum\left\{\begin{array}{c}\langle[(0,0,0),(0,0,0)],[(1,1,1),(1,1,1)],[(1,1,1),(1,1,1)]\rangle\oplus\\ \langle[(0.65,0.82,0.99),(0.11,0.18,0.25)],[(0.16,0.34,0.52),\\ (0.57,0.66,0.75)],[(0.07,0.131,0.192),(0.813,0.869,0.925)]\rangle\end{array}\right\}\\
&= \left\{\begin{array}{c}\langle[(0.65,0.82,0.99),(0.11,0.18,0.25)],[(0.16,0.34,0.52),\\ (0.57,0.66,0.75)],[(0.07,0.131,0.192),(0.813,0.869,0.925)]\rangle\end{array}\right\}
\end{aligned}
$$

Therefore, minimum value $i = 1$, corresponding to label node 2 as

$$
\begin{aligned}
&= \left\{\begin{array}{c}\langle[(0.65,0.82,0.99),(0.11,0.18,0.25)],[(0.16,0.34,0.52),\\ (0.57,0.66,0.75)],[(0.07,0.131,0.192),(0.813,0.869,0.925)]\rangle,1\end{array}\right\}\\
d_2 &= \left\{\begin{array}{c}\langle[(0.65,0.82,0.99),(0.11,0.18,0.25)],[(0.16,0.34,0.52),\\ (0.57,0.66,0.75)],[(0.07,0.131,0.192),(0.813,0.869,0.925)]\rangle\end{array}\right\}
\end{aligned}
$$

The labeled node is Gangai Konda Cholapuram and minimum provided corresponding node is Amarasundreashwarar Temple.

Minimum Node	Labeled Node	Path Node
AST	GKCT	$\langle[(0.65, 0.82, 0.99), (0.11, 0.18, 0.25)], [(0.16, 0.34, 0.52), (0.57, 0.66, 0.75)], [(0.07, 0.131, 0.192), (0.813, 0.869, 0.925)]\rangle$

Iteration 3. The node Thiruvanai Kovil Temple has two predecessors node, they are node Amarasundreashwarar Temple and node Gangai konda cholapuram Temple. Intuitionistic Triangular Fuzzy Neutrosophic Shortest Path is calculated to Thiruvanai Kovil from Amarasundreashwarar Temple and Gangai konda cholapuram. Since node 3 has two predecessors node 1 and node 2. So fix $i = 1, 2$ and $j = 3$ we apply step 2 at proposed algorithm.

$d_3 = minimum\{d_1 \oplus d_{13}, d_2 \oplus d_{23}\}$

$$= minimum \left\{ \begin{array}{r} \langle[(0,0,0),(0,0,0)],[(1,1,1),(1,1,1)],[(1,1,1),(1,1,1)]\rangle \oplus \\ \langle[(0.21,0.35,0.49),(0.49,0.65,0.81)],[(0.088,0.142,0.196), \\ (0.744,0.858,0.972)],[(0.018,0.105,0.192),(0.835,0.895,0.955)]\rangle \\ \langle[(0.65,0.82,0.99),(0.11,0.18,0.25)],[(0.16,0.34,0.52) \\ (0.57,0.66,0.75)],[(0.07,0.131,0.192),(0.813,0.869,0.925)]\rangle \oplus \\ \langle[(0.911,0.955,0.999),(0.029,0.045,0.061)],[(0.229,0.425,0.621) \\ (0.425,0.575,0.725)],[(0.013,0.157,0.301),(0.722,0.843,0.964)]\rangle \end{array} \right\}$$

$$= minimum \left\{ \begin{array}{r} \langle[(0.21,0.35,0.49),(0.49,0.65,0.81)],[(0.088,0.142,0.196), \\ (0.744,0.858,0.972)],[(0.018,0.105,0.192),(0.835,0.895,0.955)]\rangle, \\ \langle[(0.9688,0.9919,0.999),(0.1358,0.217,0.2957)],[(0.0366,0.1445,0.323), \\ (0.2422,0.3795,0.5437)],[(0.0009,0.0205,0.0577),(0.5869,0.7325,0.8917)]\rangle \end{array} \right\}$$

Using equation (2.1),we have

$$S \left\{ \begin{array}{r} \langle[(0.21,0.35,0.49),(0.49,0.65,0.81)],[(0.088,0.142,0.196), \\ (0.744,0.858,0.972)],[(0.018,0.105,0.192),(0.835,0.895,0.955)]\rangle \end{array} \right\}$$

$S(\overline{n}_1) = 0.2647$

$$S \left\{ \begin{array}{r} \langle[(0.9688,0.9919,0.999),(0.1358,0.217,0.2957)],[(0.0366,0.1445,0.323), \\ (0.2422,0.3795,0.5437)],[(0.0009,0.0205,0.0577),(0.5869,0.7325,0.8917)]\rangle \end{array} \right\}$$

$S(\overline{n}_2) = 0.0978$

Therefore minimum value $i = 2$, corresponding to label node 3 as

$$\left\{ \begin{array}{r} \langle[(0.9688,0.9919,0.999),(0.1358,0.217,0.2957)],[(0.0366,0.1445,0.323), \\ (0.2422,0.3795,0.5437)],[(0.0009,0.0205,0.0577),(0.5869,0.7325,0.8917)]\rangle, 2 \end{array} \right\}$$

$$d_3 = \left\{ \begin{array}{r} \langle[(0.9688,0.9919,0.999),(0.1358,0.217,0.2957)],[(0.0366,0.1445,0.323), \\ (0.2422,0.3795,0.5437)],[(0.0009,0.0205,0.0577),(0.5869,0.7325,0.8917)]\rangle \end{array} \right\}$$

Here, the labeled node is Thiruvanai Kovil and the minimum provided corresponding node is Gangai konda cholapuram.

Minimum Node	Labeled Node	Path Node
GKCT	TKT	$\langle[(0.9688, 0.9919, 0.999), (0.1358, 0.217, 0.2957)], [(0.0366, 0.1445, 0.323), (0.2422, 0.3795, 0.5437)], [(0.0009, 0.0205, 0.0577), (0.5869, 0.7325, 0.8917)]\rangle$

Iteration 4. The node Moovar Kovil has only node Thiruvanai Kovil as the predecessor. Intuitionistic Triangular Fuzzy Neutrosophic Shortest Path is calculated to Moovar Kovil from Thiruvanai Kovil. Since node 4 has only node 3 as

the predecessor. So fix $i = 3$ and $j = 4$ we apply step 2 at proposed algorithm.

$$d_4 = minimum\{d_3 \oplus d_{34}\}$$

$$= minimum \left\{ \begin{array}{r} \langle[(0.9688, 0.9919, 0.999), (0.1358, 0.217, 0.2957)], [(0.0366, 0.1445, 0.323), \\ (0.2422, 0.3795, 0.5437)], [(0.0009, 0.0205, 0.0577), (0.5869, 0.7325, 0.8917)]\rangle \oplus \\ \langle[(0.395, 0.465, 0.535), (0.455, 0.535, 0.615)], [(0.11, 0.22, 0.33), \\ (0.71, 0.78, 0.85)], [(0.011, 0.103, 0.195), (0.821, 0.897, 0.973)]\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{r} \langle[(0.98, 0.995, 0.999), (0.529, 0.636, 0.728)], [(0.004, 0.032, 0.107), \\ (0.172, 0.296, 0.462)], [(0.000009, 0.002, 0.011), (0.482, 0.657, 0.868)]\rangle \end{array} \right\}$$

Therefore minimum value $i = 3$, corresponding to label node 4 as

$$= \left\{ \begin{array}{r} \langle[(0.98, 0.995, 0.999), (0.529, 0.636, 0.728)], [(0.004, 0.032, 0.107), \\ (0.172, 0.296, 0.462)], [(0.000009, 0.002, 0.011), (0.482, 0.657, 0.868)]\rangle, 3 \end{array} \right\}$$

$$d_4 = \left\{ \begin{array}{r} \langle[(0.98, 0.995, 0.999), (0.529, 0.636, 0.728)], [(0.004, 0.032, 0.107), \\ (0.172, 0.296, 0.462)], [(0.000009, 0.002, 0.011), (0.482, 0.657, 0.868)]\rangle \end{array} \right\}$$

Here, the labeled node is Moovar Kovil and the minimum provided corresponding node is Thiruvanai Kovil.

Minimum Node	Labeled Node	Path Node
TKT	MKT	⟨[(0.98, 0.995, 0.999), (0.529, 0.636, 0.728)], [(0.004, 0.032, 0.107), (0.172, 0.296, 0.462)], [(0.000009, 0.002, 0.011), (0.482, 0.657, 0.868)]⟩

Iteration 5. The node Shri Suryanar Temple has only node Gangai konda cholapuram as the predecessor. Intuitionistic Triangular Fuzzy Neutrosophic Shortest Path is calculated to Shri Suryanar Temple from Gangai konda cholapuram. Since node 5 has only node 2 as the predecessor. So fix $i = 2$ and $j = 5$ we apply step 2 at proposed algorithm.

$$d_5 = minimum\{d_2 \oplus d_{25}\}$$

$$= minimum \left\{ \begin{array}{r} \langle[(0.65, 0.82, 0.99), (0.11, 0.18, 0.25)], [(0.16, 0.34, 0.52), \\ (0.57, 0.66, 0.75)], [(0.07, 0.131, 0.192), (0.813, 0.869, 0.925)]\rangle \oplus \\ \langle[(0.128, 0.316, 0.504), (0.556, 0.684, 0.812)], [(0.011, 0.172, 0.333), \\ (0.721, 0.828, 0.935)], [(0.049, 0.074, 0.099), (0.873, 0.926, 0.979)]\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{r} \langle[(0.695, 0.877, 0.995), (0.605, 0.741, 0.859)], [(0.002, 0.058, 0.173), \\ (0.411, 0.546, 0.701)], [(0.003, 0.0096, 0.019), (0.709, 0.805, 0.906)]\rangle \end{array} \right\}$$

Therefore minimum value $i = 2$, corresponding to label node 5 as

$$= \left\{ \begin{array}{r} \langle[(0.695, 0.877, 0.995), (0.605, 0.741, 0.859)], [(0.002, 0.058, 0.173), \\ (0.411, 0.546, 0.701)], [(0.003, 0.0096, 0.019), (0.709, 0.805, 0.906)]\rangle, 2 \end{array} \right\}$$

$$d_4 = \left\{ \begin{array}{r} \langle[(0.695, 0.877, 0.995), (0.605, 0.741, 0.859)], [(0.002, 0.058, 0.173), \\ (0.411, 0.546, 0.701)], [(0.003, 0.0096, 0.019), (0.709, 0.805, 0.906)]\rangle \end{array} \right\}$$

Here the labeled node is Shri Suryanar Temple and the minimum provided corresponding node is Gangai konda cholapuram.

Minimum Node	Labeled Node	Path Node
GKCT	SST	⟨[(0.695, 0.877, 0.995), (0.605, 0.741, 0.859)], [(0.002, 0.058, 0.173), (0.411, 0.546, 0.701)], [(0.003, 0.0096, 0.019), (0.709, 0.805, 0.906)]⟩

Iteration 6. The node Brihadeeswarar Temple has three predecessors node, they are node Gangai konda cholapuram, node Moovar Kovil and node Shri Suryanar Temple. Intuitionistic Triangular Fuzzy Neutrosophic Shortest Path is calculated to Brihadeeswarar Temple from Gangai konda cholapuram, Moovar Kovil and Shri Suryanar Temple. Since node 6 has three predecessors . The predecessors are node 2, node 4 and node 5. So fix $i = 2, 4, 5$ and $j = 6$ we apply step 2 at proposed algorithm.

$$d_6 = minimum\{d_2 \oplus d_{26}, d_4 \oplus d_{46}, d_5 \oplus d_{56}\}$$

$$= minimum \left\{ \begin{array}{r} \langle[(0.65, 0.82, 0.99), (0.11, 0.18, 0.25)], [(0.16, 0.34, 0.52), \\ (0.57, 0.66, 0.75)], [(0.07, 0.131, 0.192), (0.813, 0.869, 0.925)]\rangle \oplus \\ \langle[(0.59, 0.65, 0.71), (0.17, 0.35, 0.53)], [(0.17, 0.33, 0.49), \\ (0.51, 0.67, 0.83)], [(0.035, 0.109, 0.183), (0.826, 0.891, 0.956)]\rangle \\ \langle[(0.98, 0.995, 0.999), (0.529, 0.636, 0.728)], [(0.004, 0.032, 0.107), \\ (0.172, 0.296, 0.462)], [(0.000009, 0.002, 0.011), (0.482, 0.657, 0.868)]\rangle \oplus \\ \langle[(0.91, 0.95, 0.99), (0.02, 0.05, 0.08)], [(0.333, 0.435, 0.537), \\ (0.505, 0.565, 0.625)], [(0.149, 0.232, 0.315), (0.733, 0.768, 0.803)]\rangle \\ \langle[(0.695, 0.877, 0.995), (0.605, 0.741, 0.859)], [(0.002, 0.058, 0.173), \\ (0.411, 0.546, 0.701)], [(0.003, 0.0096, 0.019), (0.709, 0.805, 0.906)]\rangle \oplus \\ \langle[(0.51, 0.65, 0.79), (0.24, 0.35, 0.46)], [(0.17, 0.32, 0.47), \\ (0.6, 0.68, 0.76)], [(0.09, 0.17, 0.25), (0.69, 0.83, 0.97)]\rangle \end{array} \right\}$$

$$= minimum \left\{ \begin{array}{r} \langle[(0.856, 0.937, 0.997), (0.2613, 0.467, 0.647)], [(0.0272, 0.1122, 0.2548), \\ (0.291, 0.442, 0.6225)], [(0.0024, 0.014, 0.035), (0.671, 0.774, 0.884)]\rangle, \\ \langle[(0.998, 0.9997, 0.99999), (0.538, 0.654, 0.749)], [(0.001, 0.014, 0.057), \\ (0.087, 0.167, 0.289)], [(0.000001, 0.0005, 0.003), (0.353, 0.504, 0.697)]\rangle \\ \langle[(0.85, 0.956, 0.999), (0.699, 0.832, 0.9238)], [(0.0003, 0.0185, 0.0813), \\ (0.247, 0.371, 0.322)], [(0.0003, 0.002, 0.005), (0.489, 0.668, 0.879)]\rangle \end{array} \right\}$$

Using equation (2.1),we have

$$S \left\{ \begin{array}{r} \langle[(0.856, 0.937, 0.997), (0.2613, 0.467, 0.647)], [(0.0272, 0.1122, 0.2548), \\ (0.291, 0.442, 0.6225)], [(0.0024, 0.014, 0.035), (0.671, 0.774, 0.884)]\rangle \end{array} \right\}$$

$$S(\overline{n}_1) = 0.149$$

$$S\left\{\begin{matrix}\langle[(0.998, 0.9997, 0.99999), (0.538, 0.654, 0.749)], [(0.001, 0.014, 0.057),\\ (0.087, 0.167, 0.289)], [(0.000001, 0.0005, 0.003), (0.353, 0.504, 0.697)]\rangle\end{matrix}\right\}$$

$$S(\overline{n}_2) = 0.32662$$

$$S\left\{\begin{matrix}\langle[(0.85, 0.956, 0.999), (0.699, 0.832, 0.9238)], [(0.0003, 0.0185, 0.0813),\\ (0.247, 0.371, 0.322)], [(0.0003, 0.002, 0.005), (0.489, 0.668, 0.879)]\rangle\end{matrix}\right\}$$

$$S(\overline{n}_3) = 0.3032$$

Therefore minimum value $i = 2$, corresponding to label node 6 as

$$\left\{\begin{matrix}\langle[(0.856, 0.937, 0.997), (0.2613, 0.467, 0.647)], [(0.0272, 0.1122, 0.2548),\\ (0.291, 0.442, 0.6225)], [(0.0024, 0.014, 0.035), (0.671, 0.774, 0.884)]\rangle, 2\end{matrix}\right\}$$

$$d_3 = \left\{\begin{matrix}\langle[(0.856, 0.937, 0.997), (0.2613, 0.467, 0.647)], [(0.0272, 0.1122, 0.2548),\\ (0.291, 0.442, 0.6225)], [(0.0024, 0.014, 0.035), (0.671, 0.774, 0.884)]\rangle\end{matrix}\right\}$$

Here, the labeled node is Brihadeeswarar Temple and the minimum provided corresponding node is Gangai konda cholapuram.

Minimum Node	Labeled Node	Path Node
GKCT	BT	⟨[(0.856, 0.937, 0.997), (0.2613, 0.467, 0.647)], [(0.0272, 0.1122, 0.2548), (0.291, 0.442, 0.6225)], [(0.0024, 0.014, 0.035), (0.671, 0.774, 0.884)]⟩

Iteration 7. The node Shri Airavatesvara Temple has two predecessors node, they are node Shri Suryanar Temple and node Brihadeeswarar Temple. ITNSP is calculated to Shri Airavatesvara Temple from Shri Suryanar Temple and Brihadeeswarar Temple. Since node 7 has two predecessors node 5 and node 6. So fix $i = 5, 6$ and $j = 7$ we apply step 2 at proposed algorithm.

$$d_7 = minimum\{d_5 \oplus d_{57}, d_6 \oplus d_{67}\}$$

$$= minimum\left\{\begin{matrix}\langle[(0.695, 0.877, 0.995), (0.605, 0.741, 0.859)], [(0.002, 0.058, 0.173),\\ (0.411, 0.546, 0.701)], [(0.003, 0.0096, 0.019), (0.709, 0.805, 0.906)]\rangle\oplus\\ \langle[(0.09, 0.18, 0.27), (0.71, 0.82, 0.93)], [(0.045, 0.072, 0.099),\\ (0.869, 0.928, 0.987)], [(0.03, 0.05, 0.07), (0.93, 0.95, 0.97)]\rangle\\ \langle[(0.856, 0.937, 0.997), (0.2613, 0.467, 0.647)], [(0.0272, 0.1122, 0.2548),\\ (0.291, 0.442, 0.6225)], [(0.0024, 0.014, 0.035), (0.671, 0.774, 0.884)]\rangle\oplus\\ \langle[(0.56, 0.64, 0.72), (0.28, 0.36, 0.44)], [(0.2, 0.33, 0.46),\\ (0.59, 0.67, 0.75)], [(0.069, 0.201, 0.333), (0.737, 0.799, 0.861)]\rangle\end{matrix}\right\}$$

$$= minimum\left\{\begin{matrix}\langle[(0.722, 0.899, 0.996), (0.885, 0.953, 0.99)], [(0.00009, 0.004, 0.017),\\ (0.357, 0.507, 0.692)], [(0.00009, 0.0005, 0.0013), (0.659, 0.765, 0.8788)]\rangle,\\ \langle[(0.93, 0.977, 0.999), (0.468, 0.659, 0.802)], [(0.005, 0.037, 0.1172),\\ (0.172, 0.296, 0.467)], [(0.0002, 0.003, 0.012), (0.494, 0.618, 0.761)]\rangle\end{matrix}\right\}$$

Using equation (2.1),we have

$$S\left\{\begin{matrix}\langle[(0.722, 0.899, 0.996), (0.885, 0.953, 0.99)], [(0.00009, 0.004, 0.017), \\ (0.357, 0.507, 0.692)], [(0.00009, 0.0005, 0.0013), (0.659, 0.765, 0.8788)]\rangle\end{matrix}\right\}$$

$S(\overline{n}_1) = 0.26339$

$$S\left\{\begin{matrix}\langle[(0.93, 0.977, 0.999), (0.468, 0.659, 0.802)], [(0.005, 0.037, 0.1172), \\ (0.172, 0.296, 0.467)], [(0.0002, 0.003, 0.012), (0.494, 0.618, 0.761)]\rangle\end{matrix}\right\}$$

$S(\overline{n}_2) = 0.2665$

Therefore minimum value $i = 5$, corresponding to label node 7 as

$$\left\{\begin{matrix}\langle[(0.722, 0.899, 0.996), (0.885, 0.953, 0.99)], [(0.00009, 0.004, 0.017), \\ (0.357, 0.507, 0.692)], [(0.00009, 0.0005, 0.0013), (0.659, 0.765, 0.8788)]\rangle, 5\end{matrix}\right\}$$

$$d_3 = \left\{\begin{matrix}\langle[(0.722, 0.899, 0.996), (0.885, 0.953, 0.99)], [(0.00009, 0.004, 0.017), \\ (0.357, 0.507, 0.692)], [(0.00009, 0.0005, 0.0013), (0.659, 0.765, 0.8788)]\rangle\end{matrix}\right\}$$

The labeled node is Airavatesvara Temple and the minimum provided corresponding node is Shri Suryanar Temple.

Minimum Node	Labeled Node	Path Node
SST	AT	⟨[(0.722, 0.899, 0.996), (0.885, 0.953, 0.99)], [(0.00009, 0.004, 0.017), (0.357, 0.507, 0.692)], [(0.00009, 0.0005, 0.0013), (0.659, 0.765, 0.8788)]⟩

Since Airavatesvara Temple is the destination node.We calculate SP to destination node to source node. Since

Labeled Node	Minimum Node
Shri Airavatesvara Temple	Shri Suryanar Temple
Shri Suryanar Temple	Gangai konda cholapuram
Gangai konda cholapuram	Amarasundreashwarar Temple

Therefore, the Chola period built temples Intuitionistic Triangular Neutrosophic Fuzzy Graph Shortest Path is

$$AST \to GKCT \to SST \to SST$$

5. The shortest path on Dijkstra's algorithm

Edge weight suitable algorithm is Dijkstra's algorithm. So here we conclude same type of Shortest Path through Dijkstra's Algorithm.

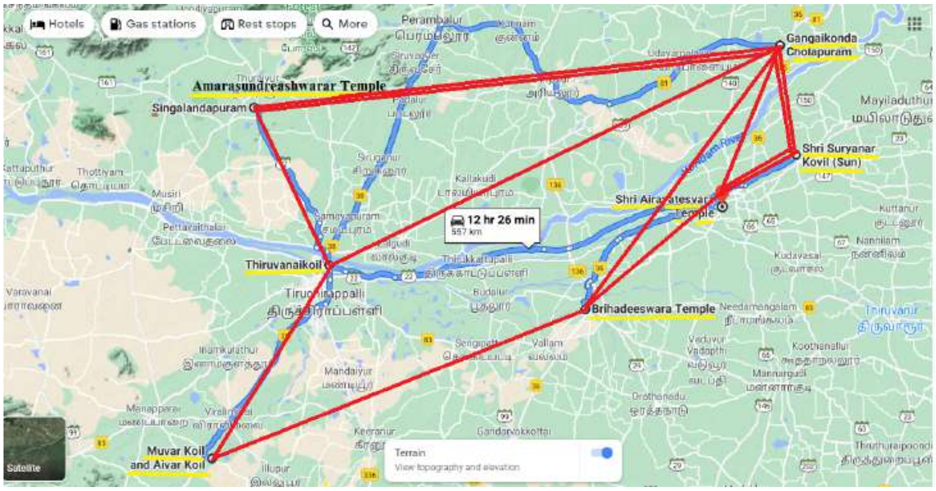


Figure 2: SP from Amarasundareshwarar Temple to Airavateswara Temple

In the above real life application, we clarify another method of Shortest Path Problem using Dijkstra's algorithm. In this Shortest Path Problem, we use direct method of Dijkstra's algorithm and we assume edge weight is Chola period temples km.

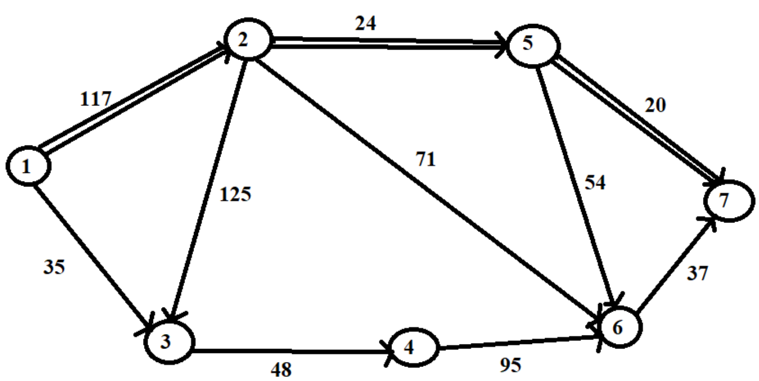


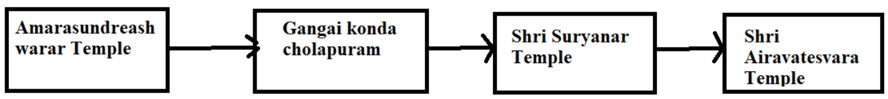
Figure 3: SP for Dijkstra's Algorithm

Here, we verify Chola period buildted temples shortest path through Dijkstra's Algorithm. We have the paths are

$$1 \to 2 \to 5 \to 7$$

Here, the intuitionistic triangular Neutrosophic fuzzy graphs and Dijkstra's Algorithm are same. The shortest path is

$$1 \to 2 \to 5 \to 7$$



DIJKSTRA'S ALGORITHM PYTHON PROGRAM

Python program has been used to verify the result of Dijktra's Algorithm. It can be accessed earily and checked properly.

```
import sys

def to_be_visited():
global visited_and_distance
v = -10

for index in range(number_of_vertices):
if visited_and_distance[index][0] == 0 and
(v < 0 or visited_and_distance[index][1]
<= visited_and_distance[v][1]):
v = index
return v

vertices = [[0,1,1,0,0,0,0],
            [0,0,1,0,1,1,0],
            [0,0,0,1,0,0,0],
            [0,0,0,0,0,1,0],
            [0,0,0,0,0,1,1],
            [0,0,0,0,0,0,1],
            [0,0,0,0,0,0,0]]
edges =  [[0,117,35,0,0,0,0],
            [0,0,125,0,24,71,0],
```

```
            [0,0,0,48,0,0,0],
            [0,0,0,0,0,95,0],
            [0,0,0,0,0,54,20],
            [0,0,0,0,0,0,37],
            [0,0,0,0,0,0,0]]

number_of_vertices = len(vertices[0])

visited_and_distance = [[0, 0]]
for i in range(number_of_vertices-1):
visited_and_distance.append([0, sys.maxsize])

for vertex in range(number_of_vertices):

to_visit = to_be_visited()
for neighbor_index in range(number_of_vertices):
if vertices[to_visit][neighbor_index] == 1 and
visited_and_distance[neighbor_index][0] == 0:
new_distance = visited_and_distance[to_visit][1] +
edges[to_visit][neighbor_index]

if visited_and_distance[neighbor_index][1] > new_distance:
visited_and_distance[neighbor_index][1] = new_distance
visited_and_distance[to_visit][0] = 1

i = 0

for distance in visited_and_distance:
print("The shortest distance of ",chr(ord('a') + i), "
from the source vertex a is:",distance[1])
i = i + 1


Output for the above program
-----------------------------

The shortest distance of  a  from the source vertex a is: 0
The shortest distance of  b  from the source vertex a is: 117
The shortest distance of  c  from the source vertex a is: 35
The shortest distance of  d  from the source vertex a is: 83
The shortest distance of  e  from the source vertex a is: 141
The shortest distance of  f  from the source vertex a is: 178
The shortest distance of  g  from the source vertex a is: 161
```

6. Conclusion

In this article, discovering Shortest Path on Chola period temples using Intuitionistic Triangular Neutrosophic Fuzzy Graph. We use Neutrosophic score function and Triangular signed distance for fuzzification of membership and non-membership function. Intuitionistic Triangular Neutrosophic Fuzzy Number Score function is used to calculate Shortest Path of Intuitionistic Triangular Neutrosophic Fuzzy Graph. A genuine application is given to act as an Intuitionistic Triangular Neutrosophic Fuzzy Graph. Finally most brief way Shortest Path on Chola period buildted temples verified with Dijkstra's algorithm through the long last Python Jupyter Notebook (form) programming.

Futuristic work

In future shortest path problem can be applied by using Kruskal's Algorithm. We can verified by applying transportation Problem and Decision making problem. All weightage fuzzy number will be incorporated to find the shortest route of traffic or human intervention practical problem.

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Invariant approximation property under group passes to extensions with a finite quotient

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Abstract. Analytic properties of invariant approximation property, studies analytic techniques from operator theory that encapsulate geometric properties of a group. also we show that the invariant approximation property passes to finite extensions.

Keywords: uniform roe algebras, invariant approximation property.

1. Introduction

The purpose of this paper is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We provide approximation property of operator algebras associated with discrete groups. There are various notions of finite dimensional approximation properties for C^*- algebras and more generally operator algebras. Some of these (approximation properties) notations will be defined in this paper, the reader is referred to [2], [3], [4], [7], [10], [11], [12], [13], and [15] for these beautiful concepts: Haagerup discovery that the reduced C^*- algebra $\mathbb{F}_n$ has the metric approximation property, Higson and Kasparov's resolution of the Baum-connes conjecture for the Haagerup groups. We study analytic techniques from operator theory that encapsulate geometric properties of a group. The approximation properties of group C^*- algebra are everywhere; it is powerful, important, backbone of countless breakthroughs.

Roe considered the discrete group of the reduced group C^*- algebra of $C_r^*(G)$ is the fixed point algebra $\{Ad\rho(t) : t \in G\}$ acting on the uniform Roe algebra $C_u^*(G)$ [14]. A discrete group G has natural coarse structure which allows us to define the the uniform Roe algebra, $C_u^*(G)$ [14]. We say that the uniform Roe algebra, $C_u^*(G)$, is the C^*- algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. The reduced C^*- algebra $C_r^*(G)$ is naturally contained in $C_u^*(G)$ [14]. According to [Roe] [14], G has the invariant approximation property (IAP) if

$$C_\lambda^*(G) = C_u^*(G)^G.$$

2. Preliminaries

In this section we shall establish the basic definitions and notations for the category of coarse metric spaces. Coarse geometry is the study of the large scale properties of spaces. The notion of large scale is quantified by means of a coarse structure.

Example 2.1 ([14])**.** Let G be a finitely generated group. Then the bounded coarse structure associated to any word metric on G is generated by the diagonals

$$\Delta_g = \{(h, hg) : h \in G\}.$$

We next recall some basic fact about uniform Roe algebra and metric property of a discrete group. Next we recall the following definitions; Let X be a discrete metric space.

Definition 2.2 ([14])**.** We say that discrete metric space X has *bounded geometry* if for all R there exists N in $\mathbb{N}$ such that for all $x \in X$, $|B_R(x)| < N$, where $B(x, r) = \{x \in X : d(y, x) \leq r\}$.

Definition 2.3 ([14])**.** A kernel $\phi : X \times X \longrightarrow \mathbb{C}$:

- is *bounded* if there, exists $M > 0$ such that $|\phi(s, t)| < M$ for all $s, t \in X$
- has *finite propagation* if there exists $R > 0$ such that $\phi(s, t) = 0$ if $d(s, t) > R$.

Let $B(X)$ be a set of bounded finite propagation kernels on $X \times X$. Each such ϕ defines a bounded operator on $\ell^2(X)$ via the usual formula for matrix multiplication

$$\phi * \zeta(s) = \sum_{r \in G} \phi(s, r)\zeta(r) \text{ for } \zeta \in \ell^2(X).$$

We shall denote the finite propagation kernels on X by $A^\infty(X)$.

Definition 2.4 ([14])**.** The uniform Roe algebra of a metric space X is the closure of $A^\infty(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X.

If a discrete group G is equipped with its bounded coarse structure introduced in Example 2.1, then one can associate with its uniform Roe algebra $C_u^*(G)$ by repeating the above. A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra $C_u^*(G)$. A group G can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C_\lambda^*(G)$ or $C_\rho^*(G)$ is a sublagebra of the uniform Roe algebra $C_u^*(G)$ of G.

Hence, any element of $\mathbb{C}[G]$ will give the finite propagation and this assignment extends to an inclusion

$$C_\lambda^*(G) \hookrightarrow C_u^*(G).$$

Next, if the metric on G is left-invariant then

$$C^*_\rho(G) \subset C^*_u(G).$$

Let d_1 be the left-invariant metric on G

$$d_1(x, y) = d_1(gx, gy) \ \forall \ g \in G.$$

Now, we choose a right invariant metric for G so that $C^*_\lambda(G) \hookrightarrow C^*_u(G)$. The right regular representation ρ gives the adjoint action on $C^*_u(G)$ defined by

$$Ad\rho(g)T = \rho(g)T\rho(g)^* = \rho(g)T\rho(g)^{-1},$$

for all $t \in G$, $T \in C^*_u(G)$. Our remarks above show that the elements of $C^*_\lambda(G)$ are invariant with respect to this action and so $C^*_\lambda(G)$ is contained in invariant subalgebra $C^*_u(G)^G$.

Lemma 2.5. *If $T \in C^*_u(G)$ has kernel $A(x, y)$, then $Ad\rho(t)T$ has kernel $A(xt, yt)$*

Proof. We have that:

$$\begin{aligned} (Ad\rho(t)T\zeta)(s) &= \rho(t)(T\rho(t)^*\zeta)(s) \\ &= T\rho(t)^*\zeta(st) \\ &= \sum_{x\in G} A(st, x)(\rho(t)^{-1}\zeta)(x) \\ &= \sum_{x\in G} A(st, x)\zeta(xt^{-1}). \end{aligned}$$

Now, $A(st, x)$ is non-zero whenever $x, y, t \in G$ such that $y = xt^{-1}$,
so $x = yt$ and we have

$$(Ad\rho(t)T\zeta)(s) = \sum_{x\in G} A(st, yt)\zeta(y)$$

Thus, $Ad\rho(t)T$ has kernel $A(st, yt)$. □

In general, if $T \in C^*_u(X)$, then $\forall \ x, y \in G$:

$$\begin{aligned} \langle Ad(\rho(t))T\delta_x, \delta_y\rangle &= \left\langle \rho(t)T\rho(t^{-1})\delta_x, \delta_y \right\rangle \\ &= \left\langle T\rho(t^{-1})\delta_x, \rho(t^{-1})\delta_y \right\rangle \\ &= \langle T\delta_{xt}, \delta_{yt}\rangle . \end{aligned}$$

So, the operator T is $Ad\rho-$ invariant if and only if

$$\forall \ x, y \in X \ \forall \ t \in G \ \ \langle T\delta_{xt}, \delta_{yt}\rangle = \langle T\delta_x, \delta_y\rangle .$$

We now define the invariant approximation: property (IAP).

Definition 2.6 ([14])**.** We say that G has the *invariant approximation property*(*IAP*) if

$$C^*_\lambda(G) = C^*_u(G)^G.$$

3. The IAP passes to extensions with a finite quotient

In this section, we show that the invariant approximation property passes to extensions. For details of extensions see [15] . Consider two groups H and N, and let G be an extension of H by N where $N \cong G/H$. Let

$$1 \longrightarrow H \overset{i}{\hookrightarrow} G \overset{\pi}{\longrightarrow} G/H \longrightarrow 1$$

be an exact sequence.

Let G be the set $G = H \times N$ and $i : H \longrightarrow G$ be given by $i(a) = (a, e)$ (for any $a \in H$), with $\pi : G \longrightarrow N$ given by $\pi(a, \gamma) = \gamma$ (for any $(a, \gamma) \in G$). We choose a set-theoretic cross-section $\sigma : N \longrightarrow G$, $1 \longmapsto 1$ of σ such that $\pi \circ \sigma = Id_{G/H}$. We define

$$f : N \times N \longrightarrow G$$

by

$$f(n_1, n_2) = \sigma(n_1)\sigma(n_2)\sigma(n_1 n_2)^{-1}, \ \ \forall\, n_1,\ n_2 \ \in \ N.$$

Let $\rho(\gamma)$ be the conjugation by $\sigma(\gamma)$ in H:

$$\rho(\gamma)(h) = \sigma(\gamma)h\sigma(\gamma)^{-1}.$$

For $\alpha \in N$,

$$Ad(\alpha) : N \longrightarrow N \text{ and } \gamma \longmapsto \alpha\gamma\alpha^{-1}.$$

Then, the function f and ρ are related as follows [5]:

$$\rho(\beta)\rho(\gamma) = Ad(f(\beta, \gamma))\rho(\beta\gamma), \tag{3.1}$$

and

$$f(\gamma_1, \gamma_2)f(\gamma_1\gamma_2, \gamma_3) = \rho(\gamma_1)f(\gamma_2, \gamma_3)f(\gamma_1, \gamma_2\gamma_3). \tag{3.2}$$

Since

$$Ad(f(\beta, \gamma))\rho(\beta\gamma) = f(\beta, \gamma)f(\beta, \gamma)^{-1}\rho(\beta\gamma) = \rho(\beta)\rho(\gamma)$$

and

$$\begin{aligned}
f(\gamma_1, \gamma_2)f(\gamma_1\gamma_2, \gamma_3) &= \sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_1\gamma_2)^{-1}\sigma(\gamma_1\gamma_2)\sigma(\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \sigma(\gamma_1)1\sigma(\gamma_1)^{-1}\sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_2)^{-1}\sigma(\gamma_1)\sigma(\gamma_2\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_3)^{-1}\sigma(\gamma_2)^{-1}\sigma(\gamma_1)\sigma(\gamma_2\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_2\gamma_3)^{-1}\sigma(\gamma_1)\sigma(\gamma_2\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)f(\gamma_2, \gamma_3)f(\gamma_1, \gamma_2\gamma_3)
\end{aligned}$$

The set group law is given by $(h_1,\gamma_1)(h_2,\gamma_2)=(h_1\rho(\gamma_1)(h_2)f(\gamma_1,\gamma_2),\gamma_1\gamma_2)$. Let G be a group. Now, we choose a set-theoretic section in

$$1\longrightarrow H\overset{i}{\hookrightarrow}G\overset{\pi}{\longrightarrow}G/H\longrightarrow 1$$

is the same as to choose coset representatives in G/H: $r_1,\cdots,r_n$. G/H is a group, but it is not true in general that

$$r_ir_j\in R=\{\text{set of coset representatives}\}$$

since r_ir_j is product in G, there is a new product on R (which is a product on G/H). Let $r_1*r_2\in G$ such that $r_1*r_2=r \leftrightsquigarrow$ the choosen coset representatives of $[r_1r_2]$. And, also

$$(Hr_1)(Hr_2)=Hr_1r_2=H\left(r_1r_2\left(r_1*r_2\right)^{-1}\right)\left(r_1*r_2\right),$$

and $(r_1r_2(r_1*r_2)^{-1})\in H$. So, r_1*r_2 is the product in G/H. To choose coset representatives, we have a set-theoretic identification:

$$G=H\times G/H \text{ (This is called \textbf{Jolissaint product})}.$$

We assume that there is a bijective

$$\phi: G\longrightarrow H\times G/H,$$

$$g=h_gr_g\longmapsto (h_g,r_g).$$

Where ϕ is a group isomorphism if $H\times G/H$ is equipped with the Jolissaint product. Coset representation, $\forall g\in G$, $\exists\ h_g\in H,\ r_g\in G/H$ such that $g=h_gr_g$ and $\forall g'\in G$, $\exists\ h_{g'}\in H,\ r_{g'}\in G/H$ such that $g'=h_{g'}r_{g'}$. Since H is normal subgroup of G, so $Hg\cong gH$. Right G action on $H\times G/H$. Consider

$$\begin{aligned} gg' &= \left(h_gr_g\right)\left(h_{g'}r_{g'}\right)\\ &= h_g\left(r_gh_{g'}r_g^{-1}\right)r_gr_{g'}\\ &= h_g\left(r_gh_{g'}r_g^{-1}\right)r_gr_{g'}\left(r_g*r_{g'}\right)^{-1}\left(r_g*r_{g'}\right), \end{aligned}$$

where $r_g*r_{g'}\in R$ and $h_g(r_gh_{g'}r_g^{-1})\in H$. $r_g*r_{g'}$ and $r_gr_{g'}$ determine the same coset, so $\exists\ s\in\ H$ such that $s(r_g*r_{g'})=r_gr_{g'}$,

$$s=r_gr_{g'}\left(r_g*r_{g'}\right)^{-1}.$$

To show that ϕ is a group homomorphism we compute:

$$\begin{aligned} gg' &\longmapsto \left\{h_g\left(r_gh_{g'}r_g^{-1}\right)r_gr_{g'}\left(r_g*r_{g'}\right)^{-1},\left(r_g*r_{g'}\right)\right\}\\ &= \left(h_g,r_g\right)*\left(h_{g'},r_{g'}\right) \end{aligned}$$

but

$$h_g\left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'}\left(r_g * r_{g'}\right)^{-1} \in H.$$

Since ϕ becomes a group isomorphism $\phi : G \longrightarrow H \times G/H$, when the space on the left is equipped with the product

$$(h_g, r_g)(h_{g'}, r_{g'}) = \left\{h_g\left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'}\left(r_g * r_{g'}\right)^{-1}, r_g * r_{g'}\right\}.$$

Therefore $H \times G/H$ is a group. While H is a subgroup of G, N is not subgroup of G, since, for example if $r_g = r_{g'} = e$, then $(h_g, e)(h_{g'}, e) = (h_g h_{g'}, e)$ or if $h_g = h_{g'} = e$, then

$$(e, r_g)(e, r_{g'}) = \left\{r_g r_{g'}\left(r_g * r_{g'}\right)^{-1}, r_g * r_{g'}\right\}.$$

Next, we consider the left G action on

$$G = G/H \times H \text{ (This is called } \textbf{Jolissaint product}\text{)}$$

We assume that there is a bijective $\phi : G \longrightarrow G/H \times H$. This is a group isomorphism when the right hand side is equipped with the Jolissaint product $g = r_g h_g \longmapsto (r_g, h_g)$. Coset representation, $\forall g \in G\ \exists\ h_g \in H,\ r_g \in G/H$ such that $g = r_g h_g$ and $\forall g' \in G\ \exists\ h_{g'} \in H,\ r_{g'} \in G/H$ such that $g' = r_{g'} h_{g'}$. To show that ϕ is a group homomorphism, we compute:

$$\begin{aligned} gg' &= (r_g h_g)\left(r_{g'} h_{g'}\right) \\ &= r_g r_{g'}\left(r_{g'}^{-1} h_g r_{g'}\right) h_{g'} \\ &= \left(r_g * r_{g'}\right)\left(r_g * r_{g'}\right)^{-1} r_g r_{g'}\left(r_{g'}^{-1} h_g r_{g'}\right) h_{g'}. \end{aligned}$$

We have

$$gg' \longmapsto (r_g, h_g) * \left(r_{g'}, h_{g'}\right).$$

Since ϕ becomes a group isomorphism $\phi : G \longrightarrow G/H \times H$, when the space on the right is equipped with the product

$$(r_g, h_g)(r_{g'}, h_{g'}) = \left\{\left(r_g * r_{g'}\right), \left(r_g * r_{g'}\right)^{-1}(r_g r_{g'})(r_{g'})^{-1} h_g r_{g'} h_{g'}\right\}.$$

Therefore, $G/H \times H$ is a group.

While H is a subgroup of G, G/H is not subgroup of G, since, for example if $r_g = r_g' = e$, then

$$(e, h_g)(e, h_{g'}) = \left\{(e * e), (e * e)^{-1}(ee')(e)^{-1} h_g e h_{g'}\right\} = \left\{e, h_g h_{g'}\right\}$$

or if $h_g = h_{g'} = e$, then

$$\begin{aligned} (r_g, e)(r_{g'}, e) &= \left\{\left(r_g * r_{g'}\right), \left(r_g * r_{g'}\right)^{-1}(r_g r_{g'})(r_{g'})^{-1} r_{g'}\right\} \\ &= \left\{\left(r_g * r_{g'}\right), \left(r_g * r_{g'}\right)^{-1}(r_g r_{g'})\right\}. \end{aligned}$$

G/H is a subgroup when the assignment $[r] \longmapsto r \ \in R \ \subset \ G$ is a group homormophsim, i.e., when $r_g * r_{g'} = r_g r_{g'}$.

Next, we show that the main result of this Chapter:

Theorem 3.1. *Let G be a discrete group. If H is a finite index normal subgroup of G with IAP, and*

$$0 \longrightarrow H \xrightarrow{i} G \longrightarrow G/H \longrightarrow 0,$$

then G has IAP.

Proof. Since $\phi : G \xrightarrow{\cong} G/H \times H$, which is a fact becomes a group isomorphism when the space on the right is equipped with **Jolissaint product** [5]. We want to understand if there is an isomorphism

$$C_u^*(G)^G \cong C_u^*(G/H \times H)^{G/H \times H}.$$

Since

$$\phi : G \xrightarrow{\cong} G/H \times H,$$

we have

$$C_\lambda^*(G) \xrightarrow{\cong} C_\lambda^*(G/H \times H).$$

We need to show that

$$C_u^*(G)^G \cong C_\rho^*(G).$$

The left coset decomposition of G

$$G = \coprod_{r \in R} rH,$$

where R is the set of left coset representatives. This space has a natural right multiplication action by H, as it preserves left cosets. R can be made into a group ($R \subset G$, a subset of G) with the $*-$ product and R is not a subgroup of G. It follows that there is a corresponding action on

$$\ell^2(G) = \bigoplus_{r \in R} \ell^2(rH),$$

where $\ell^2(rH)$ is invariant under $\rho(H)$. That is: For every $r \in R$ is the set of left coset representatives

$$\ell^2(rH) = \overline{\text{span}}\{\delta_{rh} \mid \ r \in R, \ h \in H\},$$

we have $s \in H, \rho(s)\delta_{rh} = \delta_{rhs} \ \in \ell^2(rH)$. On the other hand, the bijection ϕ gives a Hilbert spaces isomorphism $\ell^2(G) = \ell^2(G/H) \otimes \ell^2(H)$. But G/H is finite, so this is just

$$\ell^2(G) = \mathbb{C}^n \otimes \ell^2(H), n = |R| = \bigoplus_{r \in R} \ell^2(H),$$

where $\bigoplus_{r\in R} \ell^2(H)$ is the n copies of $\ell^2(H)$. The isomorphism ϕ works by means of unitary maps

$$V_r : \ell^2(rH) \longrightarrow \ell^2(H),$$
$$\delta_{rh} \longmapsto \delta_h,$$

the inverse map

$$V_r^* : \ell^2(H) \longrightarrow \ell^2(rH),$$
$$\delta_h \longmapsto \delta_{rh},$$

On $\ell^2(G)$ we can define a family of projections $P_s : \ell^2(G) \longrightarrow \ell^2(sH), s \in H$. Using the decomposition

$$G = \coprod_{r\in R} rH.$$

We can represent each function $\zeta \in \ell^2(G)$ as a linear combination $\zeta = \sum_{r\in R} \zeta_r$, where $\zeta_r \in \ell^2(rH)$ (this is understood as a subspace of $\ell^2(G)$ so that ζ_r is a function on $\ell^2(G)$ which vanishes outside rH) $P_s(\zeta) = \zeta_s$ (it seems that this works for infinite G/H as well). Note that P_s commutes with $\rho(h)$, $h \in H$ $s \in R$. So:

$$\rho(h)\zeta(t) = \sum_{r\in R} \zeta_r(th).$$

We have $(P_s\rho(h)\zeta)(t) = \rho(h)\zeta_s(t) = \zeta_s(th) = (\rho(h)P_s\zeta)(t)$. Now, take $T \in C_u^*(G)$. With respect to the decomposition

$$G = \coprod_{r\in R} rH,$$

this can be represented as

$$T = \sum_{r,r'\in R} P_r T P_{r'}$$

where $P_r T P_{r'} : \ell^2(rH) \longrightarrow \ell^2(r'H)$. In other words, T can be represented as matrix

$$\begin{pmatrix} & \vdots & \\ \cdots & P_r T P_{r'} & \cdots \\ & \vdots & \end{pmatrix}.$$

The points is that this decomposition is invariant with respect to the action of $\rho(H)$:

$$\forall\ h, h' \in H \quad P_{r'}\rho(h')T\rho(h)P_r = \rho(h')P_{r'}TP_r\rho(h).$$

Note that in particular $P_e T P_e : C_\rho^*(G) \longrightarrow C_\rho^*(G)$ and is a conditional expectation. Note also that $\forall s \in R$, the unitary operator $V_s : \ell^2(sH) \longrightarrow \ell^2(H)$ commute with $\rho(H)$

$$\rho(h')\delta_{sh} = \delta_{s(hh')} \overset{V_s}{\longmapsto} \delta_{hh'} = \rho(h')V_s\delta_{sh}.$$

We want to understand the right regular representation ρ of H in terms of the bijection $G \xrightarrow{\cong} G/H \times H$ or $G \xrightarrow{\cong} H \times G/H$. If we use left cosets of G, then

$$\phi : G \longrightarrow G/H \times H.$$

Now, we call the isomorphism $\Phi : C_u^*(G) \xrightarrow{\cong} C_u^*(G/H) \otimes C_u^*(H)$ given by

$$\Phi : T = \sum_{r,s \in R} P_r T P_s \longmapsto \sum_{r,s \in R} E_{r,s} \otimes V_r P_r T P_s V_s^*,$$

where

$$V_r : \ell^2(rH) \longrightarrow \ell^2(H)$$

and

$$P_r : \ell^2(G) = \bigoplus_{r \in R} \ell^2(rH) \longrightarrow \ell^2(rH).$$

This commutes with the action of $\rho(H)$. Note that H is a subgroup of $G/H \times H$

$$h \longmapsto (e, h).$$

We have

$$(r, h)(e, h') = ((r * e), (r * e)^{-1}(re)(e)^{-1}heh') = (r, r^{-1}rhh') = (r, hh').$$

So, (e, h') acts trivially on the first factor in $G/H \times H$. Next, we show the following important proposition, which is used for the main result (Theorem 3.1) of this Chapter.

Proposition 3.2. *The isomorphism Φ commutes with the adjoint action $Ad\rho$ of H.*

Proof. $\forall\ h \in H$

$$\begin{aligned}
\Phi(Ad\rho(h)T) &= \Phi(\sum_{r,s \in R} P_r Ad\rho(h) T P_s) \\
&= \sum_{r,s \in R} (E_{r,s} \otimes V_r P_r Ad\rho(h) T P_s V_s^*) \\
&= \sum_{r,s \in R} E_{r,s} \otimes Ad\rho(h)\,(V_r P_r T P_s V_s^*) \\
&= Ad\rho(h) \left(\sum_{r,s \in R} E_{r,s} \otimes V_r P_r T P_s V_s^* \right) \\
&= Ad\rho(h)\Phi(T).
\end{aligned}$$

$\square$

Conclusion 3.3. Since $G/H \times H$ is the right equipped with Jolissaint product [5], taking the induce action of H on both side, we have

$$C_u^*(G)^H \cong C_u^*(G/H \times H)^H \cong C_u^*(G/H)^H \otimes C_u^*(G)^H.$$

So, if H has the IAP:

$$C_u^*(G)^H \cong C_u^*(G/H) \otimes C_\lambda^*(H) = M_n(C_\lambda^*(H)),$$

then, we know

$$C_u^*(G)^G \subseteq C_u^*(G)^H \subseteq M_n(C_\lambda^*(H)).$$

Proposition 3.4. *If $T \in C_u^*(G)$ is $H-$ invariant then $\sum_{r\in R} Ad\rho(r)T$ is a $G-$ invariant .*

Proof. Take $g \in G$, such that $g = r_g h_g$, where $r_g \in R$ and $h_g \in H$. We have

$$\begin{aligned} Ad\rho(g)\left(\sum_{r\in R} Ad\rho(r)T\right) &= \sum_{r\in R} g(rTr^{-1})g^{-1} \\ &= \sum_{r\in R} (r_g h_g)\, rTr^{-1}(r_g h_g)^{-1} \\ &= \sum_{r\in R} r_g h_g r T r^{-1} h_g^{-1} r_g^{-1}. \end{aligned}$$

If we take $h_g r \in G \;\; \exists s \in R,\ h \in H$ such that $h_g r = sh$. Then

$$\begin{aligned} Ad\rho(g)\left(\sum_{r\in R} Ad\rho(r)T\right) &= \sum_{r\in R} r_g h_g r T r^{-1} h_g^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g s h T r^{-1} h^{-1} s^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g s T r^{-1} s^{-1} r_g^{-1}. \end{aligned}$$

We need to claim that $r_g s$ runs through R and $h_g r = r(r^{-1}h_g r)$. So:

$$\begin{aligned} Ad\rho(g)\left(\sum_{r\in R} Ad\rho(r)T\right) &= \sum_{r\in R} r_g h_g r T r^{-1} h_g^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g r(r^{-1}h_g r) T r^{-1} h_g^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g r T (r_g r)^{-1} \\ &= \sum_{r\in R} (r_g * r)(r_g * r)^{-1} r_g r T (r_g r)^{-1} (r_g * r)(r_g * r)^{-1} \\ &= \sum_{r\in R} (r_g * r) T (r_g * r)^{-1} \\ &= \sum_{s\in R} r_s T r_s^{-1}. \quad \square \end{aligned}$$

When we define $C_u^*(G)^G$, we consider the right action of G on $\ell^2(G)$ which induces the $Ad\rho-$ action on $C_u^*(G)$. Take $g \in G$, such that $g = r_g h_g$, where $r_g \in R$ and $h_g \in H$

$$Ad\rho(g)T{=}\rho(r_gh_g)T\rho(r_gh_g)^*{=}\rho(r_g)\rho(h_g)T\rho(h_g)^{-1}\rho(r_g)^{-1}{=}Ad\rho(r_g)(Ad\rho(h_g)T).$$

It seems that when $T \in \left(C_u^*(G)^H\right)^{G/H}$ (which still needs to be defined) then

$$Ad\rho(h_g)T = T, \text{ and } Ad\rho(r_g)\left(Ad\rho(h_g)T\right) = T.$$

So, $Ad\rho(g)T = T$. Consider $C_u^*(G)^H$. Take $r, t \in R$, $T \in C_u^*(G)^H$. We have

$$\begin{aligned} Ad\rho(rt)T &= Ad\left(\rho(r*t)(r*t)^{-1}rt\right)T \\ &= Ad\rho(r*t)\left(Ad(\rho(r*t)^{-1}rt)T\right) \\ &= Ad\rho(r*t)T. \end{aligned}$$

Conclusion 3.5. We seem to have an $R-$ action G/H on $C_u^*(G)^H$. If this is so, this could imply that

$$C_u^*(G)^G \cong \left(C_u^*(G)^H\right)^{G/H}.$$

We define $\left(C_u^*(G)^H\right)^{G/H}$:a possible action of R on $C_u^*(G)^H$. $R \subset G$, so it makes sense to consider $Ad\rho(r)T$, for any $r \in R$, $T \in C_u^*(G)$, where ρ is the right regular representation of G. Since $\rho(r)\rho(s) \neq \rho(r*s)r, s \ \in R$, then for $T \in C_u^*(G)^H$, we have:

$$\begin{aligned} Ad\rho(r)Ad\rho(s)T &= Ad\rho(r)\left(Ad\rho(s)T\right) \\ &= \rho(r)\left(\rho(s)T\rho(s)^{-1}\right)\rho(r)^{-1} \\ &= \rho(rs)T\rho(rs)^{-1} \\ &= \rho(r*s)\rho\left((r*s)^{-1}rs\right)T\rho\left((r*s)^{-1}rs\right)^{-1}\rho(r*s)^{-1} \\ &= Ad\rho(r*s)T. \end{aligned}$$

We obtain the following important proposition, which is used for the main result (Theorem 3.1) of this Chapter.

Proposition 3.6. *The group $(R, *) \cong G/H$ acts on $C_u^*(G)^H$, and the action is induced by the right regular representation ρ of G.*

We need to show that

$$C_u^*(G)^G \cong \left\{C_u^*(G)^H\right\}^{G/H}.$$

If $T \in \left(C_u^*(G)^H\right)^{G/H}$, then $T \in C_u^*(G)^G$. Since for every $g \in G$, such that $g = r_g h_g$ and

$$Ad\rho(r_gh_g)T = Ad\rho(r_g)Ad\rho(h_g)T = T$$

So, $\left(C_u^*(G)^H\right)^{G/H} \subseteq C_u^*(G)^G$. We also have $C_u^*(G)^G \subseteq C_u^*(G)^H$. If $T \in C_u^*(G)^G$ then $T \in C_u^*(G)^H$. Since, for every $g \in G$, $g = r_g h_g$. Then $Ad\rho(h_g)T = T$. We have $Ad\rho(r_g)(Ad\rho(h_g)T) = \rho(r_g)(\rho(h_g T\rho(h_g^{-1})\rho(r_g)^{-1} = \rho(r_g h_g)T\rho(r_g h_g)^{-1} = Ad\rho(g)T$. So, $C_u^*(G)^G \subseteq \left(C_u^*(G)^H\right)^{G/H}$. Which would give

$$C_u^*(G)^G \cong \left(C_u^*(G)^H\right)^{G/H}.$$

Next, we need to show that:

$$\Big((C_u^*(G/H) \otimes C_u^*(H))^H\Big)^{G/H} \cong C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H.$$

We denote by P_i the projection onto $\ell^2(Hi)$;

$$P_i : \ell^2(G) \longrightarrow \ell^2(Hi).$$

For every $r \in R$, there is also a unitary isomorphism $V_i : \ell^2(Hi) \longrightarrow \ell^2(H)$, induced by the map $hi \longmapsto h, \forall h \in H$. We have

$$(P_i\rho(r))(P_i\rho(r))^* = P_i\rho(r)\rho(r)^*P_i^* = P_iP_i^* = P_i$$

and

$$\rho(s)P_i : \ell^2(Hr) \longrightarrow \ell^2(H(r*s)),$$
$$(\rho(s)P_r)^*(\rho(s)P_r) = P_r^*\rho(r)^*\rho(r)P_r = P_r^*P_r = P_r = id_{H_r}$$

we get the unitary isomorphsim $P_i\rho(r)^* : Hs \xrightarrow{\cong} Hi$, $i = s * r^{-1}$. Then

$$\begin{aligned}\rho(r)(P_iV_i^*TV_jP_j)\rho(r)^* :&\ell^2(Hs) \xrightarrow{P_j\rho(r)^*} \ell^2(Hj) \xrightarrow{V_j} \ell^2(H) \xrightarrow{V_i^*} \\ &\ell^2(Hi) \xrightarrow{\rho(r)P_i} \ell^2(H(i*r)).\end{aligned}$$

Thus

$$\rho(r)(P_iV_i^*TV_jP_j)\rho(r)^* : \ell^2(Hs) \longrightarrow \ell^2(H(i*r)).$$

We get $E_{i*r,j*r} = Ad_{\rho_{G/H}}E_{i,j}$. Then $T \otimes E_{i,j} \longmapsto T \otimes E_{i*r,j*r}$. Therefore,

$$\Big((C_u^*(G/H) \otimes C_u^*(H))^H\Big)^{G/H} \cong C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H.$$

We know that the isomorphsim

$$\Phi : C_u^*(G) \xrightarrow{\cong} C_u^*(G/H) \otimes C_u^*(H)$$

is $H-$ equivariant so that

$$C_u^*(G)^H \cong C_u^*(G/H \times H)^H \cong C_u^*(G/H) \otimes C_u^*(H)^H.$$

The isomorphsim uses that H is a subgroup of $G/H \times H$ and acts trivially on G/H. We now need to understand the action $\rho_{G/H\times H}$ on $C_u^*(G/H) \otimes C_u^*(H)$.

We obtain the following: we want to understand the right regular representation ρ of G in terms of the bijection $G \xrightarrow{\cong} G/H \times H$ or $G \xrightarrow{\cong} H \times G/H$, we have

$$\begin{aligned} C_u^*(G)^G &\cong \left(C_u^*(G)^H\right)^{G/H} \\ &\cong \left(C_u^*(G/H) \otimes C_u^*(H)^H\right)^{G/H}. \end{aligned}$$

Taking invariants with respect to G/H.

$$\begin{aligned} \left(C_u^*(G)^H\right)^{G/H} &\cong \left(\left(C_u^*(G/H) \otimes C_u^*(H)\right)^H\right)^{G/H} \\ &\cong C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H. \end{aligned}$$

Since H has IAP. Then

$$C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H = C_u^*(G/H)^{G/H} \otimes C_\lambda^*(H).$$

Since G/H is finite group, every finite group is amenable group. Roe shows that the amenable group has IAP [14]. Thus,

$$\begin{aligned} C_u^*(G)^G &\cong \left(C_u^*(G)^H\right)^{G/H} \\ &\cong C_u^*(G/H)^{G/H} \otimes C_\lambda^*(H)^H \\ &\cong C_\lambda^*(G/H) \otimes C_\lambda^*(H). \end{aligned}$$

Next, we need to show that the following Proposition:

Proposition 3.7. *The left regular representation λ_G on $\ell^2(G)$ is isomorphic to the left regular representation $\lambda_H \otimes \lambda_{G/H}$ on $\ell^2(H) \otimes \ell^2(G/H)$.*

Proof. Let R be the set of right coset representation. We have a bijection

$$G = \coprod_{r \in R} Hr$$

which induces the Hilbert space isomorphism

$$\ell^2(G) = \coprod_{r \in R} \ell^2(Hr).$$

We denote by P_r the projection onto $\ell^2(Hr)$;

$$P_r : \ell^2(G) \longrightarrow \ell^2(Hr).$$

For every $r \in R$ there is also a unitary isomorphism $V_r : \ell^2(Hr) \longrightarrow \ell^2(H)$, induced by the map $hr \longmapsto h, \forall h \in H$. As we have seen before, the coset decomposition of G induces a bijection

$$\phi : G \xrightarrow{\cong} H \times G/H$$

and a Hilbert space isomorphism $\ell^2(G) \longrightarrow \ell^2(H) \otimes \ell^2(G/H)$. This gives a rise to the C^*- algebra isomorphism

$$\Phi : C_u^*(G) \xrightarrow{\cong} C_u^*(H) \otimes C_u^*(G/H)$$

given by

$$T \longmapsto \sum_{r',r\in R} V_{r'} P_{r'} T P_r V_r^* \otimes E_{r'r}.$$

The direct sum decomposition of $\ell^2(G)$ allows one to respect it operators in $C_u^*(G)$ as matrices of size $|R| \times |R|$ whose entries are operators

$$\ell^2(Hr) \longrightarrow \ell^2(Hr'), \text{ for } r',\ r \ \in\ R.$$

This induces an analogous matrix decomposition of element of $C_\lambda^*(G)$, and we shall now use this representation to constrict an isomorphism $\lambda_G \cong \lambda_H \otimes \lambda_{G/H}$.

We have a bijection

$$sHr \cong (sHs^{-1})sr(s*r)^{-1} \cong H(s*r),$$

$$\forall\ s, r \ \in\ R,\ \alpha(s,r) : H \longrightarrow (sHs^{-1})sr(s*r)^{-1} \in H$$

$$h \longmapsto (shs^{-1})sr(s*r)^{-1}.$$

This is a bijection, which induces a unitary isomorphism

$$U_{\alpha(s,r)} : \ell^2(H) \longrightarrow \ell^2(H)$$

given by $(U_{\alpha(s,r)}\xi)(t) = \xi(\alpha(s,r)t)$. We extend it to a map

$$H(s*r) \longrightarrow H(s*r),$$

$$h(s*r) \longmapsto (\alpha(s,r)h)(s*r).$$

We have

$$(\alpha(s,r)H)(s*r) \cong sHr,$$

where $\alpha(s,r)$ is a composition of $ad(s)$ with $\rho(sr(s*r)^{-1})$,

$$ad(s) : H \longrightarrow H$$

is a group isomorphism. And $ad(s)(h) = shs^{-1}$ and $\rho(h')(h) = hh'$. $ad(s)H$ is an isomorphism of H, while $\rho(h')$ commutes with the left action of H.

Let $g = hs \ \in\ G$, where $h \in H$, $s \in R$. When restricted to $\ell^2(Hr)$(by means of projection P_r), $\lambda_G(hs)$ can be explicitly computed as follows : Thanks to isomorphism $\ell^2(G) \longrightarrow \ell^2(H) \otimes \ell^2(G/H)$. We know that the set of linear combinations of functions on G of the form $\eta\gamma$, where $\eta \in \ell^2(H)$ and

$\gamma \in \ell^2(G/H)$ is dense in $\ell^2(G)$. We can therefore assume that $\zeta \in \ell^2(G)$ is of the form $\zeta = \eta\gamma$. Then, for every $t \in H$ $r \in$ R and $\xi \in \ell^2(Hr)$.

$$\begin{aligned}
(\lambda_G(hs)\xi(tr)) &= \xi\left(s^{-1}h^{-1}tr\right)\\
&= \xi\left(s^{-1}(h^{-1}t)ss^{-1}r(s^{-1}*r)^{-1}(s^{-1}*r)\right)\\
&= \xi\left(\alpha(s^{-1},r)(h^{-1}t)(s^{-1}*r)\right)\\
&= \eta\left(\alpha(s^{-1},r)(h^{-1}t)\right)\gamma\left((s^{-1}*r)\right).
\end{aligned}$$

Now, the operator of multiplication on the left by $\alpha(s^{-1},r) \in H$ induces a unitary isomorphism

$$U_{\alpha(s,r)} : \ell^2(H) \longrightarrow \ell^2(H)$$

given by

$$\eta \longmapsto (U_{\alpha(s^{-1},r)}\eta)(t) = \eta(\alpha(s^{-1},r)t).$$

Thus, we have $(\lambda_G(hs)\xi(tr)) = \left(\lambda_H(h)U_{(s^{-1},r)}\eta\right)(t)(\lambda_{G/H}(s)\gamma)$. Next, we need to show that the following Lemma:

Lemma 3.8. *With the above notations* $\lambda_H(h)U_{\alpha(s,r)} = U_{\alpha(s,r)}\lambda_H(ad(s)h)$.

Proof.

$$\begin{aligned}
\left(\lambda_H(h)U_{\alpha(s,r)}\zeta\right)(t) &= U_{\alpha(s,r)}\zeta(h^{-1}t)\\
&= \zeta(\alpha(s,r)(h^{-1}t))\\
&= \zeta(s(h^{-1}t)s^{-1}sr(s*r)^{-1})\\
&= \zeta(s(h^{-1}s^{-1})sts^{-1}sr(s*r)^{-1})\\
&= \zeta(ad(s)(h^{-1})\alpha(s,r)(t))\\
&= U_{\alpha(s,r)}\lambda_H\left((ad(s)h^{-1})^{-1}\zeta\right)(t)\\
&= U_{\alpha(s,r)}\lambda_H\left((ad(s)h)\zeta\right)(t).
\end{aligned}$$

We have $\lambda_H(h)U_{\alpha(s^{-1},r)}\zeta = (U_{\alpha(s^{-1},r)}\lambda_H(ad(s)h))\zeta$. □

Here the Lemma:

Lemma 3.9. *The following diagram commutes:* $r, s \in R$

$$\begin{tikzcd}
\ell^2(G) \arrow[r, "V_{s^{-1}*r}"'] & \ell^2(H)\otimes\ell^2(G/H)\\
\ell^2(g) \arrow[u, "P_{s^{-1}*r}\lambda_G(hs)P_r"] \arrow[r, "V_r"'] & \ell^2(H)\otimes\ell^2(G/H) \arrow[u, "\lambda_H(h)U(s^{-1},r)\otimes\lambda_{G/H}(h)"']
\end{tikzcd}$$

Proof. Since we have $s, r \in R$

$$\lambda_H(h)U_{\alpha(s^{-1},r)}\zeta = (U_\alpha(s^{-1},r)\lambda_H(ad(s)h))\zeta.$$

The following diagram commutes

$$\begin{tikzcd}
\ell^2(H)\otimes\ell^2(G/H) \arrow[r, "V_{s^{-1}*r}"'] & \ell^2(H)\otimes\ell^2(G/H) \arrow[r, "\cong"'] & \ell^2(H)\otimes\ell^2(G/H) \\
\ell^2(H)\otimes\ell^2(G/H) \arrow[r, "V_r"'] \arrow[u, "\lambda_H(h)U(s^{-1},r)\otimes\lambda_{G/H}(h)"'] & \ell^2(H)\otimes\ell^2(G/H) \arrow[r, "\cong"'] \arrow[u, "U(s^{-1},r)\lambda_H(ad(s)h)\otimes\lambda_{G/H}(s)"'] & \ell^2(H)\otimes\ell^2(G/H) \arrow[u, "\lambda_H(ad(s^{-1})h)\otimes\lambda_{G/H}(s)"']
\end{tikzcd}$$

We have proved:

$$\begin{aligned}
P_{s^{-1}*r}\lambda_G(hs)P_r &\cong \lambda_H(h)U(s^{-1},r)\otimes\lambda_{G/H}(h)\\
&\cong U(s^{-1},r)\lambda_H(ad(s)h)\otimes\lambda_{G/H}(h)\\
&\cong \lambda_H(ad(s^{-1})h)\otimes\lambda_{G/H}(s). \quad \square
\end{aligned}$$

On the other hand, next we need to find $(e,s)^{-1}\in H\times G/H$: The inverse of $s\in R\cong G/H$ will be denoted by $\overline{s}$. If (e,s) and $(h,\overline{s})\in H\times G/H$: we have

$$(e,s)*(h,\overline{s})=\left\{(shs^{-1})(s\overline{s})(s*\overline{s})^{-1},(s*\overline{s})\right\}.$$

If $(e,s)^{-1}=(h,\overline{s})$, then

$$\left\{(shs^{-1})(s\overline{s})(s*\overline{s})^{-1},(s*\overline{s})\right\}=(e,e).$$

If $s*\overline{s}=e=\overline{s}*s$, then $\overline{s}=s^{-1}t$, for some $t\in H\Longleftrightarrow s\overline{s}=t$ and

$$(shs^{-1})t=e\Longleftrightarrow t^{-1}=shs^{-1}\Longleftrightarrow t=sh^{-1}s^{-1},$$

thus $\overline{s}=s^{-1}t=h^{-1}s^{-1}\Longleftrightarrow h=(\overline{s}s)^{-1}$. Thus $(e,s)^{-1}=(h,\overline{s})=((\overline{s}s)^{-1},\overline{s})$. If (e,s) and $(h,r)\in H\times G/H$ and $\xi\in\ell^2(H)\otimes\ell^2(G/H)$:

$$\begin{aligned}
\left(\lambda_{H\times G/H}(e,s)\xi\right)(h,r) &= \xi\left((e,s)^{-1}(h',k')\right)\\
&= \xi\left(((\overline{s}s)^{-1},\overline{s})(h,r)\right)\\
&= \xi\left\{(\overline{s}s)^{-1}(\overline{s}h\overline{s}^{-1})(\overline{s}r)(\overline{s}*r)^{-1},(\overline{s}*r)\right\},
\end{aligned}$$

but $(\overline{s}h\overline{s}^{-1})(\overline{s}r)(\overline{s}*r)^{-1}$ is an automorphism of H and $s\in R\longmapsto\overline{s}s\in H$. But, then $\lambda_G(hs)\ell^2(Hr)$ will be isomorphic to $\lambda_H(h)\otimes\lambda_{G/H}(s)$ acting on $\ell^2(H)\otimes\ell^2(G/H)$ via the composition of the map ϕ with the isomorphism. We have the isomorphism $C^*_\lambda(H)\otimes C^*_\lambda(G/H)\cong C^*_\lambda(G)$. $\square$

We already proved $C^*_u(G)^G\cong C^*_\lambda(H)\otimes C^*_\lambda(G/H)$. By using Proposition 3.7, $C^*_u(G)^G\cong C^*_\lambda(H\times G/H)\cong C^*_\lambda(G)$. Therefore, G has IAP. $\square$

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Error estimates of two-grid method for second-order nonlinear hyperbolic equation

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Abstract. In this paper, the full discrete scheme of mixed finite element approximation is introduced for second-order nonlinear hyperbolic equation. In order to deal with the nonlinear mixed-method equations efficiently, a two-grid algorithm is considered. Numerical stability and error estimate are proved on both the coarse grid and fine grid. It is shown that the two-grid method can achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = \mathcal{O}(H^{(2k+1)/(k+1)})$. Some numerical results are provided to confirm the theoretical analysis.

Keywords: nonlinear hyperbolic equation, mixed finite element method, two-grid method, error estimate.

1. Introduction

In this paper, we consider the following nonlinear hyperbolic equation

$$u_{tt} - \nabla \cdot (K(u)\nabla u) = f, \quad (\boldsymbol{x}, t) \in \Omega \times J, \tag{1.1}$$

$$u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times J, \tag{1.2}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),\ u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $J = (0, T]$, $K(u) : \Omega \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ is a symmetric and uniformly positive definite bounded tensor.

Hyperbolic equations can demonstrate many physical processes and phenomena such as vibrations of a membrane, acoustic vibrations of a gas, hydrodynamics, displacement problems in porous media, etc. Lots of numerical methods have been developed for solving these model problems. Such as finite difference

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methods [1, 2], finite element methods [3, 4, 15, 18], mixed finite element methods [5-7] and so on. In this paper, we consider a mixed element method for nonlinear hyperbolic equation in which the coefficient K is nonlinear.

The mixed finite element method (MFEM), as a type of powerful numerical tool for solving differential problems, was extensively used in the analysis of engineering and scientific computation. In the past decades, the theoretical framework and the basic tools for the analysis of the MFEM have been developed. Perhaps the most important property of the MFEM is that it can simultaneously approximate both the scalar (pressure) and vector (flux) functions. The advantage of this approach has attracted many researchers to do research in this field. For example, there are some papers such as [8, 11, 19] on elliptic equations and parabolic equations. There are also some papers such as [5-7] on the MFEM for the linear and semilinear hyperbolic problems.

For the mixed method, the problem (1.1) is often rewritten by introducing a new variable

$$\boldsymbol{z} = -K(u)\nabla u,$$

or equivalently

$$\kappa(u)\boldsymbol{z} = -\nabla u, \tag{1.4}$$

as

$$u_{tt} + \nabla \cdot \boldsymbol{z} = f, \tag{1.5}$$

where $\kappa(u) = K^{-1}(u)$ is a square-integrable, symmetric, uniformly positive-definite tensor defined on Ω, and there exist constants K_*, $K^* > 0$, such that

$$K_*|\boldsymbol{y}|^2 \leq \boldsymbol{y}^T\kappa(u)\boldsymbol{y} \leq K^*|\boldsymbol{y}|^2, \ \boldsymbol{y} \in \mathbb{R}^2. \tag{1.6}$$

As we know, the resulting algebraic system of equations is a large systems of nonlinear equations. Therefore, it is necessary for us to study an effective algorithm for this essential system. We will consider a two-grid method inspired by Xu [9, 10]. The key feature of this method is that it can reduce the complexity of the original problem and save the computational time. Thus, many articles utilize this method to numerically solve differential equations and developed some new numerical techniques based on the idea of two-grid algorithm [11-18]. Now, the two-grid methods have been proved to be efficient discretization techniques for the complicated problems (nonsymmetric indefinite or nonlinear, etc.) of various type.

For the hyperbolic equations, Chen et al. [16] discussed a two-grid method for semilinear problem by using finite volume element method. Later on, they also investigate this method for the nonlinear case [17]. Recently, in [18], the two-grid method was presented to solve the two-dimensional nonlinear hyperbolic equation by the bilinear finite element. In this work, we use a two-grid method based on MFEM to approximate the solution of (1.1). We first solve a nonlinear MFE system on a coarse grid, then we use the known coarse grid solution and a Taylor expansion to get the solution of a linear system on the fine grid. As shown in [9, 10], the coarse mesh can be quite coarse and still maintain a good accuracy approximation. The novelty and major achievement of this paper is that we successfully extend the two-grid method to solve the nonlinear hyperbolic problems by the MFEM. Convergence rate in both time and space is proved.

This paper is organized as follows. In Section 2, we present a two-grid algorithm combined with the fully discrete MFEM for (1.1). In Section 3, we carry out the stability analysis for two-grid method. In Section 4, we deduce the error estimates for both the coarse grid and fine grid. In Section 5, we give some numerical experiments to verify the theoretical results.

Throughout this paper, let C denote a generic positive constant independent of mesh parameters with possibly different values in different contexts. Let $L^p(\Omega)$ for $1 \le p < \infty$ denote the standard Banach space defined on Ω, with norm $||\cdot||_p$. For any nonnegative integer m, let $W^{m,p}(\Omega) = \{\mu \in L^p(\Omega),\ D^\vartheta \mu \in L^p(\Omega),\ |\vartheta| \le m\}$ denote the Sobolev spaces endowed with the norm $||\mu||_{m,p}^p = \sum_{|\vartheta|\le m} ||D^\vartheta \mu||^p_{L^p(\Omega)}$. When $p = 2$, we omit the subscript.

2. The two-grid algorithm based on MFEM

Let $W = L^2(\Omega)$ and $\boldsymbol{V} = H(div;\Omega)$. The weak form for the mixed problem (1.4)-(1.5) is to seek a pair of functions: $(u, \boldsymbol{z}) : (0, T) \to W \times \boldsymbol{V}$ satisfying

$$(u_{tt}, w) + (\nabla \cdot \boldsymbol{z}, w) = (f, w), \quad \forall w \in W, \tag{2.7}$$

$$(\kappa(u)\boldsymbol{z}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, u) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.8}$$

with $u(0) = u_0$ and $u_t(0) = u_1$.

Let $\mathcal{T}_h$ be a quasi-uniform family of finite element partition of Ω into triangles or rectangles with the mesh size h. We take finite-dimensional subspaces $W_h \times \boldsymbol{V}_h \subset W \times \boldsymbol{V}$, using Raviart-Thomas ($RT$) mixed finite element space [19] of index k, where k is fixed nonnegative integer, associated with $\mathcal{T}_h$. The following inclusion holds for the RT_k spaces

$$\nabla \cdot \boldsymbol{v}_h \in W_h, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.9}$$

Let Q_h be the L^2 projection of W onto W_h such that

$$(\alpha, w_h) = (Q_h \alpha, w_h), \quad \forall w_h \in W_h,\ \alpha \in L^2(\Omega). \tag{2.10}$$

Associated with the standard mixed finite element spaces is Fortin projection $\Pi_h : (H^1(\Omega))^2 \to \boldsymbol{V}_h$, such that for $\boldsymbol{q} \in H(div, \Omega)$

$$(\nabla \cdot \Pi_h \boldsymbol{q}, w_h) = (\nabla \cdot \boldsymbol{q}, w_h), \quad \forall w_h \in W_h. \tag{2.11}$$

The following approximation properties hold for the projections Q_h and Π_h (see [19])

$$\|Q_h\alpha\|_{0,q} \le C\|\alpha\|_{0,q}, \quad 2 \le q < \infty, \tag{2.12}$$

$$\|\alpha - Q_h\alpha\|_{0,q} \le C\|\alpha\|_{r,q}h^r, \quad 0 \le r \le k+1, \tag{2.13}$$

$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\|_{0,q} \le C\|\boldsymbol{q}\|_{r,q}h^r, \quad 1/q < r \le k+1, \tag{2.14}$$

$$\|\nabla\cdot(\boldsymbol{q} - \Pi_h\boldsymbol{q})\|_{0,q} \le C\|\nabla\cdot\boldsymbol{q}\|_{r,q}h^r, \quad 0 \le r \le k+1. \tag{2.15}$$

For discretization of time variable, let

$$t^n = n\Delta t,\ n = 0, 1, \cdots, N,$$

where $\Delta t = T/N$ is the step size of time variable.

For any function φ of time, let φ^n denote $\varphi(\cdot, t^n)$. Moreover, we describe some of the notations which will be frequently used in our analysis:

$$\begin{aligned} \varphi^{n+\frac{1}{2}} = \frac{1}{2}(\varphi^{n+1} + \varphi^n), \quad \partial_t\varphi^{n+\frac{1}{2}} = \frac{1}{\Delta t}(\varphi^{n+1} - \varphi^n),\\ \partial_t\varphi^n = \frac{1}{2\Delta t}(\varphi^{n+1} - \varphi^{n-1}),\ \partial_{tt}\varphi^n = \frac{1}{(\Delta t)^2}(\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}), \end{aligned} \tag{2.16}$$

obviously, we have

$$\partial_t\varphi^n = \frac{1}{2}(\partial_t\varphi^{n+\frac{1}{2}} + \partial_t\varphi^{n-\frac{1}{2}}),\ \partial_{tt}\varphi^n = \frac{1}{\Delta t}(\partial_t\varphi^{n+\frac{1}{2}} - \partial_t\varphi^{n-\frac{1}{2}}).$$

The fully discrete scheme of (2.7)-(2.8) is as follows: find $(u_h^{n+1}, \boldsymbol{z}_h^{n+1}) \in W_h \times \boldsymbol{V}_h$ such that

$$(u_h^0, w_h) = (Q_h u_0, w_h), \quad \forall w_h \in W_h, \tag{2.17}$$

$$(\boldsymbol{z}_h^0, \boldsymbol{v}_h) = (\boldsymbol{z}^0, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.18}$$

$$\left(\frac{2}{\Delta t}\partial_t u_h^{\frac{1}{2}}, w_h\right) + (\nabla\cdot\boldsymbol{z}_h^0, w_h) = \left(f^0 + \frac{2}{\Delta t}Q_h u_1, w_h\right), \quad \forall w_h \in W_h, \tag{2.19}$$

$$(\partial_{tt}u_h^n, w_h) + (\nabla\cdot\boldsymbol{z}_h^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h, \tag{2.20}$$

$$(\kappa(u_h^{n+1})\boldsymbol{z}_h^{n+1}, \boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h, u_h^{n+1}) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.21}$$

In order to prove the existence and uniqueness of the discrete problem (2.17)-(2.21), we rewrite (2.20) as

$$\left(\frac{1}{(\Delta t)^2}u_h^{n+1}, w_h\right) = -(\nabla\cdot\boldsymbol{z}_h^n, w_h) + \left(\frac{u_h^n - u_h^{n-1}}{(\Delta t)^2}, w_h\right) + (f^n, w_h), \quad \forall w_h \in W_h. \tag{2.22}$$

Let B_u and B_z be bases of W_h and $\boldsymbol{V}_h$, respectively. So, $u_h = Y\cdot B_u$ and $\boldsymbol{z}_h = X\cdot B_z$, where X and Y are nodal variables. Let $(u_h, w_h) = (Y\cdot B_u, \alpha\cdot B_u) =$

$\alpha \cdot LY$, where L is the matrix associated with the operator whose quadratic form is the L^2 inner products. Similarly, to L, introduce matrices A, B and D,

$$\begin{aligned}&(\kappa(u_h^{n+1})\boldsymbol{z}_h^{n+1}, \boldsymbol{v}_h) = \chi \cdot AX,\\ &-(\nabla \cdot \boldsymbol{v}_h, u_h^{n+1}) = B\chi \cdot Y = B^T Y \cdot \chi,\\ &\left(\frac{1}{(\Delta t)^2} u_h^{n+1}, w_h\right) = DY \cdot \alpha,\end{aligned}$$

where $\boldsymbol{v}_h = \chi \cdot B_z$ and $w_h = \alpha \cdot B_u$. Then, the matrix form of (2.17)-(2.21), relative to the bases B_u and B_z, is

$$\begin{bmatrix} A & B^T \\ \mathbf{0} & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ G \end{bmatrix}. \tag{2.23}$$

Recalling the assumptions on $\kappa(u)$, and noting that A and D are positive definite, as required by [20], there exists a unique solution (X, Y) to the system (2.23). Therefore, we can deduce that there exists a unique solution $(u_h^{n+1}, \boldsymbol{z}_h^{n+1})$ to (2.17)-(2.21).

To speed up the scheme (2.17)-(2.21), we present two-grid algorithm for problem (2.17)-(2.21) based on another mixed finite element space $W_H \times \boldsymbol{V}_H$ ($\subset W_h \times \boldsymbol{V}_h$), having mesh size $h \ll H < 1$. The basic idea in our approach is to solve the original nonlinear problem on a coarse grid $\mathcal{T}_H(\Omega)$, and then solve a corresponding linear problem on the fine grid $\mathcal{T}_h(\Omega)$.

Now, we give the two-grid algorithm which has two steps:

Algorithm 2.1.

Step 1. On the coarse grid $\mathcal{T}_H$, find $(u_H^{n+1}, \boldsymbol{z}_H^{n+1}) \in W_H \times \boldsymbol{V}_H$, solve the following nonlinear system:

$$(u_H^0, w_H) = (Q_H u_0, w_H), \quad \forall w_H \in W_H, \tag{2.24}$$

$$(\boldsymbol{z}_H^0, \boldsymbol{v}_H) = (\boldsymbol{z}^0, \boldsymbol{v}_H), \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{2.25}$$

$$\left(\frac{2}{\Delta t}\partial_t u_H^{\frac{1}{2}}, w_H\right) + (\nabla \cdot \boldsymbol{z}_H^0, w_H) = \left(f^0 + \frac{2}{\Delta t} Q_H u_1, w_H\right), \forall w_H \in W_H, \tag{2.26}$$

$$(\partial_{tt} u_H^n, w_H) + (\nabla \cdot \boldsymbol{z}_H^n, w_H) = (f^n, w_H), \quad \forall w_H \in W_H, \tag{2.27}$$

$$(\kappa(u_H^{n+1})\boldsymbol{z}_H^{n+1}, \boldsymbol{v}_H) - (\nabla \cdot \boldsymbol{v}_H, u_H^{n+1}) = 0, \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{2.28}$$

Step 2. On the fine grid $\mathcal{T}_h$, find $(U_h^{n+1}, \boldsymbol{Z}_h^{n+1}) \in W_h \times \boldsymbol{V}_h$, solve the following linear system:

$$(U_h^0, w_h) = (Q_h u_0, w_h), \quad \forall w_h \in W_h, \tag{2.29}$$

$$(\boldsymbol{Z}_h^0, \boldsymbol{v}_h) = (\boldsymbol{z}^0, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.30}$$

$$\left(\frac{2}{\Delta t}\partial_t U_h^{\frac{1}{2}}, w_h\right) + (\nabla \cdot \boldsymbol{Z}_h^0, w_h) = \left(f^0 + \frac{2}{\Delta t} Q_h u_1, w_h\right), \quad \forall w_h \in W_h, \tag{2.31}$$

$$(\partial_{tt} U_h^n, w_h) + (\nabla \cdot \boldsymbol{Z}_h^n, w_h) = (f^n, w_h), \forall w_h \in W_h, \tag{2.32}$$

$$\begin{aligned}&(\kappa'(u_H^{n+1})\boldsymbol{z}_H^{n+1}(U_h^{n+1} - u_H^{n+1}) + \kappa(u_H^{n+1})\boldsymbol{Z}_h^{n+1}, \boldsymbol{v}_h)\\ &= (\nabla \cdot \boldsymbol{v}_h, U_h^{n+1}), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.\end{aligned} \tag{2.33}$$

3. Stability analysis

In this section, we will carry out the stability analysis for two-grid scheme (2.24)-(2.33). We suppose that $\kappa(u)$ is triple continuously differentiable with bounded derivatives up to the second order on Ω, i.e., there exists M_1, $M_2 > 0$, such that $\|\kappa_u\|_{0,\infty} \leq M_1$, $\|\kappa_{uu}\|_{0,\infty} \leq M_2$. Moreover, we also assume $\|\boldsymbol{z}\|_{0,\infty} \leq M_3$, where $M_3 > 0$. As in [6], we use the "inverse assumption", which states that there exists a constant C_0 independent of $\hbar$, such that

$$\|\nabla \cdot \varphi\| \leq C_0 \hbar^{-1} \|\varphi\|, \tag{3.34}$$

for $\varphi \in W_\hbar$, where $\hbar$ is either h or H depending on whether we work on the fine grid space or coarse grid space.

In order to derive the stability for our two-grid method, we need to obtain a stability result first for the coarse grid system (2.24)-(2.28).

Theorem 3.1. *The scheme defined by* (2.24)-(2.28) *is stable for* $\Delta t < \frac{2H}{C_0}$, *and*

$$\begin{aligned} \|u_H^{N+1}\|^2 + \|\boldsymbol{z}_H^{N+1}\|^2 &\leq C(\|u_H^1\|^2 + \|\boldsymbol{z}_H^1\|^2 + \|\partial_t u_H^{\frac{1}{2}}\|^2 \\ &+ \|\nabla \cdot \boldsymbol{z}_H^0\|^2) + C\Delta t \sum_{n=1}^{N} \max_{1\leq i\leq n} \|f^i\|^2 \end{aligned} \tag{3.35}$$

holds.

Proof. Let

$$\overline{\boldsymbol{z}}_H^0 = \frac{\Delta t}{2} \boldsymbol{z}_H^0, \quad \overline{\boldsymbol{z}}_H^n = \frac{\Delta t}{2} \boldsymbol{z}_H^0 + \Delta t \sum_{i=1}^{n} \boldsymbol{z}_H^i.$$

Summing over time levels and multiplying (2.27) by Δt, we have

$$\begin{aligned} &(\partial_t u_H^{n+\frac{1}{2}} - \partial_t u_H^{\frac{1}{2}}, w_H) + (\nabla \cdot (\overline{\boldsymbol{z}}_H^n - \overline{\boldsymbol{z}}_H^0), w_H) \\ &= \left(\Delta t \sum_{i=1}^{n} f^i, w_H \right), \quad \forall w_H \in W_H. \end{aligned} \tag{3.36}$$

We rewrite (2.28) by noting that $\boldsymbol{z}_H^{n+1} = \partial_t \overline{\boldsymbol{z}}_H^{n+\frac{1}{2}}$, so that

$$(\kappa(u_H^{n+1}) \partial_t \overline{\boldsymbol{z}}_H^{n+\frac{1}{2}}, \boldsymbol{v}_H) - (\nabla \cdot \boldsymbol{v}_H, u_H^{n+1}) = 0, \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{3.37}$$

Let $w_h = u_H^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \overline{\boldsymbol{z}}_H^{n+\frac{1}{2}}$ are the test functions in (3.36) and (3.37), then add those equations to get

$$\begin{aligned} &(u_H^{n+1} - u_H^n, u_H^{n+1} + u_H^n) + (\kappa(u_H^{n+1})(\overline{\boldsymbol{z}}_H^{n+1} - \overline{\boldsymbol{z}}_H^n), \overline{\boldsymbol{z}}_H^{n+1} + \overline{\boldsymbol{z}}_H^n) \\ &+ \Delta t(\nabla \cdot \overline{\boldsymbol{z}}_H^n, u_H^n) - \Delta t(\nabla \cdot \overline{\boldsymbol{z}}_H^{n+1}, u_H^{n+1}) \\ &= 2\Delta t \left\{ \left(\partial_t u_H^{\frac{1}{2}}, u_H^{n+\frac{1}{2}} \right) + \left(\nabla \cdot \overline{\boldsymbol{z}}_H^0, u_H^{n+\frac{1}{2}} \right) + \left(\Delta t \sum_{i=1}^{n} f^i, u_H^{n+\frac{1}{2}} \right) \right\}. \end{aligned} \tag{3.38}$$

Using the Cauchy-Schwarz inequality, the terms on the right-hand side of the previous inequality are bounded as

$$\begin{aligned}&\left(\partial_t u_H^{\frac{1}{2}}, u_H^{n+\frac{1}{2}}\right) + \left(\nabla\cdot\overline{\boldsymbol{z}}_H^0, u_H^{n+\frac{1}{2}}\right) + \left(\Delta t\sum_{i=1}^n f^i, u_H^{n+\frac{1}{2}}\right)\\ &\le C(\|\partial_t u_H^{\frac{1}{2}}\| + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\| + \|\sum_{i=1}^n f^i\|)\|u_H^{n+\frac{1}{2}}\|.\end{aligned}\tag{3.39}$$

In addition, the first two terms in the left-hand side of (3.38) are evaluated as

$$\begin{aligned}&(u_H^{n+1}-u_H^n, u_H^{n+1}+u_H^n) + (\kappa(u_H^{n+1})(\overline{\boldsymbol{z}}_H^{n+1}-\overline{\boldsymbol{z}}_H^n), \overline{\boldsymbol{z}}_H^{n+1}+\overline{\boldsymbol{z}}_H^n)\\ &\ge \|u_H^{n+1}\|^2 - \|u_H^n\|^2 + K_*(\|\overline{\boldsymbol{z}}_H^{n+1}\|^2 - \|\overline{\boldsymbol{z}}_H^n\|^2).\end{aligned}\tag{3.40}$$

Summing (3.38) from $n = 1, \cdots, N$, and using (3.39) and (3.40), we get

$$\begin{aligned}&\|u_H^{N+1}\|^2 - \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 - \|\overline{\boldsymbol{z}}_H^1\|^2 - \Delta t\left[\left(\nabla\cdot\overline{\boldsymbol{z}}_H^{N+1}, u_H^{N+1}) - (\nabla\cdot\overline{\boldsymbol{z}}_H^1, u_H^1\right)\right]\\ &\le C\Delta t\sum_{n=1}^N(\|\partial_t u_H^{\frac{1}{2}}\| + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\| + \|\sum_{i=1}^n f^i\|)\|u_H^{n+\frac{1}{2}}\|.\end{aligned}$$

Employing the Cauchy-Schwarz inequality, the inverse assumption (3.34), and choosing H and Δt such that $\Delta t < \frac{2H}{C_0}$, we obtain

$$\begin{aligned}\Delta t(\nabla\cdot\overline{\boldsymbol{z}}_H^{N+1}, u_H^{N+1}) &\le \Delta t\|\nabla\cdot\overline{\boldsymbol{z}}_H^{N+1}\|\cdot\|u_H^{N+1}\| \le \Delta t C_0 H^{-1}\|\overline{\boldsymbol{z}}_H^{N+1}\|\cdot\|u_H^{N+1}\|\\ &\le \frac{\Delta t C_0}{2H}\left(\|\overline{\boldsymbol{z}}_H^{N+1}\|^2 + \|u_H^{N+1}\|^2\right)\\ &< \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 + \|u_H^{N+1}\|^2.\end{aligned}\tag{3.41}$$

Thus, we have

$$\begin{aligned}&\|u_H^{N+1}\|^2 + \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 \le \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^1\|^2\\ &+ C\Delta t\sum_{n=1}^N(\|\partial_t u_H^{\frac{1}{2}}\| + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\| + \|\sum_{i=1}^n f^i\|)\|u_H^{n+\frac{1}{2}}\|\\ &\le \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^1\|^2 + C\Delta t\sum_{n=1}^N(\|u_H^{n+1}\|^2 + \|\partial_t u_H^{\frac{1}{2}}\|^2 + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\|^2 + \max_{1\le i\le n}\|f^i\|^2).\end{aligned}\tag{3.42}$$

Note that $\Delta t\sum_{n=1}^N \le T$, use Gronwall's lemma to get

$$\begin{aligned}\|u_H^{N+1}\|^2 + \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 \le \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^1\|^2 + C(\|\partial_t u_H^{\frac{1}{2}}\|^2 + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\|^2)\\ + C\Delta t\sum_{n=1}^N\max_{1\le i\le n}\|f^i\|^2.\end{aligned}$$

The desired inequality (3.35) follows from the above inequality, and the proof is completed. □

Following a similar analysis as that carried above for the coarse grid, we can obtain the following stability on the fine grid $\mathcal{T}_h$.

Theorem 3.2. *For the scheme (2.29)-(2.33), we have the following stable inequality*

$$\|U_h^{N+1}\|^2+\|\boldsymbol{Z}_h^{N+1}\|^2 \le C(\|u_H^1\|^2+\|\boldsymbol{z}_H^1\|^2+\|\partial_t u_H^{\frac12}\|^2+\|\nabla\cdot\boldsymbol{z}_H^0\|^2+\|U_h^1\|^2 + \|\boldsymbol{Z}_h^1\|^2+\|\partial_t U_h^{\frac12}\|^2+\|\nabla\cdot\boldsymbol{Z}_h^0\|^2)+C\Delta t\sum_{n=1}^{N}\max_{1\le i\le n}\|f^i\|^2.$$

Proof. Let

$$\overline{\boldsymbol{Z}}_h^0=\frac{\Delta t}{2}\boldsymbol{Z}_h^0,\quad \overline{\boldsymbol{Z}}_h^n=\frac{\Delta t}{2}\boldsymbol{Z}_h^0+\Delta t\sum_{i=1}^{n}\boldsymbol{Z}_h^i.$$

Similarly as in Theorem 3.1, we have (cf. (3.38)):

$$\begin{aligned}&(U_h^{n+1}-U_h^n,U_h^{n+1}+U_h^n)+(\kappa(u_H^{n+1})(\overline{\boldsymbol{Z}}_h^{n+1}-\overline{\boldsymbol{Z}}_h^n),\overline{\boldsymbol{Z}}_h^{n+1}+\overline{\boldsymbol{Z}}_h^n)+\Delta t(\nabla\cdot\overline{\boldsymbol{Z}}_h^n,U_h^n)\\&-\Delta t(\nabla\cdot\overline{\boldsymbol{Z}}_h^{n+1},U_h^{n+1})\\&=2\Delta t\{(\partial_t U_h^{\frac12},U_h^{n+\frac12})+(\nabla\cdot\overline{\boldsymbol{Z}}_h^0,U_h^{n+\frac12})-(\kappa'(u_H^{n+1})\boldsymbol{z}_H^{n+1}(U_h^{n+1}-u_H^{n+1}),\overline{\boldsymbol{Z}}_h^{n+\frac12}\\&+(\Delta t\sum_{i=1}^{n}f^i,U_h^{n+\frac12})\}.\end{aligned}$$

Following a similar analysis as that carried out for (3.42), using the boundedness assumption on $\|\boldsymbol{z}\|_{0,\infty}\le M_3$, we see that

$$\begin{aligned}&\|U_h^{N+1}\|^2+\|\overline{\boldsymbol{Z}}_h^{N+1}\|^2\\&\le\|U_h^1\|^2+\|\overline{\boldsymbol{Z}}_h^1\|^2+2\Delta t\sum_{n=1}^{N}(\|\partial_t U_h^{\frac12}\|+\|\nabla\cdot\overline{\boldsymbol{Z}}_h^0\|+\|\sum_{i=1}^{n}f^i\|)\|U_h^{n+\frac12}\|\\&+C\Delta t\sum_{n=1}^{N}\|\overline{\boldsymbol{z}}_H^{n+1}\|_{0,\infty}(\|U_h^{n+1}\|+\|u_H^{n+1}\|)\|\overline{\boldsymbol{Z}}_h^{n+\frac12}\|\\&\le\|U_h^1\|^2+\|\overline{\boldsymbol{Z}}_h^1\|^2+C\Delta t\sum_{n=1}^{N}(\|U_h^{n+1}\|^2+\|\partial_t U_h^{\frac12}\|^2+\|\nabla\cdot\overline{\boldsymbol{Z}}_h^0\|^2+\|\overline{\boldsymbol{Z}}_h^{n+1}\|^2+\|u_H^{n+1}\|^2)\\&+C\Delta t\sum_{n=1}^{N}\max_{1\le i\le n}\|f^i\|^2.\end{aligned}$$

Noting that $\Delta t\sum_{n=1}^{N}\le T$, and using Gronwall's lemma and (3.35), we derive that

$$\|U_h^{N+1}\|^2+\|\overline{\boldsymbol{Z}}_h^{N+1}\|^2\le\|U_h^1\|^2+\|\overline{\boldsymbol{Z}}_h^1\|^2+C(\|\partial_t U_h^{\frac12}\|^2+\|\nabla\cdot\overline{\boldsymbol{Z}}_h^0\|^2+\|u_H^1\|^2+\|\boldsymbol{z}_H^1\|^2 +\|\partial_t u_H^{\frac12}\|^2+\|\nabla\cdot\boldsymbol{z}_H^0\|^2)+C\Delta t\sum_{n=1}^{N}\max_{1\le i\le n}\|f^i\|^2.$$

Thus, the proof of this theorem is completed. □

4. Error analysis based on two-grid algorithm

In this section, we will prove the optimal a priori error estimate for schemes on both coarse and fine grids. As in [21], we shall use the following result

$$\|\varphi\|_{0,\infty} \leq C\hbar^{-1}\|\varphi\|. \tag{4.43}$$

The time-space norms $\|\cdot\|_{l^\infty(L^2)}$ and $\|\cdot\|_{L^p(L^2)}$ are defined as

$$\|\varphi\|_{l^\infty(L^2)} = \|\varphi\|_{l^\infty(0,T;L^2(\Omega))} = \max_{1\leq n\leq N} ||\varphi^n||_{L^2(\Omega)},$$

$$\|\varphi\|_{L^p(L^2)} = \|\varphi\|_{L^p(0,T;L^2(\Omega))} = \left(\int_0^T ||\varphi||^2_{L^2(\Omega)}\right)^{\frac{1}{p}},$$

in the case $1 \leq p < \infty$, and in the case $p = \infty$, the integral is replaced by the essential supremum.

In order to derive the error estimates for our two-grid method, we need to obtain an error estimate for the coarse grid system (2.24)-(2.28).

Theorem 4.1. *Define $(u_H^n, \boldsymbol{z}_H^n) \in W_H \times \boldsymbol{V}_H$ by (2.24)-(2.28). If the time step satisfies $\Delta t < \frac{2H}{C_0}$, then there exists a positive constant C such that*

$$||u - u_H||_{l^\infty(L^2)} + ||\boldsymbol{z} - \boldsymbol{z}_H||_{l^\infty(L^2)} \leq C((\Delta t)^2 + H^{k+1}), \tag{4.44}$$

where k is associated with the degree of the finite element polynomial.

Proof. Set $\xi^n = u_H^n - Q_H u^n$, $\eta^n = \boldsymbol{z}_H^n - \Pi_H \boldsymbol{z}^n$, $\zeta^n = u^n - Q_H u^n$ and $\delta^n = \boldsymbol{z}^n - \Pi_H \boldsymbol{z}^n$. Subtracting (2.7) from (2.27), (2.8) from (2.28), respectively, we obtain the error equations

$$(\partial_{tt}\xi^n, w_H) + (\nabla\cdot\eta^n, w_H) = (\partial_{tt}\zeta^n, w_H) + (u_{tt}^n - \partial_{tt}u^n, w_H), \forall w_H \in W_H, \tag{4.45}$$

$$(\kappa(u_H^{n+1})\eta^{n+1}, \boldsymbol{v}_H) - (\nabla\cdot\boldsymbol{v}_H, \xi^{n+1}) = (I, \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.46}$$

where

$$\begin{aligned} I =& (\kappa(u^{n+1}) - \kappa(u_H^{n+1}))\boldsymbol{z}^{n+1} - (\kappa(u^{n+1}) - \kappa(u_H^{n+1}))(\boldsymbol{z}^{n+1} - \Pi_H \boldsymbol{z}^{n+1}) \\ &+ \kappa(u^{n+1})(\boldsymbol{z}^{n+1} - \Pi_H \boldsymbol{z}^{n+1}) = \sum_{i=1}^{3} I_i. \end{aligned}$$

Using (2.17) in (4.45) yields

$$\begin{aligned} &\left(\frac{\partial_t\xi^{n+\frac{1}{2}} - \partial_t\xi^{n-\frac{1}{2}}}{\Delta t}, w_H\right) + (\nabla\cdot\eta^n, w_H) \\ =& \left(\frac{\partial_t\zeta^{n+\frac{1}{2}} - \partial_t\zeta^{n-\frac{1}{2}}}{\Delta t}, w_H\right) + (\beta_1^n, w_H), \end{aligned} \tag{4.47}$$

for any $w_H \in W_H$, where

$$\beta_1^n = u_{tt}^n - \partial_{tt} u^n = \frac{1}{6(\Delta t)^2} \int_{-\Delta t}^{\Delta t} (|t| - \Delta t)^3 \frac{\partial^4 u}{\partial t^4}(t^n + t) dt.$$

We introduce

$$\phi^0 = \frac{\Delta t}{2}\eta^0, \quad \phi^n = \frac{\Delta t}{2}\eta^0 + \Delta t \sum_{i=1}^{n} \eta^i.$$

Summing over time levels and multiplying both sides of (4.47) by Δt, we find that

$$\begin{aligned}(4.48)\quad &(\partial_t \xi^{n+\frac{1}{2}} - \partial_t \xi^{\frac{1}{2}}, w_H) + (\nabla \cdot (\phi^n - \phi^0), w_H) \\ &= (\partial_t \zeta^{n+\frac{1}{2}} - \partial_t \zeta^{\frac{1}{2}}, w_H) + \left(\Delta t \sum_{i=1}^{n} \beta_1^i, w_H\right), \quad \forall w_H \in W_H,\end{aligned}$$

where $\Delta t \sum\limits_{i=1}^{n} \eta^i = \phi^n - \phi^0$. For $t = 0$, by (2.7), we have

$$(4.49)\qquad (u_{tt}^0, w_H) + (\nabla \cdot \boldsymbol{z}^0, w_H) = (f^0, w_H), \quad \forall w_H \in W_H.$$

It is simple to see

$$\begin{aligned}(4.50)\quad \frac{1}{2\Delta t} \int_0^{\Delta t} (\Delta t - t)^2 \frac{\partial^3 u}{\partial t^3}(t) dt &= -\frac{\Delta t}{2} u_{tt}^0 + \frac{1}{\Delta t} \int_0^{\Delta t} (\Delta t - t) \frac{\partial^2 u}{\partial t^2}(t) dt \\ &= -\frac{\Delta t}{2} u_{tt}^0 - u_t^0 - \frac{1}{\Delta t} \int_0^{\Delta t} \frac{\partial u}{\partial t}(t) dt \\ &= -\frac{\Delta t}{2} u_{tt}^0 - u_t^0 - \frac{1}{\Delta t}(u^1 - u^0) \\ &= -\frac{\Delta t}{2} u_{tt}^0 - u_1 - \partial_t u^{\frac{1}{2}}.\end{aligned}$$

Using the projection operators of Q_H and Π_H, (2.11), (4.49) and (4.50), (2.26) can be transformed into the following:

$$\begin{aligned}&(\partial_t \xi^{\frac{1}{2}}, w_H) + \frac{\Delta t}{2}(\nabla \cdot \eta^0, w_H) \\ &= -(\partial_t Q_H u^{\frac{1}{2}}, w_H) - \frac{\Delta t}{2}(\nabla \cdot \Pi_H z^0, w_H) + \left(\frac{\Delta t}{2} f^0 + Q_H u_1, w_H\right) \\ (4.51)\quad &= -(\partial_t Q_H u^{\frac{1}{2}}, w_H) + \left(\frac{\Delta t}{2} u_{tt}^0, w_H\right) + (Q_H u_1, w_H) \\ &= (\partial_t \zeta^{\frac{1}{2}}, w_H) + (Q_H u_1 - u_1, w_H) + \left(\frac{\Delta t}{2} u_{tt}^0 + u_1 + \partial_t u^{\frac{1}{2}}, w_H\right) \\ &= (\partial_t \zeta^{\frac{1}{2}}, w_H) + (Q_H u_1 - u_1, w_H) - \frac{1}{2\Delta t} \int_0^{\Delta t} (\Delta t - t)^2 \left(\frac{\partial^3 u}{\partial t^3}, w_H\right) dt, \\ &\forall w_H \in W_H.\end{aligned}$$

Thus, it follows from (4.48) and (4.51) that

$$(\partial_t \xi^{n+\frac{1}{2}}, w_H) + (\nabla \cdot \phi^n, w_H) = (\partial_t \zeta^{n+\frac{1}{2}}, w_H) + (\beta_2^n, w_H), \quad \forall w_H \in W_H, \tag{4.52}$$

where

$$\beta_2^n = Q_h u_1 - u_1 + \Delta t \sum_{i=1}^{n} \beta_1^i - \frac{1}{2\Delta t} \int_0^{\Delta t} (\Delta t - t)^2 \frac{\partial^3 u}{\partial t^3}(t) dt.$$

Noting that $\eta^{n+1} = \partial_t \phi^{n+\frac{1}{2}}$, we rewrite (4.46) as follows:

$$(\kappa(u_H^{n+1}) \partial_t \phi^{n+\frac{1}{2}}, \boldsymbol{v}_H) - (\nabla \cdot \boldsymbol{v}_H, \xi^{n+1}) = (I, \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{4.53}$$

Choosing the test functions$w_H = \xi^{n+\frac{1}{2}}$ and $\boldsymbol{v}_H = \phi^{n+\frac{1}{2}}$ in (4.52) and (4.53), respectively. Then, multiplying the two resulting equations by $2\Delta t$, we have

$$\begin{aligned} &(\xi^{n+1} - \xi^n, \xi^{n+1} + \xi^n) + \Delta t (\nabla \cdot \phi^n, \xi^{n+1} + \xi^n) \\ &= 2\Delta t (\partial_t \zeta^{n+\frac{1}{2}} + \beta_2^n, \xi^{n+\frac{1}{2}}), \end{aligned} \tag{4.54}$$

$$\begin{aligned} &(\kappa(u_H^{n+1})(\phi^{n+1} - \phi^n), \phi^{n+1} + \phi^n) - \Delta t (\nabla \cdot (\phi^{n+1} + \phi^n), \xi^{n+1}) \\ &= 2\Delta t (I, \phi^{n+\frac{1}{2}}). \end{aligned} \tag{4.55}$$

Combine (4.54) and (4.55) to obtain

$$\begin{aligned} &\|\xi^{n+1}\|^2 - \|\xi^n\|^2 + (\kappa(u_H^{n+1})(\phi^{n+1} - \phi^n), \phi^{n+1} + \phi^n) + \Delta t (\nabla \cdot \phi^n, \xi^n) \\ &- \Delta t (\nabla \cdot \phi^{n+1}, \xi^{n+1}) \\ &= 2\Delta t (\partial_t \zeta^{n+\frac{1}{2}} + \beta_2^n, \xi^{n+\frac{1}{2}}) + 2\Delta t (I, \phi^{n+\frac{1}{2}}). \end{aligned} \tag{4.56}$$

Using (1.6), the third term on the left-hand side of (4.56) can be bounded as

$$(\kappa(u_H^{n+1})(\phi^{n+1} - \phi^n), \phi^{n+1} + \phi^n) \geq K_* (\|\phi^{n+1}\|^2 - \|\phi^n\|^2). \tag{4.57}$$

Next, we estimate the right-hand terms of (4.56). For the first term, using the Cauchy-Schwarz inequality, we have the following estimation

$$(\partial_t \zeta^{n+\frac{1}{2}} + \beta_2^n, \xi^{n+\frac{1}{2}}) \leq (\|\partial_t \zeta^{n+\frac{1}{2}}\| + \|\beta_2^n\|) \|\xi^{n+\frac{1}{2}}\|. \tag{4.58}$$

For the second term, by the assumptions on $\kappa(u)$ and $\boldsymbol{z}$, the inverse inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |(I_1, \phi^{n+\frac{1}{2}})| &= |((\kappa(u^{n+1}) - \kappa(Q_H u^{n+1}) + \kappa(Q_H u^{n+1}) - \kappa(u_H^{n+1}))\boldsymbol{z}^{n+1}, \phi^{n+\frac{1}{2}})| \\ &\leq C(\|\xi^{n+1}\| + \|\zeta^{n+1}\|)\|\phi^{n+\frac{1}{2}}\|, \\ |(I_2, \phi^{n+\frac{1}{2}})| &= |((\kappa(u^{n+1}) - \kappa(Q_H u^{n+1}) + \kappa(Q_H u^{n+1}) \\ &\quad - \kappa(u_H^{n+1}))(\boldsymbol{z}^{n+1} - \Pi_H \boldsymbol{z}^{n+1}), \phi^{n+\frac{1}{2}})| \\ &\leq C(\|\xi^{n+1}\|_{0,\infty} + \|\zeta^{n+1}\|_{0,\infty})\|\delta^{n+1}\| \cdot \|\phi^{n+\frac{1}{2}}\| \\ &\leq CH^{-1}(\|\xi^{n+1}\| + \|\zeta^{n+1}\|)\|\delta^{n+1}\| \cdot \|\phi^{n+\frac{1}{2}}\|, \\ |(I_3, \phi^{n+\frac{1}{2}})| &= |(\kappa(u^{n+1})(\boldsymbol{z}^{n+1} - \Pi_H \boldsymbol{z}^{n+1}), \phi^{n+\frac{1}{2}})| \\ &\leq C\|\delta^{n+1}\| \cdot \|\phi^{n+\frac{1}{2}}\|. \end{aligned}$$

Hence, by (4.43), we conclude that

$$|(I,\phi^{n+\frac{1}{2}})| \le C\left[\|\zeta^{n+1}\| + \|\xi^{n+1}\| + H^{-1}(\|\xi^{n+1}\| + \|\zeta^{n+1}\|)\|\delta^{n+1}\| + \|\delta^{n+1}\|\right]\|\phi^{n+\frac{1}{2}}\|. \tag{4.59}$$

Summing (4.56) over time levels, and using (4.57)-(4.59), we derive

$$\begin{aligned}&\|\xi^{n+1}\|^2-\|\xi^0\|^2+\|\phi^{n+1}\|^2-\|\phi^0\|^2-\Delta t\left[\left(\nabla\cdot\phi^{n+1},\xi^{n+1}\right)-\left(\nabla\cdot\phi^0,\xi^0\right)\right]\\ &\le 2\Delta t\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\| + \|\beta_2^i\|)\|\xi^{i+\frac{1}{2}}\| + C\Delta t\sum_{i=0}^{n}[\|\zeta^{i+1}\| + \|\xi^{i+1}\|\\ &+ H^{-1}(\|\xi^{i+1}\| + \|\zeta^{i+1}\|)\|\delta^{i+1}\| + \|\delta^{i+1}\|]\|\phi^{i+\frac{1}{2}}\|.\end{aligned} \tag{4.60}$$

After imposing the initial conditions (2.24) and (2.25) in (4.60), we have

$$\begin{aligned}&\|\xi^{n+1}\|^2+\|\phi^{n+1}\|^2-\Delta t\left(\nabla\cdot\phi^{n+1},\xi^{n+1}\right)\\ &\le 2\Delta t\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\| + \|\beta_2^i\|)\|\xi^{i+\frac{1}{2}}\| + C\Delta t\sum_{i=0}^{n}[\|\zeta^{i+1}\| + \|\xi^{i+1}\|\\ &+ H^{-1}(\|\xi^{i+1}\| + \|\zeta^{i+1}\|)\|\delta^{i+1}\| + \|\delta^{i+1}\|]\|\phi^{i+\frac{1}{2}}\|.\end{aligned}$$

Similar to (3.41), we have

$$\begin{aligned}\Delta t(\nabla\cdot\phi^{n+1},\xi^{n+1}) &\le \Delta t\|\nabla\cdot\phi^{n+1}\|\cdot\|\xi^{n+1}\| \le \Delta t C_0 H^{-1}\|\phi^{n+1}\|\cdot\|\xi^{n+1}\|\\ &\le \frac{\Delta t C_0}{2H}\left(\|\phi^{n+1}\|^2+\|\xi^{n+1}\|^2\right)\\ &< \|\phi^{n+1}\|^2+\|\xi^{n+1}\|^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\|\xi^{n+1}\|^2+\|\phi^{n+1}\|^2 \le \Delta t\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\|\\ &+ \|\beta_2^i\|)\|\xi^{i+\frac{1}{2}}\| + C\Delta t\sum_{i=0}^{n}[\|\zeta^{i+1}\| + \|\xi^{i+1}\|\\ &+ H^{-1}(\|\xi^{i+1}\| + \|\zeta^{i+1}\|)\|\delta^{i+1}\|\\ &+ \|\delta^{i+1}\|]\|\phi^{i+\frac{1}{2}}\| \le C\Delta t\|\xi\|_{l^\infty(L^2)}\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\| + \|\beta_2^i\|)\\ &+ C\Delta t\|\phi\|_{l^\infty(L^2)}\sum_{i=0}^{n}[\|\zeta^{i+1}\| + \|\xi^{i+1}\|\end{aligned}$$

$$
\begin{aligned}
(4.61)\qquad & + H^{-1}(\|\xi^{i+1}\| + \|\zeta^{i+1}\|)\|\delta^{i+1}\| + \|\delta^{i+1}\|] \le \frac{1}{4}\|\xi\|^2_{l^\infty(L^2)} \\
& + C\left(\Delta t\sum_{i=0}^{n}\|\partial_t\zeta^{i+\frac{1}{2}}\|\right)^2 + C\left(\Delta t\sum_{i=0}^{n}\|\beta_2^i\|\right)^2 + \frac{1}{4}\|\phi\|^2_{l^\infty(L^2)} \\
& + C\left(\Delta t\sum_{i=0}^{n}\|\xi^i\|\right)^2 + C\left(\Delta t\sum_{i=0}^{n}\|\delta^i\|\right)^2 + C\left(\Delta t\sum_{i=0}^{n}\|\zeta^i\|\right)^2,
\end{aligned}
$$

since $\|\xi^{i+\frac{1}{2}}\| \le \|\xi\|_{l^\infty(L^2)}$ and $\|\phi^{i+\frac{1}{2}}\| \le \|\phi\|_{l^\infty(L^2)}$. Taking the supremum on n on the left-hand side of (4.61), we have

$$
\begin{aligned}
& \|\xi\|^2_{l^\infty(L^2)} + \|\phi\|^2_{l^\infty(L^2)} \\
(4.62)\qquad & \le C\left(\Delta t\sum_{i=0}^{n}\|\partial_t\zeta^{i+\frac{1}{2}}\|\right)^2 + C\left(\Delta t\sum_{i=0}^{n}\|\beta_2^i\|\right)^2 + C\left(\Delta t\sum_{i=0}^{n}\|\zeta^i\|\right)^2 \\
& + C\left(\Delta t\sum_{i=0}^{n}\|\delta^i\|\right)^2 + C\left(\Delta t\sum_{i=0}^{n}\|\xi^i\|\right)^2.
\end{aligned}
$$

In the following, we analyse the right-hand side of (4.62). A direct bound shows that

$$
(4.63)\qquad \Delta t\sum_{i=0}^{n}\|\partial_t\zeta^{i+\frac{1}{2}}\| \le C\left(H^{k+1}\|u\|_{L^\infty(H^{k+1}(\Omega))} + (\Delta t)^2\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^1(L^2)}\right).
$$

By (2.13), we have

$$
\begin{aligned}
\|\beta_2^i\| &\le \Delta t\sum_{i=1}^{n}\|\beta_1^i\| + \|Q_H u_1 - u_1\| + \left\|\frac{1}{2\Delta t}\int_0^{\Delta t}(\Delta t - t)^3\frac{\partial^3 u}{\partial t^3}(t)dt\right\| \\
&\le C(\Delta t)^2\left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^\infty(L^2)} + CH^{k+1} + C(\Delta t)^2\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^\infty(L^2)} \\
&\le C(H^{k+1} + (\Delta t)^2),
\end{aligned}
$$

and hence

$$
(4.64)\qquad \Delta t\sum_{i=0}^{n}\|\beta_2^i\| \le C\|\beta_2\|_{l^\infty(L^2)} \le C(H^{k+1} + (\Delta t)^2).
$$

Using (2.13), (2.14) (4.63) and (4.64) in (4.62), and applying discrete Gronwall's inequality, we know that for Δt and H sufficiently small

$$
(4.65)\qquad \|\xi\|^2_{l^\infty(L^2)} + \|\phi\|^2_{l^\infty(L^2)} \le C((\Delta t)^4 + H^{2k+2}).
$$

Finally, by (2.13), (2.14), (4.65) and the triangle inequality, we can derive (4.44). □

Now, we can prove the following theorem for the solution of the fine grid.

Theorem 4.2. *Let* $(U_h^n, \boldsymbol{Z}_h^n) \in W_h \times \boldsymbol{V}_h$ *be the solution of the two-grid algorithm of step 2 for solving the MFE scheme* (2.29)-(2.33)*. If* $\Delta t < \frac{2h}{C_0}$*, then there is a positive constant* C *such that*

$$||u - U_h||_{l^\infty(L^2)} + ||\boldsymbol{z} - \boldsymbol{Z}_h||_{l^\infty(L^2)} \le C((\Delta t)^2 + h^{k+1} + H^{2k+1}), \tag{4.66}$$

where k *is associated with the degree of the finite element polynomial.*

Proof. Set $\rho^n = U_h^n - Q_h u^n$ and $\gamma^n = \boldsymbol{Z}_h^n - \Pi_h \boldsymbol{z}^n$. Let us first note the following error equations from (2.7)-(2.8) and (2.32)-(2.33),

$$(\partial_{tt}\rho^n, w_h) + (\nabla \cdot \gamma^n, w_h) = (\partial_{tt}\zeta^n, w_h) + (\beta_1^n, w_h),\ \forall w_h \in W_h, \tag{4.67}$$

$$(E, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, \rho^{n+1}) = 0,\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.68}$$

where β_1^n is defined by (4.47),

$$\begin{aligned} E =& \kappa'(u_H^{n+1})\boldsymbol{z}_H^{n+1}(U_h^{n+1} - u_H^{n+1}) + \kappa(u_H^{n+1})\boldsymbol{Z}_h^{n+1} - \kappa(u^{n+1})\boldsymbol{z}^{n+1} \\ &+ \kappa(u^{n+1})\Pi_h \boldsymbol{z}^{n+1} - \kappa(u^{n+1})\Pi_h \boldsymbol{z}^{n+1}, \end{aligned}$$

applying the Taylor expansions to $\kappa(u^{n+1})$ at u_H^{n+1}, i.e.

$$\kappa(u^{n+1}) = \kappa(u_H^{n+1}) + \kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1}) + \frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2,$$

where $\kappa''(u^*)$ means $\kappa''(u)$ evaluated at a point u^* between u^{n+1} and u_H^{n+1}. Then, we have

$$\begin{aligned} E =& \kappa(u^{n+1})(\Pi_h \boldsymbol{z}^{n+1} - \boldsymbol{z}^{n+1}) - \kappa(u_H^{n+1})(\Pi_h \boldsymbol{z}^{n+1} - \boldsymbol{Z}_h^{n+1}) \\ &+ \kappa'(u_H^{n+1})(U_h^{n+1} - Q_h u^{n+1} + Q_h u^{n+1} - u^{n+1})\boldsymbol{z}_H^{n+1} \\ &+ \kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1}) \\ &- \frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\Pi_h \boldsymbol{z}^{n+1} - \boldsymbol{z}^{n+1} + \boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1}) \\ &- \frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2 \boldsymbol{z}_H^{n+1}. \end{aligned} \tag{4.69}$$

By (4.68) and (4.69), we get

$$(\kappa(u_H^{n+1})\gamma^{n+1}, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, \rho^{n+1}) = (F, \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

where

$$\begin{aligned} (F, \boldsymbol{v}_h) =& (\kappa(u^{n+1})\delta^{n+1}, \boldsymbol{v}_h) - (\kappa'(u_H^{n+1})(\rho^{n+1} - \zeta^{n+1})\boldsymbol{z}_H^{n+1}, \boldsymbol{v}_h) \\ &- (\kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1}), \boldsymbol{v}_h) \\ &+ (\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1} - \delta^{n+1}), \boldsymbol{v}_h) \\ &+ (\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2 \boldsymbol{z}_H^{n+1}, \boldsymbol{v}_h) = \sum_{i=1}^{5} T_i. \end{aligned}$$

Let us define

$$\psi^0 = \frac{\Delta t}{2}\gamma^0, \quad \psi^n = \frac{\Delta t}{2}\gamma^0 + \Delta t \sum_{i=1}^{n} \gamma^i.$$

By the center difference operator $\partial_{tt}\varphi^n = \frac{1}{\Delta t}(\partial_t \varphi^{n+\frac{1}{2}} - \partial_t \varphi^{n-\frac{1}{2}})$, we obtain

$$\begin{aligned}
&\left(\frac{\partial_t \rho^{n+\frac{1}{2}} - \partial_t \rho^{n-\frac{1}{2}}}{\Delta t}, w_h\right) + (\nabla \cdot \gamma^n, w_h) \\
&= \left(\frac{\partial_t \zeta^{n+\frac{1}{2}} - \partial_t \zeta^{n-\frac{1}{2}}}{\Delta t}, w_h\right) + (\beta_1^n, w_h), \ \forall w_h \in W_h.
\end{aligned} \tag{4.70}$$

Summing over time levels of (4.70) and multiplying through by Δt, we have

$$\begin{aligned}
(\partial_t \rho^{n+\frac{1}{2}} - \partial_t \rho^{\frac{1}{2}}, w_h) + (\nabla \cdot (\psi^n - \psi^0), w_h) = (\partial_t \zeta^{n+\frac{1}{2}} - \partial_t \zeta^{\frac{1}{2}}, w_h) \\
+ \left(\Delta t \textstyle\sum_{i=1}^n \beta_1^i, w_h\right), \ \forall w_h \in W_h,
\end{aligned}$$

since $\Delta t \sum\limits_{i=1}^{n} \gamma^i = \psi^n - \psi^0$. Similar to (4.52), we have

$$(\partial_t \rho^{n+\frac{1}{2}}, w_h) + (\nabla \cdot \psi^n, w_h) = (\partial_t \zeta^{n+\frac{1}{2}}, w_h) + (\beta_2^n, w_h), \ \forall w_h \in W_h, \tag{4.71}$$

where β_2^n is defined in (4.52). Observe that $\gamma^{n+1} = \partial_t \psi^{n+\frac{1}{2}}$, therefore, we get

$$(\kappa(u_H^{n+1})\partial_t \psi^{n+\frac{1}{2}}, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, \rho^{n+1}) = (F, \boldsymbol{v}_h), \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.72}$$

Choosing $w_h = \rho^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \psi^{n+\frac{1}{2}}$ in (4.71) and (4.72), adding them and multiplying by $2\Delta t$, we find that

$$\begin{aligned}
&\|\rho^{n+1}\|^2 - \|\rho^n\|^2 + \|\kappa^{\frac{1}{2}}(u_H^{n+1})\psi^{n+1}\|^2 - \|\kappa^{\frac{1}{2}}(u_H^n)\psi^n\|^2 \\
&+((\kappa(u_H^n) - \kappa(u_H^{n+1}))\psi^n, \psi^n) \\
&+\Delta t(\nabla \cdot \psi^n, \rho^n) - \Delta t(\nabla \cdot \psi^{n+1}, \rho^{n+1}) \\
&= 2\Delta t(\partial_t \zeta^{n+\frac{1}{2}} + \beta_2^n, \rho^{n+\frac{1}{2}}) + 2\Delta t(F, \psi^{n+\frac{1}{2}}).
\end{aligned} \tag{4.73}$$

Apply the Cauchy-Schwarz inequality, it is easy to get

$$(\partial_t \zeta^{n+\frac{1}{2}} + \beta_2^n, \rho^{n+\frac{1}{2}}) \le (\|\partial_t \zeta^{n+\frac{1}{2}}\| + \|\beta_2^n\|)\|\rho^{n+\frac{1}{2}}\|. \tag{4.74}$$

Using (2.13)-(2.15), (4.44), and the assumptions on $\kappa(u)$ and $\boldsymbol{z}$, we have

$$\begin{aligned}
&|T_1| = |(\kappa(u^{n+1})\delta^{n+1}, \psi^{n+\frac{1}{2}})| \le Ch^{k+1}\|\psi^{n+\frac{1}{2}}\|, \\
&|T_2| \le |(\kappa'(u_H^{n+1})(\rho^{n+1} + \zeta^{n+1})\boldsymbol{z}_H^{n+1}, \psi^{n+\frac{1}{2}})|
\end{aligned}$$

$$
\begin{aligned}
&\leq C(\|\rho^{n+1}\| + \|\zeta^{n+1}\|)\|\boldsymbol{z}_H^{n+1}\|_{0,\infty}\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(\|\rho^{n+1}\| + h^{k+1})\|\psi^{n+\frac{1}{2}}\|, \\
|T_3| &= |(\kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1}), \psi^{n+\frac{1}{2}})| \\
&\leq C\|(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1})\| \cdot \|\psi^{n+\frac{1}{2}}\| \\
&\leq C\|u^{n+1} - u_H^{n+1}\|_{0,4}\|\boldsymbol{z}^{n+1} - \Pi_h \boldsymbol{z}^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&+ C\|u^{n+1} - u_H^{n+1}\|_{0,4}\|\boldsymbol{z}_H^{n+1} - \boldsymbol{z}^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(\|u^{n+1} - u_H^{n+1}\|_{0,4}^2 + \|\boldsymbol{z}^{n+1} - \Pi_h \boldsymbol{z}^{n+1}\|_{0,4}^2 \\
&+ \|\boldsymbol{z}_H^{n+1} - \boldsymbol{z}^{n+1}\|_{0,4}^2)\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(H^{2k+1} + h^{2k+2} + \Delta t^4)\|\psi^{n+\frac{1}{2}}\|, \qquad (4.75) \\
|T_4| &= |(\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2 \boldsymbol{z}_H^{n+1}, \psi^{n+\frac{1}{2}})| \\
&\leq C\|u^{n+1} - u_H^{n+1}\|_{0,4}^2\|\boldsymbol{z}_H^{n+1}\|_{0,\infty}\|\psi^{n+\frac{1}{2}}\| \\
&\leq CH^{2k+1}\|\psi^{n+\frac{1}{2}}\|, \\
|T_5| &= |(\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1} - \delta^{n+1}), \psi^{n+\frac{1}{2}})| \\
&\leq C\|(\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1} - \delta^{n+1})\| \cdot \|\psi^{n+\frac{1}{2}}\| \\
&\leq C\|u^{n+1} - u_H^{n+1}\|_{0,8}^2\|\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&+ C\|u^{n+1} - u_H^{n+1}\|_{0,8}^2\|\delta^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(H^{2k+1} + h^{2k+2})\|\psi^{n+\frac{1}{2}}\|.
\end{aligned}
$$

It follows from (4.75) that

$$
|(F, \psi^{n+\frac{1}{2}})| \leq C(h^{k+1} + H^{2k+1} + (\Delta t)^2 + \|\rho^{n+1}\|)\|\psi^{n+\frac{1}{2}}\|. \qquad (4.76)
$$

Using (4.74) and (4.76), and summing (4.73) over time levels, we have

$$
\begin{aligned}
&\|\rho^{n+1}\|^2 + \|\psi^{n+1}\|^2 - \Delta t(\nabla \cdot \psi^{n+1}, \rho^{n+1}) \\
&\leq 2C\Delta t \sum_{i=0}^{n}(h^{k+1} + H^{2k+1} + (\Delta t)^2 + \|\rho^{i+1}\|)\|\psi^{i+\frac{1}{2}}\| \\
&+ 2\Delta t \sum_{i=0}^{n}(\|\partial_t \zeta^{i+\frac{1}{2}}\| + \|\beta_2^i\|)\|\rho^{i+\frac{1}{2}}\|,
\end{aligned}
$$

where we used $\rho^0 = 0$ and $\psi^0 = 0$ since the initial conditions (2.29) and (2.30). In the following, similarly as the proof of (4.65), we deduce that

$$
\|\rho\|_{l^\infty(L^2)}^2 + \|\psi\|_{l^\infty(L^2)}^2 \leq C((\Delta t)^4 + h^{2k+2} + H^{4k+2}). \qquad (4.77)
$$

Thus, applying (2.13), (2.14), (4.77) and the triangle inequality, we can derive (4.66). $\square$

Remark 4.1. From Theorem 4.2, we see that the optimal error estimate is $\mathcal{O}((\Delta t)^2 + h^{k+1})$ by taking $H = \mathcal{O}(h^{\frac{k+1}{2k+1}})$, which is coincide with the error result (4.44) obtained for the original MFE system (2.17)-(2.21).

5. Numerical experiments

In the section, we consider the following second-order nonlinear hyperbolic problem:

$$
\begin{aligned}
&u_{tt} - \nabla \cdot (K(u)\nabla u) = f, \quad (\boldsymbol{x}, t) \in \Omega \times J,\\
&u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times J,\\
&u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),\ u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,
\end{aligned}
$$

where $\Omega = [0,1]^2$, $J = [0,1]$, $\boldsymbol{x} = (x_1, x_2)^T$,

$$
\boldsymbol{K}(u) = \begin{pmatrix} 1+\sin^2(u) & 0 \\ 0 & 1+\sin^2(u) \end{pmatrix},
$$

the functions f, u_0 and u_1 are chosen so that the exact solution $u(\boldsymbol{x}, t) = e^t x_1 x_2 (1-x_1)(1-x_2)$ or $u(\boldsymbol{x}, t) = e^t \sin(\pi x_1)\sin(\pi x_2)$.

We use the Raviart-Thomas spaces (RT_1) with $k = 1$. J is uniformly divided so that $\triangle t$ is a constant. We present the two-grid discretization error with coarse and fine mesh size pairs (H, h)=(1/4, 1/8), (1/9, 1/27), (1/16, 1/64) which satisfy the relation $h = H^{3/2}$. When the exact solution is chosen as $u(\boldsymbol{x}, t) = e^t x_1 x_2 (1-x_1)(1-x_2)$, we take the time step $\Delta t = 1.0e-3$, the error results, convergence rates and computational time of MEFM and two-grid method are demonstrated in Tabs. 1 and 2. When the exact solution is chosen as $u(\boldsymbol{x}, t) = e^t \sin(\pi x_1)\sin(\pi x_2)$, we couple the time step with spatial mesh as $\Delta t = h$, the numerical results of MEFM and two-grid method are presented in Tabs. 3 and 4.

Table 1: Numerical results by MFEM with $u(\boldsymbol{x}, t) = e^t x_1 x_2 (1-x_1)(1-x_2)$.

h	$\lVert u - u_h \rVert$	$\lVert \boldsymbol{z} - \boldsymbol{z}_h \rVert$	Computing time (s)
1/8	1.5826e-03	4.1845e-03	1.52
1/27	1.4147e-04	3.7821e-04	16.33
1/64	2.4891e-05	6.7930e-04	70.65
Rates	2.0	2.0	

Table 2: Numerical results by two-grid method with $u(\boldsymbol{x}, t)=e^t x_1 x_2 (1-x_1)(1-x_2)$.

$(H\ ,\ h)$	$\lVert u - U_h \rVert$	$\lVert \boldsymbol{z} - \boldsymbol{Z}_h \rVert$	Computing time (s)
(1/4,1/8)	1.6114e-03	4.5097e-03	1.71
(1/9,1/27)	1.4475e-04	4.0349e-04	9.47
(1/16,1/64)	2.5978e-05	7.2536e-04	22.54
Rates	2.0	2.0	

Table 3: Numerical results by MFEM with $u(\boldsymbol{x},t)=e^t\sin(\pi x_1)\sin(\pi x_2)$.

$h=\Delta t$	$\|\|u-u_h\|\|$	$\|\|\boldsymbol{z}-\boldsymbol{z}_h\|\|$	Computing time (s)
1/8	3.2017e-03	9.6503e-03	0.12
1/27	2.9004e-04	8.8632e-04	3.97
1/64	5.1916e-05	1.5965e-04	19.32
Rates	2.0	2.0	

Table 4: Numerical results by two-grid method with $u(\boldsymbol{x},t)=e^t\sin(\pi x_1)\sin(\pi x_2)$.

$(H\ ,\ h=\Delta t)$	$\|\|u-U_h\|\|$	$\|\|\boldsymbol{z}-\boldsymbol{Z}_h\|\|$	Computing time (s)
(1/4,1/8)	3.5235e-03	1.1218e-02	0.19
(1/9,1/27)	3.1622e-04	9.9945e-04	2.08
(1/16,1/64)	5.6992e-05	1.8016e-04	8.46
Rates	2.0	2.0	

From the numerical results in Tabs. 1-4, we observe that the proposed two methods are of second-order accuracy, which is coincided with our theoretical analysis. Moreover, we also observe that the two-grid method spends less time than the usual MFEM. Thus, we can see that two-grid algorithm is a very effective algorithm when it comes to deal with the nonlinear problems.

6. Conclusions

In this paper, we develop a two-grid mixed finite element method for a class of nonlinear hyperbolic equation. We prove the stability and the error estimate for the two-grid scheme. It is shown theoretically and numerically that when the coarse and fine mesh sizes satisfy $h=\mathcal{O}(H^{(2k+1)/(k+1)})$, the two-grid solution can achieve the same accuracy as the mixed finite element solution.

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Approximation properties of (p, q) bivariate Szász Beta type operators

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Abstract. In the present research article, we construct a new sequence of bivariate (p, q) hybrid type operators using $(p, q)-$ beta functions via Dunkl analogue. In the subsection sequence, we investigate the rate of convergence and the order of approximation for these sequences positive linear operators. Further, we study local approximation results in various class of functions. In the last section, we give the global approximation results using weight function.

Keywords: (p, q)-Bernstein operators; Rate of convergence; Order of approximation; (p, q)- beta operators; weighted spaces.

1. Introduction

The operator theory is an active research area for the last one century. Bernstein was the first who gave the first positive linear operator named as Bernstein operator to approximate the class of continuous functions over $[a, b]$. The motive of Bernstein was to give the elegant proof of Weierstrass approximation theorem using binomial distribution as follows.

$$(1) \qquad \mathcal{B}_n(f(x); x) = \sum_{k=0}^{n} b_{n,k}(x) f(x),$$

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where $b_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k}$ and f is a bounded function defined in $C([0,1])$. To improve the rate of convergence of the operators defined by (1), the q-analogues of Bernstein operators were independently given by Lupaş [19] and Phillips [21] using quantum calculus. The (p,q)-analogue of Bernstein operators was given by Mursaleen *et al.* [29] which improves the Bezier curves and radius of convergence of the complex disk due to p-parametres (see Mursaleen and Khan [26]), Khan and Lobiyal [27]. Recently, a Dunkl type generalization [17] of Szász operators [24] via post-quantum calculus was studied by Alotaibi *et al.* [15]. For more details and research motivation in Dunkl type generalizations, we mention here some research articles [4, 11, 8, 12, 20, 22, 23, 28, 29, 30, 31].

Let $f \in C[0,1]$ denote the space of all continuous functions on $[0,1]$. For all $f \in C[0,1]$, $x \geqq 0$, $\tau > -\frac{1}{2}$ and $n \in \mathbb{N}$, the (p,q)-Dunkl analogue of Szász operators [15] (see also [11]) is defined as follows:

$$(2)\quad \mathcal{D}_n^{\mu}(h;u,p,q) = \frac{1}{e_{\mu,p,q}([n]_{p,q}u)} \sum_{k=0}^{\infty} \frac{([n]_{p,q}u)^k}{\gamma_{\mu,p,q}(k)} p^{\frac{k(k-1)}{2}} f\left(\frac{p^{k+2\mu\theta_k} - q^{k+2\mu\theta_k}}{p^{k-1}(p^n-q^n)}\right),$$

where $[n]_{p,q}$ is the (p,q)-integer defined as:

$$(3)\qquad [n]_{p,q} = p^{n-1} + qp^{n-3} + \ldots + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{p-q}, & (p \neq q \neq 1), \\ \dfrac{1-q^n}{1-q}, & (p=1), \\ n, & (p=q=1), \end{cases}$$

$$(au+bv)_{p,q}^n := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} a^{n-k} b^k u^{n-k} v^k,$$

$$(1-u)_{p,q}^n = (1-u)(p-qu)(p^2-q^2u)\ldots(p^{n-1}-q^{n-1}u),$$

$$(x-y)_{p,q}^n = \begin{cases} \prod_{j=0}^{n-1}(p^jx - q^jy), & \text{if } n \in \mathbb{N}, \\ 1, & \text{if } n = 0. \end{cases}$$

The (p,q)-power basis is explained as

$$(u \oplus v)_{p,q}^n = (u+v)(pu+qv)(p^2u+q^2v)\ldots(p^{n-1}u+q^{n-1}v).$$

Furthermore, the (p,q)-analogues of the exponential function are defined by

$$e_{p,q}(u) = \sum_{k=0}^{\infty} p^{\frac{k(k-1)}{2}} \frac{u^k}{[k]_{p,q}!}, \qquad E_{p,q}(u) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{u^k}{[k]_{p,q}!};$$

Moreover, the (p,q)-Dunkl analogue of the exponential function is defined by

$$e_{\mu,p,q}(u)=\sum_{k=0}^{\infty}p^{\frac{k(k-1)}{2}}\frac{u^k}{\gamma_{\mu,p,q}(k)}, \tag{4}$$

$$\gamma_{\mu,p,q}(k)=\frac{\prod_{i=0}^{[\frac{k+1}{2}]-1}p^{2\mu(-1)^{i+1}+1}((p^2)^ip^{2\mu+1}-(q^2)^iq^{2\mu+1})\prod_{j=0}^{[\frac{k}{2}]-1}p^{2\mu(-1)^j+1}((p^2)^jp^2-(q^2)^jq^2)}{(p-q)^k}, \tag{5}$$

$$\gamma_{\mu,p,q}(k+1)=\frac{p^{2\mu(-1)^{k+1}+1}(p^{2\mu\theta_{k+1}+k+1}-q^{2\mu\theta_{k+1}+k+1})}{(p-q)}\gamma_{\mu,p,q}(k), \tag{6}$$

$$\theta_k=\begin{cases}0, & \text{for } k=2m,\ m=0,1,2,\ldots,\\ 1, & \text{for } k=2m+1,\ m=0,1,2,\ldots.\end{cases} \tag{7}$$

For $m=0,1,2,\ldots n$, the number $[\frac{m}{2}]$ denotes the greatest integer function.

In this section, we construct a class of (p,q)-Bivariate of Szász-beta operators of second kind generated by an exponential function via Dunkl generalization 1.1. This type of the construction of operators are a generalized version of the operators studied in [25].

Definition 1.1. *Let $f\in C([0,1])=\{f(t):\ f(t)=O(t^\rho),\quad t\to\infty,\quad f\in C[0,\infty)\}$ such as $x\in[0,\infty)$, $\rho>n,m$ and $n,m\in\mathbb{N}$. Then for all $0<q<p\leqq 1$, $\mu>-\frac{1}{2}$, $\nu>-\frac{1}{2}$ and $\theta_{\ell_1},\theta_{\ell_2}$ defined by (7), we define*

Let $I_1\times I_2=[0,D_n]\times[0,D_m]$ and $(x,y)\in I_1\times I_2$. Then, for a function $f\in C(I_1\times I_2)$, the (p,q)-Bivariate of Szász-beta operators of second kind generated by an exponential function via Dunkl generalization 1.1, $D^{\mu,\nu}_{n,m}(f;x,y,p_{1,2},q_{1,2})=D^{\mu,\nu}_{n,m}(f;x,y,p_1,p_2,q_1,q_2)$ are defined as follows:

$$\begin{aligned}&\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_{1,2},q_{1,2})\\&=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)\int_0^\infty\int_0^\infty\frac{t_1^{\ell_1+2\mu\theta_{\ell_1}}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}}\\&\times\frac{t_2^{\ell_2+2\nu\theta_{\ell_2}}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}}f(t_1,t_2)\mathrm{d}_{p_1,q_1}t_1,\mathrm{d}_{p_2,q_2}t_2,\end{aligned} \tag{8}$$

where

$$\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)=\frac{1}{e_{\mu,p_1,q_1}([n]_{p_1,q_1}x)}\frac{([n]_{p_1,q_1}x)^{\ell_1}}{\gamma_{\mu,p_1,q_1}(\ell_1)}p_1^{\frac{\ell_1(\ell_1-1)}{2}}\frac{1}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)},$$

$$\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)=\frac{1}{e_{\nu,p_2,q_2}([m]_{p_2,q_2}x)}\frac{([m]_{p_2,q_2}y)^{\ell_2}}{\gamma_{\nu,p_2,q_2}(\ell_2)}p_2^{\frac{\ell_2(\ell_2-1)}{2}}\frac{1}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)},$$

and $\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)$, $\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)$ *are the Beta functions of second kind in post quantum calculus and is defined by*

$$\mathcal{B}_{p,q}(\alpha,\beta)=\int_0^\infty\frac{t^{\alpha-1}}{(1\oplus pt)^{\alpha+\beta}_{p,q}}\mathrm{d}_{p,q}t,\quad \alpha,\ \beta\in\ \mathbb{N}, \tag{9}$$

$$\mathcal{B}_{p,q}(\alpha,\beta)=\frac{[\alpha-1]_{p,q}}{p^{\alpha-1}[\beta]_{p,q}}\mathcal{B}_{p,q}(\alpha-1,\beta+1),\quad \alpha,\ \beta\in\ \mathbb{N}. \tag{10}$$

Moreover, to obtain the basic estimates here we use the following relations:

$$[\ell+1+2\tau\theta_\ell]_{p,q}=q[\ell+2\tau\theta_\ell]_{p,q}+p^{\ell+2\tau\theta_\ell}, \tag{11}$$

$$[\ell+2+2\tau\theta_\ell]_{p,q}=q^2[\ell+2\tau\theta_\ell]_{p,q}+(p+q)p^{\ell+2\tau\theta_\ell}. \tag{12}$$

For more related results on (p,q)-analogues, we prefer [1, 2, 3, 16, 18, 5, 10, 7, 6]. We have the following inequalities.

Lemma 1.1. *Let* $f(t)=1,t,t^2$. *Then, the operators* $\mathcal{D}^\mu_n(\,\cdot\,;\,\cdot\,)$ *refer to* (2) *satisfy* $\mathcal{D}^\mu_n(1;x,p,q)=1$ *and the following inequalities hold:*

(13)

$$\mathcal{D}^\mu_{n,p,q}(f;x)\leqq\begin{cases}\dfrac{[n]_{p,q}}{[n-1]_{p,q}}x+\dfrac{1}{[n-1]_{p,q}}, & \textit{for } f(t)=t\\[2ex] \dfrac{[n]^2_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}x^2 & \\ \qquad+\dfrac{[n]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}\left(1+[2]_{p,q}+[1+2\tau]_{p,q}\right)x & \\ \qquad+\dfrac{[2]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}, & \textit{for } f(t)=t^2\end{cases}$$

and

$$\mathcal{D}^\mu_{n,p,q}(f;x)\geqq\begin{cases}\dfrac{q[n]_{p,q}}{[n-1]_{p,q}}x+\dfrac{1}{[n-1]_{p,q}}, & \textit{for } f(t)=t\\[2ex] \dfrac{q^3[n]^2_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}x^2 & \\ \qquad+\dfrac{q[n]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}\Bigg(q+[2]_{p,q} & \\ \qquad+q^{2+2\tau}[1-2\tau]_{p,q}\dfrac{e_{\tau,p,q}\left(\frac{q}{p}[n]_{p,q}x\right)}{e_{\tau,p,q}([n]_{p,q}x)}\Bigg)x & \\ \qquad+\dfrac{[2]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}, & \textit{for } f(t)=t^2.\end{cases}$$

Lemma 1.2. *Let $e_{i,j} = f(t_1,t_2) = t_1^i t_2^j, 0 \leq i,j \leq 2$. Then, the operators $\mathcal{D}_{n,m}^{\mu,\nu}(\cdot;\cdot)$ refer to (**??**) satisfy $\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,0};x,y,p_{1,2},q_{1,2}) = 1$ and the following inequalities hold:*

$$\begin{aligned}
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{1,0};x,y,p_{1,2},q_{1,2}) \leqq \frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}x + \frac{1}{[n-1]_{p_1,q_1}},\\
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,1};x,y,p_{1,2},q_{1,2}) \leqq \frac{[m]_{p_1,q_1}}{[n-1]_{p_1,q_1}}y + \frac{1}{[n-1]_{p_1,q_1}},\\
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0};x,y,p_{1,2},q_{1,2})\\
&\qquad \leqq \frac{[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}x^2 + \frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}\\
&\qquad (1+[2]_{p_1,q_1}+[1+2\mu]_{p_1,q_1})\,x + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}},\\
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2};x,y,p_{1,2},q_{1,2})\\
&\qquad \leqq \frac{[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}y^2 + \frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}\\
&\qquad (1+[2]_{p_2,q_2}+[1+2\nu]_{p_2,q_2})\,y + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[n-2]_{p_2,q_2}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{1,0};x,y,p_{1,2},q_{1,2}) \geqq \frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}x + \frac{1}{[n-1]_{p_1,q_1}},\\
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,1};x,y,p_{1,2},q_{1,2}) \geqq \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}y + \frac{1}{[m-1]_{p_2,q_2}},\\
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0};x,y,p_{1,2},q_{1,2})\\
&\qquad \geqq \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}x^2 + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}\\
&\qquad + \frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}\\
&\qquad \left(q_1+[2]_{p_1,q_1}+q_1^{2+2\mu}[1-2\mu]_{p_1,q_1}\frac{e_{\mu,p_1,q_1}\left(\frac{q_1}{p_1}[n]_{p_1,q_1}x\right)}{e_{\mu,p_1,q_1}([n]_{p_1,q_1}x)}\right)x,\\
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2};x,y,p_{1,2},q_{1,2})\\
&\qquad \geqq \frac{q_2^3[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}y^2 + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}\\
&\qquad + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}\\
&\qquad \left(q_2+[2]_{p_2,q_2}+q_2^{2+2\nu}[1-2\nu]_{p_2,q_2}\frac{e_{\nu,p_2,q_2}\left(\frac{q_2}{p_2}[m]_{p_2,q_2}y\right)}{e_{\nu,p_2,q_2}([m]_{p_2,q_2}y)}\right)y.
\end{aligned}$$

Proof. To prove the results of this Lemma, we use (9)–(12). Take $f(t_1,t_2)=1$. Then,

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}(e_{0,0};x,y,p_{1,2},q_{1,2})\\
&=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)\int_0^\infty\int_0^\infty\frac{t_1^{\ell_1+2\mu\theta_{\ell_1}}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}}\\
&\times\frac{t_2^{\ell_2+2\nu\theta_{\ell_2}}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}}\mathrm{d}_{p_1,q_1}t_1\mathrm{d}_{p_2,q_2}t_2\\
&=\sum_{\ell_1=0}^{\infty}\frac{1}{e_{\mu,p_1,q_1}([n]_{p_1,q_1}x)}\frac{([n]_{p_1,q_1}x)^{\ell_1}}{\gamma_{\mu,p_1,q_1}(\ell_1)}p_1^{\frac{\ell_1(\ell_1-1)}{2}}\frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}\\
&\times\sum_{\ell_2=0}^{\infty}\frac{1}{e_{\nu,p_2,q_2}([m]_{p_2,q_2}x)}\frac{([m]_{p_2,q_2}y)^{\ell_2}}{\gamma_{\nu,p_2,q_2}(\ell_2)}p_2^{\frac{\ell_2(\ell_2-1)}{2}}\frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}=1.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}(e_{1,0};x,y,p_{1,2},q_{1,2})\\
&=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)\int_0^\infty\int_0^\infty\frac{t_1^{\ell_1+2\mu\theta_{\ell_1}+1}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}}\\
&\times\frac{t_2^{\ell_2+2\nu\theta_{\ell_2}}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}}\mathrm{d}_{p_1,q_1}t_1\mathrm{d}_{p_2,q_2}t_2\\
&=\sum_{\ell_1=0}^{\infty}\frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+2,n-1)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}\sum_{\ell_2=0}^{\infty}\frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}\\
&=\frac{q_1}{[n-1]_{p_1,q_1}}\sum_{\ell=0}^{\infty}\frac{1}{p_1^{\ell_1+2\mu\theta_{\ell_1}+1}}[\ell_1+2\mu\theta_{\ell_1}]_{p_1,q_1}+\frac{1}{p_1[n-1]_{p_1,q_1}}\\
&\frac{1}{p_1[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{p_1^2[n-1]_{p_1,q_1}}\sum_{\ell_1=0}^{\infty}\left(\frac{p_1^{2\ell_1+2\mu\theta_{2\ell_1}}-q_1^{2\ell_1+2\mu\theta_{2\ell_1}}}{p_1^{2\ell_1-1}(p_1^n-q_1^n)}\right)\\
&+\frac{q_1[n]_{p_1,q_1}}{p_1^{2+2\mu}[n-1]_{p_1,q_1}}\sum_{\ell_1=0}^{\infty}\left(\frac{p_1^{2\ell_1+1+2\mu\theta_{2\ell_1+1}}-q_1^{2\ell_1+1+2\mu\theta_{2\ell_1+1}}}{p_1^{2\ell_1}(p_1^n-q_1^n)}\right).
\end{aligned}$$

Clearly, we have

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,p_1,q_1}(e_{1,0};x,y,p_{1,2},q_{1,2})\\
&\geqq\frac{1}{[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\sum_{\ell_1=0}^{\infty}\left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}}-q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n-q_1^n)}\right)\\
&=\frac{1}{[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}C_{n,p_1,q_1}(t_1;x)\\
&=\frac{1}{[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}x
\end{aligned}$$

and

$$\mathcal{D}_{n,m}^{\mu,\nu}(t_1;x,y,p_{1,2},q_{1,2}) \leqq \frac{1}{[n-1]_{p_1,q_1}} + \frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}x,$$

$$\begin{aligned}
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,1};x,y,p_{1,2},q_{1,2}) \\
&= \sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty} \mathcal{P}_{n,p_1,q_1}^{\mu,l_1}(x)\mathcal{Q}_{m,p_2,q_2}^{\nu,l_2}(y)\int_0^\infty\int_0^\infty \frac{t_1^{\ell_1+2\mu\theta_{\ell_1}}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}} \\
&\times \frac{t_2^{\ell_2+2\nu\theta_{\ell_2}+1}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}}\mathrm{d}_{p_1,q_1}t_1\mathrm{d}_{p_2,q_2}t_2 \\
&= \sum_{\ell_2=0}^{\infty}\frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+2,m-1)}{\mathcal{B}_{p_2,q_1}(\ell_2+2\nu\theta_{\ell_2}+1,m)}\sum_{\ell_1=0}^{\infty}\frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)} \\
&= \frac{q_2}{[m-1]_{p_2,q_2}}\sum_{\ell_2=0}^{\infty}\frac{1}{p_2^{\ell_2+2\nu\theta_{\ell_2}+1}}[\ell_2+2\nu\theta_{\ell_2}]_{p_2,q_2} + \frac{1}{p_2[m-1]_{p_2,q_2}} \\
&= \frac{1}{p_2[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{p_2^2[m-1]_{p_2,q_2}}\sum_{\ell_2=0}^{\infty}\left(\frac{p_2^{2\ell_2+2\nu\theta_{2\ell_2}}-q_2^{2\ell_2+2\nu\theta_{2\ell_2}}}{p_2^{2\ell_2-1}(p_2^m-q_2^m)}\right) \\
&+ \frac{q_2[m]_{p_2,q_2}}{p_2^{2+2\nu}[m-1]_{p_2,q_2}}\sum_{\ell_2=0}^{\infty}\left(\frac{p_2^{2\ell_2+1+2\nu\theta_{2\ell_2+1}}-q_2^{2\ell_2+1+2\nu\theta_{2\ell_2+1}}}{p_2^{2\ell_2}(p_2^m-q_2^m)}\right).
\end{aligned}$$

Clearly, we have

$$\begin{aligned}
&\mathcal{D}_{n,p_1,q_1}^{\mu,\nu}(e_{0,1};x,y,p_{1,2},q_{1,2}) \\
&\geqq \frac{1}{[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\sum_{\ell_2=0}^{\infty}\left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}}-q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m-q_2^m)}\right) \\
&= \frac{1}{[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}C_{m,p_2,q_2}(t_2;y) \\
&= \frac{1}{[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}y
\end{aligned}$$

and

$$\mathcal{D}_{n,m}^{\mu,\nu}(t_2;x,y,p_{1,2},q_{1,2}) \leqq \frac{1}{[m-1]_{p_2,q_2}} + \frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}y.$$

Similarly for $e_{2,0} = f(t) = t_1^2$, we have

$$\begin{aligned}
&\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0};x,y,p_{1,2},q_{1,2}) \\
&= \sum_{\ell_1=0}^{\infty}\mathcal{P}_{n,p_1,q_1}^{\mu,l_1}(x)\int_0^\infty \frac{t_1^{\ell_1+2\mu\theta_{\ell_1}+2}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}}\mathrm{d}_{p_1,q_1}t_1 \\
&= \sum_{\ell_1=0}^{\infty}\frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+3,n-2)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}\sum_{\ell_2=0}^{\infty}\frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}
\end{aligned}$$

$$= \sum_{\ell_1=0}^{\infty} \frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+3, n-2)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1, n)}$$

$$= \frac{1}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \frac{1}{p_1^{3+2\ell_1+4\mu\theta_{\ell_1}+1}} [\ell_1+2\mu\theta_{\ell_1}+1]_{p_1,q_1}[\ell_1+2\mu\theta_{\ell_1}+2]_{p_1,q_1}$$

$$= \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \frac{1}{p_1^{5+4\mu\theta_{\ell_1}}} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}} - q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n - q_1^n)} \right)^2$$

$$+ \frac{q_1(p_1+2q_1)[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \frac{1}{p_1^{4+2\mu\theta_{\ell_1}}} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}} - q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n - q_1^n)} \right)$$

$$+ \frac{(p_1+q_1)}{p_1^3[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \mathcal{P}_{n,p-1,q_1}(x).$$

Now, by separating it into even and odd terms and applying θ_{ℓ_1} from (7), i.e., taking $\ell_1 = 2r$ and $\ell_1 = 2r+1$ for all $r = 0, 1, 2, \ldots$, we have

$$\mathcal{D}_{n,m}^{\mu,\nu}(te_{2,0}; x, y, p_{1,2}, q_{1,2}) \geqq \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}} - q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n - q_1^n)} \right)^2$$

$$+ \frac{q_1(q_1+[2]_{p_1,q_1})[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}} - q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n - q_1^n)} \right)$$

$$+ \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_2}}$$

$$= \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1^2; x) + \frac{q_1(q_1+[2]_{p_1,q_1})[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1; x)$$

$$+ \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}.$$

Similarly,

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0}; x, y, p_{1,2}, q_{1,2}) \leqq \frac{[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1^2; x)$$
$$+ \frac{(1+[2]_{p_1,q_1})[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1; x) + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}.$$

Similarly, for $e_{0,2} = f(t) = t_2^2$, we have

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2}; x, y, p_{1,2}, q_{1,2}) = \sum_{\ell_2=0}^{\infty} \mathcal{Q}_{m,p_2,q_2}^{\nu,l_2}(y) \int_0^{\infty} \frac{t_2^{\ell_2+2\nu\theta_{\ell_2}+2}}{(1 \oplus p_2 t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}} \mathrm{d}_{p_2,q_2} t_2$$

$$= \sum_{\ell_2=0}^{\infty} \frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+3, m-2)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1, m)}$$

$$= \sum_{\ell_2=0}^{\infty} \frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+3,m-2)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}$$

$$= \frac{1}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \frac{1}{p_2^{3+2\ell_2+4\nu\theta_{\ell_2}+1}} [\ell_2+2\nu\theta_{\ell_2}+1]_{p_2,q_2}$$

$$[\ell_2+2\nu\theta_{\ell_2}+2]_{p_2,q_2}$$

$$= \frac{q_1^3[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \frac{1}{p_2^{5+4\nu\theta_{\ell_2}}} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)}\right)^2$$

$$+ \frac{q_2(p_2+2q_2)[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \frac{1}{p_2^{4+2\nu\theta_{\ell_2}}} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)}\right)$$

$$+ \frac{(p_2+q_2)}{p_2^3[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \mathcal{P}_{m,p-2,q_2}(y).$$

Now, by separating it into even and odd terms and applying θ_{ℓ_2} from (7), i.e., taking $\ell_2 = 2r$ and $\ell_2 = 2r+1$ for all $r = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2}; x, y, p_{1,2}, q_{1,2}) \geqq{} & \frac{q_2^3[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)}\right)^2 \\
& + \frac{q_2(q_2+[2]_{p_2,q_2})[m]_{p_2,q_2}}{[m-1]_{2_1,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)}\right) \\
& + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \\
={} & \frac{q_2^3[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2^2; x, y) \\
& + \frac{q_2(q_2+[2]_{p_2,q_2})[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2; y) \\
& + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}.
\end{aligned}$$

Similarly,

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2}; x, y, p_{1,2}, q_{1,2}) \leqq \frac{[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2^2; y)$$

$$+ \frac{(1+[2]_{p_2,q_2})[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2; y) + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}.$$

This completes the proof of Lemma 1.2. $\square$

Lemma 1.3. *Let $\Psi_{i,j}=(t_1-x)^i(t_2-x)^j$ for $i,j=1,2$, then we have following inequalities:*

1. $$\mathcal{D}_{n,m}^{\mu,\nu}(\Psi_{1,0};x,y,p_{1,2},q_{1,2})\leqq\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}-1\right)x+\frac{1}{[n-1]_{p_1,q_1}},$$ *for* $n>1,\ n\in\mathbb{N}$
2. $$\mathcal{D}_{n,m}^{\mu,\nu}(\Psi_{0,1};x,y,p_{1,2},q_{1,2})\leqq\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}-1\right)y+\frac{1}{[m-1]_{p_2,q_2}},$$ *for* $n>1,\ m\in\mathbb{N}$
3. $$\mathcal{D}_{n,m}^{\mu,\nu}(\Psi_{2,0};x,y,p_{1,2},q_{1,2})\leqq\left(\frac{[n]^2_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}-\frac{2[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}+1\right)x^2 + \frac{1}{[n-1]_{p_1,q_1}}\left(\frac{[n]_{p_1,q_1}}{[n-2]_{p_1,q_1}}\big(1+[2]_{p_1,q_1}+[1+2\mu]_{p_1,q_1}\big)-2\right)x + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}},\qquad \textit{for}\quad n>2,\ n\in\mathbb{N}.$$
4. $$\mathcal{D}_{n,m}^{\mu,\nu}(\Psi_{0,2};x,y,p_{1,2},q_{1,2})\leqq\left(\frac{[m]^2_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}-\frac{2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}+1\right)y^2 + \frac{1}{[m-1]_{p_2,q_2}}\left(\frac{[m]_{p_2,q_2}}{[m-2]_{p_2,q_2}}\big(1+[2]_{p_2,q_2}+[1+2\nu]_{p_2,q_2}\big)-2\right)y + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}},\qquad \textit{for}\quad m>2,\ m\in\mathbb{N}.$$

Definition 1.2. *Let $X,Y\subset\mathbb{R}$ be any two given intervals and the set $B(X\times Y)=\{f:X\times Y\to\mathbb{R}|f$ is bounded on $X\times Y\}$. For $f\in B(X\times Y)$, let the function $\omega_{total}(f;\cdot,*):[0,\infty)\times[0,\infty)\to\mathbb{R}$, defined for any $(\delta_1,\delta_2)\in[0,\infty)\times[0,\infty)$ by $\omega_{total}(f;\delta_1,\delta_2)=\sup_{|x-x'|\leq\delta_1,|y-y'|\leq\delta_2}\{|f(x,y)-f(x',y')|:(x,y),(x',y')\in[0,\infty)\times[0,\infty)\}$, is called the first order modulus of smoothness of the function f or the total modulus of continuity of the function f.*

In order to get the rate of convergence and degree of approximation for the operators $\mathcal{D}_{n,m}^{\mu,\nu}$, we consider $p_1=p_n$, $p_2=p_m$ and $q_1=q_n$, $q_2=q_m$ such that $0<q_n<p_n\leq 1$ and $0<q_m<p_m\leq 1$ satisfying

$$\lim_{n\to\infty}q_n^n\to a,\ \lim_{m\to\infty}q_m^m\to b,\ \lim_{n\to\infty}p_n^n\to c,\ \lim_{m\to\infty}p_m^m\to d \tag{14}$$

and

$$\lim_{n\to\infty}p_n\to 1,\ \lim_{m\to\infty}p_m\to 1,\ \lim_{n\to\infty}q_n\to 1,\ \lim_{m\to\infty}q_m\to 1, \tag{15}$$

where $0\leq a,b<c,d<1$. Here, we recall the following result due to Volkov [14]:

Theorem 1.1. *Let I and J be compact intervals of the real line. Let $L_{n,m}:C(I\times J)\to C(I\times J),(n,m)\in\mathbb{N}\times\mathbb{N}$ be linear positive operators. If*

$$\lim_{n,m\to\infty}L_{n,m}(e_{ij}) = e_{x,y},(i,j)\in\{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{n,m\to\infty} L_{n,m}(e_{20}+e_{02}) \quad = \quad e_{20}+e_{02},$$

uniformly on $I\times J$*, then the sequence* $(L_{n,m}f)$ *converges to* f *uniformly on* $I\times J$ *for any* $f\in C(I\times J)$.

Theorem 1.2. *Let* $e_{ij}(t_1,t_2)=t_1^i t_2^j$ $(0\leq i+j\leq 2, i,j\in\mathbb{N})$ *be the test functions defined on* $J_1\times J_2$ *and* $(p_n),(q_n),(p_m),(q_m)$ *be the sequences defined by* (14) *and* (15)*. If*

$$\lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}e_{ij})(t_1,t_2)=e_{ij}(t_1,t_2),(i,j)\in\{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}(e_{20}+e_{02})(t_1,t_2)=e_{20}(t_1,t_2)+e_{02}(t_1,t_2),$$

uniformly on $J_1\times J_2$*, then*

$$\lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}f)(t_1,t_2)=f(t_1,t_2),$$

uniformly for any $f\in C(J_1\times J_2)$.

Proof. Using Lemma 1.2, it is obvious for $i=j=0$

$$\lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}e_{00})(t_1,t_2)=e_{00}(t_1,t_2).$$

For $i=1$ and $j=0$, we have

$$\begin{aligned}\lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}e_{10})(t_1,t_2) &= t_1,\\ \lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}e_{10})(t_1,t_2) &= e_{10}(t_1,t_2),\end{aligned}$$

For $i=0$ and $j=1$, we have

$$\begin{aligned}\lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}e_{01})(t_1,t_2) &= t_2,\\ \lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}e_{01})(t_1,t_2) &= e_{01}(t_1,t_2),\end{aligned}$$

and

$$\begin{aligned}\lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}(e_{20}+e_{02})(t_1,t_2) &= \lim_{n,m\to\infty}\left\{\frac{p_1^{n-1}b_n}{[n]_{p_1,q_1}}x+\frac{q_1[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}}x^2\right.\\ &+ \left.\frac{p_2^{m-1}b_m}{[m]_{p_2,q_2}}y+\frac{q_2[m-1]_{p_2,q_2}}{[n]_{p_2,q_2}}y^2\right\},\\ \lim_{n,m\to\infty}(\mathcal{D}^{\mu,\nu}_{n,m}(e_{20}+e_{02})(x,y) &= e_{20}(x,y)+e_{02}(x,y).\end{aligned}$$

From Theorem 1.1, the proof of theorem 1.2 is completed. □

Theorem 1.3 ([13]). *Let $L : C([0,\infty) \times [0,\infty)) \to B([0,\infty) \times [0,\infty))$ be a linear positive operator. For any $f \in C(X \times Y)$, any $(x,y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$\begin{aligned}|(Lf)(x,y) - f(x,y)| \leq |Le_{0,0}(x,y) - 1||f(x,y)| + \Big[&Le_{0,0}(x,y)\\ &+ \delta_1^{-1}\sqrt{Le_{0,0}(x,y)(L(\cdot - x))^2(x,y)}\\ &+ \delta_2^{-1}\sqrt{Le_{0,0}(x,y)(L(* - y))^2(x,y)}\\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{0,0})^2(x,y)(L(\cdot - x))^2(x,y)(L(* - y))^2(x,y)}\Big]\\ &\times \omega_{total}(f;\delta_1,\delta_2),\end{aligned}$$

holds.

Theorem 1.4. *Let $f \in C(J_1 \times J_2)$ and $(x,y) \in J_1 \times J_2$. Then, for $(n,m) \in \mathbb{N}$ and for any $\delta_1, \delta_2 > 0$, we have*

$$|(C_{n,m}f)(x,y) - f(x,y)| \leq 4\omega_{total}(f;\delta_1,\delta_2),$$

where $\delta_1 = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1 - x)^2, x, y, p_{12}, q_{12})}$ *and* $\delta_2 = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_2 - y)^2, x, y, p_{12}, q_{12})}$.

Proof. From Theorem 1.3, we have

$$\begin{aligned}&|(\mathcal{D}_{n,m}^{\mu,\nu}f)(x,y) - f(x,y)| \leq \Big[1 + +\delta_1^{-1}\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1 - x)^2)(x,y))}\\ &+ \delta_2^{-1}\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_2 - y)^2)(x,y)}\\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1 - x)^2)(x,y))\mathcal{D}_{n,m}^{\mu,\nu}((t_2 - y)^2)(x,y)}\Big] \times \omega_{total}(f;\delta_1,\delta_2).\end{aligned}$$

On choosing $\delta = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1 - x)^2)(x,y)}$ and $\delta_2 = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_2 - y)^2)(x,y)}$, we get the required result. □

Now, we shall investigate degree of approximation for the operators $C_{n,m}$ in Lipschitz class. We consider the Lipschitz class $Lip_M(\gamma_1,\gamma_2)$ in terms of two variables as follows:

$$|f(t_1,t_2) - f(x,y)| \leq M|t_1 - x|^{\gamma_1}|t_2 - y|^{\gamma_2},$$

where $M > 0$, $0 < \gamma_1, \gamma_2 \leq 1$ and for any $(t_1,t_2), (x,y) \in J_1 \times J_2$.

Theorem 1.5. *For $f \in Lip_M(\gamma_1,\gamma_2)$, we have*

$$|\mathcal{D}_{n,m}^{\mu,\nu}(f;q_n,q_m,p_n,p_m;x,y) - f(x,y)| \leq M\delta_n^{\gamma_1/2}(x)\delta_m^{\gamma_2/2}(y),$$

where $\delta_n(x) = \mathcal{D}_{n,m}^{\mu,\nu}((t_1 - x)^2;q_n,q_m,p_n,p_m;x,y)$ *and*

$$\delta_m(y) = \mathcal{D}_{n,m}^{\mu,\nu}((t_2 - y)^2;q_n,q_m,p_n,p_m;x,y).$$

Proof. Since $f \in Lip_M(\gamma_1,\gamma_2)$, we can write

$$\begin{aligned}
&|\mathcal{D}^{\mu,\nu}_{n,m}(f;q_n,q_m,p_n,p_m;x,y)-f(x,y)|\\
&\leq \mathcal{D}^{\mu,\nu}_{n,m}(|f(t_1,t_2)-f(x,y)|;q_n,q_m,p_n,p_m;x,y)\\
&\leq M\mathcal{D}^{\mu,\nu}_{n,m}(|t_1-x|^{\gamma_1}|t_2-y|^{\gamma_2};q_n,q_m,p_n,p_m;x,y)\\
&= M\mathcal{D}^{\mu,\nu}_{n,m}(|t_1-x|^{\gamma_1};q_n,q_m,p_n,p_m;x,y)\times(|t_2-y|^{\gamma_2};q_n,q_m,p_n,p_m;x,y).
\end{aligned}$$

Using Hölder inequality with $\alpha_1=\frac{2}{\gamma_1}$, $\beta_1=\frac{2}{2-\gamma_1}$ and $\alpha_2=\frac{2}{\gamma_2}$, $\beta_2=\frac{2}{2-\gamma_2}$, respectively, we get

$$\begin{aligned}
|\mathcal{D}^{\mu,\nu}_{n,m}(f;q_n,q_m,p_n,p_m;x,y)-f(x,y)| &\leq \left\{\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^2;q_n,q_m,p_n,p_m;x,y)\right\}^{\frac{\gamma_1}{2}}\\
&\times\left\{\mathcal{D}^{\mu,\nu}_{n,m}(1;q_n,q_m,p_n,p_m;x,y)\right\}^{\frac{2}{2-\gamma_1}}\\
&\times\left\{\mathcal{D}^{\mu,\nu}_{n,m}((t_2-x_2)^2;q_n,q_m,p_n,p_m;x,y)\right\}^{\frac{\gamma_2}{2}}\\
&\times\left\{\mathcal{D}^{\mu,\nu}_{n,m}(1;q_n,q_n,p_n,p_m;x,y)\right\}^{\frac{2}{2-\gamma_2}}\\
&= M\delta_n^{\gamma_1/2}(x)\delta_m^{\gamma_2/2}(y),
\end{aligned}$$

which completes the proof of Theorem 1.5. □

Here, we discuss degree of approximation in weighted space for the operators defined by (**??**). We recall some basic notions from [**?**] as follows

$B_\rho([0,\infty)\times[0,\infty))$ is the space of all functions defined on $\mathbb{R}^2_+=[0,\infty)\times[0,\infty)$ with the condition $|f(x,y)|\leq M_f\rho(x,y)$, where M_f is a positive constant depending on f and $\rho(x,y)=1+x^2+y^2$ is a weight function. $C_\rho([0,\infty)\times[0,\infty))=\{f: f$ is a continuous function in to $B_\rho([0,\infty)\times[0,\infty))\}$ equipped with the norm $\|f\|_\rho=\sup_{(x,y)\in\mathbb{R}^2_+}\frac{|f(x,y)|}{\rho(x,y)}$ and $C^k_\rho([0,\infty)\times[0,\infty))=\{f: f\in C_\rho$ and $\lim_{x,y\to\infty}\frac{|f(x,y)|}{\rho(x,y)}<k\}$. For all $f\in C^k_\rho$, the weighted modulus of continuity is defined as

$$\omega_\rho(f;\delta_1,\delta_2)=\sup_{(x,y)\in\mathbb{R}^2_+}\ \sup_{|h_1|\leq\delta_1,|h_2|\leq\delta_2}\frac{|f(x+h_1,y+h_2)-f(x,y)|}{\rho(x,y)\rho(h_1,h_2)}$$

and

$$\begin{aligned}
&|f(t_1,t_2)-f(x,y)|\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\\
&\times\left(1+\frac{|t_1-x|}{\delta_n}\right)\left(1+\frac{|t_2-y|}{\delta_m}\right)(1+(t_1-x)^2)(1+(t_2-y)^2).
\end{aligned}\tag{16}$$

Theorem 1.6. *If the operators $\mathcal{D}^{\mu,\nu}_{n,m}(.;\ .)$ defined by (**??**) satisfying the conditions*

$$\begin{aligned}
\lim_{n,m\to\infty}\|\mathcal{D}^{\mu,\nu}_{n,m}(e_{0,0};.)-e_{0,0}\|\|&=0,\\
\lim_{n,m\to\infty}\|\mathcal{D}^{\mu,\nu}_{n,m}(e_{1,0};.)-e_{1,0}\|\|&=0,\\
\lim_{n,m\to\infty}\|\mathcal{D}^{\mu,\nu}_{n,m}(e_{0,1};.)-e_{0,1}\|\|&=0,
\end{aligned}$$

and

$$\lim_{n,m\to\infty} \|\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0}+e_{0,2};.)-(e_{2,0}+e_{0,2})|\|=0.$$

Then

$$\lim_{n,m\to\infty} \|\mathcal{D}_{n,m}^{\mu,\nu}(f;.)-f|\|=0,$$

for each $f \in C_\rho^k([0,\infty)\times[0,\infty))$.

Proof. In view of Lemma 1.2, we completes the proof of Theorem 1.6. □

Theorem 1.7. *Let* $f \in C_\rho^k([0,\infty)\times[0,\infty))$. *Then,*

$$\sup_{(x,y)\in\mathbb{R}_+^2} \frac{|\mathcal{D}_{n,m}^{\mu,\nu}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|}{(1+x^2+y^2)^3} \le K\omega_\rho(f;\delta_n,\delta_m)$$

holds for large values of n, m, *where*

$$\delta_n = o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right) \quad \textit{and} \quad \delta_m = o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right).$$

Proof. From (16) and the operators (**??**), we have

$$\begin{aligned}
&|\mathcal{D}_{n,m}^{\mu,\nu}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)| \\
&\le 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m) \\
&\times\left(1+\frac{\mathcal{D}_{n,m}^{\mu,\nu}(|t_1-x|;x,y,p_n,p_m,q_n,q_m)}{\delta_n}\right) \\
&\times\left(1+\frac{\mathcal{D}_{n,m}^{\mu,\nu}(|t_2-y|;x,y,p_n,p_m,q_n,q_m)}{\delta_m}\right) \\
&\times(1+\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m)) \\
&\times(1+\mathcal{D}_{n,m}^{\mu,\nu}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m)).
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|\mathcal{D}_{n,m}^{\mu,\nu}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)| \\
&\le 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\Bigg[1+\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m)) \\
&\frac{\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m))}}{\delta_n} \\
&\times\frac{\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m))\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^4;x,y,p_n,p_m,q_n,q_m))}}{\delta_n}\Bigg]
\end{aligned}$$

$$\begin{aligned}
(17) \qquad & \times \Bigg[1+\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m)) \\
&+\frac{\sqrt{\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m))}}{\delta_m} \\
&\times \frac{\sqrt{\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m))\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^4; x,y,p_n,p_m,q_n,q_m))}}{\delta_m}\Bigg].
\end{aligned}$$

From Lemma 1.2, we have

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x);q_n,q_m,p_n,p_m,x,y) && \leq o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)x, \\
&\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;q_n,q_m,p_n,p_m,x,y) && \leq o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)y, \\
(18) \qquad &\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^2;q_n,q_m,p_n,p_m,x,y) && \leq o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)(x^2+x), \\
&\mathcal{D}^{\mu,\nu}_{n,m}((t_2-x)^2;q_n,q_m,p_n,p_m,x,y) && \leq o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)(y^2+y).
\end{aligned}$$

Combining (17) and all identities in (18), we obtain

$$\begin{aligned}
&|\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)| \\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\Bigg[1+o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)x \\
&+\frac{\sqrt{o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)x}}{\delta_n}\times\frac{\sqrt{o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)xo\left(\right)(x^2+x)}}{\delta_n}\Bigg] \\
&\times\Bigg[1+o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)y+\frac{\sqrt{o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)y}}{\delta_m} \\
&\times\frac{\sqrt{o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)yo\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)(y^2+y)}}{\delta_n}\Bigg]. \qquad \square
\end{aligned}$$

Choosing $\delta_n=o(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}})$ and $\delta_m=o(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}})$, we find

$$|\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|$$

$$\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\left[1+\delta_n x+\sqrt{x}\frac{\sqrt{x(x^2+x)}}{\delta_n}\right]$$
$$\times\left[1+\delta_m y+\sqrt{y}\frac{\sqrt{y(y^2+y)}}{\delta_n}\right].$$

For sufficiently large values of n and m, we have

$$\sup_{(x,y)\in\mathbb{R}_+^2}\frac{|\mathcal{D}_{n,m}^{\mu,\nu}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|}{(1+x^2+y^2)^3}\leq K\omega_\rho(f;\delta_n,\delta_m),$$

where K is a positive constant independent of n, m and $\delta_n<1$, $\delta_m<1$.

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Common fixed point theorems for four self maps satisfying generalized (ψ, ϕ)-weak contraction in metric space

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Abstract. In this manuscript, we shall prove a common fixed point theorem for four weakly compatible self-maps P, Q, R and S on a metric space (M, d^*) satisfying the following generalized (ψ, ϕ)-weak contraction:

$$\psi(d^*(Ru, Sv)) \leq \psi(\triangle\,(u, v)) - \phi(\triangle\,(u, v)),$$

where

$$\begin{aligned}\triangle\,(u, v) = \max\Big\{&d^*(Ru, Sv), d^*(Ru, Pu), d^*(Sv, Qv),\\ &\frac{1}{2}[d^*(Pu, Sv) + d^*(Qv, Ru)],\\ &\frac{d^*(Pu, Ru)d^*(Qv, Sv)}{1 + d^*(Ru, Sv)}, \frac{d^*(Pu, Sv)d^*(Qv, Ru)}{1 + d^*(Ru, Sv)},\\ &d^*(Ru, Pu)[\frac{1 + d^*(Ru, Qv) + d^*(Sv, Pu)}{1 + d^*(Ru, Pu) + d^*(Sv, Qv)}]\Big\}.\end{aligned}$$

Also, we have proved common fixed point theorems for the above mentioned contraction using weakly compatible self-maps along with E.A. property and (CLR) property. An illustrative example is also provided to support our results.

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Keywords: fixed point, weakly compatible maps, E.A. property, (CLR) property, generalized (ψ, ϕ)-weak contraction.

1. Introduction

Definition 1.1. *A coincidence point of a pair of self-maps $P, Q : M \to M$ is a point $u \in M$ for which $Pu = Qu$.*

A common fixed point of a pair of self-maps $P, Q : M \to M$ is a point $u \in M$ for which $Pu = Qu = u$.

In 1996, Jungck [2] introduced the concept of weakly compatible maps to study common fixed point theorems:

Definition 1.2. *Let (M, d^*) be a metric space. A pair of self-maps $P, Q : M \to M$ is weakly compatible if they commute at their coincidence points, that is, if there exists $u \in M$ such that $PQu = QPu$, where u is coincidence point of P and Q.*

In 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 1.3. *Let (M, d^*) be a metric space. Two self-maps P and Q on M are said to satisfy the E.A. property, if there exists a sequence $\{u_n\}$ in M such that* $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t$, *for some $t \in M$.*

In 2011, Sintunavarat *et al.* [5] introduced the notion of (CLR) property as follows:

Definition 1.4. *Let (M, d^*) be a metric space. Two self-maps P and Q on M are said to satisfy the (CLR_P) property, if there exists a sequence $\{u_n\}$ in M such that* $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = Pt$, *for some $t \in M$.*

2. Main results

In this section, we prove some common fixed point theorems for weakly compatible four self maps along with (E.A.) property and (CLR) property.

Theorem 2.1. *Let (M, d^*) be a metric space and let P, Q, R and S be self-maps on M satisfying the followings:*

$$RM \subseteq QM, SM \subseteq PM, \tag{1}$$

For all $u, v \in M$, there exist right continuous functions $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\psi(0) = 0 = \phi(0)$ and $\psi(a) < a$ for $a > 0$ such that:

$$\psi(d^*(Ru, Sv) \leq \psi(\triangle\, (u, v)) - \phi(\triangle\, (u, v)), \tag{2}$$

where

$$\begin{aligned}\triangle\,(u,v) = \max\{&d^*(Ru,Sv), d^*(Ru,Pu), d^*(Sv,Qv),\\ &\frac{1}{2}[d^*(Pu,Sv)+d^*(Qv,Ru)],\\ &\frac{d^*(Pu,Ru)d^*(Qv,Sv)}{1+d^*(Ru,Sv)}, \frac{d^*(Pu,Sv)d^*(Qv,Ru)}{1+d^*(Ru,Sv)},\\ &d^*(Ru,Pu)[\frac{1+d^*(Ru,Qv)+d^*(Sv,Pu)}{1+d^*(Ru,Pu)+d^*(Sv,Qv)}]\}.\end{aligned}$$

If one of PM,QM,RM or SM is complete subspace of M, then the pair (P,R) or (Q,S) have a coincidence point. Moreover, if the pairs (P,R) and (Q,S) are weakly compatible, then P,Q,R and S have a unique common fixed point.

Proof. Let $u_0 \in M$ be an arbitrary point of M. From (2), we can construct a sequence $\{v_n\}$ in M as follows:

$$v_{2n+1} = Ru_{2n} = Qu_{2n+1}, v_{2n+2} = Su_{2n+1} = Pu_{2n+2}, \tag{3}$$

for all $n = 0,1,2,....$ Now, we define $d_n^* = d^*(v_n, v_{n+1})$. If $d_{2n}^* = 0$ for some n, then $d^*(v_{2n}, v_{2n+2}) = 0$. Then $v_{2n} = v_{2n+1}$, that is, $Su_{2n-1} = Pu_{2n} = Ru_{2n} = Qu_{2n+1}$ and P and R have a coincidence point. Similarly, if $d_{2n+1}^* = 0$, then Q and S have a coincidence point. Assume that $d_n^* \neq 0$ for each n.

On putting $u = u_{2n}$ and $v = u_{2n+1}$ in (2), we get

$$\psi(d^*(Ru_{2n}, Su_{2n+1})) \leq \psi(\triangle\,(u_{2n}, u_{2n+1})) - \phi(\triangle\,(u_{2n}, u_{2n+1})), \tag{4}$$

where

$$\begin{aligned}\triangle\,(u_{2n},u_{2n+1}) &= \max\{d^*(Ru_{2n},Su_{2n+1}), d^*(Ru_{2n},Pu_{2n}), d^*(Su_{2n+1},Qu_{2n+1}),\\ &\frac{1}{2}[d^*(Pu_{2n},Su_{2n+1})+d^*(Qu_{2n+1},Ru_{2n})],\\ &\frac{d^*(Pu_{2n},Ru_{2n})\cdot d^*(Qu_{2n+1},Su_{2n+1})}{1+d^*(Ru_{2n},Su_{2n+1})},\\ &\frac{d^*(Pu_{2n},Su_{2n+1})\cdot d^*(Qu_{2n+1},Ru_{2n})}{1+d^*(Ru_{2n},Su_{2n+1})},\\ &d^*(Ru_{2n},Pu_{2n})\frac{1+d^*(Ru_{2n},Qu_{2n+1})+d^*(Su_{2n+1},Pu_{2n})}{1+d^*(Ru_{2n},Pu_{2n})+d^*(Su_{2n+1},Qu_{2n+1})}\}\\ &= \max\{d^*(v_{2n+1},v_{2n+2}), d^*(v_{2n+1},v_{2n}), d^*(v_{2n},v_{2n+1}),\\ &\frac{1}{2}[d^*(v_{2n},v_{2n+2})+d^*(v_{2n+1},v_{2n+1})],\\ &\frac{d^*(v_{2n},v_{2n+1})\cdot d^*(v_{2n+1},v_{2n+2})}{1+d^*(v_{2n+1},v_{2n+2})},\\ &\frac{d^*(v_{2n},v_{2n+2})\cdot d^*(v_{2n+1},v_{2n+1})}{1+d^*v_{2n+1},v_{2n+2})},\\ &d^*(v_{2n+1},v_{2n})\frac{1+d^*(v_{2n+1},v_{2n+1})+d^*(v_{2n+2},v_{2n})}{1+d^*(v_{2n+1},v_{2n})+d^*(v_{2n+2},v_{2n+1})}\}\end{aligned}$$

$$= \max\{d^*_{2n+1}, d^*_{2n}, d^*_{2n+1}, \frac{1}{2}[d^*_{2n} + d^*_{2n+1} + 0], \frac{d^*_{2n} \cdot d^*_{2n+1}}{1 + d^*_{2n+1}},$$
$$0, d^*_{2n}\frac{1 + d^*_{2n} + d^*_{2n+1}}{1 + d^*_{2n} + d^*_{2n+1}}\},$$

that is

$$\triangle\, (u_{2n}, u_{2n+1}) = \max\{d^*_{2n}, d^*_{2n+1}\}. \tag{5}$$

Now, from (4), we have

$$\psi(d^*(v_{2n+1}, v_{2n+2})) \le \psi(\max\{d^*_{2n}, d^*_{2n+1}\}) - \phi(\max\{d^*_{2n}, d^*_{2n+1}\}), \tag{6}$$

Now, if $d^*_{2n+1} \ge d^*_{2n}$ for some n, then from (6), we get

$$\begin{aligned} \psi(d^*_{2n+1}) &\le \psi(d^*_{2n+1}) - \phi(d^*_{2n+1}) \\ &< \psi(d^*_{2n+1}), \end{aligned} \tag{7}$$

which is a contradiction. Thus, $d^*_{2n} > d^*_{2n+1}$ for all n, and so, from (6), we have

$$\psi(d^*_{2n+1}) \quad \le \quad \psi(d^*_{2n}) - \phi(d^*_{2n})\, for\, all\, n \in N. \tag{8}$$

Similarly,

$$\begin{aligned} \psi(d^*_{2n}) &\le \psi(d^*_{2n-1}) - \phi(d^*_{2n-1}), \\ \psi(d^*_{2n-1}) &\le \psi(d^*_{2n-2}) - \phi(d^*_{2n-2}). \end{aligned}$$

In general, we have for all $n = 1, 2, 3.\ldots$

$$\begin{aligned} \psi(d^*_n) &\le \psi(d^*_{n-1}) - \phi(d^*_{n-1}) \\ &< \psi(d^*_{n-1}). \end{aligned} \tag{9}$$

Hence, sequence $\{\psi(d^*_n)\}$ is monotonically decreasing and bounded below. Thus, there exists $s \ge 0$, such that

$$\lim_{n\to\infty} \psi(d^*_n) = s. \tag{10}$$

From (9), we deduce that

$$0 \le \phi(d^*_{n-1}) \le \psi(d^*_{n-1}) - \psi(d^*_n)$$

Taking limit as $n \to \infty$ and using (10), we get

$$\lim_{n\to\infty} \phi(d^*_{n-1}) = 0,$$

this implies that

$$\lim_{n\to\infty} \phi(d^*_{n-1}) = \lim_{n\to\infty} \phi(d^*(v_{n-1}, v_n)) = 0. \tag{11}$$

$$\lim_{n\to\infty} d_n^* = \lim_{n\to\infty} d^*(v_n, v_{n+1}) = 0. \tag{12}$$

Now, we claim that $\{v_n\}$ is a Cauchy sequence. For this, it is sufficient to show that $\{v_{2n}\}$ is a Cauchy sequence. Let, if possible, $\{v_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$, such that for each even integer $2a$ there exists even integers $2m(a) > 2n(a) > 2a$ such that

$$d^*(v_{2n(a)}, v_{2m(a)}) \geq \epsilon. \tag{13}$$

for every even integer $2a$, suppose that $2m(a)$ be the least positive integer exceeding $2n(a)$ satisfying (13), such that

$$d^*(v_{2n(a)}, v_{2m(a)-2}) < \epsilon. \tag{14}$$

From (13), we get

$$\begin{aligned}\epsilon &\leq d^*(v_{2n(a)}, v_{2m(a)}) \\ &\leq d^*(v_{2n(a)}, v_{2m(a)-2}) + d^*(v_{2m(a)-2}, v_{2m(a)-1}) + d^*(v_{2m(a)-1}, v_{2m(a)}).\end{aligned}$$

Using (12) and (14) in the above inequality, we get

$$\lim_{n\to\infty} d^*(v_{2n(a)}, v_{2m(a)}) = \epsilon. \tag{15}$$

Also, by the triangular inequality,

$$|d^*(v_{2n(a)+1}, v_{2m(a)-1})) + d^*(v_{2n(a)}, v_{2m(a)})| \leq d^*_{2m(a)-1} + d^*_{2m(a)}. \tag{16}$$

Using (12), we have

$$\lim_{n\to\infty} d^*(v_{2n(a)}, v_{2m(a)-1}) = \lim_{n\to\infty} d^*(v_{2n(a)+1}, v_{2m(a)-1}) = \epsilon. \tag{17}$$

Now, from (2), we have

$$\begin{aligned}\psi(d^*(Ru_{2n(a)}, Su_{2m(a)-1})) \leq \psi(\triangle\ (u_{2n(a)}, u_{2m(a)-1})) \\ - \phi(\triangle\ (u_{2n(a)}, u_{2m(a)-1})),\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\triangle\ (u_{2n(a)}, u_{2m(a)-1}) \\ &= \max\{d^*(Ru_{2n(a)}, Su_{2m(a)-1}), d^*(Ru_{2n(a)}, Pu_{2n(a)}), \\ &d^*(Su_{2m(a)-1}, Qu_{2m(a)-1}), \\ &\frac{1}{2}[d^*(Pu_{2n(a)}, Su_{2m(a)-1}) + d^*(Qu_{2m(a)-1}, Ru_{2m(a)})],\end{aligned}$$

$$\frac{d^*(Pu_{2n(a)}, Ru_{2n(a)}) \cdot d^*(Qu_{2m(a)-1}, Su_{2m(a)-1})}{1 + d^*(Ru_{2m(a)}, Su_{2m(a)-1})},$$
$$\frac{d^*(Pu_{2m(a)}, Su_{2m(a)-1}) \cdot d^*(Qu_{2m(a)-1}, Ru_{2n(a)})}{1 + d^*(Ru_{2m(a)}, Su_{2m(a)-1})},$$
$$d^*(Ru_{2n(a)}, Pu_{2n(a)}) \frac{1 + d^*(Ru_{2n(a)}, Qu_{2m(a)-1}) + d^*(Su_{2m(a)-1}, Pu_{2n(a)})}{1 + d^*(Ru_{2n(a)}, Pu_{2n(a)}) + d^*(Su_{2m(a)-1}, Qu_{2m(a)-1})}\}$$
$$= \max\{d^*(v_{2n(a)+1}, v_{2m(a)}), d^*(v_{2n(a)+1}, v_{2m(a)}), d^*(v_{2m(a)}, v_{2m(a)-1}),$$
$$\frac{1}{2}[d^*(v_{2n(a)}, v_{2m(a)}) + d^*(v_{2m(a)-1}, v_{2m(a)+1})],$$
$$\frac{d^*(v_{2n(a)}, v_{2n(a)+1}) \cdot d^*(v_{2m(a)-1}, v_{2m(a)})}{1 + d^*(v_{2n(a)+1}, v_{2m(a)})},$$
$$\frac{d^*(v_{2n(a)}, v_{2n(a)+1}) \cdot d^*(v_{2m(a)-1}, v_{2m(a)})}{1 + d^*(v_{2n(a)+1}, v_{2m(a)})},$$
$$d^*(v_{2n(a)+1}, v_{2n(a)}) \frac{1 + d^*(v_{2n(a)+1}, v_{2m(a)-1}) + d^*(v_{2m(a)}, v_{2n(a)})}{1 + d^*(v_{2n(a)+1}, v_{2n(a)}) + d^*(v_{2m(a)}, v_{2m(a)-1})}\}.$$

Now, taking limit as $a \to \infty$ and using equations (12), (14), (15) and (17), we get $\triangle\,(u_{2n(a)}, u_{2m(a)-1}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon + \epsilon), 0, \frac{\epsilon\cdot\epsilon}{1+\epsilon}, 0\}$, that is

$$\triangle\,(u_{2n(a)}, u_{2m(a)-1}) = \epsilon.$$

Now, by (18), we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction, since $\epsilon > 0$. Thus, $\{v_{2n}\}$ is a Cauchy sequence. So, $\{v_n\}$ is a Cauchy sequence. Now, suppose that PM is complete. Since $\{v_{2n}\}$ is contained in PM and has limit in PM say p, that is, $\lim_{n\to\infty} v_{2n} = p$. Let $q \in P^{-1}(p)$ then $Pq = p$.

Now, we shall prove that $Rq = p$.

Let, if possible, $Rq \neq p$ that is, $d^*(Rq, p) = k > 0$. On putting $u = q$,$v = u_{2n-1}$ in (2), we have

$$\psi(d^*(Rq, Su_{2n-1})) \leq \psi(\triangle\,(q, u_{2n-1})) - \phi(\triangle\,(q, u_{2n-1})), \tag{19}$$

where

$$\triangle\,(q, u_{2n-1}) = \max\{d^*(Rq, Su_{2n-1}), d^*(Rq, Pq), d^*(Su_{2n-1}, Qu_{2n-1}),$$
$$\frac{1}{2}[d^*(Pq, Su_{2n-1}) + d^*(Qu_{2n-1}, Rq)], \frac{d^*(Pq, Rq) \cdot d^*(Qu_{2n-1}, Su_{2n-1})}{1 + d^*(Rq, Su_{2n-1})},$$
$$\frac{(Pq, Su_{2n-1}) \cdot d^*(Qu_{2n-1}, Rq)}{1 + d^*(Rq, Su_{2n-1})},$$
$$d^*(Rq, Pq) \frac{1 + d^*(Rq, Qu_{2n-1}) + d^*(Su_{2n-1}, Pq)}{1 + d^*(Rq, Pq) + d^*(Su_{2n-1}, Qu_{2n-1})}\}.$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \triangle\,(q, u_{2n-1}) = \lim_{n\to\infty} \max\{d^*(Rq, Su_{2n-1}), d^*(Rq, Pq), d^*(Su_{2n-1}, Qu_{2n-1}),$$
$$\frac{1}{2}[d^*(Pq, Su_{2n-1}) + d^*(Qu_{2n-1}, Rq)], \frac{d^*(Pq, Rq)\cdot d^*(Qu_{2n-1}, Su_{2n-1})}{1 + d^*(Rq, Su_{2n-1})},$$
$$\frac{d^*(Pq, Su_{2n-1})\cdot d^*(Qu_{2n-1}, Rq)}{1 + d^*(Rq, Su_{2n-1})},$$
$$d^*(Rq, Pq)\frac{1 + d^*(Rq, Qu_{2n-1}) + d^*(Su_{2n-1}, Pq)}{1 + d^*(Rq, Pq) + d^*(Su_{2n-1}, Qu_{2n-1})}\}$$
$$= \max\{d^*(Rq, p), d^*(Rq, p), d^*(p, p), \frac{1}{2}[d^*(Pq, p) + d^*(p, Rq)],$$
$$\frac{d^*(p, Rq)\cdot d^*(p, p)}{1 + d^*(Rq, p)}, \frac{d^*(p, p)\cdot d^*(p, Rq)}{1 + d^*(Rq, p)}, d^*(Rq, p)\frac{1 + d^*(Rq, p) + d^*(p, p)}{1 + d^*(Rq, p) + d^*(p, p)}\}.$$
$$\lim_{n\to\infty} \triangle\,(q, u_{2n-1}) = d^*(p, Rq) = k.$$

Thus, from (19),we have $\psi(d^*(Rq, p)) \le \psi(k) - \phi(k)$, $\psi(k) \le \psi(k) - \phi(k)$, which is a contradiction, since $k > 0$. Thus, $Rq = Pq = p$. Hence, q is coincidence point of the pair (P, R). Since $RM \subseteq QM$, $Rq = p$ implies that, $p \in QM$. Let $w \in B^{-1}p$. Then $Bw = p$. By using the same arguments as above, we can easily verify that S $w = p = Qw$, that is, w is the coincidence point of the pair (Q, S). Similarly, we can prove the result if QM is complete subspace of M instead of PM. Now, if SM is complete then by (1), $p \in SM \subseteq PM$. In the same manner if RM is complete then $p \in RM \subseteq QM$. Now, since the pair (P, R) and (Q, S) are weakly compatible, so

$$\begin{aligned} p &= Rq = Pq = Sw = Qw,\\ Pp &= PRq = RPq = Rp,\\ Qp &= QSw = SQw = Sp. \end{aligned} \tag{20}$$

Now, we shall prove that $Sp = p$. Let, if possible, $Sp \ne p$. From (2), we have

$$\psi(d^*(p, Sp)) = \psi(d^*(Rq, Sp)) \le \psi(\triangle\,(q, p)) - \phi(\triangle\,(q, p)),$$

where

$$\triangle\,(q, p) = \max\{d^*(Rq, Sp), d^*(Rq, Pq), d^*(Sp, Qp), \frac{1}{2}[d^*(Pq, Sp) + d^*(Qp, Rq)],$$
$$\frac{d^*(Pq, Rq)\cdot d^*(Qp, Sp)}{1 + d^*(Rq, Sp)}, \frac{d^*(Pq, Sp)\cdot d^*(Qp, Rq)}{1 + d^*(Rq, Sp)},$$
$$d^*(Rq, Pq)\frac{1 + d^*(Rq, Qp) + d^*(Sp, Pq)}{1 + d^*(Rq, Pq) + d^*(Sp, Qp)}\}.$$

Using (20), we have

$$\triangle\,(q, p) = \max\{d^*(p, Sp), 0, 0, \frac{1}{2}[d^*(p, Sp) + d^*(Sp, p)], 0, \frac{d^*(Pq, Sp)\cdot d^*(Qp, Rq)}{1 + d^*(Rq, Sp)}, 0\}$$
$$\triangle\,(q, p) = d^*(p, Sp).$$

Thus, we have

$$\psi(d^*(p,Sp)) \leq \psi(d^*(p,Sp)) - \phi(d^*(p,Sp)) < \psi(d^*(p,Sp)),$$

which is a contradiction. So, $Sp = p$. Similarly, $Rp = p$. Thus, we get $Pp = Rp = Qp = Sp = p$. Hence, p is the common fixed point of P, Q, R and S. For the uniqueness, let t be another common fixed point of P, Q, R and S.

Now, we claim that $t = p$. Let, if possible $t \neq p$. From (2), we have

$$\begin{aligned}\psi(d^*(p,t)) = \psi(d^*(Rp,St)) &\leq \psi(\triangle\,(p,t)) - \phi(\triangle\,(p,t)) \\ &= \psi(d^*(p,t)) - \phi(d^*(p,t))\, since \,\triangle\,(p,t) = d^*(p,t) < \psi(d^*(p,t)),\end{aligned}$$

which is a contradiction. Thus, $t = p$, and hence the uniqueness follows. This completes the proof of the theorem. □

Theorem 2.2. *Let (M, d^*) be a metric space and P, Q, R and S be self-maps on M satisfying (1) and (2) and the followings:*

$$\text{(21)} \qquad Pairs\,(P,R)\,and\,(Q,S)\,are\,weakly\,compatible.$$

$$\text{(22)} \qquad Pair\,(P,R)\,or\,(Q,S)\,satisfy\,the\,E.A.\,property.$$

If any one of PM, QM, RM or SM is a complete subspace of M, then P, Q, R and S have a unique common fixed point.

Proof. Suppose that the pair (P, R) satisfies the E.A. property. Then, there exists a sequence $\{u_n\}$ in M, such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = p$, for some p in M. Since $RM \subseteq QM$, there exists a sequence $\{v_n\}$ in M such that $R\{u_n\} = Q\{v_n\}$. Hence, $\lim_{n\to\infty} Qv_n = p$.

We shall show that $\lim_{n\to\infty} Sv_n = p$.

Let, if possible, $Sv_n = q \neq p$. From (2), we have

$$\psi(d^*(Ru_n, Sv_n)) \leq \psi(\triangle\,(u_n,v_n)) - \phi(\triangle\,(u_n,v_n)).$$

Now, taking limit as $n \to \infty$, we get

$$\text{(23)} \qquad \lim_{n\to\infty} \psi(d^*(Ru_n,Sv_n)) \leq \lim_{n\to\infty} \psi(\triangle\,(u_n,v_n)) - \lim_{n\to\infty} \phi(\triangle\,(u_n,v_n)),$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\,(u_n,v_n) = \lim_{n\to\infty} \max\{&d^*(Ru_n,Sv_n), d^*(Ru_n,Pu_n), d^*(Sv_n,Qv_n), \\ &\frac{1}{2}[d^*(Pu_n,Sv_n) + d^*(Qv_n,Ru_n)], \\ &\frac{d^*(Pu_n,Ru_n)\cdot d^*(Qv_n,Sv_n)}{1 + d^*(Ru_n,Sv_n)},\end{aligned}$$

$$\frac{d^*(Pu_n, Sv_n) \cdot d^*(Qv_n, Ru_n)}{1 + d^*(Ru_n, Sv_n)},$$
$$d^*(Ru_n, Pu_n)\frac{1 + d^*(Ru_n, Qv_n) + d^*(Sv_n, Pu_n)}{1 + d^*(Ru_n, Pu_n) + d^*(Sv_n, Qv_n)}\}$$
$$= \max\{d^*(p,q), d^*(p,p), d^*(q,p), \frac{1}{2}[d^*(p,q) + d^*(p,p)],$$
$$\frac{d^*(p,p) \cdot d^*(p,q)}{1 + d^*(p,q)}, \frac{d^*(p,p) \cdot d^*(p,q)}{1 + d^*(p,q)},$$
$$d^*(p,p)[\frac{1 + d^*(p,p) + d^*(q,p)}{1 + d^*(p,p) + d^*(q,p)}]\}$$
$$= d^*(p,q).$$

From (23), we have

$$\psi(d^*(p,q)) \leq \psi(d^*(p,q)) - \phi(d^*(p,q)) < \psi(d^*(p,q)),$$

which is a contradiction. Therefore, $p = q$, that is $\lim_{n\to\infty} Sv_n = p$. Suppose that QM is a complete subspace of M. Then $p = Qa$ for some $a \in M$. Subsequently, we have $\lim_{n\to\infty} Sv_n = \lim_{n\to\infty} Ru_n = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qv_n = p = Qa$. Now, we shall show that $Sa = Qa$. Let, if possible $Sa \neq Qa$.

From (2), we have

$$\psi(d^*(Ru_n, Sa)) \leq \psi(\triangle\,(u_n, a)) - \phi(\triangle\,(u_n, a)).$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d^*(Ru_n, Sa)) \leq \lim_{n\to\infty} \psi(\triangle\,(u_n, a)) - \lim_{n\to\infty} \phi(\triangle\,(u_n, a)), \tag{24}$$

where

$$\lim_{n\to\infty} \triangle\,(u_n, a) = \lim_{n\to\infty} \max\{d^*(Ru_n, Sa), d^*(Ru_n, Pu_n), d^*(Sa, Qa),$$
$$\frac{1}{2}[d^*(Pu_n, Sa) + d^*(Qa, Ru_n)],$$
$$\frac{d^*(Pu_n, Ru_n) \cdot d^*(Qa, Sa)}{1 + d^*(Ru_n, Sa)},$$
$$\frac{d^*(Pu_n, Sa) \cdot d^*(Qa, Ru_n)}{1 + d^*(Rq, Sa)},$$
$$d^*(Ru_n, Pu_n)\frac{1 + d^*(Ru_n, Qa) + d^*(Sa, Pu_n)}{1 + d^*(Ru_n, Pu_n) + d^*(Sa, Qa)}\}$$
$$= \max\{d^*(p, Sa), d^*(p,p), d^*(Sa, p), \frac{1}{2}[d^*(p, Sa) + d^*(p,p)],$$
$$\frac{d^*(p,p) \cdot d^*(p, Sa)}{1 + d^*(p, Sa)}, \frac{d^*(p,p) \cdot d^*(p, Sa)}{1 + d^*(p, Sa)},$$
$$d^*(p,p)[\frac{1 + d^*(p,p) + d^*(Sa,p)}{1 + d^*(p,p) + d^*(Sa,p)}]\}$$
$$= d^*(Sa, p).$$

Thus, from (24), we have

$$\psi(d^*(p,Sa)) \leq \psi(d^*(p,Sa)) - \phi(d^*(p,Sa)) < \psi(d^*(p,Sa)),$$

which is a contradiction. Therefore, $Sa = p = Qa$. Since Q and S are weakly compatible, therefore, $QSa = SQa$, implies that, $SSa = SQa = QSa = QQa$. Since $SM \subseteq PM$, there exists $b \in M$, such that, $Sa = Pb$.

Now, we claim that $Pb = Rb$. Let, if possible, $Pb \neq Rb$. From (2), we have

$$\psi(d^*(Rb,Sa)) \leq \psi(\triangle\ (b,a)) - \phi(\triangle\ (b,a)), \tag{25}$$

where

$$\begin{aligned}\triangle\ (b,a) = \max\{&d^*(Rb,Sa), d^*(Rb,Pb), d^*(Sa,Qa),\\ &\frac{1}{2}[d^*(Pb,Sa) + d^*(Qa,Rb)],\\ &\frac{d^*(Pb,Rb)\cdot d^*(Qa,Sa)}{1+d^*(Rb,Sa)},\\ &\frac{d^*(Pb,Sa)\cdot d^*(Qa,Rb)}{1+d^*(Rb,Sa)},\\ &d^*(Rb,Pb)\frac{1+d^*(Rb,Qa)+d^*(Sa,Pb)}{1+d^*(Rb,Pb)+d^*(Sa,Qa)}\}\\ = d^*&(Rb,Sa).\end{aligned}$$

From (25), we have

$$\psi(d^*(Rb,Sa)) \leq \psi(d^*(Rb,Sa)) - \phi(d^*(Rb,Sa)) < \psi(d^*(Rb,Sa)),$$

which is a contradiction. Therefore, $Rb = Sa = Pb$. Now, since (P,R) is weakly compatible. This implies that $PRb = RPb = RRb = PPb$.

Now, we claim that Sa is common fixed point of P, Q, R and S. Let, if possible, $SSa \neq Sa$. From (2), we have

$$\psi(d^*(Sa,SSa)) = \psi(d^*(Rb,SSa)) \leq \psi(\triangle\ (b,Sa)) - \phi(\triangle\ (b,Sa)), \tag{26}$$

where

$$\begin{aligned}\triangle\ (b,Sa) = \max\{&d^*(Rb,SSa), d^*(Rb,Pb), d^*(SSa,QSa),\\ &\frac{1}{2}[d^*(Pb,SSa) + d^*(QSa,Rb)],\\ &\frac{d^*(Pb,Rb)\cdot d^*(QSa,SSa)}{1+d^*(Rb,SSa)},\\ &\frac{d^*(Pb,SSa)\cdot d^*(QSa,Rb)}{1+d^*(Rb,SSa)},\\ &d^*(Rb,Pb)\frac{1+d^*(Rb,QSa)+d^*(SSa,Pb)}{1+d^*(Rb,Pb)+d^*(SSa,QSa)}\}\\ = d^*&(Sa,SSa).\end{aligned}$$

Thus, from (26), we have

$$\psi(d^*(Sa, SSa)) \leq \psi(d^*(Sa, SSa)) - \phi(d^*(Sa, SSa)) < \psi(d^*(Sa, SSa)),$$

which is a contradiction. Therefore, $Sa = SSa = QSa$. Hence, Sa is the common fixed point of Q and S. Similarly, we can prove that Rb is common fixed point of R and P. Since $Sa = Rb$, Sa is the common fixed point of P, Q, R and S. If we assume RM is complete subspace of M, the proof is similar. Similarly we can prove the theorem for cases when PM or QM is a complete subspace of M. Since $SM \subseteq PM$ and $RM \subseteq QM$.

Now, we shall prove the uniqueness of common fixed point. If possible, let c and d be two common fixed points of P, Q, R and S, such that $c \neq d$. From (2), we have

$$\psi(d^*(c, d)) = \psi(d^*(Rc, Sd)) \leq \psi(\triangle\,(c, d)) - \phi(\triangle\,(c, d)), \tag{27}$$

where

$$\begin{aligned}\triangle\,(c, d) = \max\{&d^*(Rc, Sd), d^*(Rc, Pc), d^*(Sd, Qd),\\ &\frac{1}{2}[d^*(Pc, Sd) + d^*(Qd, Rc)],\\ &\frac{d^*(Pc, Rd) \cdot d^*(Qd, Rc)}{1 + d^*(Rc, Sd)},\\ &\frac{d^*(Pc, Sd) \cdot d^*(Qd, Rc)}{1 + d^*(Rc, Sd)},\\ &d^*(Rc, Pc)\frac{1 + d^*(Rc, Qd) + d^*(Sd, Pc)}{1 + d^*(Rc, Pc) + d^*(Sd, Qd)}\}\\ = {}&d^*(c, d).\end{aligned}$$

From (27), we have

$$\psi(d^*(c, d)) \leq \psi(d^*(c, d)) - \phi(d^*(c, d)) < \psi(d^*(c, d)),$$

which is a contradiction. Therefore, $c = d$ and this follows the uniqueness and completes the proof of the theorem. □

Theorem 2.3. *Let (M, d^*) be a metric space. Let P, Q, R and S be self maps on M satisfying (1), (2), (21) and the followings:*

$$\begin{aligned}&RM \subseteq QM \text{ and the pair } (P, R) \text{ satisfies } (CLR_P) \text{ property},\\ &SM \subseteq PM \text{ and the pair } (Q, S) \text{ satisfies } (CLR_Q) \text{ property}.\end{aligned} \tag{28}$$

Then P, Q, R and S have unique common fixed point.

Proof. Without loss of generality, assume that $RM \subseteq QM$ and the pair (P, R) satisfies the (CLR_P) property. Then, there exists a sequence $\{u_n\}$ in M such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = Pp$, for some p in M.

Since $RM \subseteq QM$, there exists a sequence $\{v_n\}$ in M such that $R\{u_n\} = Q\{v_n\}$.

Hence, $\lim_{n\to\infty} Qv_n = Pp$. Now, we shall show that $\lim_{n\to\infty} Sv_n = Pp$. Let if possible, $\lim_{n\to\infty} Sv_n = q \neq Pp$. From (2), we have

$$\psi(d^*(Ru_n, Sv_n)) \leq \psi(\triangle\ (u_n, v_n)) - \phi(\triangle\ (u_n, v_n)).$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d^*(Ru_n, Sv_n)) \leq \lim_{n\to\infty} \psi(\triangle\ (u_n, v_n)) - \lim_{n\to\infty} \phi(\triangle\ (u_n, v_n)), \tag{29}$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\ (u_n, v_n) = \lim_{n\to\infty} \max\{&d^*(Ru_n, Sv_n), d^*(Ru_n, Pu_n), d^*(Sv_n, Qv_n),\\ &\frac{1}{2}[d^*(Pu_n, Sv_n) + d^*(Qv_n, Ru_n)],\\ &\frac{d^*(Pu_n, Ru_n)\cdot d^*(QV_n, Sv_n)}{1 + d^*(Ru_n, Sv_n)},\\ &\frac{d^*(Pu_n, Sv_n)\cdot d^*(Qv_n, Ru_n)}{1 + d^*(Ru_n, Sv_n)},\\ &d^*(Ru_n, Pu_n)\frac{1 + d^*(Ru_n, Qv_n) + d^*(Sv_n, Pu_n)}{1 + d^*(Ru_n, Pu_n) + d^*(Sv_n, Qv_n)}\}\\ = \max\{&d^*(Pq, q), d^*(Pp, Pp), d^*(q, Pp), \frac{1}{2}[d^*(Pp, q) + d^*(Pp, Pp)],\\ &\frac{d^*(Pq, q)\cdot d^*(Pp, Pp)}{1 + d^*(Pp, q)}, \frac{d^*(Pp, Pp)\cdot d^*(Pp, q)}{1 + d^*(Pp, q)},\\ &d^*(Pp, Pp)[\frac{1 + d^*(Pp, Pp) + d^*(q, Pp)}{1 + d^*(Pp, Pp) + d^*(q, Pp)}]\}\\ = d^*&(Pp, q).\end{aligned}$$

From (29), we have

$$\psi(d^*(Pp, q)) \leq \psi(d^*(Pp, q)) - \phi(d^*(Pp, q)) < \psi(d^*(Pp, q)),$$

which is a contradiction. Therefore, $Pp = q$, that is, $\lim_{n\to\infty} Sv_n = Pp = q$.

Subsequently, we have $\lim_{n\to\infty} Sv_n = \lim_{n\to\infty} Ru_n = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qv_n = Pp = q$. Now, we shall show that $Rp = q$. Let, if possible, $Rp \neq q$. From (2), we have

$$\psi(d^*(Rp, Sv_n)) \leq \psi(\triangle\ (p, v_n)) - \phi(\triangle\ (p, v_n)).$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d^*(Rp, Sv_n)) \leq \lim_{n\to\infty} \psi(\triangle\ (p, v_n)) - \lim_{n\to\infty} \phi(\triangle\ (p, v_n)), \tag{30}$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\,(p,v_n) &= \lim_{n\to\infty} \max\{d^*(Rp,Sv_n), d^*(Rp,Pp), d^*(Sv_n,Qv_n),\\ &\quad \frac{1}{2}[d^*(Pp,Sv_n)+d^*(Qv_n,Rp)], \frac{d^*(Pp,Rp)\cdot d^*(QV_n,Sv_n)}{1+d^*(Rp,Sv_n)},\\ &\quad \frac{d^*(Pp,Sv_n)\cdot d^*(Qv_n,Rp)}{1+d^*(Rp,Sv_n)},\\ &\quad d^*(Rp,Pp)\frac{1+d^*(Rp,Qv_n)+d^*(Sv_n,Pp)}{1+d^*(Rp,Pp)+d^*(Sv_n,Qv_n)}\}\\ &= \max\{d^*(Rp,q), d^*(Rp,q), d^*(q,q), \frac{1}{2}[d^*(q,q)+d^*(q,Rp)],\\ &\quad \frac{d^*(q,Rp)\cdot d^*(q,q)}{1+d^*(Rp,q)}, \frac{d^*(q,q)\cdot d^*(q,Rp)}{1+d^*(Rp,q)},\\ &\quad d^*(Rp,q)[\frac{1+d^*(Rp,q)+d^*(q,q)}{1+d^*(Rp,q)+d^*(q,q)}]\}\\ &= d^*(Rp,q).\end{aligned}$$

Thus, from (30), we get

$$\psi(d^*(Rp,q)) \leq \psi(d^*(Rp,q)) - \phi(d^*(Rp,q)) < \psi(d^*(Rp,q)),$$

which is a contradiction. Therefore, $Rp = q = Pp$. Since the pair (P,R) is weakly compatible, it follows that $Pq = Rq$. Also, since $RM \subseteq QM$, there exists some r in M, such that, $Rp = Qr$, that is, $Qr = q$. Now, we show that $Sr = q$. Let, if possible $Sr \neq q$. From (2), we have

$$\psi(d^*(Ru_n,Sr)) \leq \psi(\triangle\,(u_n,r)) - \phi(\triangle\,(u_n,r)).$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d^*(Ru_n,Sr)) \leq \lim_{n\to\infty} \psi(\triangle\,(u_n,r)) - \lim_{n\to\infty} \phi(\triangle\,(u_n,r)), \tag{31}$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\,(u_n,r) &= \lim_{n\to\infty} \max\{d^*(Ru_n,Sr), d^*(Ru_n,Pu_n), d^*(Sr,Qr),\\ &\quad \frac{1}{2}[d^*(Pu_n,Sr)+d^*(Qr,Ru_n)],\\ &\quad \frac{d^*(Pu_n,Ru_n)\cdot d^*(Qr,Sr)}{1+d^*(Ru_n,Sr)}, \frac{d^*(Pu_n,Sr)\cdot d^*(Qr,Ru_n)}{1+d^*(Ru_n,Sr)},\\ &\quad d^*(Ru_n,Pu_n)\frac{1+d^*(Ru_n,Qr)+d^*(Sr,Pu_n)}{1+d^*(Ru_n,Pu_n)+d^*(Sr,Qr)}\}\\ &= \max\{d^*(q,Sr), d^*(q,q), d^*(Sr,q), \frac{1}{2}[d^*(q,Sr)+d^*(q,q)],\end{aligned}$$

$$\frac{d^*(q,q)\cdot d^*(q,Sr)}{1+d^*(q,Sr)}, \frac{d^*(q,Sr)\cdot d^*(q,q)}{1+d^*(q,Sr)},$$
$$d^*(q,q)[\frac{1+d^*(q,q)+d^*(Sr,q)}{1+d^*(q,q)+d^*(Sr,q)}]\}$$
$$= d^*(Sr,q).$$

Thus, from (31), we get

$$\psi(d^*(q,Sr)) \leq \psi(d^*(q,Sr)) - \phi(d^*(q,Sr)) < \psi(d^*(q,Sr)),$$

which is a contradiction. Therefore, $Sr = q = Qr$. Since the pair (Q,S) is weakly compatible, it follows that $Sq = Qq$. Now, we claim that $Rq = Sq$. Let, if possible, $Rq \neq Sq$. From (2), we have

$$\psi(d^*(Rq,Sq)) \leq \psi(\triangle\,(q,q)) - \phi(\triangle\,(\triangle\,(q,q)), \tag{32}$$

where

$$\triangle\,(q,q) = \max\{d^*(Rq,Sq), d^*(Rq,Pq), d^*(Sq,Qq), \frac{1}{2}[d^*(Pq,Sq)+d^*(Qq,Rq)],$$
$$\frac{d^*(Pq,Rq)\cdot d^*(Qq,Sq)}{1+d^*(Rq,Sq)}, \frac{d^*(Pq,Sq)\cdot d^*(Qq,Rq)}{1+d^*(Rq,Sq)},$$
$$d^*(Rq,Pq)\frac{1+d^*(Rq,Qq)+d^*(Sq,Pq)}{1+d^*(Rq,Pq)+d^*(Sq,Qq)}\}$$
$$= d^*(Sq,Rq).$$

From (32), we have

$$\psi(d^*(Rq,Sq)) \leq \psi(d^*(Rq,Sq)) - \phi(d^*(Rq,Sq)) < \psi(d^*(Rq,Sq)),$$

which is a contradiction. Thus, $Rq = Sq$, that is, $Pq = Rq = Sq = Qq$. Now, we shall show that $q = Sq$. Let, if possible, $q \neq Sq$. From (2), we have

$$\psi(d^*(Rp,Sq)) \leq \psi(\triangle\,(p,q)) - \phi(\triangle\,(p,q)), \tag{33}$$

where

$$\triangle\,(p,q) = \max\{d^*(Rp,Sq), d^*(Rp,Pp), d^*(Sq,Qq), \frac{1}{2}[d^*(Pp,Sq)+d^*(Qq,Rp)],$$
$$\frac{d^*(Pp,Rp)\cdot d^*(Qq,Sq)}{1+d^*(Rp,Sq)}, \frac{d^*(Pp,Sq)\cdot d^*(Qq,Rp)}{1+d^*(Rp,Sq)},$$
$$d^*(Rp,Pp)\frac{1+d^*(Rp,Qq)+d^*(Sq,Pp)}{1+d^*(Rp,Pp)+d^*(Sq,Qq)}\}$$
$$= d^*(Sq,Rp).$$

From (33), we have

$$\psi(d^*(Rp,Sq)) \leq \psi(d^*(Rp,Sq)) - \phi(d^*(Rp,Sq)) < \psi(d^*(Rp,Sq)),$$

which is a contradiction. Therefore, $q = Sq = Qq = Pq = Rq$. Hence, q is the common fixed point of P, Q, R and S.

Now, we shall prove the uniqueness of common fixed point. Let c and d be two common fixed point of P, Q, R and S. Let, if possible, $c \neq d$. From (2), we have

$$\psi(d^*(c,d)) = \psi(d^*(Rc,Sd)) \leq \psi(\triangle\,(c,d)) - \phi(\triangle\,(c,d)) = \psi(d^*(c,d)) - \phi(d^*(c,d)) \\ < \psi(d^*(c,d)),$$

which is a contradiction. Therefore, $c = d$. This proves the uniqueness of common fixed point. □

Example 2.1. Let $M = [0,1]$ be endowed with the Euclidean metric $d^*(u,v) = |u - v|$. Let the self maps P, Q, R and S be defined by

$$Ru = \frac{u}{9}, Qu = \frac{u}{6}, Su = \frac{u}{3}, Pu = u.$$

Clearly, $RM = [0, \frac{1}{9}] \subseteq [0, \frac{1}{6}] = QM$, $SM = [0, \frac{1}{3}] \subseteq [0,1] = PM$. Also, PM is complete subspace of M and pair (P,R), (Q,S) are weakly compatible.

Now,

$$\begin{aligned}
&d^*(Ru,Sv) = |\frac{u}{9} - \frac{v}{3}| = \frac{1}{9}|u - 3v|,\\
&d^*(Pu,Qv) = |u - \frac{v}{6}| = \frac{1}{6}|6u - v|,\\
&d^*(Ru,Pu) = |\frac{u}{9} - u| = \frac{8u}{9},\\
&d^*(Qv,Sv) = |\frac{v}{6} - \frac{v}{3}| = \frac{v}{6},\\
&d^*(Ru,Su) = |\frac{u}{9} - \frac{u}{3}| = \frac{2u}{9},\\
&d^*(Pu,Su) = |u - \frac{u}{3}| = \frac{2u}{3},\\
&d^*(Qu,Rv) = |\frac{u}{6} - \frac{v}{9}| = \frac{1}{18}|3u - 2v|,\\
&\frac{1}{2}[d^*(Pu,Su) + d^*(Qu,Rv)] = \frac{1}{2}[\frac{2u}{3} + \frac{1}{18}|3u - 2v|] = \frac{1}{36}|15u - 2v|,\\
&\frac{(d^*(Pv,Ru) \cdot d^*(Qu,Su)}{(1 + d^*(Rv,Su)} = \frac{\frac{8v}{9} \cdot \frac{u}{6}}{1 + \frac{1}{9}|3v - u|} = \frac{4uv}{3(9 + 3v - u)},\\
&\frac{1 + d^*(Rv,Qu) + d^*(Su,Pv)}{1 + d^*(Rv,Pv) + d^*(Su,Qu)} = \frac{1 + \frac{1}{18}(3u - 2v) + \frac{1}{9}(u - 9v)}{1 + \frac{8v}{9} + \frac{u}{6}} = \frac{|18 + 5u - 20v|}{|18 + 16v + 3u|}.
\end{aligned}$$

Let $\psi(a) = \frac{a}{2}$ and $\phi(a) = \frac{a}{4}$. Thus, we have

$$\psi(d^*(Ru,Sv)) = \psi(\frac{u}{9} - \frac{v}{3}) = \frac{1}{2}|\frac{u}{9} - \frac{v}{3}| = \frac{1}{18}|u - 3v|,$$

$$\begin{aligned}\triangle\,(u,v) = \max\{&d^*(Ru,Sv), d^*(Ru,Pu), d^*(Sv,Qv),\\ &\frac{1}{2}[d^*(Pu,Sv)+d^*(Qv,Ru)],\\ &\frac{d^*(Pu,Ru)\cdot d^*(Qv,Sv)}{1+d^*(Ru,Sv)}, \frac{d^*(Pu,Sv)\cdot d^*(Qv,Ru)}{1+d^*(Ru,Sv)},\\ &d^*(Ru,Pu)\frac{1+d^*(Ru,Qv)+d^*(Sv,Pu)}{1+d^*(Ru,Pu)+d^*(Sv,Qv)}\} = d^*(Ru,Pu),\end{aligned}$$

$$\psi(\triangle\,(u,v)) = \psi(d^*(Ru,Pu)) = \psi(\frac{8u}{9}) = \frac{1}{2}\cdot\frac{8u}{9} = \frac{4u}{9},$$

$$\phi(\triangle\,(u,v)) = \phi(d^*(Ru,Pu)) = \phi(\frac{8u}{9}) = \frac{1}{4}\cdot\frac{8u}{9} = \frac{2u}{9}.$$

Thus, we have

$$\psi(\triangle\,(u,v)) - \phi(\triangle\,(u,v)) = \frac{4u}{9} - \frac{2u}{9} = \frac{2u}{9}.$$

Hence

$$\psi(d^*(Ru,Sv)) \le \psi(\triangle\,(u,v)) - \phi(\triangle\,(u,v)).$$

This satisfies (2). If we consider the sequence $\{u_n\} = \{\frac{1}{2n}\}$, then

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{2n} = 0, \lim_{n\to\infty} Ru_n \quad = \lim_{n\to\infty}\frac{u_n}{9} = \lim_{n\to\infty}\frac{1}{2n\times 9} = 0.$$

Therefore,

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = 0, \text{ where } 0\in M.$$

So, the pair (P,R) satisfied the E.A. property. Also,

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = 0 = P(0).$$

So, the pair (P,R) satisfies the (CLR_P) property. Hence, all the conditions of above Theorems are satisfied. Therefore, P,Q,R and S must have unique common fixed point. Here 0 is the unique common fixed point of P,Q,R and S.

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Non-cancellation group of a direct product

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Abstract. The non-cancellation set of a group G, denoted by $\chi(G)$, is defined to be the set of all isomorphism classes of groups H such that $G\times\mathbb{Z}\cong H\times\mathbb{Z}$. While investigating when $\mathbb{Z}$ can be cancelled in this direct product, $\chi(G)$ has become the focus of many studies. For the semidirect product $G_i=\mathbb{Z}_{n_i}\rtimes_{\omega_i}\mathbb{Z}$, $i=1,2$, methods for computation of the non-cancellation groups $\chi(G_1\times G_2)$, $\chi(G_i^k)$, $k\in\mathbb{N}$ and $\chi(G_i,h_i)$ have been developed. We present in this study, a general method of computing $\chi(G_1\times G_2,h)$, where $h:F\hookrightarrow G_1\subseteq G_1\times G_2$ and F a finite group.

Keywords: localization, non cancellation, restricted genus, groups under a finite group.

1. Introduction

The theory of π-localization of groups, where π is a family of primes, appears to have been first discussed in [7, 8] by Mal'cev and Lazard. In the 1970s, Hilton and Mislin became interested, through their work on the localization of nilpotent spaces, in the localization of nilpotent groups. Mislin [11] define the genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N to be the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. This version of genus became known as the *Mislin genus*, and other very useful variations of this concept came into being.

In [3] Hilton and Mislin define an abelian group structure on the genus set $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroup. Throughout this study, finitely generated group with finite commutator subgroup will be called χ_0-group.

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For nilpotent groups which belong to class $\mathcal{K}$ (of semidirect products of the form $T \rtimes \mathbb{Z}^k$, where T is a finite abelian group and k is a positive integer), many computations of the genus groups appear in the literature. Indeed, the groups considered in [1, 4, 5, 14] all belong to this class. The groups used in the computation method developed in this paper are χ_o-groups belonging to $\mathcal{K}$ and will be called $\mathcal{K}_0$-groups.

For a non-nilpotent χ_o-groups, the kernel of the localizing homomorphism maybe be bigger than what it is required. So, for such χ_0-groups, the idea of genus can be generalized through non-cancellation.

For a χ_o-group G, the non-cancellation set denoted by $\chi(G)$ is defined to be the set of all isomorphism classes of groups H such that $G\times\mathbb{Z} \cong H\times\mathbb{Z}$. Scevenels and Witbooi in [15], gave an alternate description of the non-cancellation group of $\mathcal{K}_0$-groups. This enables them to make some computations. Warfield [17] proved that, if N is a nilpotent χ_o-group, then $\mathcal{G}(N) = \chi(N)$. In [18], the author showed that for a χ_o-group G the non- cancellation set $\chi(G)$ has a group structure similar to the group structure on the Mislin genus of a nilpotent χ_o-group. For any two χ_o- groups H and G, O'Sullivan in [12] proved that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ if and only if for every finite set π of primes, we have $H_\pi \cong G_\pi$ (π-localizations are isomorphic). To illuminate our understanding of genera of groups, the restricted genus of a χ_0-group under a finite group F was introduced in [10]. More precisely, for a fixed morphism $h : F \to G$, the restricted genus $\chi(G, h)$[1] is the set of isomorphism classes of morphisms $F \to H$, which are π-equivalent to h at every finite set of primes π. For a well-defined integer n depending on G, in [10] an epimorphism $\zeta : (\mathbb{Z}/n)^*/\pm 1 \to \chi(G, h)$ is established and it is shown that there exist natural epimorphisms $\chi(G, h) \to \chi(G/h(F))$ (provided $h(F)$ is normal in G) and $\chi(G, h) \to \chi(G, h \circ i))$ (provided $i : F_0 \to F$ is a morphism), which are compatible with the various involved maps ζ.

Having such homomorphisms is not always given. In [10], computation methods of $\chi(G, h)$ in the special case G is a semidirect product $T \rtimes_\omega \mathbb{Z}^k$ are used in a very particular example to provide a concrete computation of $\chi(G, h)$, where T is a finite abelian group. We extend this result to compute the restricted genus $\chi(G_1 \times G_2, h)$ of the direct product $G_1 \times G_2$, where $G_i = \mathbb{Z}_{m_i} \rtimes_{\omega_i} \mathbb{Z}$ and $h : F \hookrightarrow G_1 \times G_2$ a monomorphism, with F a finite group.

The rest of the paper is organized as follows: Section 2 is on preliminaries, Section 3 presents the group structure on the restricted genus $\chi(G, h)$ and Section 4 is on the computation method for $\chi(G_1 \times G_2, h)$.

2. Preliminaries

2.1 Definitions and notations

An interesting topic in the theory of nilpotent groups is the extraction of roots. A group G is said to be a *rational group* if n-th roots exist in G, for all positive

1. For the special case F is trivial ($F = *$), $\chi(G, * \to G) = \chi(G)$

integers n. A group which has unique extraction of roots is *torsion-free*. Roots are unique in torsion-free nilpotent groups. However, extraction of roots is not usually possible in such groups. For example, extraction of roots is not possible in the additive group of integers $\mathbb{Z}$. However, this group can be embedded in the rational group $\mathbb{Q}$. Mal'cev [9] generalized this by showing that any torsion-free nilpotent group G can be embedded in a rational nilpotent group G_0. The extraction of roots is unique in G_0 and every element of G_0 has a positive power in G. Moreover, G_0 is unique up to isomorphism.

Given any set of primes π, let π' be the set of natural numbers which are relatively prime to elements of π. Let G_π denote the subgroup of G_0 generated by G and its m-th roots whenever the prime divisors of m are in π'.

A group G is said to be π-*local* if for each $n \in \pi'$, the function $g \mapsto g^n$ of G into itself is a bijection.

The group G_π is called the π-*localization* of G and has the universal property that given any homomorphism $\phi : G \to H$, where H is a π-local group, there exists a unique $\phi_\pi : G_\pi \to H$ such that $\phi = \phi_\pi \varphi_\pi$ where φ_π is the π-*localizing homomorphism* $G \to G_\pi$. If $\pi = \{p\}$, then G_π is simply denoted by G_p.

The **genus** of a finitely generated nilpotent group G denoted by $\mathcal{G}(G)$ (known as *Mislin genus*, [11]), is the set of all isomorphism classes of finitely generated nilpotent groups H such that $G_p \cong H_p$ for every prime number p.

The set $\tau_f(G)$ of all isomorphism classes of finitely generated group H such that $G_\pi \cong H_\pi$ for every finite set of primes π is called the **restricted genus** of G.

When localizing non-nilpotent groups, it may happen that the kernel of the localizing homomorphism is bigger than what we would require. For a non-nilpotent finitely generated group G with finite commutator subgroup, the idea of the genus is generalized through *non-cancellation*, rather than considering localizations.

For groups, we know that cancellation holds in the category of finitely generated abelian groups. If G is a finitely generated abelian group, then for any abelian groups H and K, $G \oplus H \cong G \oplus K$ implies $H \cong K$. Thus, finitely generated abelian group is cancellable in the category of all abelian groups. The abelian group $\mathbb{Z}$ is cancellable in the category of abelian groups. However, it is known that $\mathbb{Z}$ is not cancellable in the category of groups in general. This was shown by an example of William Scott, which was included in [16]. Another example was given independently by Hirshon [6]. Our study of cancellation property of a group G is examined through the isomorphism of direct products $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. For a group G, the non-cancellation set $\chi(G)$ measures to what extend $\mathbb{Z}$ can be cancelled in $G \times \mathbb{Z} \cong H \times \mathbb{Z}$ for some group H. For some type of groups G, the computation of $\chi(G)$ have been the object of many studies. For $\mathcal{K}_o$-groups, methods for computation of the non-cancellation groups $\chi(G_1 \times G_2)$ and $\chi(G_i^k)$, $k \in \mathbb{N}$ were developed in [19] and [2] respectively. In these construction, the integer n(G) described below play a central role.

Given a $\mathcal{X}_0$-group G. Let n_1 be the exponent of T_G, n_2 the exponent of the group $\mathrm{Aut}(T_G)$ and n_3 the exponent of the torsion subgroup of the centre of G. Consider $n(G) = n_1n_2n_3$. The integer $n = n(G)$ has the property that the subgroup $G^{(n)} = \langle g^n \ : \ g \in G\rangle$ of G belongs to the centre of G and $G/G^{(n)}$ is a finite group.

Aspects of localization as in groups and related categories have been studied in a unified way in a categorical setting, see [13] for instance. The following subsection give a more specific presentation.

2.2 Category of groups under a finite group F

Fix a finite group F and let $h : F \longrightarrow G$ be a monomorphism. We denote by $\mathrm{Grp_F}$ the category of groups under F as in [10]. For the category $\mathrm{Grp_F}$, the objects denoted by (G_1, h_1), (G_2, h_2) are group homomorphisms $h_1 : F \longrightarrow G_2$ and $h_2 : F \longrightarrow G_2$ and a morphism in $\mathrm{Grp_F}$ is a group homomorphism $\beta : G_1 \longrightarrow G_2$ such that $\beta \circ h_1 = h_2$.

The π-localization of an object $h : F \longrightarrow G$ is the object $h_\pi : F \longrightarrow G_\pi$ where π is a set of primes. Denote by $\mathcal{X}_F$ the full subcategory of χ_o- groups under F. Then, the restricted genus $\chi(G, h)$ is the set of isomorphism classes k such that k_π is isomorphic to h_π for $k \in \mathcal{X}_F$. If F is a trivial group, then $\mathcal{X}_F$ is identified with χ_o-groups.

The restricted genus $\chi(G, h)$ has been computed in [10] and has been shown that $\chi(G, h)$ coincides with $\chi(G)$ if F is a trivial group.

Let $\mathcal{K}$ be the class of groups of the form $T \rtimes_\omega F$ where F is a finite rank free abelian group and T a finite abelian group. For a pair of relatively prime natural numbers m, u, the symbol $G(m, u)$ denotes the group $H = \mathbb{Z}_m \rtimes_\nu \mathbb{Z}$. H is a $\mathcal{K}$- group and $\mathcal{K}_\mathrm{F}$ determines a full subcategory of $\mathrm{Grp_F}$, see [10].

Let $G_i = \mathbb{Z}_{m_i} \rtimes_{\omega_i} \mathbb{Z}$ and let $h : F \hookrightarrow G_1 \times G_2$ where h is a monomorphism and F is a finite group.

In this paper we develop a general method for computing $\chi(G_1 \times G_2, h)$.

3. Group structure on the restricted genus

Recall from ([18], Section 2), to a χ_o-group G assign a natural number $n(G) = n_1n_2n_3$ where n_1 is the exponent of the torsion subgroup T_G, n_2 the exponent of $\mathrm{Aut}(T_G)$ and n_3 the exponent of the torsion of the center T_{Z_G}. It was shown in [18] that for a χ_o-group G whose subgroups H are of finite index with $T_G = T_H$, the non-cancellation set $\chi(G)$ has a group structure and is given by $\chi(G) = \mathbb{Z}_n^*/\{1, -1\}$. For a pair of relatively prime natural numbers m, u, let $H = \mathbb{Z}_m \rtimes_\nu \mathbb{Z}$, where $\nu : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$ is the automorphism of $\mathbb{Z}_m$ defined by $\nu(1)(t) = ut$. Methods of computing $\chi(H)$ and $\chi(H^r)$ for r a natural number were developed in [2], [15] and [19]. It is shown in [15] that $\chi(H) = \mathbb{Z}_d^*/\{1, -1\}$, where d is the multiplicative order of u modulo m. For a direct product H^r which can be considered to be $\mathbb{Z}_m^r \rtimes_\omega \mathbb{Z}^r$, the authors in [2] showed that there is a well defined surjective homomorphism $\Gamma : \chi(H) \longrightarrow \chi(H^r)$ given by $K \mapsto K \times H^{r-1}$ where K is a group such that $K \times \mathbb{Z} \simeq H \times \mathbb{Z}$. Thus, in order to compute the group $\chi(H^r)$, one needs only to compute the kernel of the homomorphism Γ.

Let $G_i = G(m_i, u_i)$ for some $m_i,\, u_i \in \mathbb{N}$ with $\gcd(m_i, u_i) = 1$ and let d_i be the multiplicative order of u_i modulo m_i. Consider the direct product $G = G_1 \times G_2 = (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2$ and $h : F \hookrightarrow G_1 \times G_2$ be the inclusion map. Let $t = d(G)$ be the smallest invariant factors of $\operatorname{Im}\omega$. Note that, if $\gcd(d_1, d_2) = 1$ then $t = d_1 d_2$ and if $\gcd(d_1, d_2) \neq 1$ then $t = \gcd(d_1, d_2)$. For an object $(G_1 \times G_2,\, h)$ in $\mathcal{K}_F$, we obtain an epimorphism $\Upsilon : \mathbb{Z}_t^* \longrightarrow \chi(G_1 \times G_2,\, h)$, where d depends exclusively on $\operatorname{Im}\omega$ [10]. Thus, in order to find $\chi(G_1 \times G_2,\, h)$, one only needs to find the kernel of Υ. Note that t divides the exponent of $\operatorname{Aut}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$, and so t divides n(G).

Following ([10], Lemma 3.2), we have the following Lemma:

Lemma 3.1. *Given objects (G,h) and (K,k) in $\mathcal{K}_F$ with $G = (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2$ and $K = (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}) \rtimes_\nu \mathbb{Z}^2$, then a morphism $\alpha : (G,h) \longrightarrow (K,k)$ is an isomorphism if and only if there exist group isomorphisms $\theta : (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2 \longrightarrow (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}) \rtimes_\nu \mathbb{Z}^2$ and $\beta : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ with $\theta \circ h = k$ such that for any $z \in \mathbb{Z}^2$ and $t \in (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$, $(\nu \circ \beta(z))(\theta(t)) = \theta(\omega(z)(t))$.*

Notation 3.1. We will follow the notation introduced in [10]. Let G be a χ_o-group, $n = n(G)$ and let $X(n) = \{u \in \mathbb{N} : (u,n) = 1\}$. Let $Y(G,h)$ be the set of all $u \in X(n)$ for which there exists a subgroup K of G with $[G : K] = u$ and the object $(K,\, h_K)$ is a member in $\chi(G,h)$. Let G_u be a subgroup of G such that $T_G \subseteq G_u$ and $[G : G_u] = u$ for each $u \in Y(G,h)$. Define the induced homomorphism $h_u : x \mapsto h(x)$ and a morphism $\varsigma : Y(G,h) \longrightarrow \chi(G,h)$. Let $Y^*(G,h)$ be the image of $Y(G,h)$ in $\mathbb{Z}_n^*$. From ([10], Theorem 2.5) we have $Y^*(G,h)$ is a subgroup of $\mathbb{Z}_n^*$ and $Y^*(G,h)/\pm 1 \cong \chi(G,h)$. Now for any object (G,h) of $\mathcal{K}_F$ denote by $V(G,h)$ the set of all $u \in X(t)$ for which there exist a subgroup K of G with $[G : K] = u$ and (K, h_K) is a member of $\chi(G,h)$. Let $V^*(G,h)$ be the image of $V(G,h)$ in $\mathbb{Z}_t^*$. Choose a subgroup K of G such that (K, h_K) represents a member in $\chi(G,h)$ and $[G : K] = u$. We obtain a function $V(G,h) \longrightarrow \chi(G,h)$ given by $u \mapsto [K, h_K]$. Since $t \,|\, n$ we have the following ([10], Proposition 3.4)

Proposition 3.1. *Let $n = n(G)$ and $\rho : Y(G,h) \longrightarrow \chi(G,h)$ be the epimorphism that takes a residue* $\operatorname{mod} \mathrm{n}$ *and reduces it* $\operatorname{mod}\ \mathrm{t}$. *The epimorphism $\zeta : Y^*(G,h) \longrightarrow \chi(G,h)$ factorises through the epimorphism ξ':*

$$\begin{tikzcd} Y^*(G,h) \arrow[r, "\zeta"] \arrow[d, "\rho"'] & \chi(G,h) \\ V^*(G,h) \arrow[ur, "\xi'"'] & \end{tikzcd}$$

The kernel of $V^*(G,h) \longrightarrow \chi(G,h)$ is caculated through the following theorem ([10], Theorem 3.5)

Theorem 3.1. *For $m \in V(G,h)$, the following conditions are equivalent:*

(a) $\overline{m} \in \ker[V^*(G,h) \longrightarrow \chi(G,h)]$.

(b) *There exists $\alpha \in \mathrm{Aut}\,(T)$ with $v \circ h = h$ such that $\alpha \in N_{\mathrm{Aut}\,T} \operatorname{Im} \omega$ and for an automorphism $\bigwedge : \operatorname{Im} \omega \longrightarrow \operatorname{Im} \omega$ defined by $v \mapsto \alpha v \alpha^{-1}$, we have $\det(\bigwedge) = \pm \overline{m}^{-1} \in V^*(G,h)$.*

4. Computation

1. For a pair of relatively prime natural numbers m, u, define a group $G(m,\, u) =< a,\, b \,:\, a^m = 1,\, bab^{-1} = a^u >$. The group $G(m,\, u)$ can be considered to be the semidirect product $\mathbb{Z}_m \rtimes_\omega \mathbb{Z}$ where $\omega : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$ is such that $\omega(1)$ is the automorphism of $\mathbb{Z}_m$ given by $\omega(1)(t) = ut$.

2. Let q be the product of all distinct prime divisors of m and assume that q^2 divides m.

3. Let $G_i = G(m_i, u_i)$ for some $m_i,\, u_i \in \mathbb{N}$ with $\gcd(m_1, m_2) = 1, \gcd(m_i, u_j) = 1$, $i,\, j = 1,\, 2$ and let d_i be the multiplicative order of u_i modulo m_i. Let $d = \operatorname{lcm}(d_1, d_2)$ and $m = \operatorname{lcm}(m_1, m_2)$. If $\gcd(d_1, d_2) = 1$ let $t = d_1 d_2$ and if $\gcd(d_1, d_2) \neq 1$ let $t = \gcd(d_1, d_2)$.

4. Consider the direct product $G_1 \times G_2 = (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2$ where $\omega : \mathbb{Z}^2 \longrightarrow \mathrm{Aut}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$ is such that $\omega(\epsilon_i) = \omega_i : (t_1, t_2) \mapsto (u_1^{\delta(i,1)} t_1, u_2^{\delta(i,2)} t_2)$ where $\delta(i,j)$ is the Kronecker function and $\{\epsilon_1, \epsilon_2\}$ is the standard basis of $\mathbb{Z}^2$.

5. Write $\omega(\epsilon_1) = \omega_1$ and $\omega(\epsilon_2) = \omega_2$. Each automorphism ω_i is of order d_i and $\operatorname{Im}(\omega)$ is the direct product of the cyclic subgroups $< \omega_i >$ of $\mathrm{Aut}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$. By [10], there is an epimorphism $\mathbb{Z}_t^* \longrightarrow \chi(G_1 \times G_2)$ where t is the smallest of the invariant factors of $\operatorname{Im} \omega$.

6. Let J be the subgroup of $\mathrm{Aut}(T_{G_1 \times G_2})$ generated by $\{\omega_1,\, \omega_2\}$ that is, $J = \operatorname{Im} \omega =< \omega_1, \omega_2 >$. Note that:

   - $J = \operatorname{Im} \omega =< \omega_1, \omega_2 >$ is a free $\mathbb{Z}_d$-module.
   - The determinant of an endomorphism of J is defined and is an element of $\mathbb{Z}_d$.

   Consider $J_* = N_{\mathrm{Aut}(T_{G_1 \times G_2})} J$. For any $\alpha \in \mathrm{Aut}(T_{G_1 \times G_2})$, let $\bigwedge_\alpha$ be the inner automorphism such that $\bigwedge_\alpha \,:\, v \mapsto \alpha\, v\, \alpha^{-1}$.

7. Let q_2 be a multiple of q such that $q_2 q$ divides m. Let $e_1 = (1,0)$, $e_2 = (0,1)$ be elements of $T_{G_1 \times G_2}$ and let $F = \{a q_2 e_2 \,:\, a \in \mathbb{Z}\}$ be a subgroup of $T_{G_1 \times G_2}$ and $h :\, F \hookrightarrow G_1 \times G_2$ be the inclusion map.

4.1 Inner automorphism of $\mathrm{Aut}\,(T_{G_1\times G_2})$

Fix $\alpha \in J^*$ such that $\alpha(x) = x$ for all $x \in F$. There exists a 2×2 matrix (α_{ij}) of integers such that $\alpha(e_i) = \sum_{j=1}^{2} \alpha_{ji} e_j$. Suppose that $\bigwedge$ is the inner automorphism of J determined by α.

Proposition 4.1. *For the matrix (α_{ij}), α_{ii} is a unit modulo m for $i = 1, 2$.*

Proof. We note that $\alpha(0, q_2) = (0, q_2)$ since $(0, q_2) \in F$. Also $\alpha(0, q_2) = (q_2\alpha_{12}, q_2\alpha_{22})$. Thus $(0, q_2) = (q_2\alpha_{12}, q_2\alpha_{22})$ and we have that m divides α_{12}. In particular q divides α_{12} while α_{22} is a unit modulo m. Therefore the matrix of α is of the form $\begin{pmatrix} \alpha_{11} & tq \\ \alpha_{21} & u \end{pmatrix}$ where t, $u \in \mathbb{Z}$. Now $\det(\alpha) = \alpha_{11}u - \alpha_{21}tq$.

We claim that α_{11} is a unit modulo m. Suppose that α_{11} is not a unit modulo m then let p be a common prime divisor of α_{11} and m. Then p divides q and p divides $\det(\alpha)$ which is a contradiction since $\det(\alpha)$ is a unit modulo m (α is an automorphism). Thus α_{11} is a unit modulo m. □

Proposition 4.2. *The inner automorphism $\bigwedge$ of J coincides with the identity automorphism of J.*

Proof. For the inner automorphism $\bigwedge$ induced by α, there exists a matrix $(\bigwedge_{ij})$ of integers such that $\bigwedge \omega_i = \omega_1^{\bigwedge_{1i}} \omega_2^{\bigwedge_{2i}}$ for each i. Let $\bigwedge \omega_i = v_i$. Then $\bigwedge \omega_i = \alpha\omega_i\alpha^{-1} = v_i$ and $\alpha\omega_i = v_i\alpha$. On one hand $\alpha\omega_i(e_i) = \alpha(u_ie_i) = \sum_{j=1}^{2} u_i^{\delta(i,j)} \alpha_{ji} e_j$ and on the other hand $v_i\alpha(e_i) = v_i\left(\sum_{j=1}^{2} \alpha_{ji}e_j\right) = \sum_{j=1}^{2} \alpha_{ji} u_i^{\delta(i,j)\bigwedge_{ji}} e_j$. That is, $\sum_{j=1}^{2} u_i^{\delta(i,j)} \alpha_{ji} e_j = \sum_{j=1}^{2} \alpha_{ji} u_i^{\delta(i,j)\bigwedge_{ji}} e_j$.

For $j = i$, we have from , that α_{ii} is a unit modulo m, therefore $u_i^{\bigwedge_{ii}} \equiv u_i \bmod m$. Thus $\bigwedge_{ii} \equiv 1 \bmod d$ and consequently $\bigwedge_{ii} \equiv 1 \bmod t$.

For the case $j \neq i$, we have $\alpha\omega_i(e_j) = \alpha(e_j) = \sum_{k=1}^{2} \alpha_{kj} e_k$ and $v_i\alpha(e_j) = \sum_{k=1}^{2} \alpha_{kj} u_k^{\bigwedge_{ki}} e_k$. Therefore $\sum_{k=1}^{2} \alpha_{kj} e_k = \sum_{k=1}^{2} \alpha_{kj} u_k^{\bigwedge_{ki}} e_k$. For the case $k = j$. Since α_{jj} is a unit modulo m then $u_j^{\bigwedge_{ji}} \equiv 1 \bmod m$, that is, $\bigwedge_{ji} \cong 0 \bmod d$. Consequently $\bigwedge_{ji} \equiv 0 \bmod t$. Thus $\det(\bigwedge) = 1$ and $\bigwedge$ is coincides with the identity automorphism on J. □

Proposition 4.3. *Let $G_i = G(m_i, u_i)$, $i = 1, 2$ and $(G_1 \times G_2, h)$ be an object of $\mathcal{K}_F$. Let d_i be the multiplicative order of u_i modulo m_i. Then, $\chi(G_1 \times G_2, h) \cong \mathbb{Z}_t^*/\pm 1$, where ($t = d_1d_2$, if $(d_1, d_2) = 1$) or ($t = (d_1, d_2)$, otherwise).*

Proof. The Proposition follows from Theorem 3.1 and Proposition 4.2. □

This construction can be generalized to compute $\chi(G_1 \times \cdots \times G_n, h)$ and $\chi(G_i^k, l)$. This will be done in our future work.

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On the semiring variety generated by $B^0, (B^0)^*, N_2, T_2, Z_2, W_2$

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Abstract. In this paper, we study the semiring variety generated by $B^0, (B^0)^*, N_2$, T_2, Z_2, W_2. We prove that this variety is finitely based and prove that the lattice of subvarieties of this variety is a distributive lattice of order 1014. Moreover, we deduce this variety is hereditarily finite based.

Keywords: semiring, variety, lattice, identity, hereditarily finite based.

1. Introduction

A semiring is an algebra with two associative binary operations $+, \cdot$, in which $+$ is commutative and $\cdot$ distributive over $+$ from the left and right. Such an algebra is a common generalization of both rings and distributive lattice. It has broad applications in information science and theoretical computer science (see [5], [6]). In this paper, we shall investigate some small-order semirings which will paly a crucial role in subsequent follows.

The semiring B with addition and multiplication table

+	a	b	c
a	a	b	c
b	b	b	b
c	c	b	c

·	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

Eight 2-element semirings with addition and multiplication table

*. Corresponding author

Semiring	+		·		Semiring	+		·	
L_2	0	1	0	0	R_2	0	1	0	1
	1	1	1	1		1	1	0	1
M_2	0	1	0	1	D_2	0	1	0	0
	1	1	1	1		1	1	0	1
N_2	0	1	0	0	T_2	0	1	1	1
	1	1	0	0		1	1	1	1
Z_2	0	0	0	0	W_2	0	0	0	0
	0	0	0	0		0	0	0	1

For any semiring S, we denote by S^0 the semiring obtained from S by adding an extra element 0 and where $a = 0 + a = a + 0, 0 = 0a = a0$ for every $a \in S$. For any semiring S, S^* will denote the (multiplicative) left-right dual of S. Pastijn et al. [4, 10] studied the semiring variety generated by B^0 and $(B^0)^*$. They showed that the lattice of subvarieties of this variety is distributive and contains 78 varieties precisely. Moreover, each of these is finitely based. It is obvious that the variety generated by L_2, R_2, M_2, D_2 is properly contained in the variety generated by B^0 and $(B^0)^*$, that is, $\mathbf{HSP}(L_2, R_2, M_2, D_2) \subsetneqq \mathbf{HSP}(B^0, (B^0)^*)$. In 2016, Shao and Ren [14] studied the variety generated by $L_2, R_2, M_2, D_2, N_2, T_2$. They showed that the lattice of subvarieties of this variety is distributive and contains 64 varieties precisely. Moreover, each of these is finitely based. Recently, Ren and Zeng [13] studied the variety generated by $B^0, (B^0)^*, N_2, T_2$. They proved that the lattice of subvarieties of this variety is a distributive lattice of order 312 and that each subvarieties of its is finitely based. It is easy to check

$$\mathbf{HSP}(B^0, (B^0)^*) \subsetneqq \mathbf{HSP}(B^0, (B^0)^*, N_2, T_2) \subsetneqq \mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2).$$

So, the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2)$ is a proper subvariety of the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. The main purpose of this paper is to study the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. We show that the lattice of subvarieties of this variety is a distributive lattice of order 1014. Moreover, we show this variety is hereditarily finitely based.

2. Preliminaries

Let $\mathbf{V}$ be a variety, $\mathcal{L}(\mathbf{V})$ denote the lattice of subvarieties of $\mathbf{V}$ and $\mathrm{Id}_{\mathbf{V}}(X)$ denote the set of all identities defining $\mathbf{V}$. If $\mathbf{V}$ can be defined by finitely many identities, then we say that $\mathbf{V}$ is *finitely based*. In other words, $\mathbf{V}$ is said to be finitely based if there exists a finite subset Σ of $\mathrm{Id}_{\mathbf{V}}(X)$ such that for any $p \approx q \in \mathrm{Id}_{\mathbf{V}}(X)$, $p \approx q$ can be derived from Σ, i.e., $\Sigma \vdash p \approx q$. Otherwise, we say that $\mathbf{V}$ is *nonfinitely based*. Recall that $\mathbf{V}$ is said to be *heredirarily finitely based* if all members of $\mathcal{L}(\mathbf{V})$ are finitely based. If a variety $\mathbf{V}$ is finitely based and $\mathcal{L}(\mathbf{V})$ is a finite lattice, then $\mathbf{V}$ is hereditarily finite based (see [13]).

The variety of all semirings is denoted by **SR**. A semiring is called an *additively idempotent semiring* (ai-semiring for short) if its additive reduct is a semilattice, i.e., a commutative idempotent semigroup. It is also called a *semilattice-order semigroup* (see [3], [8], [12]). The variety of all ai-semirings is denoted by **AI**. Let X denote a fixed countably infinite set of variables and X^+ the free semigroup on X. A *semiring identity* (**SR**-identity for short) is an expression of the form $u \approx v$, where u and v are terms with $u = u_1 + \cdots + u_k$, $v = v_1 + \cdots + v_\ell$, where $u_i, v_j \in X^+$ (An ai-semiring identity denoted by **AI**-identity). Let $\underline{k}$ denote the set $\{1, 2, \ldots, k\}$ for a positive integer k, Σ be a set of identities which include the identities determining **AI** and $u \approx v$ be an **AI**-identity. It is easy to check that the ai-semiring variety defined by $u \approx v$ coincides with the ai-semiring variety defined by the identities $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$. Thus, in order to show that $u \approx v$ is derivable from Σ, we only need to show that $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$ can be derived from Σ.

To solve the word problem for the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$, the following notions and notations are needed. Let q be an element of X^+. Then

- the *head* of q, denoted by $h(q)$, is the first variable occurring in q;
- the *tail* of q, denoted by $t(q)$, is the last variable occurring in q;
- the *content* of q, denoted by $c(q)$, is the set of variables occurring in q;
- the *length* of q, denoted by $|q|$, is the number of variables occurring in q counting multiplicities;
- the *initial part* of q, denoted by $i(q)$, is the word obtained from q by retaining only the first occurrence of each variable;
- the *final part* of q, denoted by $f(q)$, is the word obtained from q by retaining only the last occurrence of each variable.

The basis for each one of N_2, T_2, Z_2, W_2 can be found from [11] (See Table 1).

Table 1. Bases for N_2, T_2, Z_2, W_2

Semiring	Basis
N_2	$xy \approx zt, x + x^2 \approx x$
T_2	$xy \approx zt, x + x^2 \approx x^2$
Z_2	$x + y \approx z + u, xy \approx x + y$
W_2	$x + y \approx z + u, x^2 \approx x, xy \approx yx$

By [14, Lemma 1.1] and the bases for Z_2, W_2 in the above Table 1, we have

Lemma 2.1. *Let $u \approx v$ be a nontrivial* **SR**-*identity, where $u = u_1+u_2+\cdots+u_m$, $v = v_1+ v_2 + \cdots + v_n$, $u_i, v_j \in X^+$, $i \in \underline{m}, j \in \underline{n}$. Then:*

(*i*) *N_2 satisfies $u \approx v$ if and only if $\{u_i \in u \,|\, |u_i| = 1\} = \{v_i \in v \,|\, |v_i| = 1\}$;*

(*ii*) *T_2 satisfies $u \approx v$ if and only if $\{u_i \in u \,|\, |u_i| \geq 2\} \neq \phi$, $\{v_i \in v \,|\, |v_i| \geq 2\} \neq \phi$;*

(*iii*) *Z_2 satisfies $u \approx v$ if and only if $(\forall x \in X) \,|\, u \neq x$, $v \neq x$;*

(*iv*) *W_2 satisfies $u \approx v$ if and only if $m = n = 1, c(u_1) = c(v_1)$ or $m, n \geq 2$.*

Suppose that $u = u_1 + \cdots + u_m, u_i \in X^+, i \in \underline{m}$. Let 1 be a symbol which is not in X and Y an arbitray subset of $\bigcup_{i=1}^{i=m} c(u_1)$. For any u_i in u, if $c(u_i) \subseteq Y$, put $h_Y(u_i) = 1$. Otherwise, we shall denote by $h_Y(u_i)$ the first variable occurring in the word obtained from u_i by deleting all variables in Y. The set $\{h_Y(u_i) | u_i \in u\}$ is written $H_Y(u)$. Dually, we have the notations $t_Y(u_i)$ and $T_Y(u_i)$. In particular, if $Y = \emptyset$, then $h_Y(u_i) = h(u_i)$ and $t_Y(u_i) = t(u_i)$. Moreover, if $c(u_i) \cap Y \neq \emptyset$ for every u_i in u, then we write $D_Y(u) = \emptyset$. Otherwise, $D_Y(u)$ is the sum of all terms u_i in u such that $c(u_i) \cap Y = \emptyset$. By [4, Lemma 2.4 and its dual, Lemma 2.5 and 2.6], we have

Lemma 2.2. *Let $u \approx u + q$ be an* ***AI***-*identity, where $u = u_1 + \cdots + u_m, u_i, q \in X^+, i \in \underline{m}$. If $u \approx u + q$ holds in* **HSP**$(B^0, (B^0)^*)$*, then $c(q) \subseteq \bigcup_{i=1}^{i=m} c(u_i)$ and for the set $Z = \bigcup_{i=1}^{i=m} c(u_i) \setminus c(q)$ and for any subset Y of Z, $H_Y(D_Z(u)) = H_Y(D_Z(u) + q)$ and $T_Y(D_Z(u)) = T_Y(D_Z(u) + q)$.*

For other notations and terminology used in this paper, the read is referred to [1, 4, 7].

3. Equational basis of HSP$(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$

In [13], Ren and Zeng studied the join **W** of semiring variety **HSP**$(B^0, (B^0)^*)$ and semiring variety **HSP**(N_2, T_2) and obtained the following result.

Lemma 3.1 ([13]). *$\mathcal{L}(\mathbf{W})$ is a 312-element distributive lattice and* **W** *is determined by*

$$2x \approx x; \tag{1}$$
$$x^2y \approx xy; \tag{2}$$
$$xy^2 \approx xy; \tag{3}$$
$$(xy)^2 \approx xy; \tag{4}$$
$$xyzt \approx xzyt; \tag{5}$$
$$x + yz \approx x + yz + x^2; \tag{6}$$
$$x + yz \approx x + yz + xyz; \tag{7}$$
$$x + yz \approx x + yz + yzx; \tag{8}$$
$$x + yz \approx x + yz + yxz. \tag{9}$$

In the following Theorem, we shall give an Equational basis of $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. From Lemma 2.1, Z_2 and W_2 dose not satisfy the identity $2x \approx x$, that is, Z_2 and W_2 are not ai-semirings. In deed, we have

Theorem 3.1. *The semiring variety* $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ *is determined by the identities (2)-(9) and the following identity*

$$x + y \approx x + 2y. \tag{10}$$

Proof of Theorem 3.1. From [4, 10] and Lemma 2.1, both $\mathbf{W}$ and $\mathbf{HSP}(Z_2, W_2)$ satisfy the identities (2)-(10) and so dose $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$.

Next, we shall show that every identity that holds in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ can be derived from (2)-(10) and the identities determining $\mathbf{SR}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m$, $v = v_1 + v_2 + \cdots + v_n$, $u_i, v_j \in X^+$, $1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 2.1 (iv), we only need to consider the following two cases:

Case 1 $m = n = 1$ and $C(u_1) = C(v_1)$. Now that $L_2, R_2, T_2, Z_2 \models u_1 \approx v_1$, it follows that $H(u_1) = H(v_1)$, $T(u_1) = T(v_1)$, $|u_1| \geq 2$ and $|v_1| \geq 2$. Hence $u_1 \overset{(2),(3),(5)}{\approx} v_1$.

Case 2 $m, n \geq 2$. It is easy to verify that $u \approx v$ and the identity (10) can imply the identities $u \approx u + v_j$, $v \approx v + u_i$ for all i, j such that $1 \leq i \leq m, 1 \leq j \leq n$. Conversely, the latter $m + n$ identities can imply $u \approx u + v \approx v$. Thus, to show that $u \approx v$ is derivable from (2)-(10) and the identities determining $\mathbf{SR}$, we only need to show that the simpler identities $u \approx u + v_j$, $v \approx v + u_i$ for all i, j such that $1 \leq i \leq m, 1 \leq j \leq n$. Hence, we need to consider the following two cases:

Case 2.1 $u \approx u + q$, where $|q| = 1$. Since $N_2 \models u \approx u + q$, there exists $u_s = q$. Thus $u + q \approx u' + u_s + q \approx u' + u_s + u_s \overset{(10)}{\approx} u' + u_s \approx u$.

Case 2.2 $u \approx u + q$, where $|q| \geq 2$. By (2), (3) and (5), we have

$$q \approx i(q)q \approx i(q)qf(q) \approx i(q)f(q)$$

and so

$$q \approx i(q)f(q). \tag{11}$$

Note that $c(q) = c(i(q)) = c(f(q))$. Since $u \approx u + q$ holds in T_2, it follows from Lemma 2.1 (ii) that there exists u_i in u such that $|u_i| \geq 2$. Put $Z = (\bigcup_{i=1}^{i=m} c(u_i)) \setminus c(q)$. Assume that $D_Z(u) = u_1 + \cdots + u_k$. Then $\bigcup_{i=1}^{i=k} c(u_i) = c(q)$. Moreover, we have

$$\begin{aligned} u &\approx u + u_i + D_Z(u) && \text{(by (10))} \\ &\approx u + u_i + D_Z(u) + u_1^2 && \text{(by (6))} \\ &\approx u + u_i + D_Z(u) + u_1^2 + u_1^2 u_2 \cdots u_k. && \text{(by (8))} \end{aligned}$$

Denote p for $u_1^2 u_2 \cdots u_k$. Thus $c(p) = c(q)$ and we have derived the identity

$$u \approx u + p. \tag{12}$$

Now that $|p| > 1$, by (4), we have

$$p^2 \approx p. \tag{13}$$

Suppose that $i(q) = x_1 x_2 \cdots x_\ell$. We shall show by induction on j that for every $1 \le j \le \ell$, $u \approx u + x_1 x_2 \cdots x_\ell p$ is derivable from (2)-(10) and the identities defining **SR**.

From Lemma 2.2, there exists u_{i_1} in $D_Z(u)$ with $c(u_{i_1}) \subseteq c(q)$ such that $h(u_{i_1}) = h(q) = x_1$. Furthermore,

$$\begin{aligned}
u &\approx u + u_{i_1} + p && \text{(by (12))}\\
&\approx u + u_{i_1} + p + u_{i_1}p && \text{(by (7))}\\
&\approx u + u_{i_1} + p + x_1 u_{i_1} p && \text{(by (2))}\\
&\approx u + u_{i_1} + p + x_1 u_{i_1} p + x_1 p u_{i_i} p && \text{(by (9))}\\
&\approx u + u_{i_1} + p + x_1 u_{i_1} p + x_1 p. && (by (5), (13))
\end{aligned}$$

Therefore,

$$u \approx u + x_1 p. \tag{14}$$

Assume that, for some $1 < j \le \ell$,

$$u \approx u + x_1 x_2 \cdots x_{j-1} p \tag{15}$$

is derivable from (2)-(10) and the identities defining **SR**. By Lemma 2.2, there exists u_i in $D_Z(u)$ with $c(u_i) \subseteq c(q)$ such that $u_i = u_{i_1} x_j u_{i_2}$ and $c(u_{i_1}) \subseteq \{x_1, x_2, \ldots, x_{j-1}\}$. It follows that

$$\begin{aligned}
u &\approx u + u_i + p\\
&\approx u + u_i + p + u_i p && \text{(by (7))}\\
&\approx u + u_i + p + u_{i_1} x_j u_{i_2} p\\
&\approx u + u_i + p + u_{i_1} x_j u_{i_2} p + u_{i_1} x_j p u_{i_2} p && \text{(by (9))}\\
&\approx u + u_i + p + u_{i_1} x_j u_{i_2} p + u_{i_1} x_j p. && \text{(by (5),(13))}
\end{aligned}$$

Consequently

$$u \approx u + u_{i_1} x_j p. \tag{16}$$

Moreover, we have

$$\begin{aligned}
u &\approx u + x_1 x_2 \cdots x_{j-1} p + u_{i_1} x_j p && \text{(by (15),(16))}\\
&\approx u + x_1 x_2 \cdots x_{j-1} p + u_{i_1} x_j p + x_1 x_2 \cdots x_{j-1} u_{i_1} x_j p p && \text{(by (9))}\\
&\approx u + x_1 x_2 \cdots x_{j-1} p + u_{i_1} x_j p + x_1 x_2 \cdots x_{j-1} x_j p. && \text{(by (2),(5),(13))}
\end{aligned}$$

Hence $u \approx u + x_1x_2\cdots x_{j-1}x_jp$. Using induction we have

$$u \approx u + i(q)p. \tag{17}$$

Dually,

$$u \approx u + pf(q). \tag{18}$$

Thus

$$\begin{array}{rcll} u & \approx & u + p + i(q)p + pf(q) & \text{(by (12),(17),(18))} \\ & \approx & u + p + i(q)p + pf(q) + i(q)ppf(q) & \text{(by (7))} \\ & \approx & u + p + i(q)p + pf(q) + i(q)f(q) & \text{(by (2),(3),(4),(5))} \\ & \approx & u + p + i(q)p + pf(q) + q. & \text{(by (11))} \end{array}$$

It follows that $u \approx u + q$.

4. The lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$

In this section we characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$. Throughout this section, $t(x_1, \ldots, x_n)$ denotes the term t which contains no other variables than $x_1, \ldots, x_n$ (but not necessarily all of them). Let $S \in \mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ and $E^+(S)$ denote the set $\{a \in S \,|\, 2a = a\}$, where the elements of $E^+(S)$ is said to be *additive idempotent* of $(S, +)$. Notice that $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ satisfies the identities

$$2(x + y) \approx 2x + 2y; \tag{19}$$

$$2xy \approx (x + x)(y + y). \tag{20}$$

By (19) and (20), it is easy to verify that $E^+(S) = \{2a \,|\, a \in S\}$ forms a subsemiring of S. To characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$, we need to consider the following

$$\varphi : \mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)) \to \mathcal{L}(\mathbf{W}), \mathbf{V} \mapsto \mathbf{V} \cap \mathbf{W}. \tag{21}$$

It is easy to prove that $\varphi(\mathbf{V}) = \{E^+(S) \,|\, S \in \mathbf{V}\}$ for each member $\mathbf{V}$ of $\mathcal{L}(\mathbf{W})$. If $\mathbf{V}$ is the subvariety of $\mathbf{W}$ determined by the identities

$$u_i(x_{i_1}, \ldots, x_{i_n}) \approx v_i(x_{i_1}, \ldots, x_{i_n}), i \in \underline{k},$$

then $\widehat{\mathbf{V}}$ denotes the subvariety of $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ determined by the identities

$$u_i(2x_{i_1}, \ldots, 2x_{i_n}) \approx v_i(2x_{i_1}, \ldots, 2x_{i_n}), i \in \underline{k}. \tag{22}$$

Lemma 4.1. *Let $\mathbf{V}$ be a member of $\mathcal{L}(\mathbf{W})$. Then, $\widehat{\mathbf{V}} = \mathbf{V} \vee \mathbf{HSP}(Z_2, W_2)$.*

Proof of Lemma 4.1. Since $\mathbf{V}$ satisfies the identities (22), it follows that $\mathbf{V}$ is a subvariety of $\widehat{\mathbf{V}}$. And both Z_2 and W_2 are members of $\widehat{\mathbf{V}}$ and so the join $\mathbf{V} \vee \mathbf{HSP}(Z_2, W_2) \subseteq \widehat{\mathbf{V}}$. To show the converse inclusion, it suffices to show that every identity that is satisfied by $\mathbf{V} \vee \mathbf{HSP}(Z_2, W_2)$ can be derived by the identities holding in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ and $u_i(2x_{i_1}, \ldots, 2x_{i_n}) \approx v_i(2x_{i_1}, \ldots, 2x_{i_n}), i \in \underline{k}$, if $\mathbf{V}$ is the subvariety of $\mathbf{W}$ determined by $u_i(x_{i_1}, \ldots, x_{i_n}) \approx v_i(x_{i_1}, \ldots, x_{i_n}), i \in \underline{k}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 2.1 (8), we only need to consider the following two cases.

Case 1 $m, n \geq 2$. By identity (10), $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ satisfies the identities

$$2u \approx u; \tag{23}$$

$$2v \approx v. \tag{24}$$

Since $u \approx v$ holds in $\mathbf{W}$, we have that it is derivable from the collection Σ of $u_i \approx v_i, i \in \underline{k}$ and the identities determining $\mathbf{W}$. From [1, Exercise II.14.11], it follows that there exist $t_1, t_2, \ldots, t_\ell \in P_f(X^+)$ such that

- $t_1 = u, t_\ell = v$;
- For any $i = 1, 2, \ldots, \ell - 1$, there exist $p_i, q_i, r_i \in P_f(X^+)$ (where p_i, q_i and r_i may be empty words), a semiring substitution φ_i and an identity $u'_i \approx v'_i \in \Sigma$ such that

$$\begin{array}{l} t_i = p_i\varphi_i(w_i)q_i + r_i, t_{i+1} = p_i\varphi_i(s_i)q_i + r_i, \\ \text{where either } w_i = u'_i, s_i = v'_i \text{ or } w_i = v'_i, s_i = u'_i. \end{array}$$

Let Σ' denote the set $\{2u \approx 2v \,|\, u \approx v \in \Sigma\}$. For any $i = 1, 2, \ldots, \ell - 1$, we shall show that $2t_i \approx 2t_{i+1}$ is derivable from Σ' and the identities holding in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. In deed, we have

$$\begin{array}{rcl} 2t_i &=& 2(p_i\varphi_i(w_i)q_i + r_i) \\ &\approx& 2(p_i\varphi_i(w_i)q_i) + 2r_i \\ &\approx& p_i(\varphi_i(2w_i))q_i + 2r_i \\ &\approx& p_i(\varphi_i(2s_i))q_i + 2r_i \\ && (\text{since } 2w_i \approx 2s_i \in \Sigma' \text{ or } 2s_i \approx 2w_i \in \Sigma') \\ &\approx& 2(p_i\varphi_i(s_i)q_i) + 2r_i \\ &\approx& 2(p_i\varphi_i(s_i)q_i + r_i) \\ &=& 2t_{i+1}. \end{array}$$

Further,

$$2u = 2t_1 \approx 2t_2 \approx \cdots \approx 2t_\ell = 2v.$$

This implies the identity

$$2u \approx 2v. \tag{25}$$

We now have

$$u \overset{(24)}{\approx} 2u \overset{(25)}{\approx} 2v \overset{(24)}{\approx} v. \tag{26}$$

Case 2 $m = n = 1$ and $C(u) = C(v)$. Since $Z_2 \models u_1 \approx v_1$, $u_1 \neq x, v_1 \neq x$, for every $x \in X$. Since $u_1 \approx v_1$ holds in $\mathbf{W}$, we have that it is derivable from the collection Σ of $u_i \approx v_i, i \in \underline{k}$ and the identities definging $\mathbf{W}$. From [1, Exercise II.14.11], it follows that there exist $t_1, t_2, \ldots, t_\ell \in P_f(X^+)$ such that

- $t_1 = u_1, t_\ell = v_1$;
- For any $i = 1, 2, \ldots, \ell - 1$, there exist $p_i, q_i \in P_f(X^+)$ (where p_i and q_i may be empty words), a semiring substitution φ_i and an identity $u_i' \approx v_i' \in \Sigma$ (where u_i' and v_i' are words) such that

$$\begin{aligned} &t_i = p_i\varphi_i(w_i)q_i, t_{i+1} = p_i\varphi_i(s_i)q_i,\\ &\text{where either } w_i = u_i', s_i = v_i' \text{ or } w_i = v_i', s_i = u_i'. \end{aligned}$$

By Lemma 3.1, $u_1 \approx v_1$ can be derived from (2), (3), (4) and (5), moreover, by Lemma 3.1, it can be derived from monomial identities holding in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. This completes the proof. $\square$

Lemma 4.2. *The following equality holds:*

$$\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)) = \bigcup_{\mathbf{V} \in \mathcal{L}(\mathbf{W})} [\mathbf{V}, \widehat{\mathbf{V}}]. \tag{27}$$

There are 312 *intervals in* (27)*, and each interval is a congruence class of the kernel of the complete epimorphism* φ *in* (21)*.*

Proof of Lemma 4.2. First, we shall show that equality (27) holds. It is easy to see that

$$\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)) = \bigcup_{\mathbf{V} \in \mathcal{L}(\mathbf{W})} \varphi^{-1}(\mathbf{V}).$$

So it suffices to show that

$$\varphi^{-1}(\mathbf{V}) = [\mathbf{V}, \widehat{\mathbf{V}}], \tag{28}$$

for each member $\mathbf{V}$ of $\mathcal{L}(\mathbf{W})$. If $\mathbf{V}_1$ is a member of $[\mathbf{V}, \widehat{\mathbf{V}}]$, then it is routine to verity that $\mathbf{V} \subseteq \{E^+(S) \mid S \in \mathbf{V}_1\} \subseteq \mathbf{V}$. This implies that $\{E^+(S) \mid S \in \mathbf{V}_1\} = \mathbf{V}$ and so $\varphi(\mathbf{V}_1) = \mathbf{V}$. Hence, $\mathbf{V}_1$ is a member of $\varphi^{-1}(\mathbf{V})$ and so

$[\mathbf{V}, \widehat{\mathbf{V}}] \subseteq \varphi^{-1}(\mathbf{V})$. Conversely, if $\mathbf{V}_1$ is a member of $\varphi^{-1}(\mathbf{V})$, then $\mathbf{V} = \varphi(\mathbf{V}_1) = \{E^+(S) \mid S \in \mathbf{V}_1\}$ and so $\varphi^{-1}(\mathbf{V}) \subseteq [\mathbf{V}, \widehat{\mathbf{V}}]$. This shows that (27) holds.

From Lemma 3.1, we know that $\mathcal{L}(\mathbf{W})$ is a lattice of order 312. So there are 312 intervals in (27). Next, we show that φ a complete epimorphism. On the one hand, it is easy to see that φ is a complete $\wedge$-epimorphism. On the other hand, let $(\mathbf{V}_i)_{i\in I}$ be a family of members of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$. Then, by (21), we have that $\varphi(\mathbf{V}_i) \subseteq \mathbf{V}_i \subseteq \widehat{\varphi(\mathbf{V}_i)}$ for each $i \in I$. Further,

$$\bigvee_{i\in I}\varphi(\mathbf{V}_i) \subseteq \bigvee_{i\in I}\mathbf{V}_i \subseteq \bigvee_{i\in I}\widehat{\varphi(\mathbf{V}_i)} \subseteq \widehat{\bigvee_{i\in I}\varphi(\mathbf{V}_i)}.$$

This implies that $\varphi(\bigvee_{i\in I}\mathbf{V}_i) = \bigvee_{i\in I}\varphi(\mathbf{V}_i)$. Thus, φ is a complete $\vee$-homomorphism and so φ is a complete epimorphism. By (28), we deduce that each interval in (21) is a congruence class of the kernel of the complete epimorphism φ. $\square$

In order to characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$, by Lemma 4.2, we only need to describe the interval $[\mathbf{V}, \widehat{\mathbf{V}}]$ for each member $\mathbf{V}$ of $\mathcal{L}(\mathbf{W})$. Next, we have

Lemma 4.3. *Let $\mathbf{V}$ be a member of $\mathcal{L}(\mathbf{W})$. Then, $\mathbf{V} \vee \mathbf{HSP}(Z_2)$ is the subvariety of $\widehat{\mathbf{V}}$ determined by the identity*

$$xy \approx 2xy. \tag{29}$$

Proof of Lemma 4.3. It is easy to see that both $\mathbf{V}$ and $\mathbf{HSP}(Z_2)$ satisfy the identity (29) and so does $\mathbf{V} \vee \mathbf{HSP}(Z_2)$. In the following we prove that every identity that is satisfied by $\mathbf{V}\vee\mathbf{HSP}(Z_2)$ is derivable from (29) and the identities holding in $\widehat{\mathbf{V}}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \leq i \leq m, 1 \leq j \leq n$. We only need to consider the following cases.

Case 1. $m = n = 1$. Since Z_2 satisfies $u_1 \approx v_1$, it follows that $|u_1| \neq 1$ and $|v_1| \neq 1$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $2u_1 \approx 2v_1$. Hence $u_1 \overset{(29)}{\approx} 2u_1 \approx 2v_1 \overset{(29)}{\approx} v_1$.

Case 2. $m = 1$, $n \geq 2$. Since Z_2 satisfies $u_1 \approx v$, it follows that $|u_1| \neq 1$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $2u_1 \approx 2v$. Hence $u_1 \overset{(29)}{\approx} 2u_1 \approx 2v \overset{(10)}{\approx} v$.

Case 3. $m \geq 2$, $n = 1$. Similar to case 2.

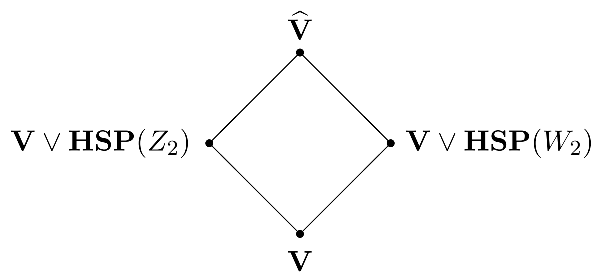
Case 4. $m, n \geq 2$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $2u \approx 2v$. Hence $u \overset{(10)}{\approx} 2u \approx 2v \overset{(10)}{\approx} v$. $\square$

Lemma 4.4. *Let $\mathbf{V}$ be a member of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*))$. Then $\mathbf{V} \vee \mathbf{HSP}(W_2)$ is the subvariety of $\widehat{\mathbf{V}}$ determined by the identity*

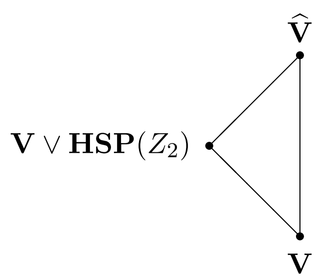
$$x^2 \approx x. \tag{30}$$

Proof of Lemma 4.4. It is easy to see that both $\mathbf{V}$ and $\mathbf{HSP}(W_2)$ satisfy the identity (30) and so does $\mathbf{V} \vee \mathbf{HSP}(W_2)$. So it suffices to show that every identity that is satisfied by $\mathbf{V} \vee \mathbf{HSP}(W_2)$ is derivable from (30) and the identities holding in $\widehat{\mathbf{V}}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $u^2 \approx v^2$. Hence, $u \overset{(30)}{\approx} u^2 \approx v^2 \overset{(30)}{\approx} v$. $\square$

Lemma 4.5. *Let $\mathbf{V} \in \mathcal{L}(\mathbf{W})$. Then the interval $[\mathbf{V}, \widehat{\mathbf{V}}]$ of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$ is given in Fig.1*



Case. 1 $N_2, T_2 \notin \mathbf{V}$



Case. 2 $N_2 \in \mathbf{V}$ or $T_2 \in \mathbf{V}$

Fig.1 *The interval* $[\mathbf{V}, \widehat{\mathbf{V}}]$

Proof of Lemma 4.5. Suppose that $\mathbf{V}_1$ is a member of $[\mathbf{V}, \widehat{\mathbf{V}}]$ such that $\mathbf{V}_1 \neq \widehat{\mathbf{V}}$ and $\mathbf{V}_1 \neq \mathbf{V}$. Then, there exists a nontrivial identity $u \approx v$ holding in $\mathbf{V}_1$ such that it is not satisfied by $\widehat{\mathbf{V}}$. Also, we have that $\mathbf{V}_1$ dose not satisfy the identity $2x \approx x$. By Lemma 4.1, we only need to consider the following two cases.

Case 1 $\mathbf{HSP}(Z_2) \models u \approx v, \mathbf{HSP}(W_2) \not\models u \approx v$. Then, $u \approx v$ satisfies one of the following three cases:

- $m = n = 1$, $c(u_1) \neq c(v_1)$, $|u_1| \neq 1$ and $|v_1| \neq 1$;
- $m = 1, n > 1$ and $|u_1| \neq 1$;

- $m > 1, n = 1$ and $|v_1| \neq 1$.

It is easy to see that, in each of the above cases, $u \approx v$ can imply the identity $xy \approx 2xy$. By Lemma 4.3, we have that $\mathbf{V}_1$ is a subvariety of $\mathbf{V} \vee \mathbf{HSP}(Z_2)$. On the other hand, since $\mathbf{V}_1 \models xy \approx 2xy$ and $\mathbf{V}_1 \not\models 2x \approx x$, it follows that Z_2 is a member of $\mathbf{V}_1$ and so $\mathbf{V} \vee \mathbf{HSP}(Z_2)$ is a subvariety of $\mathbf{V}_1$. Thus, $\mathbf{V}_1 = \mathbf{V} \vee \mathbf{HSP}(Z_2)$.
Case 2 $\mathbf{HSP}(Z_2) \not\models u \approx v, \mathbf{HSP}(W_2) \models u \approx v$. Then, $u \approx v$ satisfies one of the following two cases:

- $m = n = 1$, $c(u_1) = c(v_1)$ and $|u_1| = 1$;
- $m = n = 1$, $c(u_1) = c(v_1)$ and $|v_1| = 1$.

If $N_2, T_2 \notin \mathbf{V}$, then, in each of the above cases, $u \approx v$ can imply the identity $x \approx x^2$. By Lemma 4.4, $\mathbf{V}_1$ is a subvariety of $\mathbf{V} \vee \mathbf{HSP}(W_2)$. On the other hand, since $\mathbf{V}_1 \models x \approx x^2$ and $\mathbf{V}_1 \not\models x \approx 2x$, it follows that W_2 is a member of $\mathbf{V}_1$ and so $\mathbf{V} \vee \mathbf{HSP}(W_2)$ is a subvariety of $\mathbf{V}_1$. Thus, $\mathbf{V}_1 = \mathbf{V} \vee \mathbf{HSP}(W_2)$.

If $N_2 \in \mathbf{V}$, then, by Lemma 2.1 (i), $|u_1| = |v_1| = 1$, a contradiction. Thus, $\mathbf{V}_1 = \widehat{\mathbf{V}}$.

If $T_2 \in \mathbf{V}$, then, by Lemma 2.1 (ii), $|u_1| \geq 2, |v_1| \geq 2$, a contradiction. Thus, $\mathbf{V}_1 = \widehat{\mathbf{V}}$. □

Theorem 4.1. *$\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$ is a distributive lattice of order* 1014*.*

Proof of Theorem 4.1. By (27) and Lemma 4.5, we can show that $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$ has exactly 1014 elements. Suppose that $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_3$ are members of $\mathcal{L}(\mathbf{W})$ such that $\mathbf{W}_1 \vee \mathbf{W}_2 = \mathbf{W}_1 \vee \mathbf{W}_3$ and $\mathbf{W}_1 \wedge \mathbf{W}_2 = \mathbf{W}_1 \wedge \mathbf{W}_3$. Then, by Lemma 4.2

$$\varphi(\mathbf{W}_1) \vee \varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_1) \vee \varphi(\mathbf{W}_3)$$

and

$$\varphi(\mathbf{W}_1) \wedge \varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_1) \wedge \varphi(\mathbf{W}_3).$$

Since $\mathcal{L}(\mathbf{W})$ is distributive, it follows that $\varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_3)$. Write $\mathbf{V}$ for $\varphi(\mathbf{W}_2)$. Then both $\mathbf{W}_2, \mathbf{W}_3$ are members of $[\mathbf{V}, \widehat{\mathbf{V}}]$. By Fig.1, we deduce that $\mathbf{W}_2 = \mathbf{W}_3$. □

By Theorem 3.1, 4.1 and [13, Corollary 1.2], we now immediately deduce

Corollary 4.1. $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ *is hereditarily finitely based.*

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Fixed point theorem for (ϕ, F)-contraction on C^*-algebra valued partial metric spaces

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Abstract. Recently, a new type of mapping called (ϕ, F)- contraction was introduced in the literature as a generalization of the concepts of contractive mappings. This present article extends the new notion in C^*-algebra valued partial metric spaces and establishing the existence and uniqueness of fixed point for them. Non-trivial examples are further provided to support the hypotheses of our results.

Keywords: fixed point, C^*-algebra valued partial metric spaces, C^*-algebra valued partial (ϕ, F)- contraction.

1. Introduction

Metric fixed point theory has its roots in methods from the late 19th century, when successive approximations were used to establish the existence and uniqueness of solutions to equations, and especially differential equations. This approach is particularly associated with the work of Picard, although it was Stefan Banach who in 1922 in [2] developed the ideas involved in an abstract setting.

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Banach's contraction principle is a fundamental result in fixed point theory. Due to its importance, several authors have obtained many interesting extensions and generalizations (see [1, 3, 5, 10, 12]).

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, C^*-algebra valued metric spaces were introduced by Ma et al. [11] as a generalization of metric spaces they proved certain fixed point theorems, by giving the definition of C^*-algebra valued contractive mapping analogous to Banach contraction principle.

In this paper, inspired by the work done in [6, 9], we introduce the notion of C^*-algebra valued partial (ϕ, F)-contraction and establish some new fixed point theorems for mappings in the setting of complete C^*-algebra valued partial metric spaces. Moreover, an illustrative example is presented to support the obtained results.

2. Preliminaries

Throughout this paper, we denote $\mathbb{A}$ an unital C^*-algebra with linear involution $*$, such that for all $x, y \in \mathbb{A}$,

$$(xy)^* = y^* x^*, \quad x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$ if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x.

Using positive element, we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows:

$$x \preceq y \text{ if and only if } y - x \succeq \theta,$$

where θ means the zero element in $\mathbb{A}$.

We denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ by $\mathbb{A}_+$ and $|x| = (x^* x)^{\frac{1}{2}}$.

Remark 2.1. When $\mathbb{A}$ is an unital C^*-algebra, then for any $x \in \mathbb{A}_+$ we have

$$x \preceq I \Longleftrightarrow \|x\| \leq 1.$$

Definition 2.2 ([8])**.** Let X be a non-empty set. A mapping $p : X \times X \to \mathbb{A}$ is called a C^*-algebra valued metric on X if the following conditions are satisfied:

(i) $\theta \preceq p(x, y)$ for all $x, y \in X$ and $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$

(ii) $p(x, y) = p(y, x)$ for all $x, y \in X$;

(iii) $p(x, x) \preceq p(x, y)$ for all $x, y \in X$

(iv) $p(x, y) \preceq p(x, z) + p(z, y) - p(z, z)$ for all $x, y, z \in X$.

Then $(X, \mathbb{A}_+, p)$ is called a C^*-algebra valued partial metric space.

If we take $\mathbb{A} = \mathbb{R}$, then the new notion of C^*-algebra valued partial metric space becomes equivalent to the definition of the real partial metric space.

Example 2.3. Let$X = [0, 1]$ and $x \in \mathbb{A}$ be a nonzero element.

Define $p(s, t) = \max\{1 + s, 1 + t\}xx^*$. Then we can easily show that $p : X \times X \to \mathbb{A}$ is a C^*-algebra valued partial metric.

Example 2.4. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ with the usual norm is a real Banach space.

Let $p : X \times X \to \mathbb{R}^2$ be given as follows:

$$p(x, y) = (|x - y|, |x - y|).$$

Then, $(X, \mathbb{R}^2, p)$ is a complete C^*-algebra valued partial metric.

Definition 2.5 ([7])**.** Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$.

(1) $\{x_n\} \subset X$ converges to x whenever for every $\varepsilon > 0$ there is a natural number N such that for all $n > N$,

$$\|p(x_n, x) - p(x, x)\| \le \varepsilon.$$

We denote it by

$$\lim_{n\to\infty} p(x_n, x) - p(x, x) = \theta.$$

(2) $\{x_n\}$ is a partial Cauchy sequence respect to $\mathbb{A}$, whenever $\varepsilon > 0$ there is a natural number N such that

$$\begin{aligned}&(p(x_n, x_m) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))((p(x_n, x_m)\\ &- \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))^* \preceq \varepsilon^2,\end{aligned}$$

for all $n, m > N$;

(3) $(X, \mathbb{A}_+, p)$ is said to be complete with respect to $\mathbb{A}$ if every partial Cauchy sequence with respect to $\mathbb{A}$ converges to a point x in X such that

$$\lim_{n\to\infty} (p(x_n, x) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x, x)) = \theta.$$

From given C^*-algebra-valued partial metric, we can obtain a C^*- algebra-valued metric. Put

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y).$$

Then, p^s is a C^*- algebra -valued metric.

Lemma 2.6 ([7])**.** *Let $(X, \mathbb{A}, p)$ be a C^*- algebra- valued partial metric space.*

(1) $\{x_n\}$ *is a partial Cauchy sequence in* $(X, \mathbb{A}, p)$ *if and only if it is Cauchy in the* C^**- algebra -valued metric* $(X, \mathbb{A}, p^s)$*.*

(2) *A* C^**- algebra- valued partial metric space* $(X, \mathbb{A}, p)$ *is complete if and only if* C^**- algebra- valued metric space* $(X, \mathbb{A}, p^s)$ *is complete. Furthermore,*

$$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow \lim_{n\to\infty}(2p(x_n, x) - p(x_n, x_n) - p(x, x)) = \theta$$

or

$$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow \lim_{n\to\infty} p(x_n, x) - p(x_n, x_n) = \theta,\ \lim_{n\to\infty} p(x_n, x) - p(x, x) = \theta.$$

Lemma 2.7 ([7])**.** *Assume that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$ *in a* C^*- *algebra valued partial metric space* $(X, \mathbb{A}, p)$*. Then*

$$\lim_{n\to\infty}(p(x_n, y_n) - p(x_n, x_n)) = p(x, y) - p(x, x)$$

and

$$\lim_{n\to\infty}(p(x_n, y_n) - p(y_n, y_n)) = p(x, y) - p(y, y).$$

Definition 2.8 ([14])**.** Let the function $\phi : A^+ \to A^+$ be positive if having the following constraints:

(i) ϕ is continuous and nondecreasing;

(ii) $\phi(a) = \theta$ if and only if $a = \theta$;

(iii) $\lim_{n\to\infty} \phi^n(a) = \theta$.

Definition 2.9 ([14])**.** Suppose that A and B are C^*-algebra. A mapping $\phi : A \to B$ is said to be C^*- homomorphism if:

(i) $\phi(ax + by) = a\phi(x) + b\phi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$;

(ii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$;

(iii) $\phi(x^*) = \phi(x)^*$ for all $x \in A$;

(iv) ϕ maps the unit in A to the unit in B.

Definition 2.10 ([14])**.** Let A and B be C^*-algebra spaces and let $\phi : A \to B$ be a homomorphism, then ϕ is called an $*-$ homomorphism if it is one to one $*-$ homomorphism. A C^*-algebra A is $*-$isomorphic to a C^*-algebra B if there exists $*-$ isomorphism of A onto B.

Lemma 2.11 ([13])**.** *Let* A *and* B *be* C^**-algebra spaces and* $\phi : A \to B$ *is a* C^*- *homomorphism for all* $x \in A$ *we have*

$$\sigma(\phi(x)) \subset \sigma(x),\ \ \|\phi(x)\| \le \|\phi\|.$$

Corollary 2.12 ([14]). *Every C^*- homomorphism is bounded.*

Corollary 2.13 ([14]). *Suppose that ϕ is C^*- isomorphism from A to B, then $\sigma(\phi(x)) = \sigma(x)$ and $\|\phi(x)\| = \|\phi\|$ for all $x \in A$.*

Lemma 2.14 ([14]). *Every $*-$ homomorphism is positive.*

The following definition was given by D. Wardowski in [4].

Definition 2.15 ([10]). Let $F\colon\ \mathbb{R}_+ \to \mathbb{R}$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping satisfying:

(i) F is strictly increasing, for $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty.$$

(iii) $\liminf_{s\to\alpha^+} \phi(s) > 0$, for all $s > 0$.

(iv) There exists $k \in]0, 1[$ such that $\lim_{x\to 0} x^k F(x) = 0$. A mapping $T : X \to X$ is said to be an (ϕ, F)-contraction in partial metric space if

$$\forall x, y \in X; p(Tx, Ty) \geq 0 \Rightarrow \phi(p(x, y)) + F(p(Tx, Ty) \leqslant F(p(x, y)).$$

Definition 2.16 ([10]). Let (X, p) be a complete partial metric space. A mapping $T : X \to X$ is called an $(\phi, F)-$ contraction on (X, p) if there exists F and ϕ defined in Definition 2.15 such that

$$(p(Tx, Ty) > 0 \Rightarrow F(p(Tx, Ty) + \phi(p(x, y)) \leqslant F(p(x, y)),$$

for all $x, y \in X$ for which $Tx \neq Ty$.

Theorem 2.17. *Let (X, p) be a complete partial metric space and $T : X \to X$ be an $(\phi, F)-$ contraction. Then T has a unique fixed point.*

3. Main result

Aspired by Wardowski in [10], we introduce the notion of $(\phi, F) - C^*$-valued partial contraction.

Definition 3.1. Let $F\colon \mathbb{A}_+ \to \mathbb{A}_+$ a function satisfying:

(i) F is continuous and nondecreasing.

(ii) $F(T) = \theta$ if and only if $T = \theta$.

1. A mapping $T : X \to X$ is said to be a (ϕ, F) C^* valued partial contraction of type (I) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$ homomorphism such that

(1) $\forall x, y \in X; (p(Tx, Ty) \succeq \theta \Rightarrow F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(p(x, y)).$

2. A mapping $T : X \to X$ is said to be a (ϕ, F) C^* valued partial contraction of type (II) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$ homomorphism satisfying:
   (a) $\phi(a) \prec a$ for $a \in \mathbb{A}_+$.
   (b) Either $\phi(a) \preceq p(x, y)$ or $p(x, y) \preceq \phi(a)$, where $a \in \mathbb{A}_+$ and $x, y \in X$.
   (c) $F(a) \prec \phi(a)$. Such that
$$(p(Tx, Ty) \succeq \theta \Rightarrow F(p(Tx, Ty) + \phi(p(x, y)) \preceq F(M(x, y)),$$
   where $M(x, y) = a_1 p(x, y) + a_2[p(Tx, y) + p(Ty, x)] + a_3[p(Tx, x) + p(Ty, y)]$, with $a_1, a_2, a_3 \geq 0$, $a_1 + 2a_2 + 2a_3 \leq 1$.
3. T is said to be (ϕ, F)- Kannan-type C^*- valued contraction if there exist ϕ satisfy (a), (b) and (c) such that $p(Tx, Ty) \succeq \theta$, we have
$$F(p(Tx, Ty) + \phi(p(x, y)) \preceq F(\frac{p(x, Tx) + p(y, Ty)}{2}).$$
4. T is said to be (ϕ, F)- Reich-type C^*- valued partial contraction if there exist ϕ satisfy (a), (b) and (c) such that $p(Tx, Ty) \succeq \theta$, we have
$$F(p(Tx, Ty) + \phi(p(x, y)) \preceq F(\frac{p(x, y) + p(x, Tx) + p(y, Ty)}{3}).$$

Example 3.2. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ Then $\mathbb{A}$ is a C^*- algebra with norm $\|.\| : \mathbb{A} \to \mathbb{R}$ defined by
$$\|(x, y)\| = (x^2 + y^2)^{\frac{1}{2}}.$$
Define a C^*- algebra valued partial metric $p : X \times X \to \mathbb{A}$ on X by $p(x, y) = (x + y, x + y)$, with ordering on $\mathbb{A}$ by
$$(a, b) \preceq (c, d) \ \Leftrightarrow a \leq c \ \text{ and } \ b \leq d.$$
A mapping $T : X \to X$ given by $Tx = x - \frac{1}{2}x^2$ is continuous with respect to $\mathbb{A}$. Let $F : \mathbb{A}_+ \to \mathbb{A}_+$. Defined by $F(x, y) = (x, y)$.

It is clear that F satisfies (i) and (ii).

We have $F(p(Tx, Ty)) = p(Tx, Ty) = (x - \frac{1}{2}x^2 + y - \frac{1}{2}y^2, x - \frac{1}{2}x^2 + y - \frac{1}{2}y^2)$ and $F(p(Tx, Ty)) - F(p(x, y)) \leq -(\frac{1}{4}(x + y)^2, \frac{1}{4}(x + y)^2)$. Therefore, T is a C^*-algebra valued partial F-contraction with $\phi(x, y) = (\frac{1}{4}(x + y)^2, \frac{1}{4}(x + y)^2)$.

Example 3.3. Let $X = [0, 1] \cup \{2, 3, 4, ...\}$ and $\mathbb{A} = \mathbb{C}$ with a norm $|| \ z \ ||=| \ z \ |$ be a C^*- algebra. We define $\mathbb{C}^+ = \{z = (x, y) \in \mathbb{C};\ x = Re(z) \geq 0,\ y = Im(z) \geq 0\}$.

The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ is the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.

Let $p : X \times X \to \mathbb{C}$

$$p(x,y) = \begin{cases} (|x-y|\,, |x-y|), & \text{if } x, y \in [0,1], x \neq y, \\ (x+y, x+y), & \text{if at least one of } x \text{ or } y \notin [0,1] \text{ and } x \neq y, \\ (0,0), & \text{if } x = y. \end{cases}$$

Then, $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued metric space.

Let $F : \mathbb{C}^+ \to \mathbb{C}$ be defined as

$$F(t) = \begin{cases} t, & \text{if } t \in [0,1], \\ t^2, & \text{if } t > 1. \end{cases}$$

It is clear that F satisfies (i) and (ii) Let $T : X \to X$ be defined as

$$T(x) = \begin{cases} x - \frac{1}{2}x^2, & \text{if } x \in [0,1], \\ x - 1, & \text{if } x \in \{2, 3, 4, ...\}. \end{cases}$$

Without loss of generality, we assume that $x > y$ and discuss the following cases:
Case 1. $(x \in [0;1])$. Then

$$\begin{aligned} F(p(Tx,Ty)) &= ((x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2), (x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2)) \\ &= ((x-y) - \frac{1}{2}(x-y)(x+y), (x-y) - \frac{1}{2}(x-y)(x+y)) \\ &\leq ((x-y) - \frac{1}{2}((x-y))^2, (x-y) - \frac{1}{2}((x-y))^2) \\ &= p(x,y) - \frac{1}{2}(p(x,y))^2 \\ &= F(p(x,y)) - \frac{1}{2}(p(x,y))^2. \end{aligned}$$

Then, there exists ϕ such $\phi(x,y) = \frac{1}{2}(p(x,y))^2$ and $\forall x, y \in X$, $p(Tx,Ty) \geq 0 \Rightarrow \phi(x,y) + F(p(Tx,Ty)) \leq F(p(x,y))$.
Case 2. $(x \in \{3, 4, ...\})$, then

$$p(Tx,Ty) = p(x-1, y - \frac{1}{2}y^2) \;\text{ if } y \in [0,1]$$

or

$$\begin{aligned} p(Tx,Ty) = (x - 1 + y - \frac{1}{2}y^2, x - 1 + y - \frac{1}{2}y^2) \leq (x + y - 1, x + y - 1), \\ p(Tx,Ty) = p(x-1, y-1) \text{ if } y \in \{2, 3, 4, ...\} \end{aligned}$$

or

$$p(Tx,Ty) = (x + y - 2, x + y - 2) < (x + y - 1, x + y - 1).$$

Consequently,

$$\begin{aligned}F(p(Tx,Ty)) &= (p(Tx,Ty))^2 \leq ((x+y-1)^2,(x+y-1)^2)\\ &< ((x+y-1)(x+y+1),(x+y-1)(x+y+1))\\ &= ((x+y)^2-1,(x+y)^2-1) < ((x+y)^2-\frac{1}{2},(x+y)^2-\frac{1}{2})\\ &= F(p(x,y))-\frac{1}{2}.\end{aligned}$$

Case 3. $(x=2)$, then $y\in[0,1]$, $Tx=1$ and

$$p(Tx,Ty) = (1-(y-\frac{1}{2}y^2),1-(y-\frac{1}{2}y^2)).$$

So, we have $F(p(Tx,Ty)) \leq F(1) = 1$. Again, $p(x,y) = (2+y,2+y)$. So, $1 = F(p(Tx,Ty)) \leq F(p(x,y)) - \frac{1}{2}$.

Theorem 3.4. *Let $(X,\mathbb{A},p)$ be a complete C^*-algebra valued partial metric space and let $T : X \to X$ be a (ϕ,F) C^*- valued partial contraction mapping of type (I). Then T has a unique fixed point $x* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to $x*$.*

Proof. First, let us observe that T has at most one fixed point. Indeed if

$$x_1^*; x_2^* \in X,\ \ Tx_1^* = x_1^* \neq x_2^* = Tx_2^*$$

then, we get

$$\phi(p(x,y)) \preceq F(p(x_1^*;x_2^*)) - F(p(Tx_1^*;Tx_2^*)) = \theta$$

which is a contradiction.

In order to show that thas a fixed point let $x_0 \in X$ be arbitrary and fixed we define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$; $x_{n+1} = Tx_n, n = 0,1,2,\dots$ denote $p_n = p(x_{n+1};x_n), n = 0,1,2,\dots$ if there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$ then $Tx_{n_0} = x_{n_0}$ and the proof is finished.

Suppose now, that $x_{n+1} \neq x_n$, for every $n \in X$ then $p_n \succ \theta$, for all $n \in \mathbb{N}$ and using (1) the following holds, for every $n \in \mathbb{N}$

$$F(p_n) \preceq F(p_{n-1}) - \phi(p_{n-1}) \prec F(p_{n-1}). \tag{2}$$

Hence, F is non decreasing and so the sequence (p_n) is monotonically decreasing in $\mathbb{A}_+$. So, there exists $\theta \preceq t \in \mathbb{A}_+$ such that

$$p(x_n,x_{n+1}) \to t \text{ as } n \to \infty.$$

From (2) we obtain $\lim_{n\to\infty} F(p_n) = \theta$ that together with (ii) gives

$$\lim_{n\to\infty} p_n = \theta. \tag{3}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p)$. By Lemma 2.6 it is sufficient To prove that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p^s)$, we have proved $\lim_{n\to\infty} p_n = \theta$. Keeping in mind that $\theta \preceq p(x_n, x_n) \preceq p(x_n, x_{n+1})$, we get

$$\lim_{n\to\infty} p(x_n, x_n) = \theta. \tag{4}$$

Also, $\theta \preceq p(x_{n+1}, x_{n+1}) \preceq p(x_n, x_{n+1})$ this implies

$$\lim_{n\to\infty} p(x_{n+1}, x_{n+1}) = \theta. \tag{5}$$

Assume that $\{x_n\}$ is not a Cauchy sequence in $(X, \mathbb{A}, p^s)$. Then, exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$\|p^s(x_{m_k}, x_{n_k})\| > \varepsilon.$$

Now, corresponding to m_k , we can choose n_k such that it is the smallest integer with $n_k > m_k$ and satisfying above inequality. Hence, $\|p^s(x_{m_k}, x_{n_k-1})\| \leq \varepsilon$. So, we have

$$\begin{aligned} \varepsilon &\leq \|p^s(x_{m_k}, x_{n_k})\| \leq \|p^s(x_{m_k}, x_{n_k-1} + p^s(x_{n_k-1}, x_{n_k}) - p^s(x_{n_k-1}, x_{n_k-1})\| \\ &\leq \|p^s(x_{m_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{n_k})\| \leq \varepsilon + \|p^s(x_{n_k-1}, x_{n_k})\|. \end{aligned} \tag{6}$$

We know that

$$p^s(x_{n_k-1}, x_{n_k}) = 2p(x_{n_k-1}, x_{n_k}) - p^s(x_{n_k-1}, x_{n_k-1}) - p^s(x_{n_k}, x_{n_k}). \tag{7}$$

Using (3), (4), (5) and (7) we have

$$\varepsilon \preceq \lim_{k\to\infty} \|p^s(x_{n_k-1}, x_{n_k})\| < \varepsilon + \theta.$$

This implies

$$\lim_{k\to\infty} \|p^s(x_{m_k}, x_{n_k})\| = \varepsilon. \tag{8}$$

Again,

$$\begin{aligned} \|p^s(x_{n_k}, x_{m_k})\| &\leq \|p^s(x_{n_k}, x_{n_k-1}) + p^s(x_{n_k-1}, x_{m_k}) - p^s(x_{n_k-1}, x_{n_k-1})\| \\ &\leq \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k})\| \\ &\leq \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k-1}) \\ &+ p^s(x_{m_k-1}, x_{m_k}) - p^s(x_{m_k-1}, x_{m_k-1})\| \\ &\leq \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k-1})\| + \|p^s(x_{m_k-1}, x_{m_k})\|. \end{aligned} \tag{9}$$

Also,

$$\begin{aligned} \|p^s(x_{n_k-1}, x_{m_k-1})\| &\leq \|p^s(x_{n_k-1}, x_{n_k}) + p^s(x_{n_k}, x_{m_k-1}) - p^s(x_{n_k}, x_{m_k})\| \\ &\leq \|p^s(x_{n_k-1}, x_{n_k})\| + \|p^s(x_{n_k}, x_{m_k-1})\| \\ &\leq \|p^s(x_{n_k-1}, x_{nk})\| + \|p^s(x_{n_k}, x_{m_k}) \\ &+ p^s(x_{m_k}, x_{mk-1}) - p^s(x_{m_k}, x_{m_k})\| \\ &\leq \|p^s(x_{n_k-1}, x_{nk})\| + \|p^s(x_{n_k}, x_{m_k})\| + \|p^s(x_{m_k}, x_{m_k-1})\|. \end{aligned} \tag{10}$$

Letting $k \to \infty$ in (9) and (10) and using (4) and (8) we have

$$\lim_{k\to\infty} \|p^s(x_{n_k-1}, x_{m_k-1})\| = \varepsilon.$$

Thus,

$$\begin{aligned}\lim_{k\to\infty} \|p(x_{n_k-1}, x_{m_k-1})\| &= \frac{1}{2}\lim_{k\to\infty} \|2p^s(x_{n_k-1}, x_{m_k-1}) \\ &\quad - p^s(x_{n_k-1}, x_{n_k-1}) - p^s(x_{m_k-1}, x_{m_k-1})\| \\ &= \frac{1}{2}\lim_{k\to\infty} \|p^s(x_{n_k-1}, x_{m_k-1})\| = \frac{\varepsilon}{2}.\end{aligned}$$

Since $p(x_{n_k-1}, x_{m_k-1})$, $p(x_{n_k}, x_{m_k}) \in \mathbb{A}_+$ and

$$\lim_{k\to\infty} \|p(x_{n_k-1}, x_{m_k-1})\| = \lim_{k\to\infty} \|p(x_{n_k}, x_{m_k})\| = \frac{\varepsilon}{2}$$

there is exists $s \in \mathbb{A}_+$ with $\|s\| = \varepsilon$ such that

$$\lim_{k\to\infty} \|p(x_{n_k-1}, x_{m_k-1})\| = \lim_{k\to\infty} \|p(x_{n_k}, x_{m_k})\| = s \tag{11}$$

by (7) we have

$$F(s) = \lim_{k\to\infty} F(p(x_{n_k}, x_{m_k})) \preceq \lim_{k\to\infty} F(p(x_{n_{k-1}}, x_{m_{k-1}})).$$

Therefore, $F(s) \prec F(s)$. Thus, $F(s) = \theta$ and so $s = \theta$ which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p^s)$ and so $\{x_n\}$ is partially Cauchy in the complete C^*-algebra-valued partial metric space $(X, \mathbb{A}, p)$. Hence, there exist $z \in X$ such that $\lim_{n\to\infty} p(x_n, z) - p(x_n, x_n) = \theta$.

Using (4), we get $\lim_{n\to\infty} p(x_n, z) = \theta$ and thus $p(z, z) = \theta$

Now, we shall show that z is fixed point of T. Using (1), we get $\theta \preceq F(p(Tz, Tz)) \prec F(p(z, z)) = F(\theta) = \theta$. Thus, $F(p(Tz, Tz)) = \theta$ which implies $p(Tv, Tv) = \theta$. On the other hand, $F(p(x_n, Tz)) \prec F(p(x_{n-1}, z))$.

Letting $n \to \infty$ and using the concept of continuity of the function of T. We have $p(z, Tz) = \theta$. Hence, by Definition 2.2, we have $p(z, z) = p(Tz, Tz) = p(z, Tz) = \theta$, then $Tz = z$, which completes the proof. □

Example 3.5. Considering all cases in Example 3.3, we conclude that inequality (1) remains valid for F and T constructed as above and consequently by an application of Theorem 3.3, T has a unique fixed point. It is seen that 0 is the unique fixed point of T.

Theorem 3.6. *Let $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued partial metric space.*

Let $T : X \to X$ be a (ϕ, F) of type (II), i.e, there exist F and ϕ two $*-$homomorphisms such that for any $x, y \in X$ we have

$$p(Tx, Ty) \succeq \theta \Rightarrow F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(M(x, y)),$$

where $M(x, y) = a_1 p(x, y) + a_2[p(Tx, y) + p(Ty, x)] + a_3[p(Tx, x) + p(Ty, y)]$, with $a_1, a_2, a_3 \geq 0$,$a_1 + 2a_2 + 2a_3 \leq 1$. Then, T has a fixed point.

Proof. Let $x_0 \in X$ and define $x_1 = Tx_0, x_2 = Tx_1, ..., x_n = Tx_{n-1}$. We have

$$\begin{aligned} F(p(x_{n+2}, x_{n+1})) &= F(p(Tx_{n+1}, Tx_n)) \preceq F(M(x_{n+1}, x_n)) + \phi(p(x_{n+1}, x_n)) \\ &= F(a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ &\quad + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]) - \phi(p(x_{n+1}, x_n)). \end{aligned}$$

Then, we have

$$\begin{aligned} F(p(x_{n+2}, x_{n+1})) &\preceq F(a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ &\quad + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]). \end{aligned}$$

Using the strongly monotone property of F, we have

$$\begin{aligned} p(x_{n+2}, x_{n+1}) &\preceq a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ &\quad + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]. \end{aligned}$$

That is

$$(1 - a_2 - a_3)p(Tx_{n+1}, Tx_n) \preceq (a_1 + a_2 + a_3)p(x_{n+1}, x_n).$$

Therefore,

$$p(x_{n+2}, x_{n+1}) \preceq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} p(x_{n+1}, x_n).$$

Which implies that

$$p(x_{n+2}, x_{n+1}) \preceq p(x_{n+1}, x_n).$$

Since

$$\frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1.$$

Therefore, $\{p(x_{n+1}, x_n)\}$ is monotone decreasing sequence. There exists, $u \in \mathbb{A}_+$ such that $d(x_{n+1}, x_n) \to u$ as $n \to \infty$. Taking $n \to \infty$ in

$$\begin{aligned} F(p(x_{n+2}, x_{n+1})) &\preceq F(a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ &\quad + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]). \end{aligned}$$

Using the continuities of F and ϕ, we have

$$F(u) \preceq F((a_1 + 2a_2 + 2a_3)u) - \phi(u)$$

which implies that $F(u) \preceq F(u) - \phi(u)$ since $a_1 + 2a_2 + 2a_3 \leq 1$ and F is strongly monotonic increasing wich is a contradiction unless $u = \theta$. Hence,

$$p(x_{n+1}, x_n) \to \theta \text{ as } n \to \infty. \tag{12}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence.

If $\{x_n\}$ is not a Cauchy sequence then there exists $c \in \mathbb{A}$ such that $\forall n_0 \in \mathbb{N}$, $\exists n, m \in \mathbb{N}$ with $n > m \geq n_0, F(c) \preceq p(x_n, x_m)$. Therefore, there exists sequences $\{m_k\}$ and $\{n_k\}$ in $\mathbb{N}$ such that for all positive integers k, $n_k > m_k > k$ and $p(x_{n_{(k)}}, x_{m_{(k)}}) \succeq \phi(c)$ and $p(x_{n_{(k)}-1}, x_{m_{(k)}}) \preceq \phi(c)$ then

$$\phi(c) \preceq p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq p(x_{n_{(k)}}, x_{n_{(k)}-1}) + p(x_{n_{(k)}-1}, x_{m_{(k)}})$$

that is $\phi(c) \preceq p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq p(x_{n_{(k)}}, x_{n_{(k)}-1}) + \phi(c)$ letting $k \to \infty$, we have

$$\lim_{k\to\infty} p(x_{n_{(k)}}, x_{m_{(k)}}) = \phi(c) \tag{13}$$

again

$$p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}}, x_{n_{(k)}+1}) + p(x_{n_{(k)}+1}, x_{m_{(k)}}) - p(x_{n_{(k)}+1}, x_{n_{(k)}+1})]$$

and

$$p(x_{n_{(k)}+1}, x_{m_{(k)}+1}) \preceq [p(x_{n_{(k)}+1}, x_{n_{(k)}}) + p(x_{n_{(k)}}, x_{m_{(k)}+1}) - p(x_{n_{(k)}}, x_{n_{(k)}})]$$

letting $k \to \infty$ in above inequalities, we have

$$\lim_{k\to\infty} p(x_{n_{(k)+1}}, x_{m_{(k)+1}}) = \phi(c). \tag{14}$$

Again

$$p(x_{n_{(k)}}, x_{m_{(k)}+1}) \preceq [p(x_{n_{(k)}}, x_{m_{(k)}}) + p(x_{m_{(k)}}, x_{m_{(k)}+1})]$$

and

$$p(x_{n_{(k)}+1}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}+1}, x_{n_{(k)}}) + p(x_{n_{(k)}}, x_{m_{(k)}}) - p(x_{n_{(k)}}, x_{n_{(k)}})].$$

Further,

$$p(x_{n_{(k)}+1}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}+1}, x_{n_{(k)}}) + p(x_{n_{(k)}}, x_{m_{(k)}})]$$

and

$$p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}}, x_{n_{(k)}+1}) + p(x_{n_{(k)}+1}, x_{m_{(k)}})].$$

Letting $k \to \infty$ in the above four inequalities we have

$$\lim_{k\to\infty} p(x_{n_{(k)}}, x_{m_{(k)}+1}) = \phi(c), \tag{15}$$

$$\lim_{k\to\infty} p(x_{n_{(k)}+1}, x_{m_{(k)}}) = \phi(c). \tag{16}$$

Using (12), (13), (15) and (16) we have

$$\begin{aligned}\lim_{k\to\infty} M(x_{n_{(k)}}, x_{m_{(k)}}) = \lim_{k\to\infty} & a_1 p(x_{n_{(k)}}, x_{m_{(k)}}) + a_2[p(x_{n_{(k)}}, x_{m_{(k)}}) \\ & + p(x_{m_{(k)}}, x_{m_{(k)}+1})] + a_3[p(x_{n_{(k)}}, x_{m_{(k)}+1}) \\ & + p(x_{m_{(k)}}, x_{n_{(k)}+1})] = (a_1 + 2a_2)\phi(c). \end{aligned} \tag{17}$$

Clearly $x_{m_k} \preceq x_{n_k}$. Putting $x = x_{n_{(k)}}$,$y = x_{m_{(k)}}$

$$F(p(x_{n_{(k)}+1}, x_{m_{(k)}+1})) = F(p(Tx_{n_{(k)}}, Tx_{m_{(k)}})) \preceq F(M(x_{n_{(k)}}, x_{m_{(k)}})) - \phi(x_{n_{(k)}}, x_{m_{(k)}}).$$

Letting $k \to \infty$ in the above inequality using (13), (14) and (17) and the continuities of F and ϕ, we have $F(\phi(c)) \preceq F((a_1 + 2a_2)\phi(c)) - \phi(\phi(c))$ that is $F(\phi(c)) \preceq F(\phi(c)) - \phi(\phi(c))$, (since $(a_1 + 2a_2) < 1$) and F is strongly monotonic increasing. Which a contradiction by virtue of a proprety of ϕ. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Since T is continuous and $Tx_n \to Tz$ as $n \to \infty$ that is $\lim_{n\to\infty} x_{n+1} = Tz$, that is $z = Tz$. Hence, z is a fixed point of T. □

Example 3.7. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{C}$ with a norm $\| z \| = | z |$ be a C^*- algebra.

We define $\mathbb{C}^+ = \{z = (x, y) \in \mathbb{C}; x = \mathcal{R}\mathrm{e}\,(z) \geq 0, y = \mathcal{I}\mathrm{m}\,(z) \geq 0\}$.

The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ is the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.

Let $p : X \times X \to \mathbb{C}$. Suppose that $p(x, y) = (| x - y |, | x - y |)$ for $x, y \in X$. Then, $(X, \mathbb{C}, p)$ is a C^*- algebra valued metric space with the required properties of Theorem 3.6.

Let $F, \phi : \mathbb{C}^+ \to \mathbb{C}^+$ such that they can defined as follows: for $t = (x, y) \in \mathbb{C}^+$,

$$F(t) = \begin{cases} (x, y), & \text{if } x \leq 1, y \leq 1, \\ (x^2, y), & \text{if } x > 1, y \leq 1, \\ (x, y^2), & \text{if } x \leq 1, y > 1, \\ (x^2, y^2), & \text{if } x > 1,\ y > 1 \end{cases}$$

and for $s = (s_1, s_2) \in \mathbb{C}^+$ with $v = \min\{s_1, s_2\}$,

$$\phi = \begin{cases} \left(\dfrac{v^2}{2}, \dfrac{v^2}{2}\right), & \text{if } v \leq 1, \\ \left(\dfrac{1}{2}, \dfrac{1}{2}\right), & \text{if } v > 1. \end{cases}$$

Then, F and ϕ have the propreties mentioned in Definitions 2.8 and 2.9. Let $T : X \to X$ be defined as follows: $T(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{16}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$ Then, T has the required properties mentioned in Theorem 3.6.

Let $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{8}$ and $a_3 = \frac{1}{8}$. It can be verified that $F(p(Tx, Ty)) \preceq F(M(x, y)) - \phi(p(x, y))$, for all $x, y \in X$ with $y \preceq x$ the conditions of Theorem 3.6 are satisfied. Here, it is seen that 0 is a fixed point of T.

Theorem 3.8. *Let $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued partial metric space. Let $T : X \to X$ be a (ϕ, F)- Kannan-type C^*- valued partial contraction.Then T has a unique fixed point.*

Proof. Since T is a (ϕ, F)- Kannan-type C^*- valued partial contraction, then exist F and ϕ such that

$$F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(\frac{p(x, Tx) + p(y, Ty)}{2}) \preceq F(M(x, y)),$$

where $M(x, y) = a_1 p(x, y) + a_2[p(Tx, y) + p(Ty, x)] + a_3[p(Tx, x) + p(Ty, y)]$ with $a_1 = 0$, $a_2 = 0$ and $a_3 = \frac{1}{2}$. As in the proof of Theorem 3.6, T has a fixed point. □

Theorem 3.9. *Let $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued partial metric space. Let $T : X \to X$ be a (ϕ, F)- Reich-type C^*- valued partial contraction. Then T has a unique fixed point.*

Proof. By taking $a_1 = \frac{1}{3}$, $a_2 = 0$ and $a_3 = \frac{1}{3}$, we have

$$F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(M(x, y)) = F(\frac{p(x, y) + p(x, Tx) + p(y, Ty)}{3}).$$

As in the proof of Theorem 3.6 T has a fixed point. □

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The structure of (θ_1, θ_2)-isoclinism classes of groups

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Abstract. In 1940, Philip Hall introduced the concept of isoclinism among all groups, and it is generalized to a more general notion called isologism. This concept is isoclinism with respect to a given variety of groups. The equivalence relation of isologism partitions the class of all groups into families.

In this article, we introduce a kind of isoclinism with respect to θ-centre, $Z^{\theta}(G)$, and right θ-commutator subgroup $K^{\theta}(G)$, for some automorphism θ of the group G, and we investigate some of its properties.

Keywords: right and left θ-commutator, central automorphism, absolute centre, θ-centre, θ-commutator subgroup.

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1. Introduction

One of the most classical notions playing a fundamental role in classifying groups is the notion of isomorphism among all groups. However, in many cases this notion is too strong. For instance, in the case of finite groups one would like to consider abelian groups being classified as a one family.

P. Hall in 1940 introduced the concept of isoclinism [3]. This is an equivalence relation on the class of all groups, which is weaker than isomorphism and such that all abelian groups fall into one equivalence class, namely they are equivalent to the trivial group. Roughly speaking two groups are isoclinic if and only if there exists an isomorphism between their central quotients, which induces an isomorphism between their commutator subgroups.

In [2], the second and third authors introduced and studied the concept of *right* and *left* α*-commutator*, as follows:

Definition 1.1. *For arbitrary elements* x *and* y *in a given group* G *and* $\alpha \in \mathrm{Aut}(G)$*, we say* x *and* y *commute under the automorphism* α *whenever* $yx = xy^\alpha$ *or* $y^{\phi_x} = y^\alpha$*, where* ϕ_x *is the inner automorphism induced by* x*.*

Moreover, $[x, y]_\alpha = x^{-1}y^{-1}xy^\alpha$ *is called right* α*-commutator of* x *and* y*. Also,* ${}_\alpha[x, y] = (x^{-1})^\alpha y^{-1}xy$ *is called left* α*-commutator of* x *and* y*.*

For $n \geqslant 3$, we may define inductively right and left α-commutator of weight n as follows:

$$[x_1, x_2, \cdots, x_n]_\alpha = [[x_1, x_2, \cdots, x_{n-1}]_\alpha, x_n]_\alpha,$$

$${}_\alpha[x_1, x_2, \cdots, x_n] = {}_\alpha[{}_\alpha[x_1, x_2, \cdots, x_{n-1}], x_n],$$

for all $x_i \in G$ and $1 \leqslant i \leqslant n$. It is clear that, if α is the identity automorphism of G or x_i's are in $C_G(\alpha)$ then we have ordinary commutator $[x_1, x_2, \cdots, x_n]$ of weight n, where

$$C_G(\alpha) = \{x \in G \mid [x, \alpha] = x^{-1}x^\alpha = x^{-1}\alpha(x) = 1\},$$

is the centralizer of α in G.

For a given group G and automorphisms α and β in $\mathrm{Aut}(G)$ we consider, $\alpha^\beta = \beta^{-1}\alpha\beta$. The following lemma is very useful in our further investigations.

Lemma 1.1. *Let* x, y *and* z *be elements of a group* G *and* $\alpha, \beta \in \mathrm{Aut}(G)$*. Then the following identities hold:*

(i) $[x, y]_\alpha = [x, y][y, \alpha]$;

(ii) $[x, x]_\alpha = [x, \alpha]$;

(iii) $([x, y]_\alpha)^\alpha = [x^\alpha, y^\alpha]_\alpha$;

(iv) $[x, y^{-1}]_\alpha = [x, y]_\alpha^{-(y^\alpha)^{-1}}$;

(v) $([x, y]_{\alpha^\beta})^\beta = [x^\beta, y^\beta]_\alpha$;

(*vi*) $[xy, z]_\alpha = ([x, z]_\alpha)^y [y, z^\alpha]$;

(*vii*) $[x, yz]_\alpha = [x, z]_\alpha ([x, y]_\alpha)^{z^\alpha}$;

(*viii*) $([[x, y^{-1}]_\alpha, z]_\alpha)^{y^\alpha} = [x, y^{-1}, z]^y [z^y, \alpha]$.

Proof. All parts follow using the definition of right α-commutator and the above notation. □

One can easily see that $[x, y]_\alpha^{-1} = {}_\alpha[y, x]$, hence we may state similar relations, as the above lemma, for left α-commutator. Here we work with right α-commutators in the rest of article.

Remark 1. For an automorphism α of a group G, the action $\psi : G \times G \to G$ given by $\psi(x, y) = y^{-1}xy^\alpha$, partitions the group G into α-conjugacy classes, which we denote it by x_α^G, i.e.

$$x_\alpha^G = \{y^{-1}xy^\alpha \mid y \in G\}.$$

Note that the number of α-conjugacy classes is equal with the number of ordinary conjugacy classes, which are invariant under α and it is also equal to the number of irreducible characters which are invariant under α (see [7, 9] for more details).

Now, we recall that the following subgroup is called *α-centre* of the group G

$$Z^\alpha(G) = \bigcap_{x \in G} C_G^\alpha(x) = \{y \in G \mid [x, y]_\alpha = 1, \forall x \in G\},$$

where $C_G^\alpha(x) = \{y \in G \mid [x, y]_\alpha = 1\}$ and $|x_\alpha^G| = [G : C_G^\alpha(x)]$ (see [1, 9] for more information). One can easily check that $Z^\alpha(G) = Z(G) \cap C_G(\alpha)$ and so $Z^\alpha(G) \trianglelefteq G$. Also, $L(G) = \bigcap_{\alpha \in \mathrm{Aut}(G)} Z^\alpha(G)$, and hence

$$L(G) \subseteq Z^\alpha(G) \subsetneqq Z(G),$$

as $[x, y]_\alpha = [x, y][y, \alpha] = 1$, for all $x \in G$ and $y \in Z^\alpha(G)$, while $[y, x]_\alpha = [y, x][x, \alpha] \neq 1$.

Now, one may define *α-commutator* subgroup of G as follows

$$K^\alpha(G) = \langle [x, y]_\alpha \mid x, y \in G \rangle.$$

Clearly, Lemma 1.2 (i) and (ii) imply that $G' \subseteq K^\alpha(G) \subseteq K(G)$, where $K(G)$ is the autocommutator subgroup of G (see [4]). Note that, Lemma 1.2 (iii) implies that $K^\alpha(G)$ is an α-invariant subgroup of G.

Let α be an automorphism of the group G and for any $x \in G$, then α is called *class preserving* if $x^\alpha \in x^G$. Clearly, if α is class preserving automorphism of a group G then $x^\alpha = x^g$ for some $g \in G$, and hence $[g, x]_\alpha = 1$. This topic has been studied by many authors (see [5, 6, 10], for more details).

2. Main results

Clearly α-commutator subgroup $K^{\alpha}(G)$ of an abelian group G is always normal in G, for any automorphism $\alpha \in \mathrm{Aut}(G)$. In the following, we show that $K^{\alpha}(G)$ is a normal subgroup in a non abelian group G, for any automorphism α of G.

One may define the action of a group G on $\mathrm{Aut}(G)$ given by $\alpha^g = \alpha^{\varphi_g} = \varphi_{g^{-1}} \circ \alpha \circ \varphi_g$ and the action of $\mathrm{Aut}(G)$ on G given by $g^{\alpha} = \alpha(g)$, for all $g \in G$, $\alpha \in \mathrm{Aut}(G)$ and $\varphi_g \in \mathrm{Inn}(G)$ (see also [8]).

Theorem 2.1. *Let α be any automorphism of a given group G, then $K^{\alpha}(G)$ is always a normal subgroup of G.*

Proof. Take α to be any automorphism of the group G and for any $x, y, g \in G$, Lemma 1.2 (i) implies that

$$\begin{aligned} [x,y]_{\alpha}^g = [x,y]^g[y,\alpha]^g &= [x^g, y^g][y,\alpha]^g \\ &= [x^g, y^g]g^{-1}y^{-1}\alpha(y)g \\ &= [x^g, y^g][g,y]_{\alpha}[\alpha(y), g] \in K^{\alpha}(G). \end{aligned}$$

Hence, $K^{\alpha}(G) \trianglelefteq G$. □

Here the notion of (θ_1, θ_2)-isoclinism between two groups is introduced and we study some of its properties.

Definition 2.1. *Let G_1 and G_2 be two groups, θ_1 and θ_2 be suitable automorphisms of G_1 and G_2, respectively, such that there exist $\alpha : \frac{G_1}{Z^{\theta_1}(G_1)} \to \frac{G_2}{Z^{\theta_2}(G_2)}$ and $\beta : K^{\theta_1}(G_1) \to K^{\theta_2}(G_2)$ so that the following diagram is commutative*

$$\begin{tikzcd} \frac{G_1}{Z^{\theta_1}(G_1)} \times \frac{G_1}{Z^{\theta_1}(G_1)} \arrow[r, "\alpha\times\alpha"] & \frac{G_2}{Z^{\theta_2}(G_2)} \times \frac{G_2}{Z^{\theta_2}(G_2)} \\ (g_1Z^{\theta_1}(G_1), g_2Z^{\theta_1}(G_1)) \arrow[r, mapsto] \arrow[d, "\rho"'] & (g_1'Z^{\theta_2}(G_2), g_2'Z^{\theta_2}(G_2)) \arrow[d, "\psi"] \\ {[g_1, g_2]_{\theta_1}} \arrow[r] & {[g_1', g_2']_{\theta_2}} \\ K^{\theta_1}(G_1) \arrow[r, "\beta"] & K^{\theta_2}(G_2) \end{tikzcd}$$

where $\alpha(g_iZ^{\theta_1}(G_1)) = g_i'Z^{\theta_2}(G_2)$ and $g_i' \in \alpha(g_i)Z^{\theta_2}(G_2)$ for every $g_i \in G_1$ and $g_i' \in G_2$ $(i = 1, 2)$. Moreover, $\beta([g_1, g_2]_{\theta_1}) = [g_1', g_2']_{\theta_2}$, i.e. the commutative diagram is compatible.

Then the pair (α, β) is called (θ_1, θ_2)-isoclinism from G_1 to G_2 and denoted by $G_1 \overset{(\theta_1,\theta_2)}{\sim} G_2$. In this case, G_1 and G_2 are called (θ_1, θ_2)-isoclinic.

Observe that the above notion generalizes the concept of isoclinism (see [3]). In fact, if θ_1 and θ_2 are identities, then the above definition is the concept of ordinary isoclinism of groups.

Example 2.2. (i) There are no automorphisms θ_1 and θ_2 of the groups $\mathbb{Z}_4$ and $\mathbb{Z}_6$, respectively, such that $\mathbb{Z}_4 \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_6$. As, for any automorphisms θ_1 and θ_2 of $\mathbb{Z}_4$ and $\mathbb{Z}_6$, we have $|\frac{\mathbb{Z}_4}{Z^{\theta_1}(\mathbb{Z}_4)}| = 2$ and $|\frac{\mathbb{Z}_6}{Z^{\theta_2}(\mathbb{Z}_6)}| = 3$.

(ii) Consider the cyclic groups $\mathbb{Z}_4(x)$ and $\mathbb{Z}_8(y)$ of orders 4 and 8 with generators x and y, and take the automorphisms $\theta_1 : x \mapsto x^3$ and $\theta_2 : y \mapsto y^5$. Then one can easily check that $Z^{\theta_1}(\mathbb{Z}_4(x)) = \{1, x^2\}$, $K^{\theta_1}(\mathbb{Z}_4(x)) = \{1, x^2\}$. Also, $Z^{\theta_2}(\mathbb{Z}_8(y)) = \{1, y^2, y^4, y^6\}$ and $K^{\theta_2}(\mathbb{Z}_8(y)) = \{1, y^4\}$. Now, it is easy to verify that $\frac{\mathbb{Z}_4(x)}{Z^{\theta_1}(\mathbb{Z}_4(x))} \cong \frac{\mathbb{Z}_8(y)}{Z^{\theta_2}(\mathbb{Z}_8(y))}$ and $K^{\theta_1}(\mathbb{Z}_4(x)) \cong K^{\theta_2}(\mathbb{Z}_8(y))$, hence $\mathbb{Z}_4(x) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_8(y)$.

(iii) Assume $D_8 = \langle x, y : x^4 = y^2 = 1, x^y = x^{-1}\rangle$ and $Q_8 = \langle x, y : x^4 = 1, x^2 = y^2, x^y = x^{-1}\rangle$ are Dihedral and Quaternion groups of orders 8. Also, take the automorphisms θ_1 and θ_2 both given by: $x \mapsto x^3, y \mapsto x^2y$ of D_8 and Q_8, respectively. One can calculate that $Z^{\theta_1}(D_8) \cong Z^{\theta_2}(Q_8) = \{1, x^2\}$ and $K^{\theta_1}(D_8) \cong K^{\theta_2}(Q_8) = \{1, x^2\}$. Hence, $D_8 \overset{(\theta_1,\theta_2)}{\sim} Q_8$.

Now, the question arises that; "In what cases, there exist some suitable automorphisms θ_1 and θ_2 in arbitrary finite cyclic groups, which force them to be (θ_1, θ_2)-isoclinic?"

In the following, we give a complete answer to the above question, for finite cyclic groups.

Remark 2. (i) Let $\mathbb{Z}_m(x_1)$ and $\mathbb{Z}_n(x_2)$ be cyclic groups with a common divisor p^r of m and n, where p is a prime number and $r \geqslant 2$.

Assume $m{=}p^r p_2^{r_2} \cdots p_s^{r_s}$ and $n{=}p^r q_2^{r'_2} \cdots q_t^{r'_t}$. Clearly $\theta_1 : x_1 \mapsto x_1^{p^{r-1}p_2^{r_2}\cdots p_s^{r_s}+1}$ and $\theta_2 : x_2 \mapsto x_2^{p^{r-1}q_2^{r'_2}\cdots q_t^{r'_t}+1}$ are automorphisms of cyclic groups of orders m and n, respectively.

As m and $\frac{m}{p} + 1$ are co-prime, then $K^{\theta_1}(\mathbb{Z}_m) = \langle x_1^{\frac{m}{p}}\rangle$ and

$$Z^{\theta_1}(\mathbb{Z}_m) = \{x_1^p, x_1^{2p}, \cdots, x_1^{\frac{m}{p}p} = 1\}.$$

The same argument implies that $|K^{\theta_1}(\mathbb{Z}_m)|{=}|\frac{\mathbb{Z}_m}{Z^{\theta_1}(\mathbb{Z}_m)}|{=}|K^{\theta_2}(\mathbb{Z}_n)| = |\frac{\mathbb{Z}_n}{Z^{\theta_2}(\mathbb{Z}_n)}| = p$, and hence $\mathbb{Z}_m(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_n(x_2)$. Such as $\mathbb{Z}_{12}$ and $\mathbb{Z}_{20}$.

(ii) If the orders of cyclic groups are with different prime decomposition factors, then they can not be (θ_1, θ_2)-isoclinic, for any automorphisms θ_1 and θ_2. Such as $\mathbb{Z}_6$ and $\mathbb{Z}_{35}$.

(iii) Consider the cyclic groups $\mathbb{Z}_{m_1}(x_1)$ and $\mathbb{Z}_{m_2}(x_2)$ with $(m_1, m_2) = p$. Clearly, if $(\frac{k_i m_i}{p}, m_i) = 1$, for $i = 1, 2$ and $1 \leq k_i < p$, then

$$\theta : x_i \mapsto x_i^{\frac{k_i m_i}{p}+1},$$

is an automorphism of the cyclic group $\mathbb{Z}_{m_i}(x_i)$. Now,

$$K^{\theta_i}(\mathbb{Z}_{m_i}(x_i)) = \langle [x_i, \theta_i] = x_i^{-1}x^{\theta_i} \rangle = \langle x^{\frac{k_i m_i}{p}} \rangle,$$

which is a cyclic group of order p, for $i = 1, 2$; i.e. $K^{\theta_1}(\mathbb{Z}_{m_1}(x_1)) \cong K^{\theta_2}(\mathbb{Z}_{m_2}(x_2))$.

On the other hand, we have

$$Z^{\theta_i}(\mathbb{Z}_{m_i}(x_i)) = \{x_i^r \mid [x_i, x_i^r]_{\theta_i} = [x_i^r, \theta_i] = x_i^{\frac{rk_i m_i}{p}} = 1\}.$$

Hence, $p|r$ and $|Z^{\theta_i}(\mathbb{Z}_{m_i}(x_i))| = \frac{m_i}{p}$, which implies that

$$\frac{\mathbb{Z}_{m_1}(x_1)}{Z^{\theta_1}(\mathbb{Z}_{m_1}(x_1))} \cong \frac{\mathbb{Z}_{m_2}(x_2)}{Z^{\theta_2}(\mathbb{Z}_{m_2}(x_2))}$$

and so $\mathbb{Z}_{m_1}(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_{m_2}(x_2)$.

Using the technique of Remark 2 (iii), we have the following examples.

Example 2.3. (i) Consider $\mathbb{Z}_{15}(x_1)$ and $\mathbb{Z}_{21}(x_2)$. One notes that $(\frac{15}{3}+1, 15) \neq 1$, while $(\frac{30}{3}+1, 15) = 1$. Also, $(\frac{21}{3}+1, 21) = 1$. Hence, $\theta_1 : x_1 \mapsto x_1^{11}$ and $\theta_2 : x_2 \mapsto x_2^8$ are automorphisms of $\mathbb{Z}_{15}(x_1)$ and $\mathbb{Z}_{21}(x_2)$, respectively. These automorphisms guaranty that $\mathbb{Z}_{15}(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_{21}(x_2)$.

(ii) Consider $\mathbb{Z}_6(x_1)$ and $\mathbb{Z}_{15}(x_2)$. we observe that $(\frac{6}{3}+1, 6) \neq 1$ and $(\frac{15}{3}+1, 15) \neq 1$, while $(\frac{12}{3}+1, 6) = 1$ and $(\frac{30}{3}+1, 15) = 1$. Hence, the automorphisms $\theta_1 : x_1 \mapsto x_1^5$ and $\theta_2 : x_2 \mapsto x_2^{11}$ will do the job and so $\mathbb{Z}_6(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_{15}(x_2)$.

(iii) $\mathbb{Z}_6 \overset{(\theta_1,\theta_2)}{\nsim} \mathbb{Z}_{10}$, since there are no suitable automorphisms, as the above.

In case of 1-isoclinism, P. Hall [3] showed that in every family there exists a group S with the property that $Z(S) \subseteq \gamma_2(S)$. Such a group is called *stemgroup*. In the case of finite groups, the stemgroups in a given family are characterized by the fact that they are just the groups of smallest order in that family. They play an essential role in classification problem.

Clearly, (θ_i, θ_j)-isoclinism forms an equivalence relation on the pair of groups. Hence, such relation partitions the group into equivalence classes, or family of (θ_i, θ_j)-isoclinism of groups.

Here, we introduce α-stemgroup in the case of (θ_i, θ_j)-isoclinism of groups.

Definition 2.2. *Let $\mathcal{C}$ be a family of (θ_i, θ_j)-isoclinism of groups. If there exists a group S with the property that $G_r \overset{(\theta_i,\theta_j)}{\sim} S$ and $Z^\alpha(S) \subseteq K^\alpha(S)$, where $G_r \in \mathcal{C}$ and α is an automorphism of the group S. Then such a group S is said to be α-stem group. In finite case, the α-stem group S has the least possible order among all other groups in the family.*

Example 2.4. Consider a famiy of finite cyclic groups, which their orders have a common prime divisor p^r, where $r \geqslant 2$. Then it is clear that $\alpha : x \mapsto x^{p+1}$ is an automorphism of $\mathbb{Z}_{p^2}(x)$ and $K^\alpha(\mathbb{Z}_{p^2}(x)) = Z^\alpha(\mathbb{Z}_{p^2}(x)) = \langle x^p \rangle$. Therefore Remark 2 and Definition 2.5 imply that $\mathbb{Z}_{p^2}(x)$ is the α-stem group.

The above example shows that in a family of finite cyclic groups, for which p^r, $(r \geqslant 2)$, is a common divisor of their orders, the cyclic group $\mathbb{Z}_{p^2}$ is α-stem group with smallest order in such family of groups.

Our final result gives a useful criterion for two groups to be (θ_1, θ_2)-isoclinic.

Proposition 2.1. *Let $A \leq Z^{\theta_1}(G)$ and $B \leq Z^{\theta_2}(H)$. Also, assume $\alpha : G/A \to H/B$ and $\beta : K^{\theta_1}(G) \to K^{\theta_2}(H)$ are isomorphisms so that $\alpha(gZ^{\theta_1}(G)) = hZ^{\theta_2}(H)$ and $\beta([g, g']_{\theta_1}) = [h, h']_{\theta_2}$, for all $g, g' \in G$ and $h, h' \in H$. Then G and H are (θ_1, θ_2)-isoclinic.*

Proof. We must show that α induces an isomorphism from $G/Z^{\theta_1}(G)$ onto $H/Z^{\theta_2}(H)$.

Since $G/Z^{\theta_1}(G) \simeq (G/A)/(Z^{\theta_1}(G)/A)$ and $H/Z^{\theta_1}(H) \simeq (H/B)/(Z^{\theta_2}(H)/B)$, it is sufficient to show that $\alpha(Z^{\theta_1}(G)/A) = Z^{\theta_2}(H)/B$. So for any $g \in Z^{\theta_1}(G)$, we have $[g', g]_{\theta_1} = 1$ for all g' in G. Then there exists h in H such that $[h', h]_{\theta_2} = 1$ for all $h' \in H$, as β is an isomorphism. Thus $h \in Z^{\theta_2}(H)$ and $\alpha(Z^{\theta_1}(G)/A) \leq Z^{\theta_2}(H)/B$.

On the other hand, if $h_0 \in Z^{\theta_2}(H)$ is an arbitrary element, then there exists an element $g_0 \in G$ such that $\alpha(g_0 A) = h_0 B$, as α is surjective. Now, $\beta([g, g_0]_{\theta_1}) = [h, h_0]_{\theta_2}$ and hence $g_0 \in Z^{\theta_1}(G)$, as β is isomorphism. Therefore $\alpha(Z^{\theta_1}(G)/A) \geq Z^{\theta_2}(H)/B$, which completes the proof. □

The following corollary is obtained by replacing $G = H_1$, $H = H_2$, $A = Z^{\theta_1}(G_1)$ and $B = Z^{\theta_2}(G_2)$ in the above proposition.

Corollary 2.1. *Let (α, β) be (θ_1, θ_2)-isoclinism between two groups G_1 and G_2 and H_i be a characteristic subgroup of G_i for $i = 1, 2$. If $Z^{\theta_1}(G_1) \leq H_1 \leq G_1$ and $\alpha(H_1/Z^{\theta_1}(G_1)) = H_2/Z^{\theta_2}(G_2)$, then H_1 and H_2 are also (θ_1, θ_2)-isoclinic, where $Z^{\theta_2}(G_2) \leq H_2 \leq G_2$.*

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Hyper BCK-hashing algorithm: employing encoding system based on logical algebra in enhancing the secure hash algorithms

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Abstract. Cryptographic algorithms perform essential functions to generate data from digital form to comprehensible patterns such that the permitted user is the only one who can understand the message. In this study, we propose Hyper BCK-Hashing (HBCK-HASHING) Algorithm based on a hyper BCK-valued function and hash function (SHA-2). It targets to enhance the Secure Hash algorithms (SHA-2) with an algorithm of hyper BCK-valued function which based on the redundant encoding to maximize the security level of the cryptographic process of n-ary block codes (U) through maximize the quantity of information with the fewest number of visible characteristics. The redundant encoding based on making a unique - identified HBCK-algebra (H) for n-ary block codes (U) with applying the hyper BCK-valued function on (H) to generate n-ary block codes(U_H). In addition, we perform the computational Secure Hash algorithms on (U_H) to map the size of n-ary block codes (U_H) into a fixed size. The proposed algorithm was evaluated by using the avalanche effect parameter in comparison with the Secure Hash algorithm (512 and 256). Experimental outcomes indicate that the HBCK-HASHING algorithm shows a significant-high.

Keywords: Hyper BCK-algebras, N-ary block codes, secure Hash algorithm(SHA-2), avalanche effect.

1. Introduction

1.1 Logical algebras and its applications on block codes and Hyper structure approach

Logic algebra indicates a conveying for characteristics and conditions from logic to algebra. Logic algebra fulfills methods for the main assignment of artificial intelligence in elucidating the basics of keeping a computer simulates a human in dealing with data. There are numerous attempts to study emerging characteristics of logic algebras like [1, 2, 3, 4]. Recently, there are abundant research papers studied the relationship between logical algebras and block codes. Block codes mirror an essential class of error-correcting codes Which considered effective to encode data in blocks. Error-control codes allow increasing the security of data transmission over noisy communication channels. Luis Hernandez Encinas [5] introduced the notion of R_0- valued function with related features and examined the generating of binary block codes by R_0- valued function. Cristina Flaut [6] examined the relationships between binary block-codes and Hilbert algebras. Also, she suggested other characteristics associated with Hilbert algebras. Samy M. Mostafa et al. [7, 8, 9] offered an efficient method to produce a KU-algebra from binary block code and introduced the notion of KU- valued function with producing binary code from KU- function. Also, they constructed codes by soft sets PU-valued functions. A.B. Saeid et al. [10] presented an algorithm to generate BCK-algebra from n-ary block code. Numerous applications of Hyper structures are employed in pure and applied sciences. Hyper structures approach adapted to the logical algebraic structure BCK-algebra and consisted the concept of Hyper BCK-algebra.

Y. B. Jun et al. [11] clarified that the generalization of BCK-algebra is Hyper BCK-algebra. Authors defined Hyper BCK-algebra and studied some relevant properties. Atamewouetsafacksurdive et al. [12] stated the concept of hyper BCK-function with some properties related, and generated binary codes by the hyper BCK-function through an algorithm allows constructing a hyper BCK-algebra from binary block code.

In the following, we introduce some results associated with hyper BCK-algebras and algebraic structures applications in coding theory that will be applied effectively in the study.

1.2 Secure Hash Algorithms (SHA-512 and SHA-256)

National Institute of Standards and Technology (NIST) announced Secure Hash Algorithms which indicates SHA. It was developed in 1993 as a federal information processing standard. [14]. After discovering a few weaknesses, an insecure hash algorithm called SHA-0 was withdrawn. SHA-1 procedure has a hash value of 20 bytes (160 bits). SHA-2 is a more powerful version than its ancestors (SHA-0, SHA-1). SHA-256 is a member of the SHA-2 group, yielding alike functionality with more security like SHA-384 and SHA-512. It is an iterative and one-way function. SHA-512 is a member of SHA with a message digest 512- bit of length less than 2128. When the length of any message less than 2128 bits is an input to a hash algorithm, the result is a fixed message digest size (512). Also, SHA-256 is a version of SHA with a 256- bit message digest of length less than 264. When the length of any message less than 264 bits is input to a hash algorithm, the result is a fixed message digest size (256). These algorithms allow the purpose of information's integrity. Any modification in the message will make a modified message digest with a high probability [15].A cryptographic hash function directs to ensure different features, which provides high value for message safety. The hash function requires to satisfy the following features [17, 18]:

1. Compression: hash function maps the input message of uncertain finite-size to a value of fixed size.

2. Security of calculation: the hash value of an input message is simple to compute.

3. Pre-image resistance (one-way): it is obstinate to obtain only one input message which hashed to a determined hash value.

4. Weak collision resistance: it is obstinate to detect other messages that have an equal hash value.

5. Strong collision resistance: it is obstinate to detect two separated input messages hashed to the alike hash value.

Currently, countless applications through unrestricted networks require end-to-end protected connections to support authentication and data privacy [1]. Consequently, Cryptography algorithms are necessary for information security.

One of the cryptography algorithms families that the encrypter and decrypter utilize the same secret key is Symmetric-Key Cryptography. These algorithms depend on the agreement on a key from the sender and receiver before transferring their information. These algorithms use a unique key for encryption and decryption. Some popular patterns of Symmetric-Key encryption algorithms are Advanced Encryption Standard, Data Encryption Standard, Rivest Cipher 5, 3DES, Blowfish, etc.

1.3 Applications of Secure Hash Algorithms

To create a protected cryptographic process, the described algorithm must be trusted, time-examined, and peer-reviewed extensively. A hash function is an algorithm that receives input data and forms a data digest. In this paper, we utilized SHA-2 (SHA-256 and SHA-512). One of the most important reasons for using SHA-2 in our implementation (SHA-256 and SHA-512) is, providing more outcomes (512b and 256b sequentially) than SHA-1 (160b), such that the increased output intensity of SHA-2 is the main reason behind attack defense. Next, present the most vital applications of SHA.

1.3.1 BlockChain Technology

Blockchain technology is an extremely and advanced invention. It empowers digital data to remain distributed but not replicated [19]. Blockchain controls the modern crypto-currency named Bitcoin (digital gold). The expression of "blockchain" indicates structures of data, systems, or networks. It is a listing of ordered blocks, every block includes transactions and communicated to prior one, carrying the hashed value from prior block. Consequently, the transaction history cannot be removed without removing the contents of chain [20]. This is the main reason for saving blockchain from hackers.

Information stored on the blockchain, encrypted by applying HASH functions [22]. Bitcoin utilizes SHA-256. It is one of the most secure functions since every encrypted data give a fully different hash value. The encryption level is a firm such that brute attacks demand various endeavors and still find different input values. Blockchain has principal features as follows [21].

1. Decentralization. Third parties are not needed to confirm activities. Agreement algorithms are employed to keep data on blockchain networks.

2. Persistency. Valid Transactions are quickly, and invalid transactions are not accepted. Therefore, it is infeasible to remove transactions that have already happened.

3. Anonymity: On a blockchain network, the user communicates with others through a produced address. So, the real identification of the user is not represented during the communication.

4. Auditability. Every transaction on the blockchain network indicates the prior transaction. So, each transaction is confirmed easily and followed.

1.3.2 Internet of Things(IoT)

Internet of Things is great employment of the Internet to manage devices that are utilized daily, identified (things) through sensors within the Internet. IOT is defined as a network system of associated various devices (things) that empower us to interact using the protocol of machine-to-machine transmission [23].

Multiple safety vulnerabilities have been identified in the associated devices. Many users have concerned about safety issues, they worry about their data being removed or stolen, or misused [24].Advanced Encryption System (AES-256), SHA-1. SHA-256, etc. are security tools employed in IoT systems to secure the data [25]. IoT is a principle for future Internet development. IoT has managed and the base of emergent technologies like WoT defined as the Web of Things [26]. WOT technology is designed to perform our lives simpler and best. The accelerated growth of IoT led to appear various obstacles, like the vulnerability to cyber-attacks [24, 30]. It is difficult to make safe IoT devices because several security systems are broken to make IoT devices small in size, easy to use, and cheap. One technique that can be arranged to increase the security of IoT is the utilization of blockchain technology [27, 28, 29, 31]. Ronglin Hao et al. [32] an algebraic fault attack on the SHA-256 compression function introduced under the word-oriented random fault pattern. Throughout the attack, the automated Segmentation, Targeting and Positioning (STP) Model is employed, which forms binary representations for the word-based operations in the SHA-256 compression function and then requests a Satisfiability Problem (SAT) solver to resolve the equations. M. Sumathi et al. [33] announced a software framework for the implementation of data security algorithms. AES, RC5 and SHA algorithms have been used in this investigation and examined their implementations in Quartus – II software. They designed the encryption and decryption using Verilog HDL and simulated using ModelSim. With these algorithms, SHA-256 is more cooperative for preparing long data and it produces extraordinary security. The system meets all conditions and the results confirmed its reliability for data transmission. Fırat Artuğer and Fatih Özkaynak [34] offered a new technique to improve the performance of chaos-based substitution box structures. Substitution box structures have a special role in block cipher algorithms since they are the only non-linear elements in substitution permutation network designs. The analysis outcomes explain that the recommended approach can increase the performance standards. The quality of these results is that chaos-based designs may give chances for other applications in addition to the arrest of side-channel attacks.

2. HBCK-HASHING Algorithm

We describe the steps of HBCK – HASHING algorithm, initiated by the step of preparing N-ary block codes (U) as input message to generate square associated matrix of U by using specific notations. Next, we describe a multiplication operation $i \circ j = \beta_{ij}$ towards making HBCK-algebra $(H, \circ, 0)$. Subsequently, we construct N-ary block codes U_H with code words of length q for every HBCK-valued function such that U_H have U inside with redundancy. Moreover, we apply the steps of the secure hash algorithms (SHA-2), starting from the step of Appending bits, Length, and Initialize hash buffer step. Then, divide the message into blocks. Lastly, output the final value as a cipher text.

Step 1. Pre-processing the input N-ary block codes $U=\{d_1,\dots,d_m\}$. Consider a finite set $L'_n = \{1,2,\dots, n-1\}$. After lexicographic order, ascending order U of length q. Let $d_i = d_{i1}, d_{i2}, \dots, d_{iq}, d_{ij} \in L'_n$ and d_{ij} ordered descending.

Step 2. Constructing the associated matrix $T \in t_r(L_n)$ *of hyper BCK-algebra of U.* We generate an associated matrix T of N-ary codes U such that $T \in t_r(L_n)$, $r = m + q + 1$. we define the following equation 2.1:

$$\begin{cases} \beta_{s0} = s, \beta_{0t} = 0, s \in \{0, 1, 2, \dots, r-1\}, \\ \beta_{st} = 0, \text{if } s \leq t, \\ \text{for } q<s \leq r-1, \text{we suppose } \beta_{st} = d_{(q+i)}; \\ \qquad i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, q\}, \\ \text{for } q<s \leq r-1, q<t<r-1, s>t, \beta_{st} = 1. \end{cases} \tag{2.1}$$

If T is the related matrix of U defined on L'_n and $L_r = \{0, 1, 2, \dots, r-1\}$ is a non-empty set. Then, by using the previous schemes 2.1, we defined on L_r the following operation $i\circ$ j $= \beta_{ij}$.

Step 3. Applying the HBCK-valued function on T to get U_H. We construct N-ary block codes $L_r = \{d_0,d_1,\dots,d_r\}$ with length q for every HBCK- function such that U_H have U inside with redundancy. Suppose that we have the following:

Finite hyper BCK-algebra (H ,$\circ$,0) with elements (n), finite non-empty set (L) and L_n as a finite set, where $H = \{r_0,r_1,\dots,r_{n-1}\}, L = \{a_0,\ a_1,\dots,a_{m-1}\}$.

The map $f : L \to H$ is a hyper BCK-function, and the generalized function cutted of f is

$$\begin{aligned} &f_{rj} : L \to L_n; r_j \in H, f_{r_j}(a_i) = k \\ &\Leftrightarrow r_j \circ f(a_i) = (a_i) = \begin{cases} [0, r_k], \\ (0, r_k], \quad \forall r_j, r_k \in H, a_i \in L \\ \{r_k\}. \end{cases} \end{aligned} \tag{2.2}$$

$k, j, i \in \{0, 1, 2, \dots.., n-1\}$.

We suppose $\forall r \in H$, the generalized cut function f_r: L $\to L_n$. Every generalized cut fun, we construct the following code word d_r, with digits belong to the set $L_n$as the following:

$$d_r = d_0, d_1, \dots, d_{m-1}, d_i = j, j \in L_n \Leftrightarrow f_r(a_i) = j; r \circ f(a_i) = \begin{cases} [0, r_j], \\ (0, r_j], \\ \{r_j\}. \end{cases} \tag{2.3}$$

Enlightenment:

1. $(L_r, \circ, 0)$ is a unique identified hyper BCK-algebras since it was obtained by using T, which was a unique identified by U.
2. Let $d_X = \{X_1, X_2, \dots, X_q\}, d_Y = \{Y_1, Y_2, \dots, Y_q\} \in U_H$. We define the relation of order $\leq_c$ on U_H by the following $d_x \leq_c d_y \Leftrightarrow x_i \leq y_i, i \in \{1, 2, \dots, q\}$.
3. On H we define the following:

$$x \circ y = \begin{cases} \theta, & \text{if } x \leq_c y, \forall x, y \in H, \\ (\theta, y], & \text{if } x >_c y, y \neq 0, x, y \in H, \\ \{X\}, & \text{if } y = 0, \\ \{\theta\}, & \text{if } x = 0. \end{cases} \tag{2.4}$$

Where, it gets a hyper BCK-algebra structure.Next steps, we have an exchange between applying SHA-512 or SHA-256, in case of selecting one of them. The following stages concerning applying steps of SHA-512.

Step 4. Appending bits on U_H. It consists of a single 1-bit accompanied by the required amount of 0-bits so that its range is matching to 896 modulo 1024 [range = 896(mod 1024)]. Padding is always added to the N-ary block codes U_H, even if U_H is already of the desired range.

Step 5. Appending length on U_H. A block of 128 bits [unsigned 128-bit integer].

Step 6. Initialize hash buffer. Buffer of 512-bit is utilized to operate in-between and last result of HBCK-HASHING algorithm. Registers of eight 64-bit (a, b, c, d, e, f, g, h) represents the buffer. These records are initialized to the next 64-bit integers (hexadecimal values): a = 6A09E667F3BCC908, b = BB67AE8584CAA73B, c = 3C6EF372FE94F82B, d = A54FF53A5F1D36F1, e = 510E527FADE682D1, f = 9B05688C2B3E6C1F, g = 1F83D9ABFB41BD6B, h = 5BE0CD19137E2179.

Step 7. Divide the message into blocks of 1024-bit with 80 rounds. The module of 80 rounds is identified g. Every round income the input of 512-bit buffer (H_i), and appraises the fillings of the buffer. The value of the eightieth round is joined to the input to the first round (H_{i-1}) to create Hi, the increase is made separately for every of the eight-word in the buffer with each of the similar words in H_{i-1} using addition modulo 264.

Step 8. Output the final desired cipher text.

3. Structure of HBCK-HASHING Algorithm

To understand the proposed HBCK-HASHING algorithm, it is essential to present the model of construction as shown in Figure 1. This model shows the structure of the proposed algorithm through HBCK-valued function as a pre-processing stage that applied on the associated matrix T of the input N-ary block codes U. This function changes the input N-ary block code U to N-ary block codes U_H with redundancy. It aims to maximize the quantity of information with the fewest number of visible characteristics during enlarging the size of U from n×m to r× r with the same length q [35].

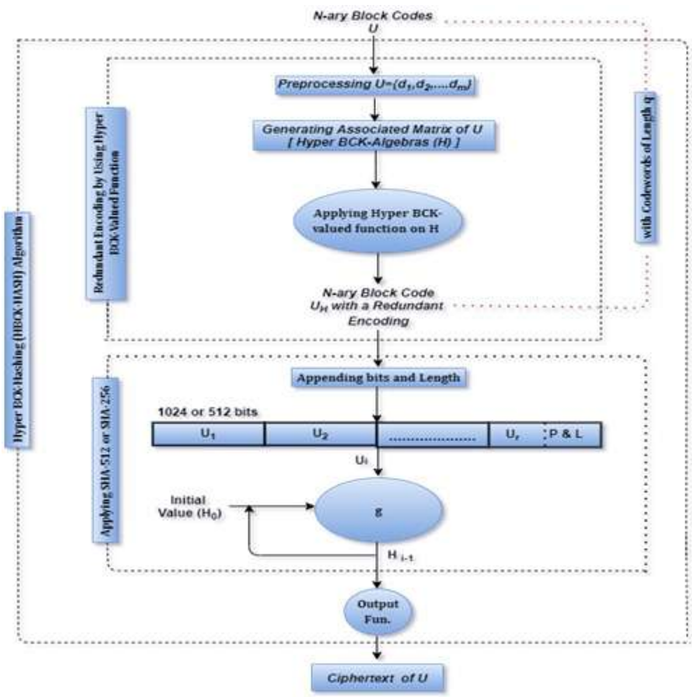

Figure 1: Model of HBCK-HASHING construction

Besides, the structure of the model demonstrates the subsequent steps which including adding padding and length to the N-ary block codes U_H with dividing the U_H into blocks of 1024-bit (in case of using SHA-512) and512-bit (in case of using SHA-256) to get the cipher text value of the N-ary block codes U. The compression Function g, in the construction model, represents

$$g : \{0,1\}^s \times \{0,1\}^{|U_i|} \to \{0,1\}^s. \tag{3.1}$$

Receives an input code H_i (i = 0, ..., r -2) of size S bits and U_i (i = 0, ..., r-1) of size U_i bits, to get the renewed cipher text variable H_i (i = 1, ..., r-1) of size S bits. Consequently, to support the rule of input code of uncertain length, the construction requires padding to transform the input code into a padded code of length a multiple of U_i bits. Simple padding makes unsafe constructed Cipher text. So that, the construction utilizes a padding function, which attaches the value of code length S at the end of U_i to produce the expanded code U_i.

$$H^i = H^{i-1} + g_U^i(H^{i-1}), \tag{3.2}$$

where, g is the compression function of SHA, + is word by word edition mod 264 and H is the cipher text of U.

4. Evaluation parameter

We evaluated the strength of HBCK-HASHING by calculating Avalanche Effect for every N-ary block codes. It has computed over small changes on the plaintext that contains 20 digits. These should provide a meaningful difference in cipher text. Particularly, changing an only bit in the plaintext, fixing the key, should change every bit in cipher text with probability (¿ 50%) ([16]).

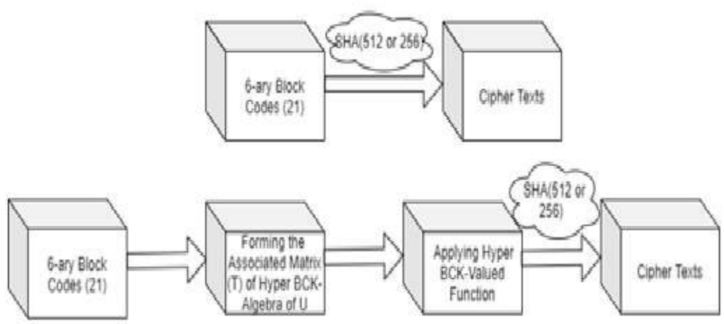

Figure 2: Process of Cryptographing (21) 6-ary block codes

We selected (21) 6-ary block codes with ascending ordered after lexicographic order and descending ordered for bits of each block inside the 6-ary block code. Each block code with4 code words of length 5. i.e., 20 bits in each 6-ary block code. Figure2 shows the process of Cryptographing (21) 6-ary block codes by using SHA-512 or SHA-256 directly, and also with applying HBCK-HASHING to compare the cipher texts and calculate the avalanche effect as an evaluation parameter. We implement HBCK-HASHING on P1 in the case of picking up SHA-512 through constructing a unique identified HBCK-algebra(H)and applying the function of HBCK f: L $\rightarrow$ H given by

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

to generate 6-ary block codes with a redundant encoding $U_H =$ 00000, 10000, 11000, 11100, 11110, 43221, 53321, 54321, 5431,as stated by step 1, step 2, and step 3 of HBCK-HASHING algorithm. In addition, we implement the Secure Hash Algorithm 512 on U_H to generate the first cipher text of P1. On the same way, we implemented HBCK-HASHING algorithm on another (20) 6-block codes by using the same HBCK-valued function f: L $\rightarrow$ H , and calculated the Avalanche Effect as shown in 4. To calculate the Avalanche Effect of (21) 6-ary block codes, we compared the cipher text of them after applying HBCK-HASHING, as shown in 5 with the cipher texts of (21)6-ary block code after applying the secure hash algorithm 512, as shown in 5, through the division of Number of flipped bits in the cipher text after applying HBCK-HASHING over number of bits in the cipher texts, as shown in 4.

6-ary Block Codes	No. of Flipped Bits in Cipher Texts of 6-ary Block Codes after Applying HBCK-HASHING	No. of Total Bits in Cipher Texts	Avalanche Effect(%)
P1	121	128	94.53125
P2	117	128	91.40625
P3	118	128	92.1875
P4	120	128	93.75
P5	124	128	96.875
P6	117	128	91.40625
P7	119	128	92.96875
P8	124	128	96.875
P9	119	128	92.96875
P10	115	128	89.84375
P11	114	128	89.0625
P12	119	128	92.96875
P13	115	128	89.84375
P14	116	128	90.625
P15	122	128	95.3125
P16	121	128	94.53125
P17	121	128	94.53125
P18	121	128	94.53125
P19	121	128	94.53125
P20	121	128	94.53125
P21	124	128	96.875

Table 1: Value of Avalanche Effect of (21) 6-ary block codes after applying HBCK-HASHING in case of using SHA-512.

we perform HBCK-HASHING algorithm, in the case of picking up SHA-256, on P1 and compared the cipher text of P1 after applying HBCK-HASHING, as shown in 5 with the cipher text of the same 6-ary block codes(P1) after the implementation of SHA-256, as shown in 5. Further, we measured the

Avalanche Effect, as shown in 4. On the same way, we implemented HBCK-HASHING algorithm on another (20) 6-block codes by using the same hyper BCK-valued function f: L $\rightarrow$ H, and calculated the Avalanche Effect as shown in 4. To calculate the Avalanche Effect of (21) 6-ary block codes, we compared the cipher text of them after applying HBCK-HASHING, as shown in 5with the cipher texts of (21)6-ary block code after applying SHA-512, as shown in 5, through the division of Number of flipped bits in the cipher text after applying HBCK-HASHING over number of bits in the cipher texts, as shown in 4.

6-ary Block Codes	No. of Flipped Bits in Cipher Texts of 6-ary Block Codes after Applying HBCK-HASHING	No. of Total Bits in Cipher Texts	Avalanche Effect(%)
P1	60	64	93.75
P2	60	64	93.75
P3	60	64	93.75
P4	62	64	96.875
P5	58	64	90.625
P6	60	64	93.75
P7	60	64	93.75
P8	60	64	93.75
P9	59	64	92.1875
P10	62	64	96.875
P11	63	64	98.4375
P12	59	64	92.1875
P13	61	64	95.3125
P14	57	64	89.0625
P15	62	64	96.875
P16	60	64	93.75
P17	61	64	95.3125
P18	60	64	93.75
P19	60	64	93.75
P20	60	64	93.75
P21	60	64	93.75

Table 2: Value of Avalanche Effect of 6-ary block codes (U) after applying HBCK-HASHING in case of using SHA-256.

5. Experimental results and analysis

In the following, we have promising results regarding the algorithm of HBCK-HASHING, in case of using SHA-512. 5 shows cipher texts of (21) 6-ary block codes after applying the algorithm and 5 shows cipher texts of (21) 6-ary block codes after applying SHA-512 only.we calculate the Avalanche Effect of (21) 6-ary block codes by computing number of flipped bits in cipher texts, as shown in 4 and representing the values of Avalanche Effect on a graph of (21)6-ary block codes, as shown in Figure 3. We noticed that the maximum value of Avalanche Effect was 96.875% in P5, P8 and P21, where the number of flipped bits in the cipher texts increased to 124, and the least value of Avalanche Effect was 89.0625% in P11, where the number of flipped bits decreased to 114. Addition, the trending line of all values of Avalanche Effect lies between 92% and 94% as shown in Figure 4.

The increasing of Avalanche Effect probabilities lead to increase the security level and the complexity of break through the system.

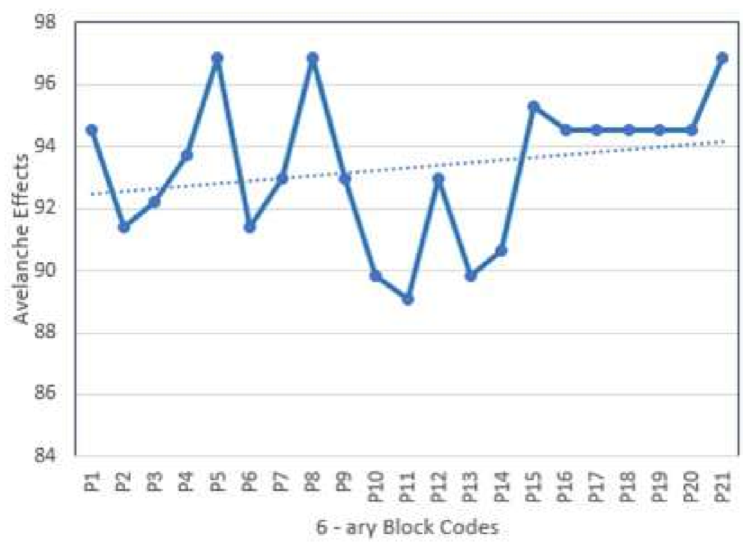

Figure 3: Avalanche Effect of 6-ary block codes (U) after applying HBCK-HASHING in case of using 512

Similarly, in the case of joining SHA-256 with HBCK-HASHING. 5 shows cipher texts of (21) 6-ary block codes after HBCK-HASHING, in case of using SHA-256, and 5 shows cipher texts of (21) 6-ary block codes subsequent implementing SHA-256 only. After computing the Avalanche Effect of (21) 6-ary block codes , as shown in 4 , and representing the values of Avalanche Effect on a graph of (21) 6-ary block codes, as shown in Figure 4. In the case of attaching SHA-256, especially in 4,the highest percentage of Avalanche Effect is 98.4375 in P11, wherever the quantity of flipped bits in the cipher texts following utilizing HBCK-HASHING raised to 63, and the smallest percentage of Avalanche Effect was 89.0625 in P14, wherever the number of flipped bits reduced to 57. In addition, the trending range of all values of Avalanche Effect rest between 93% and 95% as shown in Figure 4.

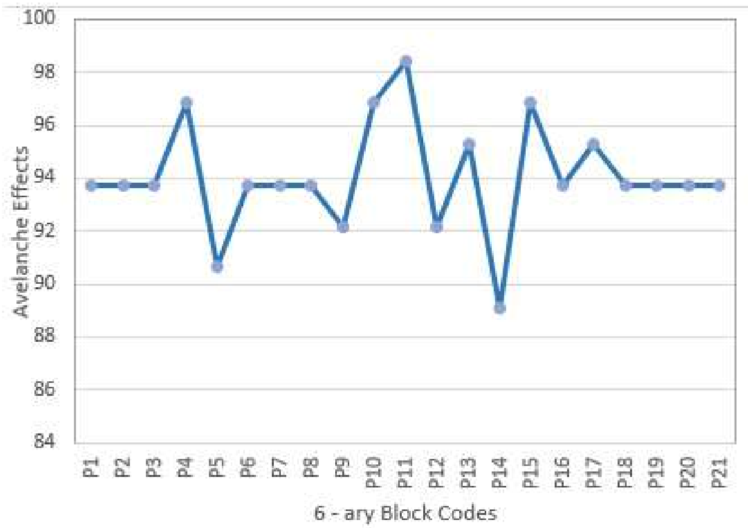

Figure 4: Avalanche Effect of 6-ary block codes (U) after applying HBCK-HASHING in case of using SHA-256.

	6-ary Block Codes (U)	6-ary Block Codes with a Redundant Encoding (U_H)	Cipher Texts of 6-ary Block Codes with a Redundant Encoding by Using SHA (512)
P1	4322153321 5432155431	000001000011000 111001111043221 533215432155431	dc228680e90ec2f6a285518e5ee23e5611b0872 bb20f05d559d524aa1dbf2c474ea259eaa917c74 5cf12c68ec1408f40854e4fbc76cbc7e1e3ffa416 1178463b
P2	**5**322153321 5432155431	000001000011000 1110011110**5**3221 533215432155431	5e779c9152f7af033cec0d01bab8a74954c448bf 43d5c3be58187a1c77c29cb489f3466b95892ff0 43d5c3be58187a1c77c29cb489f3466b95892ff0 8b54e3e03
P3	4**4**22153321 5432155431	000001000011000 1110011110 4**4**221 533215432155431	a9bab493ff75ff08506e0670a8252065aed839ed 78184a70fee3bd285d0e274b796eb991bf3ef666 58cd262511790e3f928532ada54e4a8e5cbda12 2ae4da427
P4	43**5**2153321 5432155431	000001000011000 111001111043**5**21 533215432155431	b921b1265dd55fdc4e461c4be657afb1bc3c796 b5ccd4678c848b81c96e6dcb691afb50e190043 e1dc504882094a8fc4c1c14aaaa131ab133cb222 73bc0d51af
P5	432**4**153321 5432155431	000001000011000 1110011110432**4**1 533215432155431	182922bf9a7fcdadf82ec275fd0d83586989c78e 58f864e49ecf944eb9c46fdbb4914f574f52eacd2 a6416acdc68b64f442d561f04c59b7476f3def1d 749d45c
P6	4322**5**53321 5432155431	000001000011000 11100111104322**5** 533215432155431	1dc0c82129a07348c123e97c6e4c912c6dd1ea9 53dfbd76a8f9eaee28f31c58394c50d9bbe3ac80 57ac2008b5fef45ff04146343a1671cd2a1c6c2b a608aba86
P7	43221**6**3321 5432155431	000001000011000 111001111043221 **6**33215432155431	5e228a9948c55bb44900c811758ed4bbde9cbd2 1484d16ae706a2993cf1b7dc2304ee182cf76060 85f8bf973e0a44d1646d23ae34c752b9c5140de 1c8498615
P8	432215**4**3321 5432155431	000001000011000 111001111043221	7bf8d3a918bf70eab23308c379e64671468c9dd 102b1fdc401d098f3111550e05e8df82cb373f0d

		5**4**3215432155431	60838c84fd60f7bffe4c9bf625512ee0649687ec d2e7fbf08
P9	4322153**6**21 5432155431	000001000011000 111001111043221 53**6**215432155431	87e21431a04822155db9e3595b2e5998fc52747 9cd4a8c08b3600de59b2ad24f4c6ab55df03b29c 6bbff88dcc355b3bdbef62e9c8e30d651ba387cc 8ac3da53c
P10	43221533**4**1 5432155431	000001000011000 111001111043221 533**4**15432155431	d615749b41bd75d9967accf44e8ca754ed93a47 7a8274c35620f976466ca985893ffbdddfdefcc1 8440ebd3012c2e0830fbbe158e7a4fc0aec33a05 df073cb89
P11	432215332**5** 5432155431	000001000011000 111001111043221 5332**5**5432155431	b8795773db70c087ab42aa679724a6e72201b14 1772cb8b4c530f31cd87c3571bf91e3169d617b 69d45749c92b7bd599c65bb4765740fee97f0c6 ac691e7190
P12	4322153321 **3**432155431	000001000011000 111001111043221 53321**3**432155431	1a3f80d12198d196290f83653aae952e2bed62a b5b6f70b6334b5d20c6629e247de28ad657bb87 54a331f17a2c9e23f2a067c3ec9ba2ead740b022 9b3828f148
P13	4322153321 5**5**32155431	000001000011000 111001111043221 533215**5**32155431	726b1e4b5ec8a9f803726336c9784b02cec50ad 72dbd57a13853526b449d8e05c1bba4c779206f 20bc7a00a79bc36807b0a2dfebd59a0756237c3 d19fba54f2
P14	4322153321 54**2**2155431	000001000011000 111001111043221 5332154**2**2155431	d7e30f3505cc4281a35666379e4cd11cab617bfc 726f6b705c2cfef33ac18b4723aa5dd330a77c5c 8874b1328155a4d2b18007ce7c26683be82f331 7af6382fc
P15	4322153321 543**5**155431	000001000011000 111001111043221 53321543**5**155431	7c9cf687d470ba75efdf63ebd4a3797249f28fce0 c9af5a46081e16a7652741ac597f5531ebbafd69 13d3619895bf6c837e071ac6ecc90538d40c6c3 185ef084
P16	4322153321 5432**3**55431	000001000011000 111001111043221 533215432**3**55431	5d37540ddae4f63eb8b2ecc60735e07150dd5cc 5ca2d723514041543d7804af93e5b681978fa4e bedf1305fef5d4624804d17039ee52ca67027886 dfa42e409e
P17	4322153321 54321**6**5431	000001000011000 111001111043221 5332154321**6**5431	3f88ff4f061ffed2caab9226c377bafb8a83c2c92 7487650deeb38c9e0b021c6c0eba016bc0d1d10 4e9e0f9ea0c3a6cdeaf4449f58d15368c6cc1393 bbd822d5
P18	4322153321 543215**4**431	000001000011000 111001111043221 53321543215**4**431	51a56ddffaea5df9ab884e07ff5d5ba12591bd2d 4883cbdadf0d0a018b0eff4b5799232dfcd92ab4 0f743ddc615c4c8507fdc84ce67aa83aee7939b6 48ad5897
P19	4322153321 5432155**3**31	000001000011000 111001111043221 533215432155**3**31	6d3e924bee5c38e20c01aa7b2a37321821a8a1c eb353b7b08bf40ab756fc87c24c2eda761c1efcb 4a0d59bd2676243db2f244f2e0469dc2477a4a5 56b735b2fa
P20	4322153321 54321554**2**1	000001000011000 111001111043221 5332154321554**2**1	8e9b1debb8fc55df828969595cbd69c1a53571d 138d5f4221c5b2d4416afad15b34504b86a18fd b4921beee6b94673b244f170b4406be9c2a85e5 6bae6b2936
P21	4322153321 543215543**2**	000001000011000 111001111043221 53321543215543**2**	1d6598a24e464832be5da6fb679272c5f35916b b0c612715236275fc4af33cdafb8d46e595f5fa7 04ec1655448918c4ca3bb1ac1c7d855cf9131a9f f49c2a8b4

Table 3: Cipher texts of 6-ary block codes by using HBCK-HASHING algorithm in the case of using SHA-512

	6-ary Block Codes (U)	Cipher texts
P1	4322153321 5432155431	8dff689bfca583e6734665c695ce8db3163909380af6bbd72d6d716da ff7f5c5ec9e913ea89b630be957eedbc20246e7ee3d07345d7fc526f81 49cd72391d73d
P2	**5**322153321 5432155431	8ac6818ec798fd2511525516b2ebadd6434d485d5fff70b6657befb67 f5c7b1d547e0c07a7225236392729046cd617ea5e1418d12c1b9041a e9beb7cfe99f205
P3	4**4**22153321 5432155431	3ef9b78088b2ac2109f363fd3c81e1bb7d413c1c4055b72ffce42e772 239c916bc5a151ebc37822da23e741be300529152703d81a484f8b11 acdddc653518e1c
P4	43**5**2153321 5432155431	872f5ade38a10c8998e07bf29556ebbb239e4cb4e5f3f2c09a30b4f2f0 7443daa26abe1f0cce4b5c360c68c53db221231b2c95a10294225047 46b82a27fbe73d
P5	432**4**153321 5432155431	56dd4f42da49f779b73faec92f62fe23556d21e70376f08cfa390c1fd1 3446b5205c75a29351508778512e06fe53373913cdffcc9899633f983 9c4384417bfcb
P6	4322**5**53321 5432155431	d0e0c0ab89bce182aee4736f053d013ce209911bce82d8810b9812c7 d308e5e2cc464ae241c898e22cc6a7a8f9d10f2ea4724ccb36b0cd344 74175e6177e98c2
P7	43221**6**3321 5432155431	4f26287098b573d4e56c8dfba84afe1d778d5432939b9ec89cd7629bb bbfb4026f998122ad8e1435acffc496ce828ee2f4b6eccbb09f260ed76 e6a7d0eebc79d
P8	432215**4**321 5432155431	ecaeaf191c7375c85982365a31a4544215d2a18cb1f6a52695fa8b0c8 2a9213e00a36303fc17221e4c6121a8a168c5ba642144c400b653e1fc a6644019b5c39f
P9	4322153**6**21 5432155431	2ec54eb37462d87c01608e009b460aed68aa243e5cd41d0fb807d05fb 3dde000a3da2ab02be006dfb2c2b9592bc0981c0ac9ae599d26bf0b1a cbc084f97b5505
P10	43221533**4**1 5432155431	6ced46ed192444df99e8def15733ef9f1daea1107037fb0d184045765 49a9e7c1adbb5dd97898fda9744ec732a226aca533bd4b2da9a281f5d cdc97649f4b35f
P11	432215332**5** 5432155431	dc99a1583fb7f1eca60212a4aadc6cd2a2064904636c93b8acfc6abd4 25e6f31492d216bcbf0425ec51b2bc524486e096d6e5506bac8e7d4c9 2427cb4a881fa7
P12	4322153321 **3**432155431	bce45b51b6b57c003815439c1ceb938df4fdc4ef565ddb01211b045f5 53cb364d85e986a4cfab9d30c405403c816ba37935b6cf77412397f6c dc64f431fe0387
P13	4322153321 5**5**32155431	b201af458c161044b203fe38ad2be39fb649a0943c2e43e65b2f1cf8b d277d77eb85146220b00a36bc8e726560a6e804e046f331f79aa6533 8c5829175ce22fc
P14	4322153321 54**2**2155431	dffb5af0dc513e321caef73feeede9fc7204420b278b4365f70addb7abe 1b1471fa4e508f733eb3cdb161ea84c40b8f41c4e58c6651541353e2d 089d7212d780
P15	4322153321 543**5**155431	37f794ec81ff5963f329ab167a6e2a9210fdd77ac5222bcf3e578a8f14 0816bebb5efbb32dd7f0bf84498d9eba0cfbf3def9512eed9b0d82303e 3c2a23ec9819
P16	4322153321 5432**3**55431	9a762eba83a1249a2842ca5b668a26e9522a3118fef745c29b716e3db c37402842e7ea8ec6324ccabc2c41386a459563452a6d5c1af831c84a cd26d3f5a032f6
P17	4322153321 54321**6**5431	3b2e53f31d329878685ccc9a11972dc65fad1b872d520b1d10ba499f9 2a54fbafdc879ec135f041f936ad3b3a5f518acfb780441b99527e183b 4ca2cb6b66b62
P18	4322153321 543215**4**431	d2a5b248e9aec7ad19a100832f3555d8accb70f98b2befb092e2def9aa a90e2d501f6b25ed6275d573e36b9700ab133fd860d87bf563b31cc7 50d67acfca620f
P19	4322153321 5432155**3**31	5e231249bad007f2008330441281f8d7c4e5f212a6b22be64948470b a1147cc33dd4849e1278c8ca214aa5e13d2daabd5c5d8e75ffefdf7335 390205e1a99139
P20	4322153321 54321554**2**1	2c958aa35c81a73ceaa79fe89e9326099e7643289646a8638f6045156 775f5cd2ccd0018b742aba1c8151c6bfc91458f1d9f15b1b23da4c96b

P21	4322153321 543215543**2**	2e2f78046baf77 025de2e020e7f67796e31c417cd6921416620c4d0fb31815883c98e5f f8e633655e6e74f8246811a9c591f2aceab9b9219175dd5bd738f578c 69e4a1f4a75d6a

Table 4: Cipher texts of 6-ary block codes by using SHA-512

	6-ary Block Codes (U)	6-ary Block Codes with a Redundant Encoding (U_H)	Cipher Texts of 6-ary Block Codes with a Redundant Encoding by Using SHA (256)
P1	4322153321 5432155431	000001000011000 111001111043221 533215432155431	c443e90b301d4b313ed9f15135550a8f52cfcd1 e1f271b50df3ee20651c39c31
P2	**5**322153321 5432155431	000001000011000 1110011110**5**3221 533215432155431	1a5f25ab34f89787fb650b632d523969b889967 6ff67e0c9545bb1b211e9f90b
P3	4**4**22153321 5432155431	000001000011000 11100111104**4**221 533215432155431	8e24467b41ec0b64ae9301c90d97d0b6a1ba4f1 a234b9c4e9898cbff8a56fc7d
P4	43**5**2153321 5432155431	000001000011000 111001111043**5**21 533215432155431	15e571900d38fe7cefd68226891c3ee95067253 c975e831aea6201c85d3a44f1
P5	432**4**153321 5432155431	000001000011000 1110011110432**4**1 533215432155431	6579148a6c847e16ccfd06cb4d6aaa2e117dac6 97e18dcade7e9c52f7cd0efc4
P6	4322**5**53321 5432155431	000001000011000 11100111104322**5** 533215432155431	b56ce6bf6f383011285fb14170553112108d94f 8a836a325cf2a29f6f20a46a7
P7	43221**6**3321 5432155431	000001000011000 111001111043221 **6**33215432155431	b5ba82dbefb13902140c9f29214defc17d5d34b f4e5c202f053e72a54eb1687d
P8	432215**4**3321 5432155431	000001000011000 111001111043221 5**4**3215432155431	e8b36d7c026609584b524b1af624e3c2d43ea8a 29dd28eb03ad9046744a36bd1
P9	4322153**6**21 5432155431	000001000011000 111001111043221 53**6**215432155431	638371974ee28e81a6dcede48c739111dcc41ef 4946de646615b15dda4206fbd
P10	43221533**4**1 5432155431	000001000011000 111001111043221 533**4**15432155431	edebbc31e7d02f45d459fdedf6ab6cfcf47e4761 7d47b8c7a3ad5d1d37f968d1
P11	432215332**5** 5432155431	000001000011000 111001111043221 5332**5**5432155431	fbebdf9b610ca90e3e47d6098557f3d623b34e3 9f4d9d7c5699cbcc8e0157d49
P12	4322153321 **3**432155431	000001000011000 111001111043221 53321**3**432155431	54a7a28ca927b7c6bbf8df27572d887a7fcd994 436f89543818dc533b96d192f
P13	4322153321 5**5**32155431	000001000011000 111001111043221 533215**5**32155431	4c1ea6391bcb8ae9580b2d9cc4cb9e1c8a85778 b6c3413e0fcb3c42a784544ff
P14	4322153321 54**2**2155431	000001000011000 111001111043221 5332154**2**2155431	e9faf3a45b983203a57a81fd1092447b083ffc0e d8d90cc1e07b785d7c0c701f
P15	4322153321 543**5**155431	000001000011000 111001111043221 53321543**5**155431	0472c022839276a1c8a9c2e06cce663e57a6985 d00163e49de43bb49531f0c68
P16	4322153321 5432**3**55431	000001000011000 111001111043221 533215432**3**55431	cd8034033603933af5d5f15b8d06e06ebc6fef0 65498d115eb90a15ce41b376f
P17	4322153321 54321**6**5431	000001000011000 111001111043221 5332154321**6**5431	d75972576f0b7168e053332a824f8010aaaff16 59592642e2177a87a214171a7

P18	4322153321 543215**4**431	000001000011000 111001111043221 533215432154**4**431	c8c49737513ae92869caee869864ad73cd38552 bb0e93a0d9a4c0e278b6cedf6
P19	4322153321 5432155**3**31	000001000011000 111001111043221 533215432155**3**31	7b63c9f8810434bdcc388f1acb0bb80cbd98dd8 8a418ebbae3f12cb104a8899d
P20	4322153321 54321554**2**1	000001000011000 111001111043221 5332154321554**2**1	cf1460cf18c4f7cdc94420cf87390133b24bc074 5085c6b01e87130072edaeae
P21	4322153321 543215543**2**	000001000011000 111001111043221 53321543215543**2**	75a6a221dbb141d6ff31a9224508c4db28b5f03 e4251d6dc6c856632157ed98b

Table 5: Cipher texts of 6-ary block codes by using HBCK-HASHING algorithm in the case of using SHA- 256

	6-ary Block Codes (U)	Cipher texts
P1	4322153321 5432155431	615cdff092a7b8bc9eead08549ea60c12776e2dc9cd9818 c187e29b6f16f685e
P2	**5**322153321 5432155431	3f8f2609bc1a7c2c1efea2efa82fd744c4f407f312af611cb e87b3d0f1a0371d
P3	4**4**22153321 5432155431	247324ce62b2922a4472bcc1e532d74e3cfb029d955a0ff 390ac62f83c22c94e
P4	43**5**2153321 5432155431	575d73533b24b2d6c1be9d2844d3e42c174e7ee4ba43eff df1d5839f7594b71f
P5	432**4**153321 5432155431	9557f68e556c735534fe1631188b540582e4d49a7cdca03 c222377c6f11a530e
P6	4322**5**53321 5432155431	476451d83f3e15f940687fab0ecffc6ca6664b0809241d4 9e3a1909e57e48376
P7	43221**6**3321 5432155431	0e69e71b6a96cc120ca694a4d2732e3a0643ef919d56888 d77a4847c09737b9a
P8	432215**4**321 5432155431	57481836aabf4dbe63dad0e49642bc74d0edaa972baac8e 33777475bd8d095c2
P9	4322153**6**21 5432155431	52ffbc60a39bddf7ac2e5ad9e2d443b8cff807595df3d6d9 91895a3ba4b3904a
P10	43221533**4**1 5432155431	92983647186b1f3fee7651fb9fb110553c9df676e5bf5adf e17964aff4ec2ec1
P11	432215332**5** 5432155431	53aa730d5d39758401b968b6e8e1e0ce731172fd975eae 58816338ba149698be
P12	4322153321 **3**432155431	16ce0b01b68906d484fdf732ad63bf4f05e111e238413c3 31685685a79354db7
P13	4322153321 5**5**32155431	3b24ca7468d9baf78affb2ad02d0182056e260b5d6c827f 097285a5d741dbf8e
P14	4322153321 54**2**2155431	d1b2f630399cc0e1cd89d1495957ef9862aec559f236b8c 191122ea57a01ffd5
P15	4322153321 543**5**155431	6768bb0ce52c1004fc38d5ceff78245932a78783963fad0 003e4db5c8291d345
P16	4322153321 5432**3**55431	c696a68e8731c21dc5c34356865a79bd25c5788730aefe8 df754f0783c95a90b
P17	4322153321 54321**6**5431	53d111ae10641bf1ede879fdb569b3c7c1d03768fadf267 0e4e79a496a03f8ff
P18	4322153321 543215**4**431	6f0f830a57f359cd03b13874cb9864e646f8dfe2f236315e c7bda9e2c5ea9b0a
P19	4322153321 5432155**3**31	625d8332bafcc1f8130449a89ac0bd9a040b04181719ec1 6c988ca842ac498bf
P20	4322153321 54321554**2**1	339f041f012a6b016ae073158c7f5ff15ac594a495abf20d a142a61721301eed
P21	4322153321 543215543**2**	e2a00587de1c3434e014082ce6e2da337daad81b5e91e5 5bab430deaf80c0aa1

Table 6: Cipher texts of 6-ary block codes by using SHA-256

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A note on b-generalized derivations in rings with involution

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Abstract. Let R be a ring with involution $*$. The purpose of this paper is to investigate the special type of mappings defined on $(R, *)$. In fact it is shown that these mappings are actually the b-generalized derivation defined on R.

Keywords: prime ring, derivation, b-generalized derivation, involution.

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1. Introduction

Throughout this article, R be a prime ring with involution and Q_r will be the right Martindale quotient ring of R. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol $x \circ y$ will stand for the anti-commutator $xy+yx$. R is said to be 2-torsion free if whenever $2x = 0$; with $x \in R$ implies $x = 0$. R is prime if $xRy = (0)$, where $x, y \in R$, implies $x = 0$ or $y = 0$ and is called a semiprime ring in case $xRx = (0)$ implies $x = 0$. A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$, for all $x \in R$. Following Brešar [9], an additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Basic examples are derivations and generalized inner derivations are maps of type $x \mapsto ax + xb$ for some $a, b \in R$.

Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. Many results in this direction can be found in [1, 5, 6, 7, 8, 9, 10, 15, 16, 17]. Very recently Koşan and Lee [14] introduced the new concept of left b-generalized derivation as follows: Let $d : R \to Q_r$ be an additive mapping and $b \in Q_r$. An additive mapping $F : R \to Q_r$ is called a left b-generalized derivation, with an associated mapping d, if $F(xy) = F(x)y + bxd(y)$, for all $x, y \in R$. Moreover, it is prove that if R is a prime ring, then d is a derivation of R. In the present paper, this mapping F will be called a b-generalized derivation with an associated derivation d. It is easy to see that every generalized derivation is a 1-generalized derivation. For instance for any $x \in R$, the mapping $x :\to ax + bxc$ for $a, b, c \in Q$ is a b-generalized derivation of R, which is known as inner b-generalized derivation of R.

An additive mapping $* : R \to R$ is called an involution if $*$ is an anti-automorphism of order 2, that is $(x^*)^* = x$, for all $x \in R$. A ring equipped with an involution $*$ is called an involution ring. Very recently, many authors have studied certain additive mappings like derivations, generalized derivations in the setting of rings with involution (see [2, 3, 4, 11, 12] for references). They not only characterized these mappings but also found that there is a close connection between the commutativity of R and these mappings. Here our emphasis will be more in the direction of the a special type of mapping defined on R, which were first studied in [18].

In fact, our motivations comes from [[13], Theorem 4.1.2], which stated as: Let R be a simple ring with involution of characteristic not 2, such that $dim_Z R > 4$. Let $d : R \to R$ be such that $d(xx^*) = d(x)x^* + xd(x^*)$, for all $x \in R$. Then d is a derivation of R. We prove the following results.

Theorem 1.1. *Let R be a 2-torsion free semiprime $*-$ring with involution such that R has a commutator which is not a zero divisor. If there exists an additive mapping $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that*

$F(xx^*) = F(x)x^* + bxd(x^*)$, for all $x \in R$, where b is a fixed element of R. Then F is a b-generalized derivation.

Theorem 1.2. *Let R be a 2-torsion free semiprime $*-$ring and let R has a commutator which is not a zero divisor. If there exists an additive mapping $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $F(xy^*x) = F(x)y^*x + bxd(y^*)x + bxy^*d(x)$, for all $x, y \in R$ and b is a fixed element of R. Then F is a b-generalized derivation.*

2. Main results

To prove the above results we need the following lemma.

Lemma 2.1. *Let R be a 2-torsion free ring and let $F : R \to R$ be an additive mapping associated with a nonzero derivation $d : R \to R$ such that $F(x^2) = F(x)x + bxd(x)$, where b is a fixed element of R. Then, for all $x, y, z \in R$, the following statements hold:*

(i) $F(xy + yx) = F(x)y + F(y)x + bxd(y) + byd(x)$;

(ii) $F(xyx) = F(x)yx + bxd(y)x + bxyd(x)$;

(iii) $F(xyz+zyx) = F(x)yz+F(z)yx+bxd(y)z+bxyd(z)+bzd(y)x+bzyd(x)$;

(iv) $\delta(x, y)[x, y] = 0$, *where* $\delta(x, y) = F(xy) - F(x)y - bxd(y)$.

Proof. (i) We have

$$F(x^2) = F(x)x + bxd(x), \quad \text{for all} \quad x \in R. \tag{1}$$

Replacing x by $x + y$ and using (1), we get

$$F(xy + yx) = F(x)y + F(y)x + bxd(y) + byd(x), \quad \text{for all} \quad x, y \in R. \tag{2}$$

(ii) Taking $y = xy + yx$ in (2) and using it, we arrive at

$$\begin{aligned} F(x^2y + yx^2) + 2F(xyx) = {} & F(x)xy + F(x)yx + F(x)yx \\ & + F(y)x^2 + bxd(y)x + byd(x)x \\ & + bxd(x)y + bx^2d(y) + bxd(y)x + bxy(x) \\ & + bxyd(x) + byxd(x), \quad \text{for all} \quad x, y \in R. \end{aligned} \tag{3}$$

Replacing x by x^2 in (2) and using (3) and the fact that R is 2-torsion free, we obtain

$$F(xyx) = F(x)yx + bxd(y)x + bxyd(x), \quad \text{for all} \quad x, y \in R. \tag{4}$$

There by proving (ii).

(iii) Replacing x by $x+z$ in (4) and using (4), we get

$$F(xyz+zyx)=F(x)yz+F(z)yx+bxd(y)z+bzd(y)x+bxyd(z)+bzyd(x), \tag{5}$$

for all $x, y \in R$. Thus proves (iii).

(iv) On substituting $xy-yx$ in place of z in (5), we get $\delta(x,y)[x,y]=0$, for all $x, y \in R$, where $\delta(x,y)=F(xy)-F(x)y-bxd(y)$. This completes the proof of Lemma. □

Proof of Theorem 1.1. We have

$$F(xx^*)=F(x)x^*+bxd(x^*), \quad \text{for all} \quad x \in R. \tag{6}$$

On linearizing (6), we get

$$F(xy^*+yx^*)=F(x)y^*+F(y)x^*+byd(x^*)+bxd(y^*), \quad \text{for all} \quad x, y \in R. \tag{7}$$

Taking $y=x^*$ in (7), we have

$$F(x^2+(x^*)^2)+F(x)x+F(x^*)x^*+bx^*d(x^*)+bxd(x), \quad \text{for all} \quad x \in R.$$

This can be further written as

$$B(x)+B(x^*)=0, \quad \text{for all} \quad x \in R, \tag{8}$$

where $B(x)=F(x^2)-F(x)x-bxd(x)$, for all $x \in R$. Replacing y by xy^*+yx^* in (7), we obtain

$$F(x(y+y^*)x^*)=-B(x)y^*+F(x)(y+y^*)x^*+bxd(y+y^*)x^*, \quad \text{for all} \quad x, y \in R.$$

Using $y-y^*$ for y, we get

$$B(x)y=B(x)y^*, \quad \text{for all} \quad x, y \in R. \tag{9}$$

In view of [[19], Lemma 1], we get $B(x) \in Z(R)$, for all $x \in R$. Taking $y=y^*$ in (7), we arrive at

$$F(xy+y^*x^*)=F(x)y+F(y^*)x^*+by^*d(x^*)+bxd(y), \quad \text{for all} \quad x, y \in R. \tag{10}$$

Replacing y by xy in (10), we get

$$F(x^2y+y^*(x^*)^2)=F(x)xy+F(y^*x^*)x^*+bxd(x)y+bx^2d(y)+by^*x^*d(x^*), \tag{11}$$

for all $x, y \in R$. Taking $x=x^2$ in (10), we obtain

$$F(x^2y+y^*(x^*)^2)=F(x^2)y+F(y^*)(x^*)^2+bx^2d(y)+by^*d((x^*)^2), \tag{12}$$

for all $x, y \in R$. Using (11) and (12), we get

$$(F(x^2)-F(x)x-bxd(x))y+(F(y)x^*-F(y^*x^*)+by^*d(x^*))x^*=0, \quad \text{for all} \quad x, y \in R.$$

Replacing y by x, we have

$$(F(x^2)-F(x)x-bxd(x))x-(F((x^*)^2-F(x^*)x^*-bx^*d(x^*))x^* = 0, \text{ for all } x \in R.$$

This implies that

$$B(x)x - B(x^*)x^* = 0, \quad \text{for all } \ x \in R.$$

By (8), we arrive at

$$B(x)(x + x^*) = 0, \quad \text{for all } \ x \in R. \tag{13}$$

Taking $y = x$ in (9), we get

$$B(x)(x - x^*) = 0, \quad \text{for all } \ x \in R. \tag{14}$$

Thus in view of (13) and (14), we get $2B(x)x = 0$, for all $x \in R$. Since R is 2-torsion free, we obtain

$$B(x)x = 0, \quad \text{for all } \ x \in R. \tag{15}$$

Since $B(x)$ is in $Z(R)$, this implies that $xB(x) = 0$, for all $x \in R$. Linearizing (15), we get

$$B(x)y + B(y)x + \sigma(x, y)x + \sigma(x, y)y = 0, \quad \text{for all } \ x, y \in R, \tag{16}$$

where $\sigma(x, y) = F(xy + yx) - F(x)y - F(y)x - bxd(y) - byd(x)$, for all $x, y \in R$. Taking $x = -x$ in (16), we have

$$B(x)y - B(y)x + \sigma(x, y)x - \sigma(x, y)y = 0, \quad \text{for all } \ x, y \in R. \tag{17}$$

Using (16) and (17), we arrive at $B(x)y + \sigma(x, y)x = 0$, for all $x, y \in R$. Right multiplying by $B(x)$, we get $B(x)yB(x) + \sigma(x, y)xB(x) = 0$, for all $x, y \in R$. This implies that $B(x)yB(x) = 0$, for all $x, y \in R$. Since R is a semiprime ring, we obtain $B(x) = 0$, for all $x \in R$. This implies that

$$F(x^2) = F(x)x + bxd(x), \quad \text{for all } \ x \in R. \tag{18}$$

Let u, v be fixed element of R such that $w[u, v] = 0$ or $[u, v]w = 0$. Then in view of Lemma 2.1 (iv) and hypothesis

$$\delta(u, v) = 0, \tag{19}$$

we have to show that $\delta(x, y) = 0$, for all $x, y \in R$. Again in view of Lemma 2.1 (iv), we have

$$\delta(x, y)[x, y] = 0. \tag{20}$$

Replacing x by $x+u$ and using (20), we get

$$\delta(x,y)[u,y]+\delta(u,y)[x,y]=0, \quad \text{for all} \quad x,y\in R. \tag{21}$$

On substituting y by $y+v$ and using (19) and (20), we have

$$\delta(x,y)[u,v]+\delta(x,v)[u,y]+\delta(x,v)[u,v]+\delta(u,y)[x,v]=0, \quad \text{for all} \quad x,y\in R. \tag{22}$$

Taking $x=u$ in (22) and making use of (19), we obtain $2\delta(u,y)[u,v]=0$, for all $y\in R$. Since R is 2-torsion free and using the given assumption, we have

$$\delta(u,y)=0, \quad \text{for all} \quad y\in R. \tag{23}$$

Again replacing y by v in (21) and using (19), we get $\delta(x,v)[u,v]=0$, for all $x,y\in R$. Since $[u,v]$ is not a zero divisor, we get

$$\delta(x,v)=0, \quad \text{for all} \quad x\in R. \tag{24}$$

Thus by (22), (23) and (24) we get $\delta(x,y)[u,v]=0$, for all $x,y\in R$. This implies that $\delta(x,y)=0$, for all $x,y\in R$ i.e., $F(xy)=F(x)y+bxd(y)$, for all $x,y\in R$, which completes the proof.

Proof of Theorem 1.2. By the given hypothesis, we have

$$F(xy^*x)=F(x)y^*x+bxd(y^*)x+bxy^*d(x), \quad \text{for all} \quad x,y\in R. \tag{25}$$

On substituting x by $x+z$ and on solving, we have

$$\begin{aligned} F(xy^*z+zy^*x) &= F(x)y^*z+F(z)y^*x+bxd(y^*)z \\ &\quad +bzd(y^*)x+bxy^*d(z)+bzy^*d(x), \end{aligned} \tag{26}$$

for all $x,y,z\in R$. Replacing z by x^2 in (26), we get

$$\begin{aligned} F(xy^*x^2+x^2y^*x) &= F(x)y^*x^2+F(x^2)y^*x+bxd(y^*)x^2+bx^2d(y^*)x \\ &\quad +bxy^*d(x)x+bxy^*xd(x)+bx^2y^*d(x), \quad \text{for all} \quad x,y\in R. \end{aligned} \tag{27}$$

Taking y as x^*y+yx^* in (25), we obtain

$$\begin{aligned} F(xy^*x^2+x^2y^*x) &= F(x)y^*x^2+F(x)xy^*x+bxd(y^*)x^2 \\ &\quad +bxy^*d(x)x+bxd(x)y^*x \\ &\quad +bx^2d(y^*)x+bxy^*xd(x)+bx^2y^*d(x), \quad \text{for all} \quad x,y\in R. \end{aligned} \tag{28}$$

On comparing (27) and (28), we arrive at

$$(F(x^2)-F(x)x-bxd(x))y^*x=0, \quad \text{for all} \quad x,y\in R. \tag{29}$$

This can be further written as

$$\phi(x)y^*x=0, \quad \text{for all} \quad x,y\in R, \tag{30}$$

where $\phi(x) = F(x^2) - F(x)x - bxd(x)$. Replacing y by y^*x^* in (30), we get $\phi(x)xyx = 0$. Now replacing y by $z\phi(x)$, we get $\phi(x)xz\phi(x)x = 0$, for all $x, z \in R$. Using the semiprimeness of R, we obtain

$$\phi(x)x = 0, \quad \text{for all} \quad x \in R. \tag{31}$$

Taking $x = x + y$, we get

$$\phi(x)y + \beta(x, y)x + \phi(y)x + \beta(x, y)y = 0, \quad \text{for all} \quad x, y \in R, \tag{32}$$

where $\beta(x, y) = F(xy + yx) - F(x)y - F(y)x - bxd(y) - byd(x)$. Replacing x by $-x$ in (32) and making use of (32), we get

$$2(\phi(x)y + \beta(x, y)x) = 0, \quad \text{for all} \quad x, y \in R.$$

Since R is 2-torsion free, we arrive at

$$\phi(x)y + \beta(x, y) = 0, \quad \text{for all} \quad x, y \in R. \tag{33}$$

Multiplying (33) by $\phi(x)$ on the right side, we get

$$\phi(x)y\phi(x) + \beta(x, y)x\phi(x) = 0, \quad \text{for all} \quad x, y \in R. \tag{34}$$

Taking $y = y^*$ in (30), we get $\phi(x)yx = 0$, for all $x, y \in R$. This further implies that $x\phi(x)yx\phi(x) = 0$, for all $x, y \in R$. Thus by the semiprimeness of R, we get $x\phi(x) = 0$, for all $x \in R$. Using this in (34), we obtain $\phi(x) = 0$, for all $x \in R$. Hence $F(x^2) = F(x)x + bxd(x)$, for all $x \in R$. Now, following on similar lines as after (18), we get the required result.

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Quasi-metric hyper dynamical systems

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Abstract. We start this paper by introducing the concept of quasi-metric hypergroups. We show that the product of two quasi-metric hypergroups is a quasi-metric hypergroup. Quasi-metric hyperdynamical systems are defined, and a method for constructing quasi-metric hyperdynamical systems via two given quasi-metric hyperdynamical systems, is deduced. Attracting sets for quasi-metric hyperdynamical systems are considered. A method for constructing quasi-metric hyperdynamical systems with attracting sets via two given quasi-metric hyperdynamical systems with attracting sets, is presented.
Keywords: quasi-metric hypergroup, quasi-metric hyperdynamical system, time hypergroup, attracting set.

1. Introduction

The notion of quasi-metric spaces has been studied first by Stolenberg as an extension of the metric spaces [7, 8]. Quasi-metric spaces are more compatible with some of realistic structures. In fact symmetry is an ideal property for soft computing, but in the realistic case we do not have such property, for example according to traffic rules the time for going from point a to point b in a city is not equal to the time for going from b to a. In section three we introduce quasi-metric hypergroups by adding a kind of continuity to the join operation via a

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quasi-metric on it, and we show that there is a quasi-metric on the product of two quasi-metric hypergroups to make it a quasi-metric hypergroup. In section four we define quasi-metric hyperdynamical systems via two conditions. The first condition makes it an evolution operator and the second condition put continuity on its time evolution. We show that the product of two quasi-metric hyperdynamical systems is a quasi-metric hyperdynamical system. In section five we consider attracting sets for a quasi-metric hyperdynamical system as an invariant set of it which attracts any bounded set which is near it. We show that if two given quasi-metric hyperdynamical systems have attracting sets, then their product has an attracting set.

2. Basic notions

According to Stolenberg definition [7] if M is a non-empty set, then a function $q: M \times M \longrightarrow [0, \infty)$ is called a quasi-metric if it satisfies the following conditions.

(1) $q(x, y) = 0 \Leftrightarrow x = y$.

(2) $q(x, z) \leq q(x, y) + q(y, z)$, for all $x, y, z \in M$

In this case (M, q) is called a quasi-metric space or a q-metric space.

The topology $\tau(q)$ induced by a quasi-metric q on M is the topology determined by the basis consisting of all r-balls $B^q_r(p)= \{m \in M\colon \quad q(p, m) < r \}$ where $p \in M$ and $r \in [0, \infty)$ (see, [7, 6]).

A join operation on the nonempty set H ([1, 2]) is a mapping from $H \times H$ to the set $P_*(H)$ which is the set of all nonempty subsets of H. If $x, y \in H$, then we denote their joins by xy. A join operation on H creates an operation from $P_*(H) \times P_*(H)$ to $P_*(H)$ by $(X, Y) \mapsto XY$, where $XY = \bigcup_{(x,y) \in X \times Y} xy$. For simplicity $\{x\}Y$ and $Y\{x\}$ are denoted by xY and Yx respectively.

H with a join operation is called a hypergroup ([4]) if for all $x, y, z \in H$ we have $x(yz) = (xy)z$ and $xH = Hx = H$.

3. Quasi-metric hypergroups

We assume H is a hypergroup and q is a quasi-metric on it.

Definition 3.1. (H, q) *is said to be a quasi-metric hypergroup or a q-hypergroup if for given* $x, y \in H$ *and for all* $r > 0$ *and* $z \in xy$*, there is* $d > 0$ *such that* $hs \cap B^q_r(z) \neq \emptyset$*, for all* $h \in B^q_d(x)$ *and* $s \in B^q_d(y)$.

Now, we show that the product of two quasi-metric hypergroups is a quasi-metric hypergroup.

Theorem 3.1. *If* (H_1, q_1) *and* (H_2, q_2) *are two quasi-metric hypergroups, then* $(H_1 \times H_2, q)$ *is a quasi-metric hypergroup, where* q *is the following map:*

$$q : (H_1 \times H_2, q) \longrightarrow [0, \infty),$$
$$((h_1, h_2), (s_1, s_2)) \longmapsto q_1(h_1, s_1) + q_2(h_2, s_2),$$

and the join operation of $H_1 \times H_2$ is the following operation:

$$\begin{aligned}&. : (H_1 \times H_2) \times (H_1 \times H_2) \longrightarrow P_*(H_1 \times H_2),\\&((h_1, h_2), (s_1, s_2)) \longmapsto (h_1 s_1) \times (h_2 s_2).\end{aligned}$$

Proof. We first show that $(H_1 \times H_2, q)$ is a quasi-metric space. We see that:

$$\begin{aligned}&q((h_1, h_2), (s_1, s_2)) = 0 \Leftrightarrow\\&q_1(h_1, s_1) + q_2(h_2, s_2) = 0 \Leftrightarrow\\&q_1(h_1, s_1) = 0 \quad \text{and} \quad q_2(h_2, s_2) = 0 \Leftrightarrow\\&h_1 = s_1 \quad \text{and} \quad h_2 = s_2.\end{aligned}$$

For $(h_1, h_2), (t_1, t_2), (s_1, s_2) \in H_1 \times H_2$ we have:

$$\begin{aligned}&q((h_1, h_2), (s_1, s_2)) = q_1(h_1, s_1) + q_2(h_2, s_2)\\&\le q_1(h_1, t_1) + q_1(t_1, s_1) + q_2(h_2, t_2) + q_2(t_2, s_2)\\&\le q((h_1, h_2), (t_1, t_2)) + q((t_1, t_2), (s_1, s_2)).\end{aligned}$$

Thus, $(H_1 \times H_2, q)$ is a quasi-metric space. For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in H_1 \times H_2$ we have:

$$\begin{aligned}&(x_1, x_2)((y_1, y_2)(z_1, z_2)) = (x_1(y_1 z_1), x_2(y_2 z_2))\\&= ((x_1 y_1) z_1, (x_2 y_2) z_2) = ((x_1, x_2)(y_1, y_2))(z_1, z_2),\end{aligned}$$

and

$$(x_1, x_2) H_1 \times H_2 = x_1 H_1 \times x_2 H_2 = H_1 \times H_2 = H_1 x_1 \times H_2 x_2 = H_1 \times H_2(x_1, x_2).$$

Hence $H_1 \times H_2$ is a hypergroup.

Let $(x_1, x_2), (y_1, y_2) \in H_1 \times H_2$, $r > 0$, and $(z_1, z_2) \in (x_1, x_2)(y_1, y_2)$ be given. Since $z_1 \in x_1 y_1$ and $z_2 \in x_2 y_2$, then there exist $d_1 > 0$ and $d_2 > 0$ such that $h_1 s_1 \cap B^{q_1}_{\frac{r}{2}}(z_1) \neq \emptyset$ and $h_2 s_2 \cap B^{q_2}_{\frac{r}{2}}(z_2) \neq \emptyset$, for all $h_1 \in B^{q_1}_{d_1}(x_1)$, $s_1 \in B^{q_1}_{d_1}(y_1)$, $h_2 \in B^{q_2}_{d_2}(x_2)$, and $s_2 \in B^{q_2}_{d_2}(y_2)$. The definition of q implies that $B^{q_1}_{\frac{r}{2}}(z_1) \times B^{q_2}_{\frac{r}{2}}(z_2) \subseteq B^q_r(z_1, z_2)$, so, for all $(h_1, h_2) \in B^{q_1}_{d_1}(x_1) \times B^{q_2}_{d_2}(x_2)$ and $(s_1, s_2) \in B^{q_1}_{d_1}(y_1) \times B^{q_2}_{d_2}(y_2)$, we have $(h_1 s_1 \times h_2 s_2) \cap B^q_r(z_1, z_2) \neq \emptyset$. If we take $d = \min\{d_1, d_2\}$ then $B^q_d(x_1, x_2) \subseteq B^{q_1}_{d_1}(x_1) \times B^{q_2}_{d_2}(x_2)$ and $B^q_d(y_1, y_2) \subseteq B^{q_1}_{d_1}(y_1) \times B^{q_2}_{d_2}(y_2)$. Thus, for all $(h_1, h_2) \in B^q_d(x_1, x_2)$ and $(s_1, s_2) \in B^q_d(y_1, y_2)$ we have $(h_1 s_1 \times h_2 s_2) \cap B^q_r(z_1, z_2) \neq \emptyset$. Hence $(H_1 \times H_2, q)$ is a quasi-metric hypergroup. □

4. Quasi-metric hyperdynamical systems

Let (M, q_1) be a q-metric space and let (H, q_2) be a q-hypergroup. Moreover, let $\varphi : H \times M \longrightarrow M$ be a mapping. With these assumptions we have the next definition.

Definition 4.1. *The* 5*-tuple* $(M, q_1, \varphi, H, q_2)$ *is said to be a quasi-metric hyper-dynamical system, if it satisfies the following conditions:*

(*i*) *If* $h_1, h_2 \in H$ *and* $m \in M$*, then* $\varphi(h_1, \varphi(h_2, m)) \in \varphi(h_1 h_2, m)$*, where* $\varphi(h_1 h_2, m) = \{\varphi(h, m) : \quad h \in h_1 h_2\ \}$;

(*ii*) $\varphi : H \times M \longrightarrow M$ *is a* (q_2, q_1) *continuous map i.e., for all* $V \in \tau_{q_1}$*, there exist* $W \in \tau_{q_2}$ *and* $Z \in \tau_{q_1}$ *such that* $\varphi(W \times Z) \subseteq V$*.*

Now, we assume that $(M, q_1, \varphi, H, d_1)$ and (N, q_2, ψ, S, d_2) are two quasi-metric hyperdynamical systems. We take quasi-metrics q and d on $M \times N$ and $H \times S$ as in Theorem 3.1 respectively. Moreover, we take the join operation on $H \times S$ as in Theorem 3.1. With these assumptions we have the next theorem.

Theorem 4.1. $(M \times N, q, \varphi \times \psi, H \times S, d)$ *is a quasi-metric hyperdynamical system.*

Proof. If $(h_1, s_1),\ (h_2, s_2) \in H \times S$ and $(m, n) \in M \times N$, then

$$\begin{aligned}
&(\varphi \times \psi)((h_1, s_1), (\varphi \times \psi)(h_2, s_2), (m, n)) = (\varphi \times \psi)((h_1, s_1), (\varphi(h_2, m), \psi(s_2, n))) \\
&= (\varphi(h_1, \varphi(h_2, m)), \psi(s_1, \psi(s_2, n))) \in \varphi(h_1 h_2, m) \times \psi(s_1 s_2, n) \\
&= (\varphi \times \psi)(h_1 h_2 \times s_1 s_2, (m, n)).
\end{aligned}$$

Let the nonempty set $V \in \tau_q$ be given. Then, there is $(m, n) \in V$ and $r > 0$ such that $B_r^q(m, n) \subseteq V$. Since $B_{\frac{r}{2}}^{q_1}(m) \in \tau_{q_1}$ and $B_{\frac{r}{2}}^{q_2}(n) \in \tau_{q_2}$, then there exist $W_1 \in \tau_{d_1}$, $Z_1 \in \tau_{q_1}$, $W_2 \in \tau_{d_2}$, and $Z_2 \in \tau_{q_2}$ such that $\varphi(W_1, Z_1) \subseteq B_{\frac{r}{2}}^{q_1}(m)$ and $\psi(W_2, Z_2) \subseteq B_{\frac{r}{2}}^{q_2}(n)$. Hence we have

$$\begin{aligned}
(\varphi \times \psi)(W_1 \times W_2, Z_1 \times Z_2) &= \varphi(W_1, Z_1) \times \psi(W_2, Z\psi_2) \\
&\subseteq B_{\frac{r}{2}}^{q_1}(m) \times B_{\frac{r}{2}}^{q_2}(n) \subseteq B_r^q(m, n) \subseteq V.
\end{aligned}$$

Thus, $\varphi \times \psi$ is a (d, q) continuous map. □

5. Attracting sets

We begin this section by recalling the definition of time hypergroup (see, [5]). A hypergroup H is called a time hypergroup with zero time $e \in H$ ([5]) if $(H, <)$ is a partially ordered hypergroup with the following two properties:

(1) $h \in he$ and $h \in eh$, for all $h \in H$;

(2) If $h \in H$, then $h < e$ or $h > e$ or $h = e$.

In this section we assume that H is a time hypergroup, and $(M, q_1, \varphi, H, q_2)$ is a quasi-metric hyperdynamical system.

A subset A of M is called an invariant set for $(M, q_1, \varphi, H, q_2)$ if $\varphi(H, A) \subseteq A$.

If $A, B \subseteq M$ the distance of A to B is defined by

$$\operatorname{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} q_1(a, b).$$

Definition 5.1. *A non-empty invariant subset A is said to be an attracting set for $(M, q_1, \varphi, H, q_2)$ if there is an open subset B of M containing A such that for each bounded subset C of B we have:* $\lim_{n\to\infty} \operatorname{dist}(\varphi(h^n, C), A) = 0$ *for each $h > e$.*

We see that any attracting set of $(M, q_1, \varphi, H, q_2)$ attracts any bounded subset which is near it, under the iterations of positive times.

Suppose H and S are two time hypergroups with the zero times e_1 and e_2 and partial orders $<_1$ and $<_2$ respectively. On $H \times S$ we define a partial order $<$ by: $(h, s) < (e_1, e_2)$ if $h <_1 e_1$, and $s <_2 e_2$; $(e_1, e_2) < (r, t)$ if $e_1 <_1 r$, or $e_2 <_2 t$; $(h, s) < (r, t)$ if $(h, s) < (e_1, e_2) < (r, t)$.

We see that $H \times S$ with this partial order is a time hypergroup. Now, we also assume that $(M, q_1, \varphi, H, d_1)$ and (N, q_2, ψ, S, d_2) are two quasi-metric hyperdynamical systems with attracting sets A_1 and A_2 respectively. Moreover, we assume that $(M \times N, q, \varphi \times \psi, H \times S, d)$ is the quasi-metric hyperdynamical system which is constructed via the assumptions of Theorem 4.1. With these assumptions we have the next theorem.

Theorem 5.1. *$A_1 \times A_2$ is an attracting set for $(M \times N, q, \varphi \times \psi, H \times S, d)$.*

Proof. There exist open sets U and V in M and N corresponding to $(M, q_1, \varphi, H, d_1)$ and (N, q_2, ψ, S, d_2) for A_1 and A_2 respectively. We show that the open set $U \times V$ satisfies the condition of Definition 5.1 for $A_1 \times A_2$. If C is a bounded set in $U \times V$, then it means there is $r > 0$, and $(m, n) \in M \times N$ such that $C \subseteq B_r^q(m, n)$. Thus, $C \subseteq B_r^{q_1}(m) \times B_r^{q_2}(n)$. Hence $C \subseteq C_1 \times C_2$, where $C_1 = B_r^{q_1}(m) \cap U$ and $C_2 = B_r^{q_2}(n) \cap V$. If $h = (h_1, h_2) > (e_1, e_2)$, then $h_1 > e_1$ and $h_2 > e_2$, and we have:

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{dist}((\varphi \times \psi)(h^n, C), A_1 \times A_2) \leq \lim_{n\to\infty} \operatorname{dist}((\varphi \times \psi)(h^n, C_1 \times C_2), A_1 \times A_2)\\
&= \lim_{n\to\infty} \operatorname{dist}(\varphi(h_1^n, C_1) \times \psi(h_2^n, C_2), A_1 \times A_2)\\
&= \lim_{n\to\infty} \operatorname{dist}(\varphi(h_1^n, C_1), A_1) + \lim_{n\to\infty} \operatorname{dist}(\psi(h_2^n, C_2), A_2) = 0.
\end{aligned}$$

Thus, the invariant set $A_1 \times A_2$ is an attracting set for $(M \times N, q, \varphi \times \psi, H \times S, d)$. □

Conclusion

We have considered quasi-metric hyperdynamical systems as an extension of metric dynamical systems. In fact we extend the phase spaces to quasi-metric spaces and we also extend the time sets to hypergroups. By using of time hypergroup we have considered attracting sets for the quasi-metric hyperdynamical systems. If H and S are two times hypergroups, then we have introduced a partial order on $H \times S$ which make it a time hypergroup and able us to construct new attracting sets via the product of their quasi-metric hyperdynamical systems. One must pay attention to this point that we can define another partial order on $H \times S$ which make it a time hypergroup, but Theorem 5.1 is not valid for it. For example by refer to [5] one can see the partial order $<$ on $H \times S$ which has been defined by: $(h_1, s_1) < (h_2, s_2)$ if $h_1 <_1 h_2$, and when $h_1 = h_2$ then $s_1 <_2 s_2$, where $<_1$ and $<_2$ are partial orders on H and S respectively. With this partial order Theorem 5.1 is not valid.

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Efficient block approach for the numerical integration of higher-order ordinary differential equations with initial values

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Abstract. This paper considered an innovative procedure to numerically approximate higher–order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs). The proposed method is a one-step, self-starting Block integrator method employed to approximate higherorder (Third, Fourth, and Fifth-order) IVPs without reduction to lower order. The method was developed through collocation and interpolation approach. The basic properties of the method such as convergence, consistency, zero stability, order and error constant are well investigated. The accuracy of the method over existing methods are validated by numeral experiments. The method produces more interesting and superior results when compared to some existing numerical methods in terms of accuracy and absolute errors.

Keywords: accuracy, block method, collocation and interpolation, higher-order ODEs, Initial Value Problems (IVPs).

1. Introduction

The pursuit of translating various scientific, engineering, modeling and real life problems to differential equations which can be ordinary differential equations or partial differential equations in nature has given rise to the development of several numerical methods to provide an approximate solution to resulting higher order differential equations coupled with its initial or boundary conditions that may be rigorous to solve or has no analytical solutions. Researchers have suggested methods of solving higher order ODEs directly (Jena and Mohanty 2019, Yap and Ismail 2015, Waeleh et al., 2011, Suleiman et al., 2011). Oftentimes, developing numerical methods has been to convalesce the efficiency

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and convergence of the method. Adeyefa & Kuboye (2020)

$$
\begin{aligned}
y''(x) &= f(x, y, y', y''), y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2,\\
y'^{v}(x) &= f(x, y, y', y'', y'''), y(x_0) = y_0, y'(x_0) = y_1, y''(x_0)\\
&= y_2, y'''(x_0) = y_3,\\
y^{v}(x) &= f(x, y, y', y'', y''', y'^{v}), y(x_0) = y_0, y'(x_0) = y_1, y'(x_0)\\
&= y_2, y'''(x_0) = y_3, y'^{v}(x_0) = y_4.
\end{aligned} \tag{1}
$$

In literature, researchers have proposed block methods for the direct integration of (1) without necessarily reducing it to system of first order ODEs (Khataybeh et al., 2019, Adoghe et al., 2016, Agboola et al., 2015, Kuboye and Omar 2015a, Kuboye and Omar 2015b, Hussain et al. 2015). These authors developed different method to handle various higher-order of ODEs with focus only on p^{th} order of equation. Adeyefa & Kuboye (2020), developed a method that is capable of handling two different orders of ODEs using the block approach. The developed method was capable of solving second and third-order ODEs. Our interest in this work is to develop a method that will be capable of handling three different systems of p^{th}, $(p+1)^{th}$ and $(p+2)^{th}$ orders IVPs for direct solution of ODEs where $p = 3$.

2. The method

We consider the formulation of our proposed method for p^{th}, $(p+1)^{th}$ and

$$
y(x) = \sum_{j=0}^{k+9} a_j x^j \tag{2}
$$

$(p+2)^{th}$ by adopting a power series of the form: for $k = 1$, as the approximate solution of (1), interpolating (2) at $x = x_{n+\tau}, \tau = \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and collocating the third, fourth, and fifth derivative of (2) at $x = x_{n+\varsigma}, \varsigma = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, x = x_{n+\varepsilon}, \varepsilon = 0$ and $x = x_{n+\varpi}, \varpi = 1$, respectively. This subsequently resulted into systems of linear equations of the form:

$$
\begin{aligned}
&\sum_{j=0}^{k+9} a_j x^{n+\tau} = y_{n+\tau}, \quad \sum_{j=2}^{k+9} j(j-1)(j-2) a_j x_{n+\varsigma}^{j-2} = f_{n+\varsigma},\\
&\sum_{j=3}^{k+9} j(j-1)(j-2)(j-3) a_j x_{n+\varepsilon}^{j-5} = g_{n+\varepsilon},\\
&\sum_{j=4}^{k+9} j(j-1)(j-2)(j-3)(j-4) a_j x_{n+\omega}^{j-5} = m_{n+\omega},
\end{aligned} \tag{3}
$$

where f, g and m are the third, fourth, and fifth derivatives of (2) respectively. Gaussian's elimination method is applied to find the values of as' in (3) and then substituted into (2) to produce a continuous implicit method:

$$
\begin{aligned}
(4)\quad &\alpha_{\frac{2}{5}}(t)y_{n\frac{2}{5}} + \alpha\frac{3}{5}(t)y_{n+\frac{3}{5}} + \alpha_{\frac{4}{5}}(t)y_{n+\frac{4}{5}} = h^3(\beta_0(t)f_n + \beta_{\frac{1}{5}}(t)f_{n+\frac{1}{5}} + \beta_{\frac{2}{5}}(t)f_{n+\frac{2}{5}} \\
&+\beta_{\frac{3}{5}}(t)f_{n+\frac{3}{5}} + \beta_{\frac{4}{5}}(t)f_{n+\frac{4}{5}} + \beta_1(t)f_{n+1}) + h^4(\gamma_0(t)g_n) + h^5(\omega_{\frac{1}{5}}(t)m_{n+\frac{1}{5}}),
\end{aligned}
$$

for $t = \frac{x-x_n}{h}$, $\begin{pmatrix} \alpha_{\frac{2}{5}} \\ \alpha_{\frac{3}{5}} \\ \alpha_{\frac{4}{5}} \end{pmatrix}(t) = [\Phi]\begin{pmatrix} t^0 \\ t^1 \\ t^2 \end{pmatrix}$,

$$
\begin{pmatrix} \beta_0 \\ \beta_{\frac{1}{5}} \\ \beta_{\frac{2}{5}} \\ \beta_{\frac{3}{5}} \\ \beta_{\frac{4}{5}} \\ \beta_1 \end{pmatrix}(t)=h^3[\gamma]\begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \end{pmatrix}(\gamma_0)(t)=h^4[\Psi]\begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \end{pmatrix}, \left(\omega_{\frac{1}{5}}\right)(t)=h^5[\Pi]\begin{pmatrix} t^0 \\ t^1 \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ t^6 \\ t^7 \\ t^8 \\ t^9 \\ t^{10} \end{pmatrix}
$$

where

$$
\gamma = \begin{pmatrix}
\frac{499699}{37800000} & -\frac{8987}{315000} & -\frac{2099}{630000} & -\frac{457}{33750} & \frac{559}{2520000} & \frac{61}{1575000} \\
-\frac{20688119}{453600000} & \frac{99251}{756000} & \frac{49547}{1512000} & \frac{31447}{567000} & \frac{4327}{6048000} & \frac{383}{2700000} \\
\frac{90163}{90163} & -\frac{15367}{43200} & \frac{4397}{120960} & -\frac{15917}{226800} & \frac{1067}{345600} & -\frac{107}{302400} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{208879}{8640} & \frac{1075}{24} & -\frac{575}{24} & \frac{425}{108} & -\frac{125}{192} & \frac{7}{120} \\
\frac{924719}{6912} & -\frac{924719}{576} & \frac{78025}{576} & -\frac{19075}{864} & \frac{8375}{2304} & -\frac{187}{576} \\
-\frac{1872425}{6048} & \frac{1170875}{2016} & -\frac{22625}{72} & \frac{152875}{3024} & -\frac{2375}{288} & \frac{1475}{2016} \\
\frac{2194375}{6048} & \frac{3659375}{5376} & \frac{246875}{672} & -\frac{1403125}{24192} & \frac{3125}{336} & -\frac{625}{768} \\
-\frac{3855625}{18144} & \frac{9640625}{24192} & -\frac{1296875}{6048} & \frac{171875}{5184} & -\frac{15625}{3024} & \frac{10625}{24192} \\
\frac{2050625}{41472} & -\frac{640625}{6912} & \frac{171875}{3456} & -\frac{78125}{10368} & \frac{15625}{15625} & -\frac{625}{6912}
\end{pmatrix},
$$

$$\Psi = \begin{pmatrix} \frac{281}{450000} \\ -\frac{11941}{5400000} \\ \frac{431}{86400} \\ 0 \\ \frac{1}{24} \\ -\frac{917}{720} \\ \frac{3787}{576} \\ -\frac{1075}{72} \\ \frac{625}{36} \\ -\frac{4375}{432} \\ \frac{8125}{3456} \end{pmatrix}, \Pi = \begin{pmatrix} -\frac{43750}{13} \\ \frac{39}{35000} \\ -\frac{373}{126000} \\ 0 \\ 0 \\ \frac{1}{2} \\ -\frac{137}{48} \\ \frac{375}{56} \\ -\frac{10625}{1344} \\ \frac{3125}{672} \\ -\frac{625}{576} \end{pmatrix}$$

and

$$\Phi = \begin{pmatrix} 3 & -8 & 6 \\ -\frac{25}{2} & 30 & \frac{35}{1} \\ \frac{25}{2} & -25 & \frac{25}{2} \end{pmatrix}.$$

Evaluating (4) at $x = x_{n+\eta}, \eta = 1, \frac{1}{5}, 0$ to produce a discrete scheme of the form:

$$\begin{pmatrix} y_{n+1} \\ y_{n+\frac{1}{5}} \\ y_n \end{pmatrix} - \begin{pmatrix} 3 & 1 & 3 \\ -3 & -3 & -8 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} y_{n+\frac{4}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{2}{5}} \end{pmatrix}$$

$$= h^3[A] \begin{pmatrix} f_{n+1} \\ f_{n+\frac{4}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{2}{5}} \\ f_n \end{pmatrix} + h^4 \begin{pmatrix} \frac{301}{900000} \\ \frac{31}{100000} \\ \frac{281}{450000} \end{pmatrix} (g_n) + h^5 \begin{pmatrix} -\frac{167}{1050000} \\ -\frac{31}{210000} \\ -\frac{13}{43750} \end{pmatrix} \left(m_{n+\frac{1}{5}}\right) \tag{5}$$

for

$$A = \begin{pmatrix} \frac{3546}{75600000} & -\frac{482}{25200000} & -\frac{1464}{37800000} \\ \frac{293745}{75600000} & \frac{3585}{25200000} & \frac{8385}{37800000} \\ \frac{259420}{75600000} & -\frac{123340}{25200000} & -\frac{511840}{37800000} \\ \frac{509220}{75600000} & \frac{62660}{25200000} & -\frac{125940}{37800000} \\ -\frac{999030}{75600000} & -\frac{310290}{25200000} & -\frac{107844}{37800000} \\ \frac{537899}{75600000} & \frac{166267}{25200000} & \frac{499699}{37800000} \end{pmatrix}.$$

The first two derivative of equation (4) yields equations (6) and (7) respectively.

$$\begin{aligned}&\alpha'_{\frac{2}{5}}(t)y_{n+\frac{2}{5}}+\alpha'_{\frac{3}{5}}(t)y_{n+\frac{3}{5}}+\alpha'_{\frac{4}{5}}(t)=h^2(\beta'_0(t)f_n+\beta'_{\frac{1}{5}}(t)f_{n+\frac{1}{5}}+\beta'_{\frac{2}{5}}(t)f_{\frac{2}{5}}\\&+\beta'_{\frac{3}{5}}(t)f_{n+\frac{3}{5}}+\beta'_{\frac{4}{5}}(t)f_{n+\frac{4}{5}}+\beta'_1 f_{n+1})+h^3(\gamma_0(t)g_n)+h^4(\omega'_{\frac{1}{5}}(t)m_{n+\frac{1}{5}})\end{aligned}\tag{6}$$

$$\begin{aligned}&\alpha''_{\frac{2}{5}}(t)y_{n+\frac{2}{5}}+\alpha''_{\frac{3}{5}}(t)y_{n+\frac{3}{5}}+\alpha''_{\frac{4}{5}}(t)=h(\beta''_0(t)f_n+\beta''_{\frac{1}{5}}(t)f_{n+\frac{1}{5}}+\beta''_{\frac{2}{5}}(t)f_{\frac{2}{5}}\\&+\beta''_{\frac{3}{5}}(t)f_{n+\frac{3}{5}}+\beta''_{\frac{4}{5}}(t)f_{n+\frac{4}{5}}+\beta''_1 f_{n+1})+h^2(\gamma_0(t)g_n)+h^3(\omega''_{\frac{1}{5}}(t)m_{n+\frac{1}{5}})\end{aligned}\tag{7}$$

Evaluating equations (6) and (7) at $x = x_{n+\theta}, \theta = 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0$ and $x = x_{n+\psi}, \psi = 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0$ to produce the derivatives as:

$$\begin{aligned}&\begin{pmatrix} y'_{n+1} \\ y'_{n+\frac{4}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{1}{5}} \\ y'_n \end{pmatrix} - \frac{1}{h}[B]\begin{pmatrix} y_{n+\frac{4}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{2}{5}} \end{pmatrix} = h^2[C]\begin{pmatrix} f_{n+1} \\ f_{n+\frac{4}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{2}{5}} \\ f_n \end{pmatrix}\\ &+h^3\begin{pmatrix} -\frac{3040044}{453600000} \\ -\frac{32768}{151200000} \\ -\frac{104412}{453600000} \\ -\frac{332556}{453600000} \\ -\frac{238980}{151200000} \\ -\frac{1003044}{453600000} \end{pmatrix}(g_n) + h^4\begin{pmatrix} \frac{1419840}{453600000} \\ -\frac{154464}{151200000} \\ \frac{48816}{453600000} \\ \frac{159264}{453600000} \\ \frac{112224}{151200000} \\ \frac{505440}{453600000} \end{pmatrix}\begin{pmatrix} m_{n+\frac{1}{5}} \end{pmatrix}\end{aligned}\tag{8}$$

where

$$B = \begin{pmatrix} \frac{5670000000}{453600000} & \frac{113400000}{151200000} & \frac{1134000000}{453600000} & -\frac{1134000000}{453600000} & -\frac{1134000000}{151200000} & -\frac{5670000000}{453600000} \\ -\frac{9072000000}{453600000} & -\frac{1512000000}{151200000} & 0 & \frac{4536000000}{453600000} & \frac{3024000000}{151200000} & \frac{13608000000}{453600000} \\ \frac{3402000000}{453600000} & \frac{37800000}{151200000} & -\frac{1134000000}{453600000} & -\frac{3402000000}{453600000} & -\frac{189000000}{151200000} & -\frac{7938000000}{453600000} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1280844}{453600000} & -\frac{31384}{151200000} & \frac{11712}{453600000} & \frac{1965}{453600000} & \frac{1478}{151200000} & \frac{64344}{453600000} \\ \frac{17710725}{453600000} & \frac{613125}{151200000} & -\frac{229575}{453600000} & -\frac{135375}{453600000} & \frac{85425}{151200000} & -\frac{324525}{453600000} \\ \frac{2211460}{453600000} & \frac{692000}{151200000} & -\frac{235120}{453600000} & \frac{528400}{453600000} & -\frac{5128200}{151200000} & \frac{25157600}{453600000} \\ -\frac{63855900}{453600000} & \frac{6749500}{151200000} & -\frac{2390700}{453600000} & -\frac{5080500}{453600000} & \frac{118270}{151200000} & \frac{14864100}{453600000} \\ \frac{120281100}{453600000} & \frac{12957000}{151200000} & \frac{4144800}{453600000} & \frac{13047000}{453600000} & \frac{9920700}{151200000} & \frac{59550600}{453600000} \\ -\frac{64267369}{453600000} & \frac{694975}{151200000} & -\frac{220903}{453600000} & -\frac{7086781}{453600000} & -\frac{5072955}{151200000} & -\frac{20688119}{453600000} \end{pmatrix}$$

$$\begin{pmatrix} y''_{n+1} \\ y''_{n+\frac{4}{5}} \\ y''_{n+\frac{3}{5}} \\ y''_{n+\frac{2}{5}} \\ y''_{n+\frac{1}{5}} \\ y''_{n} \end{pmatrix} - \frac{1}{h^2}[D] \begin{pmatrix} y_{n+\frac{4}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{2}{5}} \end{pmatrix} = h[E] \begin{pmatrix} f_{n+1} \\ f_{n+\frac{4}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n} \end{pmatrix}$$

$$+ h^2 \begin{pmatrix} -\frac{146874}{9072000} \\ \frac{697116}{18144000} \\ -\frac{89796}{9072000} \\ \frac{253596}{18144000} \\ -\frac{49476}{9072000} \\ \frac{181020}{18144000} \end{pmatrix} (g_n) + h^3 \begin{pmatrix} \frac{688788}{9072000} \\ -\frac{32860}{18144000} \\ \frac{42516}{9072000} \\ -\frac{12124}{18144000} \\ \frac{25236}{9072000} \\ -\frac{107424}{18144000} \end{pmatrix} \left(m_{n+\frac{1}{5}} \right) \tag{9}$$

where

$$D = \begin{pmatrix} \frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000} \\ -\frac{453600000}{9072000} & -\frac{907200000}{18144000} & -\frac{453600000}{9072000} & -\frac{907200000}{18144000} & -\frac{453600000}{9072000} & -\frac{907200000}{18144000} \\ \frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000} \end{pmatrix}$$

$$E = \begin{pmatrix} \frac{604455}{9072000} & -\frac{68136}{18144000} & \frac{7431}{9072000} & -\frac{16296}{18144000} & \frac{3111}{9072000} & -\frac{12840}{18144000} \\ \frac{1954455}{9072000} & \frac{1701795}{18144000} & -\frac{106185}{9072000} & \frac{150915}{18144000} & -\frac{50025}{9072000} & \frac{112035}{18144000} \\ \frac{4632890}{9072000} & \frac{463840}{18144000} & \frac{217850}{9072000} & -\frac{2788640}{18144000} & -\frac{635590}{9072000} & -\frac{2546720}{18144000} \\ -\frac{30561900}{9072000} & \frac{14308260}{18144000} & -\frac{1756140}{9072000} & \frac{3577380}{18144000} & -\frac{3190380}{9072000} & \frac{1313700}{18144000} \\ \frac{58092735}{9072000} & -\frac{27561720}{18144000} & \frac{3544095}{9072000} & -\frac{9953400}{18144000} & \frac{1310655}{9072000} & -\frac{12908280}{18144000} \\ -\frac{31093835}{9072000} & \frac{14784761}{18144000} & -\frac{1907051}{9072000} & \frac{5401241}{18144000} & -\frac{1066571}{9072000} & \frac{3155705}{18144000} \end{pmatrix}.$$

The fusing of equations (5), (8) and (9) in matrix form by matrix inversion gives a block method written explicitly as

$$
\begin{aligned}
I\begin{bmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \end{bmatrix} &= L\begin{bmatrix} y_{n-\frac{1}{5}} \\ y_{n-\frac{2}{5}} \\ y_{n-\frac{3}{5}} \\ y_{n-\frac{4}{5}} \\ y_{n} \end{bmatrix} + h(K)\begin{bmatrix} y'_{n-\frac{1}{5}} \\ y'_{n-\frac{2}{5}} \\ y'_{n-\frac{3}{5}} \\ y'_{n-\frac{4}{5}} \\ y'_{n} \end{bmatrix} + h^2(R)\begin{bmatrix} y''_{n-\frac{1}{5}} \\ y''_{n-\frac{2}{5}} \\ y''_{n-\frac{3}{5}} \\ y''_{n-\frac{4}{5}} \\ y''_{n} \end{bmatrix} \\
&+ h^3(P)\begin{bmatrix} f_{n-\frac{1}{5}} \\ f_{n-\frac{2}{5}} \\ f_{n-\frac{3}{5}} \\ f_{n-\frac{4}{5}} \\ f_{n} \end{bmatrix} + h^3(V)\begin{bmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} + h^4(T)\begin{bmatrix} g_{n-\frac{1}{5}} \\ g_{n-\frac{2}{5}} \\ g_{n-\frac{3}{5}} \\ g_{n-\frac{4}{5}} \\ g_{n} \end{bmatrix} + h^5(Z)\begin{bmatrix} m_{n-\frac{1}{5}} \\ m_{n-\frac{2}{5}} \\ m_{n-\frac{3}{5}} \\ m_{n-\frac{4}{5}} \\ m_{n} \end{bmatrix}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
I\begin{bmatrix} y'_{n+\frac{1}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{4}{5}} \\ y'_{n+1} \end{bmatrix} &= L\begin{bmatrix} y'_{n-\frac{1}{5}} \\ y'_{n-\frac{2}{5}} \\ y'_{n-\frac{3}{5}} \\ y'_{n-\frac{4}{5}} \\ y'_{n} \end{bmatrix} + h(K)\begin{bmatrix} y''_{n-\frac{1}{5}} \\ y''_{n-\frac{2}{5}} \\ y''_{n-\frac{3}{5}} \\ y''_{n-\frac{4}{5}} \\ y''_{n} \end{bmatrix} + h^2(\bar{N})\begin{bmatrix} f_{n-\frac{1}{5}} \\ f_{n-\frac{2}{5}} \\ f_{n-\frac{3}{5}} \\ f_{n-\frac{4}{5}} \\ f_{n} \end{bmatrix} \\
&+ h^2(\bar{W})\begin{bmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} + h^3(\bar{D})\begin{bmatrix} g_{n-\frac{1}{5}} \\ g_{n-\frac{2}{5}} \\ g_{n-\frac{3}{5}} \\ g_{n-\frac{4}{5}} \\ g_{n} \end{bmatrix} + h^4(\bar{E})\begin{bmatrix} m_{n-\frac{1}{5}} \\ m_{n-\frac{2}{5}} \\ m_{n-\frac{3}{5}} \\ m_{n-\frac{4}{5}} \\ m_{n} \end{bmatrix}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
I\begin{bmatrix} y''_{n+\frac{1}{5}} \\ y''_{n+\frac{2}{5}} \\ y''_{n+\frac{3}{5}} \\ y''_{n+\frac{4}{5}} \\ y'_{n+1} \end{bmatrix} &= L\begin{bmatrix} y''_{n-\frac{1}{5}} \\ y''_{n-\frac{2}{5}} \\ y''_{n-\frac{3}{5}} \\ y''_{n-\frac{4}{5}} \\ y''_{n} \end{bmatrix} + h(\hat{S})\begin{bmatrix} f_{n-\frac{1}{5}} \\ f_{n-\frac{2}{5}} \\ f_{n-\frac{3}{5}} \\ f_{n-\frac{4}{5}} \\ f_{n} \end{bmatrix} \\
&+ h(\hat{Q})\begin{bmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} + h^2(\hat{Z})\begin{bmatrix} g_{n-\frac{1}{5}} \\ g_{n-\frac{2}{5}} \\ g_{n-\frac{3}{5}} \\ g_{n-\frac{4}{5}} \\ g_{n} \end{bmatrix} + h^3(\hat{P})\begin{bmatrix} m_{n-\frac{1}{5}} \\ m_{n-\frac{2}{5}} \\ m_{n-\frac{3}{5}} \\ m_{n-\frac{4}{5}} \\ m_{n} \end{bmatrix}
\end{aligned} \tag{12}
$$

(see Appendix for the values of the terms).

Equations (10) to (12) are applied as the block integrators to provide solutions to problems.

3. Investigation of the basic properties of the block method

We investigate the order, error constant, and zero stability as well as the consistency of the block method as demonstrated below:

3.1 Order of accuracy and Local Truncation Error (LTE)

We describe a linear difference operators related to the block in (10), described as

$$\begin{aligned} L[y(x):h] = \sum_{j=0}^{k}[a_j y(x_n+jh) - h^3\beta_j f(x_n+jh) \\ - h^4\gamma_j g(x_n+jh) - h^5\omega_j m(x_n+jh)], \end{aligned} \tag{13}$$

where $y(x)$ is continuously differentiable function on $[a,b]$. Expanding (10) by Taylor series, collecting their terms in powers of h produces

$$L[y(x):h] = C_0 y(x) + C_1 hy'(x) C_2 hy''(x) + \cdots + C_q h^q(x) + (h^{q+1}),$$

where C_i are constants. Then, if the first C_{p+2} disappears, we have $C_o = C_1 = C_2 = \cdots = C_p = C_{p+2} = 0$ and $C_{p+3} \neq$. Thus, where p is called the order of the method. Then, $C_{p+3}h^{p+3}y^{p+3}(x)$ is the principal Local Truncation Error if the order p and error constant C_{p+3} are known. Lambert (1973). Therefore, for our block method,$p = [8,8,8,8,8]^T$ and

$$\begin{aligned} C_{p+3} = \Bigg[\frac{5323}{413437500000000}, -\frac{26141}{58179453100}, -\frac{17291}{53634509200}, \\ -\frac{22341}{8172494600}, -\frac{23451}{817965200}\Bigg]^T. \end{aligned}$$

3.2 Zero stability

The block method (10) is said to be zero stable if no root of the first characteristic polynomial $\rho(r)$ has modulus greater than one and if every root of modulus one has $\rho(r) = det(rA^0 - A^1) = r^m(r-1)$ multiplicity not greater than the order of the differential equation (Lambert 1973). It follows that the integrators are normalized to give the first characteristic polynomial $\rho(r)$ as with

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The roots of $\rho(r) = 0$ satisfy $|r_j| \leq 1$. Hence, the block method is zero-stable.

3.3 Consistency

A block method is said to be consistent if it has order $p \geq 1$. Hence, our block method is consistent, since$p = 8$ A linear multistep method is said to be convergent, if it is consistent and zero stable (see Lambert, 1973). Hence, our method is convergent satisfying the above condition.

4. Numerical experiments

Experiment 1

$y''' = 3\sin x \ y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$. The experiment above which is a special third-order initial value problem was solved by Kuboye & Omar (2015a) using a block method of step-length $k = 7$ order $p = 8$ and $h = 0.1$

The analytical solution:

$y(x) = 3\cos x + \frac{x^2}{2} - 2$ proposed block method is also used to solve the same experiment. The results are displayed in Table 1 and comparison of their absolute errors shown in Fig. 1.

Experiment 2

$y''' = y'(2xy'' + y'), y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = 0, h = 0.01.$

Analytical solution

$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Experiment 2 solved by our new block method is a non-linear third-order problem previously solved by Adoghe et al., (2016) using Taylor's series approach with step-length $k = 5$,order $p = 9$ and $h = 0.1$. The numerical results are displayed in Table 2 and their absolute errors shown in Fig. 2.

Experiment 3

$y''' = y'' - y' + y\ y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0,01;\ 0 \le x \le 1$

Analytical solution

$y(x) = \cos x$ Experiment 3 above is a linear third-order initial value problem solved by our proposed block method and Khataybeh et al., (2019). Table 3 displayed the numerical results and the comparison of their absolute errors can be seen in Fig. 3.

Experiment 4

$y'^{v} = (y')^2 - y(y'') - 4x^2 + e^x\left(1 - 4x + x^2\right)$
$y(0) = 1, y'(0) = 1, y''(0) = 3,$
$y'''(0) = 1, h = 0.01; 0 \le x \le 1.$

Analytical solution

$y(x) = x^2 + e^x$ The experiment above is a non-linear third-order problem solved by Kuboye & Omar (2015b) where a 6-step bock method of order $p = 7$ with $h = 0.01$ was developed. Our proposed method is used to solve the same experiment and our results are compared together as displayed in Table 4 and the absolute error for both methods can be seen in Fig. 4.

Experiment 5

$y^{v} = -(\cos x + \sin x),$
$y(0) = 1, y'(0) = 1, y''(0) = -2$
$y'''(0) = 1, y'^{v}(0) = 2, h = 0.1; 0 \le x \le 1.$

Exact solution

$y(x) = 2x - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cos x - \sin x$

The special fifth order experiment above was solved by Jena & Mohanty (2019) with a step -length of $k = 7$,order $p = 7$ and $h - 0.1$. We also used our proposed method to solve the same experiment with the results generated compared together (see Table 5) and the graphical performance of their absolute error demonstrated in Fig. 5.

5. Numerical results and comparison

Table 1: Comparison of the new block method results with the results of Kuboye & Omar (2015a)

x	Analytical Results	Numerical Results	Error in New Method	Error in Adoghe et al.,2016
0.1	0.99001249583 407730	0.990012495834 079360	2.0600E - 15	1.7430E- 14
0.2	0.96019973352 372489	0.96019973352 3687974	2.9051E - 14	1.0824E- 13
0.3	0.91100946737 681806	0.91100946737 6764056	5.4004E- 14	2.7111E- 13
0.4	0.84318298200 865525	0.84318298200 8574846	8.0404E- 14	5.0792E- 13
0.5	0.75774768567 111815	0.7577476856 71090550	2.7600E- 14	8.1645E- 13
0.6	0.65600684472 903489	0.65600684472 9236145	2.0125E- 13	1.1997E- 12
0.7	0.53952656185 346528	0.53952656185 4197778	7.3249E- 13	1.6543E- 12
0.8	0.41012012804 149626	0.41012012804 3217003	1.7207E- 12	1.6746E- 10
0.9	0.26982990481 199337	0.2698299048 15340641	3.3472E- 12	3.3363E- 10
1.0	0.12090691760 441915	0.12090691761 0237425	5.8182E- 12	5.0017E- 10

Table 2: Comparison of the new block method results with the results of Adoghe et al., (2016)

x	Analytical Results	Numerical Results	Error in New Method	Error in Adoghe et al.,2016
0.1	0.91482908192 435238	0.91482908192 4352375	5.0000E-18	0.9148E +00
0.2	0.85859724183 983017	0.85859724183 9830167	3.0000E-18	0.8585E +00
0.3	0.83014119242 399690	0.83014119242 3996896	4.0000E-18	0.8301E+00
0.4	0.82817530235 872968	0.82817530235 8729681	1.0000E-18	0.8281E +00
0.5	0.8512787292 9987185	0.85127872929 9871852	2.0000E-18	0.8512E +00
0.6	0.89788119960 949103	0.8978811996 09491024	6.0000E-18	0.8978E +00
0.7	0.96624729252 952348	0.9662472925 29523477	3.0000E-18	0.9662E +00
0.8	1.05445907150 753240	1.0544590715 07532390	1.0000E-17	0.1054E +01
0.9	1.16039688884 305034	1.1603968888 43050330	1.0000E-17	0.1160E +01
1.0	1.28171817154 095476	1.281718171 54095475	1.0000E-17	0.1281E +01

Table 3: Comparison of the new block method results with the results of Khataybeh et al., (2019)

x	Analytical Results	Numerical Results	Error in New Method	Khataybeh et al., (2019)
0.1	0.99995000041 6665278	0.99995000041 6665282	4.0000E-18	5.84E-18
0.2	0.99980000666 6577778	0.99980000666 6577783	5.0000E-18	2.85E-17
0.3	0.99955003374 8987516	0.99955003374 8987521	5.0000E-18	6.86E-17
0.4	0.99920010666 0977940	0.99920010666 0977944	4.0000E-18	1.23E-16
0.5	0.99875026039 4966247	0.99875026039 4966250	3.0000E-18	1.83E-16
0.6	0.99820053993 5204166	0.99820053993 5204169	3.0000E-18	2.24E-16
0.7	0.99755100025 3279575	0.99755100025 3279578	3.0000E-18	1.97E-16
0.8	0.99680170630 2619385	0.99680170630 2619386	1.0000E-18	3.01E-17
0.9	0.99595273301 1994253	0.99595273301 1994252	1.0000E-18	8.53E-16
1.0	0.99500416527 8025766	0.995004165278 025764	2.0000E-18	3.45E-15

Table 4: Comparison of the new block method results with the results of Kuboye & Omar (2015b)

x	Analytical Results	Numerical Results	Error in New Method	Error in Kuboye & Omar (2015b)
0.01	0.000200000000000000	0.000200000000000000	0.00000000E+00	2.220446E-16
0.02	0.000400000000000000	0.000400000000000000	0.00000000E+00	0.000000E+00
0.03	0.000600000000000001	0.000600000000000001	0.00000000E+00	0.000000E+00
0.04	0.000800000000000003	0.000800000000000003	0.00000000E+00	0.000000E+00
0.05	0.001000000000000008	0.001000000000000008	0.00000000E+00	0.000000E+00
0.06	0.001200000000000021	0.001200000000000010	1.12757026E-17	0.000000E+00
0.07	0.001400000000000045	0.001400000000000020	2.51534904E-17	1.043610E-14
0.08	0.001600000000000087	0.001600000000000042	4.53196508E-17	3.463896E-14
0.09	0.001800000000000157	0.001800000000000089	6.83047369E-17	7.882583E-14
0.10	0.002000000000000267	0.002000000000000168	9.84455573E-17	1.505462E-13

Table 5: Comparison of the new block method results with the results of Jena & Mohanty (2019)

x	Analytical Results	Numerical Results	Error in New Method	Error in Jena & Mohanty (2019)
0.1	1.000259932402929800	1.000259932402929800	0.0000E+00	4 .8849E-15
0.2	1.000279921603659100	1.000279921603659300	2.2204E-16	9.1038E-15
0.3	1.000299910004500700	1.000299910004500900	2.2204E-16	1.7763E-15
0.4	1.000319897605462100	1.000319897605462400	2.2204E-16	7.7049E-14
0.5	1.000339884406551900	1.000339884406552100	2.2204E-16	9.9920E-15
0.6	1.000359870407777600	1.000359870407777900	2.2204E-16	1.1990E-14
0.7	1.000379855609147000	1.000379855609147400	4.4408E-16	1.9095E-14
0.8	1.000399840010668800	1.000399840010669500	6.6613E-16	1.11022E-15
0.9	1.000419823612350500	1.000419823612351400	8.8817E-16	5.9952E-15
1.0	1.000439806414200400	1.000439806414201500	1.1102E-15	6.1062E-14

6. Conclusion

A new single step self-starting implicit block method has been presented in this paper for solving higher-order (Third, Fourth, and Fifth-order) initial value problems of ordinary differential equations. The solutions of the numerical experiments obtained are accurate and superior to the existing methods in terms of performance and accuracy as demonstrated in Tables (1 to 5) and Figures (1 to 5). Hence, this method has given rise to more accurate and superior convergent results which is a good approach to provide solution to higher order initial value problems.

Appendix

Appendix 1

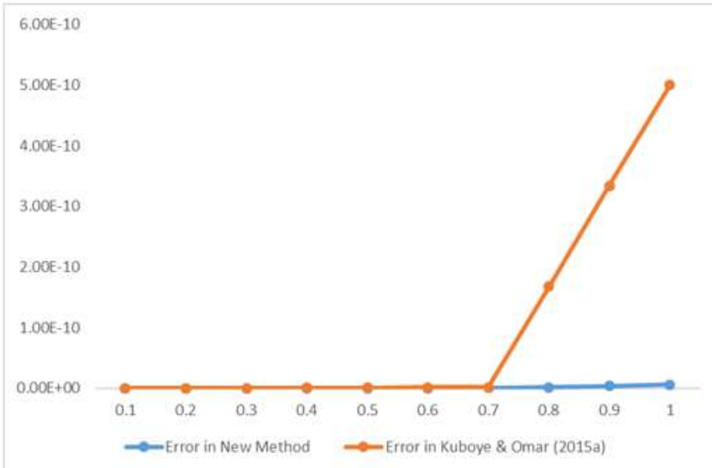


Figure 1: Comparison of Absolute Error with Error in Kuboye & Omar (2015a)

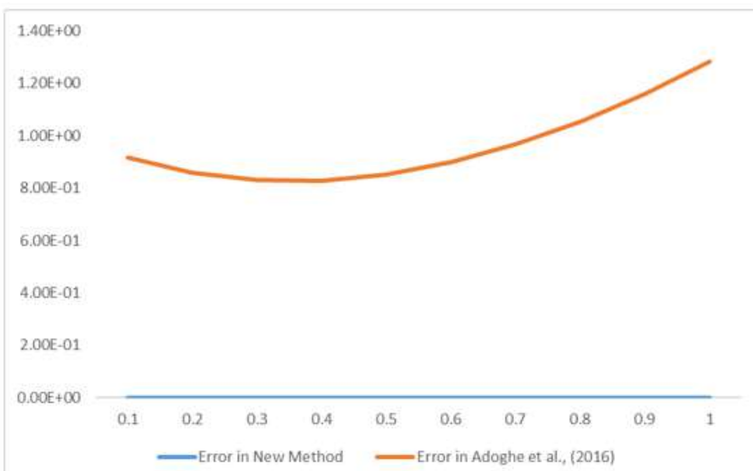


Figure 2: Comparison of Absolute Error with Adoghe et al., (2016)

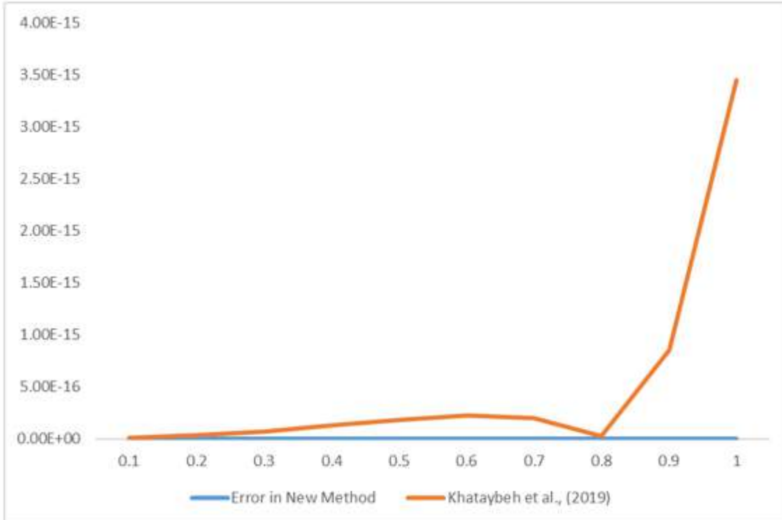


Figure 3: Comparison of Absolute Error with Khataybeh et al., (2019)

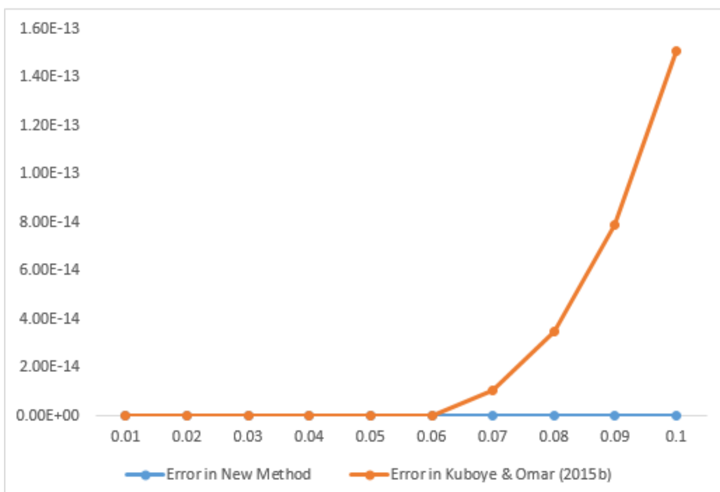
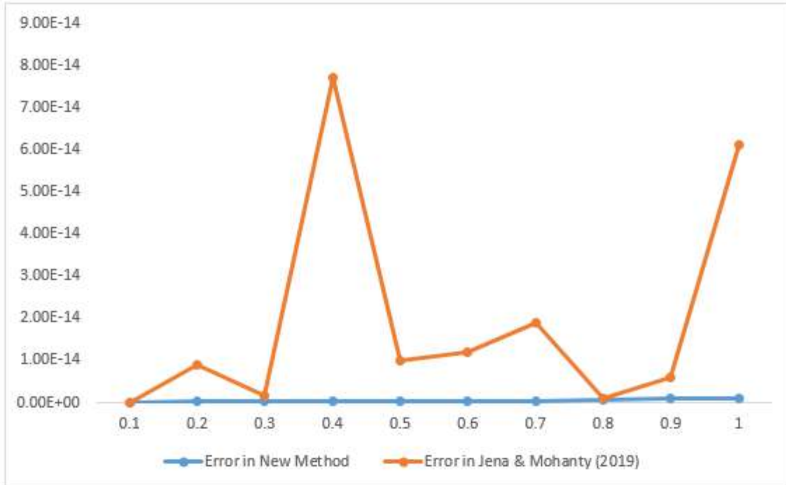


Figure 4: Comparison of Absolute Error with Kuboye & Omar (2015b)

Appendix 2

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, K = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{50} \\ 0 & 0 & 0 & 0 & \frac{2}{25} \\ 0 & 0 & 0 & 0 & \frac{9}{50} \\ 0 & 0 & 0 & 0 & \frac{8}{25} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{493123}{504000000} \\ 0 & 0 & 0 & 0 & -\frac{315043}{35437500} \\ 0 & 0 & 0 & 0 & -\frac{961011}{56000000} \\ 0 & 0 & 0 & 0 & -\frac{13664}{421875} \\ 0 & 0 & 0 & 0 & -\frac{9841}{207360} \end{bmatrix}, V = \begin{bmatrix} \frac{377}{90000} & \frac{778}{23625} & \frac{21789}{280000} & \frac{1984}{13125} & \frac{725}{3024} \\ -\frac{33017}{15120000} & -\frac{613}{39375} & -\frac{2349}{80000} & -\frac{1088}{23625} & \frac{475}{8064} \\ \frac{13}{36000} & \frac{916}{354375} & \frac{387}{70000} & \frac{1664}{118125} & \frac{575}{18144} \\ -\frac{403}{6720000} & -\frac{29}{67500} & -\frac{81}{89600} & -\frac{64}{39375} & \frac{125}{96768} \\ \frac{509}{94500000} & \frac{38}{984375} & \frac{81}{1000000} & \frac{64}{421875} & \frac{1}{3360} \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1291}{18000000} \\ 0 & 0 & 0 & 0 & -\frac{227}{421875} \\ 0 & 0 & 0 & 0 & -\frac{2187}{2000000} \\ 0 & 0 & 0 & 0 & -\frac{32}{15625} \\ 0 & 0 & 0 & 0 & -\frac{53}{17280} \end{bmatrix},$$



$$Z=\begin{bmatrix}0&0&0&0&\frac{71}{1575000}\\0&0&0&0&\frac{64}{196875}\\0&0&0&0&\frac{243}{350000}\\0&0&0&0&\frac{256}{196875}\\0&0&0&0&\frac{1}{504}\end{bmatrix},\ \bar{N}=\begin{bmatrix}0&0&0&0&-\frac{1472753}{64800000}\\0&0&0&0&-\frac{140279}{3543750}\\0&0&0&0&-\frac{356253}{56000000}\\0&0&0&0&-\frac{84284}{1771875}\\0&0&0&0&-\frac{39191}{145152}\end{bmatrix},\ \bar{W}=\begin{bmatrix}\frac{16549}{216000}&\frac{4301}{23625}&\frac{2133}{7000}&\frac{1664}{4725}&-\frac{10222}{12096}\\-\frac{11923}{302400}&-\frac{1723}{23625}&-\frac{4563}{56000}&-\frac{1088}{23625}&-\frac{425}{1728}\\\frac{14803}{2268000}&\frac{874}{70875}&\frac{33}{1400}&\frac{4352}{70875}&\frac{2425}{18144}\\-\frac{937}{864000}&-\frac{97}{47250}&-\frac{783}{224000}&-\frac{4}{23625}&-\frac{1625}{48384}\\\frac{3683}{37800000}&\frac{109}{590625}&\frac{27}{87500}&\frac{128}{590625}&-\frac{41}{12096}\end{bmatrix},$$

$$\bar{D}=\begin{bmatrix}0&0&0&0&-\frac{7369}{5400000}\\0&0&0&0&-\frac{212}{84375}\\0&0&0&0&-\frac{801}{200000}\\0&0&0&0&-\frac{304}{84375}\\0&0&0&0&-\frac{25}{1728}\end{bmatrix},\bar{E}=\begin{bmatrix}0&0&0&0&\frac{1279}{1575000}\\0&0&0&0&\frac{316}{196875}\\0&0&0&0&\frac{891}{350000}\\0&0&0&0&\frac{512}{196875}\\0&0&0&0&-\frac{1}{126}\end{bmatrix},\ \hat{S}=\begin{bmatrix}0&0&0&0&-\frac{1762949}{6048000}\\0&0&0&0&\frac{2599}{21000}\\0&0&0&0&-\frac{86047}{224000}\\0&0&0&0&\frac{15142}{23625}\\0&0&0&0&-\frac{19361}{5376}\end{bmatrix},\hat{Z}=$$

$$\begin{bmatrix}0&0&0&0&-\frac{1111}{72000}\\0&0&0&0&\frac{1}{250}\\0&0&0&0&-\frac{159}{8000}\\0&0&0&0&\frac{32}{1125}\\0&0&0&0&-\frac{11}{64}\end{bmatrix},\ \hat{Q}=\begin{bmatrix}\frac{57517}{67200}&\frac{57}{350}&\frac{24687}{22400}&-\frac{424}{525}&-\frac{6375}{896}\\-\frac{14249}{33600}&\frac{131}{1050}&-\frac{2979}{11200}&\frac{376}{525}&-\frac{4625}{1344}\\\frac{3037}{43200}&-\frac{1}{75}&\frac{263}{1600}&\frac{112}{675}&\frac{125}{192}\\-\frac{1571}{134400}&\frac{3}{1400}&-\frac{801}{44800}&\frac{46}{525}&\frac{375}{1792}\\\frac{353}{336000}&\frac{1}{5250}&\frac{171}{112000}&-\frac{8}{2625}&-\frac{181}{2688}\end{bmatrix},\ \hat{P}=\begin{bmatrix}0&0&0&0&\frac{731}{84000}\\0&0&0&0&-\frac{2}{2625}\\0&0&0&0&\frac{297}{28000}\\0&0&0&0&-\frac{32}{2625}\\0&0&0&0&\frac{55}{672}\end{bmatrix}$$

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On a class of Lorentzian paracontact metric manifolds

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Abstract. In this present paper, we consider a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds admitting a pseudo-projective curvature tensor $\overline{W}(X,Y)$. We study and have shown that the scalar curvature of Lorentzian para-Kenmotsu manifold is constant if and only if the time like vector field ξ is harmonic, whenever the LP-Kenmotsu manifold satisfying $R(X,Y)\cdot\overline{W}=0$ is not an Einstein manifold. Further we have shown that Lorentzian para-Kenmotsu manifolds admitting an irrotational pseudo-projective curvature tensor and a conservative pseudo-projective curvature tensor are an Einstein manifolds of constant scalar curvature. At the end, we construct an example of a 3-dimensional LP-Kenmotsu manifold admitting a pseudo-projective curvature tensor which verifies the results discussed in the present work.

Keywords: Lorentzian para-Kenmotsu manifolds, pseudo-projective curvature tensor, harmonic vector field, irrotational and conservative vector fields.

1. Introduction

In 1989, Matsumoto [8] introduced the notion of Lorentzian paracontact metric manifolds and defined Lorentzian para-Sasakian (LP-Sasakian) manifolds, which are regarded as a special kind of these Lorentzian paracontact manifolds. Further, these manifolds have been widely studied by many geometers such as

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De, Matsumoto and Shaikh [7], Matsumoto and Mihai [9], Mihai and Rosca [10], Mihai, Shaikh and De [11], Venkatesha and Bagewadi [16], Venkatesha, Pradeep Kumar and Bagewadi [17] and obtained several results on these manifolds.

In 1995, Sinha and Sai Prasad [15] defined a class of almost paracontact metric manifolds namely para-Kenmotsu (briefly P-Kenmotsu) and special para-Kenmotsu (briefly SP-Kenmotsu) manifolds in similar to P-Sasakian and SP-Sasakian manifolds. In 2018, Abdul Haseeb and Rajendra Prasad defined a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds [1] and they studied ϕ-semi-symmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [13].

On the other hand, in 1970 [12], Pokhariyal and Mishra introduced new tensor fields, called the Weyl-projective curvature tensor W_2 of type $(1, 3)$ and the tensor field E on a Riemannian manifold. In our earlier work, we consider LP-Kenmotsu manifolds admitting the Weyl-projective curvature tensor W_2 and shown that these manifolds admitting a Weyl-flat projective curvature tensor, an irrotational Weyl-projective curvature tensor and a conservative Weyl-projective curvature tensor are an Einstein manifolds of constant scalar curvature [14].

The idea of Weyl-projective curvature tensor has been extended by Bhagawat Prasad [6], and in 2002 he defined the pseudo-projective curvature tensor $\overline{W}$ on a Riemannian manifold M_n of dimension n as:

$$\begin{aligned} \overline{W}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad - \frac{r}{n}\Big[\frac{a}{n-1} + b\Big][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

where a and b are constants such that $a, b \neq 0$. In the above expression $R(X,Y)$ is known to be the Riemannian curvature tensor, S is the Ricci tensor and r is the scalar curvature with respect to the Levi-Civita connection.

The pseudo-projective curvature tensor on a Riemannian manifold was widely studied by Bagewadi *et al.,* [2], Bagewadi and Venkatesha [3, 4] and by many geometers. In 2008, Bagewadi *et al.,* [5] have extended these concepts to Lorentzian paracontact structures and studied LP-Sasakian manifolds admitting this tensor field of particular type. They have shown that the LP-Sasakian manifold is an Einstein manifold if the pseudo projective curvature tensor admitted by the manifold is irrotational.

Motivated by these studies, in the present paper, we explore the geometrical significance of LP-Kenmotsu manifolds admitting the pseudo-projective curvature tensor. The present paper is organized as follows: Section 2 is equipped with some prerequisites about Lorentzian para-Kenmotsu manifolds. In section 3, we consider Lorentzian para-Kenmotsu manifolds admitting $R(X,Y)\cdot\overline{W} = 0$ and shown that it is an η-Einstein manifold of constant scalar curvature $n(n-1)$. As a special case, we have shown that the scalar curvature of Lorentzian para-Kenmotsu manifold is constant if and only if the time like vector field ξ is harmonic, whenever the LP-Kenmotsu manifold satisfying $R(X,Y)\cdot\overline{W}=0$ is not an Einstein manifold.

In the sections 4 and 5, we study geometrical properties of these manifolds, and in particular, we have shown that Lorentzian para-Kenmotsu manifolds admitting an irrotational pseudo-projective curvature tensor and a conservative pseudo-projective curvature tensor are an Einstein manifolds of constant scalar curvature. Finally, in section 6, we construct an example of a 3-dimensional LP-Kenmotsu manifold admitting pseudo-projective curvature tensor which verifies the results discussed in the present work.

2. Preliminaries

An n-dimensional differentiable manifold M_n admitting a (1, 1) tensor field ϕ, contravariant vector field ξ, a 1-form η and the Lorentzian metric $g(X,Y)$ satisfying

$$\phi^2X = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2}$$

and

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = \eta(X), \ rank\ \phi = n-1, \tag{3}$$

for arbitrary vector fields X, Y on M_n, is called Lorentzian almost paracontact manifold [8].

In a Lorentzian almost paracontact manifold, for any vector fields X, Y on M_n, we have

$$\Phi(X,Y) = \Phi(Y,X), \tag{4}$$

where $\Phi(X,Y) = g(X,\phi Y)$ is a symmetric (0, 2) tensor field.

A Lorentzian almost paracontact manifold M_n is called Lorentzian para-Kenmotsu manifold if [1]

$$(\nabla_X\phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{5}$$

for all X, $Y \in \chi(M_n)$, where $\chi(M_n)$ is the set of all differentiable vector fields on M_n and ∇ is known to be the operator of covariant differentiation with respect to the Lorentzian metric g.

In a Lorentzian para-Kenmotsu manifold, the following relations hold good [1]:

$$\nabla_X\xi = -\phi^2X = -X - \eta(X)\xi, \tag{6}$$

$$(\nabla_X\eta)Y = -g(X,Y) - \eta(X)\eta(Y), \tag{7}$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{8}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{11}$$

and

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{12}$$

for any vector fields X, Y and Z on M_n.

By putting $Z = \xi$ in (1) and on simplification by using (3), (10) and (11), we get

$$\overline{W}(X,Y)\xi = [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big][\eta(Y)X - \eta(X)Y]. \tag{13}$$

The above expression can be written as:

$$\overline{W}(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{14}$$

where

$$k = [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big].$$

3. Pseudo-projective semisymmetric LP-Kenmotsu manifolds

Let us consider an LP-Kenmotsu manifold (M_n, g) satisfying the condition [3, 4]

$$R(X,Y)\cdot \overline{W} = 0, \tag{15}$$

for any arbitrary vector fields X, Y on M_n. Then the manifold M_n is called as the pseudo-projective semisymmetric LP-Kenmotsu manifold (or) simply called as $\overline{W}$-semisymmetric LP-Kenmotsu manifold.

On the other hand, we have

$$\begin{aligned}(R(X,Y)\cdot \overline{W})(U,V)Z =& R(X,Y)\overline{W}(U,V)Z - \overline{W}(R(X,Y)U,V)Z \\ &- \overline{W}(U,R(X,Y)V)Z - \overline{W}(U,V)R(X,Y)Z,\end{aligned} \tag{16}$$

for any vector fields X, Y, Z, U, $V \in \chi(M_n)$. Then, from (15) and (16), we have

$$\begin{aligned}&g(R(\xi,Y)\overline{W}(U,V)Z,\xi) - g(\overline{W}(R(\xi,Y)U,V)Z,\xi)\\ &- g(\overline{W}(U,R(\xi,Y)V)Z,\xi) - g(\overline{W}(U,V)(R(\xi,Y)Z,\xi)) = 0.\end{aligned} \tag{17}$$

By virtue of (8) and (9), we get each term of the above expression as:

$$\begin{aligned}&(a)\ g(R(\xi,Y)\overline{W}(U,V)Z,\xi) = -\overline{W}'(U,V,Z,Y) - \eta(Y)\eta(\overline{W}(U,V)Z),\\ &(b)\ g(\overline{W}(R(\xi,Y)U,V)Z,\xi) = g(Y,U)\eta(\overline{W}(\xi,V)Z) - \eta(U)\eta(\overline{W}(Y,V)Z),\\ &(c)\ g(\overline{W}(U,R(\xi,Y)V)Z,\xi) = \eta(V)\eta(\overline{W}(U,Y)Z) - g(Y,V)\eta(\overline{W}(U,\xi)Z),\\ &(d)\ g(\overline{W}(U,V)(R(\xi,Y)Z,\xi)) = g(Y,Z)\eta(\overline{W}(U,V)\xi)\\ &\qquad\qquad - \eta(Z)\eta(\overline{W}(U,V)Y) = 0,\end{aligned} \tag{18}$$

for arbitrary vector fields U, V, Z, $Y \in \chi(M_n)$, where

$$\overline{W}'(U,V,Z,Y) = g(\overline{W}(U,V)Z,Y).$$

By substituting the above values from (18) in (17), we obtain that

$$\begin{aligned} &-\overline{W}'(U,V,Z,Y) - \eta(Y)\eta(\overline{W}(U,V)Z) - g(Y,U)\eta(\overline{W}(\xi,V)Z) \\ &+\eta(U)\eta(\overline{W}(Y,V)Z) - g(Y,V)\eta(\overline{W}(U,\xi)Z) + \eta(V)\eta(\overline{W}(U,Y)Z) \\ &- g(Y,Z)\eta(\overline{W}(U,V)\xi) + \eta(Z)\eta(\overline{W}(U,V)Y) = 0. \end{aligned} \tag{19}$$

Clearly it follows from (13) that

$$\eta(\overline{W}(U,V)\xi) = 0, \tag{20}$$

where U, $V \in \chi(M_n)$.

Now, by using (20) in (19), we get

$$\begin{aligned} &-\overline{W}'(U,V,Z,Y) - \eta(Y)\eta(\overline{W}(U,V)Z) - g(Y,U)\eta(\overline{W}(\xi,V)Z) \\ &+\eta(U)\eta(\overline{W}(Y,V)Z) - g(Y,V)\eta(\overline{W}(U,\xi)Z) + \eta(V)\eta(\overline{W}(U,Y)Z) \\ &+\eta(Z)\eta(\overline{W}(U,V)Y) = 0, \end{aligned} \tag{21}$$

for any vector fields U, V, Z, $Y \in \chi(M_n)$.

Let $\{e_i = 1 : i = 1,2,3,\cdots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold.

By putting $U = Y = e_i$ in (21) we get that

$$\begin{aligned} &\overline{W}'(e_i,V,Z,e_i) + g(e_i,e_i)\eta(\overline{W}(\xi,V)Z) + \eta(V)\eta(\overline{W}(e_i,\xi)Z) \\ &-\eta(V)\eta(\overline{W}(e_i,e_i)Z) - \eta(Z)\eta(\overline{W}(e_i,V)e_i) = 0. \end{aligned} \tag{22}$$

On further simplification of the above equation, we have

$$\overline{W}'(e_i,V,Z,e_i) + g(e_i,e_i)\eta(\overline{W}(\xi,V)Z) - \eta(Z)\eta(\overline{W}(e_i,V)e_i) = 0, \tag{23}$$

as $\eta(\overline{W}(e_i,e_i)Z) = 0$.

Now, by taking summation over $1 \le i \le n$ in (23), we get

$$\sum_{i=1}^{n}\epsilon_i\overline{W}'(e_i,V,Z,e_i) + (n-1)\eta(\overline{W}(\xi,V)Z) - \eta(Z)\sum_{i=1}^{n}\epsilon_i\eta(\overline{W}(e_i,V)e_i) = 0, \tag{24}$$

where $\epsilon_i = g(e_i,e_i)$.

Now, by using (9) and (1), the terms of the above expression are obtained as:

$$
\begin{aligned}
&(a)\ \sum_{i=1}^{n}\epsilon_i\overline{W}'(e_i,V,Z,e_i)=[a+(n-1)b]S(V,Z)\\
&\qquad\qquad\qquad -\frac{r}{n}\Big[a+(n-1)b\Big]g(V,Z),\\
&(b)\ \eta(\overline{W}(\xi,V)Z)=\Big[-a+\frac{r}{n}\Big(\frac{a}{n-1}+b\Big)\Big][g(V,Z)+\eta(V)\eta(Z)\\
&\qquad\qquad\qquad -bS(V,Z)-b(n-1)\eta(V)\eta(Z)],\\
&(c)\ \sum_{i=1}^{n}\epsilon_i\eta(\overline{W}(e_i,V)e_i)=[a-b]\Big[\frac{r}{n}-(n-1)\Big]\eta(V).
\end{aligned}
\tag{25}
$$

By substituting the above values in (24), we get

$$aS(V,Z)-a(n-1)g(V,Z)+b[r-n(n-1)]\eta(V)\eta(Z)=0, \tag{26}$$

which can be written as

$$S(V,Z)=(n-1)g(V,Z)-\frac{b}{a}[r-n(n-1)]\eta(V)\eta(Z), \tag{27}$$

for any vector fields V and Z on M_n. Thus, we have the following assertion.

Theorem 3.1. *An LP-Kenmotsu manifold (M_n,g) satisfying the condition $R(X,Y)\cdot\overline{W}=0$ is an η-Einstein manifold.*

Further, by taking $Z=\xi$ in (27) and on simplification by using (3) and (11), we obtain that $r=n(n-1)$ and this leads to the following assertion.

Corollary 3.1. *An LP-Kenmotsu manifold (M_n,g) satisfying the condition $R(X,Y)\cdot\overline{W}=0$ is of constant scalar curvature $n(n-1)$.*

Now, let us consider a special case in which the LP-Kenmotsu manifold admitting $R(X,Y)\cdot\overline{W}=0$ is not an Einstein manifold. Then, from (27) it follows that $r\neq n(n-1)$; otherwise it is an Einstein manifold.

On differentiating (27) covariantly along X and then on using (7), we get

$$
\begin{aligned}
(\nabla_X S)(V,Z)=&-\frac{b}{a}dr(X)\eta(V)\eta(Z)\\
&-\frac{b}{a}[r-n(n-1)][g(X,V)\eta(Z)+g(X,Z)\eta(V)+2\eta(X)\eta(V)\eta(Z)].
\end{aligned}
\tag{28}
$$

By putting $X=Z=e_i$ in the above expression and on taking summation for $1\le i\le n$, we obtain that

$$dr(V)=\frac{b}{a}\Big[dr(\xi)-[r-n(n-1)\Psi]\Big]\eta(V), \tag{29}$$

where $\Psi=1+\sum_{i=1}^{n}\epsilon_i g(e_i,e_i)$.

On replacing V with ξ in the above expression (29), we get that

$$dr(\xi) = \frac{b}{a+b}[r - n(n-1)]\Psi. \tag{30}$$

From (29) and (30) we obtain

$$dr(V) = \frac{b}{a+b}[n(n-1) - r]\Psi\eta(V). \tag{31}$$

If r is constant then (31) yields either $r = n(n-1)$ or $\Psi = 0$. But as $r \neq n(n-1)$, we must have $\Psi = 0$, which means that the vector field ξ is harmonic.

Again, if $\Psi = 0$, then from (31) it follows that r is constant. Thus we can state the following:

Theorem 3.2. *If the LP-Kenmotsu manifold admitting the condition $R(X,Y) \cdot \overline{W} = 0$ is not an Einstein manifold, then the scalar curvature of the manifold is constant if and only if the time like vector field ξ is harmonic.*

4. Irrotational pseudo-projective curvature tensor in LP-Kenmotsu manifolds

Definition 4.1. *The rotation (curl) of pseudo-projective curvature tensor $\overline{W}$ on a Riemannian manifold is given by [2]*

$$\begin{aligned} Rot\ \overline{W} &= (\nabla_U \overline{W})(X,Y)Z + (\nabla_X \overline{W})(U,Y)Z \\ &\quad + (\nabla_Y \overline{W})(X,U)Z - (\nabla_Z \overline{W})(X,Y)U, \end{aligned} \tag{32}$$

for all X, Y, U, $Z \in \chi(M_n)$.

In virtue of Bianchi's second identity, we have

$$(\nabla_U \overline{W})(X,Y)Z + (\nabla_X \overline{W})(U,Y)Z + (\nabla_Y \overline{W})(X,U)Z = 0. \tag{33}$$

Therefore, (32) reduces to

$$Rot\ \overline{W} = -(\nabla_Z \overline{W})(X,Y)U, \tag{34}$$

for all X, Y, U, $Z \in \chi(M_n)$.

Now, let us suppose that the pseudo-projective curvature tensor is irrotational. Then *curl* $\overline{W} = 0$ and so by (34) we get

$$-(\nabla_Z \overline{W})(X,Y)U = 0,$$

which implies the following:

$$\nabla_Z(\overline{W}(X,Y)U) = \overline{W}(\nabla_Z X, Y) + \overline{W}(X, \nabla_Z Y)U + \overline{W}(X,Y)\nabla_Z U \tag{35}$$

for any arbitrary vector fields X, Y, U, $Z \in \chi(M_n)$.

By replacing $U = \xi$ in (35), we have

$$\nabla_Z(\overline{W}(X,Y)\xi) = \overline{W}(\nabla_Z X, Y)\xi + \overline{W}(X, \nabla_Z Y)\xi + \overline{W}(X,Y)\nabla_Z \xi. \tag{36}$$

Using (14) in (36) and on simplifying by making use of (6), we get

$$\overline{W}(X,Y)\phi^2 Z = -k[g(Z,\phi Y)X - g(Z,\phi X)Y], \tag{37}$$

which on further simplification by using (2) and (14), we get

$$\overline{W}(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y], \tag{38}$$

for any vector fields X, Y, $Z \in \chi(M_n)$. Thus, we can state:

Lemma 4.1. *If the pseudo-projective curvature tensor $\overline{W}$ in an LP-Kenmotsu manifold is irrotational, then $\overline{W}$ is given by the expression* (38).

Further, in view of (1) and (38) we get

$$\begin{aligned} aR(X,Y)W = {} & [a + (n-1)b][g(Y,W)X - g(X,W)Y] \\ & - b[S(Y,W)X - S(X,W)Y], \end{aligned} \tag{39}$$

where X, Y, $Z \in \chi(M_n)$.

Let $\{e_i = 1 : i = 1, 2, 3, \cdots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then, by putting $Y = Z = e_i$ in (39), we get that

$$\begin{aligned} aR(X,e_i)W = {} & [a + (n-1)b][\eta(W)X - g(X,W)e_i] \\ & - b[S(e_i,W)X - S(X,W)e_i]. \end{aligned} \tag{40}$$

By taking the inner product of (40) with W and on taking summation over $1 \le i \le n$ we get

$$S(X,W) = (n-1)g(X,W). \tag{41}$$

This proves that the manifold is Einstein.

Finally, by taking $X = W = e_i$ in (41) and on taking summation from 1 to n we obtain

$$r = n(n-1). \tag{42}$$

Hence we can state that:

Theorem 4.1. *If the pseudo-projective curvature tensor in an LP-Kenmotsu manifold is irrotational, then the manifold is Einstein and the scalar curvature under such conditions is given by $n(n-1)$.*

5. Conservative pseudo-projective curvature tensor in LP-Kenmotsu manifolds

On differentiating (1) with respect to U, we get

$$(\nabla_U \overline{W})(X,Y)Z = a(\nabla_U R)(X,Y)Z + b[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y] - \frac{dr(U)}{n}\Big[\frac{a}{n-1} + b\Big][g(Y,Z)X - g(X,Z)Y], \tag{43}$$

which on contraction with respect to U becomes

$$(\operatorname{div} \overline{W})(X,Y)Z = a[(\operatorname{div} R)(X,Y)Z] + b[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - \frac{1}{n(n-1)}[a + (n-1)b][g(Y,Z)dr(X) - g(X,Z)dr(Y)], \tag{44}$$

for arbitrary vector fields X, Y, Z, $U \in \chi(M_n)$.

Let us suppose that the pseudo-projective curvature tensor is conservative, i. e., div $\overline{W} = 0$. Then, (44) can be written as:

$$(a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] = \frac{1}{n(n-1)}[a + (n-1)b][g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{45}$$

By putting $X = \xi$ in (45), we have

$$(a+b)[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] = \frac{1}{n(n-1)}[a + (n-1)b][g(Y,Z)dr(\xi) - g(\xi,Z)dr(Y)]. \tag{46}$$

On the other hand, since ξ is a Killing vector and the scalar curvature r remains invariant, we have $dr(\xi) = 0$.

Also, we have

$$(\nabla_\xi S)(Y,Z) = \xi S(Y,Z) - S(\nabla_\xi Y, Z) - S(Y, \nabla_\xi Z),$$

and

$$(\nabla_Y S)(\xi,Z) = \nabla_Y S(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z),$$

for any vector fields Y, $Z \in \chi(M_n)$.

By virtue of the above, the relation (46) becomes

$$(a+b)[-\nabla_Y(S(\xi,Z)) + S(\nabla_Y \xi, Z) + S(\xi, \nabla_Y Z)] = \frac{1}{n(n-1)}[a + (n-1)b][-\eta(Z)dr(Y)], \tag{47}$$

which on using (6) reduces to

$$(a+b)[-\nabla_Y\{(n-1)\eta(Z)\} + S(-\phi^2 Y, Z) + (n-1)\eta(\nabla_Y Z)] = \frac{1}{n(n-1)}[a + (n-1)b][-\eta(Z)dr(Y)], \tag{48}$$

and further it is simplified to

$$
\begin{aligned}
&(a+b)[-(n-1)\nabla_Y\{\eta(Z)\} - S(\phi Y, \phi Z) + (n-1)\eta(\nabla_Y Z)] \\
&= \frac{1}{n(n-1)}[a+(n-1)b][-\eta(Z)dr(Y)],
\end{aligned} \tag{49}
$$

for arbitrary vector fields Y, $Z \in \chi(M_n)$.

By putting $Z = \phi Z$ in (49), we get

$$(a+b)[-S(\phi Y, \phi^2 Z) + (n-1)\eta(\nabla_Y(\phi Z))] = 0. \tag{50}$$

If $a + b \neq 0$, then (50) becomes

$$S(\phi Y, Z) = (n-1)g(\phi Y, Z). \tag{51}$$

By putting $Z = \phi Z$ in (51), we get

$$S(\phi Y, \phi Z) = (n-1)g(\phi Y, \phi Z), \tag{52}$$

and this implies that

$$S(Y, Z) = (n-1)g(Y, Z), \tag{53}$$

which on contracting gives

$$r = n(n-1), \text{ where} \tag{54}$$

$$r = \sum_{i=1}^{3} \epsilon_i \; S(e_i, e_i) \text{ and } \epsilon_i = g(e_i, e_i), \text{ which is constant.} \tag{55}$$

So, one can state that:

Theorem 5.1. *An LP-Kenmotsu manifold admitting a conservative pseudo-projective curvature tensor is an Einstein manifold and it is of constant scalar curvature.*

6. Example

Example 6.1. We consider a 3-dimensional manifold $M_3 = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M_3 given by

$$e_1 = x\frac{\partial}{\partial x} = \xi, \quad e_2 = x\frac{\partial}{\partial y}, \quad e_3 = x\frac{\partial}{\partial z}. \tag{56}$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M_3 and hence form a basis of $\chi(M_3)$.

The Lorentzian metric $g(X, Y)$ is defined by:

$$g(e_i, e_j) = \begin{cases} -1, & \text{if } \ i = j = 1 \\ 1, & \text{if } \ i = j = 2 \text{ or } 3 \\ 0, & \text{if } \ i \neq j; \ i, \ j = 1, 2, 3. \end{cases}$$

Let η be the 1-form defined by:

$$\eta(Z) = g(Z, e_1), \text{for any } Z \in \chi(M_3).$$

Let ϕ be a (1, 1)-tensor field on M_3 defined by:

$$\phi(e_1) = 0, \phi(e_2) = -e_2, \phi(e_3) = -e_3 \ \text{ and } \ \phi^2(e_1) = 0, \phi^2(e_2) = e_2, \phi^2(e_3) = e_3.$$

The linearity of ϕ and $g(X, Y)$ yields that

$$\eta(e_1) = -1, \ \phi^2(Z) = Z + \eta(Z)e_1 \ \text{ and } \ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for any vector fields $X, \ Y, \ Z \in \chi(M_3)$. Thus, for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact structure on M_3.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then, we have [14]

$$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_2, e_3] = 0.$$

The Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{57}$$

By using the above Koszul's formula and on taking $e_1 = \xi$, we get the following [14]:

$$\begin{aligned} & \nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0, \\ & \nabla_{e_2} e_1 = -e_2, \nabla_{e_2} e_2 = -e_1, \nabla_{e_2} e_3 = 0, \\ & \nabla_{e_3} e_1 = -e_3, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = -e_1. \end{aligned} \tag{58}$$

From the above calculations, we see that the manifold under consideration satisfies all the properties of Lorentzian para-Kenmotsu manifold i.e., $\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi$ and $(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X$, for all $e_1 = \xi$. Thus, the manifold M_3 under consideration with the structure (ϕ, ξ, η, g) is a 3-dimensional Lorentzian para-Kenmotsu manifold [14].

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{59}$$

Then, by using (58) and (59), the non-vanishing components of the curvature tensor are obtained as [14]:

$$R(e_1,e_2)e_1 = e_2, R(e_1,e_2)e_2 = e_1, R(e_1,e_3)e_1 = e_3,$$
$$R(e_1,e_3)e_3 = e_1, R(e_2,e_3)e_2 = -e_3, R(e_2,e_3)e_3 = e_2. \tag{60}$$

With the help of above expressions of the curvature tensors, it follows that

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{61}$$

This proves that the 3-dimensional manifold M_3 under consideration is an LP-Kenmotsu manifold and it admits a pseudo-projective curvature tensor.

Let X, Y and Z be any three vector fields given by:

$$X = a_1e_1 + a_2e_2 + a_3e_3, Y = b_1e_1 + b_2e_2 + b_3e_3, Z = c_1e_1 + c_2e_2 + c_3e_3; \tag{62}$$

where a_i, b_i, c_i are all non-zero real numbers, for all $i = 1, 2, 3$.

By putting $Z = \xi = e_1$ in (61) and on using (62), we get that

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y = a_1b_2e_2 + a_1b_3e_3 - a_2b_1e_2 - a_3b_1e_3.$$

Further, in view of (61) and (62), we get

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y &= (c_1e_2 + c_2e_1)(a_1b_2 - a_2b_1) \\ &+ (a_1b_3 - a_3b_1)(c_1e_3 + c_3e_1) + (a_2b_3 - a_3b_2)(c_3e_2 - c_2e_3) \end{aligned}$$

and hence from (1) we have

$$\begin{aligned} \overline{W}(X,Y)Z = [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big]&(c_1e_2 + c_2e_1)(a_1b_2 - a_2b_1) \\ + (a_1b_3 - a_3b_1)(c_1e_3 + c_3e_1) &+ (a_2b_3 - a_3b_2)(c_3e_2 - c_2e_3), \end{aligned} \tag{63}$$

and

$$\begin{aligned} &\overline{W}(X,Y)\xi \\ &= [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big](a_1b_2e_2 + a_1b_3e_3 - a_2b_1e_2 - a_3b_1e_3). \end{aligned} \tag{64}$$

Hence, we can say that $\overline{W}(X,Y)Z = 0$ (or) $\overline{W}(X,Y)\xi = 0$, only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

This proves that the manifold M_3 under consideration is an LP-Kenmotsu manifold and it admits a flat pseudo-projective curvature tensor, provided the above condition is satisfied.

Further, by using (60), we obtain the Ricci tensors and scalar curvatures as follows: $S(e_1,e_1) = -2$, $S(e_2,e_2) = 2$, $S(e_3,e_3) = 2$ and $r = 6$, where

$$S(X,Y) = \sum_{i=1}^{3} \epsilon_i \; g(R(e_i,X)Y, e_i),$$

$$r = \sum_{i=1}^{3} \epsilon_i \; S(e_i,e_i) \text{ and } \epsilon_i = g(e_i,e_i).$$

The above arguments verifies the results discussed in sections 4 and 5.

7. Conclusions

The present work explores the geometrical significance of a new class of Lorentzian paracontact metric manifolds namely Lorentzian para-Kenmotsu manifolds whenever a pseudo-projective curvature tensor admitted by these manifolds exhibits the physical phenomena, *i.e.*, the curvature tensor is either irrotational or conservative.

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Roughness of soft sets over a semigroup

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Abstract. In this study we discuss the concept of rough soft sets over a semigroup. Basic results of the lower and upper approximations of soft semigroups , soft ideals, soft bi-ideals and soft interior ideals over a semigroup with a congruence relation are introduced. Finally, topological structures of rough soft sets are presented.
Keywords: soft sets, soft semigroups, soft ideals, rough soft sets.

1. Introduction

A lot of concepts like fuzzy sets [9], rough sets [14], soft sets [8] are presented to deal mathematically with uncertain knowledge. In 1982, Pawlak [14] introduced the rough set theory as an extension of ordinary set theory, in which a pair of ordinary sets namely the lower approximation and upper approximation are associated to a subset of a universe. A connection between algebraic systems and rough sets are studied by some authors. One of these algebraic systems is semigroup theory, which is the main interest of this study. Kuroki in [7], presented the concept of a rough ideal in a semigroup. Since Molodtsov [8] intoduced the theory of soft sets, the literature of soft algebraic systems has grown rapidly. For example, Aktas and Cagman [1] initiated the concept of soft groups, Ali et al. [2] applied the notion of soft sets to the semigroup theory, and

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introduced the concept of soft semigroups and soft ideals. Following, the present authors defined several types of soft ideals over a semigroup in [10, 11, 12]. In 2010, Feng et al. [3] studied the relation between soft sets and rough sets by introducing the concept of rough soft set. Ghosh and Samanta [4] discussed the main properties of rough soft sets and defined rough soft groups. In this paper, we deal with rough soft set theory by giving the universal set the structure of a semigroup and examine some basic properties of rough soft sets with illustrative examples. Then, we study the roughness of soft semigroups, soft left (right) ideals, soft bi-ideals and soft interior ideals over a semigroup. Finally, some topological spaces induced by rough soft sets are studied.

2. Preliminaries

A nonempty subset $\phi \neq B \subseteq S$ of a semigroup S is called a bi-ideal (an interior ideal) of S if it is subsemigroup of S and $BSB \subseteq B(SBS \subseteq B)$ (see, [5]).

Definition 2.1 ([8]). *Let E be a set of parameters, $P(S)$ the power set of S and $A \subseteq E$. The pair (F, A) is called a soft set over S, where F is a mapping $F : A \longrightarrow P(S)$.*

The soft set (F, A) is called a nonempty soft set,$(F, A) \neq \phi$, if and only if $F(a) \neq \phi$, for all $a \in A$. Here, we fix A as the set of parameters and denote the set of all soft sets over a semigroup S by $\mathcal{T}(S)$.

Definition 2.2 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$, then (G, A) is called a soft subset of (F, A), denoted by $(G, A) \sqsubseteq (F, A)$ if $G(a) \subseteq F(a)$, for all $a \in A$. The two sets (F, A) and (G, A) are equal iff $(G, A) \sqsubseteq (F, A)$ and $(F, A) \sqsubseteq (G, A)$.*

Definition 2.3 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$, the intersection of (F, A) and (G, A) is the soft set $(F \sqcap G, A)$ such that $F \sqcap G(a) = F(a) \cap G(a)$, for all $a \in A$.*

Definition 2.4 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$, the union of (F, A) and (G, A) is the soft set $(F \sqcup G, A)$ such that $F \sqcup G(a) = F(a) \cup G(a)$ for all $a \in A$.*

Definition 2.5 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$. The soft product $(F, A) \bullet (G, A)$ is defined as the soft set (FG, A) where $FG(a) = F(a)G(a)$, for all $a \in A$.*

Definition 2.6 ([2]). *A soft set (F, A) over a semigroup S is called a soft semigroup if $(F, A) \bullet (F, A) \sqsubseteq (F, A)$.*

Definition 2.7 ([2]). *A soft set (F, A) over S is called a soft left [right] ideal over S, if $(S, A) \bullet (F, A) \sqsubseteq (F, A)$ $[(F, A) \bullet (S, A) \sqsubseteq (F, A)]$, where (S, A) is a soft set over S defined by $S(a) = S$ $\forall a \in A$. A soft set (F, A) is called a soft ideal if it is both a soft left and a soft right ideal over S.*

Proposition 2.1 ([2]). *A soft set (F, A) is a soft semigroup (ideal) over S if and only if $\forall a \in A$, $F(a) \neq \phi$ is a subsemigroup (an ideal) of S.*

Definition 2.8 ([2, 11]). *A soft set (F, A) over a semigroup S is called a soft bi-ideal (interior ideal) if and only if $\forall a \in A$, $F(a) \neq \phi$ is a bi-ideal (interior ideal) of S.*

Definition 2.9 ([13]). *A collection $\mathcal{T}$ of soft sets over S is called a soft topology on S if:*

(i) $(\phi, A), (S, A) \in \mathcal{T}$ *where* $\phi(a) = \phi$ *and* $S(a) = S$, *for all* $a \in A$,

(ii) the intersection of any two soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$,

(iii) the union of any number of soft sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The triplet $(S, \mathcal{T}, A)$ is called a soft topological space over S.

3. Approximation of soft sets over a semigroup

Let (S, θ) be a Pawlak approximation space (PAS), that is, θ is an equivalence relation on a semigroup S. The lower approximation $\theta_\star(X)$ and upper approximation $\theta^\star(X)$ of $X \subseteq S$ are defined by[14]

$$\theta_\star(X) = \{x \in S : [x] \subseteq X\},$$

$$\theta^\star(X) = \{x \in S : [x] \cap X \neq \phi\}.$$

Definition 3.1 ([3]). *Let (S, θ) be PAS and (F, A) be a soft set over S. The lower approximation $(\underline{F}, A)$ and upper approximation $(\overline{F}, A)$ of (F, A) are soft sets over S defined by*

$$\underline{F}(a) = \theta_\star(F(a)) = \{x \in S : [x] \subseteq F(a)\},$$
$$\overline{F}(a) = \theta^\star(F(a)) = \{x \in S : [x] \cap F(a) \neq \phi\},$$

for all $a \in A$. If $(\underline{F}, A) = (\overline{F}, A)$, then (F, A) is called definable; otherwise (F, A) is called a rough soft set.

The following properties of rough soft sets are due to [3]. We shall give a proof for completeness.

Theorem 3.1. *Suppose that (S, θ) is PAS. If $(F, A), (G, A) \in \mathcal{T}(S)$, then the following hold:*

(1) $(\underline{F}, A) \sqsubseteq (F, A) \sqsubseteq (\overline{F}, A)$;

(2) $(\overline{F \sqcup G}, A) = (\overline{F}, A) \sqcup (\overline{G}, A)$;

(3) $(\underline{F \sqcap G}, A) = (\underline{F}, A) \sqcap (\underline{G}, A)$;

(4) *If* $(F, A) \sqsubseteq (G, A) \Longrightarrow (\underline{F}, A) \sqsubseteq (\underline{G}, A)$ *and* $(\overline{F}, A) \sqsubseteq (\overline{G}, A)$;

(5) $(\overline{F \sqcap G}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A)$;

(6) $(\underline{F \sqcup G}, A) \sqsupseteq (\underline{F}, A) \sqcup (\underline{G}, A)$;

(7) $(\underline{\underline{F}}, A) = (\underline{F}, A)$;

(8) $(\overline{\underline{F}}, A) = (\underline{F}, A)$.

Proof. (1) Let $u \in \underline{F}(a)$ then $u \in [u] \subseteq F(a)$, for all $a \in A$. Thus, $(\underline{F}, A) \sqsubseteq (F, A)$. If $u \in F(a) \Rightarrow [u] \cap F(a) \neq \phi$, that is, $u \in \theta^\star(F(a)) = \overline{F}(a)$. Hence, $(F, A) \sqsubseteq (\overline{F}, A)$.

(2) Let $u \in \overline{F \sqcup G}(a) = \theta^\star((F \sqcup G)(a)) = \theta^\star(F(a) \cup G(a)) = \theta^\star(F(a)) \cup \theta^\star(G(a)) = \overline{F}(a) \cup \overline{G}(a)$. Thus, $(\overline{F \sqcup G}, A) = (\overline{F}, A) \sqcup (\overline{G}, A)$.

(3) Let $u \in \underline{F \sqcap G}(a) = \theta_\star((F \sqcap G)(a)) = \theta_\star(F(a) \cap G(a)) = \theta_\star(F(a)) \cap \theta_\star(G(a)) = \underline{F}(a) \cap \underline{G}(a)$. Thus, $(\underline{F \sqcap G}, A) = (\underline{F}, A) \sqcap (\underline{G}, A)$.

(4) Let $u \in \underline{F}(a) = \theta_\star(F(a)) \subseteq \theta_\star(G(a)) = \underline{G}(a)$, since $F(a) \subseteq G(a)$, for all $\in A$. Hence, $(\underline{F}, A) \sqsubseteq (\underline{G}, A)$. Similarly, we show that $(\overline{F}, A) \sqsubseteq (\overline{G}, A)$.

(5) Let $u \in \overline{F \sqcap G}(a) = \theta^\star(F \sqcap G(a)) = \theta^\star(F(a) \cap G(a)) \subseteq \theta^\star(F(a)) \cap \theta^\star(G(a)) = \overline{F}(a) \cap \overline{G}(a)$. Therefore, $(\overline{F \sqcap G}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A)$.

(6) Let $u \in \underline{F \sqcup G}(a) = \theta_\star(F \sqcup G(a)) = \theta_\star(F(a) \cup G(a)) \supseteq \theta_\star(F(a)) \cup \theta_\star(G(a)) = \overline{F}(a) \cup \overline{G}(a)$. Therefore, $(\underline{F \sqcup G}, A) \sqsupseteq (\underline{F}, A) \sqcup (\underline{G}, A)$.

(7), (8) see items (4,6) in Theorem 3 [3]. □

Definition 3.2 ([7]). *An equivalence relation θ on a semigroup S is called a congruence on S if $(a, b) \in \theta$ implies $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$, for all $x \in S$. A congruence θ on S is called complete if $[x][y] = [xy]$, for all $x, y \in S$.*

Theorem 3.2. *Let θ be a congruence relation on S. If (F, A) and (G, A) are nonempty soft sets over S, then*

$$(\overline{F}, A) \bullet (\overline{G}, A) \sqsubseteq (\overline{FG}, A).$$

Proof. By applying Theorem 2.2 in [7], we have

$$\begin{aligned}\overline{F}(a)\overline{G}(a) &= \theta^\star(F(a))\theta^\star(G(a)) \\ &\subseteq \theta^\star(F(a)G(a)) = \theta^\star(FG(a)) = \overline{FG}(a).\end{aligned}$$

Then, $\overline{F}(a)\overline{G}(a) \subseteq \overline{FG}(a)$, for all $a \in A$. Thus, we have $(\overline{F}, A) \bullet (\overline{G}, A) \sqsubseteq (\overline{FG}, A)$. □

The following theorem comes as a direct application of Theorem 2.3 in [7].

Theorem 3.3. *Let θ be a complete congruence relation on S. If (F, A) and (G, A) are nonempty soft sets over S, then*

$$(\underline{F}, A) \bullet (\underline{G}, A) \sqsubseteq (\underline{FG}, A).$$

Proposition 3.1 ([2]). *A soft set (F, A) is a soft semigroup over S if and only if $\forall a \in A$, $F(a) \neq \phi$ is a subsemigroup of S.*

Theorem 3.4. *Let θ be a congruence relation on a semigroup S. Then:*

(1) *(F, A) is a soft semigroup $\Longrightarrow$ $(\overline{F}, A)$ is a soft semigroup;*

(2) *(F, A) is a soft ideal $\Longrightarrow$ $(\overline{F}, A)$ is a soft ideal.*

Proof. (1) Assume that (F, A) is a soft semigroup over S, then $F(a)$ is a subsemigroup of S, for all $a \in A$. Since $(\overline{F}, A)$ is nonempty soft set then Theorem 3.2 implies that

$$(\overline{F}, A) \bullet (\overline{F}, A) \sqsubseteq (\overline{FF}, A).$$

That is, $\overline{F}(a)\overline{F}(a) \subseteq \overline{FF}(a)$, for all $a \in A$. By Theorem 3.1, we obtain

$$\overline{F}(a)\overline{F}(a) \subseteq \overline{FF}(a) = \theta^\star(FF(a)) = \theta^\star(F(a)F(a)) \subseteq \theta^\star(F(a)) = \overline{F}(a).$$

Thus, $\overline{F}(a)$ is a subsemigroup of S, for all $a \in A$. Therefore, $(\overline{F}, A)$ is a soft semigroup over S.

(2) Assume that (F, A) is a soft left ideal over S, then $F(a)$ is a left ideal of S, for all $a \in A$. Note that $\theta^\star(S) = S$. Then, Theorem 3.2 implies

$$(\overline{S}, A) \bullet (\overline{F}, A) \sqsubseteq (\overline{SF}, A).$$

That is, $\overline{S}(a)\overline{F}(a) \subseteq \overline{SF}(a)$, for all $a \in A$. By Theorem 3.1, we obtain

$$\overline{S}(a)\overline{F}(a) \subseteq \overline{SF}(a) = \theta^\star(SF(a)) \subseteq \theta^\star(F(a)) = \overline{F}(a).$$

Thus, $\overline{F}(a)$ is a left ideal of S, for all $a \in A$. Therefore, $(\overline{F}, A)$ is a soft left ideal over S. In a similar way, it can be shown that $(\overline{F}, A)$ is a soft right ideal over S whenever (F, A) is. This completes the proof. □

Theorem 3.4 shows that every soft semigroup (soft ideal) (F, A) over a semigroup S can be extended to the largest soft semigroup (soft ideal) $(\overline{F}, A)$. Generally, the converse of the above theorem does not hold, as shown in the following example.

Example 3.1. *Let $S = \{a, b, c, d\}$ be a semigroup with the following table* [7]:

*	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	a

Suppose S is partitioned by a congruence relation θ into the classes: $\{a\}, \{d\}, \{b, c\}$ Let $A = \{e_1, e_2\}$ and (F, A) be a soft set over S defined by

$$F(e_1) = \{b\}, \quad F(e_2) = S.$$

Then, the upper approximation $(\overline{F}, A)$ of (F, A) is defined as follows:

$$\overline{F}(e_1) = \theta^\star(\{b\}) = \{b, c\}, \quad \overline{F}(e_2) = \theta^\star(S) = S.$$

It is clear that $(\overline{F}, A)$ is a soft ideal over S while (F, A) is not.

The below result is an application to Theorem 3.2 in [7].

Theorem 3.5. *Let θ be a complete congruence relation on S. Then*

(1) (F, A) *is a soft semigroup* $\Longrightarrow$ $(\underline{F}, A) \neq \phi$ *is a soft semigroup;*

(2) (F, A) *is a soft ideal* $\Longrightarrow$ $(\underline{F}, A) \neq \phi$ *is a soft ideal.*

Proof. (1) Suppose that (F, A) is a soft semigroup over S, then $F(a)$ is a subsemigroup of S, for all $a \in A$. Since $(\underline{F}, A)$ is nonempty soft set then Theorem 3.3 implies that

$$(\underline{F}, A) \bullet (\underline{F}, A) \sqsubseteq (\underline{FF}, A).$$

That is, $\underline{F}(a)\underline{F}(a) \subseteq \underline{FF}(a)$, for all $a \in A$. By Theorem 3.1, we obtain

$$\underline{F}(a)\underline{F}(a) \subseteq \underline{FF}(a) = \theta_\star(FF(a)) = \theta_\star(F(a)F(a)) \subseteq \theta_\star(F(a)) = \underline{F}(a).$$

Thus, $\underline{F}(a)$ is a subsemigroup of S, for all $a \in A$. Therefore, $(\underline{F}, A)$ is a soft semigroup over S.

(2) Assume that (F, A) is a soft left ideal over S, then $F(a)$ is a left ideal of S, for all $a \in A$. Note that $\theta_\star(S) = S$. Then, Theorem 3.1 implies

$$(\underline{S}, A) \bullet (\underline{F}, A) \sqsubseteq (\underline{SF}, A).$$

That is, $\underline{S}(a)\underline{F}(a) \subseteq \underline{SF}(a)$, for all $a \in A$. By Theorem 3.3, we obtain

$$\underline{S}(a)\underline{F}(a) \subseteq \underline{SF}(a) = \theta_\star(SF(a)) \subseteq \theta_\star(F(a)) = \underline{F}(a).$$

Thus, $\underline{F}(a)$ is a a left ideal of S, for all $a \in A$. Therefore, $(\underline{F}, A)$ is a soft left ideal over S. In a similar way, $(\underline{F}, A)$ is a soft right ideal over S whenever (F, A) is. This completes the proof. □

Theorem 3.6. *Let θ be a congruence relation on S. Then, $(\overline{F}, A)$ is a soft bi-ideal over S if (F, A) is a soft bi-ideal.*

Proof. Let (F, A) be a soft bi-ideal over S then (F, A) is a soft semigroup over S and by theorem 3.4, $(\overline{F}, A)$ is a soft semigroup. By applying Theorem 3.2 more times, we have

$$\begin{aligned}
\overline{F}(a)\overline{S}(a)\overline{F}(a)) &= \theta^\star(F(a))\theta^\star(S(a))\theta^\star(F(a)) \\
&\subseteq \theta^\star(F(a))\theta^\star(S(a)F(a)) \\
&\subseteq \theta^\star(F(a)S(a)F(a)) \\
&\subseteq \theta^\star(F(a)) = \overline{F}(a).
\end{aligned}$$

Thus $\overline{F}(a)$ is a bi-ideal of S, for all $a \in A$. Therefore,$(\overline{F}, A)$ is a soft bi-ideal over S. □

Theorem 3.7. *Let θ be a complete congruence relation on S. Then, $(\underline{F}, A) \neq \phi$ is a soft bi-ideal over S if (F, A) is a soft bi-ideal.*

Proof. It is immediate by applying Theorem 3.3 and Theorem 3.5. □

Theorem 3.8. *Let θ be a congruence relation on S. Then, $(\overline{F}, A)$ is a soft interior ideal over S if (F, A) is a soft interior ideal.*

Proof. Let (F, A) be a soft interior ideal over S then (F, A) is a soft semigroup over S and by theorem 3.4, $(\overline{F}, A)$ is a soft semigroup. By applying Theorem 3.2 , we have

$$\begin{aligned} S\overline{F}(a)S = \overline{S}(a)\overline{F}(a)\overline{S}(a) &= \theta^\star(S(a))\theta^\star(F(a))\theta^\star(S(a)) \\ &\subseteq \theta^\star(S(a))\theta^\star(F(a)S(a)) \\ &\subseteq \theta^\star(S(a)F(a)S(a)) \\ &\subseteq \theta^\star(F(a)) = \overline{F}(a). \end{aligned}$$

Thus, $\overline{F}(a)$ is an interior ideal of S, for all $a \in A$. Therefore, $(\overline{F}, A)$ is a soft interior ideal over S. □

Theorem 3.9. *Let θ be a complete congruence relation on S. Then, $(\underline{F}, A) \neq \phi$ is a soft interior ideal over S if (F, A) is a soft interior ideal.*

Proof. It is immediate by applying Theorem 3.3 and Theorem 3.5. □

Theorem 3.10. *Let θ be a congruence relation on S. (F, A) and (G, A) are a soft right ideal and a soft left ideal over S, respectively, then:*

(i) $(\overline{FG}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A)$;

(ii) $(\underline{FG}, A) \sqsubseteq (\underline{F}, A) \sqcap (\underline{G}, A)$.

Proof. (i) By hypotheses, $F(a)$ is a right ideal of S and $G(a)$ is a left ideal of S, for all $a \in A$. Then, we have

$$F(a)G(a) \subseteq F(a)S \subseteq F(a),\ F(a)G(a) \subseteq SG(a) \subseteq G(a).$$

Thus, $F(a)G(a) \subseteq F(a) \cap G(a)$, for all $a \in A$. Theorem 3.1 verifies that

$$(\overline{FG}, A) \sqsubseteq (\overline{F \sqcap G}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A).$$

(ii) Similar to item (i). This completes the proof. □

Assume θ and ρ are congruence relations on S. Then, the composition

$$\theta \circ \rho = \{(x, y) : (x, z) \in \theta,\ (z, y) \in \rho\}$$

is a congruence relation on S iff $\theta \circ \rho = \rho \circ \theta$ (see, [7]).

Theorem 3.11. *Let θ and ρ be congruence relations on S such that $\theta \circ \rho = \rho \circ \theta$. If (F, A) is a soft semigroup over S, then*

$$(\overline{F}_\theta, A) \bullet (\overline{F}_\rho, A) \sqsubseteq (\overline{F}_{\theta\circ\rho}, A).$$

Proof. Let $c \in \overline{F}_\theta(a)\overline{F}_\rho(a)$ then there exist $x \in \overline{F}_\theta(a) = \theta^\star(F(a))$ and $y \in \overline{F}_\rho(a) = \rho^\star(F(a))$ such that $c = xy \in \theta^\star(F(a))\rho^\star(F(a)) \subseteq S$. From definition of the upper approximation, there exist $z, w \in S$ such that

$$z \in [x]_\theta \cap F(a), \quad w \in [y]_\rho \cap F(a).$$

That is, $(z, x) \in \theta$ and $(w, y) \in \rho$. Since θ and ρ are congruence relations on S, then $(zw, xw) \in \theta$ and $(xw, xy) \in \rho$. This implies that $(zw, xy) \in \theta \circ \rho$. Since $F(a)$ is a subsemigroup of S, for all $a \in A, zw \in F(a)$. Therefore we have $zw \in [xy]_{\theta\circ\rho} \cap F(a)$ which implies $xy \in (\theta \circ \rho)^\star(F(a)) = \overline{F}_{\theta\circ\rho}(a)$. Thus, we have

$$(\overline{F}_\theta, A) \bullet (\overline{F}_\rho, A) \sqsubseteq (\overline{F}_{\theta\circ\rho}, A).$$

4. Topological structures on rough soft sets

In this section, some topological spaces induced by rough soft sets are discussed. Throughout this section, S is a semigroup, θ is an equivalence relation on S.

4.1 Topological semigroups Vs soft sets

Topologies versus rough sets are studied by M. Kondo [6].

Proposition 4.1 ([6])**.** *$T_\theta = \{X \subseteq S : \theta_\star(X) = X\}$ is a topology on S.*

Furthermore, we show that the pair (S, T_θ) is a topological semigroup.

Theorem 4.1. *If θ is a complete congruence relation on S, (S, T_θ) is a topological semigroup.*

Proof. Let $x, y \in S$ and $U \in T_\theta$ be an open set containing the element $p = xy$. Then, $xy \in U = \theta_\star(U)$ which implies $[xy] \subseteq U$. By completeness of θ, we have $[x][y] = [xy] \subseteq U$. Since, $[x], [y]$ are open sets containing x, y respectively such that $[x][y] \subseteq U$ then we conclude that the multiplication $. : S \times S \to S$ of S is a continuous mapping. Therefore, (S, T_θ) is a topological semigroup. □

Theorem 4.2. *Let θ be a complete congruence relation on S, and (F, A) be a soft semigroup over S. Then, for all $a \in A$, the pair $(F(a), T_{F(a)})$ is a topological semigroup, where $T_{F(a)}$ is the relative topology on $F(a)$ induced from T_θ.*

Proof. By Theorem 4.1,(S, T_θ) is a topological semigroup. Since (F, A) is a soft semigroup over S, then $F(a)$ is a subsemigroup of S, for all $a \in A$. Let $x, y \in F(a)$ and $W \in T_{F(a)}$ containing the element xy. From definition of $T_{F(a)}$, there exists an open set $U \in T_\theta$ such that $xy \in W = F(a) \cap U$. Hence, $xy \in U = \theta_\star(U)$ that is, $[xy] \subseteq U$. Since the classes $[x], [y] \in T_\theta$, the two sets $F(a) \cap [x]$, $F(a) \cap [y]$ are open sets in $F(a)$ and containing x, y respectively. Then, we obtain

$$(F(a) \cap [x])(F(a) \cap [y]) \subseteq F(a) \cap [x][y] = F(a) \cap [xy] \subseteq F(a) \cap U = W.$$

Thus, for all $a \in A$, the pair $(F(a), T_{F(a)})$ is a topological semigroup. □

4.2 Soft topologies Vs rough soft sets

Let (F, A) be a soft set over S. Denote

$$\mathcal{T}_B = \{(F, A) \in \mathcal{T}(S) : (\underline{F}, A) = (\overline{F}, A)\},$$

$$\mathcal{T}_E = \{(F, A) \in \mathcal{T}(S) : (\underline{F}, A) = (F, A)\},\ \mathcal{T}_L = \{(\underline{F}, A) : (F, A) \in \mathcal{T}(S)\}.$$

By using items (7),(8) in Theorem 3.1, the proof of the following result is a clear matter.

Proposition 4.2. $\mathcal{T}_B = \mathcal{T}_E = \mathcal{T}_L$.

Notatios:. The complement of $(F, A) \in \mathcal{T}(S)$ is a soft set $(H, A) \in \mathcal{T}(S)$ such that $H(a) = F(a)^c = S - F(a)$, for all $a \in A$.

Proposition 4.3. *Let (S, θ) be PAS. Then, for all $(F, A) \in \mathcal{T}(S)$,*

$$(\underline{F}, A) = (F, A) \Longleftrightarrow (H, A) = (\underline{H}, A).$$

Proof. $(\Rightarrow)$ Suppose that $(\underline{F}, A) = (F, A)$. That is, $F(a) = \theta_\star(F(a))$, for all $a \in A$. From Propostion 5 in [6], we have

$$H(a) = F(a)^c = \theta_\star(F(a)^c) = \theta_\star(H(a)) = \underline{H}(a).$$

This means that $(H, A) = (\underline{H}, A)$.

$(\Leftarrow)$ Proof of the converse statement is similar. □

Theorem 4.3. *Let (S, θ) be PAS, then:*

(i) $(S, \mathcal{T}_E, A)$ is a soft topological space.

(ii) $(F, A) \in \mathcal{T}_E \Longleftrightarrow (H, A) \in \mathcal{T}_E$.

Proof. (i) Since $\underline{S}(a) = \theta_\star(S(a)) = \theta_\star(S) = S$, for all $a \in A$, then the whole soft set $(S, A) \in \mathcal{T}_E$. Similarly the empty soft set $(\phi, A) \in \mathcal{T}_E$. Assume that (F, A) and (G, A) are arbitrary elements in $\mathcal{T}_E$, then, by Theorem 3.1, we have

$$\underline{F \sqcap G}(a) = \underline{F} \sqcap \underline{G}(a) = \underline{F}(a) \cap \underline{G}(a) = F(a) \cap G(a) = F \sqcap G(a)$$

Thus, $(F, A) \sqcap (G, A)$ is an element in $\mathcal{T}_E$.

Let $\{(F_i, A) : i \in J\} \subset \mathcal{T}_E$ be a family of soft open sets over S, then, by item(1) in Theorem 3.1, we have

$$(\underline{\sqcup_{i\in J} F_i}, A) \sqsubseteq (\sqcup_{i\in J} F_i, A).$$

Since $(\underline{F_i}, A) = (F_i, A)\, \forall i \in J$ and by Propostion 2 in [6], we have

$$\sqcup_{i\in J} F_i(a) = \bigcup_{i\in J} F_i(a) \subseteq \theta_\star(\bigcup_{i\in J} F_i(a)) = \theta_\star(\sqcup_{i\in J} F_i(a)) = \underline{\sqcup_{i\in J} F_i}(a),$$

for all $a \in A$. Then, $(\underline{\sqcup_{i\in J} F_i}, A) = (\sqcup_{i\in J} F_i, A)$. So $(\sqcup_{i\in J} F_i, A)$ is an element in $\mathcal{T}_E$. Therefore, $(S, \mathcal{T}_E, A)$ is a soft topological space.

(ii) Let $(F, A) \in \mathcal{T}_E \Leftrightarrow (\underline{F}, A) = (F, A)$. Then, Proposition 4.3 implies that $(\underline{H}, A) = (H, A)$ and so $(H, A) \in \mathcal{T}_E$. Thus, (F, A) is soft closed.

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On the primary-like dimension of modules

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Abstract. Let R be a ring and let M be a left R-module. In this article, we introduce and study the primary-like dimension of M was defined to be the supremum of the lengths of all strong-like chains of primary-like submodules of M and denoted by $P.L.dim(M)$.

Keywords: primary-like dimension, virtually-like Noetherian, virtually-like Artinian.

1. Introduction

In this paper, all rings are associative rings with identity, and all modules are unital and left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If Q is a submodule of M, then we denote the left annihilator of a factor module M/N of M by $(Q : M)$. We call M faithful if $(0 : M) = 0$. Recall that a left R-module M is said to be prime if $Ann(Q) = Ann(M)$ for every nonzero submodule Q of M. A proper submodule Q of M is called a prime submodule if the quotient module M/Q is a prime module, i.e., if $IN \subseteq Q$, where N is a submodule of M and I is an ideal of R, then either $N \subseteq Q$ or $IM \subseteq Q$. The collection of all prime submodules of M is denoted by $Spec(M)$. This notion of prime submodule was first introduced and systematically studied in [4] and recently it has received a good deal of attention from several authors, see, for example, [1, 2, 10, 11, 15, 18, 20] and many others. There is already a generalization of classical Krull dimension for modules via prime dimension. In fact, the notion of prime dimension of a module $dim(M)$ over a commutative ring $dim(M)$ (denoted by $dim(M)$), was introduced by Marcelo and Masqué [14], as the maximum length of the chains of prime submodules of M (see also [13, 19] for some known results about the prime dimension of modules). A submodule Q of M is said to be primary-like if $Q \neq M$ and whenever $rm \in Q$ (where $r \in R$ and $m \in M$) implies $r \in (Q : M)$ or $m \in radQ$ [5, 6]. An R-module M is said to be primeful if either $M = (0)$ or $M \neq (0)$ and the map $\psi : Spec(M) \longrightarrow Spec(R/Ann(M))$ defined by $Q \longmapsto (Q : M)/Ann(M)$ is surjective[12]. If M/Q is a primeful over R, then $\sqrt{(Q : M)} = (radQ : M)$ [12, Proposition 5.3]. It is easily seen that, if Q is a primary-like submodule of Q such that M/Q is a primeful over R, then $(Q : M)$ is a primary ideal of R and so $P = \sqrt{(Q : M)}$ is a prime ideal of R, and in this case Q is called a P-primary-like submodule of M. The primary-like spectrum of M denoted by $Spec_L(M)$ is defined to be the set of all primary-like submodules Q of M, where M/Q is primeful. In this article, when we say that Q is a primary-like submodue of M, it means that Q is primary-like submodule of M, where M/Q is primeful; i. e. $Q \in Spec_L(M)$. Let M be a left R-module and Q, Q' be two submodules of M. We say that Q is strongly-like properly contained in Q', and write $Q \subset_{sl} Q'$, if $Q \subset Q'$ and also $\sqrt{(Q : M)} \subset \sqrt{(Q' : M)}$. In this case, we also say that Q' strongly-like properly contains Q. Also, $Q \subseteq_{sl} Q'$ means that $Q \subset_{sl} Q'$ or $Q = Q'$. A submodule Q of M will be called virtually maximal primary-like if Q is primary-like and there is no primary-like submodule Q' such that $Q \subset_{sl} Q'$.

Let R be a ring and M be a left R-module such that every primary-like submodule of M is contained in a virtually maximal primary-like submodule. We define, by transfinite induction, sets X_α of primary-like submodules of M. To start with, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined, for all ordinals $\beta < \alpha$, let X_α be the set of those primary-like submodules Q in M such that all primary-like submodules strongly-like properly containing Q belong to $\bigcup_{\beta<\alpha} X_\beta$. (In particular, X_0 is the set of virtually maximal primary-like submodules of M.) If some X_γ contains all primary-like submodules of M, we say that $P.L.dim(M)$ exists, and we set $P.L.dim(M)$-the primary-like dimension of M-equal to the smallest such γ. We write $P.L.dim(M) = \gamma$ as an abbreviation for the statement that $P.L.dim(M)$ exists and equals γ.

In Section 2, we introduce the notion of a virtual-like chain condition on submodules of a module. In Section 3, the meaning of the primary-like dimension of modules and related topics are studied.

2. Virtual-like chain conditions

In this section we introduce the notion of virtual-like chain condition on submodules of a module.

Definition 2.1. *Let R be a ring and M be a left R-module. A submodule Q of M will be called:*

(1) *maximal primary-like if Q is a primary-like submodule of M and there is no primary-like submodule Q' of M such that $Q \subset Q'$;*

(2) *virtually maximal primary-like if Q is a primary-like submodule of M and there is no primary-like submodule Q' of M such that $Q \subset_{sl} Q'$ (i.e., Q is a primary-like submodule of M and for any primary-like submodule Q' of M, such that $Q \subseteq Q'$, we have $\sqrt{(Q : M)} = \sqrt{(Q' : M)}$);*

(3) *virtually maximal if the factor module M/Q is a homogeneous semisimple module (see also [16], for definition).*

Example 2.1. Let $M = \mathbb{Q} \bigoplus \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the cyclic group of order p. Then $Spec(M) = \{\mathbb{Q} \bigoplus 0, 0 \bigoplus \mathbb{Z}_p\}$ by [17, Example 2.6]. Clearly, if N is a submodule of M such that $N \nsubseteq \mathbb{Q} \bigoplus 0$ or $N \nsubseteq 0 \bigoplus \mathbb{Z}_p$, then N does not satisfy the primeful property. Also, If $N \subseteq 0 \bigoplus \mathbb{Z}_p$, then $(N : M) = 0$ and so N dose not satisfy the primeful property. Consider the only remaining case $N \subseteq \mathbb{Q} \bigoplus 0$. In this case, if $(N : M) = p\mathbb{Z}$, then $N = \mathbb{Q} \bigoplus 0$ and so $\mathbb{Q} \bigoplus 0 \in Spec_L(M)$. If $(N : M) = 0$, then N does not satisfy the primeful property. The finial case is $0 \subset (N : M) \subset p\mathbb{Z}$. In this case if N is a primary-like submodule satisfying the primeful property, then $(N : M) = p^i\mathbb{Z}$ for some $i \geq 1$, since $(N : M)$ is a primary ideal of R. Assume $i \neq 1$ and $(0, b) \in M \backslash \mathbb{Q} \bigoplus 0$. Now, $p(0, b) = (0, 0)$, follows $p \in p^i\mathbb{Z}$ which is a contradiction. Therefore, $Spec_L(M) = \{\mathbb{Q} \bigoplus 0\}$. Hence, $\mathbb{Q} \bigoplus 0$ is maximal primary-like and virtually maximal primary-like submodule.

Definition 2.2. *Let R be a ring and M be a left R-module. Then, the chain $Q_1 \subseteq_{sl} Q_2 \subseteq_{sl} Q_3 \subseteq_{sl} \cdots$ of submodules of M is called a strong-like ascending chain. Also, the chain $Q_1 \ {}_{sl}{\supseteq}\ Q_2 \ {}_{sl}{\supseteq}\ Q_3 \ {}_{sl}{\supseteq}\ \cdots$ of submodules of M is called a strong-like descending chain.*

Definition 2.3. *Let R be a ring. A left R-module M is said to satisfy the virtual-like ascending chain condition on submodules (or to be virtually-like Noetherian or virtual-like acc) if for every strong-like chain $Q_1 \subseteq_{sl} Q_2 \subseteq_{sl} Q_3 \subseteq_{sl} \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$. Also, a left R-module M is said to satisfy the virtual-like descending chain condition on submodules (or to be virtually-like Artinian or virtual-like dcc) if for every strong-like chain $Q_1 \ {}_{sl}{\supseteq}\ Q_2 \ {}_{sl}{\supseteq}\ Q_3 \ {}_{sl}{\supseteq} \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$.*

It is clear that every Noetherian (respectively, Artinian) module is virtually-like Noetherian (respectively, virtually-like Artinian). In general, the converse is not true. See the following example

Example 2.2. 1) Let R be a commutative Noetherian (respectively, Artinian) ring. Then, every R-module is virtually-like Noetherian (respectively, virtually-like Artinian).

2) For a prime number p, $\mathbb{Z}(p^\infty)$ as a $\mathcal{Z}$-module is virtually-like Noetherian, since every proper submodule of $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is primary-like. However $Spec_L(\mathbb{Z}(p^\infty)) = Spec(\mathbb{Z}(p^\infty)) = \emptyset$. But it is not a Noetherian $\mathbb{Z}$-module.

3) For $\mathbb{Z}$-module $\mathbb{Q}$, $Spec(\mathbb{Q}) = \{0\}$ and $Spec_L(\mathbb{Q}) = \emptyset$, because $\mathbb{Q}$ have no submodules satisfying the primeful property. Therefore, $\mathbb{Q}$ as a $\mathbb{Z}$-module is virtually-like Artinian, but it is not an Artinian $\mathbb{Z}$-module.

4) For a vector space V over a field F, $Spec_L(V) = Spec(V)$= the set of all proper vector subspaces of V. Hence, every vector space over a field is both virtually-like Noetherian and virtually-like Artinian.

Proposition 2.1. *Let M be a left R-module and Q be a proper submodule of M. Then*

$$\begin{tikzcd}
Q \text{ is max and } M/Q \text{ is primeful} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & Q \text{ is max primary-like} \arrow[d, Rightarrow] \\
Q \text{ is virtually max} \arrow[r, Rightarrow] & Q \text{ is max virtually primary-like} \arrow[d, Rightarrow] \\
 & Q \text{ is primary-like}
\end{tikzcd}$$

Proof. Assume that Q is maximal. Then M/Q is a simple module, and it follows that Q is a maximal primary-like submodule. Also, it is clear that every maximal submodule of M is virtually maximal but, the converse is not true (for example, every proper submodule of a homogeneous semisimple module is virtually maximal but it is not necessarily maximal). Clearly, if Q is a maximal primary-like submodule of M, then Q is virtually maximal primary-like. Now, if Q is virtually maximal, then M/Q is a homogeneous semisimple module. Clearly, for every proper submodule Q' of M, $\sqrt{(Q:M)} = \sqrt{(Q':M)}$ and it follows that Q is a virtually maximal primary-like submodule. Finally, it is clear that every virtually maximal primary-like submodule is primary-like. □

Let M be a left R-module and $N, L \leq M$. We say that N is strongly properly contained in L, and write $N \subset_s L$, if $N \subset L$ and also $(N:M) \subset (L:M)$. A submodule Q of is said to be virtually maximal prime if Q is a prime submodule of M and there is no prime submodule Q' of Q' such that $Q \subset_s \subset Q'$ (i.e., Q is a prime submodule of M and for any prime submodule Q' of M, such that $Q \subseteq Q'$, we have $(Q:M) = (Q':M)$). A left R-module M is said to satisfy the virtual ascending chain condition on submodules (or to be virtually Noetherian or virtual acc) if for every strong chain $Q_1 \subseteq_s Q_2 \subseteq_s Q_3 \subseteq_s \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$. Also, a left R-module M is said to satisfy the virtual descending chain condition on submodules (or to be virtually Artinian or virtual dcc) if for every strong chain $Q_1 \ {}_s{\supseteq}\ Q_2 \ {}_s{\supseteq}\ Q_3 \ {}_s{\supseteq} \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$ (see [3]).

Proposition 2.2. *Let R be a ring. Then, the following statements are equivalent:*

1) *R has acc (respectively, dcc) on two-sided ideals;*

2) *each R-module is virtually-like Noetherian (respectively, virtually-lkie Artinian);*

3) *the left R-module R is virtually-like Noetherian (respectively, virtually-like Artinian);*

4) *the left R-module R is virtually Noetherian* (*respectively, virtually Artinian*)*;*

5) *each R-module is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

6) *each R-module is virtually Noetherian* (*respectively, virtually Artinian*)*.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clrear.
(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows from [3, Proposition 2.1]. $\square$

Corollary 2.1. *Let R be a commutative ring. Then, the following statements are equivalent:*

1) *R is Noetherian* (*respectively, Artinian*)*;*

2) *each R-module is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

3) *the R-module R is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

4) *the R-module R is virtually Noetherian* (*respectively, virtually Artinian*)*;*

5) *each R-module is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

6) *each R-module is virtually Noetherian* (*respectively, virtually Artinian*)*.*

Proof. Follows from Proposition 2.2. $\square$

Definition 2.4. *An R-module M is said to satisfy the virtual-like maximum condition* (*respectively, virtual-like minimum condition*) *on submodules if every nonempty set of submodules of M contains a maximal* (*respectively, minimal*) *element with respect to strong inclusion $\subseteq_{sl}$* (*respectively, ${}_{sl}\supseteq$*)*.*

Proposition 2.3. *An R-module M is virtually-like Noetherian* (*respectively, virtually-like Artinian*) *if and only if M satisfies virtual-like maximum condition* (*respectively, virtual-like minimum condition*) *on submodules.*

Proof. Is clear. $\square$

Proposition 2.4. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of modules. Then, M_2 is virtually-like Noetherian (respectively, virtually-like Artinian) if and only if M_1 and M_2 are virtually-like Noetherian (respectively, virtually-like Artinian).*

Proof. Is clear. $\square$

Corollary 2.2. *Let N be a submodule of an R-module M. Then, M satisfies the strong-like ascending (respectively, descending) chain condition if and only if so do N and M/N.*

Proof. Apply Proposition 2.4 to the sequence

$$0 \to N \overset{\subseteq}{\to} M \to M/N \to 0. \qquad \square$$

Corollary 2.3. *Let $M_1, M_2, \cdots, M_n$ be modules. Then, the direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ satisfies the strong-like ascending (respectively, descending) chain condition on submodules if and only if so does each M_i.*

Proof. Use induction on n. If $n = 2$, apply Proposition 2.4 to the following sequence

$$0 \to M_1 \overset{\iota_1}{\to} M_1 \oplus M_2 \overset{\pi_2}{\to} M_2 \to 0. \qquad \square$$

3. Primary-like dimension for modules

In this section, we introduce and study a new generalization of the Krull dimension for modules.

Definition 3.1. *Let R be a ring and M be a left R-module such that every primary-like submodule of M is contained in a virtually maximal primary-like submodule. We define, by transfinite induction, sets X_α of primary-like submodules of M. To start with, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined, for all ordinals $\beta < \alpha$, let X_α be the set of those primary-like submodules Q in M such that all primary-like submodules strongly-like properly containing Q belong to $\bigcup_{\beta<\alpha} X_\beta$. (In particular, X_0 is the set of virtually maximal primary-like submodules of M.) If some X_γ contains all primary-like submodules of M, we say that $P.L.dim(M)$ exists, and we set $P.L.dim(M)$-the primary-like dimension of M-equal to the smallest such γ. We write $P.L.dim(M) = \gamma$ as an abbreviation for the statement that $P.L.dim(M)$ exists and equals γ.*

Proposition 3.1. *Let R be a ring and M be a left R-module with the virtual-like acc on primary-like submodules. Then $P.L.dim(M)$ exists.*

Proof. Define the sets X_γ of primary-like submodules as in the definition above of primary-like dimension. Since there is a bound the cardinalities of these sets (e.g., 2^{cardM}), the transfinite chain $X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$ cannot be properly increasing forever. Hence, there exists an ordinal γ such that $X_\gamma = X_{\gamma+1}$. If $P.L.dim(M)$ dose not exist, then X_γ dose not contain all the primary-like submodules of M. Using the virtual-like acc on primary-like submodules, there is a primary-like submodule Q of M virtually maximal with respect to the property $Q \notin X_\gamma$. Hence, all primary-like submodules strongly-like properly containing Q lie in X_γ. But, then $Q \in X_{\gamma+1} = X_\gamma$, a contradiction. $\square$

Corollary 3.1. *Let R be aring and M be a left R-module such that the set $\{P \in Spec(R) | P = \sqrt{(Q : M)}, Q \in Spec_L(M)\}$ has acc. Then $P.L.dim(M)$ exists.*

Proof. Follows from Proposition 3.1. □

Lemma 3.1. *Let M be an R-module for which $P.L.dim(M)$ exists. Then, for any submodule N of M, $P.L.dim(M/N)$ exists and is no larger than $P.L.dim(M)$.*

Proof. Note submodule Q/N of M/N is primary-like if and only if submodule Q of M is primary-like and $N \subseteq Q$. □

Corollary 3.2. *Let M be an R-module for which $P.L.dim(M)$ exists. If Q and Q' are primary-like submodules of M such that $Q \subset_{sl} Q'$, then $P.L.dim(M/Q') \leq P.L.dim(M/Q)$.*

Proof. Follows from Lemm 3.1. □

Theorem 3.1. *Let M be a left R-module. Then, $P.L.dim(M)$ exists if and only if M has virtual-like acc on primary-like submodules.*

Proof. Suppose that $P.L.dim(M) = \gamma$, where γ is an ordinal number. If $Q_1 \subset_{sl} Q_2 \subset_{sl} Q_3 \subset_{sl} \cdots$ is a strong-like assenting chain of primary-like submodules of M, then by Lemma 3.1 and Corollary 3.2, we have

$$\cdots < P.L.dim(M/Q_3) < P.L.dim(M/Q_2) < P.L.dim(M/Q_1) < \gamma,$$

which is impossible. Therefore, M has virtual-like acc on primary-like submodules. The converse is immediate from Proposition 3.1. □

Suppose that the module M contains a primary-like submodule Q. Then, the virtual-like height of Q, denoted by $vl.ht(Q)$, is the greatest nonnegative integer n such that there exists a strong-like chain of primary-like submodules of M

$$Q_0 \subset_{sl} Q_1 \subset_{sl} \cdots \subset_{sl} Q_n = Q,$$

and $vl.ht(Q) = \infty$ if no such n exists.

A prime ring R is called left bounded if for each regular element r in R there exists an ideal I of R and a regular element s such that $Rs \subseteq I \subseteq Rr$. A general ring R is called left fully bounded if every prime homomorphic image of R is left bounded. A ring R is called a left FBN-ring if R is left fully bounded and left Noetherian. It is well known that if R is a PI-ring (ring with polynomial identity) and P is a prime ideal of R, then the ring R/P is (left and right) bounded and (left and right) Goldie [18, 13.6.6].

Proposition 3.2. *Let R be a PI-ring (or an FBN-ring) and let M be an R-module such that every primary-like submodule of M is contained in a maximal submodule of M. If $P.L.dim(M) = n < \infty$, then for each primary-like submodule Q of M such that $vl.ht(Q) = n$, the factor module M/Q is homogeneous semisimple.*

Proof. Suppose that Q is a primary-like submodule of M with $vl.ht(Q) = n$ and Q' is a maximal submodule of M such that $Q \subseteq Q'$. Since $P.L.dim(M) = n$, so that $P = \sqrt{(Q : M)} = \sqrt{(Q' : M)}$ is a maximal ideal of R and M/Q' is a faithful simple R/P-module. The ring R/P is left bounded, left Goldie, thus, [7, Proposition 8.7] gives that R/P embeds as a left R-module in a finite direct sum of copies of M/Q'. It follows that the ring R/P is left Artinian, and, hence, R/P is simple Artinian. Thus, the left R/P-module M/Q is a direct sum of isomorphic simple modules. It follows that M/Q is a homogeneous semisimple R-module. □

Corollary 3.3. *Let R be a PI-ring and M be a finitely generated R-module such that $P.L.dim(M) = n < \infty$. Then, for each primary-like submodule Q of M such that $vl.ht(Q) = n$, the factor module M/Q is homogeneous semisimple.*

Proof. It follows from Proposition 3.2. □

Lemma 3.2. *Let M be an R-module. Then, $P.L.dim(M) = 0$ if and only if $Spec_L(M) \neq \emptyset$; and every primary-like submodule of M is a virtually maximal primary-like submodule.*

Proof. Is clear. □

A ring R is called a left FBN-ring if R is left fully bounded and left Noetherian.

A submodule Q of M is said to be virtually maximal prime if Q is a prime submodule of M and there is no prime submodule Q' of M such that $Q \subset_s Q'$ (i.e., Q is a prime submodule of M and for any prime submodule Q' of M, such that $Q \subseteq Q'$, we have $(Q : M) = (Q' : M)$).

Lemma 3.3. *Let R be a PI-ring (or an FBN-ring) and let M be an R-module in which every proper submodule is contained in a maximal submodule. Then, for each proper submodule Q of M such that M/Q is primeful, the following statements are equivalent.*

1) *Q is a virtually maximal submodule.*

2) *Q is a virtually maximal prime submodule.*

3) *Q is a virtually maximal primary-like submodule.*

Proof. (1) ⇒ (2) ⇒ (3) is clear.

(3) ⇒ (1) Assume that Q is a virtually maximal primary-like submodule of M. Then, there exists a maximal submodule Q' of M such that $Q \subset Q'$. It follows that $\sqrt{(Q : M)} = \sqrt{(Q' : M)} = P$ and M/Q' is a simple R/P-module. Since R is a PI-ring (or an FBN-ring), then the ring R/P is a left bounded, left Goldie ring. Now, by [7, Proposition 8.7] we have that R/P embeds as a left R-module in a finite direct sum of copies of M/Q'. It follows that the ring R/P

is left Artinian, and, hence, R/P is simple Artinian. Thus, the left R/P-module M/Q is a direct sum of isomorphic simple modules. It follows that M/Q is a homogeneous semisimple R-module; i.e., Q is a virtually maximal submodule of M. □

Corollary 3.4. *Let R be a PI-ring (or an FBN-ring) and let M be an R-module in which every proper submodule is contained in a maximal submodule and $Spec_L(M) \neq \emptyset$. Then, for each proper submodule Q of M such that M/Q is primeful, the following statements are equivalent.*

1) *Q is a virtually maximal submodule.*
2) *Q is a virtually maximal prime submodule.*
3) *Q is a virtually maximal primary-like submodule.*
4) *$P.L.dim(M) = 0$.*

Proof. Follows from Lemmas 3.2 and 3.3. □

4. Conclusion

In this paper, we introduced the notion of virtual-like ascending and descending chains condition on submodules of a module where every Noetherian (respectively, Artinian) module is virtually-like Noetherian (respectively, virtually-like Artinian) and it is shown that the converse is not generally true Example 2.2.

The connections between maximal, virtually maximal, maximal primary-like, maximal virtually primary-like and primary-like submodules are investigated Proposition 2.1. Also, exact sequences of modules, the quotient structure and the direct sum of modules are considered and studied under this concept Proposition 2.4 and Corollaries 2.2 and 2.3.

Moreover, the primary-like dimension of a module is defined and shown that it there exists for every left R-module with the virtual-like acc on primary-like submodules Proposition 3.1. Furthermore, links of the primary-like dimension of a module and the related quotient structure and also primary-like submodules are investigated and it is shown that existence of the primary-like dimension of a module is depended to existence of virtual-like acc on primary-like submodules Theorem 3.1. And the connection between the finiteness of the primary-like dimension of modules and homogeneity and semi-simplicity of the related factor modules Proposition 3.2. Finally the connection between virtually maximal, virtually maximal prime and virtually maximal primary-like submodules in R-modules with a PI-ring (or an FBN-ring) R is indicated Proposition 3.3.

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A characterization of PSL$(4, p^2)$ by some character degree

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Abstract. Let G be a finite group and cd(G) be the set of irreducible character degree of G. In this paper we prove that if p is a prime number, then the simple group PSL$(4,p^2)$ are uniquely determined by its order and some its character degrees.

Keywords: character degrees, order, projective special linear group.

1. Introduction

All groups considered are finite and all characters are complex characters. Let G be a group. Denote by Irr(G) the set of all irreducible characters of G. Let cd(G) be the set of all irreducible character degree of G.

Many authors were recently concerned with the following question:

What can be said about the structures of a finite group G , if some information is known about the arithmetical structure of the degree of the irreducible characters of G (see, [17,18]). A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors.

Yan et al. [17] and [18] proved that all simple k_3-group and the Mathieu groups are uniquely determined by their orders and some its character degrees.

Also, Khosravi et al. in [9] and [10] proved that the simple groups PSL$(2, p)$ and PSL$(2, p^2)$ are uniquely determined by its order and its largest and second largest irreducible character degrees, where p is an odd prime. Also, Hung

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and Thomson in [13] proved that the simple group PSL$(4, q)$ whit $q \geq 13$ are determined by the set of their character degrees.

Let p be an odd prime number. In [14] the authors proved that the simple group $PSL(4, p)$ is uniquely by its order and some character degrees.

The goal of this paper is to introduce a new characterization for the finite group PSL$(4, p^2)$, where p is prime, by its order and some its character degrees. In fact we prove the following theorem.

Theorem 1.1 (Main Theorem)**.** *Let $p > 7$ be a prime. If G is a finite group such that the following statements hold, then G is isomorphic to $PSL(4, p^2)$.*
(i) $|G| = |PSL(4, p^2)|$.
(ii) $kp^{12} \in cd(G)$ *if only if* $k = 1$*, where k is an integer number.*
(iii) $p^2(p^4 + p^2 + 1)$ *is the smallest nonlinear character degree of G.*
(iv) $\{p^2(p^2 + 1)^2(p^4 + 1), (p^2 + 1)(p^4 + 1)\} \subset cd(PSL(4, p^2))$.

2. Notation and preliminary

We know that if p is an odd prime, then

$$|PSL(4, p^2)| = \frac{p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)}{(4, p^2 - 1)}$$

and let Φ_k denote the kth cyclotomic polynomial evaluated at p^2. In particular,

$$\Phi_1 = p^2 - 1, \Phi_2 = p^2 + 1, \Phi_3 = p^4 + p^2 + 1, \Phi_4 = p^4 + 1.$$

The data in [18] gives the character degree of $PSL(4, q)$. From there, we are able to extract the character degree of $PSL(4, p^2)$.These degrees are given in Table 1. The word “possible” in the second column means that the condition for the existence of corresponding degree in fairly complicated

$$\{p^{12}, p^2\Phi_3, p^2\Phi_2^2\Phi_4, \Phi_2\Phi_4\} \subset cd(PSL(4,^2 p))$$

and the smallest nonlinear character degrees of $PSL(4, p^2)$ is $p^2\Phi_3$.

If n is an integer and r is a prime number, then we write $r^\alpha || n$, when $r^\alpha | n$ but $r^{\alpha+1} \mid n$. All other notations are standard and we refer to [1].

If $N \trianglelefteq G$ and $\theta \in Irr(N)$, then the inertia group of θ in G is $I_G(\theta)$=$\{g \in G \mid \theta^g$=$\theta\ \}$.

Lemma 2.1 (Thompson, [14], Lemma 2.3)**.** *Suppos that p is a prime and $p \mid \chi(1)$ for every nonlinear $\chi \in Irr(G)$. Then, G has a normal p-complement.*

Lemma 2.2 (Ghallgher's Theorem, [8], Corollary 6.17)**.** *Let $N \trianglelefteq G$ and let $\chi \in Irr(G)$ be such that $\chi_N = \theta \in Irr(N)$. Then, the characters $\beta\chi$ for $\beta \in Irr(\frac{G}{N})$ are irreducible and distinct for distinct β and are all of the irreducible constituents of θ^G.*

Lemma 2.3 (Ito's Theorem, [3], Corollary 6.15)**.** *Let $A \trianglelefteq G$ be abelian. Then, $\chi(1)$ divides $|G : A|$ for all $\chi \in Irr(G)$.*

Lemma 2.4 ([3], Theorems 6.2, 6.8, 11.29)**.** *Let $N \trianglelefteq G$ and let $\chi \in Irr(G)$. Let θ be an irreducible constituent of χ_N, and suppose $\theta_1 = \theta,..,\theta_t$ are the distinct conjugates of θ in G. Then, $\chi_N = e\sum_{i=1}^{t} e_i\chi_i$, where $e=[\chi_N, \theta]$ and $t=[G{:}I_G(\theta)]$. Also, $\theta(1)|\chi(1)$ and $\chi(1)/\theta(1)||G{:}N|$.*

Lemma 2.5 ([17], Lemma)**.** *Let G be nonsolvable group. Then, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple group and $|G/K|||Out(K/H)|$.*

Lemma 2.6 ([3], Lemma 12.3 and Theorem 12.4)**.** *Let $N \trianglelefteq G$ be maximal such that G/N is solvable and nonabelian. Then, one of the following holds.*

(i) G/N is a r-group for some prime r. If $\chi \in Irr(G)$ and $r \mid \chi(1)$, then $\chi\tau \in Irr(G)$ for all $\tau \in Irr(G/N)$.

(ii) G/N is a Frobenius group with an elementary abelian Frobenius kennel F/N.

Thus, $|G : F| \in cd\ (G)$, $|F : N| = r^{\alpha}$, where a is the smallest integer such that $|G : F| \mid r^{\alpha} - 1$. For every $\psi \in Irr(F)$, either $|G : F|\psi(1) \in cd(G)$ or $|F : N||\psi(1)^2$. If no proper multiple of $|G : F|$ is in $cd(G)$, then $\chi(1)||G : F|$ for all $\chi \in Irr(G)$ such that $r \mid \chi(1)$.

Lemma 2.7 ([16], Lemma 2.3)**.** *In the context of (ii) of Lemma 2.5, we have*

(i) If $\chi \in Irr(G)$ such that lcm$(\chi(1),|G : F|)$ does not divide any character degree of G, then $r^{\alpha} \mid \chi(1)^2$

(ii) If $\chi \in Irr(G)$ such that no proper multiple of $\chi(1)$ is a degree of G, then either $|G : F| \mid \chi(1)$ or $r^{\alpha}|\chi(1)^2$. Moreover if $\chi(1)$ is divisible by no nontrivial proper character degree in G then $|G : F| = \chi(1)$ or $r^{a}|\chi(1)^2$.

3. Proof of the main theorem

In this section we present the proof of Main theorem. In fact, we prove this theorem by two steps:

Step 1. First we prove that G is a nonsolvable group.We show that $G' = G''$. Assume by contradiction that $G' \neq G''$ and let $N \trianglelefteq G$ be maximal such that G/N is solvable and nonabelian.By Lemma 2.6, G/N is an r-group for some prime r or G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N.

Case 1. G/N is an r-group for some prime r. Since G/N is nonabelian, there is $\psi \in Irr(G/N)$ such that $\psi(1) = r^a > 1$. From the classification of prime power degree representations of quasi-simple group in [12], we deduce that $\psi(1) = r^a$ must be equal to the degree of the Steinberg character of H of degree p^{12} and thus $r^a = p^{12}$, which implies that $r = p$. By Lemma 2.1, G possesses a nontrivial irreducible character χ with $p \mid \chi(1)$. Lemma 2.4 implies that $\chi_N \in Irr(N)$.

Using Ghallagher's lemma, we deduce that $\chi(1)\psi(1) = p^{12}\chi(1)$ is a character degree of G, which is impossible with the condition (ii) of main theorem.

Case 2. G/N is a Frobenius group whit an elementary abelian Frobenius kernel F/N. Thus according to Lemma 2.6, $|G : F| \in cd(G)$, $|F : N| = r^a$, where a is the smallest integer such that $|G : F||r^a - 1$. Let χ be a character of G of degree p^{12}. As no proper multiple of p^{12} is in $cd(G)$, Lemma 2.6 implies that either $|G : F||p^{12}$ or $r = p$. We consider two following subcases.

(a) $|G : F||p^6$. Then, $|G : F| \in cd(G)$, by the assumption of the theorem, this implies that no multiple of $|G : F|$ is in $cd(G)$. Therefore, by Lemma 2.6, for every $\psi \in Irr(G)$ either $\psi(1)|p^{12}$ or $r|\psi(1)$. Taking ψ to be characters of degree $p^2\Phi_3$ and $p^2\Phi_2^2\Phi_4$, we obtain that $r|\psi(1)$.This implies that r divides both $p^2\Phi_3$ and $p^2\Phi_2^2\Phi_4$. This leads us to a contradiction since (Φ_3 , $\Phi_2^2\Phi_4$)=1.

(b) $r = p$. Thus $|F : N| = p^a$ and $|G : F||p^a - 1$. Let χ be a character of G of degree $p^2\Phi_2^2\Phi_4$ and ψ be a character of degree $\Phi_2\Phi_4$). It follows that $\psi(1)|\chi(1)$ so that by Lemma 2.7, $|G : F| = p^2\Phi_2^2\Phi_4$ or $p^a|p^4\Phi_2^2\Phi_4^2$ which implies that $a \leq 4$, $|G : F| \leq p^4 - 1$. This leads us to a contradiction since $min\{\chi(1)|\chi(1) > 1, \chi \in Irr(G)\} = p^2\Phi_3$.

Therefore, G is not a solvable group.

Step 2. Now, we prove that G is isomorphic to $PSL(4, p^2)$.

By the above discussion and using Lemma 2.5, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of m copies of a nonabelian simple group S and $|G/K|||Out(\mathrm{K/H})|$. Also, p is a prime divisor of $|G|$ such that $p^{12}\||G|$

First, we prove that $p \nmid |G/K|$. On the contrary, let $p||G/K|$. We know that $Out(K/H) \cong Out(S) \wr S_m$, which implies that $p||S_m|$ or $p||Out(S)|$. If $P||S_m|$, then $m \geq p$ and so $p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1) \geq |K/H| \geq 60^p$, which is impossible. Hence $p||Out(S)|$. According to the orders of automorphism group of alternating group and sporadic simple group, we implies that S is a simple group of Lie type over $GF(q)$, where $q = p_0^f$. By assumption, $p||Out(S)| = dfg$, where d, f, and $g \leq 3$ are the orders of diagonal, field, and graph automorphisms of S respectively. Using [2], we know that if S is a simple group of Lie type over $GF(q)$, then $q(q^2 - 1) \leq S$ and so if $p|f$, then $2^p(2^{2p} - 1) \leq q(q^2 - 1) \leq |S| \leq p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$, which is a contradiction. Hence $p|d$. Since $p > 7$, we get that $S = A_n(q)$ and $d = (n + 1, q - 1)$ or $S =^2 A_n(q)$ and $d = (n + 1, q + 1)$. In each case we get that $p|q - 1$ and $n \geq 6$ or $p|q + 1$ and $n \geq 6$. Then, $p^{13}||S|$, which is a contradiction. Therefore, $p \nmid |G/K|$.

Now, we prove that $p \nmid |H|$. On the contrary, let $p||H|$. So there exist twelve possibilities, $p^i\||H|$ where $1 \leq i \leq 12$.

Case 1. First, suppose that $p\||H|$. Using the classification of finite simple group we determine all simple groups S such that $p^5||S|^5$. Now, we consider two subcases:

(i) Let $m = 1$. Then, $p^{11}||S|$ and $|S||p^{11}(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$, then $p \leq n$ and $n!|p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$. Which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple groups, we get that, there is no Lie group satisfying these conditions.

Since the proofs for the other simple groups are similar, we state the proof only for a few of them for convenience.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions.

If $S \cong B_n(q)$, where $n \geq 2$, then $p|q^{2j}-1$, for some $1 \leq j \leq n$. Therefore, $p \leq q^n+1$. Then, since $q^{2i-1} \leq q^{2i}-1$, we get that

$$q^{n^2}.q^{2(1+2+...+n)-n} \leq |S| < p^{23} \leq (q^n+1)^{23} \leq q^{23n+23},$$

which implies that $2n^2 < 23(n+1)$. Therefore, $n \in \{2,3,4,...,12\}$. First let $n=2$. Then, $p^{11}|q^4(q^2-1)(q^4-1)$.It implies that $p^{11}|(q-1)^2$ or $p^{11}|(q+1)^2$ or $p^{11}|q^2+1$, and so $p^{11} < 2q^2$. On the other hand $q^4|(p-1)^3$ or $q^4|(p+1)^3$ or $q^4|(p^2+1)^2$ or $q^4|(p^2+p+1)$ or $q^4|(p^2-p+1)$, and so $q^4 < p^5$. Therefore, easily we get a contradiction. If $n \in \{3,4,5,...,12\}$, similarly we get a contradiction. If $S \cong C_n(q)$, where $n \geq 4$, then withe the same manner we get a contradiction.

If $S \cong A_n(q)$, then similarly to the above, we get $n \in \{1,2,...,15\}$. For example, let $n=5$. Then,

$$p^{11}|(q-1)^5(q+1)^3(q^2+q+1)^2(q^2-q+1)(q^4+q^3+q^2+q+1)$$

so, $p^{11} < 5q^4$. On the other hand $q^{15}|(p-1)^3(p+1)^3(p^2+1)^2(p^2-p+1)(p^2+p+1)$ so $q^{15} < p^7$. Therefore, we get a contradiction. For other case, similarly we get a contradiction. If $S \cong^2 A_n(q)$, with the same manner we get a contradiction.

If $S \cong D_n(q)$, where $n \geq 4$, then $p^{11}||S|$, Therefore, $p|q^{2i}-1$, for some $1 \leq i \leq n-1$ or $p|(q^n-1)$. Therefore, $p < q^n$, and since $q^{2i-1} < q^{2i}-1$, we get that

$$q^{n(n-1)}q^{n-1}(q^{2(1+2+...+(n-1)-(n-1))} < |S| < p^{23}$$

and so $q^{(2n(n-1)} < |S| < p^{23}$. On the other hand, $p < q^n$ and hence $2(n-1) < 23$. Therefore, $n \in \{4,5,6,...,12\}$. Let $n=6$. Then, $p^{11}|(q-1)^6(q+1)^6(q^2+q+1)^2(q^2-q+1)^2(q^2+1)^2(q^4+1)(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$ and so $p^{11} < q^7$. On the other hand

$$q^{30}|(p-1)^3(p+1)^3(p^2+1)^2(p^2+p+1)(p^2-p+1)$$

and so, $q^{30} < p^7$. Therefore, we get a contradiction. Fore some other cases, similarly we get a contradiction. If $S \cong^2 D_n(q)$, with the same manner we get a contradiction.

If $S \cong G_2(q)$, then $p^{11}||S|$, and hence $p^{11} < q^3$. On the other hand,

$$q^6|(p-1)^3(p+1)^3(p^2+1)^2(p^2+p+1)(p^2-p+1)$$

so, $q^6 < p^7$. Therefore, we get a contradiction. If $S \cong F_4(q),^2F_4(q), E_6(q), E_7(q)$ or $E_8(q)$, we get a contradiction similarly.

If $S \cong^2 B_2(q)$, where $q = 2^{2n+1}$, then $p^{11}|q-1$ or $p^{11}|q^2+1$. If $p^{11}|q-1$, then $|S| < p^{23} < (q-1)^5$, wiche is impossible. If $p^{11}|(q^2+1)$, then $p^{11}|(q^2+1)/5$, so $p^{11} < q^2$. On the other hand

$$q^2|4(p-1)^3(p+1)^3(\frac{p^2+1}{2})^2(p^2+p+1)(p^2-p+1)$$

therefore, $q^2|16(p-1)^3$ or $q^2|16(p+1)^3$, so $q < p^3$, which is impossible.

If $S \cong^2 G_2(q)$, where $q = 3^{2n+1}$, then $p^{11}||S|$, therefore $p^{11}|q-1$ or $p^{11}|q+1$ or $p^{11}|q^2-q+1$ or$p^{11}|q^2+q+1$, it follows that $p^{11} < q^2$. On the other hand, $q^3|6(p-1)^3(p+1)^3$ or $q^3|(p^2+1)/2$ or $q^3|(p^2+p+1)$ or$q^3|(p^2-p+1)$, it follows that $q^3 < p^7$, which is impossible.

Therefore, $m \neq 1$.

(ii) $m = 11$. Then, $p||S|$ and $|S|^{11}|p^{11}(p^4-1)(p^6-1)(p^8-1)$.

Similarly, to the previous case we get a contradiction.

Case 2. Suppose that $p^2|||H|$. Therefore, $p^{10}||K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 2, 5, 10\}$. Now we consider four subcases:

(i) Let $m = 1$. Then, $p^{10}||S|$ and $|S||p^{10}(p^4-1)(p^6-1)(p^8-1)$. We claim that there is no simple group satisfying these conditions.

If $S \cong A_n$, then $p < n$ and $n!|p^{10}(p^4-1)(p^6-1)(p^8-1)$, which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

Similarl to case 1, we deduce that, there is no nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$, satisfying the above conditions.

Hence, $m \neq 1$.

(ii) Let $m = 2$

Similarly to last case, we deduce $S \ncong A_n$. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the order of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 2$

(iii) Let $m = 5$. Then, $p^2||S|$ and $|S|^5|p^{10}(p^4-1)(p^6-1)(p^8-1)$. Using the classification of finite simple group, we show that, there is no simple group satisfying these conditions. If S is a nonabelian simple group of Lie type over a field of characteristic p, using the order of the simple group, we get that, there is no simple group satisfying the above conditions.

If $S \cong A_n$, then $p \leq n$ and $(n!)^5|p^{10}(p^4-1)(p^6-1)(p^8-1)$, which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence, $m \neq 5$.

(iv) Let $m = 10$. Then, $p||S|$ and $|S|^{10}|p^{12}(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$, then $p \leq n$ and $(n!)^{10}|p^{12}(p^4-1)(p^6-1)(p^8-1)$, which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple groups, we get that, the only possibility cases are $A_1(p)$ and $A_2(p)$.

(A) If $S \cong A_1(p)$, then $p^{10}(p^2-1)^{10}|p^{12}(p-1)^3(p+1)^3(p^2+1)^2(p^2+p+1)(p^2-p+1)$, therefore $(p-1)^7(p+1)^7|(p^2+1)^2(p^2+p+1)(p^2-p+1)$, which is impossible.

(B) If $S \cong A_2(p)$, then $|S|^{10} \leq p^{12}(p^4-1)(p^6-1)(p^8-1)$, which is impossible.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 10$.

Case 3. If $p^3|||H|$. Therefore, $p^9||K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 3, 9\}$. Now we consider three subcases:

(i) Let $m = 1$. Then, $p^9|||S|$ and $|S||p^3(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence, $m \neq 1$.

(ii) Let $m = 3$. Then, $p^3||S|$ and $|S|^3|p^3(p^4-1)(p^6)(p^8-1)$

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above condition.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 3$.

(iii) Let $m = 9$. Then, $p||S|$ and $|S|^9|p^{11}(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above condition.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence, $m \neq 9$.

Case 4. If $p^4 \| |H|$. Therefore, $p^8 || K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 2, 4, 8\}$. Now we consider two subcases:

(i) Let $m = 1$. Then, $p^8 \| |S|$ and $|S| | p^4(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$, then similar to Case 1, we get a contradiction. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence, $m \neq 1$.

(ii) Let $m = 2$. Then, $p^6 \| |S|$ and $|S|^2 | p^6(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence, $m \neq 2$.

(iii) Let $m = 4$. Then, $p^3 || S|$ and $|S|^3 | p^9(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence, $m \neq 4$.

(iv) Let $m = 8$. Then, $p \| |S|$ and $|S|^8 | p^{11}(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence, $m \neq 8$.

Case 5. If $p^5 \| |H|$. Therefore, $p^7 | |K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 7\}$.

(i) Let $m = 1$. Then, $p^7 \| |S|$ and $|S| | p^5(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction. (ii)Let $m = 7$. Then, $p \| |S|$ and $|S|^7 | p^{11}(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$, then similar to Case 1, we get a contradiction. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction. Where $6 \leq i \leq 11$, then withe the same manner we get contradiction.

If $i = 12$ then $p^{12} | |H|$, choos $\chi \in Irr(G)$, such that $\chi(1) = p^{12}$. Let θ be an irreducible constituent of χ_H, then $\chi(1)/\theta(1) | |G : H|$, which implies that $\theta(1) = p^{12}$. Therefore, $\chi_H = \theta$ and by Gallagher's theorem $\beta\chi \in Irr(G)$, for each $\beta \in Irr(G/H)$. Hence $p^{12}\beta(1) \in \mathrm{cd(G)}$, which is contradiction.

By the above discussion, we get that $p^{12} | |K/H|$. Since $p^{12} \| |G|$, it follows that K/H is a nonabelian simple group say S, such that $p^{12} \| |S|$ and $|S| | p^{12}(p^4-1)(p^6-1)(p^8-1)$ or $K/H \cong S \times S$ and $p^6 \| |S|$ and $|S|^2 | p^{12}(p^4-1)(p^6-1)(p^8-1)$ or $K/H \cong \prod_{i=1}^{3} S$ and $|S|^4 | p^{12}(p^4-1)(p^6-1)(p^8-1)$ or $K/H \cong \prod_{i=1}^{4} S$ and $p^3 \| |S|$ and $|S|^4 | p^{12}(p^4-1)(p^6-1)(p^8-1)$ or $K/H \cong \prod_{i=1}^{6} S$ and $p^2 \| |S|$ and $|S|^6 | p^{12}(p^4-1)(p^6-1)(p^8-1)$ or $K/H \cong \prod_{i=1}^{12} S$ and $p \| |S|$ and $|S|^{12} | p^{12}(p^4-1)(p^6-1)(p^8-1)$.

Now, using the classification of finite simple groups and similar to the above argument, we get $K/H \cong PSL(4, p^2)$. Therefore, $|H||G/K| = 1$, and hence, $H = 1$ and $G/K = 1$. Hence $G \cong PSL(4, p^2)$, and the main theorem is proved.

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Separation coordinates in a Hamiltonian quartic system

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Abstract. The separability of Hamiltonian integrable systems has been the object of a considerable amount of attention in the last decades. Over the years several techniques have been proposed to deal with this difficult problem. In this paper we make use of the method of the Kowalewski's Conditions. To illustrate the effectiveness of the method we consider the Hénon-Heiles system known as HH4 1:6:8. This system is integrable in two cases. For one of them, separated only in some particular cases, we provide the separation coordinates in the generic form. The other case remains unsolved.
Keywords: integrable systems, separation of coordinates, integration in quadratures.

1. Introduction

Hénon-Heiles (HH) systems are Hamiltonian systems in $\mathbb{R}^4$ endowed with the standard symplectic form $dp_1 \wedge dx + dp_2 \wedge dy$. The Hamiltonian function has the form

$$H = \frac{1}{2}({p_1}^2 + {p_2}^2) + V(x, y),$$

where V is a polynomial function. There are four nontrivial integral cases with quartic potential that can be "generalized" adding inverse terms without destroying the integrability in the Liouville sense. This means that every one of these systems possesses an integral of the motion called K. The most general forms of H and K, for all the integrable HH systems, have been given by Hietarinta [3].

Once proved the Liouville integrability of these systems, the question arises of an explicit integration of the equations of motion. The most efficient way to bring the systems to quadratures is to find coordinates that separate the Hamilton-Jacobi equation. This is such a difficult task that, after decades of efforts, only one of the four quartic systems has been separated in the generic form [2]. In this paper we will deal with the so called HH4 1:6:8 system only (1, 6, 8 are the coefficients of the quartic monomials). The Hamiltonian function is

$$H = \frac{{p_1}^2}{2} + \frac{{p_2}^2}{2} + x^4 + 6\,x^2y^2 + 8\,y^4 + \omega\,\left(x^2 + 4\,y^2\right) + \frac{a}{y^2} + \frac{b^2}{x^2} - \frac{c^2}{2\,x^6} + ey, \tag{1}$$

where $\omega,\ a,\ b,\ c$ and e are arbitrary constants.

The function K for this system is quite complicated [3]:

$$
\begin{aligned}
K = {}& {p_1}^4 + 2\,{p_1}^2 \left(2x^4 + 12\,x^2y^2 + 2\omega\,x^2 + 2\frac{b^2}{x^2} - \frac{c^2}{x^6} + 2ey\right) \\
& - 4xp_1p_2\left(4\,x^2y + e\right) + 4\,x^4{p_2}^2 \\
& + 4\,\frac{b^4}{x^4} + 8\,b^2x^2 + 16\,b^2y^2 + 4\,\omega^2x^4 + 8\,\omega\,x^6 + 16\,\omega\,x^4y^2 + 4\,x^8 \\
& + 16\,x^6y^2 + 16\,x^4y^4 + 8\,\frac{ax^4}{y^2} - c^2\left(4\,\frac{b^2}{x^8} - \frac{c^2}{x^{12}} + 4\frac{\omega}{x^4} + 4\,\frac{1}{x^2} + 24\,\frac{y^2}{x^4}\right) \\
& + 2e\left(4\,\frac{b^2y}{x^2} - ex^2 - 4\,x^4y - 8\,x^2y^3 - 4\,\omega\,x^2y\right).
\end{aligned} \tag{2}
$$

The reader can easily check that the Poisson bracket of H and K is

$$
\{H,K\} = -\frac{4e\left(2ax^8p_1 - 3\,c^2xy^3p_2 + 6\,c^2y^4p_1\right)}{x^7y^3}
$$

and this lets us with two cases of integrability:

- Case I: $a = c = 0$
- Case II: $e = 0$.

The first case has been solved only under the additional hypothesis $be = 0$ [13] and $e = 2\sqrt{2}\,b$ [12]; the separation coordinates for the generic case remain unknown.

Case II has been studied in the particular case $e = c = ab = 0$ [8]. The authors wrote, about adding the term in x^{-6} or the linear term: "it would be interesting to extend our approach to these cases although we anticipate serious technical difficulties". The aim of this paper is to show that these difficulties can be bypassed looking at the problem from a different perspective. Using the method of the Kowalewski Conditions (KC) we will be able to provide the separation coordinates, for Case II, in the generic form.

2. The method of the vector field Z

Let's introduce quickly the method adopted in the following calculations. A comprehensive presentation, with all the necessary proofs that are omitted here, can be found in [7] and [10].

Separable Hamiltonian systems come equipped with a torsionless recursive tensor N (Nijenhuis tensor), compatible with the Poisson tensor P, i.e. forming a so called PN manifold. If the manifold is 4-dimensional and N has two functionally independent eigenvalues, then they are the separation coordinates of the system (under suitable hypotheses, see below and [5]).

The explicit calculation of the tensor N can be quite cumbersome except in some simple cases [11]. Nevertheless, the essential remark is that N acts on the vector fields tangent to the Lagrangian foliation given by $H = c_1$ and $K = c_2$, so that one can simply calculate the eigenvalues of the restriction of N to the bi-dimensional foliation. This restricted tensor, given a basis on the leaves, reduces to a 2x2 matrix M called the Control Matrix. In the basis associated with the flows of the Hamiltonian vector fields X_H and X_K, this matrix has the form $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$. The Kowalewski Conditions (KC), introduced by F. Magri in [6], characterize the entries of the matrix M. These functions verify four differential constraints if and only if[1] the eigenvalues of M are separation coordinates:

$$\begin{aligned} X_H(m_3) &= X_K(m_1) \\ X_H(m_4) &= X_K(m_2) \\ X_H(m_1m_3 + m_3m_4) &= X_K(m_1^2 + m_2m_3) \\ X_H(m_2m_3 + m_4^2) &= X_K(m_1m_2 + m_2m_4) \end{aligned} \tag{3}$$

and the involutivity of the trace and the determinant if we want the eigenvalues of M to be canonical coordinates:

$$\{m_1 + m_4, m_1m_4 - m_2m_3\} = 0. \tag{4}$$

This is a system of 5 differential equations in 4 unknown functions and it is, in general, difficult to solve. A possible strategy to attack this problem is outlined in the following steps:

1. We start looking for two "Fundamental Functions" F and G verifying

   $$X_H(G) = X_K(F) \tag{5}$$

   and

   $$dF \wedge dG \wedge dH \wedge dK \neq 0.$$

   We can see (F, G, H, K) as non-canonical coordinates associated to the Lagrangian foliation. We use these coordinates to write the Control Matrix in the simplified form:

   $$M = \begin{pmatrix} AF + B & 1 \\ AG + C & D \end{pmatrix} \tag{6}$$

   where A, B, C and D are constant of the motion. In this way the first two equations in (3) are automatically satisfied and the constants A, B, C, D have to be chosen in such a way that the other equations are verified too. One could say that the method of the Fundamental Functions reduces the problem to the search of two functions only: F and G. An example of application of this method can be found in [9].

1. There are some additional technical conditions that are clearly verified in the present case. The complete Theorem can be found in [6] or [9].

2. The next step consists in introducing a "potential function" V and the canonical vector field Z associated to $V:\ Z = X_V$. The functions F and G can be generated by V in the following way:

$$F = Z(H) \qquad G = Z(K) \tag{7}$$

and equation (5) is still verified for any choice of V [10].

3. Unfortunately the method of the potential function seems excessively restrictive and many interesting problems don't fall under this scheme (several examples are given in [10]). The set of all possible fields Z must be enlarged. The idea is to use the constants $a,\ b$ and c present in the Hamiltonian functions as variables, and turn the symplectic system into a Poisson one in $\mathbb{R}^7$ with coordinates $(p_1, p_2, x, y, a, b, c)$. This is easily obtained adding three lines and columns of zeros to the matrix representing the standard Poisson tensor and extending the canonical vector field X_f, associated to a function f, to $\widetilde{X}_f$:

$$\widetilde{X}_f = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, \frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, 0, 0, 0\right)^T.$$

In this framework, the vector field Z can be extended with extra terms in this way

$$Z = X_V + w_1\frac{\partial}{\partial a} + w_2\frac{\partial}{\partial b} + w_3\frac{\partial}{\partial c} \tag{8}$$

where $w_1,\ w_2$ and w_3 are constants.

At the end of the process, the problem is reduced to the determination of a single function V and, eventually, a few constants $w_1,\ w_2$ and w_3.
We are now ready to solve the generic Case II.

3. The separation coordinates for Case II

According to the discussion in the Introduction we replace $e = 0$ in (1) and (2). Our problem is to calculate the separation coordinates of this system without imposing any additional restriction to the remaining constants $a,\ b$ and c.

Writing Z in the extended form (8), we can calculate the Fundamental Functions with (7) and finally obtain the Control Matrix in the simplified form (6).

Now, we have to replace $m_1, \dots, m_4$ into the KC (3). The first two equations are verified for any choice of the potential function and constants [10]. The second couple of KC are verified with $V = c/(2x^2)$ and the constant $w_1 = 0,\ w_2 = b/2$ and $w_3 = c$. Therefore, the vector field Z has the simple form

$$Z = \frac{c}{x^3}\frac{\partial}{\partial p_1} + \frac{b}{2}\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}. \tag{9}$$

This vector field contains all the essential information needed to separate the system. Finally we still have to choose the constants of motion A, B, C and D in order to verify (4). The results can be summarized in the following

Theorem 3.1. *Consider the integrable Hamiltonian system* (1)-(2) *with* $e = 0$. *Let* Z *be the vector field in* (9) *and* F *and* G *the functions in* (7). *Then, the Control Matrix of the system takes the form*

$$M = \begin{pmatrix} -16F + 8H & 1 \\ -16G + 16K & 8H \end{pmatrix} \tag{10}$$

i.e. the eigenvalues of (10) *are canonical separation coordinates for both* H *and* K.

Proof. The functions F and G can be calculated directly with (7):

$$F = \frac{b^2x^4 + p_1cx^3 - c^2}{x^6}$$

and

$$\begin{aligned} G = \frac{1}{x^{12}}\Big[&8\,b^2x^{14} + 8\,cx^{13}p_1 + \left(16\,b^2y^2 - 16\,cyp_2\right)x^{12} + 8\,c\left(6y^2 + \omega\right)p_1x^{11} \\ &+ \left(4\,b^2{p_1}^2 - 8\,c^2\right)x^{10} + 4\,cx^9{p_1}^3 + \left(-48\,c^2y^2 - 8\,\omega c^2 + 8\,b^4\right)x^8 \\ &+ 8\,b^2cx^7p_1 - 4\,c^2x^6{p_1}^2 - 12\,c^2b^2x^4 - 4\,c^3x^3p_1 + 4\,c^4\Big]. \end{aligned}$$

Replacing these functions in (10) one can find the explicit form of $m_1, \ldots, m_4$. According to the results in [6], it is enough to prove that these functions verify the KC (3), as well as the condition of canonicity (4). All these conditions can be easily checked with a software like Maple. □

Remark 3.1. Different Control Matrices can be obtained using more complicated entries, for instance quadratic functions in F and G:

$$M' = \begin{pmatrix} -16F^2 + G & F \\ -16FG + 16KF & G \end{pmatrix}.$$

The eigenvalues of M' provide a different set of separation coordinates. These coordinates reduce to the ones found by Ravoson et al. [8] in the case $a = c = 0$.

A similar method can be applied to Case I too and provides an alternative way to calculate the separation coordinates for the degenerate cases $be = 0$ [10]. In [12] we find the separation coordinates under the particular condition $e = 2\sqrt{2}\,b$. The idea was to guess the form of the potential function V taking example from these particular cases. However the application of the theory to the general case presents some difficulties: it seems that neither linear nor quadratic functions in F and G verify all conditions (3) and (4). Finding separation coordinates for Case I in the generic form remains an open problem.

On the other hand, separation coordinates in Case II could be found without any additional condition on the coefficients and the potential function is as simple as $V = c/(2x^2)$. This system represents, in our opinion, one of the most convincing examples of the effectiveness of the method of the KC. The complete separation of the system goes beyond the scopes of the paper and requires additional work. Nevertheless this paper could be considered as a first step in that direction.

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A mathematical model of COVID-19 transmission dynamics with treatment and quarantine

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Abstract. Corona-virus disease (COVID-19) is caused by the novel-virus (SARS-COV2). This disease mainly targets human respiratory system. COVID-19 (Coronavirus) has affected day to day life and is slowing down the global economy. This pandemic has affected thousands of peoples, who are either sick or are being killed due to the spread of this disease. In this paper we developed an eight compartmental model with quarantine and treatment of COVID-19. After proposing the model, we analysed the qualitative behaviors of the model, like the disease free and endemic equilibrium points and their stability analysis. Moreover, we obtained the basic reproduction number using next-generation matrix method and we performed the sensitivity analysis to identify the most affecting parameters in terms of disease control and spreed. To investigate the detail effect of each major parameters, we performed numerical simulation. We obtained that using both quarantine and treatment is best way to combating COVID-19 in the community. Therefore, stakeholders and policy makers should work both quarantine and treatment simultaneously in combating the pandemic from the population.

Keywords: COVID-19, mathematical modeling, numerical simulation, stability analysis, quarantine and treatment.

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1. Introduction

Since the outbreak in Wuhan, China, December, 2019, coronavirus disease (COVID-19) caused by the novel coronavirus, has now become a global pandemic as declared by World Health Organization (WHO) [1] and the world is presently battling with it [1, 2, 3]. The most common symptoms of COVID-19 are fever, fatigue, and dry cough [1]. Some patients may have ache and discomfort, nasal congestion, runny nose, sore throat, or diarrhea [3]. Such symptoms occurs 2-14 days after exposure, most usually about 5 days [4].

The pandemic can be transmitted directly or indirectly from an infectious person to a healthy person through the eyes, nose, mouth, and sometimes through the ears through moisture content when coughing or sneezing [3]. According to the data reported by WHO (World Health Organization), on 13 August 2020, the reported laboratory confirmed that the number of affected humans reached more than 25.9 million including more than 0.86 million death cases and more than 18.2 million recovers are recorded [5]. The government of different countries have been implementing diverse control measures such as imposing strict, mandatory lockdowns other measures such as individuals maintaining individual social distancing, avoiding crowded events, imposing a maximum number on individuals in any religious and social, the use of face masks while in public, use of sanitizers in any contact many in the markets and etc [6, 7, 8] to mitigate the spread of this pandemic.

Mathematical models have long been used as tools in gaining insight into the dynamics of infectious diseases [9, 10]. Several mathematical models have already been formulated for the population dynamics of COVID-19 in several countries [4, 11, 12, 6, 13, 14]. From this studies, Tang et al. [15] considered, an SEIR-type mathematical model to estimate the transmission risk of COVID-19 and its implication. The study in [6] , formulated a model for novel coronavirus disease 2019 (COVID-19) in Lagos, Nigeria and shown the effect of control measures, specifically the common social distancing, use of face mask and case detection on the dynamics of COVID-19. Khan et.al,[16], formulated a fractional mathematical model for the dynamics of COVID-19 with quarantine and isolation. D.K Mamo [13], developed SHEIQRD coronavirus pandemic spread model. He Identified that isolation of exposed and infected individuals, reduction of transmission, and stay-at-home return rate can mitigate COVID-19 pandemic. In this study, we developed a model by incorporating the hospitalize/quarantine and home treatment subclasses as well as home quarantine subclasses.

2. Model description and formulation

In this study the total population, $N(t)$, at time, t is divided into eight subpopulations; Susceptible,$S(t)$,Stay-home susceptables, $S_h(t)$, Exposed, $E(t)$, Asymptomatic, $A(t)$, Infected, $I(t)$, home Treatment, $T(t)$, Hospitalized/quarantine,

$Q(t)$ and Recovered, $R(t)$. The Susceptible are recruited into the population at a constant rate, Π. It is assumed that β_1 and β_2 are the contact rate of susceptible individuals with asymptomatic and infected individuals respectively and they move to the exposed compartment. We also assumed that susceptible individuals stay at home at a rate of υ and at a rate of τ peoples move from stay at home for due to different reasons and susceptible to the pandemic. Finishing the incubation period, the exposed individuals becomes infected at a rate of γ. From this $\alpha\gamma$ proportion become asymptomatic and the rest $(1-\alpha)\gamma$ become infectious. Through diagnosis $\sigma\delta$ proportion asymptomatic individuals got positive and join quarantine/hospitalized. The rest $(1-\sigma)\delta$ proportion of asymptomatic individuals recover from the disease. Also from infected individuals, $c\varepsilon$ fraction of individuals move to hospitalized. The others are taking treatment at their home at a rate $(1-c)\varepsilon$. However, when the pandemic for the treated individuals become savior $\phi\rho$ fraction move the quarantine/hospitalized. The remaining fractions recovers with the home treatment. Infected individuals recover at a rate of ω and quarantine individuals recover from the pandemic a rate k. The asymptomatic, infectious, treated and quarantine individuals die due to the disease at a rate $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ respectively. The whole population have an average death rate of μ. For more information, Table 2 shows the description of model parameters. The flow diagram of the model is shown in Figure 1 below. Therefore, based on the above asumptions, the model is governed by the

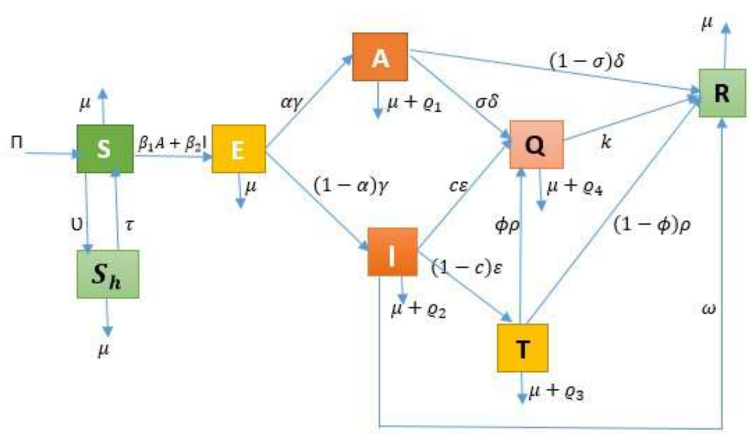


Figure 1: Compartmental flow diagram of the pandemic COVID 19 transmission

following system of differential equation:

$$
\begin{cases}
\dfrac{dS}{dt} = \Pi + \tau S_h - (\beta_1 A + \beta_2 I)\, S - (\upsilon + \mu) S, \\
\dfrac{dS_h}{dt} = \upsilon\, S - (\tau + \mu)\, S_h, \\
\dfrac{dE}{dt} = (\beta_1 A + \beta_2 I)\, S - (\gamma + \mu)\, E, \\
\dfrac{dA}{dt} = \alpha\, \gamma\, E - (\varrho_1 + \delta + \mu)\, A, \\
\dfrac{dI}{dt} = (1 - \alpha)\, \gamma\, E - (\varrho_2 + \omega + \varepsilon + \mu)\, I, \\
\dfrac{dT}{dt} = (1 - c)\, \varepsilon\, I - (\varrho_3 + \rho + \mu)\, T, \\
\dfrac{dQ}{dt} = \sigma\, \delta\, A + c\varepsilon I + \phi\rho T - (\varrho_4 + k + \mu)\, Q, \\
\dfrac{dR}{dt} = \omega I + (1 - \sigma)\, \delta\, A + kQ + (1 - \phi)\rho\, T - \mu\, R,
\end{cases}
\tag{1}
$$

with the initial condition

$$
S(0) = S_0 \geq 0,\; E(0) = E_0 \geq 0 \;\; I(0) = I_0 \geq 0,\; R(0) = R_0 \geq 0. \tag{2}
$$

3. Model analysis

3.1 Invariant region

In this section, a region in which solutions of the model are uniformlly bounded is the proper subset of $\Omega \in \mathcal{R}^8_+$. The total population at any time t is given by $N = S + S_h + E + A + I + T + Q + R$ and $\frac{dN}{dt} = \Pi - \varrho_1 A - \varrho_2 I - \varrho_3 T - \varrho_4 Q - \mu N$. In the absence of mortality due to COVID-19 pandemic, it becomes

$$
\frac{dN}{dt} \leq \Pi - \mu N. \tag{3}
$$

Solving equation (3), we obtain $0 \leq N \leq \frac{\Pi}{\mu}$. Therefore, the feasible solution set of the system in equation (1) is the region given by:

$$
\Omega = \left\{ (S, S_h, E, A, I, T, Q, R) \in \mathcal{R}^8_+ : N \leq \frac{\Pi}{\mu} \right\}. \tag{4}
$$

3.2 Positivity of solutions

Theorem 3.1. *If the initial conditions of the model are nonnegative in the feasible set* Ω*, then the solution set* $(S(t), S_h(t), E(t), A(t), I(t), Q(t), T(t), R(t))$ *of system* (1) *is positive for future time* $t \geq 0$.

Proof. We let $\tau = sup\{t > 0 : S_0(\zeta) \geq 0,\ S_{h0}(\zeta) \geq 0,\ E_0(\zeta) \geq 0, A_0(\zeta) \geq 0, I_0(\zeta) \geq 0, T_0(\zeta) \geq 0, Q_0(\zeta) \geq 0, R_0(\zeta) \geq 0$ for all $\zeta \in [0,t]\}$. Since $S_0(t) \geq 0,\ S_{h0}(t) \geq 0,\ E_0(t) \geq 0,\ A_0(t) \geq 0, I_0(t) \geq 0, T_0(t) \geq 0, Q_0(t) \geq 0, R_0(t) \geq 0$ then $\tau > 0$. If $\tau < \infty$, then automaticaly $S_0(t)$ or $S_{h0}(t)$ or $E_0(t)$ or $A_0(t)$ or $I_0(t)$ or $T_0(t)$ or $Q_0(t)$ or $R_0(t)$ is equal to zero at τ. Taking the first equation of the model (1)

$$\frac{dS}{dt} = \Pi - (\beta_1 A + \beta_2 I)\, S - (\upsilon + \mu) S. \tag{5}$$

Then, using the variation of constants formula the solution of equation (5) at τ is given by:

$$\begin{aligned} S(\tau) = S(0) \exp &\left[- \int_0^{\tau} ((\beta_1 A + \beta_2 I)\, S + (\upsilon + \mu) S)\, (S) dS \right] \\ &+ \int_0^{\tau} \Pi. \exp \left[- \int_S^{\tau} ((\beta_1 A + \beta_2 I)\, S + (\upsilon + \mu) S)\, (\zeta) d\zeta \right] dS > 0. \end{aligned}$$

Moreover, since all the variables are positive in $[0, \tau]$, hence, $S(\tau) > 0$. It can be shown in a similar way that $S_h(\tau) > 0, E(\tau) > 0, A(\tau) > 0\ I(\tau) > 0, T(\tau) > 0, Q(\tau) > 0$ and $R(\tau) > 0$. Which is a contradiction. Hence, $\tau = \infty$. Therefore, all the solution sets are positive for $t \geq 0$. □

3.3 COVID-19 Free Equilibrium Point (CFEP)

COVID-19 free equilibrium point is the state at which the infection is not present in the population and note that it has been eradicated. In the case of COVID 19 free the compartments $E = I = A = 0$. Hence, equating zero for the remaining equations in (1) leads the COVID-19 free equilibrium point and given by:

$$E_0 = \left(\frac{\pi}{\mu}, \frac{\upsilon \pi}{\mu(\gamma + \mu)}, 0, 0, 0, 0, 0, 0\right). \tag{6}$$

3.4 Basic reproduction number

To analyze the stability of the equilibrium points, the basic reproduction number $\mathcal{R}_0$ of the model is important. It is obtained using the next-generation matrix method [17, 18]. The first step is rewrite the model equations, starting with newly infective classes:

$$\begin{cases} \frac{dE}{dt} = (\beta_1 A + \beta_2 I)\, S - (\gamma + \mu)\, E, \\ \frac{dA}{dt} = \alpha\, \gamma\, E - (\varrho_1 + \delta + \mu)\, A, \\ \frac{dI}{dt} = (1 - \alpha)\, \gamma\, E - (\varrho_2 + \omega + \varepsilon + \mu)\, I, \\ \frac{dT}{dt} = (1 - c)\, \varepsilon\, I - (\varrho_3 + \rho + \mu)\, T, \\ \frac{dQ}{dt} = \sigma\, \delta\, A + c\varepsilon I + \phi \rho T - (\varrho_4 + k + \mu)\, Q, \\ \frac{dR}{dt} = \omega I + (1 - \sigma)\, \delta\, A + kQ + (1 - \phi) \rho\, T - \mu\, R. \end{cases} \tag{7}$$

Then, by the principle of next-generation matrix, the Jacobian matrices at DFE is given by

$$\mathcal{F} = \begin{pmatrix} 0 & \frac{\beta_1 \Pi}{\mu} & \frac{\beta_2 \Pi}{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{V} = \begin{pmatrix} \gamma + \mu & 0 & 0 & 0 & 0 & 0 \\ -\alpha\,\gamma & \psi_1 & 0 & 0 & 0 & 0 \\ -\left(1-\alpha\right)\gamma & 0 & \psi_2 & 0 & 0 & 0 \\ 0 & 0 & -\left(1-c\right)\epsilon & \psi_3 & 0 & 0 \\ 0 & -\sigma\,\delta & -c\epsilon & 0 & \psi_4 & 0 \\ 0 & -\left(1-\sigma\right)\delta & -\omega & -\phi\,\rho & -k & \mu \end{pmatrix},$$

where

$$\begin{aligned} &\psi_1 = \varrho_1 + \delta + \mu, \psi_2 = \varrho_2 + \omega + \epsilon + \mu, \\ &\psi_3 = \varrho_3 + \theta + \mu + \varphi + \epsilon + \mu, \psi_4 = \varrho_4 + k + \rho + \mu. \end{aligned}$$

Therefore, the basic reproduction number is the spectral radius of the next-generation matrix $\mathcal{F}\mathcal{V}^{-1}$, is given us

$$\mathcal{R}_0 = \frac{\left(\left(1-\alpha\right)\left(\delta+\varrho_1+\mu\right)\beta_2 + \left(\epsilon+\omega+\varrho_2+\mu\right)\alpha\,\beta_1\right)\gamma\,\Pi}{\mu\left(\gamma+\mu+\upsilon\right)\left(\delta+\varrho_1+\mu\right)\left(\epsilon+\omega+\varrho_2+\mu\right)}. \tag{8}$$

Which is a threshold parameter that represents the average number of infection caused by one infectious individual when introduced in the susceptible population [17] in its infectious life time.

3.5 Local stability of DFEP

Theorem 3.2. *The DFEP point is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix, evaluated at the disease-free equilibrium E_0, we get:

$$J = \begin{pmatrix} -\mu-\upsilon & \tau & 0 & -\frac{\beta_1(\tau+\mu)\Pi}{(\tau+\mu+\upsilon)\mu} & -\frac{\beta_2(\tau+\mu)\Pi}{(\tau+\mu+\upsilon)\mu} & 0 & 0 & 0 \\ \upsilon & -\tau-\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma-\mu & \frac{\beta_1(\tau+\mu)\Pi}{(\tau+\mu+\upsilon)\mu} & \frac{\beta_2(\tau+\mu)\Pi}{(\tau+\mu+\upsilon)\mu} & 0 & 0 & 0 \\ 0 & 0 & \alpha\,\gamma & -\psi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(1-\alpha\right)\gamma & 0 & -\psi_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(1-c\right)\epsilon & -\psi_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma\,\delta & c\epsilon & \phi\,\rho & -\psi_4 & 0 \\ 0 & 0 & 0 & \left(1-\sigma\right)\delta & \omega & \left(1-\phi\right)\rho & k & -\mu \end{pmatrix},$$

where

$$\psi_1 = \varrho_1 + \delta + \mu, \psi_2 = \varrho_2 + \omega + \epsilon + \mu,$$
$$\psi_3 = \varrho_3 + \theta + \mu + \varphi + \epsilon + \mu, \psi_4 = \varrho_4 + k + \rho + \mu.$$

The first five eigenvalues are listed as:

$$-\mu, -(\tau + \mu), -\psi_3, -\psi_4, -\mu.$$

The other eigenvalues are obtained from the characteristic polynomial:

$$\mathcal{P}(\lambda) = \lambda^3 + \varphi_1 \lambda^2 + \varphi_2 \lambda + \varphi_3 = 0. \tag{9}$$

where

$$\varphi_1 = \psi_1 + \psi_2 + \gamma + \mu,$$
$$\varphi_2 = \frac{-\Pi\,\alpha\,\gamma\,\beta_1 + (1-\alpha)\Pi\;\gamma\,\beta_2 + \gamma\,\mu\,\psi_1 + \gamma\,\mu\,\psi_2 + \psi_1\mu^2 + \psi_2\mu^2 + \psi_2\mu\,\psi_1}{\mu},$$
$$\varphi_3 = -\frac{\Pi\,\alpha\,\gamma\,\beta_1\psi_2 - \Pi\,\alpha\,\gamma\,\beta_2\psi_1 + \Pi\,\gamma\,\beta_2\psi_1 - \gamma\,\mu\,\psi_1\psi_2 - \mu^2\psi_1\psi_2}{\mu}.$$

To check the positivity of the eigenvalues, We used Routh-Hurwitz criteria and by this principle equation (9) has strictly negative real root iff $\psi_1 > 0$, $\psi_2 > 0$ and $\psi_3 > 0$. Clearly we see that $\psi_1 > 0$ and $\psi_2 > 0$ because they are the sum of positive parameters. Then taking the third equation,

$$\psi_3 = (\varepsilon + \rho + \mu)\,(\delta + \mu)\,[1 - \mathcal{R}_0] > 0$$

Hence the DFEP is locally asymptotically stable if $\mathcal{R}_0 < 1$. □

3.6 Global stability of DFEP

In this section, we investigate global asymptotic stability of the disease free equilibrium using the theorem of Castillo-Chavez [19, 14]. We rewrite model in equation (1) as:

$$\begin{cases} \frac{dZ}{dt} = F(Z, Y), \\ \frac{dY}{dt} = G(Z, Y), G(Z, 0) = 0, \end{cases} \tag{10}$$

where Z=$(S, S_h, R) \in \mathcal{R}^3$ denotes uninfected populations and $Y = (E, A, I, T, Q)$ $\in \mathcal{R}^5$ denotes the infected population. $E_0 = (Z^*, 0)$ represents the DFEP of this system. List two conditions as:

(i) For $\frac{dZ}{dt} = F(Z, 0), Z^*$ is globally asymptotically stable.

(ii) $\frac{dY}{dt} = D_Y G(Z, 0)Y, -\hat{G}(Z, Y), \hat{G}(Z, Y) \geq 0 \quad \text{for all} \quad (Z, Y) \in \Omega.$

If DFEP satisfies the above two conditions, we conclude that E_0 is globally asymptotically stable and according to Castillo-Chavez [19] and the following theorem holds.

Theorem 3.3. *The equilibrium point $E_0 = (Z^*, 0)$ of the system (10) is globally asymptotically stable if $\mathcal{R}_0 < 1$ and the conditions (i) and (ii) are satisfied.*

Proof. We start the proof by defining new variables and dividing the system into subsystems. $Z = (S, R, Q)$ and $Y = (E, A)$. From equation (10) we have two functions $G(Z, Y)$ and $F(Z, Y)$ given by:

$$F(X,Y) = \begin{pmatrix} \Pi + \varphi S_h - (\beta_1 A + \beta_2 I)\, S - (\upsilon + \mu) S \\ \upsilon\, S - (\tau + \mu)\, S_h \\ \omega I + (1-\sigma)\,\delta\, A + kQ + (1-\phi)\rho\, T - \mu\, R \end{pmatrix}$$

and

$$G(Z,Y) = \begin{pmatrix} (\beta_1 A + \beta_2 I)\, S - (\gamma + \mu)\, E \\ \alpha\,\gamma\, E - (\varrho_1 + \delta + \mu)\, A \\ (1-\alpha)\,\gamma\, E - (\varrho_2 + \omega + \varepsilon + \mu)\, I \\ (1-c)\,\varepsilon\, I - (\varrho_3 + \rho + \mu)\, T \\ \sigma\,\delta\, A + c\varepsilon I + \phi\rho T - (\varrho_4 + k + \mu)\, Q \end{pmatrix}.$$

Now, we consider the reduced system $\frac{dZ}{dt} = F(Z, 0)$ from condition (i)

$$\begin{cases} \frac{dS}{dt} = \Pi + \tau S_h - (\upsilon + \mu) S, \\ \frac{dS_h}{dt} = \upsilon\, S - (\tau + \mu)\, S_h, \\ \frac{dR}{dt} = -\mu R. \end{cases} \tag{11}$$

We note that this asymptomatic dynamics is independent of the initial conditions in Ω , therefore the convergence of the solutions of the reduced system equation (11) is global in Ω. We compute

$$G(Z, Y) = D_Y G(Z^*, 0) Y - \hat{G}(Z; Y)$$

and show that $\hat{G}(Z; Y) \geq 0$. Now,

$$D_Y G(Z^*, 0)$$
$$= \begin{pmatrix} -\gamma - \mu & \frac{\beta_1(\tau+\mu)\Pi}{(\tau+\mu+\upsilon)\mu} & \frac{\beta_2(\tau+\mu)\Pi}{(\tau+\mu+\upsilon)\mu} & 0 & 0 \\ \alpha\,\gamma & -\varrho_1 - \delta - \mu & 0 & 0 & 0 \\ (1-\alpha)\,\gamma & 0 & -\varrho_2 - \varphi - \omega - \epsilon - \mu & 0 & 0 \\ 0 & 0 & (1-c)\,\epsilon & -\varrho_3 - \rho - \mu & 0 \\ 0 & \sigma\,\delta & c\epsilon & \phi\,\rho & -\varrho_4 - k - \mu \end{pmatrix}.$$

And, we get

$$\hat{G}(X, Y) = \begin{pmatrix} \left(\frac{\Pi}{\mu} - S\right)(\beta_1 A + \beta_2 I) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, since $\frac{\Pi}{\mu} = S^0 \geq S$, Hence, it is clear that $\hat{G}(Z,Y) \geq 0$ for all (Z,Y) $\in \Omega$. Therefore, by LaSalle's invariance principle [20] this proves that DFE is globally asymptotically stable for $\mathcal{R}_0 < 1$. From this result, we can say that the model exhabits forward bifurication. In other words, for $\mathcal{R}_0 < 1$ the DFEP and EEP does not co-exist. $\square$

3.7 The endemic equilibrium point (EEP)

For endemic equilibrium point of the model we denote it by E^* and $E^* = (S^*, S_h^*, E^*, A^*, I^*, T^*, Q^*, R^*) \geq 0$. The COVID-19 pandemic model has a unique endemic equilibrium and it can be obtained by equating each equation of the model equal to zero. i.e

$$\frac{dS}{dt} = \frac{dS_h}{dt} = \frac{dA}{dt} = \frac{dI}{dt} = \frac{dT}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = 0.$$

Then, we obtain

$$\begin{cases} S^* = \frac{(\gamma+\mu)[\delta(\varepsilon+\omega+\varrho_2+\mu)+\mu(\omega+\varrho_1+\varrho_2+\mu)+\varepsilon(\varrho_1+\mu)+\varrho_1(\omega+\varrho_2)]}{\gamma[(1-\alpha)(\delta+\varrho_1+\mu)\beta_2+(\varepsilon+\omega+\varrho_2+\mu)\alpha\beta_1]} \\ S_h^* = \frac{\upsilon(\gamma+\mu)[\delta(\varepsilon+\omega+\varphi+\varrho_2+\mu)+\mu(\omega+\varrho_1+\varrho_2+\mu)+\varepsilon(\varrho_1+\mu)+\varrho_1(\omega+\varrho_2)]}{\gamma(\tau+\mu)[(1-\alpha)(\delta+\varrho_1+\mu)\beta_2+(\varepsilon+\omega+\varrho_2+\mu)\alpha\beta_1]} \\ E^* = \frac{\xi_1}{\gamma\xi_2} \\ A^* = \frac{\alpha\xi_1}{(\varrho_1+\delta+\mu)\xi_2} \\ I^* = \frac{(1-\alpha)\xi_1}{(\varrho_2+\omega+\varepsilon+\mu)\xi_2} \\ T^* = \frac{(\alpha-1)(c-1)\varepsilon\xi_1}{(\varrho_2+\omega+\varepsilon+\mu)(\rho+\varrho_3+\mu)\xi_2} \\ Q^* = \frac{\sigma\delta A^*+c\varepsilon I^*+\phi\rho T^*}{\varrho_4+k+\mu} \\ R^* = \frac{\omega I^*+(1-\alpha)\delta A^*+kQ^*+(1-\phi)\rho T^*}{\mu}, \end{cases} \tag{12}$$

where

$$\begin{aligned} \xi_1 &= (\delta+\varrho_1+\mu)(\alpha-1)\Pi\gamma\beta_2 - (\varepsilon+\omega+\varrho_2+\mu)\Pi\alpha\gamma\beta_1 \\ &\quad + \mu(\gamma+\mu)[\delta(\varepsilon+\omega+\varrho_2+\mu)+\varepsilon(\varrho_1+\mu)+\mu(\omega+\varrho_1+\varrho_2+\mu) \\ &\quad + \varrho_1(\omega+\varphi+\varrho_2)], \\ \xi_2 &= (\gamma+\mu)[\delta\beta_2(\alpha-1)-\alpha\varepsilon\beta_1] - (\omega+\varrho_2+\mu)(\gamma+\mu)\alpha\beta_1 \\ &\quad + (\alpha-1)(\gamma+\mu)(\varrho_1+\mu)\beta_2. \end{aligned}$$

3.8 Sensitivity analysis

We used the normalized forward sensitivity index definition to go through sensitivity analysis on the basic parameters [21] as done in [22, 23]. The Normalized forward sensitivity index of a variable, $\mathcal{R}_0$, that depends differentiably on a parameter, p, is defined as: $\Lambda_p^{\mathcal{R}_0} = \frac{\partial\mathcal{R}_0}{\partial p} \times \frac{p}{\mathcal{R}_0}$ for p represents all the basic parameters. Here, we have $\mathcal{R}_0 = \frac{((1-\alpha)(\delta+\varrho_1+\mu)\beta_2+(\epsilon+\omega+\varrho_2+\mu)\alpha\beta_1)\gamma\Pi}{\mu(\gamma+\mu+\upsilon)(\delta+\varrho_1+\mu)(\epsilon+\omega+\varrho_2+\mu)}$. For the sensitivity index of $\mathcal{R}_0$ to the parameters:

$$\Lambda_{\beta_1}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \beta_1} \times \frac{\beta_1}{\mathcal{R}_0} = \frac{(\epsilon + \omega + \varrho_2 + \mu)\, \alpha\, \beta_1}{(1-\alpha)\,(\delta + \varrho_1 + \mu)\, \beta_2 + (\epsilon + \omega + \varrho_2 + \mu)\, \alpha\, \beta_1} > 0,$$
$$\Lambda_{\beta_2}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \beta_2} \times \frac{\beta_2}{\mathcal{R}_0} = \frac{(1-\alpha)\,(\delta + \varrho_1 + \mu)\, \beta_2}{(1-\alpha)\,(\delta + \varrho_1 + \mu)\, \beta_2 + (\epsilon + \omega + \varrho_2 + \mu)\, \alpha\, \beta_1} > 0,$$
$$\Lambda_{\alpha}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \alpha} \times \frac{\alpha}{\mathcal{R}_0} = \frac{\alpha\,\left(-\,(\delta + \varrho_1 + \mu)\, \beta_2 + (\epsilon + \omega + \varrho_2 + \mu)\, \beta_1\right)}{(1-\alpha)\,(\delta + \varrho_1 + \mu)\, \beta_2 + (\epsilon + \omega + \varrho_2 + \mu)\, \alpha\, \beta_1} < 0.$$

Similarly, we can work for the other parameters. The sensitivity indices of the basic reproductive number with respect to main parameters are found in Table 1.

Table 1: Sensitivity indecies table.

Parameter symbol	**Sensitivity indecies**
β_1	+ve
β_2	+ve
γ	+ve
σ_1	-ve
σ_2	-ve
k	-ve
ε	-ve
δ	-ve
ω	-ve
μ	-ve

4. Numerical simulations

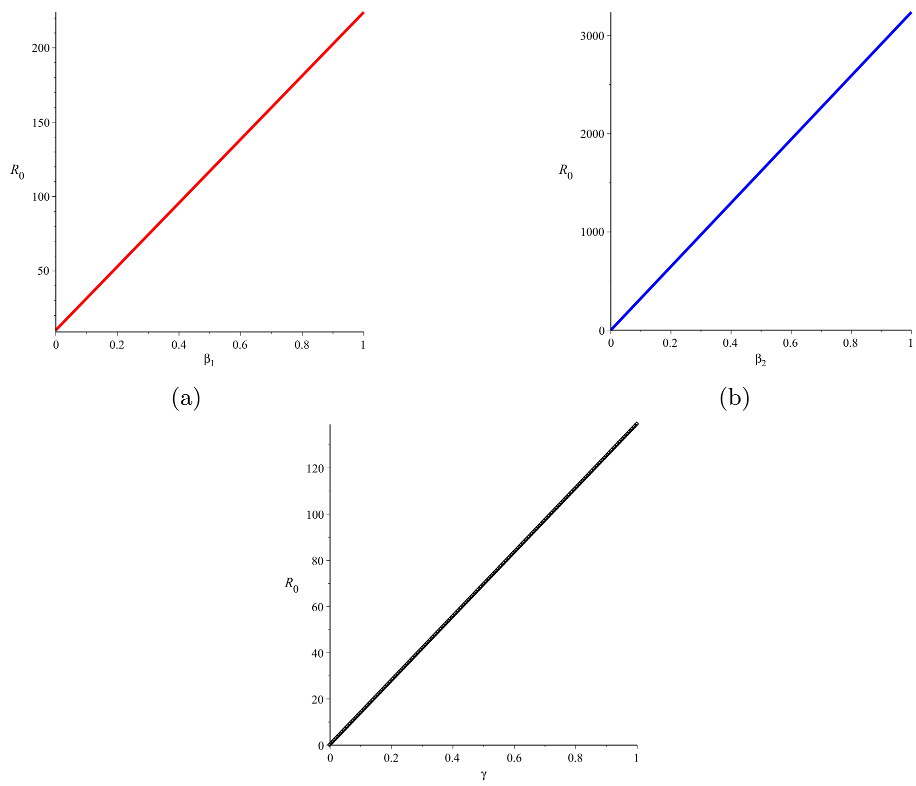
Analytic studies cannot be complete without numerical verification of the results. In this section, we present computer simulation of some solutions of the system (1). Besides verification of our analytical outcomes, these numerical simulations are very significant from practical point of view. To illustrate the results, we used parameter values in the Table 2 .

From Figure 2, we find the positve indices parameters. These parameters ($\beta_1,\ \beta_2,\ \ and\ \ \gamma$) show that they have great impact on expanding the disease in the community if their values are increasing. This is because that the basic reproduction number increases as their values increase, so that the average number of secondary cases of infection increases in the community. Therfore, stakeholders should take action to decrease the effect of the pandemic.

Figure 3, shows those parameters in which their sensitivity indices are negative ($\delta, \omega, \varepsilon, k,\ \ and\ \ \mu$) and the increment of the parameters have an effect of minimizing the burden of the disease in the community. Therefore, research advice

Table 2: Description of parameters of the model (1)

Parameter	**Description**	Value	Source
Π	Ricuirement rate of individuals	150	Assumed
β_1	Transmission rate from asymptomatic to susceptible individuals	0.00000115	[16]
β_2	Transmission rate from infected to susceptible individuals	0.003	[16]
ρ	Individuals who leave from treatment subpopulation	0.2	[16]
δ	Proportion of exposed individuals leaving the compartment	0.2	[16]
ε	Individuals who leave leave from the infected subpopulation	0.001	[16]
τ	Proportion of exposed individuals who join infected compartment	0.07	[13, 24]
υ	Proportion of exposed individuals who join infected compartment	0.005	[13]
μ	Natural death rate the population	0.016	[13]
k	Recovery rate of individuals under quarantine	0.2	[16]
ϱ_1	Induced death rate of asymptomatic individuals	0.002	Assumed
ϱ_2	Induced death rate of infected individuals	0.0002	[16, 24]
ϱ_3	Induced death rate of individuals under treatment	0.0303	Assumed
ϱ_4	Induced death rate of individuals under quarantine	0.0103	[16]
γ	Exposed individuals that become infectious	0.143	[16, 24]
ϕ	Proportion of individuals under treatment who join quarantine	0.3	[16]
c	Proportion of infected individuals who join quarantine	0.5	[16]
ω	Fraction of infected individuals that are immune	0.00023	[16]
σ	Fraction of asymptomatic individuals that are immune	0.01	[16]
α	Fraction of exposed individuals that become asymptomatic	0.1	[16]



(a) (b)

(c)

Figure 2: The positive indices parametres

for stakeholders to work on increasing negative indices parameters to fight the pandemic persistence.

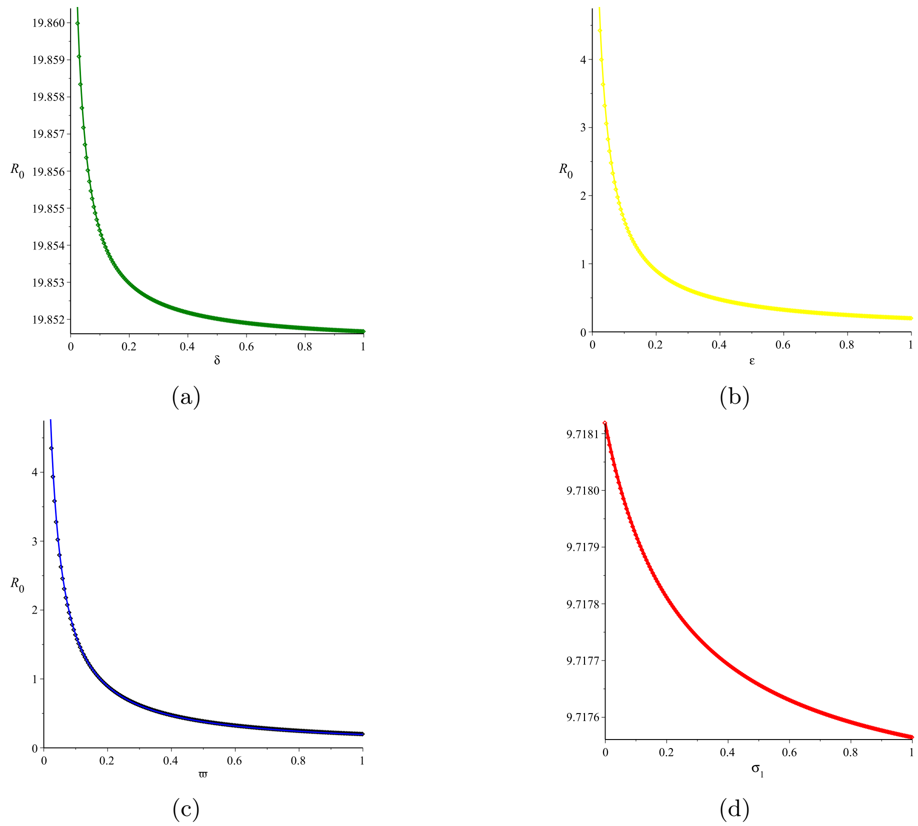


Figure 3: The negative indices parameters

4.1 Impact of γ on infected population

From Figure 4, as we incease the rate of the number of exposed population to infected and asymptomatic stage increases the number of total infected individuals in the population. Thus, the closing of government offices fully or partially was an important decision to control the spread of the pandemic.

4.2 Impact of hospitalizing and treatment (ε) on infected population

As we see from the Figure 5, by increasing the value of ε, the number of infected people is decreasing due to an increase number of hospitalize/quarantine and treantment of infectives at home. This is due to the reason that infectious individuals plays an important role in the infection generation, and therefore, the people should use every control mechanisms and should be educated to avoid the interaction with such people and ready for testing. Therefore, the government should work testing and diagnosis to reduce the infectious number from the population by quarantine/ hospitalize or and treantment of infectives at home.

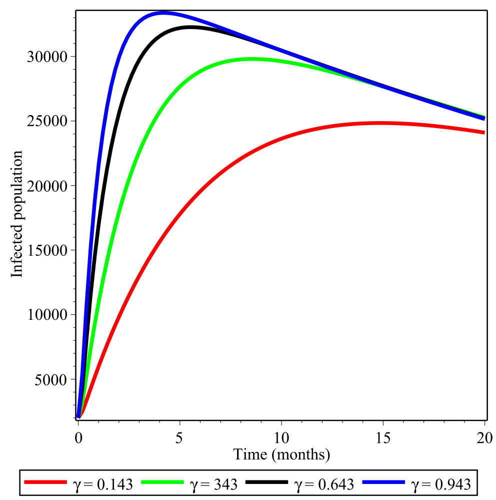


Figure 4: Impact of γ on Infected population

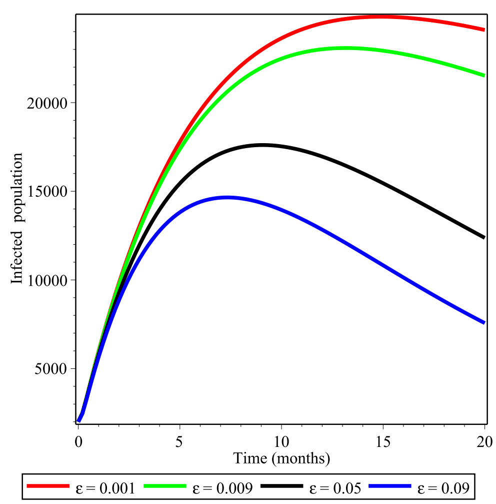


Figure 5: Impact of ε on Infected population

Figure 6, presents the dynamics of the mode with and without quarantine/hospitalize and treatment. From the figure, one can see that, using quarantine/hospitalize and treatment, it is possible to increase the number of recovered individuals. Therefore, here stakeholders should work on using those combating ways to fight the pandemic. A comparison figure is shown to see the effects on th number of total recovered individuals, as seen in the Figure 7. It is evident from figure that from the individual management techniques hospitalize/quarantine infective individuals is better than taking treatment at home. However, instead of using them separately, it is best to use the integration of both techniques to produce a big number of recovered population from the pandemic.

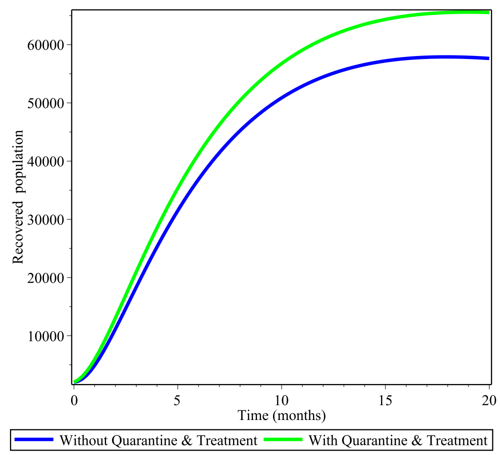


Figure 6: Comparison between with and without quarantine & Treatment

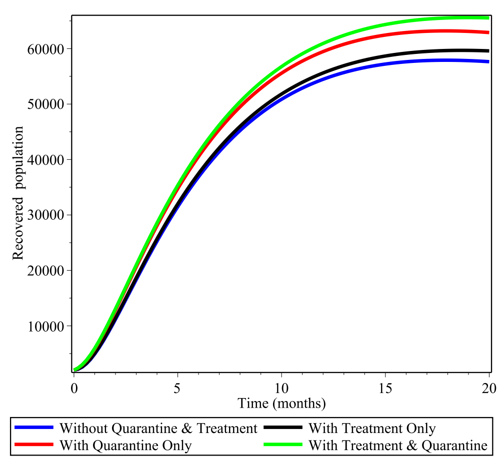


Figure 7: Comparison between quarantine only, Treatment only, with and without quarantine & Treatment

5. Conclusions

In this paper an SEAIR deterministic model with quarantine and treatment for the transmission dynamics of the pandemic COVID-19 was formulated. The mathematical results for the model were shown. The basic reproduction number $\mathcal{R}_0$ was computed and the stability of equilibria points was investigated. Using Castillo-Chavez theorem, the disease free equilibrium point globally asymptotically stable whenever the $\mathcal{R}_0 < 1$ was proven. We consider some parameters and their effect on the model graphically, which can be regarded as the controls for disease eradication. Also we show the effect of using quarantine and treatment in geting better number of recovered individuals. Therefore, as it is shown in the figure, stakeholders should apply both quarantine and treatment simultaneously in cambating the pandemic.

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Ruled hypersurfaces in nonflat complex space forms satisfying Fischer-Marsden equation

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Abstract. In this paper, we prove that there exist no ruled hypersurfaces in a nonflat complex space form satisfying the Fischer-Marsden equation. This answers partially an open question posed by Venkatesha et al. in (Ann. Univ. Ferrara, 67 (2021), 203–216).
Keywords: Fischer-Marsden equation, complex space form, ruled hypersurface.

1. Introduction

It is well known that there exist no Einstein real hypersurfaces in a nonflat complex space form $M^n(c)$ (cf. [2, 12]). Here by a nonflat complex space form $M^n(c)$ we refer to a complete and connected Kähler manifold with constant holomorphic sectional curvature $c \neq 0$ of complex dimension $n \geq 2$. It is complex analytically isometric to a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$. In geometry of real hypersurface, it has been an active and interesting problem for a long time to research the existence and classification of some geometric conditions which generalize Einstein condition. For example, in 1979, Kon in [10] introduced pseudo-Einstein hypersurfaces and later they became an important research subject (see many references related with these hypersurfaces in [2, 12]). In 2009, Cho and Kimura in [4] first initiated the study of Ricci soliton on real hypersurfaces. Here by a Ricci soliton defined on a Riemannian manifold (M, g), we mean a triple (g, V, λ) (or shortly, a metric g) satisfying

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} = \lambda g, \tag{1}$$

where V is a non-zero vector field, $\mathcal{L}$ is the Lie derivative and λ is a constant. When V is a Killing vector field, then a Ricci soliton becomes an Einstein metric. In particular, if V is the gradient of a smooth function f, then (1) becomes

$$\mathrm{Hess} f + \mathrm{Ric} = \lambda g, \tag{2}$$

and it is called a gradient Ricci soliton, where Hess denotes the Hessian operator. Ricci solitons are fixed points of the Ricci flow and play very important roles in modern differential geometry.

It was proved by Cho and Kimura in [5] that there exist no Hopf hypersurfaces which admits a gradient Ricci soliton in a nonflat complex space form. Some other studies involving Ricci solitons on real hypersurfaces can be seen in [1, 8, 11]. These results motivate many other research in which some other extensions of Einstein metrics were discussed. Next we exhibit some of them. A Riemannian manifold (M, g) is said to admit a Miao-Tam critical metric if on M there exists a smooth function f such that

$$\mathrm{Hess}f - (\triangle f)g - f\mathrm{Ric} = g. \tag{3}$$

Note that (3) reduces to an Einstein metric when f is a nonzero constant, just like that case in a gradient Ricci soliton. Applying Cho and Kimura's techniques in [5], Chen in [3] proved that there exist no Hopf real hypersurfaces with Miao-Tam critical metric in a nonflat complex space form. Similarly, a Riemannian manifold (M, g) is said to admit an m-quasi-Einstein metric if on M there exists a smooth function f such that

$$\mathrm{Hess}f - \frac{1}{m}\mathrm{d}f \otimes \mathrm{d}f + \mathrm{Ric} = \lambda g, \tag{4}$$

where m denotes a positive constant. Note that (4) reduces to still an Einstein metric if f is a constant. Applying those techniques in [5], Cui and Chen in [6] proved that there exist no Hopf real hypersurfaces with m-quasi Einstein metric in nonflat complex space forms. A Riemannian manifold (M, g) is said to admit Fischer-Marsden metric if on M there exists a smooth function f such that

$$\mathrm{Hess}f - f\mathrm{Ric} = (\Delta f)g, \tag{5}$$

The well known Fischer-Marsden conjecture states that a compact Riemannian manifold is Einstein if it admits a non-trivial solution to equation (5) (cf. [7]). In view of this, Fischer-Marsden equation (5) is also a nice extension of Einstein metrics. Applying those techniques in [5], Venkatesha et al. in [13] proved that there exist no complete Hopf hypersurfaces satisfying Fischer-Marsden equation in a nonflat complex space form. In addition, Venkatesha et al. in [13] proposed an open question:

Are there real hypersurfaces in nonflat complex space forms satisfying Fischer-Marsden equation?

The present paper aims to investigate the above problem on a special hypersurface. We prove that there exist no ruled hypersurfaces in a nonflat complex space form satisfying Fischer-Marsden equation. The proof of this result is given in the last section of the paper.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of g respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,\ \overline{\nabla}_X N = -AX, \tag{6}$$

for any $X, Y \in \mathfrak{X}(M)$, where A denotes the shape operator of M in $M^n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N,\ JN = -\xi. \tag{7}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0, \tag{8}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi), \tag{9}$$

for any $X, Y \in \mathfrak{X}(M)$. If the structure vector field ξ is principal, that is, $A\xi = \alpha\xi$ at each point, where $\alpha = \eta(A\xi)$, then M is called a Hopf hypersurface and α is called Hopf principal curvature.

Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla} J = 0$) of $M^n(c)$ and using (6), (7) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{10}$$

$$\nabla_X \xi = \phi AX, \tag{11}$$

for any $X, Y \in \mathfrak{X}(M)$. Let R be the Riemannian curvature tensor of M. Because $M^n(c)$ is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as the following:

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{12}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{13}$$

for any $X, Y \in \mathfrak{X}(M)$. From the Gauss equation, the Ricci operator is given by

$$Q = \frac{c}{4}((2n+1)\mathrm{id} - 3\eta \otimes \xi) + (\mathrm{trace}A)A - A^2. \tag{14}$$

3. Main results

Taking a regular curve γ in a nonflat complex space form $M^n(c)$ with tangent vector field X. There is a unique complex projective or hyperbolic hyperplane at each point of γ such that it cuts γ so as to be orthogonal to both X and JX.

The union of these hyperplanes is said to be a ruled real hypersurface ([9, 12]). A ruled hypersurface cannot be Hopf and has some interesting characterizations. For example, a real hypersurface in a nonflat complex space form is ruled if and only if $g(AX, Y) = 0$, for any vector fields X and Y orthogonal to ξ (cf. [9]). It follows that

$$\begin{aligned} A\xi &= \alpha\xi + \beta U, \\ AU &= \beta\xi, \\ AZ &= 0, \ \forall\, Z \in \{\xi, U\}^{\perp}, \end{aligned} \tag{15}$$

where $\alpha = g(A\xi, \xi)$, β is a smooth nowhere vanishing function and U is a unit vector field parallel to $\phi\nabla_{\xi}\xi$. Putting (15) into (14) we have

$$\begin{aligned} Q\xi &= \left(\frac{1}{2}(n-1)c - \beta^2\right)\xi, \\ QU &= \left(\frac{1}{4}(2n+1)c - \beta^2\right)U, \\ QZ &= \frac{1}{4}(2n+1)cZ, \ \forall\, Z \in \{\xi, U\}^{\perp}. \end{aligned} \tag{16}$$

It follows directly that the scalar curvature is $r = (n^2-1)c - 2\beta^2$. We collect some necessary properties of ruled hypersurfaces (cf. [9]) in the following lemma.

Lemma 3.1. *On a ruled hypersurface the following relations are valid:*

$$\begin{aligned} \nabla_U\phi U &= \left(\frac{c}{4\beta} - \beta\right)U, \ \nabla_{\phi U}U = 0, \\ U(\beta) = 0, \ \phi U(\beta) &= \beta^2 + \frac{c}{4}, \ W(\beta) = 0, \ \forall\, W \in \{\xi, U, \phi U\}^{\perp}. \end{aligned} \tag{17}$$

Lemma 3.2. *On a real hypersurface in a noflat complex space form satisfying Fischer-Marsden equation, the following relation is valid:*

$$\begin{aligned} &\left(\frac{1}{2(n-1)}Y(fr) - \frac{c}{4}Y(f)\right)X - \left(\frac{1}{2(n-1)}X(fr) - \frac{c}{4}X(f)\right)Y \\ &+ (X(f)QY - Y(f)QX) + f((\nabla_X Q)Y - (\nabla_Y Q)X) - \frac{c}{2}g(X, \phi Y)\phi Df \\ &+ \frac{c}{4}(\phi X(f)\phi Y - \phi Y(f)\phi X) + AX(f)AY - AY(f)AX = 0, \end{aligned} \tag{18}$$

for any vector field X, Y, where Df denotes the gradient of function f.

Proof. Note that the Fischer-Marsden equation (5) can be transformed into the following

$$\nabla_X Df = (\Delta f)X + fQX,$$

for any vector field X. Contracting the above equality over X gives that $\Delta f = -\frac{fr}{2(n-1)}$. Putting this into the above equality gives

$$\nabla_X Df = -\frac{fr}{2(n-1)}X + fQX.$$

Taking the derivative of this equality we obtain

$$\nabla_Y\nabla_X Df = -\frac{1}{2(n-1)}Y(fr)X - \frac{fr}{2(n-1)}\nabla_Y X + Y(f)QX + f\nabla_Y(QX),$$

for any vector fields X, Y. Applying this equality and previous one in definition of the curvature tensor we have

$$\begin{aligned} R(X,Y)Df =& \frac{1}{2(n-1)}(Y(fr)X - X(fr)Y) \\ &+X(f)QY - Y(f)QX + f(\nabla_X Q)Y - f(\nabla_Y Q)X. \end{aligned} \tag{19}$$

On the other hand, replacing Z by Df in (12) we get

$$\begin{aligned} R(X,Y)Df =& \frac{c}{4}(Y(f)X - X(f)Y + \phi Y(f)\phi X - \phi X(f)\phi Y) \\ &+\frac{c}{2}g(X,\phi Y)\phi Df + AY(f)AX - AX(f)AY. \end{aligned}$$

Comparing the above equality with (19) gives (18). □

With the help of (15), (16) and Lemma 3.1, by a direct calculation we have

$$(\nabla_\xi Q)U - (\nabla_U Q)\xi = -2\beta\xi(\beta)U - \beta^2\nabla_\xi U.$$

Note that we have applied $\nabla_\xi U \in \{\xi, U\}^\perp$ due to $g(\nabla_\xi U, \xi) = 0$ and $g(\nabla_\xi U, U) = 0$. Form now on, suppose that a real hypersurface in a nonflat complex space form satisfies Fischer-Marsden equation. In (18), replacing X and Y by ξ and U, respectively, we obtain an equality. Taking the ξ-component of this equality gives

$$\frac{1}{2(n-1)}U(fr) - \frac{c}{4}U(f) - \left(\frac{n-1}{2}c - \beta^2\right)U(f) + \beta^2 U(f) = 0.$$

Substituting the scalar curvature $r = (n^2-1)c - 2\beta^2$ into the above equality and applying Lemma 3.1, we get

$$\left(\frac{2n-3}{n-1}\beta^2 + \frac{3}{4}c\right)U(f) = 0.$$

Suppose that there exists a point p on the hypersurface such that $U(f) \neq 0$ at p and hence on an open neighborhood Ω around p. Thus, working on Ω we obtain $\frac{2n-3}{n-1}\beta^2 + \frac{3}{4}c = 0$. Then β is a constant. Applying (17) again we obtain $\beta^2 + \frac{c}{4} = 0$. Putting this into the previous one reduces to either $n = 0$ or $c = 0$, a contradiction. Therefore, $U(f) = 0$ holds on the whole of the hypersurface.

With the help of (15), (16) and Lemma 3.1, by a direct calculation we have

$$(\nabla_\xi Q)\phi U - (\nabla_{\phi U} Q)\xi = \frac{2n+1}{4}c\nabla_\xi\phi U - Q\nabla_\xi\phi U + 2\beta\left(\beta^2 + \frac{c}{4}\right)\xi.$$

In (18), replacing X and Y by ξ and ϕU, respectively, we obtain an equality. Taking the ξ-component of this equality gives

$$\begin{aligned}&\frac{1}{2(n-1)}\phi U(fr)-\frac{c}{4}\phi U(f)-\left(\frac{n-1}{2}c-\beta^2\right)\phi U(f)\\&+\frac{2n+1}{4}cfg(\nabla_\xi\phi U,\xi)-fg(Q\nabla_\xi\phi U,\xi)+2f\beta(\beta^2+\frac{c}{4})=0.\end{aligned}$$

Substituting the scalar curvature $r=(n^2-1)c-2\beta^2$ into the above equality and applying Lemma 3.1, we get

$$\left(\frac{n-2}{n-1}\beta^2+\frac{3}{4}c\right)\phi U(f)+\frac{n-3}{n-1}f\beta^3-\frac{n+1}{4(n-1)}cf\beta=0. \tag{20}$$

With the help of (15), (16) and Lemma 3.1, by a direct calculation we have

$$(\nabla_U Q)\phi U-(\nabla_{\phi U}Q)U=\beta\left(\beta^2+\frac{c}{2}\right)U.$$

In (18), replacing X and Y by U and ϕU, respectively, we obtain an equality. Applying the fact $U(f)=0$ and taking the U-component of this equality gives

$$\begin{aligned}&\frac{1}{2(n-1)}\phi U(fr)-\frac{c}{4}\phi U(f)-\left(\frac{2n+1}{4}c-\beta^2\right)\phi U(f)\\&+f\beta\left(\beta^2+\frac{c}{2}\right)-\frac{c}{4}\phi U(f)-\frac{c}{2}\phi U(f)=0.\end{aligned}$$

Substituting the scalar curvature $r=(n^2-1)c-2\beta^2$ into the above equality and applying Lemma 3.1, we get

$$\left(\frac{n-2}{n-1}\beta^2-\frac{3}{4}c\right)\phi U(f)+\frac{n-3}{n-1}f\beta^3+\frac{n+1}{2(n-1)}cf\beta=0. \tag{21}$$

Subtracting (20) from (21) we obtain $\phi U(f)=\frac{1}{2}f\beta$ because of $c\neq 0$. Substituting this into (20) we get

$$\frac{3n-8}{2(n-1)}\beta^2+\frac{3}{8}c-\frac{n+1}{4(n-1)}c=0.$$

This means that β is a constant, and hence from Lemma 3.1 we have $\beta^2+\frac{c}{4}=0$. Putting this into the above equality we arrive at a contradiction. Therefore, we obtain the following result.

Theorem 3.1. *There are no ruled hypersurfaces in nonflat complex space forms satisfying Fischer-Marsden equation.*

Remark 3.1. Hopf and ruled hypersurfaces are ones of the most classical real hypersurfaces in a nonflat complex space form. Except for this two types of real hypersurfaces, the existence and classification problems of general non-Hopf real hypersurfaces satisfying Fischer-Marsden equation are still open questions.

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A unified generalization and refinement of Hermite-Hadamard-type and Simpson-type inequalities via s-convex functions

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Abstract. In this paper, by introducing the incomplete beta function, we establish a multi-parameter integral inequality via s-convex functions, which provides a unified generalization and refinement of Hermite-Hadamard-type and Simpson-type inequalities. As applications, we illustrate that a number of Hermite-Hadamard-type and Simpson-type inequalities can be derived from the special cases of the main result.

Keywords: Hermite-Hadamard-type inequalities, Simpson-type inequalities, s-convex function, generalization, refinement, incomplete beta function.

1. Introduction

The theory of inequalities has been greatly developed since Jensen introduced the concept of convex functions 100 years ago. There are a large number of inequalities which are established by the convexity of functions (see, [1, 2, 3, 4]). Among these results, the Hermite-Hadamard inequality is one of the best known results in the literature, which is stated as follows:

Let $f : [a, b] \to \mathbb{R}$ be a convex function with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Nowadays, the Hermite-Hadamard inequality has been studied extensively both in theory and in practical applications, see, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references cited therein.

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Recently, it has attracted our attention that an extraordinary generalization of Hermite-Hadamard-type inequality was posted by Deng and Wu in [18], in which the Hermite-Hadamard type inequality was generalized by the way of n-time differentiable functions, as follows:

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \\
&\leq \frac{(|n-2|+1)(b-a)^n}{2(|n-2|+3)n!} \\
&\times\left[\frac{(\,(n+1)\,|n-2|+n)\left|f^{(n)}(a)\right|^q+(|n-2|+2)\left|f^{(n)}(b)\right|^q}{(n+2)(\,|n-2|+1\,)}\right]^{\frac{1}{q}},
\end{aligned}
\tag{2}
$$

where $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is convex on $[a,b]$, $n \geq 1$, $q \geq 1$.

The main role of above-mentioned inequality is to provides an estimation to the difference between the middle and rightmost terms in the Hermite-Hadamard inequality (1). This result also leads us to pay attention to another famous inequality, called Simpson's inequality, which gives the estimate of the error term in the quadrature formula [19], i.e.,

$$
\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^4}{2880}\|f^{(4)}\|_\infty,
\tag{3}
$$

where $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\|f^{(4)}\|_\infty=\sup_{x\in(a,b)}|f^{(4)}(x)|<\infty$.

Motivated by the above-mentioned results, in this paper, by introducing more parameters, we establish a multi-parameter integral inequality via s-convex functions, which provides a unified generalization and refinement of inequalities (2) and (3). The methods used are mainly based on the representation of integral using the incomplete beta function and the extension of convexity via the s-convex functions.

The remaining parts of this paper are organized as follows. In Section 2, we present some definitions and lemmas which are essential in the proof of the main results. In Sections 3, we establish our main result, in which a unified generalization and refinement of inequalities (2) and (3) is proved. In Sections 3 and 4, we explain the applications of our main result with two aspects corresponding to the two types of integral inequalities, we show that a lot of Hermite-Hadamard-type and Simpson-type inequalities can be derived respectively when some suitable values are assigned to the parameters.

2. Definitions and lemmas

We begin with introducing some essential definitions and lemmas in preparation for the proof of our main result.

Definition 2.1 ([5]). *Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be convex on I if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{4}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

In [20], Hudzik and Maligranda introduced the class of functions which are s-convex in the second sense, as follows:

Definition 2.2. *A real-valued function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{5}$$

holds for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

It can be easily observed that for $s = 1$ s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

Below are two lemmas, we will give a representation of integral via the incomplete beta function and establish an integral identity.

Lemma 2.1. *Let $\sigma > 0,\ \upsilon > 0,\ \zeta \geq 0,\ 0 < \tau \leq 1$. Then we have*

$$\begin{aligned}
\mathcal{N}(\tau,\sigma,\upsilon,\zeta) &:= \int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}|\zeta - x|dx \\
&= \begin{cases} \zeta B_\tau(\sigma,\upsilon) - B_\tau(\sigma+1,\upsilon), & \zeta \geq \tau, \\ 2\zeta B_\zeta(\sigma,\upsilon) - 2B_\zeta(\sigma+1,\upsilon) & \\ \qquad\qquad -\zeta B_\tau(\sigma,\upsilon) + B_\tau(\sigma+1,\upsilon), & 0 \leq \zeta < \tau \end{cases}
\end{aligned} \tag{6}$$

where $B_t(\kappa,\iota)$ $(0 < t < 1)$ and $B_1(\kappa,\iota)$ denote respectively the incomplete beta function and the beta function [21], i.e.,

$$B_t(\kappa,\iota) = \int_0^t x^{\kappa-1}(1-x)^{\iota-1}dx, \quad \kappa,\iota > 0,$$
$$B_1(\kappa,\iota) = \int_0^1 x^{\kappa-1}(1-x)^{\iota-1}dx, \quad \kappa,\iota > 0.$$

Proof. We compute the integral $\mathcal{N}(\tau,\sigma,\upsilon,\zeta)$ by discussing separately two cases of $\zeta \geq \tau$ and $0 \leq \zeta < \tau$, it follows that

$$\begin{aligned}
\mathcal{N}(\tau,\sigma,\upsilon,\zeta) &= \int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}|\zeta - x|dx \\
&= \begin{cases} \displaystyle\int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx, & \zeta \geq \tau, \\ \displaystyle\int_0^\zeta x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx & \\ \qquad\qquad \displaystyle+\int_\zeta^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(x-\zeta)dx, & 0 \leq \zeta < \tau. \end{cases} \\
&= \begin{cases} \displaystyle\int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx, & \zeta \geq \tau, \\ \displaystyle 2\int_0^\zeta x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx & \\ \qquad\qquad \displaystyle-\int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx, & 0 \leq \zeta < \tau \end{cases} \\
&= \begin{cases} \displaystyle\int_0^\tau \zeta x^{\sigma-1}(1-x)^{\upsilon-1}dx & \\ \qquad\qquad \displaystyle-\int_0^\tau x^{\sigma}(1-x)^{\upsilon-1}dx, & \zeta \geq \tau, \\ \displaystyle 2\int_0^\zeta \zeta x^{\sigma-1}(1-x)^{\upsilon-1}dx & \\ \qquad\qquad \displaystyle-2\int_0^\zeta x^{\sigma}(1-x)^{\upsilon-1}dx & \\ \qquad\qquad \displaystyle-\int_0^\tau \zeta x^{\sigma-1}(1-x)^{\upsilon-1}dx & \\ \qquad\qquad \displaystyle+\int_0^\tau x^{\sigma}(1-x)^{\upsilon-1}dx, & 0 \leq \zeta < \tau \end{cases} \\
&= \begin{cases} \zeta B_\tau(\sigma,\upsilon) - B_\tau(\sigma+1,\upsilon), & \zeta \geq \tau, \\ 2\zeta B_\zeta(\sigma,\upsilon) - 2B_\zeta(\sigma+1,\upsilon) & \\ \qquad\qquad -\zeta B_\tau(\sigma,\upsilon) + B_\tau(\sigma+1,\upsilon), & 0 \leq \zeta < \tau. \end{cases}
\end{aligned}$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let $f : [a,b] \to \mathbb{R}$, and let $f^{(n)}$ $(n \geq 1)$ be integrable on $[a,b]$. Then for $\mu, \lambda \in \mathbb{R}$ and $\theta \in [0,1]$, we have the following identity:*

$$\frac{(b-a)^n}{n!}\left[\int_0^\theta x^{n-1}(n\lambda - x)f^{(n)}(xa+(1-x)b)dx \right.$$
$$\left. - \int_\theta^1 (x-1)^{n-1}(n\mu + x - 1)f^{(n)}(xa+(1-x)b)dx\right]$$

$$
\begin{aligned}
(7) \qquad &= \mu f(a) + \lambda f(b) - (\lambda+\mu-1) f(\theta a + (1-\theta)b) - \frac{1}{b-a}\int_a^b f(x)dx \\
&- \sum_{k=1}^{n-1} \frac{(b-a)^k}{(k+1)!}\Big[\lambda(k+1)\theta^k - \theta^{k+1} + \mu(k+1)(\theta-1)^k + (\theta-1)^{k+1}\Big] \\
&\times f^{(k)}(\theta a + (1-\theta)b).
\end{aligned}
$$

Proof. Using integration by parts for $n-1$ times, we obtain

$$
\begin{aligned}
I &= \int_0^\theta x^{n-1}(n\lambda - x) f^{(n)}(xa + (1-x)b)dx \\
&- \int_\theta^1 (x-1)^{n-1}(n\mu + x - 1) f^{(n)}(xa + (1-x)b)dx \\
&= \sum_{j=1}^{n-1} \frac{f^{(n-j)}(\theta a + (1-\theta)b)}{(b-a)^j} \frac{n!}{(n+1-j)!}\Big[(\theta^{n+1-j} - (n+1-j)\lambda\theta^{n-j}) \\
&- ((\theta-1)^{n+1-j} + (n+1-j)\mu(\theta-1)^{n-j})\Big] + \frac{n!}{(b-a)^{n-1}} \\
&\times \left[\int_0^\theta (\lambda - x) f'(xa + (1-x)b)dx - \int_\theta^1 (\mu + x - 1) f'(xa + (1-x)b)dx\right].
\end{aligned}
$$

Again, using integration by parts again yields

$$
\begin{aligned}
I &= \sum_{j=1}^{n-1} \frac{f^{(n-j)}(\theta a + (1-\theta)b)}{(b-a)^j} \frac{n!}{(n+1-j)!}\Big[(\theta^{n+1-j} - (n+1-j)\lambda\theta^{n-j}) \\
&- ((\theta-1)^{n+1-j} + (n+1-j)\mu(\theta-1)^{n-j})\Big] - \frac{n!}{(b-a)^{n+1}}\int_a^b f(x)dx \\
&+ \frac{n!\Big(\mu f(a) + \lambda f(b) - (\lambda+\mu-1) f(\theta a + (1-\theta)b\Big)}{(b-a)^n}.
\end{aligned}
$$

Performing a substitution $j \to n-k$ gives

$$
\begin{aligned}
I &= \sum_{k=1}^{n-1} \frac{f^{(k)}(\theta a + (1-\theta)b)}{(b-a)^{n-k}} \frac{n!}{(k+1)!}\Big[(\theta^{k+1} - (k+1)\lambda\theta^k) \\
&- ((\theta-1)^{k+1} + (k+1)\mu(\theta-1)^k)\Big] - \frac{n!}{(b-a)^{n+1}}\int_a^b f(x)dx \\
&+ \frac{n!\Big(\mu f(a) + \lambda f(b) - (\lambda+\mu-1) f(\theta a + (1-\theta)b\Big)}{(b-a)^n}.
\end{aligned}
$$

Multiplying both side of the above equation by $(b-a)^n/n!$ leads to the desired identity (7). This completes the proof of Lemma 2.2. □

3. Main result

Our main result is stated in the following theorem, which provides a unified generalization and refinement of inequalities (2) and (3).

Theorem 3.1. *Let $f : [0, \infty) \to \mathbb{R}$ be n-time differentiable function and $a, b \in [0, \infty)$ with $a < b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is s-convex on $[a, b]$, $n \geq 1$, $q \geq 1$, $0 < s \leq 1$ and $\theta, \lambda, \mu \in [0, 1]$, then*

$$\Bigg|\mu f(a)+\lambda f(b)-(\lambda+\mu-1)f(\theta a+(1-\theta)b)-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{(b-a)^k}{(k+1)!}$$
$$\times\left[\lambda(k+1)\theta^k-\theta^{k+1}+\mu(k+1)(\theta-1)^k+(\theta-1)^{k+1}\right]f^{(k)}(\theta a+(1-\theta)b)\Bigg|$$
$$\leq \frac{(b-a)^n}{n!}\Bigg[(\mathcal{N}(\theta,n,1,n\lambda))^{1-\frac{1}{q}}\Big(\mathcal{N}(\theta,n+s,1,n\lambda)|f^{(n)}(a)|^q \tag{8}$$
$$+\mathcal{N}(\theta,n,s+1,n\lambda)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}+(\mathcal{N}(1-\theta,n,1,n\mu))^{1-\frac{1}{q}}$$
$$\times\Big(\mathcal{N}(1-\theta,n,s+1,n\mu)|f^{(n)}(a)|^q+\mathcal{N}(1-\theta,n+s,1,n\mu)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{(b-a)^n}{n!}(\mathcal{N}(\theta,n,1,n\lambda)+\mathcal{N}(1-\theta,n,1,n\mu))^{1-\frac{1}{q}} \tag{9}$$
$$\times\Bigg[\Big(\mathcal{N}(\theta,n+s,1,n\lambda)+\mathcal{N}(1-\theta,n,s+1,n\mu)\Big)|f^{(n)}(a)|^q$$
$$+\Big(\mathcal{N}(\theta,n,s+1,n\lambda)+\mathcal{N}(1-\theta,n+s,1,n\mu)\Big)|f^{(n)}(b)|^q\Bigg]^{\frac{1}{q}},$$

where $\mathcal{N}(\theta, n, 1, n\lambda)$, $\mathcal{N}(\theta, n+s, 1, n\lambda)$, $\mathcal{N}(\theta, n, s+1, n\lambda)$, $\mathcal{N}(1-\theta, n, 1, n\mu)$, $\mathcal{N}(1-\theta, n, s+1, n\mu)$, $\mathcal{N}(1-\theta, n+s, 1, n\mu)$ are given by the formula (6).

Proof. Let

$$\mathcal{H}(\theta,n,\lambda,\mu) := \mu f(a)+\lambda f(b)-(\lambda+\mu-1)f(\theta a+(1-\theta)b)$$
$$-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{(b-a)^k}{(k+1)!}$$
$$\times\left[\lambda(k+1)\theta^k-\theta^{k+1}+\mu(k+1)(\theta-1)^k+(\theta-1)^{k+1}\right]f^{(k)}(\theta a+(1-\theta)b).$$

Then, form Lemma 2.2, one has

$$|\mathcal{H}(\theta,n,\lambda,\mu)| = \frac{(b-a)^n}{n!}\left|\int_0^\theta x^{n-1}(n\lambda-x)f^{(n)}(xa+(1-x)b)dx\right.$$

$$- \int_{\theta}^{1} (x-1)^{n-1}(n\mu + x - 1) f^{(n)}(xa + (1-x)b)dx \Big|$$
$$\leq \frac{(b-a)^n}{n!} \left[\int_0^{\theta} \Big| x^{n-1}(n\lambda - x) f^{(n)}(xa + (1-x)b) \Big| dx \right.$$
$$\left. + \int_{\theta}^{1} \Big| (x-1)^{n-1}(n\mu + x - 1) f^{(n)}(xa + (1-x)b) \Big| dx \right].$$

Using the Hölder integral inequality, we obtain

$$|\mathcal{H}(\theta, n, \lambda, \mu))| \leq \frac{(b-a)^n}{n!} \left[\Big(\int_0^{\theta} |x^{n-1}(n\lambda - x)| dx \Big)^{1-\frac{1}{q}} \right.$$
$$\times \Big(\int_0^{\theta} |x^{n-1}(n\lambda - x)| |f^{(n)}(xa + (1-x)b)|^q dx \Big)^{\frac{1}{q}}$$
$$+ \Big(\int_{\theta}^{1} |(x-1)^{n-1}(n\mu + x - 1)| dx \Big)^{1-\frac{1}{q}}$$
$$\left. \times \Big(\int_{\theta}^{1} |(x-1)^{n-1}(n\mu + x - 1)| |f^{(n)}(xa + (1-x)b)|^q dx \Big)^{\frac{1}{q}} \right].$$

Further, utilizing the s-convexity of $|f^{(n)}(x)|^q$, we deduce that

$$|\mathcal{H}(\theta, n, \lambda, \mu))| \leq \frac{(b-a)^n}{n!} \left[\Big(\int_0^{\theta} |x^{n-1}(n\lambda - x)| dx \Big)^{1-\frac{1}{q}} \right.$$
$$\times \Big(\int_0^{\theta} \Big(x^{n+s-1} |n\lambda - x| |f^{(n)}(a)|^q + x^{n-1}(1-x)^s |n\lambda - x| |f^{(n)}(b)|^q \Big) dx \Big)^{\frac{1}{q}}$$
$$+ \Big(\int_{\theta}^{1} |(x-1)^{n-1}(n\mu + x - 1)| dx \Big)^{1-\frac{1}{q}}$$
$$\times \Big(\int_{\theta}^{1} \Big((1-x)^{n-1} x^s |(n\mu + x - 1)| |f^{(n)}(a)|^q$$
$$\left. + (1-x)^{n+s-1} |(n\mu + x - 1)| |f^{(n)}(b)|^q \Big) dx \Big)^{\frac{1}{q}} \right]$$
$$= \frac{(b-a)^n}{n!} \left[\Big(\int_0^{\theta} |x^{n-1}(n\lambda - x)| dx \Big)^{1-\frac{1}{q}} \right.$$
$$\times \Big(\int_0^{\theta} x^{n+s-1} |n\lambda - x| |f^{(n)}(a)|^q dx + \int_0^{\theta} x^{n-1}(1-x)^s |n\lambda - x| |f^{(n)}(b)|^q dx \Big)^{\frac{1}{q}}$$
$$+ \Big(\int_0^{1-\theta} x^{n-1} |(n\mu - x)| dx \Big)^{1-\frac{1}{q}} \Big(\int_0^{1-\theta} x^{n-1}(1-x)^s |(n\mu - x)| |f^{(n)}(a)|^q dx$$
$$\left. + \int_0^{1-\theta} x^{n+s-1} |(n\mu - x)| |f^{(n)}(b)|^q dx \Big)^{\frac{1}{q}} \right].$$

Now, by means of the integral representation via the incomplete beta function described in Lemma 2.1, we obtain

$$|\mathcal{H}(\theta,n,\lambda,\mu))| \le \frac{(b-a)^n}{n!}\Bigg[(\mathcal{N}(\theta,n,1,n\lambda))^{1-\frac{1}{q}}\Big(\mathcal{N}(\theta,n+s,1,n\lambda)|f^{(n)}(a)|^q$$
$$+\mathcal{N}(\theta,n,s+1,n\lambda)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}+(\mathcal{N}(1-\theta,n,1,n\mu))^{1-\frac{1}{q}}$$
$$\times\Big(\mathcal{N}(1-\theta,n,s+1,n\mu)|f^{(n)}(a)|^q+\mathcal{N}(1-\theta,n+s,1,n\mu)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\Bigg],$$

which is the desired inequality (8).

Now, let us turn to the proof of inequality (9). In fact, the inequality (9) can be derived directly from the following discrete Hölder inequality

$$x_1^{1-\frac{1}{q}}x_2^{\frac{1}{q}}+y_1^{1-\frac{1}{q}}y_2^{\frac{1}{q}} \le (x_1+y_1)^{1-\frac{1}{q}}(x_2+y_2)^{\frac{1}{q}} \quad (q\ge 1)$$

with a choice of

$$\begin{aligned}
x_1 &= \mathcal{N}(\theta,n,1,n\lambda),\\
x_2 &= \mathcal{N}(1-\theta,n,1,n\mu),\\
y_1 &= \mathcal{N}(\theta,n+s,1,n\lambda)|f^{(n)}(a)|^q+\mathcal{N}(\theta,n,s+1,n\lambda)|f^{(n)}(b)|^q,\\
y_2 &= \mathcal{N}(1-\theta,n,s+1,n\mu)|f^{(n)}(a)|^q+\mathcal{N}(1-\theta,n+s,1,n\mu)|f^{(n)}(b)|^q.
\end{aligned}$$

The proof of Theorem 3.1 is complete. □

4. Applications to the establishing of Hermite-Hadamard-type inequalities

In this section, we illustrate that some Hermite-Hadamard-type inequalities can be derived from the special cases of Theorem 3.1.

Putting $\lambda=\mu=\frac{1}{2}$ and $\theta=1$ in the inequalities of Theorem 3.1, we obtain a generalization of Hermite-Hadamard-type inequality (2), as follows:

Corollary 4.1. *Let $f:[0,\infty)\to\mathbb{R}$ be n-time differentiable function and $a,b\in[0,\infty)$ with $a<b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is s-convex on $[a,b]$, $n\ge 1$, $q\ge 1$, $0<s\le 1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\
&\le \frac{(b-a)^n}{n!}\Big(\mathcal{N}\Big(1,n,1,\frac{n}{2}\Big)\Big)^{1-\frac{1}{q}} \qquad (10)\\
&\times\left[\Big(\mathcal{N}\Big(1,n+s,1,\frac{n}{2}\Big)\Big)|f^{(n)}(a)|^q+\Big(\mathcal{N}\Big(1,n,s+1,\frac{n}{2}\Big)\Big)|f^{(n)}(b)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

where $\mathcal{N}(1,n,1,\frac{n}{2})$, $\mathcal{N}(1,n+s,1,\frac{n}{2})$, $\mathcal{N}(1,n,s+1,\frac{n}{2})$ *are given by the formula* (6).

If we take $n=1$ and $n=2$ in inequality (10) respectively, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)}{4\times(2s+2^{1-s})^{-\frac{1}{q}}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{(s+1)(s+2)}\right]^{\frac{1}{q}}, \tag{11}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{2\times 6^{1-\frac{1}{q}}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}. \tag{12}$$

Setting $s=1$ in (10), we get

Corollary 4.2. *Let* $f:[0,\infty)\to\mathbb{R}$ *be* n*-time differentiable function and* $a,b\in[0,\infty)$ *with* $a<b$. *If* $f^{(n)}$ *is integrable and* $|f^{(n)}|^q$ *is convex on* $[a,b]$, $n\geq 1$, $q\geq 1$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\
&\leq\frac{(b-a)^n}{n!}\Big(\mathcal{N}\Big(1,n,1,\frac{n}{2}\Big)\Big)^{1-\frac{1}{q}}\\
&\times\left[\Big(\mathcal{N}\Big(1,n+1,1,\frac{n}{2}\Big)\Big)|f^{(n)}(a)|^q+\Big(\mathcal{N}\Big(1,n,2,\frac{n}{2}\Big)\Big)|f^{(n)}(b)|^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{13}$$

For $n=1$, inequality (13) becomes

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{14}$$

For $n=2$, inequality (13) reduces to

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{12}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{15}$$

Especially, for $n\geq 2$, a simple computation gives

$$\mathcal{N}\Big(1,n,1,\frac{n}{2}\Big)=\frac{n-1}{2(n+1)},$$

$$\mathcal{N}\Big(1,n+1,1,\frac{n}{2}\Big)=\frac{n^2-2}{2(n+1)(n+2)},$$

$$\mathcal{N}\Big(1,n,2,\frac{n}{2}\Big)=\frac{n}{2(n+1)(n+2)},$$

and then substituting them into inequality (13), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq \frac{(n-1)(b-a)^n}{2(n+1)!}\left[\frac{(n^2-2)\left|f^{(n)}(a)\right|^q+n\left|f^{(n)}(b)\right|^q}{(n-1)(n+2)}\right]^{\frac{1}{q}}, \tag{16}$$

which is equivalent to the inequality (2) ($n \geq 2$) that we have mentioned at the beginning section.

5. Applications to the establishing of Simpson-type inequalities

In this section, we show that some Simpson-type inequalities can be derived from the special cases of Theorem 3.1.

Putting $\lambda = \mu = \frac{1}{6}$ and $\theta = \frac{1}{2}$ in the inequalities of Theorem 3.1, we obtain a generalization of Simpson's inequality (3), as follows:

Corollary 5.1. *Let $f : [0, \infty) \to \mathbb{R}$ be n-time differentiable function and $a, b \in [0, \infty)$ with $a < b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is s-convex on $[a, b]$, $n \geq 1$, $q \geq 1$, $0 < s \leq 1$, then*

$$\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\Big(-\frac{1}{2}\Big)^k+\Big(\frac{1}{2}\Big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\right|$$
$$\leq \frac{(b-a)^n}{n!}\Big(2\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}} \tag{17}$$
$$\times\Big(\mathcal{N}\Big(\frac{1}{2},n+s,1,\frac{n}{6}\Big)+\mathcal{N}\Big(\frac{1}{2},n,s+1,\frac{n}{6}\Big)\Big)^{\frac{1}{q}}\Big[|f^{(n)}(a)|^q+|f^{(n)}(b)|^q\Big]^{\frac{1}{q}},$$

where $\mathcal{N}(\frac{1}{2},n,1,\frac{n}{6})$, $\mathcal{N}(\frac{1}{2},n+s,1,\frac{n}{6})$, $\mathcal{N}(\frac{1}{2},n,s+1,\frac{n}{6})$ are given by the formula (6).

If we take $n = 1$ in inequality (17), we obtain

$$\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{18}$$
$$\leq \frac{5(b-a)}{36}\left[\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{5\times 6^s(s+1)(s+2)}\right]^{\frac{1}{q}}\Big[|f'(a)|^q+|f'(b)|^q\Big]^{\frac{1}{q}}.$$

Setting $s = 1$ in (17), we get

Corollary 5.2. *Let $f : [0,\infty) \to \mathbb{R}$ be n-time differentiable function and $a, b \in [0,\infty)$ with $a < b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is convex on $[a,b]$, $n \geq 1$, $q \geq 1$, then*

$$
\begin{aligned}
&\bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx \\
&-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\big(-\frac{1}{2}\big)^k+\big(\frac{1}{2}\big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\bigg| \\
&\leq \frac{(b-a)^n}{n!}\Big(2\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}} \\
&\times\Big(\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)+\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)\Big)^{\frac{1}{q}}\Big[|f^{(n)}(a)|^q+|f^{(n)}(b)|^q\Big]^{\frac{1}{q}}.
\end{aligned}
\tag{19}
$$

Choosing $n = 1$ in inequality (19) yields

$$
\begin{aligned}
&\bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\bigg| \\
&\leq \frac{5(b-a)}{36}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{20}
$$

Choosing $n = 2$ in inequality (19), we obtain

$$
\begin{aligned}
&\bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\bigg| \\
&\leq \frac{(b-a)^2}{81}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{21}
$$

Especially, for $n \geq 3$, a simple computation gives

$$
\begin{aligned}
\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big) &= \frac{n-2}{6(n+1)}\left(\frac{1}{2}\right)^n, \\
\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big) &= \frac{n^2-n-3}{3\,(n+2)\,(n+1)}\left(\frac{1}{2}\right)^{n+2}, \\
\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big) &= \frac{n^2+n-5}{3\,(n+2)\,(n+1)}\left(\frac{1}{2}\right)^{n+2},
\end{aligned}
$$

and then substituting them into inequality (19), we get

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right.\\
&\left.-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\Big(-\frac{1}{2}\Big)^k+\Big(\frac{1}{2}\Big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\right|\\
&\le\frac{(n-2)(b-a)^n}{3\times 2^n(n+1)!}\left[\frac{|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{2}\right]^{\frac{1}{q}}. \qquad (22)
\end{aligned}$$

For $n=3$, inequality (22) reduces to

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{(b-a)^3}{576}\left[\frac{|f'''(a)|^q+|f'''(b)|^q}{2}\right]^{\frac{1}{q}}. \qquad (23)
\end{aligned}$$

For $n=4$, inequality (22) becomes

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{(b-a)^4}{2880}\left[\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}\right]^{\frac{1}{q}}. \qquad (24)
\end{aligned}$$

Inequality (24) is just the Simpson's inequality (3) that we have mentioned in the introduction section.

6. Concluding remarks

In Sections 3 and 4, we have shown the applications of our main result in establishing some generalizations of Hermite-Hadamard-type and Simpson-type inequalities, respectively. Here, we demonstrate that our main result given by Theorem 3.1 can also generate some refined inequalities of Hermite-Hadamard and Simpson type when some suitable values are assigned to the parameters. For example, if we take $s=1,\theta=\frac{1}{2}$ and $\lambda=\mu=\frac{1}{6}$ in Theorem 3.1, then, under the assumptions of Corollary 5.2, we have the following refinement of Simpson's inequalities:

$$\begin{aligned}
&\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right.\\
&\left.-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\Big(-\frac{1}{2}\Big)^k+\Big(\frac{1}{2}\Big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\right|
\end{aligned}$$

$$\leq \frac{(b-a)^n}{n!}\Big(\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}} \tag{25}$$
$$\times\Bigg[\Big(\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)|f^{(n)}(a)|^q+\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}$$
$$+\Big(\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)|f^{(n)}(a)|^q+\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{(b-a)^n}{n!}\Big(2\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}} \tag{26}$$
$$\times\Big(\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)+\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)\Big)^{\frac{1}{q}}\Big[|f^{(n)}(a)|^q+|f^{(n)}(b)|^q\Big]^{\frac{1}{q}}.$$

Especially, when $n \geq 3$, the above inequalities reduce to the following refined inequalities of Simpson type.

$$\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx$$
$$-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\big(-\frac{1}{2}\big)^k+\big(\frac{1}{2}\big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\Bigg|$$
$$\leq \frac{(n-2)(b-a)^n}{3\times 2^{n+1}(n+1)!}\times\Bigg[\Big(\frac{n^2-n-3}{2n^2-8}|f^{(n)}(a)|^q+\frac{n^2+n-5}{2n^2-8}|f^{(n)}(b)|^q\Big)^{\frac{1}{q}} \tag{27}$$
$$+\Big(\frac{n^2+n-5}{2n^2-8}|f^{(n)}(a)|^q+\frac{n^2-n-3}{2n^2-8}|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{(n-2)(b-a)^n}{3\times 2^n(n+1)!}\Bigg[\frac{|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{2}\Bigg]^{\frac{1}{q}}. \tag{28}$$

As a direct consequence, a refinement of Simpson's inequality can be derived by taking $n=4$ in the above inequalities, i.e.,

$$\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Bigg|$$
$$\leq \frac{(b-a)^4}{5760}\Bigg[\Bigg(\frac{3}{8}|f^{(4)}(a)|^q+\frac{5}{8}|f^{(4)}(b)|^q\Bigg)^{\frac{1}{q}} \tag{29}$$
$$+\Bigg(\frac{5}{8}|f^{(4)}(a)|^q+\frac{3}{8}|f^{(4)}(b)|^q\Bigg)^{\frac{1}{q}}\Bigg]$$
$$\leq \frac{(b-a)^4}{2880}\Bigg[\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}\Bigg]^{\frac{1}{q}}. \tag{30}$$

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Oscillation criteria of fractional damped differential equations

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Abstract. In this paper, we prove some properties of oscillation for a class of fractional damped differential equations using generalized Riccati transformation and inequality technique, we prove some new oscillatory criteria. Recent results in the literature are generalized and significant improved. Example is shown to illustrate our main results.
Keywords: oscillatory criteria, fractional derivative, fractional damped differential equation.

1. Introduction

Consider the oscillation of the following fractional damped differential equations

$$[r(t)(D_{-}^{\alpha}y)^{\eta}(t)]' - p(t)(D_{-}^{\alpha}y)^{\eta}(t) - q(t)f(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv) = 0, \tag{1.1}$$

where $\alpha \in (0,1)$ is a constant, and $\eta > 0$ is a quotient of odd positive integers. The differential operator $D_{-}^{\alpha}y$ is the Liouville right-sided fractional derivation

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of order α for y defined by $(D_{-}^{\alpha}y)(t) = -\frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt}\int_t^{\infty}(v-t)^{-\alpha}y(v)dv$ for $t \in R_+ := (0, +\infty)$. Here Γ is the gamma function $\Gamma(t) = \int_0^{\infty} v^{t-1}e^{-v}dv$ for $t \in R_+$. We assume that conditions hold:

(H_1) $r(t), p(t)$ and $q(t)$ are positive continuous functions on $[t_0, \infty)$ for a certain $t_0 > 0$. The function$f : R \to R$ is a continuous function such that $\frac{f(u)}{u^{\eta}} \geq G$ for a certain constant for $G > 0$ and for all $u \neq 0$.

By a solution of (1.1) we mean a nontrivial function $y \in C(R_+, R)$ such that $\int_t^{\infty}(v-t)^{-\alpha}y(v)dv \in C^1(R_+, R), r(t)(D_{-}^{\alpha}y)^{\eta}(t) \in C^1(R_+, R)$ and (1.1) hold for $t > 0$. We focus on those solutions of (1.1) which exist on R_+ such that $\sup\{|y(t)| : t > t_*\} > 0$ for any $t_* \geq 0$. A solution y of (1.1) is said to be called oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Due to its important applications on many fields, such as viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc, $(see, for example,$ $[1-12]$), in last decade, a lot of attentions has been focused on the study of the stability and properties of solutions for fractional differential equations, see, for example, [13-21].

In particular, Chen [2] studied oscillatory properties of solutions to the following fractional differential equations

$$(*) \qquad [r(t)(D_{-}^{\alpha}y)^{\eta}(t)]' - q(t)f(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv) = 0,$$

for $t > 0$, where $D_{-}^{\alpha}y$ denotes the Liouville right-sided fractional derivative of order α with the form

$$(D_{-}^{\alpha}y)(t) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt}\int_t^{\infty}(v-t)^{-\alpha}y(v)dv,$$

for $t \in R_+ := (0, \infty)$. By using Riccati transformation technique the authors obtained some sufficient conditions, which guarantee that every solution of the equation is oscillatory.

Zhang [18] considered the oscillation of the nonlinear fractional damped fractional differential equations

$$[a(t)(D_{-}^{\alpha}x(t))^{\gamma}]' + p(t)(D_{-}^{\alpha}x(t))^{\gamma} - q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi) = 0, t \in [t_0, \infty),$$

where $D_{-}^{\alpha}x(t)$ denotes the Liouville right-sided fractional derivative of order α of x. By using a generalized Riccati function and the inequality technique, he established some new criteria.

Qi and Huang [19] studied the oscillation behavior of the equation:

$$(a(t)[r(t)D_{-}^{\alpha}x(t)]')' + p(t)[r(t)D_{-}^{\alpha}x(t)]' - q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi) = 0,$$

where $D_{-}^{\alpha}x(t)$ also denotes the Liouville right-sided fractional derivative and established some sufficient conditions for the oscillation of the equation.

However, as much as we know, very little is known on the oscillation of fractional damped differential equations. Only a few of papers have been published on the oscillation theory of fractional damped differential equations, such as [3,4,16-18].

In this paper, we will establish some new oscillation criteria for (1.1), by a class of new function $\Phi(t,s,l)$ and $H(t)$, generalized Riccati transformation and inequality technique.

2. Preliminaries

In this section,we present the definitions of fractional integrals, fractional derivatives and function Φ, which are used throughout this article. We also, give several lemmas, which are useful in establishing our results.

Definition 2.1 (KiLbas et al. [7])**.** The Liouville right-sided fractional integral of order $\beta > 0$ of a function $g : R_{+} \to R$ on the half-axis R_{+} is given by

$$(I_{-}^{\beta}g)(t) := \frac{1}{\Gamma(\beta)}\int_{t}^{\infty}(v-t)^{\beta-1}g(v)dv, \tag{2.1}$$

for $t > 0$, provided the right-hand side is pontwise defined on R_{+}, where Γ is the gamma function.

Definition 2.2 (Kilbas et al. [7])**.** The Liouville right-sided fractional derivative of order $\beta > 0$ of a function $g : R_{+} \to R$ on the half-axis R_{+} is given by

$$\begin{aligned}(D_{-}^{\beta}g)(t) &= (-1)^{[\beta]}\frac{d^{[\beta]}}{dt^{[\beta]}}(I_{-}^{[\beta]-\beta}g)(t)\\ &= (-1)^{[\beta]}\frac{1}{\Gamma([\beta]-\beta)}\cdot\frac{d^{[\beta]}}{dt^{[\beta]}}\int_{t}^{\infty}(v-t)^{[\beta]-\beta-1}g(v)dv,\end{aligned} \tag{2.2}$$

for $t > 0$, provided the right-hand side is pointwise defined on R_{+},where$[\beta] := \min\{z \in Z : z \geq \beta\}$ is the ceilling function.

Definition 2.3 (Sun et al. [15])**.** We say that a function $\Phi = \Phi(t,s,l)$ belongs to the function class Y, denoted by $\Phi \in Y$, if $\Phi \in (E,R)$, where $E = \{(t,s,l) : t_0 \leq l \leq s \leq t < \infty\}$,which satisfies $\Phi(t,t,l) = 0, \Phi(t,l,l) = 0, \Phi(t,s,l) \neq 0$ for $l < s < t$, and has the partial derivative $\frac{\partial\Phi}{\partial s}$ on E such that $\frac{\partial\Phi}{\partial s}$ is locally integrable with respect to s in E.

Definition 2.4 (Sun et al. [15])**.** Let $\Phi \in Y, g \in C^1([t_0,+\infty),R)$, the operator $T[*;l,t]$ is defined by

$$T[g;l,t] = \int_{l}^{t}\Phi^2(t,s,l)g(s)ds, \tag{2.3}$$

for $t \geq s \geq l \geq t_0$ and $g(s) \in C[t_0, \infty)$,and the function $\varphi = \varphi(t, s, l)$ is defined by

$$\frac{\partial \Phi(t,s,l)}{\partial s} = \varphi(t,s,l)\Phi(t,s,l). \tag{2.4}$$

It is easy to verify that $T[*; l, t]$ is a linear operator and satisfies

$$T[g'; l, t] = -2T[g\varphi; l, t]. \tag{2.5}$$

Lemma 2.1. Let y be a solution of (1.1) and

$$A(t) := \int_t^\infty (v-t)^{-\alpha} y(v)dv, B(t) = e^{\int_t^\infty \frac{p(s)}{r(s)}ds}, \tag{2.6}$$

for $\alpha \in (0,1)$ and $t > 0$, then

$$[A(t)B(t)]' = -\Gamma(1-\alpha)(D_-^\alpha y)(t)B(t) - \frac{p(t)}{r(t)}A(t)B(t), \tag{2.7}$$

for $\alpha \in (0,1)$ and $t > 0$.

Proof. From(2.6) and(2.2), for $\alpha \in (0,1)$ and $t > 0$, we obtain

$$\begin{aligned}
[A(t)B(t)]' &= A'(t)B(t) + A(t)B'(t) \\
&= \Gamma(1-\alpha) \cdot \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty (v-t)^{-\alpha} y(v)dv.B(t) \\
&- \int_t^\infty (v-t)^{-\alpha} y(v)dv \cdot \frac{p(t)}{r(t)}.B(t) \\
&= -\Gamma(1-\alpha)[(-1)^{[\alpha]} \frac{1}{\Gamma([\alpha]-\alpha)} \cdot \frac{d^{[\alpha]}}{dt^{[\alpha]}} \int_t^\infty (v-t)^{[\alpha]-\alpha-1} y(v)dv]B(t) \\
&- \frac{p(t)}{r(t)}A(t)B(t) \\
&= -\Gamma(1-\alpha)(D_-^\alpha y)(t)B(t) - \frac{p(t)}{r(t)}A(t)B(t).
\end{aligned}$$

The proof is complete. □

Lemma 2.2 (Hardy et al. [15])**.** If X and Y are nonnegative, then

$$mXY^{m-1} - X^m \leq (m-1)Y^m,$$

for $m > 1$, where the equality holds if and only if $X = Y$.

3. Main result

Theorem 3.1. Suppose that (H_1) and

$$\int_{t_0}^{\infty} r^{-\frac{1}{\eta}}(s)B^{1-\frac{1}{\eta}}(s)ds = \infty \tag{3.1}$$

hold. Furthermore, assume that there exists a positive function $b(t) \in C^1[t_0, \infty)$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^{\eta}}]ds = \infty, \tag{3.2}$$

where $b'_+(s) = \max\{b'(s), 0\}$, then every solution of (1.1) is oscillatory.

Proof. Suppose that y is a non-oscillatory solution of (1.1). Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of(1.1). Then, there exists $t_1 \in [t_0, \infty)$ such that

$$y(t) > 0 \text{ and } A(t)B(t) > 0, \tag{3.3}$$

for $t \in [t_1, \infty)$, where $A(t), B(t)$ is defined as in (2.6). Therefore, it follows from (1.1) that

$$\begin{aligned} [r(t)(D_-^{\alpha}y)^{\eta}(t)B(t)]' &= [r(t)(D_-^{\alpha}y)^{\eta}(t)]'B(t) - p(t)(D_-^{\alpha}y)^{\eta}(t)B(t) \\ &= q(t)f(A(t))B(t) > 0, \end{aligned} \tag{3.4}$$

for $t \in [t_1, \infty)$.

Thus, $r(t)(D_-^{\alpha}y)^{\eta}(t)B(t)$ is strictly increasing on $[t_1, \infty)$ and is eventually of one sign. Since $r(t) > 0, B(t) > 0$ for $t \in [t_1, \infty)$ and $\eta > 0$ is a quotient of odd positive integers,we see that $(D_-^{\alpha}y)(t)$ is eventually of one sign. We now claim

$$(D_-^{\alpha}y)(t) < 0,$$

for $t \in [t_1, \infty)$.

If not, then $(D_-^{\alpha}y)(t)$ is eventually positive and there exists $t_2 \in [t_1, \infty)$ such that $(D_-^{\alpha}y)(t_2) > 0$. Since $r(t)(D_-^{\alpha}y)^{\eta}(t)B(t)$ is strictly increasing on $[t_1, \infty)$, it is clear that

$$r(t)(D_-^{\alpha}y)^{\eta}(t)B(t) \geq r(t_2)(D_-^{\alpha}y)^{\eta}(t_2)B(t_2) := a_1 > 0,$$

for $t \in [t_2, \infty)$. Therefore, from (2.4), we have

$$\begin{aligned} -\frac{[A(t)B(t)]'}{\Gamma(1-\alpha)B(t)} &= -\frac{-\Gamma(1-\alpha)(D_-^{\alpha}y)(t)B(t) - \frac{p(t)}{r(t)}A(t)B(t)}{\Gamma(1-\alpha)B(t)} \\ &= (D_-^{\alpha}y)(t) + \frac{A(t)p(t)}{\Gamma(1-\alpha)r(t)} \\ &\geq (D_-^{\alpha}y)(t) \geq (\frac{a_1}{r(t)B(t)})^{\frac{1}{\eta}} = a_1^{\frac{1}{\eta}} r^{-\frac{1}{\eta}}(t)B^{-\frac{1}{\eta}}(t), \end{aligned}$$

and, then, we have

$$-\frac{[A(t)B(t)]'}{\Gamma(1-\alpha)} \geq a_1^{\frac{1}{\eta}} r^{-\frac{1}{\eta}}(t) B^{1-\frac{1}{\eta}}(t),$$

for $t \in [t_2, \infty)$.

Integrating both sides of the last inequality from t_2 to t, we have

$$\int_{t_2}^{t} r^{-\frac{1}{\eta}}(s) B^{1-\frac{1}{\eta}}(s) ds \leq -\frac{A(t)B(t) - A(t_2)B(t_2)}{a_1^{\frac{1}{\eta}} \Gamma(1-\alpha)} \leq \frac{A(t_2)B(t_2)}{a_1^{\frac{1}{\eta}} \Gamma(1-\alpha)} < \infty,$$

for $t \in [t_2, \infty)$.

Letting $t \to \infty$, we see

$$\int_{t_2}^{\infty} r^{-\frac{1}{\eta}}(s) B^{1-\frac{1}{\eta}}(s) ds \leq \frac{A(t_2)B(t_2)}{a_1^{\frac{1}{\eta}} \Gamma(1-\alpha)} < \infty.$$

This contradicts (3.1). Hence, (3.5) holds. Define the function $w(t)$ by the generalized Riccati substitution

$$w(t) = b(t) \cdot \frac{-r(t)(D_-^{\alpha} y)^{\eta}(t) B(t)}{(A(t))^{\eta}}, \tag{3.6}$$

for $t \in [t_1, \infty)$.

Then, we have $w(t) > 0$ for $t \in [t_1, \infty)$. From (3.6),(1.1),(3.4) and (H_1), it follows that

$$\begin{aligned}
w'(t) = {} & b'(t) \cdot \frac{-r(t)(D_-^{\alpha} y)^{\eta}(t) B(t)}{(A(t))^{\eta}} + b(t)[\frac{-r(t)(D_-^{\alpha} y)^{\eta}(t) B(t)}{(A(t))^{\eta}}]' \\
\leq {} & b_+'(t) \cdot \frac{-r(t)(D_-^{\alpha} y)^{\eta}(t) B(t)}{(A(t))^{\eta}} \\
& + b(t) \cdot \frac{\begin{array}{c}[-r(t)(D_-^{\alpha} y)^{\eta}(t) B(t)]'(A(t))^{\eta} \\ + r(t)(D_-^{\alpha} y)^{\eta}(t) B(t) \eta (A(t))^{\eta-1}(-\Gamma(1-\alpha)(D_-^{\alpha} y)(t))\end{array}}{(A(t))^{2\eta}} \\
= {} & \frac{b_+'(t)}{b(t)} w(t) + b(t)[\frac{-q(t) f(A(t)) B(t)}{(A(t))^{\eta}} \\
& + r(t)(D_-^{\alpha} y)^{\eta}(t) \cdot \frac{\eta B(t)[-\Gamma(1-\alpha)(D_-^{\alpha} y)(t)]}{(A(t))^{\eta+1}} \\
\leq {} & \frac{b_+'(t)}{b(t)} w(t) - G q(t) b(t) B(t) - \eta \Gamma(1-\alpha) b(t) r(t) B(t) [\frac{w(t)}{b(t) r(t) B(t)}]^{1+\frac{1}{\eta}} \\
= {} & -G q(t) b(t) B(t) + \frac{b_+'(t)}{b(t)} w(t) - \eta \Gamma(1-\alpha)[b(t) r(t) B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t), \qquad (3.7)
\end{aligned}$$

for $t \geq t_1$, where $b_+'(t)$ is defined as in Theorem 3.1.

Taking

$$m = 1 + \frac{1}{\eta}, \quad X = \frac{[\eta\Gamma(1-\alpha)]^{\frac{1}{m}} w(t)}{[b(t)r(t)B(t)]^{\frac{1}{\eta+1}}}$$

and

$$Y = \frac{[\frac{b'_+(t)}{b(t)}]^{\eta}[b(t)r(t)B(t)]^{\frac{1}{m}}}{m^{\eta}[\eta\Gamma(1-\alpha)]^{\frac{\eta}{m}}},$$

from (3.7) and Lemma 2.2, we conclude that

$$w'(t) \le -Gq(t)b(t)B(t) + \frac{r(t)B(t)[b'_+]^{\eta+1}}{(\eta+1)^{\eta+1}[\Gamma(1-\alpha)b(t)]},$$

for $t \in [t_1, \infty)$.

Integrating both sides of the last inequality from t_1 to t, we have

$$\int_{t_1}^{t} [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^{\eta}}]ds \le w(t_1) - w(t) < w(t_1),$$

for $t \in [t_1, \infty)$.

Letting

$$\lim_{t\to\infty} \sup \int_{t_1}^{t} [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^{\eta}}]ds < w(t_1) < \infty,$$

which contradicts (3.2). The proof is complete. □

Remark 3.1. Theorem 3.1 in [2] is a special case of Theorem 3.1 with $p(t) = 0$, respectively. Theorem 3.1 improves and extend the results of Theorem 3.1.

Theorem 3.2. Suppose that (H_1) and (3.1) hold. Let $T_0 \ge t_0$,then there exist a and b such that $b > a > T_0$. Let

$$D(a,b) = \{U(t) \in C^1[a,b] : U(t) \ne 0, t \in (a,b), U(a) = U(b) = 0\}.$$

If there exist a function $H(t) \in D(a,b)$ such that the following condition that holds:

$$\int_a^b Gb(s)q(s)B(s)ds > \int_a^b \frac{[H^{\eta}(s)(H'(s) + \frac{b'_+(s)}{(\eta+1)b(s)})]^{\eta+1} b(s)r(s)B(s)}{[\Gamma(1-\alpha)H^{\eta+1}(s)]^{\eta}} ds. \tag{3.8}$$

Then equation (1.1) is oscillatory.

Proof. Suppose that $y(t)$ is a non-oscillatory solution of (1.1), Without loss of generality, we may suppose that $y(t)$ is an eventually positive solution of (1.1). We proceed as in proof of Theorem 3.1 to get that (3.7) holds.

Multiplying both sides of (3.7) by $H^{\eta+1}(t)$ and integrating from a to b, by $H(a) = H(b) = 0$, we obtain

$$\begin{aligned}\int_a^b H^{\eta+1}(s)w'(s)ds &\le -\int_a^b Gb(s)r(s)B(s)H^{\eta+1}(s)ds \\ &+ \int_a^b \frac{b'_+(s)H^{\eta+1}(s)w(s)}{b(s)}ds \\ &- \int_a^b \eta\Gamma(1-\alpha)H^{\eta+1}(s)[b(s)r(s)B(s)]^{-\frac{1}{\eta}}w^{1+\frac{1}{\eta}}(s)ds \end{aligned} \tag{3.9}$$

and then we get

$$\begin{aligned}\int_a^b Gb(s)r(s)B(s)H^{\eta+1}(s)ds &\le \int_a^b (\eta+1)H^{\eta}(s)H'(s)w(s)ds \\ &+ \int_a^b \frac{b'_+(s)H^{\eta+1}(s)w(s)}{b(s)}ds \\ &- \int_a^b \eta\Gamma(1-\alpha)H^{\eta+1}(s)[b(s)r(s)B(s)]^{-\frac{1}{\eta}}w^{1+\frac{1}{\eta}}(s)ds \\ &= \int_a^b [(\eta+1)H^{\eta}(s)w(s)(H'(s) + \frac{b'_+(s)H(s)}{(\eta+1)b(s)}) \\ &- \eta\Gamma(1-\alpha)H^{\eta+1}(s)[b(s)r(s)B(s)]^{-\frac{1}{\eta}}w^{1+\frac{1}{\eta}}(s)]ds. \end{aligned} \tag{3.10}$$

Taking

$$m = 1 + \frac{1}{\eta}, \quad X = \frac{[\eta\Gamma(1-\alpha)H^{\eta+1}(s)]^{\frac{1}{m}}w(s)}{[b(s)r(s)B(s)]^{\frac{1}{\eta+1}}}$$

and

$$Y = \frac{[(\eta+1)H^{\eta}(s)(H'(s) + \frac{b'_+(s)H(s)}{(\eta+1)b(s)})]^{\eta}[b(s)r(s)B(s)]^{\frac{1}{m}}}{m^{\eta}[\eta\Gamma(1-\alpha)H^{\eta+1}(s)]^{\frac{\eta}{m}}},$$

by (3.10) and Lemma2.2, we conclude that

$$\int_a^b Gb(s)q(s)B(s)ds \le \int_a^b \frac{[H^{\eta}(s)(H'(s)+\frac{b'_+(s)H(s)}{(\eta+1)b(s)})]^{\eta+1}b(s)r(s)B(s)}{[\Gamma(1-\alpha)H^{\eta+1}(s)]^{\eta}}ds, \tag{3.11}$$

which contradicts the condition (3.8). The proof is complete. □

Remark 3.2. Theorem 3.2 is new because we introduce a new class of functions $H(t)$.

Theorem 3.3. Suppose that (H_1) and (3.1) hold. There exist a function $\Phi \in Y$. Such that

$$\limsup_{t\to\infty} \int -l^t[Gq(s)b(s)B(s) - \frac{\frac{1}{\eta}[2\varphi + \frac{b'_+(s)}{b(s)}]^{\eta+1} b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^\eta}]\Phi^2(t,s,l)ds > 0 \tag{3.12}$$

for each $l \geq T_0 \geq t_0$, where operator T defined by (2.3) and the function $\varphi = \varphi(t,s,l)$ is defined by (2.4). Then every solution y of (1.1) is oscillatory.

Proof. Suppose that y is a non-oscillatory solution of (1.1). Without loss of generality,we can assume that$y$is an eventually positive solution of (1.1). Similarly in the proof of Theorem 3.1 to get (3.7) hold, and then we have

$$Gq(t)b(t)B(t) \leq -w'(t) + \frac{b'_+(t)}{b(t)}w(t) - \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t). \tag{3.13}$$

Applying $T[*;T_0,t]$ to (3.13), we have

$$\begin{aligned} &T[Gq(t)b(t)B(t);T_0,t] \\ &\leq T[-w'(t) + \frac{b'_+(t)}{b(t)}w(t) - \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t);T_0,t] \\ &= 2T[w(t)\varphi(t,s,l);T_0,t] + T[\frac{b'_+(t)}{b(t)}w(t) \\ &- \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t);T_0,t] \\ &= T[(2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)})w(t) \\ &- \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t);T_0,t], \end{aligned} \tag{3.14}$$

for $t \in [T_0, \infty)$.

Taking

$$m = 1 + \frac{1}{\eta}, \quad X = \frac{[\eta\Gamma(1-\alpha)]^{\frac{\eta}{1+\eta}} w(t)}{[b(t)r(t)B(t)]^{\frac{1}{\eta+1}}}$$

and

$$Y = \frac{[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^\eta [b(t)r(t)B(t)]^{\frac{\eta}{\eta+1}}}{(\frac{1+\eta}{\eta})^\eta [\eta\Gamma(1-\alpha)]^{\frac{\eta^2}{1+\eta}}},$$

by (3.14) and Lemma 2.2, we conclude that

$$T[Gq(t)b(t)B(t);T_0,t] \leq T[\frac{\frac{1}{\eta}[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^{\eta+1} b(t)r(t)B(t)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^\eta};T_0,t].$$

Noting that (2.3) and then we have

$$T[Gq(t)b(t)B(t) - \frac{\frac{1}{\eta}[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^{\eta+1}b(t)r(t)B(t)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}; T_0, t] \leq 0.$$

Letting $t \to +\infty$, we have

$$\lim_{t\to+\infty}\sup T[Gq(t)b(t)B(t) - \frac{\frac{1}{\eta}[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^{\eta+1}b(t)r(t)B(t)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}; T_0, t] \leq 0,$$

and then

$$\lim_{t\to\infty}\sup\int_l^t [Gq(s)b(s)B(s) - \frac{\frac{1}{\eta}[2\varphi + \frac{b'_+(s)}{b(s)}]^{\eta+1}b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}]\Phi^2(t,s,l)ds \leq 0.$$

Which is a contradiction to (3.12). The proof is complete. □

If we chose $\Phi(t,s,l) = \rho(s)(t-s)^{\alpha}(s-l)^{\beta}$ for $\alpha, \beta > \frac{1}{2}$ and $\rho(t) \in C^1([t_0,\infty), (0,\infty))$, then, we have

$$\varphi(t,s,l) = \frac{\rho'(s)}{\rho(s)} + \frac{\beta t - (\alpha+\beta)s + \alpha l}{(t-s)(s-l)}.$$

Thus, by Theorem 3.3, we have the following corollary.

Corollary 3.4. Suppose that(H_1) and (3.1) hold. For each $l \geq t_0$, there exist a function $\rho(t) \in C^1([t_0,\infty), R)$ and two constants $\alpha, \beta > \frac{1}{2}$, such that

$$\begin{aligned}&\lim_{t\to\infty}\sup\int_l^t \rho^2(s)(t-s)^{2\alpha}(s-l)^{2\beta}[Gq(s)b(s)B(s)\\ &-\frac{\frac{1}{\eta}[2(\frac{\rho'(s)}{\rho(s)} + \frac{\beta t-(\alpha+\beta)s+\alpha l}{(t-s)(s-l)}) + \frac{b'_+(s)}{b(s)}]^{\eta+1}b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}]\Phi^2(t,s,l)ds] > 0.\end{aligned} \tag{3.15}$$

All solutions of (1.1) is oscillatory.

Define

$$R(t) = \int_l^t \frac{p(s)}{r(s)}ds, t \geq l \geq t_0.$$

If we chose $\Phi(t,s,l) = \rho(s)(R(t)-R(s))^{\alpha}(R(s)-R(l))^{\beta}$ for $\alpha, \beta > \frac{1}{2}$ and $\rho(t) \in C^1([t_0,\infty), (0,\infty))$, then, we have

$$\varphi(t,s,l) = \frac{\rho'(s)}{\rho(s)} + \frac{p(t)[\beta R(t) - (\alpha+\beta)R(s) + \alpha R(l)]}{r(s)(R(t)-R(s))(R(s)-R(l))}.$$

Thus, by Theorem 3.3, we have the following Theorem.

Theorem 3.5. Suppose that(H_1) and (3.1) hold. For each $l \geq t_0$, there exist a function $\rho(t) \in C^1([t_0, \infty), R)$ and two constants $\alpha, \beta > \frac{1}{2}$, such that

$$\lim_{t\to\infty} \sup \int_l^t \rho^2(s)(R(t)-R(s))^{2\alpha}(R(s)-R(l))^{2\beta}[Gq(s)b(s)B(s) - \frac{\frac{1}{\eta}[2(\frac{\rho'(s)}{\rho(s)} + \frac{\beta R(t)-(\alpha+\beta)R(s)+\alpha R(l)}{(R(t)-R(s))(R(s)-R(l))}) + \frac{b'_+(s)}{b(s)}]^{\eta+1} b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^\eta}]\Phi^2(t,s,l)ds] > 0. \tag{3.16}$$

The every solution of (1.1) is oscillatory.

Remark 3.4. Theorems 3.3-3.5 are new because we introduce a class of kernel functions $\Phi = \Phi(t,s,l)$ which is basically a product $H(t,s)H(s,l)$ for a kernel $H(t,s)$ of Philos'type. On the other hand, when Eq. (1.1) becomes Eq. (*), conditions (3.12), (3.15), (3.16) become simpler, and they are stronger (in many case) than many exist oscillation conditions. Theorems 3.3, 3.4 improve and extend the results Theorems 3.2, 3.3 in [2].

4. Examples

Example 4.1. Consider the fractional differential equation

$$[\frac{1}{t^6}(D_-^{\frac{1}{2}}y)^\eta(t)]' - \frac{1}{t^7}(D_-^{\frac{1}{2}}y)^\eta(t) - \frac{1}{t^2}(\int_t^\infty (v-t)^{-\frac{1}{2}}y(v)dv) = 0, t > 0, \tag{4.1}$$

where $\alpha = \frac{1}{2}, \eta > 0$ is a quotient of odd positive integers and $(\eta+1)^{\eta+1}(\Gamma(\frac{1}{2}))^\eta > 1$. In (4.1),$r(t) = t^{-6}, p(t) = t^{-7}, q(t) = t^{-2}, f(u) = u$. Take $t_0 > 0, G = 1$. Since

$$B(s) = exp(-\int_{t_0}^t \frac{p(s)}{r(s)}ds) = exp(-\int_{t_0}^t \frac{1}{s}ds) = \frac{t_0}{t},$$
$$\int_{t_0}^\infty r^{-\frac{1}{\eta}}(s)B^{1-\frac{1}{\eta}}(s)ds = \int_{t_0}^\infty s^{\frac{6}{\eta}}(\frac{t_0}{s})^{1-\frac{1}{\eta}}ds = t_0^{1-\frac{1}{\eta}}\int_{t_0}^\infty s^{\frac{7}{\eta}-1}ds = \infty,$$

we find that(H_1) and (3.1) hold. We will apply Theorem 3.1, and it remains to satisfy the condition (3.2), taking $b(s) = s^2$, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \sup \int_{t_0}^t [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^\eta}]ds \\
&= \lim_{t\to\infty} \sup \int_{t_0}^t [s^2 \cdot \frac{1}{s^2} \cdot \frac{t_0}{s} - \frac{s^{-6} \cdot \frac{t_0}{s} \cdot (2s)^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(\frac{1}{2}) \cdot s^2]^\eta}]ds \\
&= \lim_{t\to\infty} \sup \int_{t_0}^t [\frac{t_0}{s} - \frac{s^{-\eta-6} \cdot 2^{\eta+1} \cdot t_0}{(\eta+1)^{(\eta+1)}[\Gamma(\frac{1}{2})]^\eta}]ds = \infty
\end{aligned}$$

which implies that (3.2) hold. Therefore, by Theorem3.1 every solution of(4.1) is oscillatory.

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Hedges in quasi-pseudo-MV algebras

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Abstract. In this paper, we introduce the notions of multiplicative interior operators (mi-operators, for short), additive closure operators (ac-operators, for short) and hedges in quasi-pseudo-MV algebras which will generalize the related contents in pseudo-MV algebras. First we discuss the relationship between mi-operators and ac-operators in a quasi-pseudo-MV algebra and investigate the properties of mi-operators in quasi-pseudo-MV algebras. Second we define and study hedges in quasi-pseudo-MV algebras. We also show that mi-operators are hedges. Finally, the properties of filters and weak filters in a quasi-pseudo-MV algebra with hedge are discussed.

Keywords: quasi-pseudo-MV algebras, Hedges, multiplicative interior operators, filters.

1. Introduction

Quasi-pseudo-MV algebras (qpMV-algebras, for short) were introduced in [4] as the generalizations of both pseudo-MV algebras [9] and quasi-MV algebras [11]. Considering that qpMV-algebras may play an important role in studying many-valued fuzzy logic and quantum computational logic, many properties of qpMV-algebras have been investigated in [3, 4, 5, 6, 7].

The notions of hedges were defined as operators acting on fuzzy subsets by Zadeh in order to describe linguistic hedges such as "very", "more or less", "much", and so on [18]. In his paper, some examples were given to handle how to define hedges as operators. However, any sort of axiomatization was not considered. In [1], authors defined a hedge as operator on a complete lattice. The

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hedge in their definition is a mapping f which needs satisfy four axioms: (1) $f(1)=1$, (2) $f(x)\le x$, (3) $f(x\to y)\le f(x)\to f(y)$ and (4) $f(f(x))=x$. Authors also pointed out that this definition of hedge was indeed a truth function of logical connective "very true". On the other hand, the concepts of very true operators (vt-operators, for short) were introduced by Hajek on BL-algebras and the algebraic structures were called BL_{vt}-algebras [10]. A vt-operator is a mapping f which also contains four axioms: (1) $f(1)=1$, (2) $f(x)\le x$, (3) $f(x\to y)\le f(x)\to f(y)$ and (4′) $f(x\vee y)\le f(x)\vee f(y)$. A comparison of these two definitions indicates that they have the same axioms except the last one. Hence, a mapping which satisfies the axioms (1)(2)(3) is called a weak vt-operator [13, 15]. Moreover, on any commutative residuated lattice, Liu and Wang defined a vt-operator which is a weak vt-operator with the axiom (4′) and a hedge which is a weak vt-operator with the axiom (4), respectively [13]. Consequently, the concepts of vt-operators and hedges had been extended to other logical algebras such as pseudo-MV algebras [12], basic algebras [2], MTL-algebras [17], equality algebras [16], pseudo-BCK algebras [8] and so on. We need to point out that although authors named after vt-operators on some algebras, these operators are defined to satisfy the axiom (4), in other words, they are indeed "hedges" following the idea in [13]. In addition, the notions of multiplicative interior operators and additive closure operators were introduced to MV-algebras as the generalizations of topological Boolean algebras [14]. Independent of their original motivation, any multiplicative interior operator is a hedge in an MV-algebra from the purely algebraic viewpoint. Thus, it is natural to ask whether the concepts of multiplicative interior operators and hedges can be generalized to a qpMV-algebra for the more general results and new applications.

In this paper, we introduce the notions of multiplicative interior operators, additive closure operators and hedges on a qpMV-algebra and investigate the new algebraic structure. The paper is organized as follows. In Section 2, we recall some definitions and results which will be used in what follows. In Section 3, we introduce the notions of multiplicative interior operators (mi-operators, for short) and additive closure operators (ac-operators, for short) in qpMV-algebras which will generalize the related contents in pseudo-MV algebras. We discuss the relationship between mi-operators and ac-operators and investigate some properties of mi-operators in qpMV-algebras. In Section 4, we define and study hedges in qpMV-algebras. We also show that mi-operators are hedges. The properties of filters and weak filters in a qpMV-algebra with hedge are discussed.

2. Preliminary

In this section, we recall some definitions and results which will be used in the following. We list the definition and the related properties of a quasi-pseudo-MV algebra and recall the hedges on residuated lattices.

Definition 2.1 ([4]). *An algebra $\boldsymbol{A} = \langle A; \oplus, ^-, ^\sim, 0\rangle$ of type $\langle 2,1,1,0\rangle$ is called a* quasi-pseudo-MV algebra *(qpMV-algebra, for short), if it satisfies the following axioms, for any $x, y, z \in A$,*

($QPMV1$) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
($QPMV2$) $0^- = 0^\sim$;
($QPMV3$) $x \oplus 0 = 0 \oplus x$;
($QPMV4$) $x \oplus 0^- = 0^- = 0^- \oplus x$;
($QPMV5$) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
($QPMV6$) $x^{-\sim} = x = x^{\sim -}$;
($QPMV7$) $y \oplus (x^- \oplus y)^\sim = (y \oplus x^\sim)^- \oplus y = x \oplus (y^- \oplus x)^\sim = (x \oplus y^\sim)^- \oplus x$;
($QPMV8$) $(x \oplus 0)^- = x^- \oplus 0$ *and* $(x \oplus 0)^\sim = x^\sim \oplus 0$;
($QPMV9$) $x \oplus y \oplus 0 = x \oplus y$.

A *qp*MV-algebra in which the binary operation $\oplus$ is commutative and the unary operations $^-$ and $^\sim$ coincide, is a quasi-MV algebra (*q*MV-algebra, for short). On the other hand, a *qp*MV-algebra satisfying the axiom $x \oplus 0 = x$ is a pseudo-MV algebra (*ps*MV-algebra, for short).

On any *qp*MV-algebra $\mathbf{A}$, we can define some operations, for any $x, y \in A$:

$$\begin{aligned} x \odot y &= (x^- \oplus y^-)^\sim; \\ x \vee y &= x \oplus (y^- \oplus x)^\sim; \\ x \wedge y &= (x^- \vee y^-)^\sim; \\ x \to y &= x^- \oplus y; \\ x \rightsquigarrow y &= y \oplus x^\sim. \end{aligned}$$

We can also define a relation $x \leq y$ iff $x \vee y = y \oplus 0$, or equivalently, $x \wedge y = x \oplus 0$. This is a quasi-ordering relation [4]. Moreover, if $x \leq y$ and $y \leq x$, then $x \oplus 0 = y \oplus 0$.

Below we list some properties of these operations and the relation. The proofs can be seen in [3, 4].

Proposition 2.1. *Let $\boldsymbol{A}$ be a qpMV-algebra. Then, for any $x, y, z \in A$,*

(P1) $0 \oplus 0 = 0$ *and* $1 \oplus 0 = 1$;
(P2) $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$;
(P3) $1 \to x = 1 \rightsquigarrow x$;
(P4) $x \vee y = (x \vee y) \oplus 0 = (x \oplus 0) \vee y = x \vee (y \oplus 0)$,
$x \wedge y = (x \wedge y) \oplus 0 = (x \oplus 0) \wedge y = x \wedge (y \oplus 0)$;
(P5) $x \to y = (x \to y) \oplus 0 = (x \oplus 0) \to y = x \to (y \oplus 0)$,
$x \rightsquigarrow y = (x \rightsquigarrow y) \oplus 0 = (x \oplus 0) \rightsquigarrow y = x \rightsquigarrow (y \oplus 0)$;
(P6) $x \leq y$ *iff* $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$;
(P7) $0 \leq x \leq 1$;
(P8) $x \leq 1 \to x$ *and* $1 \to x \leq x$;
(P9) $x \leq y$ *iff* $y^- \leq x^-$ *iff* $y^\sim \leq x^\sim$;
(P10) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$ *and* $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;
(P11) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$;

(P12) $x \leq y \to x$ *and* $x \leq y \rightsquigarrow x$;
(P13) $x \odot y \leq z$ *iff* $x \leq y \to z$ *iff* $y \leq x \rightsquigarrow z$;
(P14) *If* $x \leq y$, *then* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
(P15) *If* $x \leq y$, *then* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;
(P16) $(x \vee y)^- = x^- \wedge y^-$ *and* $(x \vee y)^\sim = x^\sim \wedge y^\sim$,
$(x \wedge y)^- = x^- \vee y^-$ *and* $(x \wedge y)^\sim = x^\sim \vee y^\sim$;
(P17) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$ *and* $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$,
$z \to (x \vee y) = (z \to x) \vee (z \to y)$ *and* $z \rightsquigarrow (x \vee y) = (z \rightsquigarrow x) \vee (z \rightsquigarrow y)$;
(P18) $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$ *and* $z \rightsquigarrow (x \wedge y) = (z \rightsquigarrow x) \wedge (z \rightsquigarrow y)$,
$(x \wedge y) \to z = (x \to z) \vee (y \to z)$ *and* $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$.

Given that **A** is a *qp*MV-algebra and consider the set $R(A) = \{x \in A | x \oplus 0 = x\}$. Then, we have that $R(A)$ is a non-empty subset of A by Proposition 2.1 and elements in $R(A)$ are called *regular*. Moreover, $\mathbf{R_A} = \langle R(A); \oplus, ^-, ^\sim, 0\rangle$ is a pseudo-MV subalgebra of **A**. We recall that a *qp*MV-algebra in which $0 = 1$ is called *flat*. Then, we can show the following result.

Theorem 2.1 ([4])**.** *For any qpMV-algebra* $\boldsymbol{A}$*, there exist a pseudo-MV algebra* $\boldsymbol{M}$ *and a flat qpMV-algebra* $\boldsymbol{F}$ *such that* $\boldsymbol{A}$ *can be embedded into the direct product* $\boldsymbol{M} \times \boldsymbol{F}$.

Let **A** be a *qp*MV-algebra. A non-empty subset F of A is called a *filter* of **A**, if, for any $x, y \in A$, the following conditions are satisfied (F1) $1 \in F$; (F2) if $x, y \in F$, then $x \odot y \in F$; (F3) if $x \in F$ and $y \in A$ with $x \leq y$, then $y \in F$. A non-empty subset F of A is called a *weak filter* of **A**, if, for any $x, y \in A$, the following conditions are satisfied (WF1) $1 \in F$; (WF2) if $x, y \in F$, then $x \odot y \in F$; (WF3) if $x \in F$ and $y \in A$, then $y \oplus x \in F$ and $x \oplus y \in F$. Moreover, a (weak) filter F is called *normal*, if $x \to y \in F$ iff $x \rightsquigarrow y \in F$, for any $x, y \in A$. Finally, we recall that a congruence θ on **A** is called *filter congruence*, if $\langle x \odot 1, y \odot 1\rangle \in \theta$ can imply $\langle x, y\rangle \in \theta$, for any $x, y \in A$.

The filter is the dual notion of an ideal in any *qp*MV-algebra. In [6], we have proved that there exists a bijective correspondence between normal ideals and ideal congruences on a *qp*MV-algebra. On the basis of the proof, we can get the following result.

Theorem 2.2. *Let* $\boldsymbol{A}$ *be a qpMV-algebra,* F *be a normal filter of* $\boldsymbol{A}$ *and* θ *be a filter congruence on* $\boldsymbol{A}$*. Then,*

(1) $f(F) = \{\langle x, y\rangle \in A^2 | x \to y \in F \, and \; y \to x \in F\}$ *is a filter congruence on* $\boldsymbol{A}$*;*

(2) $g(\theta) = \{x \in A | \langle x, 1\rangle \in \theta\}$ *is a normal filter of* $\boldsymbol{A}$*;*

(3) $g(f(F)) = F$*;*

(4) $f(g(\theta)) = \theta$*.*

3. Interior and closed operators

MV-algebras with multiplicative interior operators (interior MV-algebras) or additive closure operators (closure MV-algebras) were introduced in [14]. In fact, a multiplicative interior operator (or an additive closure operator) on an MV-algebra generalizes that of a topological interior operator (or closure operator) on a Boolean algebra. In this section, we generalize these notions to *qp*MV-algebras.

Definition 3.1. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* $f : A \to A$ *be a mapping. Then,* f *is called a* multiplicative interior operator *(*mi-operator*, for short) on* $\boldsymbol{A}$*, if the following conditions are satisfied, for any* $x, y \in A$*,*

$(MI1)$ $f(1) = 1$*;*

$(MI2)$ $f(x) \leq x$*;*

$(MI3)$ $f(x \odot y) = f(x) \odot f(y)$*;*

$(MI4)$ $f(f(x)) = f(x)$*.*

The pair $(\boldsymbol{A}, f)$ *is called an* interior *qp*MV-algebra*. For any* $x \in A$*, the element* $f(x)$ *is called the* interior *of* x*. An element* $x \in A$ *is called* open*, if* $f(x) = x$*.*

Similarly, we have the following definition.

Definition 3.2. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* $g : A \to A$ *be a mapping. Then,* g *is called an* additive closure operator *(*ac-operator*, for short) on* $\boldsymbol{A}$*, if the following conditions are satisfied, for any* $x, y \in A$*,*

$(AC1)$ $g(0) = 0$*;*

$(AC2)$ $x \leq g(x)$*;*

$(AC3)$ $g(x \oplus y) = g(x) \oplus g(y)$*;*

$(AC4)$ $g(g(x)) = g(x)$*.*

The pair $(\boldsymbol{A}, g)$ *is called a* closure *qp*MV-algebra*. For any* $x \in A$*, the element* $g(x)$ *is called the* closure *of* x*. An element* $x \in A$ *is called* closed*, if* $g(x) = x$*.*

Proposition 3.1. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* f *be an mi-operator on* $\boldsymbol{A}$*. Then, the mappings* $f^-_\sim$ *defined by* $f^-_\sim(x) = (f(x^\sim))^-$ *and* $f^\sim_-$ *defined by* $f^\sim_-(x) = (f(x^-))^\sim$*, for any* $x \in A$*, are ac-operators on* $\boldsymbol{A}$*.*

Proof. We only check the case of $f^-_\sim$. The other can be proved similarly.

(AC1) We have $f^-_\sim(0) = (f(0^\sim))^- = (f(1))^- = 1^- = 0$.

(AC2) Since $f^-_\sim(x) = (f(x^\sim))^-$ and $f(x^\sim) \leq x^\sim$ by (MI2), we have $x \leq f^-_\sim(x)$ by (P10).

(AC3) We have $f^-_\sim(x \oplus y) = (f((x \oplus y)^\sim))^- = (f(x^\sim \odot y^\sim))^- = (f(x^\sim) \odot f(y^\sim))^- = (f(x^\sim))^- \oplus (f(y^\sim))^- = f^-_\sim(x) \oplus f^-_\sim(y)$.

(AC4) We have $f^-_\sim(f^-_\sim(x)) = f^-_\sim((f(x^\sim))^-) = (f(f(x^\sim)))^- = (f(x^\sim))^- = f^-_\sim(x)$. □

Dually, we get the following result.

Proposition 3.2. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* g *be an ac-operator on* $\boldsymbol{A}$*. Then, the mappings* $g^-_\sim$ *defined by* $g^-_\sim(x) = (g(x^\sim))^-$ *and* $g^\sim_-$ *defined by* $g^\sim_-(x) = (g(x^-))^\sim$*, for any* $x \in A$*, are mi-operators on* $\boldsymbol{A}$*.*

As shown above, there exist the corresponding relations between mi-operators and ac-operators on a *qp*MV-algebra, thus we will only discuss mi-operators in the rest.

Proposition 3.3. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* f *be an mi-operator on* $\boldsymbol{A}$*. Then, for any* $x, y \in A$,

(1) f *keeps regular elements, i.e., if* $x = x \oplus 0$*, then* $f(x) = f(x) \oplus 0$*;*
(2) $f(0) = 0$*;*
(3) $f(x^-) \leq (f(x))^-$ *and* $f(x^\sim) \leq (f(x))^\sim$*;*
(4) *If* $x \leq y$*, then* $f(x) \leq f(y)$*;*
(5) $f(x \to y) \leq f(x) \to f(y)$ *and* $f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y)$*.*

Proof. (1) Since $x = x \oplus 0$ iff $x = x \odot 1$, for any $x \in A$, we get the result by (MI3) and (MI1).

(2) Since $0 \leq f(0) \leq 0$ by (MI2), we have $f(0) \oplus 0 = 0 \oplus 0$. Note that 0 is a regular element, it turns out that $f(0) = 0$ by (1).

(3) By (MI2), we have $f(x) \leq x$, so $x^- \leq (f(x))^-$. Using (MI2) again, we have $f(x^-) \leq x^-$, it turns out that $f(x^-) \leq (f(x))^-$ by the transitivity. The other can be proved similarly.

(4) If $x \leq y$, then $x \odot 1 = x \wedge y = y \odot (y \rightsquigarrow x)$. On the one hand, we have $f(x \odot 1) = f(x) \odot f(1) = f(x) \odot 1 \geq f(x)$ by (MI3) and (MI1). On the other hand, we have $f(x \wedge y) = f(y \odot (y \rightsquigarrow x)) = f(y) \odot f(y \rightsquigarrow x) \leq f(y) \odot 1 \leq f(y)$. Hence, $f(x) \leq f(y)$.

(5) Since $(x \to y) \odot x = x \wedge y \leq y$, we have $f((x \to y) \odot x) = f(x \to y) \odot f(x) \leq f(y)$ by (MI3) and (4), so $f(x \to y) \leq f(x) \to f(y)$. The other can be proved similarly. □

Proposition 3.4. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* f *be an mi-operator on* $\boldsymbol{R_A}$*. Then,* f *can be extended to an mi-operator on* $\boldsymbol{A}$*.*

Proof. For any $x \in A$, define $\bar{f}(x) = \begin{cases} f(x), & x \in R(A), \\ f(x \oplus 0), & x \in A \setminus R(A). \end{cases}$ Then, $\bar{f}$ is an mi-operator on **A**. Indeed, $\bar{f}(1) = f(1) = 1$, so the condition (MI1) is true. Now, we check the conditions (MI2)-(MI4).

(MI2) If $x \in R(A)$, then $\bar{f}(x) = f(x) \leq x$. If $x \notin R(A)$, then $\bar{f}(x) = f(x \oplus 0) \leq x \oplus 0 \leq x$.

(MI3) If $x, y \in R(A)$, then $\bar{f}(x \odot y) = f(x \odot y) = f(x) \odot f(y) = \bar{f}(x) \odot \bar{f}(y)$. If $x \in R(A)$ and $y \notin R(A)$, then $\bar{f}(x \odot y) = f(x \odot y) = f(x \odot (y \oplus 0)) = f(x) \odot f(y \oplus 0) = \bar{f}(x) \odot \bar{f}(y)$. If $x \notin R(A)$ and $y \in R(A)$, the proof is similar as above. If $x, y \notin R(A)$, then $\bar{f}(x \odot y) = f(x \odot y) = f((x \oplus 0) \odot (y \oplus 0)) = f(x \oplus 0) \odot f(y \oplus 0) = \bar{f}(x) \odot \bar{f}(y)$.

(MI4) If $x \in R(A)$, then $\bar{f}(\bar{f}(x)) = f(f(x)) = f(x) = \bar{f}(x)$. If $x \notin R(A)$, then $\bar{f}(x) = f(x \oplus 0) = f(f(x \oplus 0))$ and $\bar{f}(\bar{f}(x)) = \bar{f}(f(x \oplus 0)) = f(f(x \oplus 0) \oplus 0) = f(f(x \oplus 0))$, so $\bar{f}(\bar{f}(x)) = \bar{f}(x)$. □

In [14], authors showed that for a complete MV-algebra, every topological closure operator on the Boolean algebra of additively idempotent elements can be extended to a closure operator on the whole MV-algebra. Since the set of additively idempotent elements in a pseudo-MV algebra is also a Boolean algebra [9], we can extend the result to a complete pseudo-MV algebra. Suppose that $\mathbf{M} = \langle M; \oplus, ^{-}, ^{\sim}, 0, 1\rangle$ is a pseudo-MV algebra and denote $B(M) =$ the set of additive idempotent elements in M. Then, $\mathbb{B}(\mathbf{M}) = \langle B(M); \vee, \wedge, 0, 1\rangle$ is a Boolean algebra, where $x \vee y = x \oplus (y^{-} \oplus x)^{\sim}$ and $x \wedge y = (x^{-} \vee y^{-})^{\sim}$, for any $x, y \in B(M)$.

Lemma 3.1. *Let $\boldsymbol{M}$ be a interior complete pseudo-MV algebra and f be a topological interior operator on $\mathbb{B}(\boldsymbol{M})$. Then, there is an mi-operator on $\boldsymbol{M}$ such that its restriction on $B(M)$ is equal to f.*

Proposition 3.5. *Let $\boldsymbol{A}$ be a qpMV-algebra and $\boldsymbol{R_A}$ be its interior complete pseudo-subalgebra of $\boldsymbol{A}$. If f is a topological interior operator on the Boolean algebra $\mathbb{B}(\boldsymbol{R_A})$, then there is an mi-operator on $\boldsymbol{A}$ such that its restriction on $B(R(A))$ is equal to f.*

Proof. Follows from Proposition 3.4 and Lemma 3.1. □

Proposition 3.6. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be mi-operators on $\boldsymbol{A}$. Then, $f_1 \leq f_2$ iff $f_1 f_2|_{R(A)} = f_1|_{R(A)}$.*

Proof. Suppose that $f_1 \leq f_2$. Then, for any $x \in A$, we have $f_1(x) \leq f_2(x)$. By (MI4) and Proposition 3.3(4), $f_1(x) = f_1(f_1(x)) \leq f_1(f_2(x)) = (f_1 f_2)(x)$. Meanwhile, since $f_2(x) \leq x$, it follows that $(f_1 f_2)(x) = f_1(f_2(x)) \leq f_1(x)$ using Proposition 3.3(4) again. Thus, $f_1(x) \oplus 0 = f_1 f_2(x) \oplus 0$. By Proposition 2.1(1), if $x \in R(A)$, then $(f_1 f_2)(x) = f_1(x)$, i.e., $f_1 f_2|_{R(A)} = f_1|_{R(A)}$. Conversely, if $f_1 f_2|_{R(A)} = f_1|_{R(A)}$, then, for any $x \in A$, we have $f_1(x) \leq f_1(x \oplus 0) = (f_1 f_2)(x \oplus 0) \leq f_2(x \oplus 0) \leq f_2(x)$, so $f_1 \leq f_2$. □

Following the proof of Proposition 3.6, we can get the result.

Proposition 3.7. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be isotone mappings on $\boldsymbol{A}$. If f_1 and f_2 restricted on $\boldsymbol{R_A}$ are mi-operators, then $f_1 \leq f_2$ iff $f_1 f_2|_{R(A)} = f_1|_{R(A)}$.*

Proposition 3.8. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be isotone mappings on $\boldsymbol{A}$. If f_1 and f_2 restricted on $\boldsymbol{R_A}$ are mi-operators, then the following conditions are equivalent:*

(1) $f_1 f_2|_{R(A)} = f_2 f_1|_{R(A)}$;
(2) $f_1 f_2|_{R(A)}$ *and* $f_2 f_1|_{R(A)}$ *are mi-operators;*
(3) $f_1 f_2 f_1 f_2|_{R(A)} = f_1 f_2|_{R(A)}$ *and* $f_2 f_1 f_2 f_1|_{R(A)} = f_2 f_1|_{R(A)}$.

Proof. (1)⇒(2) For any $x \in R(A)$, we have $(f_1 f_2)(1) = 1$ and $(f_1 f_2)(x) \leq f_2(x) \leq x$. Moreover, $(f_1 f_2)(x \odot y) = f_1(f_2(x) \odot f_2(y)) = (f_1 f_2)(x) \odot (f_1 f_2)(y)$

and $(f_1f_2)((f_1f_2)(x)) = (f_1f_2f_1f_2)(x)) = (f_1f_1)(f_2f_2)(x) = (f_1f_2)(x)$. Hence, $f_1f_2|_{R(A)}$ is an mi-operator. The case of $f_2f_1|_{R(A)}$ can be proved similarly.

(2)⇒(3) Since $f_1f_2 \leq f_1f_2$ and $f_2f_1 \leq f_2f_1$, we have the result by Proposition 3.7.

(3)⇒(1) On the one hand, for any $x \in R(A)$, we have $(f_1f_2)(x) = (f_1f_2f_1f_2)(x) \leq (f_2f_1f_2)(x) \leq (f_2f_1)(x)$. On the other hand, for any $x \in R(A)$, we have $(f_2f_1)(x) = (f_2f_1f_2f_1)(x) \leq (f_1f_2f_1)(x) \leq (f_1f_2)(x)$. Hence, we get $f_1f_2|_{R(A)} = f_2f_1|_{R(A)}$. □

Let $\mathbf{A}$ be a qpMV-algebra and f be any mi-operator on $\mathbf{A}$. We denote the set of all open elements of A by $O_f(\mathbf{A}) = \{x \in A | f(x) = x\}$.

Theorem 3.1. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be mi-operators on $\boldsymbol{A}$. If $O_{f_1}(\boldsymbol{A}) = O_{f_2}(\boldsymbol{A})$, then $f_1|_{R(A)} = f_2|_{R(A)}$.*

Proof. For any $x \in A$, since $f_1(f_1(x)) = f_1(x)$, we have $f_1(x) \in O_{f_1}(\mathbf{A}) = O_{f_2}(\mathbf{A})$, it follows that $f_2(f_1(x)) = f_1(x)$. Similarly, we have $f_1(f_2(x)) = f_2(x)$. Since $f_1(x) \leq x$, we get $f_2(f_1(x)) \leq f_2(x)$, it turns out that $f_1(x) \leq f_2(x)$. Meanwhile, since $f_2(x) \leq x$, we get $f_1(f_2(x)) \leq f_1(x)$, it turns out that $f_2(x) \leq f_1(x)$. Hence, $f_1(x) \oplus 0 = f_2(x) \oplus 0$ which means that $f_1(x \oplus 0) = f_2(x \oplus 0)$ and then we get $f_1|_{R(A)} = f_2|_{R(A)}$. □

4. Hedges in quasi-pseudo-MV algebras

In this section, we introduce the notion of hedge in a qpMV-algebra and show some basic properties of it. We also investigate some properties of (weak) filters in qpMV-algebras with hedges and discuss the relationship between normal filters and filter congruences on qpMV-algebras with hedges.

Definition 4.1. *Let $\boldsymbol{A}$ be a qpMV-algebra and $h : A \to A$ be a mapping. Then, h is called a* weak hedge *in $\boldsymbol{A}$, if the following conditions are satisfied, for any $x, y \in A$,*

$(H1)$ $h(1) = 1$;

$(H2)$ $h(x) \leq x$;

$(H3)$ $h(x \to y) \leq h(x) \to h(y)$ and $h(x \rightsquigarrow y) \leq h(x) \rightsquigarrow h(y)$.

If a weak hedge h satisfies (H4) $h(h(x)) = h(x)$, then it is called a hedge *in $\boldsymbol{A}$. The pair $(\boldsymbol{A}, h)$ is called a* qpMV-algebra with hedge*. Moreover, if a hedge h keeps regular elements, then it is called a* strong hedge *in $\boldsymbol{A}$ and the pair $(\boldsymbol{A}, h)$ is called a* qpMV-algebra with strong hedge*.*

Example 4.1. Let $\mathbf{F}$ be a flat qpMV-algebra and $h : F \to F$ be a mapping satisfying $h(1) = 1$ and $h(x) \leq x$, for any $x \in F$. Then, h is a weak hedge in $\mathbf{F}$. In fact, since $1 = 0$ and $x \oplus y = x \oplus y \oplus 0 = x \oplus y \oplus 1 = 1$, we have that $x \to y$, $x \rightsquigarrow y$, $h(x) \to h(y)$ and $h(x) \rightsquigarrow h(y)$ are equal to 1. Hence, the condition (H3) is satisfied. Moreover, if the condition (H4) is also satisfied, then it is a hedge in $\mathbf{F}$ and also a strong hedge in $\mathbf{F}$.

Example 4.2. Let $\mathbf{A}$ be a qpMV-algebra. It is easy to see that the identity mapping $\mathbf{Id}_A$ is a hedge in $\mathbf{A}$. That is to say that any qpMV-algebra can be regarded as a qpMV-algebra with hedge.

Example 4.3. Let $\mathbf{A}$ be a qpMV-algebra and satisfy $x \leq y$ or $y \leq x$, for any $x, y \in A$. We define a mapping $h : A \to A$ by $h(1) = 1$ and $h(x) = 0$, for any $x < 1$. Then, h is a hedge in $\mathbf{A}$.

Example 4.4. Let $\mathbf{A}$ be a qpMV-algebra in which the operations are defined as follows:

$\oplus$	0	a	b	1
0	0	b	b	1
a	b	1	1	1
b	b	1	1	1
1	1	1	1	1

and

$'$	
0	1
a	a
b	b
1	0

. In fact, it is a quasi-MV algebra [11]. Define $h(1) = 1$, $h(0) = 0$, $h(a) = h(b) = a$. Then, h is a hedge in $\mathbf{A}$.

Example 4.5. Let $(\mathbf{M}, h_1)$ be a pseudo-MV algebra with hedge and $(\mathbf{F}, h_2)$ be a flat qpMV-algebra with hedge. Then, $\mathbf{M} \times \mathbf{F}$ is a qpMV-algebra. If we define $h(\langle x, y\rangle) = (h_1(x), h_2(y))$, for any $\langle x, y\rangle \in M \times F$, then h is a hedge in $\mathbf{M} \times \mathbf{F}$.

Remark 4.1. Following from Proposition 3.3, it is immediate to see that any mi-operator on a qpMV-algebra is a hedge. However, the converse is not true in general. In Example 4.4, we calculate $h(b \odot 1) = h(b \oplus 0) = a$ and $h(b) \odot h(1) = a \odot 1 = a \oplus 0 = b$, which imply that $h(b \odot 1) \neq h(b) \odot h(1)$, so h is not an mi-operator on $\mathbf{A}$.

Proposition 4.1. *Let $\boldsymbol{A}$ be a qpMV-algebra and h be a weak hedge in $\boldsymbol{A}$. Then, for any $x, y \in A$,*

(1) $h(0) \oplus 0 = 0$;
(2) *If* $h(x) = 1$, *then* $x \oplus 0 = 1$;
(3) *If* $x \leq y$, *then* $h(x) \leq h(y)$;
(4) *If* $h(x) \leq h(y)$, *then* $h(x) \leq y$;
(5) $h(x^-) \leq (h(x))^-$ *and* $h(x^\sim) \leq (h(x))^\sim$;
(6) $h(x) \odot h(y) \leq h(x \odot y)$;
(7) $h(x \oplus 0) \oplus 0 = h(x) \oplus 0$.
If h is a hedge in $\boldsymbol{A}$, then
(8) $h(x) \leq h(y)$ *iff* $h(x) \leq y$;
(9) $Im(h) = \mathrm{Fix}_h(A) = \{x \in A | h(x) = x\}$;
(10) *If h is surjective, then* $h = \mathrm{Id}_A$.

Proof. (1) By (H2), we have $h(0) \leq 0$. And $0 \leq h(0)$, it turns out that $h(0) \oplus 0 = 0$.

(2) Since $1 = h(x) \leq x$ and $x \leq 1$, we have $x \oplus 0 = 1$.

(3) Since $x \leq y$, we have $x \to y = 1$. By (H1) and (H3), we get $1 = h(1) = h(x \to y) \leq h(x) \to h(y)$. Note that, $h(x) \to h(y)$ is a regular element and $h(x) \to h(y) \leq 1$, we have $h(x) \to h(y) = 1$, so $h(x) \leq h(y)$.

(4) Since $h(x) \leq h(y)$ and $h(y) \leq y$ by (H2), we have $h(x) \leq y$.

(5) Since $x^- \leq (1 \to x)^- = (1 \to x) \to 0$, we have $h(x^-) \leq h((1 \to x) \to 0) \leq h(1 \to x) \to h(0) \leq h(x) \to h(0) = h(x) \to (h(0) \oplus 0) = h(x) \to 0 \leq (h(x))^-$ using (H3) and (1). The other can be proved similarly.

(6) Since $x \odot y \leq x \odot y$, we have $x \leq y \to (x \odot y)$, it turns out that $h(x) \leq h(y \to (x \odot y)) \leq h(y) \to h(x \odot y)$ by (3) and (H3), so $h(x) \odot h(y) \leq h(x \odot y)$.

(7) Since $x \leq x \oplus 0$ and $x \oplus 0 \leq x$, we have $h(x) \leq h(x \oplus 0)$ and $h(x \oplus 0) \leq h(x)$ by (3), so $h(x) \oplus 0 = h(x \oplus 0) \oplus 0$.

(8) If $h(x) \leq y$, then $h(x) = h(h(x)) \leq h(y)$ by (3). The converse follows from (4).

(9) For any $x \in Im(h)$, then there exists $a \in A$ such that $x = h(a)$, it follows that $h(x) = h(h(a)) = h(a) = x$, so $x \in \mathrm{Fix}_h(A)$. Conversely, for any $x \in \mathrm{Fix}_h(A)$, then $h(x) = x$, we have $x \in Im(h)$.

(10) If h is surjective, then $Im(h) = A = \mathrm{Fix}_h(A)$ by (9), it follows that $h(x) = x = \mathrm{Id}_A(x)$, for any $x \in A$, so $h = \mathrm{Id}_A$. $\square$

Proposition 4.2. *Let $\boldsymbol{A}$ be a qpMV-algebra and h be a hedge in $\boldsymbol{R_A}$. Then, h can be extended to a hedge in $\boldsymbol{A}$.*

Proof. For any $x \in A$, define $\bar{h}(x) = \begin{cases} h(x), & x \in R(A); \\ h(x \oplus 0), & x \in A \setminus R(A). \end{cases}$ Then, $\bar{h}$ is a hedge in $\mathbf{A}$. Indeed, $\bar{h}(1) = h(1) = 1$, so the condition (H1) is true. Now, we check the conditions (H2)-(H4).

(H2) For any $x \in A$, if $x \in R(A)$, then $\bar{h}(x) = h(x) \leq x$. If $x \notin R(A)$, then $\bar{h}(x) = h(x \oplus 0) \leq x \oplus 0 \leq x$. Hence, $\bar{h}(x) \leq x$.

(H3) We only prove the first one. The other can be proved similarly. If $x, y \in R(A)$, then $\bar{h}(x \to y) = h(x \to y) \leq h(x) \to h(y) = \bar{h}(x) \to \bar{h}(y)$. If $x \in R(A)$ and $y \notin R(A)$, then $\bar{h}(x \to y) = h(x \to y) = h(x \to (y \oplus 0)) \leq h(x) \to h(y \oplus 0) = \bar{h}(x) \to \bar{h}(y)$. If $x \notin R(A)$ and $y \in R(A)$, the proof is similar as above. If $x, y \notin R(A)$, then $\bar{h}(x \to y) = h(x \to y) = h((x \oplus 0) \to (y \oplus 0)) \leq h(x \oplus 0) \to h(y \oplus 0) = \bar{h}(x) \to \bar{h}(y)$.

(H4) If $x \in R(A)$, then $\bar{h}(\bar{h}(x)) = h(h(x)) = h(x) = \bar{h}(x)$. If $x \notin R(A)$, then $\bar{h}(x) = h(x \oplus 0) = h(h(x \oplus 0))$ and $\bar{h}\bar{h}(x) = \bar{h}(h(x \oplus 0)) = h(h(x \oplus 0) \oplus 0) = h(h(x \oplus 0))$, so $\bar{h}(\bar{h}(x)) = \bar{h}(x)$. $\square$

Definition 4.2. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge and F be a (weak) filter of $\boldsymbol{A}$. Then, F is called an* (weak) h-filter *of $(\boldsymbol{A}, h)$, if $h(F) \subseteq F$. In addition, if F is a (weak) h-filter of $(\boldsymbol{A}, h)$ and satisfies $x \to y \in F$ iff $x \rightsquigarrow y \in F$, for any $x, y \in A$, then it is called a* normal (weak) h-filter *of $(\boldsymbol{A}, h)$.*

Definition 4.3. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge and θ be a congruence on $\boldsymbol{A}$. Then, θ is called a* congruence *on $(\boldsymbol{A}, h)$, if $\langle x, y \rangle \in \theta$ implies $\langle h(x), h(y) \rangle \in \theta$, for any $x, y \in A$. In addition, if θ is a congruence on $(\boldsymbol{A}, h)$ and $\langle x \odot 1, y \odot 1 \rangle \in \theta$ can imply $\langle x, y \rangle \in \theta$, for any $x, y \in A$, then it is called a* filter congruence *on $(\boldsymbol{A}, h)$.*

Theorem 4.1. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge. Then, there exists a bijection between normal h-filters and filter congruences on $(\boldsymbol{A}, h)$.*

Proof. Let F be a normal h-filter of $(\mathbf{A}, h)$. Then, $\theta_F = \{\langle x, y\rangle \in A^2 | x \to y \in F \text{ and } y \to x \in F\}$ is a filter congruence on $\mathbf{A}$ by Proposition 2.2. Moreover, since F is a h-filter of $(\mathbf{A}, h)$, we have $h(x \to y) \in F$ and $h(y \to x) \in F$. By (H3), we have $h(x \to y) \leq h(x) \to h(y)$ and $h(y \to x) \leq h(y) \to h(x)$, it follows that $h(x) \to h(y) \in F$ and $h(y) \to h(x) \in F$, so $\langle h(x), h(y)\rangle \in \theta_F$. Conversely, let θ be a filter congruence on $(\mathbf{A}, h)$. Then, $F_\theta = \{x \in A | \langle x, 1\rangle \in \theta\}$ is a normal filter of $\mathbf{A}$ using Proposition 2.2 again. Moreover, for any $x \in F_\theta$, we have $\langle h(x), 1\rangle = \langle h(x), h(1)\rangle \in \theta$, so $h(x) \in F_\theta$. The left is obtained by Proposition 2.2. □

Let $(\mathbf{A}, h)$ be a qpMV-algebra with hedge and F be a normal h-filter of $(\mathbf{A}, h)$. Then, $A/F = \{x/F | x \in A\}$ where $x/F = \{y \in A | x \to y \in F \text{ and } y \to x \in F\}$ is a *quotient set* with respect to F. We define some operations as follows: $(x/F) \oplus (y/F) = (x \oplus y)/F$, $(x/F)^- = x^-/F$ and $(x/F)^\sim = x^\sim/F$. Then, $\mathbf{A}/F = \langle A/F; \oplus, ^-, ^\sim, 0/F, 1/F\rangle$ is a pseudo-MV algebra by [6].

Theorem 4.2. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge and F be a normal h-filter of $(\boldsymbol{A}, h)$. Define $\bar{h} : A/F \to A/F$ by $\bar{h}(x/F) = h(x)/F$, for any $x \in A$. Then, $(\boldsymbol{A}/F, \bar{h})$ is a pseudo-MV algebra with hedge.*

Proof. It is easy to see that $\bar{h}$ is well-defined. Now, we check that the conditions (H1-H4) are satisfied. Obviously, $\bar{h}(1/F) = h(1)/F = 1/F$ and $\bar{h}(x/F) = h(x)/F \leq x/F$. For any $x/F, y/F \in A/F$, we have $\bar{h}(x/F \to y/F) = \bar{h}((x \to y)/F) = h(x \to y)/F \leq (h(x) \to h(y))/F = h(x)/F \to h(y)/F = \bar{h}(x/F) \to \bar{h}(y/F)$. Similarly, we have $\bar{h}(x \rightsquigarrow y) \leq \bar{h}(x) \rightsquigarrow \bar{h}(y)$. Finally, we have $\bar{h}(\bar{h}(x/F)) = h(h(x))/F = h(x)/F = \bar{h}(x/F)$. □

Proposition 4.3. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with strong hedge. Then,* $\ker(h)$ *is a weak h-filter of $(\boldsymbol{A}, h)$.*

Proof. Obviously, $1 \in \ker(h)$. For any $x, y \in \ker(h)$, then $h(x) = h(y) = 1$, we have $h(x \odot y) \geq h(x) \odot h(y) = 1 \odot 1 = 1$, so $h(x \odot y) = 1$ and then $x \odot y \in \ker(h)$. Let $x \in \ker(h)$ and $y \in A$. Then, $1 = h(x) \leq h(x \oplus y)$, we have $h(x \oplus y) = 1$, so $x \oplus y \in \ker(h)$. Similarly, we have $y \oplus x \in \ker(h)$. Hence, $\ker(h)$ is a weak filter of $(\mathbf{A}, h)$. Moreover, for any $x \in \ker(h)$, we have $h(h(x)) = h(1) = 1$, so $h(x) \in \ker(h)$. Hence, $\ker(h)$ is a weak h-filter of $(\mathbf{A}, h)$. □

Since any mi-operator is a strong hedge in a qpMV-algebra, we have the following result.

Corollary 4.1. *Let $(\boldsymbol{A}, f)$ be an interior qpMV-algebra. Then,* $\ker(f)$ *is a weak f-filter of $(\boldsymbol{A}, f)$.*

Definition 4.4. *Let* $(\boldsymbol{A}, h_1)$ *and* $(\boldsymbol{B}, h_2)$ *be qpMV-algebras with hedges and* $\varphi : A \to B$ *be a mapping. Then,* φ *is called a* qpMV-algebra with hedge homomorphism, *if it satisfies the following conditions, for any* $x, y \in A$,

$(HH1)$ $\varphi(1) = 1$;

$(HH2)$ $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$;

$(HH3)$ $\varphi(x^-) = (\varphi(x))^-$;

$(HH4)$ $\varphi(x^\sim) = (\varphi(x))^\sim$;

$(HH5)$ $\varphi(h_1(x)) = h_2(\varphi(x))$.

Proposition 4.4. *Let* $(\boldsymbol{A}, h_1)$ *and* $(\boldsymbol{B}, h_2)$ *be qpMV-algebras with hedges and* φ *be a homomorphism from* $(\boldsymbol{A}, h_1)$ *to* $(\boldsymbol{B}, h_2)$. *Then, the following conditions are equivalent:*

(1) $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$;

(2) $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$;

(3) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$;

(4) $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$;

(5) $\varphi(x \to y) = \varphi(x) \to \varphi(y)$;

(6) $\varphi(x \rightsquigarrow y) = \varphi(x) \rightsquigarrow \varphi(y)$.

Proof. (1) $\Rightarrow$ (2) We have $\varphi(x \vee y) = \varphi(y \oplus (x^- \oplus y)^\sim) = \varphi(y) \oplus \varphi((x^- \oplus y)^\sim) = \varphi(y) \oplus (\varphi(x^- \oplus y))^\sim = \varphi(y) \oplus (\varphi(x)^- \oplus \varphi(y))^\sim = \varphi(x) \vee \varphi(y)$.

(2) $\Rightarrow$ (3) We have $\varphi(x \wedge y) = \varphi((x^- \vee y^-)^\sim) = (\varphi(x^- \vee y^-))^\sim = (\varphi(x)^- \vee \varphi(y)^-)^\sim = \varphi(x) \wedge \varphi(y)$.

(3) $\Rightarrow$ (1) Since $x \oplus y = x \oplus (y \wedge x^\sim)$, we have $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y \wedge x^\sim) = \varphi(x) \oplus (\varphi(y) \wedge \varphi(x)^\sim) = (\varphi(x) \oplus \varphi(y)) \wedge (\varphi(x) \oplus \varphi(x)^\sim) = \varphi(x) \oplus \varphi(y)$.

(1) $\Leftrightarrow$ (4) Since $x \odot y = (x^- \oplus y^-)^\sim$ and $x \oplus y = (x^- \odot y^-)^\sim$, we get the result.

(1) $\Leftrightarrow$ (5) Since $x \to y = x^- \oplus y$ and $x \oplus y = x^\sim \to y$, we get the result.

(1) $\Leftrightarrow$ (6) Analogously. $\square$

Recall that a *h-subalgebra* $(\mathbf{S}, h)$ of a qpMV-algebra with hedge $(\mathbf{A}, h)$, if $\mathbf{S}$ is a subalgebra of $\mathbf{A}$ and $h(S) \subseteq S$.

Theorem 4.3. *Let* $(\boldsymbol{A}, h_1)$ *and* $(\boldsymbol{B}, h_2)$ *be qpMV-algebras with hedges and* φ *be a homomorphism from* $(\boldsymbol{A}, h_1)$ *to* $(\boldsymbol{B}, h_2)$. *Then,*

(1) *If* $(\boldsymbol{S}, h_1)$ *is a* h_1*-subalgebra of* $(\boldsymbol{A}, h_1)$, *then* $(\varphi(\boldsymbol{S}), h_2)$ *is a* h_2*-subalgebra of* $(\boldsymbol{B}, h_2)$;

(2) *If* φ *is surjective and* F *is a weak* h_1*-filter of* $(\boldsymbol{A}, h_1)$, *then* $\varphi(F)$ *is a weak* h_2*-filter of* $(\boldsymbol{B}, h_2)$;

(3) *If* F *is a (weak)* h_2*-filter of* $(\boldsymbol{B}, h_2)$, *then* $\varphi^{-1}(F)$ *is a (weak)* h_1*-filter of* $(\boldsymbol{A}, h_1)$;

(4) $\ker(\varphi) = \{x \in A | \varphi(x) = 1\}$ *is a normal weak* h_1*-filter of* $(\boldsymbol{A}, h_1)$.

Proof. (1) It is easy to show that $\varphi(\mathbf{S})$ is a subalgebra of $\mathbf{B}$. Moreover, for any $\varphi(x) \in \varphi(S)$ where $x \in S$, we have $h_2(\varphi(x)) = \varphi(h_1(x)) \in \varphi(S)$.

(2) Since $1 \in F$, we have $1 = \varphi(1) \in \varphi(F)$. Let $x, y \in \varphi(F)$. Then, there exist $m, n \in F$ such that $\varphi(m) = x$ and $\varphi(n) = y$, it turns out that $x \odot y = \varphi(m) \odot \varphi(n) = \varphi(m \odot n) \in \varphi(F)$. Now, let $x \in \varphi(F)$ and $y \in B$. Because φ is surjective, there exist $m \in F$ and $n \in A$ such that $x = \varphi(m)$ and $y = \varphi(n)$. We have $x \oplus y = \varphi(m) \oplus \varphi(n) = \varphi(m \oplus n) \in \varphi(F)$. Similarly, $y \oplus x = \varphi(n \oplus m) \in \varphi(F)$. Hence, $\varphi(F)$ is a weak filter of $(\mathbf{B}, h_2)$. For any $\varphi(x) \in \varphi(F)$ where $x \in F$, we have $h_2(\varphi(x)) = \varphi(h_1(x)) \in \varphi(F)$, so $\varphi(F)$ is a weak h_2-filter of $(\mathbf{B}, h_2)$.

(3) We only prove the case of filters. The case of weak filters can be proved similarly. Obviously, $\varphi(1) = 1 \in F$, so $1 \in \varphi^{-1}(F)$. For any $x, y \in \varphi^{-1}(F)$, then there exist $a, b \in F$ such that $\varphi(x) = a$ and $\varphi(y) = b$, it follows that $\varphi(x \odot y) = \varphi(x) \odot \varphi(y) = a \odot b \in F$, so $x \odot y \in \varphi^{-1}(F)$. Let $x \in \varphi^{-1}(F)$ and $y \in A$ with $x \leq y$. Then, there exists $a \in F$ such that $\varphi(x) = a$ and $a = \varphi(x) \leq \varphi(y)$. Because $a \in F$, we have $\varphi(y) \in F$, so $y \in \varphi^{-1}(F)$. For any $x \in \varphi^{-1}(F)$, there exists $a \in F$ such that $\varphi(x) = a$, then we have $\varphi(h_1(x)) = h_2(\varphi(x)) = h_2(a) \in F$, so $h_1(x) \in \varphi^{-1}(F)$. Hence, $\varphi^{-1}(F)$ is a h_1-filter of $(\mathbf{A}, h_1)$.

(4) Obviously, $1 \in \ker(\varphi)$. For any $x, y \in \ker(\varphi)$, we have $\varphi(x \odot y) = \varphi(x) \odot \varphi(y) = 1 \odot 1 = 1$, so $x \odot y \in \ker(\varphi)$. Let $x \in \ker(\varphi)$ and $y \in A$. Then, $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y) = 1 \oplus \varphi(y) = 1$ and $\varphi(y \oplus x) = \varphi(y) \oplus \varphi(x) = \varphi(y) \oplus 1 = 1$, it follows that $x \oplus y \in \ker(\varphi)$ and $y \oplus x \in \ker(\varphi)$. For $x \in \ker(\varphi)$, we have $\varphi(h_1(x)) = h_2(\varphi(x)) = h_2(1) = 1$, so $h_1(x) \in \ker(\varphi)$. Hence, $\ker(\varphi)$ is a weak h_1-filter of $(\mathbf{A}, h_1)$. Finally, for any $x, y \in A$, we have $x \to y \in \ker(\varphi)$ iff $\varphi(x \to y) = 1$ iff $\varphi(x) \to \varphi(y) = 1$ iff $\varphi(x) \leq \varphi(y)$ iff $\varphi(x) \rightsquigarrow \varphi(y) = 1$ iff $\varphi(x \rightsquigarrow y) = 1$ iff $x \rightsquigarrow y \in \ker(\varphi)$. So $\ker(\varphi)$ is normal. $\square$

Corollary 4.2. *Let $(\boldsymbol{A}, h_1)$ and $(\boldsymbol{B}, h_2)$ be qpMV-algebras with strong hedges and φ be a homomorphism from $(\boldsymbol{A}, h_1)$ to $(\boldsymbol{B}, h_2)$. Then,*

(1) *$\varphi^{-1}(\ker(h_2))$ is a weak h_1-filter of $(\boldsymbol{A}, h_1)$;*

(2) *If φ is surjective, $\varphi(\ker h_1)$ is a weak h_2-filter of $(\boldsymbol{B}, h_2)$.*

Let $(\mathbf{A}, h)$ be a *qp*MV-algebra with hedge and F be a normal h-filter of $(\mathbf{A}, h)$. Then, $(\mathbf{A}/F, \bar{h})$ is a pseudo-MV algebra with hedge by Theorem 4.2. Define $\pi : A \to A/F$ by $x \mapsto x/F$, for any $x \in A$. Then, we have the following result.

Proposition 4.5. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge and F be a normal h-filter of $(\boldsymbol{A}, h)$. Then,*

(1) *π is a homomorphism from $(\boldsymbol{A}, h)$ to $(\boldsymbol{A}/F, \bar{h})$ and $\ker \pi = F$;*

(2) *$\pi^{-1}(\ker \bar{h}) \subseteq h^{-1}(F)$;*

(3) *$\pi(\ker h) \subseteq \ker(\bar{h})$.*

Proof. (1) It is easy to check that π is a homomorphism $(\mathbf{A}, h)$ to $(\mathbf{A}/F, \bar{h})$. For any $x \in \ker(\pi)$, then $\pi(x) = x/F = 1/F$, it turns out that $1 \to x \in F$. Since $1 \to x \leq x$, we have $x \in F$, so $\ker(\pi) \subseteq F$. For any $x \in F$, then $1 \to x \in F$

and $x \to 1 = 1 \in F$, we have $1 \in x/F$, so $1/F \subseteq x/F$. Conversely, for any $y \in x/F$, then $y \to x \in F$ and $x \to y \in F$. Because $x \in F$, we have $y \in F$, it turns out that $y \to 1 = 1 \in F$ and $1 \to y \in F$, so $y \in 1/F$ and $x/F \subseteq 1/F$. Thus, $1/F = x/F$ which means that $x \in \ker(\pi)$, we have $F \subseteq \ker(\pi)$. Hence, $\ker(\pi) = F$.

(2) For any $x \in \pi^{-1}(\ker \bar{h})$, then $\pi(x) \in \ker(\bar{h})$, so $1/F = \bar{h}(\pi(x)) = \pi(h(x)) = h(x)/F$ and then $1 \to h(x) \in F$. Since $1 \to h(x) \leq h(x)$, we have $h(x) \in F$, so $x \in h^{-1}(F)$. Hence, $\pi^{-1}(\ker \bar{h}) \subseteq h^{-1}(F)$.

(3) For any $x \in \pi(\ker(h))$, there exists $m \in \ker(h)$ such that $\pi(m) = x$, then we have $\bar{h}(x) = \bar{h}(\pi(m)) = \pi(h(m)) = \pi(1) = 1/F$, so $x \in \ker(\bar{h})$ and then $\pi(\ker h) \subseteq \ker(\bar{h})$. □

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Congruence-free restriction semigroups

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Abstract. Restriction semigroups are common generalizations of ample semigroups and inverse semigroups. The main aim of this paper is to probe restriction semigroups with certain congruence properties. In this paper we give some characterizations of restriction semigroups each of whose proper $(2, 1, 1)$-congruences are reduced, so called H-reduced restriction semigroups. In particular, the classification of congruence-free restriction semigroups is obtained; that is, it is proved that a restriction semigroup is congruence-free if and only if it is either a simple group or an H-reduced restriction semigroup without nontrivial reduced restriction monoid $(2, 1, 1)$-congruences. These results extend and enrich the related results of inverse semigroups.
Keywords: restriction semigroup, fundamental restriction semigroup, ample semigroup, congruence.

1. Introduction

Inverse semigroups play an important role in the theory of semigroups. Many authors have tried to generalize inverse semigroups. Restriction semigroups are non-regular generalizations of inverse semigroups. They are semigroups equipped with two additional unary operators which satisfy certain identities. In particular, each inverse semigroup determines a restriction semigroup in which the unary operations assign the idempotents aa^{-1} and $a^{-1}a$, respectively, to

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any element a. The class of restriction semigroups is just the variety of algebras generated by these restriction semigroups obtained from inverse semigroups, see [8]. Restriction semigroups (formerly, called weakly E-ample semigroups) have arisen from a number of mathematical perspectives. For a detailed introduction of the history and basic properties of restricted semigroups, please refer to [13] and [18].

So far, a number of important results of the rich structure theory of inverse semigroups have been recast in the broader setting of restriction semigroups; see [11, 9, 10, 21, 25, 16]. In theory of inverse semigroups, congruences play an important role. Because restriction semigroups are generalizations of inverse semigroups, it is natural to probe the congruence theory of restriction semigroups. This is the main aim of this paper. It is an important property that any quotient of an inverse semigroup over a congruence is also inverse. This property is a key to study the congruence theory of inverse semigroups in the present ways. Unfortunately, the quotient of a restriction semigroup over a general congruence need not be still a restriction one (see [15]). So, we only consider the $(2,1,1)$-congruences on a restriction semigroup. Indeed, we are inspired by the results of El Qallali in [4] on congruences on an ample semigroup, formerly called type-A semigroups. This is because any ample semigroup is a special restriction semigroup.

We proceed as follows: after some preliminaries, in Section 3, we obtain some trace characterizations of $(2,1,1)$-congruences on a restriction semigroup. In Section 4, we consider restriction semigroups all of whose proper $(2,1,1)$-congruences are reduced, called H-reduced restriction semigroups. It is interesting that an H-reduced restriction semigroup must be an ample semigroup. Moreover, we determine when a restriction semigroup is H-reduced (Theorem 4.1). This result extends those of Tucci in [26] on inverse semigroups all of whose proper homomorphic images are groups. Section 5 is devoted to congruence-free restriction semigroups. So-called a *congruence-free restriction semigroup* is a restriction semigroup whose $(2,1,1)$-congrunces are only the identity relation and the universal relation. Such semigroups are analogue of congruence-free inverse semigroups. For congruence-free inverse semigroups, see [22, 27]. In [23], Munn further researched congruence-free regular semigroups. Indeed, any congruence-free inverse semigroup is fundamental; for fundamental inverse semigroups, see [20, 24]. It is proved that a semigroup S is a congruence-free restriction semigroup if and only if S is either a simple group, or an H-reduced restriction semigroup without nontrivial reduced $(2,1,1)$-congruences (Theorem 5.1). Our results enrich and extend the related results on inverse semigroups, or ample semigroups.

2. Preliminaries

We recall some concepts and notations, which are used in the sequel without mentions.

2.1 Restriction semigroups

A *left restriction semigroup* is defined to be an algebra of type $(2,1)$, more precisely, an algebra $S = (S, \cdot, {}^+)$ where $(S, \cdot)$ is a semigroup and ${}^+$ is a unary operator such that the following identities are satisfied:

$$(2.1)\qquad \begin{array}{l} (x^+)^+ = x^+,\ x^+x = x,\ x^+y^+ = y^+x^+, \\ (x^+y)^+ = x^+y^+,\ (xy)^+ = (xy^+)^+,\ xy^+ = (xy^+)^+x. \end{array}$$

A *right restriction semigroup* is dually defined, that is, it is an algebra $(S, \cdot, *)$ satisfying the duals of the identities (2.1). If $S = (S, \cdot, {}^+, *)$ is an algebra of type $(2,1,1)$ where $S = (S, \cdot, {}^+)$ is a left restriction semigroup and $S = (S, \cdot, *)$ is a right restriction semigroup and the identities

$$(2.2)\qquad (x^+)^* = x^+,\ (x^*)^+ = x^*$$

hold, then it is called a *restriction semigroup*. By definition, the defining properties of a restriction semigroup are left-right dual. Therefore in the sequel dual definitions and statements will not be explicitly formulated. It is well known that in a restriction semigroup, we always have

$$(2.3)\qquad (xy)^+ = (xy^+)^+ \text{ and } (xy)^* = (x^*y)^*$$

(for example, see [13]).

Among restriction semigroups, the notions of subalgebra, homomorphism, congruence and factor algebra are understood in type $(2,1,1)$, which is emphasised by using the expressions $(2,1,1)$-subsemigroup, $(2,1,1)$-morphism, $(2,1,1)$-congruence and $(2,1,1)$-factor semigroup, respectively. A restriction semigroup with identity element 1 and such that $1^+ = 1 = 1^*$ is also called a *restriction monoid*.

Let S be a restriction semigroup. By (2.2), we have

$$\{x^+ : x \in S\} = \{x^* : x \in S\}.$$

This set is called the *set of projections of* S and denoted by $P(S)$. Again by (2.1) and its dual, $P(S)$ is a $(2,1,1)$-subsemigroup of S which is indeed a semilattice. We call a restriction semigroup to be *reduced* if $P(S)$ is a singleton. In this case, the unique element of $P(S)$ is the identity element of S. As in [16], we define

$$\mathfrak{C} = \{(u,v) \in S \times S : u^+ = v^+, u^* = v^*\}.$$

2.2 Ample semigroups

The relations $\mathcal{R}^*$ and $\mathcal{L}^*$ are generalizations of the usual Green's relations $\mathcal{R}$ and $\mathcal{L}$, respectively. Elements a and b of a semigroup T is related by $\mathcal{R}^*$ (respectively, $\mathcal{L}^*$) if and only if they are related by $\mathcal{R}$ (respectively, $\mathcal{L}$) in some oversemigroup of T. Equivalently, we have

$$(a,b) \in \mathcal{R}^* \text{ if and only if } xa = ya \Leftrightarrow xb = yb \text{ for any } x, y \in T^1$$

and

$$(a, b) \in \mathcal{L}^* \text{ if and only if } ax = ay \Leftrightarrow bx = by \text{ for any } x, y \in T^1.$$

A semigroup T is an *ample semigroup* if the following conditions are satisfied:

(i) for any $a \in T$, the $\mathcal{R}^*$-class R_a^* of T containing a exists uniquely one idempotent a^+;

(ii) for any $a \in T$, the $\mathcal{L}^*$-class L_a^* of T containing a exists uniquely one idempotent a^+;

(iii) the set $E(T)$ of idempotents of T becomes a commutative subsemigroup; that is, $E(T)$ is a semilattice under the multiplication of T;

(iv) for any $a \in T, e \in E(T)$, $ea = a(ea)^*$ and $ae = (ae)^+a$.

Ample semigroups are formerly called as type A semigroups. It is well known that any inverse semigroup is an ample semigroup and any ample semigroup can be viewed as a subsemigroup of some inverse semigroup. Indeed, an inverse semigroup is just an ample semigroup being regular.

For an ample semigroup T, we have that $e^+ = e = e^*$ for all $e \in E(T)$. By definition, it is easy to see that T is a restriction semigroup with unary operators:

$$^+ : T \to T;\ a \mapsto a^+$$

and

$$^* : T \to T;\ a \mapsto a^*,$$

and in this case,

(i) $P(T) = E(T)$;

(ii) $(2, 1, 1)$-congruences are just admissible congruences on T;

(iii) $(2, 1, 1)$-homomorphisms are just admissible homomorphisms on T;

(iv) any reduced $(2, 1, 1)$-congruence is indeed a cancellative monoid congruence;

(v) $\mathfrak{C} = \mathcal{H}^*$, where $\mathcal{H}^* = \mathcal{L}^* \sqcap \mathcal{R}^*$.

Consequently, any ample semigroup is a restriction semigroup S in which for any $a \in S$, $a^+\mathcal{R}^*a\mathcal{L}^*a^*$.

In what follows, we view an ample semigroup as a restriction semigroup with the unary operations as above.

Recall that a left (right) ideal J of a semigroup T is a *left* (*right*) $*$-*ideal* of T if $J = \sqcup_{x\in J} L_x^*$ $(J = \sqcup_{y\in J} R_y^*)$, where L_x^* (R_x^*) is the $\mathcal{L}^*$-class (the $\mathcal{R}^*$-class) of S containing a. Moreover, an ideal of T is a $*$-*ideal* of T if it is both a left $*$-ideal and a right $*$-ideal.

2.3 Unary polynomials

Given a set X of variables, by a *term* in X we mean a formal expression built up from the elements of X by means of the operational symbols— the binary operational sysmbol $\cdot$ and the unary operational symbols $^+$ and *— in finitely many steps. For example, the left and right hand sides of equalities in (2.1)-(2.3) are terms in variables x, y. If we work with an associative binary operation then we delete the unnecessary parenthesis from terms. If S is a restriction semigroup then we introduce a nullary operational symbols for every element s in S, and for simplicity, denote it also by s. By a *polynomial* of S we mean an expression obtained in a way similar to terms, but from variables and these operational symbols. A polynomial can also be interpreted in the way that such nullary operational symbols are substituted for certain variables in a term. For simplicity, later on we just say that elements of S are substituted for the variables. As it is usual for semigroups, we allow to substitute also $1 \in S^1$ for several, but not all, variables to indicate that the variables in question be deleted from the term. For example, if 1 is substituted for variable y in the terms xyz and $zy^*(x^*y)^+$ then the terms obtained are xz and $z(x^*)^+$, respectively. A *unary polynomial* of S is a polynomial with at most one variable. Their set is denoted by $\mathcal{P}_1(S)$.

If $\mathbf{t} = \mathbf{t}(x_1, x_2, \cdots, x_n)$ is a term or $\mathbf{p} = \mathbf{p}(x_1, x_2, \cdots, x_n)$ is a polynomial in the variables $x_1, x_2, \cdots, x_n$, and we substitute elements $s_1, s_2, \cdots, s_n$ of S^1 with $\{s_1, \cdots, s_n\} \cap S \neq \emptyset$ for the variables, then we can evaluate the expression so obtained in S^1. The result is an element of S which is denoted by $\mathbf{t}^S(s_1, s_2, \cdots, s_n)$ and $\mathbf{p}^S(s_1, s_2, \cdots, s_n)$, respectively. Notice that the evaluation is compatible with the interpretation of the substitution of $1 \in S^1$ for variables. The *polynomial function* of S corresponding to the polynomial $\mathbf{p}$ is the mapping

$$\mathbf{p}^S : S^n \to S, \ (s_1, s_2, \cdots, s_n) \mapsto \mathbf{p}(s_1, s_2, \cdots, s_n),$$

which is also denoted by $\mathbf{p}^S(x_1, x_2, \cdots, x_n)$.

An identity is a formal equality $\mathbf{t} = \mathbf{u}$ of two terms, considered with a common set of variables. A restriction semigroup *satisfies the identity* $\mathbf{t} = \mathbf{u}$ if

$$\mathbf{t}^S(s_1, s_2, \cdots, s_n) = \mathbf{u}^S(s_1, s_2, \cdots, s_n),$$

for any $s_1, s_2, \cdots, s_n \in S$.

Let τ be a relation on a restriction semigroup S. If $c, d \in S$ are such that

$$c = \mathbf{p}^S(a), \ d = \mathbf{p}^S(b),$$

for some $\mathbf{p} \in \mathcal{P}_1(S)$, where either (a, b) or (b, a) belongs to τ, we say that c is connected to d by a *polynomial τ-transition*, in notation, $c \xrightarrow{\mathbf{p}} d$. We denote by $\tau^{\#}$ the $(2, 1, 1)$-congruence on S generated by τ.

A well-known universal algebraic fact implies the following description, due to Szendrei (see [25]).

Lemma 2.1. *Let S be a restriction semigroup and τ a relation on S. Then for any $c, d \in S$, $c\tau^{\#}d$ if and only if $c = d$ or there is a sequence*

$$c = c_1 \xrightarrow{p} c_2 \xrightarrow{p} \cdots \xrightarrow{p} c_n = d$$

of polynomial τ-transitions.

3. Congruences

In this section, we need to obtain some characterizations of $(2,1,1)$-congruence on restricted semigroups. Let S be a restriction semigroup. For a $(2,1,1)$-congruence ρ on S, we have the restriction $\rho|_{P(S)}$ of ρ to $P(S)$ which is called the *projection trace* of ρ, denoted by $Ptr\rho$. It is easy to see that $Ptr\rho$ is a congruence on $P(S)$.

Definition 3.1. *A congruence τ on $P(S)$ is* projection-normal *if for any $e, f \in P(S)$ and $x \in S$, $(ex)^*\tau(fx)^*$ and $(xe)^+\tau(xf)^+$ whenever $e\tau f$.*

Corollary 3.1. *If ρ is a $(2,1,1)$-congruence on S, then $Ptr\rho$ is projection-normal.*

Proof. Let $e, f \in P(S)$ and $x \in S$. If $e\rho f$, then $ex\rho fx, xe\rho xf$, so that

$$(ex)^*\rho(fx)^*,\ (xe)^+\rho(xf)^+,$$

therefore $Ptr\rho$ is projection-normal. □

Lemma 3.1. *Let τ be a projection-normal congruence on $P(S)$ and $u, v \in S$. Then the following statements are equivalent:*

(i) $u^*\tau v^*, ue = ve$ *for some* $e \in P(S), e\tau u^*$;

(ii) $u^+\tau v^+, fu = fv$ *for some* $f \in P(S), f\tau u^+$.

Proof. (i)⇒(ii). Because S is a restriction semigroup, $ue = ve$ implies that $(ue)^+u = (ve)^+v$ and $(ue)^+ = (ve)^+$. And, by the normality of τ, $e\tau u^*$ implies that $(ue)^+\tau(uu^*)^+ = u^+$; similarly, $(ve)^+\tau v^+$. Together with the foregoing proof: $(ue)^+ = (ve)^+$, we have $u^+\tau v^+$ and (ii) holds.

(ii)⇒(i). It is similar as (i)⇒(ii). □

Proposition 3.1. *For a projection-normal congruence τ on $P(S)$, the relation*

$$\tau_{\min} = \{(u, v) \in S \times S : u^*\tau v^*, ue = ve \text{ for some } e \in P(S), e\tau u^*\}$$

is the smallest $(2,1,1)$-congruence on S such that $Ptr\tau_{\min} = \tau$.

Proof. It is routine to check that $\tau_{\min}$ is an equivalence relation. Let $u, v, t \in S$ with $(u, v) \in \tau_{\min}$, then $u^*\tau\ v^*, ue = ve$ for some $e \in P(S)$ and $e\tau\ u^*$, so that $tue = tve$. Moreover,

$$(tu)^* = (tu)^*u^*\tau(tu)^*e = (tue)^* = (tve)^* = (tv)^*e$$

and $(tv)^* = (tv)^*v^*\tau(tv)^*e$. Therefore, $(tu)^*\tau(tv)^*$. Notice that

$$(tu)^*e = (tue)^* = (tve)^* = (tv)^*e,$$

we observe that

$$(tu)(tu)^*e = tue = tve = (tv)(tv)^*e = (tv)(tu)^*e.$$

Together with $(tu)^*e \in P(S)$, we have now proved that $(tu, tv) \in \tau_{\min}$. On the other side, we have

$$ue = ve \Rightarrow uet = vet \Rightarrow ut(et)^* = vt(et)^*.$$

By the normality of τ, $e\tau u^*$ implies that $(et)^*\tau(u^*t)^* = (ut)^*$, so that $(et)^*\tau(ut)^*$. Similarly, $(et)^*\tau(vt)^*$. Therefore $(ut)^*\tau(vt)^*$. We have now proved that $(ut, vt) \in \tau_{\min}$. Therefore, $\tau_{\min}$ is congruence.

Also, $(u^*)^* = u^*\tau v^* = (v^*)^*$, $u^*e = (ue)^* = (ve)^* = u^*e$ and $e\tau u^* = (u^*)^*$. By definition, these three formula can derive that $u^*\tau_{\min}v^*$. Similarly, by Lemma 3.1, $u^+\tau_{\min}v^+$. Consequently, $\tau_{\min}$ is indeed a $(2, 1, 1)$-congruence.

For any $e, f \in P(S)$, if $e\tau f$, then by the normality of τ, $(eu)^*\tau(fu)^*$ and $(ue)^+\tau(uf)^+$. Notice that $ef\tau e$ and $eef = fef$, we can observe that $e\tau_{\min}f$. Conversely, if $e\tau_{\min}f$ then by definition, $e\tau f$. Hence, $Ptr\tau_{\min} = \tau$.

Suppose now that ρ is a $(2, 1, 1)$-congruence on S such that $\mathrm{Ptr}\rho = \tau$, and $(u, v) \in \tau_{\min}$ for some $u, v \in S$, then $u^*\tau v^*, ue = ve$ for some $e \in P(S), e\tau u^*$. It follows that $(u^*, e), (v^*, e) \in \rho$. Therefore, $u = uu^*\rho ue = ve\rho vv^* = v$. Hence $\tau_{\min} \subseteq \rho$ and $\tau_{\min}$ is the smallest $(2, 1, 1)$-congruence on S such that $Ptr\tau_{\min} = \tau$. □

By Lemma 3.1, the following corollary is an immediate consequence of Proposition 3.1.

Corollary 3.2. *The congruence $\tau_{\min}$ of Proposition 3.4 has also the following from:*

$$\tau_{\min} = \{(u, v) \in S \times S : u^+\tau v^+, fu = fv \text{ for some } f \in P(S), f\tau u^+\}.$$

By a *projection separating* $(2, 1, 1)$-*congruence* on S, we mean a $(2, 1, 1)$-congruence ρ on S in which for any $e, f \in P(S)$, if $e\rho f$, then $e = f$. Gould [11] pointed out that for a restriction semigroup S, the relation

$$\begin{aligned}\mu_S &= \{(u, v) \in S \times S : u^+ = v^+ \text{ and } (eu)^* = (ev)^* \text{ for all } e \in P(S)\}\\ &= \{(u, v) \in S \times S : u^* = v^* \text{ and } (uf)^+ = (vf)^+ \text{ for all } f \in P(S)\}\end{aligned}$$

is the greatest projection separating $(2,1,1)$-congruence on S and $\mu_S \subseteq \mathfrak{C}$. Sometime, we write also μ_S as $\mu(S)$. By definition, a $(2,1,1)$-congruence ρ on S is projection-separating if and only if $Ptr\rho = id_{P(S)}$ where $id_{P(S)}$ denotes the identity relation on $P(S)$.

For a projection-normal congruence τ on $P(S)$, we define

$$\tau_{\max} = \{(u,v) \in S \times S : (eu)^*\tau(ev)^* \text{ and } (ue)^+\tau(ve)^+ \text{ for any } e \in P(S)\}.$$

Lemma 3.2. *Let τ be a projection-normal congruence on $P(S)$. Then for any $u, v \in S$, the following statements are equivalent:*

(i) $(u,v) \in \tau_{\max}$;

(ii) $(eu)^*\tau(fv)^*$ *and* $(ue)^+\tau(vf)^+$, *for any* $e, f \in P(S)$ *with* $e\tau f$;

(iii) $(u\tau_{\min}, v\tau_{\min}) \in \mu_{S/\tau_{\min}}$.

Proof. (i)$\Rightarrow$(ii). For any $e, f \in P(S)$ with $e\tau f$, we have $(ev)^*\tau(fv)^*$ by normality. If $(u,v) \in \tau_{\max}$, then $(eu)^*\tau(ev)^*$ so that $(eu)^*\tau(fv)^*$; similarly, $(ue)^+\tau(vf)^+$.

(ii)$\Rightarrow$(i). It is clear.

(i)$\Leftrightarrow$(iii). It follows from the following implications:

$$\begin{aligned}
(u,v) \in \tau_{\max} &\Leftrightarrow (eu)^*\tau(ev)^* \text{ and } (ue)^+\tau(ve)^+ \text{ for any } e \in P(S);\\
&\Leftrightarrow (eu)^*\tau_{\min} = (ev)^*\tau_{\min} \text{ and } (ue)^+\tau_{\min} = (ve)^+\tau_{\min}\\
&\qquad \text{for any } e \in P(S);\\
&\Leftrightarrow ((eu)\tau_{\min})^* = ((ev)^{\tau}_{\min})^* \text{ and } ((ue)\tau_{\min})^+ = ((ve)\tau_{\min})^+\\
&\qquad \text{for any } e \in P(S);\\
&\Leftrightarrow (e\tau_{\min} \cdot u\tau_{\min})^* = (e\tau_{\min} \cdot b\tau_{\min})^* \text{ and}\\
&\qquad (u\tau_{\min} \cdot e\tau_{\min})^+ = (v\tau_{\min} \cdot e\tau_{\min})^+ \text{ for all } e \in P(S);\\
&\Leftrightarrow (u\tau_{\min}, v\tau_{\min}) \in \mu(S/\tau_{\min}).
\end{aligned}$$

We complete the proof. $\square$

Proposition 3.2. *Let τ be a projection-normal congruence on $P(S)$. Then, $\tau_{\max}$ is the greatest $(2,1,1)$-congruence on S such that $Ptr\tau_{\max} = \tau$.*

Proof. It is routine to check that $\tau_{\max}$ is an equivalence relation. Let $u, v, t \in S$ with $(u,v) \in \tau_{\max}, e \in P(S)$. Then $(eu)^*\tau(ev)^*$ and by the normality of τ, it follows that

$$(eut)^* = ((eu)^*t)^*\tau((ev)^*t)^* = (evt)^*.$$

Notice that $(te)^\dagger \in P(S)$, we have

$$(ute)^+ = (u(te)^+)^+\tau(v(te)^+)^+ = (vte)^+.$$

Therefore, $(ut, vt) \in \tau_{\max}$; similarly, $(tu, tv) \in \tau_{\max}$. Hence, $\tau_{\max}$ is a congruence.

It is obvious that $\tau \subseteq \tau_{\max}$. Let $e, f, g \in P(S)$. If $f\tau_{\max}g$, then $ef\tau eg$. Take in turn $e = f$ and $e = g$ to get $f\tau fg, gf\tau g$. As $fg = gf$, now $f\tau g$. Thus, $Ptr\tau_{\max} = \tau$.

If $(u, v) \in \tau_{\max}$, then $(u\tau_{\min}, v\tau_{\min}) \in \mu(S/\tau_{\min})$. But, $\mu(S/\tau_{\min})$ and $\tau_{\min}$ are $(2,1,1)$-congruence, so $(u^*\tau_{\min}, v^*\tau_{\min}) \in \mu(S/\tau_{\min})$ and $(u^+\tau_{\min}, v^+\tau_{\min}) \in \mu(S/\tau_{\min})$. By Lemma 3.2, these show that $(u^*, v^*) \in \tau_{\max}$ and $(u^+, v^+) \in \tau_{\max}$. Therefore, $\tau_{\max}$ is a $(2,1,1)$-congruence.

Finally, we let ρ be a $(2,1,1)$-congruence on S such that $Ptr\rho = \tau$. If $(u, v) \in \rho$, then for any $e \in P(S)$, $(eu, ev) \in \rho$ and $(ue, ve) \in \rho$. It follows that $((eu)^*, (ev)^*), ((ue)^+, (ve)^+) \in \rho$. Thus $(eu)^*\tau(ev)^*, (ue)^+\tau(ve)^+$. Hence $\rho \subseteq \tau_{\max}$ and the proof is completed. □

In what follows, we call a $(2,1,1)$-congrunce ρ on S a *reduced* $(2,1,1)$-*congruence* if S/ρ is a reduced restriction monoid. The following proposition gives a characterization of reduced $(2,1,1)$-congruences.

Proposition 3.3. *Let ρ be a $(2,1,1)$-congruence on S. Then ρ is a reduced $(2,1,1)$-congruence on S if and only if $Ptr\rho = P(S) \times P(S)$.*

Proof. Suppose that ρ is a reduced $(2,1,1)$-congruence on S, then S/ρ is a reduced restriction monoid. This means that $|P(S/\rho)| = 1$. Obviously, for any $e, f \in P(S)$, $e\rho = f\rho$. Thus $P(S) \times P(S) \subseteq Ptr\rho$. Hence $Ptr\rho = P(S) \times P(S)$.

Conversely, suppose that $Ptr\rho = P(S) \times P(S)$, then for any $e, f \in P(S)$, $e\rho = f\rho$. This shows that $|\{e\rho : e \in P(S)\}| = 1$. On the other hand, if $a\rho$ $(a \in S)$ is a projection of S/ρ, then as ρ is a $(2,1,1)$-congruence on S, $a\rho = (a\rho)^+ = a^+\rho$. So, $P(S/\rho) = \{e\rho : e \in P(S)\}$. Therefore $|P(S/\rho)| = 1$, and so S$/\rho$ is a reduced restriction monoid. Hence ρ is a reduced $(2,1,1)$-congruence on S. □

Denote $\omega = P(S) \times P(S)$. It is obvious that ω is a normal congruence on $P(S)$. So, by Proposition 3.3, $\omega_{\min}$ and $\omega_{\max}$ are both reduced $(2,1,1)$-congruences. Again by Propositions 3.1 and 3.2, we have the following corollary.

Corollary 3.3. *Let S be a restriction semigroup. Then*

(i) $\omega_{\min}$ *is the smallest reduced* $(2,1,1)$-*congruence on S;*

(ii) $\omega_{\max} = S \times S$.

Evidently, the identity relation Δ on $P(S)$ is a normal congruence on $P(S)$. It is not difficult to see that for a restriction semigroup S, we have

(i) $\Delta_{\min}$ is the identity relation on S;

(ii) $\Delta_{\max} = \mu_S$.

Proposition 3.4. *Let S be a restriction semigroup. If ρ is a $(2,1,1)$-congruence on S, then $P(S/\rho) = \{e\rho : e \in P(S)\}$.*

Proof. Obviously, $\{e\rho : e \in P(S)\} \subseteq P(S/\rho)$. If $a\rho$ $(a \in S)$ is a projection of S/ρ, then $a\rho = (a\rho)^+ = a^+\rho$, so that $P(S/\rho) \subseteq \{e\rho : e \in P(S)\}$. Therefore, $P(S/\rho) = \{e\rho : e \in P(S)\}$. $\square$

4. H-reduced restriction semigroups

In this section, we give the definition of H-reduced restricted semigroups.

Definition 4.1. *A semigroup S is an* H-reduced restriction semigroup *if*

(i) *S is not a reduced restriction monoid;*

(ii) $|S| \geq 2$*;*

(iii) *any $(2,1,1)$-congruence ρ on S is either the identical relation or a reduced $(2,1,1)$-congruence.*

Notice that a restriction semigroup is reduced if and only if its set of projections is a singleton. So, it is easy to know that for any H-reduced restriction semigroup S, we have always $|P(S)| \geq 2$.

By a 0-$\mathcal{J}^*$*-simple semigroup*, we mean a semigroup with zero element 0 and satisfying the conditions as follows:

(i) $S^2 \neq \{0\}$;

(ii) S and $\{0\}$ are the only $*$-ideals of S.

And, we call a 0-$\mathcal{J}^*$-simple semigroup having no zero element to be a $\mathcal{J}^*$*-simple semigroup*. Equivalently, a semigroup S with zero element is 0-$\mathcal{J}^*$-simple if and only if $S^2 \neq \{0\}$ and

$$\mathcal{J}^* = \{(0,0)\} \sqcup (S\backslash\{0\}) \times (S\backslash\{0\});$$

if and only if $S^2 \neq \{0\}$ and $a\mathcal{J}^*b$ for any nonzero elements a, b of S. Also, it is easy to see that a semigroup is $\mathcal{J}^*$-simple if and only if $\mathcal{J}^*$ is the universal relation on S.

Take after Gould, we call a restriction semigroup S to be *fundamental* if the maximum projection-separating $(2,1,1)$-congruence μ is the identity relation. In [11], Gould proved that any fundamental restriction semigroup is isomorphic to some full $(2,1,1)$-subsemigroup of the Munn semigroup on its projection semilattice. According to a result of Fountain in [6], any full subsemigroup of an inverse semigroup must be an ample semigroup. Because any Munn semigroup is an inverse semigroup, this shows that any fundamental restriction semigroup is always an ample semigroup.

By Definition 4.1, we have the following corollary.

Corollary 4.1. *Any H-reduced restriction semigroup is a* 0-$\mathcal{J}^*$*-simple ample semigroup which is fundamental.*

Proof. Let S be an H-reduced restriction semigroup. If the projection separating $(2,1,1)$-congruence μ_S is not the identity relation, then μ_S is a reduced $(2,1,1)$-congruence, and by Proposition 3.4, $|P(S/\mu_S)| = |\{e\mu_S : e \in P(S)\}|$. But μ_S is projection-separating, so $|\{e\mu_S : e \in P(S)\}| = |P(S)|$. Therefore $1 = |P(S/\mu_S)| = |P(S)|$, so that $P(S)$ is a singleton. It follows that S is a reduced restriction semigroup, contrary to Definition 4.1. Thus μ_S is the identity relation on S, so that S is a fundamental restriction semigroup. Now by the foregoing arguments before Corollary 4.1, S is an ample semigroup.

Now let U be a $*$-ideal of S and $U \neq S$. Then by [14, Lemma 2.2], the Rees congruence $R_U := U \times U \sqcup id_S$ is a $(2,1,1)$-congruence on S, where id_S is the identity relation on S.

(i) When the Rees congruence R_U is the identity relation. In this case, $U = \{0\}$.

(ii) If R_U is not the identity relation, then by hypothesis, R_U is a reduced $(2,1,1)$-congruence, and so S/R_U is a trivial semigroup, since S/R_U is a restriction semigroup with zero element and the projection set of a reduced restriction semigroup is a singleton. Therefore $U = S$.

However S has only two $*$-ideals: $\{0\}$ and S. This means that S is a 0-$\mathcal{J}^*$-simple semigroup. □

We arrive now at the main result of this section.

Theorem 4.1. *Let S be a restriction semigroup such that $|P(S)| > 1$. Then S is an H-reduced restriction semigroup if and only if the following statements hold:*

(FA) *S is a fundamental ample semigroup;*

(HR) *for any $e, f, h \in P(S)$ with $e > f$, there is a sequence*

$$e = e_1 \xrightarrow{p} e_2 \xrightarrow{p} \cdots \xrightarrow{p} e_n = h$$

of polynomial τ-transitions, where

(i) *$e_1, e_2, \cdots, e_n \in P(S)$;*

(ii) *$\tau = \{(e, f)\}$.*

Proof. Suppose that Conditions (FA) and (HR) hold. Let ρ be a $(2,1,1)$-congruence on S, and $\rho \neq S \times S$. We consider the following two cases:

(1) If $Ptr\rho = id_{P(S)}$, then ρ is a projection-separating $(2,1,1)$-congruence, so $\rho \subseteq \mu_S$. But S is fundamental, then $\mu_S = id_S$ and thus ρ is the identity congruence on S.

(2) Assume that $Ptr\rho \neq id_{P(S)}$. Then there is $e, h \in P(S)$ such that $e \neq h$ and $(e, h) \in \rho$. It follows that $(e, eh) \in \rho$.

(a) If $e = eh$, then $e < h$. Now by Lemma 2.1, Condition (HR) implies that for any $g \in P(S)$, $(h, g) \in \tau^{\#}$ where $\tau = \{(h, e)\}$. But $\tau \subseteq \rho$, so $\tau^{\#} \subseteq \rho$. Accordingly, $(g, h) \in \rho$. This means that $P(S) \times P(S) \subseteq \rho$. Now by Proposition 3.3, ρ is a reduced $(2, 1, 1)$-congruence on S.

(b) Assume that $e \neq eh$. We have that $eh < e$. Applying the arguments on e, h to e, eh, we can get that ρ is a reduced $(2, 1, 1)$-congruence on S.

Consequently, S is an H-reduced restriction semigroup.

Conversely, suppose that S is an H-reduced restriction semigroup. Notice that μ_S is a $(2, 1, 1)$-congruence on S. By hypothesis, $\mu_S = id_S$ or μ_S is reduced.

(A) If the first case holds, then S is a fundamental restriction semigroup. So, S is isomorphic to a full subsemigroup of the Munn semigroup on $P(S)$. But the Munn semigroup is an inverse semigroup, so any full subsemigroup of the Munn semigroup is always an ample semigroup. Therefore S is a fundamental ample semigroup.

(B) If the second case is true, then $Ptr\mu_S$ is the universal relation on $P(S)$. But μ_S is projection-separating, so $|P(S)| = 1$, contrary to the hypothesis that $|P(S)| \geq 2$.

However, S is a fundamental ample semigroup.

Let $e, f, h \in P(S)$ be such that $e > f$. Consider the relation $\tau = \{(e, f)\}$ on S, we know easily that $\tau^{\#}$ is not the identity on S. By definition, $\tau^{\#}$ is a reduced $(2, 1, 1)$-congruence on S. It follows that $(e, h) \in \tau^{\#}$. Now by Lemma 2.1, there is a sequence

$$e = c_1 \xrightarrow{p} c_2 \xrightarrow{p} \cdots \xrightarrow{p} c_n = h$$

of polynomial τ-transitions. Let $p_i \in \mathcal{P}_1(S)$ with $i = 1, 2, \cdots, n$ and such that

$$\begin{aligned} c_1 =& p_1^S(a_1), p_1^S(b_1) = c_2 = p_2^S(a_2), p_2^S(b_2) = c_3 = p_3^S(a_3), \cdots \\ & p_{n-1}^S(b_{n-1}) = c_{n-1} = p_n^S(a_n), p_n^S(b_n) = c_n, \end{aligned} \tag{4.1}$$

where either (a_i, b_i) or (b_i, a_i) belong to τ. Now let $q_i(x) = (p_i(x))^+$. Obviously, $q_i(x) \in \mathcal{P}_1(S)$. Notice that $e = e^+ = c_1^+$ and $h = h^+ = c_n^+$. By (4.1), we can obtain that

$$\begin{aligned} c_1^+ =& (p_1^S(a_1))^+, (p_1^S(b_1))^+ = c_2^+ = (p_2^S(a_2))^+, (p_2^S(b_2))^+ = c_3^+ = (p_3^S(a_3))^+, \cdots \\ & (p_{n-1}^S(b_{n-1}))^+ = c_{n-1}^+ = (p_n^S(a_n))^+, (p_n^S(b_n))^+ = c_n^+; \end{aligned}$$

that is,

$$\begin{aligned} e = c_1^+ =& q_1^S(a_1), q_1^S(b_1) = c_2^+ = q_2^S(a_2), q_2^S(b_2) = c_3^+ = q_3^S(a_3), \cdots \\ & q_{n-1}^S(b_{n-1}) = c_{n-1}^+ = q_n^S(a_n), q_n^S(b_n) = c_n^+ = h, \end{aligned} \tag{4.2}$$

where $q_k^S(x) = (p_k^+(x))^+ \in \mathcal{P}_1(S)$ for $k = 1, 2, \cdots, n$. It results Condition (HR). □

By a *proper congruence* on S, we mean a congruence ρ on S with $\rho \neq S \times S$.

Let S be an inverse semigroup. It is easy to see that any congruence on S is always a $(2,1,1)$-congruence on S. Notice that for any congruence ρ on S, ρ is a group congruence on S if and only if $E(S) \times E(S) \subseteq \rho$. We can observe that ρ is a group congruence if and only if ρ is reduced. Now, the following corollary is an immediate consequence of Theorem 4.1, which is essentially the main result in [26].

Corollary 4.2. *Let S be an inverse semigroup which is not a group. Then every proper congruence of S is a group congruence if and only if S is a fundamental inverse semigroup satisfying Condition (HR).*

5. Congruence-free restriction semigroups

In this section, we shall discuss congruence-free restriction semigroups.

Definition 5.1. *A restriction semigroup S is* congruence-free *if any* $(2,1,1)$-*congruence on S is either the universal congruence or the identity congruence.*

Let S be a congruence-free restriction semigroup. Notice that the universal relation is a reduced restriction monoid. By definition, any $(2,1,1)$-congruence on a congruence-free restriction semigroup S is either the identity relation or a reduced $(2,1,1)$-congruence. So, S is either a reduced restriction semigroup or an H-reduced restriction semigroup. On the other hand, also by definition, the greatest projection separating $(2,1,1)$-congruence μ_S is the identity relation on S. So, S is a fundamental restriction semigroup. Furthermore, S is a full $(2,1,1)$-subsemigroup of the Munn semigroup on $P(S)$, so that S is an ample semigroup. Assume, in addition, that S is a reduced restriction semigroup. Obviously, S is a monoid with identity 1. Consider that an ample semigroup may be viewed as a restriction semigroup in which for any element a, $a^+\mathcal{R}^*a\mathcal{L}^*a^*$, this shows that for any $a \in S$, $a\mathcal{H}^*1$. That is, S is an $\mathcal{H}^*$-class containing an idempotent 1. By a result of Fountain in [7], S is a cancellative monoid. Therefore we have the following corollary.

Corollary 5.1. *If S is a congruence-free restriction semigroup, then S is either a cancellative monoid or an H-reduced restriction semigroup.*

Lemma 5.1. *Let S be a restriction semigroup. Then every (2, 1, 1)-congruence on S is either a projection separating* $(2,1,1)$*-congruence or a reduced* $(2,1,1)$-*congruence if and only if S satisfies Condition* (HR)*.*

Proof. Suppose that S satisfies (HR). Indeed, in the proof of the sufficiency of Theorem 4.1, we have proved that any proper $(2,1,1)$-congruence on S is either projection-separating or reduced. It results the sufficiency.

Conversely, suppose that every $(2,1,1)$-congruence on S is either a projection-separating $(2,1,1)$-congruence or a reduce $(2,1,1)$-congruence. For $e,f,g \in P(S)$ with $e > f$, we consider the relation $\tau = \{e,f\}$. It is easy to see that $\tau^\#$ is not a projection-separating $(2,1,1)$-congruence on S, since the projection trace of a projection-separating congruence on S is the identity relation on $P(S)$. Furthermore, $\tau^\#$ is a reduced $(2,1,1)$-congruence on S. Again by the proof of the necessity of Theorem 4.1, we may obtain that S satisfies Condition (HR). The proof is finished. $\square$

Lemma 5.2. *Let T be a cancellative monoid with identity 1. Then T is a congruence-free restriction semigroup if and only if T is a simple group.*

Proof. Suppose that T is a congruence-free restriction semigroup, and denote by $U(T)$ the set of all units of T. Then $U(T)$ is a subgroup of T, and $T\backslash U(T)$ is an ideal of T. It is easy to see that $\rho = (T\backslash U(T)) \times (T\backslash U(T)) \sqcup id_{U(T)}$ is a $(2,1,1)$-congruence on T. But T is congruence-free, so ρ is the identity relation on T. It follows that $T\backslash U(T)$ is the zero element of T. This means that $T = U(T)^0$ (the semigroup obtained from $U(T)$ by adjoining a zero). Thus $T = U(T)$ since T is cancellative. Moreover by [19, Proposition 8.2 (i), p.32], T is a simple group.

Conversely, by [19, Proposition 8.2 (i), p.32], it is clear that a simple group is a congruence-free restriction semigroup. $\square$

We now give the main result of this section.

Theorem 5.1. *A semigroup S is a congruence-free restriction semigroup if and only if S is either a simple group or an H-reduced restriction semigroup without nontrivial reduced $(2,1,1)$-congruences.*

Proof. Suppose that S is congruence-free. By Corollary 5.1, S is either a cancellative monoid or an H-reduced restriction semigroup. If S is a cancellative monoid, then by Lemma 5.2, S is a simple group. If S is an H-reduced restriction semigroup, then any $(2,1,1)$-congruence on S is either the identity relation or a reduced $(2,1,1)$-congruence (including the universal relation), so that S has no nontrivial reduced $(2,1,1)$-congruences.

Conversely, if S is an H-reduced restriction semigroup without nontrivial reduced $(2,1,1)$-congruences, then S has only the identity relation and the universal relation. It follows that S is congruence-free. Assume that S is a simple group. By [19, Proposition 8.2 (i), p.32], any congruence on S is of the form: $\rho_N = \{(g,h) \in S \times S : gh^{-1} \in N\}$ where N is a normal subgroup of S. This shows that S is congruence-free. $\square$

By definition, a restriction semigroup is inverse if and only if it is regular. The following corollary is an easy consequence of Theorem 5.1 and essentially the main result of Munn in [22].

Corollary 5.2. *A semigroup S is a congruence-free inverse semigroup if and only if S is either a simple group or a fundamental inverse semigroup satisfying Condition* (HR) *and without nontrivial group congruences.*

The following example is due to Tucci; for detail, see [26].

Example 5.1. Let $\mathbb{N}$ be the set of all non-negative integers. On $S = \mathbb{N} \times \mathbb{N}$, define a multiplication by

$$(m, n)(p, q) = (m - n + \max(n, p), q - p + \max(n, p)).$$

It is well known that under the above multiplication, S is an inverse semigroup. Indeed, S is the bicyclic semigroup. By [26, Corollary 7], S is a congruence-free restriction semigroup.

6. Conclusion

With the development of semigroup theory, restriction semigroups have become a hot topic in semigroup theory. This paper is based on Tucci 's inverse semigroups all of whose proper homomorphic images are groups in [26]. Moreover, EI Qallali's results in [4] on congruences on an ample semigroups give us great inspiration. In this paper, we discuss the properties of some congruences on restriction semigroups and obtain the classification of congruence-free restriction semigroups. Finally, we hope these conclusions will be helpful to the study of restriction semigroups.

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A new single step hybrid block algorithm for solving fourth order ordinary differential equations directly

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Abstract. In this paper, a one-step hybrid block method with generalized three off-step points for solving general fourth order ordinary differential equations is developed using power series of order eight as a basis function. The technique employed for the derivation of this method are to interpolate the power series at x_n and all off-step points and to collocate the fourth derivative of the basis function at all points in the selected interval. The method derived is proven to be zero stable, consistent and then convergent. The performance of the method is tested by solving linear and non-linear fourth order initial value problems.

Keywords: one step, hybrid method, block method, fourth order differential equation, power series, three generalized off step points.

1. Introduction

In this article, we consider the numerical solution of the fourth order IVPs of the form

$$y'''' = f(x, y, y', y'', y'''),\ x \in [a, b] \tag{1}$$

with initial conditions

$$y(a) = \omega_0,\ y'(a) = \omega_1, y''(a) = \omega_2, y'''(a) = \omega_3.$$

Generally, equation (1) can be solved by converting it into system of four equations of first-order IVPs and then appropriate numerical method is applied. Another alternative approach, for solving equation (1) directly which can avoid computational burden has been discussed by Awoyemi, (1992). This approach

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has been widely used for solving high order IVPs by many researchers. Some of these researchers are Omar, et al. (2015), Jator (2010), Fasasi et al. (2014) and Raft Abdelrahim (2021). Recently, hybrid block method for solving equation (1) directly have been proposed by Kayode et al. (2014), Adesanya (2012), Omar et al. (2004), Kayode (2008a) and Kuboye et al. (2015a). Avoiding the disadvantages in reduction method and employing the features of both block and hybrid methods which include generating numerical solutions simultaneously (Lambert, 1973) and overcoming zero stability barrier of linear multistep method are the aims of this new paper.

2. Methodology

In order to derive the new hybrid block method, the power series in equation (2) below is used.

$$y(x) = \sum_{i=0}^{d+c-1} a_i \left(\frac{x-x_n}{h}\right)^i,\ n = 0,1,2,...,N-1,\ x \in [x_n, x_{n+1}], \tag{2}$$

where $c = 5$ and $d = 4$ represent the number of collocation and interpolation points, $h = x_n - x_{n-1}$ and $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$. Interpolating (2) at x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and collocating the forth derivative of (2) at all points in the interval i.e at x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and x_{n+1}. This leads to system equations as shown below:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & s_1^6 & s_1^7 & s_1^8 \\
1 & s_2 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 & s_2^7 & s_2^8 \\
1 & s_3 & s_3^2 & s_3^3 & s_3^4 & s_3^5 & s_3^6 & s_2^7 & s_3^8 \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_1}{h^4} & \frac{360s_1^2}{h^4} & \frac{840s_1^3}{h^4} & \frac{1680s_1^4}{h^4} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_2}{h^4} & \frac{360s_2^2}{h^4} & \frac{840s_2^3}{h^4} & \frac{1680s_2^4}{h^4} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_3}{h^4} & \frac{360s_3^2}{h^4} & \frac{840s_3^3}{h^4} & \frac{1680s_3^4}{h^4} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4}
\end{pmatrix}
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix}
=
\begin{pmatrix} y_n \\ y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ f_n \\ f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}. \tag{3}$$

System above is solved by Gaussian Elimination Method to find $a_i^\prime s, i = 0(1)8$. Secondly, substituting the values of $a_i^\prime s$ into equation (2) yields a continuous implicit scheme of the form:

$$y(x) = \alpha_0(x) + \sum_{i=1}^{3}(\alpha_{s_i}(x)y_{n+S_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\beta_i(x)f_{n+i}. \tag{4}$$

The first, second and third derivatives of equation (4) are

$$y'(x) = \frac{\partial}{\partial x}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial}{\partial x}(\alpha_{s_i}(x)y_{n+s_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\frac{\partial}{\partial x}\beta_i(x)f_{n+i},$$

$$y''(x) = \frac{\partial^2}{\partial x^2}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial^2}{\partial x^2}(\alpha_{s_i}(x)y_{n+s_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\frac{\partial^2}{\partial x^2}\beta_i(x)f_{n+i},$$

$$y'''(x) = \frac{\partial^3}{\partial x^3}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial^3}{\partial x^3}(\alpha_{s_i}(x)y_{n+s_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\frac{\partial^3}{\partial x^3}\beta_i(x)f_{n+i},$$

where,

$$\alpha_0 = \frac{(x_n - x + hs_3)(x_n - x + hs_1)(x_n - x + hs_2)}{(h^3 s_1 s_2 s_3)},$$

$$\alpha_{s_1} = \frac{(x - x_n)(x_n - x + hs_3)(x_n - x + hs_2)}{(h^3 s_1(s_1 - s_3)(s_1 - s_2)},$$

$$\alpha_{s_2} = \frac{(x - x_n)(x - x_n - hs_3)(x_n - x + hs_1)}{(h^3 s_2(s_2 - s_3)(s_1 - s_2))},$$

$$\alpha_{s_3} = \frac{(x - x_n)(x_n - x + hs_2)(x_n - x + hs_1)}{(h^3 s_3(s_2 - s_3)(s_1 - s_3)},$$

$$\begin{aligned}
\beta_0 = {} & -\frac{(x - x_n)(x_n - x + hs_1)(x_n - x + hs_2)(x_n - x + hs_3)}{(5040h^4 s_1 s_2 s_3)}(3h^4 s_1^4 - 5h^4 s_1^3 s_2, \\
& - 5h^4 s_1^3 s_3 - 8h^4 s_1^3 - 5h^4 s_1^2 s_2^2 + 15h^4 s_1^2 s_2 s_3 + 20h^4 s_1^2 s_2 - 5h^4 s_1^2 s_3^2 + 20h^4 s_1^2 s_3 \\
& - 5h^4 s_1 s_2^3 + 15h^4 s_1 s_2^2 s_3 + 20h^4 s_1 s_2^2 + 15h^4 s_1 s_2 s_3^2 - 120h^4 s_1 s_2 s_3 - 5h^4 s_1 s_3^3 \\
& + 20h^4 s_1 s_3^2 + 3h^4 s_2^4 - 5h^4 s_2^3 s_3 - 8h^4 s_2^3 - 5h^4 s_2^2 s_3^2 + 20h^4 s_2^2 s_3 - 5h^4 s_2 s_3^3 \\
& + 20h^4 s_2 s_3^2 - 8h^4 s_3^3 + 3h^3 s_1^3 x - 3h^3 s_1^3 x_n - 5h^3 s_1^2 s_2 x + 5h^3 s_1^2 s_2 x_n + 12x^3 x_n \\
& - 5h^3 s_1^2 s_3 x + 5h^3 s_1^2 s_3 x_n - 8h^3 s_1^2 x + 8h^3 s_1^2 x_n - 5h^3 s_1 s_2^2 x + 5h^3 s_1 s_2^2 x_n + 3h^4 s_3^4 \\
& - 15h^3 s_1 s_2 s_3 x_n + 20h^3 s_1 s_2 x - 20h^3 s_1 s_2 x_n - 5h^3 s_1 s_3^2 x + 5h^3 s_1 s_3^2 x_n + 3hs_3 x^3 \\
& - 20h^3 s_1 s_3 x_n + 3h^3 s_2^3 x - 3h^3 s_2^3 x_n - 5h^3 s_2^2 s_3 x + 5h^3 s_2^2 s_3 x_n - 8h^3 s_2^2 x - 3x_n^4 \\
& - 5h^3 s_2 s_3^2 x + 5h^3 s_2 s_3^2 x_n + 20h^3 s_2 s_3 x - 20h^3 s_2 s_3 x_n + 3h^3 s_3^3 x - 3h^3 s_3^3 x_n - 3x^4 \\
& + 8h^3 s_3^2 x_n + 3h^2 s_1^2 x^2 - 6h^2 s_1^2 x x_n + 3h^2 s_1^2 x_n^2 - 5h^2 s_1 s_2 x^2 + 10h^2 s_1 s_2 x x_n \\
& - 5h^2 s_1 s_3 x^2 + 10h^2 s_1 s_3 x x_n - 5h^2 s_1 s_3 x_n^2 - 8h^2 s_1 x^2 + 16h^2 s_1 x x_n - 8h^2 s_1 x_n^2 \\
& - 6h^2 s_2^2 x x_n + 3h^2 s_2^2 x_n^2 - 5h^2 s_2 s_3 x^2 + 10h^2 s_2 s_3 x x_n - 5h^2 s_2 s_3 x_n^2 - 8h^2 s_2 x^2 \\
& - 8h^2 s_2 x_n^2 + 3h^2 s_3^2 x^2 - 6h^2 s_3^2 x x_n + 3h^2 s_3^2 x_n^2 - 8h^2 s_3 x^2 + 16h^2 s_3 x x_n - 8h^2 s_3 x_n^2 \\
& + 3hs_1 x^3 - 9hs_1 x^2 x_n + 9hs_1 x x_n^2 - 3hs_1 x_n^3 + 3hs_2 x^3 - 9hs_2 x^2 x_n + 9hs_2 x x_n^2 \\
& + 3h^2 s_2^2 x^2 - 9hs_3 x^2 x_n + 9hs_3 x x_n^2 - 3hs_3 x_n^3 + 6hx^3 - 18hx^2 x_n + 18hx x_n^2 \\
& + 20h^3 s_1 s_3 x - 3hs_2 x_n^3 - 5h^2 s_1 s_2 x_n^2 + 16h^2 s_2 x x_n + 15h^3 s_1 s_2 s_3 x - 8h^3 s_3^2 x \\
& + 8h^3 s_2^2 x_n - 18x^2 x_n^2 + 12x x_n^3 - 6hx_n^3),
\end{aligned}$$

$$
\begin{aligned}
\beta_{s_1} &= -\frac{((x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3))}{(5040h^4s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3h^4s_1^4+3x^4\\
&- 3h^4s_1^3s_3 - 6h^4s_1^3 - 3h^4s_1^2s_2^2 + 5h^4s_1^2s_2s_3 + 8h^4s_1^2s_2 - 3h^4s_1^2s_3^2 + 8h^4s_1^2s_3\\
&+ 5h^4s_1s_2^2s_3 + 8h^4s_1s_2^2 + 5h^4s_1s_2s_3^2 - 20h^4s_1s_2s_3 - 3h^4s_1s_3^3 + 8h^4s_1s_3^2 - 3h^4s_2^4\\
&+ 5h^4s_2^3s_3 + 8h^4s_2^3 + 5h^4s_2^2s_3^2 - 20h^4s_2^2s_3 + 5h^4s_2s_3^3 - 20h^4s_2s_3^2 - 3h^4s_3^4\\
&+ 3h^3s_1^3x - 3h^3s_1^3x_n - 3h^3s_1^2s_2x + 3h^3s_1^2s_2x_n - 3h^3s_1^2s_3x + 3h^3s_1^2s_3x_n\\
&+ 6h^3s_1^2x_n - 3h^3s_1s_2^2x + 3h^3s_1s_2^2x_n + 5h^3s_1s_2s_3x - 5h^3s_1s_2s_3x_n + 8h^3s_1s_2x\\
&- 3h^3s_1s_3^2x + 3h^3s_1s_3^2x_n + 8h^3s_1s_3x - 8h^3s_1s_3x_n - 3h^3s_2^3x + 3h^3s_2^3x_n + 6hx_n^3\\
&+ 5h^3s_2s_3^2x - 5h^3s_2s_3^2x_n - 20h^3s_2s_3x + 20h^3s_2s_3x_n - 3h^3s_3^3x + 3h^3s_3^3x_n\\
&- 8h^3s_3^2x_n + 3h^2s_1^2x^2 - 6h^2s_1^2xx_n + 3h^2s_1^2x_n^2 - 3h^2s_1s_3x^2 + 8h^2s_2x_n^2\\
&+ 6h^2s_1s_3xx_n - 3h^2s_1s_3x_n^2 - 6h^2s_1x^2 + 12h^2s_1xx_n - 6h^2s_1x_n^2 - 3h^2s_2^2x^2\\
&- 3h^2s_2^2x_n^2 + 5h^2s_2s_3x^2 - 10h^2s_2s_3xx_n + 5h^2s_2s_3x_n^2 + 8h^2s_2x^2 - 16h^2s_2xx_n\\
&+ 6h^2s_3^2xx_n - 3h^2s_3^2x_n^2 + 8h^2s_3x^2 - 16h^2s_3xx_n + 8h^2s_3x_n^2 + 3hs_1x^3 - 9hs_1x^2x_n\\
&- 3hs_2x^3 + 9hs_2x^2x_n - 9hs_2xx_n^2 + 3hs_2x_n^3 - 3hs_3x^3 + 9hs_3x^2x_n\\
&+ 9hs_1xx_n^2 - 6hx^3 - 3h^2s_3^2x^2 - 3h^2s_1s_2x^2 + 6h^2s_1s_2xx_n - 3h^2s_1s_2x_n^2 - 3hs_1x_n^3\\
&- 5h^3s_2^2s_3x_n + 8h^3s_2^2x - 8h^3s_2^2x_n + 18hx^2x_n - 18hxx_n^2 - 12x^3x_n + 18x^2x_n^2\\
&- 3h^4s_1^3s_2 - 3h^4s_1s_2^3 + 5h^3s_2^2s_3x + 8h^3s_3^2x + 6h^2s_2^2xx_n + 3hs_3x_n^3 + 8h^4s_3^3\\
&- 6h^3s_1^2x - 12xx_n^3 - 8h^3s_1s_2x_n - 9hs_3xx_n^2 + 3x_n^4),\\
\beta_{s_2} &= \frac{(x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3)}{(5040h^4s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(3h^2s_1s_2x_n^2+20h^4s_1^2s_3\\
&- 5h^4s_1^3s_3-8h^4s_1^3+3h^4s_1^2s_2^2-5h^4s_1^2s_2s_3-8h^4s_1^2s_2-5h^4s_1^2s_3^2+3h^4s_3^4+3h^4s_1^3s_2\\
&+ 3h^4s_1s_2^3-5h^4s_1s_2^2s_3-8h^4s_1s_2^2-5h^4s_1s_2s_3^2+20h^4s_1s_2s_3-5h^4s_1s_3^3-18hx^2x_n\\
&+ 20h^4s_1s_3^2-3h^4s_2^4+3h^4s_2^3s_3+6h^4s_2^3+3h^4s_2^2s_3^2-8h^4s_2^2s_3+3h^4s_2s_3^3+18hxx_n^2\\
&- 8h^4s_2s_3^2-8h^4s_3^3+3h^3s_1^3x-3h^3s_1^3x_n+3h^3s_1^2s_2x-3h^3s_1^2s_2x_n-3x^4+12x^3x_n\\
&+ 5h^3s_1^2s_3x_n-8h^3s_1^2x+8h^3s_1^2x_n+3h^3s_1s_2^2x-3h^3s_1s_2^2x_n-5h^3s_1s_2s_3x-18x^2x_n^2\\
&+ 5h^3s_1s_2s_3x_n-8h^3s_1s_2x+8h^3s_1s_2x_n-5h^3s_1s_3^2x+5h^3s_1s_3^2x_n-6hx_n^3+12xx_n^3\\
&- 20h^3s_1s_3x_n-3h^3s_2^3x+3h^3s_2^3x_n+3h^3s_2^2s_3x-3h^3s_2^2s_3x_n+6h^3s_2^2x-8h^2s_3x^2\\
&+ 3h^3s_2s_3^2x-3h^3s_2s_3^2x_n-8h^3s_2s_3x+8h^3s_2s_3x_n+3h^3s_3^3x-3h^3s_3^3x_n-8h^2s_1x_n^2\\
&+ 8h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2+3h^2s_1s_2x^2-6h^2s_1s_2xx_n+3h^4s_1^4\\
&- 5h^2s_1s_3x^2+10h^2s_1s_3xx_n-5h^2s_1s_3x_n^2-8h^2s_1x^2+16h^2s_1xx_n-3x_n^4-8h^3s_3^2x\\
&- 3h^2s_2^2x^2+6h^2s_2^2xx_n-3h^2s_2^2x_n^2+3h^2s_2s_3x^2-6h^2s_2s_3xx_n+3h^2s_2s_3x_n^2-6h^3s_2^2x_n\\
&+ 6h^2s_2x^2-12h^2s_2xx_n+6h^2s_2x_n^2+3h^2s_3^2x^2-6h^2s_3^2xx_n+3h^2s_3^2x_n^2+6hx^3+3hs_3x^3\\
&- 8h^2s_3x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3-3hs_2x^3-3hs_3x_n^3+3hs_2x_n^3\\
&+ 20h^3s_1s_3x-5h^3s_1^2s_3x+9hs_2x^2x_n-9hs_2xx_n^2-9hs_3x^2x_n+9hs_3xx_n^2+16h^2s_3xx_n),
\end{aligned}
$$

$$\beta_{s_3}=\frac{-(x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3))}{(5040h^4s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(3h^4s_1^4-5h^4s_1^3s_2$$
$$+3h^4s_1^3s_3-8h^4s_1^3-5h^4s_1^2s_2^2-5h^4s_1^2s_2s_3+20h^4s_1^2s_2+3h^4s_1^2s_3^2-8h^4s_1^2s_3$$
$$-5h^4s_1s_2^3-5h^4s_1s_2^2s_3+20h^4s_1s_2^2-5h^4s_1s_2s_3^2+20h^4s_1s_2s_3+3h^4s_1s_3^3-6hx_n^3$$
$$-8h^4s_1s_3^2+3h^4s_2^4+3h^4s_2^3s_3-8h^4s_2^3+3h^4s_2^2s_3^2-8h^4s_2^2s_3+3h^4s_2s_3^3-8h^4s_2s_3^2$$
$$-3h^4s_3^4+6h^4s_3^3+3h^3s_1^3x-3h^3s_1^3x_n-5h^3s_1^2s_2x+5h^3s_1^2s_2x_n+3h^3s_1^2s_3x-3x^4$$
$$-3h^3s_1^2s_3x_n-8h^3s_1^2x+8h^3s_1^2x_n-5h^3s_1s_2^2x+5h^3s_1s_2^2x_n-5h^3s_1s_2s_3x-3x_n^4$$
$$+5h^3s_1s_2s_3x_n+20h^3s_1s_2x-20h^3s_1s_2x_n+3h^3s_1s_3^2x-3h^3s_1s_3^2x_n-8h^3s_1s_3x$$
$$+8h^3s_1s_3x_n+3h^3s_2^3x-3h^3s_2^3x_n+3h^3s_2^2s_3x-3h^3s_2^2s_3x_n-8h^3s_2^2x+8h^3s_2^2x_n$$
$$+3h^3s_2s_3^2x-3h^3s_2s_3^2x_n-8h^3s_2s_3x+8h^3s_2s_3x_n-3h^3s_3^3x+3h^3s_3^3x_n+6h^3s_3^2x$$
$$-6h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2-5h^2s_1s_2x^2+10h^2s_1s_2xx_n+6hx^3$$
$$-5h^2s_1s_2x_n^2+3h^2s_1s_3x^2-6h^2s_1s_3xx_n+3h^2s_1s_3x_n^2-8h^2s_1x^2+16h^2s_1xx_n$$
$$-8h^2s_1x_n^2+3h^2s_2^2x^2-6h^2s_2^2xx_n+3h^2s_2^2x_n^2+3h^2s_2s_3x^2-6h^2s_2s_3xx_n+12xx_n^3$$
$$+3h^2s_2s_3x_n^2-8h^2s_2x^2+16h^2s_2xx_n-8h^2s_2x_n^2-3h^2s_3^2x^2+6h^2s_3^2xx_n-3h^2s_3^2x_n^2$$
$$+6h^2s_3x^2-12h^2s_3xx_n+6h^2s_3x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3$$
$$+3hs_2x^3-9hs_2x^2x_n+9hs_2xx_n^2-3hs_2x_n^3-3hs_3x^3+9hs_3x^2x_n-9hs_3xx_n^2$$
$$+3hs_3x_n^3-18hx^2x_n+18hxx_n^2+12x^3x_n-18x^2x_n^2),$$

$$\beta_1=-\frac{(x-x_n)(x_n-x+hs_3)(x_n-x+hs_1)(x_n-x+hs_2)}{(5040h^4(s_3-1)(s_2-1)(s_1-1))}(3h^4s_1^4-5h^4s_1^3s_2$$
$$-5h^4s_1^3s_3-5h^4s_1^2s_2^2+15h^4s_1^2s_2s_3-5h^4s_1^2s_3^2-5h^4s_1s_2^3+15h^4s_1s_2^2s_3-3hs_3x_n^3$$
$$-5h^4s_1s_3^3+3h^4s_2^4-5h^4s_2^3s_3-5h^4s_2^2s_3^2-5h^4s_2s_3^3+3h^4s_3^4+3h^3s_1^3x-3h^3s_1^3x_n$$
$$-5h^3s_1^2s_2x+5h^3s_1^2s_2x_n-5h^3s_1^2s_3x+5h^3s_1^2s_3x_n-5h^3s_1s_2^2x+5h^3s_1s_2^2x_n-3x_n^4$$
$$+15h^3s_1s_2s_3x-15h^3s_1s_2s_3x_n-5h^3s_1s_3^2x+5h^3s_1s_3^2x_n+3h^3s_2^3x-3h^3s_2^3x_n$$
$$+5h^3s_2^2s_3x_n-5h^3s_2s_3^2x+5h^3s_2s_3^2x_n+3h^3s_3^3x-3h^3s_3^3x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n$$
$$+3h^2s_1^2x_n^2-5h^2s_1s_2x^2+10h^2s_1s_2xx_n-5h^2s_1s_2x_n^2-5h^2s_1s_3x^2+10h^2s_1s_3xx_n$$
$$+3h^2s_2^2x^2-6h^2s_2^2xx_n+3h^2s_2^2x_n^2-5h^2s_2s_3x^2+10h^2s_2s_3xx_n-5h^2s_2s_3x_n^2$$
$$-6h^2s_3^2xx_n+3h^2s_3^2x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3+3hs_2x^3$$
$$-9hs_2x^2x_n+3h^2s_3^2x^2-5h^2s_1s_3x_n^2-5h^3s_2^2s_3x+9hs_2xx_n^2-3hs_2x_n^3+3hs_3x^3$$
$$+15h^4s_1s_2s_3^2-9hs_3x^2x_n+9hs_3xx_n^2-3x^4+12x^3x_n-18x^2x_n^2+12xx_n^3).$$

Equation (4) is evaluated at the non-interpolating point x_{n+1} while its derivatives are evaluated at all points in the selected interval to produce the discreet schemes. Discreet schemes and its derivatives at x_n are combined on a block of the form

$$A^{[3]_4}Y_m^{[3]_4} = \sum_{i=1}^{4} B_i^{[3]_4}R_i^{[3]_4} + h^4[D^{[3]_4}R_5^{[3]_4} + E^{[3]_4}R_6^{[3]_4}], \tag{5}$$

where

$$A^{[3]_4} = \begin{pmatrix} \frac{-(s_2-1)(s_3-1)}{s_1(s_1-s_2)(s_1-s_3)} & \frac{(s_1-1)(s_3-1)}{s_2(s_1-s_2)(s_2-s_3)} & \frac{-(s_1-1)(s_2-1)}{s_3(s_1-s_3)(s_2-s_3)} & 1 \\ \frac{-(s_2 s_3)}{hs_1(s_1-s_2)(s_1-s_3)} & \frac{(s_1 s_3)}{(hs_2(s_1-s_2)(s_2-s_3))} & \frac{-(s_1 s_2)}{hs_3(s_1-s_3)(s_2-s_3)} & 0 \\ \frac{(2s_2+2s_3)}{h^2 s_1(s_1-s_2)(s_1-s_3)} & \frac{-(2s_1+2s_3)}{h^2 s_2(s_1-s_2)(s_2-s_3)} & \frac{(2s_1+2s_2)}{h^2 s_3(s_1-s_3)(s_2-s_3)} & 0 \\ \frac{-6}{h^3 s_1(s_1-s_2)(s_1-s_3)} & \frac{6}{h^3 s_2(s_1-s_2)(s_2-s_3)} & \frac{-6}{h^3 s_3(s_1-s_3)(s_2-s_3)} & 0 \end{pmatrix},$$

$$Y_m^{[3]_4} = \begin{pmatrix} y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ y_{n+1} \end{pmatrix}, B_1^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{((s_1-1)(s_2-1)(s_3-1))}{(s_1 s_2 s_3)} \\ 0 & 0 & 0 & \frac{-(s_1 s_2+s_1 s_3+s_2 s_3)}{(hs_1 s_2 s_3)} \\ 0 & 0 & 0 & \frac{(2(s_1+s_2+s_3))}{(h^2 s_1 s_2 s_3)} \\ 0 & 0 & 0 & \frac{-6}{(h^3 s_1 s_2 s_3)} \end{pmatrix}, R_1^{[3]_4} = \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},$$

$$B_2^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2^{[3]_4} = \begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}, B_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_2^{[3]_4} = \begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix}, B_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, R_3^{[3]_4} = \begin{pmatrix} y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_n \end{pmatrix} \ R_5^{[3]_4} = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix},$$

$$R^{[3]_4} = \begin{pmatrix} f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}, D^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & D_{14}^{[3]_4} \\ 0 & 0 & 0 & D_{24}^{[3]_4} \\ 0 & 0 & 0 & D_{34}^{[3]_4} \\ 0 & 0 & 0 & D_{44}^{[3]_4} \end{pmatrix},$$

$$E^{[3]_4} = \begin{pmatrix} E_{11}^{[3]_4} & E_{12}^{[3]_4} & E_{13}^{[3]_4} & E_{14}^{[3]_4} \\ E_{21}^{[3]_4} & E_{22}^{[3]_4} & E_{23}^{[3]_4} & E_{24}^{[3]_4} \\ E_{31}^{[3]_4} & E_{32}^{[3]_4} & E_{33}^{[3]_4} & E_{34}^{[3]_4} \\ E_{41}^{[3]_4} & E_{42}^{[3]_4} & E_{43}^{[3]_4} & E_{44}^{[3]_4} \end{pmatrix}.$$

The elements of $D^{[3]_4}$ and $E^{[3]_4}$ are given in Appendix A.

Multiplying equation (5) by inverse of $A^{[3]_4}$ to have a hybrid block method of the form

$$I^{[3]_4} Y_m^{[3]_4} = \sum_{i=1}^{4} \bar{B}_i^{[3]_4} R_i^{[3]_4} + h^4[\bar{D}^{[3]_4} R_5^{[3]_2} + \bar{E}^{[3]_2} R_6^{[3]_4}], \tag{6}$$

where

$$I^{[3]_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \bar{B}_1^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \bar{B}_2^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & s_1 h \\ 0 & 0 & 0 & s_2 h \\ 0 & 0 & 0 & s_3 h \\ 0 & 0 & 0 & h \end{pmatrix},$$

$$\bar{B}_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{s_1^2}{2}h \\ 0 & 0 & 0 & \frac{s_2^2}{2}h \\ 0 & 0 & 0 & \frac{s_3^2}{2}h \\ 0 & 0 & 0 & \frac{1}{2}h \end{pmatrix},\ \bar{B}_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{s_1^3}{6}h \\ 0 & 0 & 0 & \frac{s_2^3}{6}h \\ 0 & 0 & 0 & \frac{s_3^3}{6}h \\ 0 & 0 & 0 & \frac{1}{6}h \end{pmatrix},\ \bar{D}^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_4} \end{pmatrix}$$

$$\bar{E}^{[3]_4} = \begin{pmatrix} \bar{E}_{11}^{[3]_4} & \bar{E}_{12}^{[3]_4} & \bar{E}_{13}^{[3]_4} & \bar{E}_{14}^{[3]_4} \\ \bar{E}_{21}^{[3]_4} & \bar{E}_{22}^{[3]_4} & \bar{E}_{23}^{[3]_4} & \bar{E}_{24}^{[3]_4} \\ \bar{E}_{31}^{[3]_4} & \bar{E}_{32}^{[3]_4} & \bar{E}_{33}^{[3]_4} & \bar{E}_{34}^{[3]_4} \\ \bar{E}_{41}^{[3]_4} & \bar{E}_{42}^{[3]_4} & \bar{E}_{43}^{[3]_4} & \bar{E}_{44}^{[3]_4} \end{pmatrix}$$

and the non-zero terms of $\bar{D}^{[3]_4}$ and $\bar{E}^{[3]_4}$ are given by

$$\bar{D}_{14}^{[3]_4} = -\frac{(s_1^4(28s_1s_2 + 28s_1s_3 - 168s_2s_3 - 8s_1^2s_2 - 8s_1^2s_3 - 8s_1^2 + 3s_1^3 + 28s_1s_2s_3))}{(5040s_2s_3)},$$

$$\bar{D}_{14}^{[3]_4} = \frac{(s_2^4(168s_1s_3 - 28s_1s_2 - 28s_2s_3 + 8s_1s_2^2 + 8s_2^2s_3 + 8s_2^2 - 3s_2^3 - 28s_1s_2s_3))}{(5040s_1s_3)},$$

$$\bar{D}_{14}^{[3]_4} = -\frac{(s_3^4(28s_1s_3 - 168s_1s_2 + 28s_2s_3 - 8s_1s_3^2 - 8s_2s_3^2 - 8s_3^2 + 3s_3^3 + 28s_1s_2s_3))}{(5040s_1s_2)},$$

$$\bar{D}_{14}^{[3]_4} = \frac{((8s_1 + 8s_2 + 8s_3 - 28s_1s_2 - 28s_1s_3 - 28s_2s_3 + 168s_1s_2s_3 - 3))}{(5040s_1s_2s_3)},$$

$$\bar{E}_{11}^{[3]_4} = \frac{(s_1^4(14s_1s_2 + 14s_1s_3 - 42s_2s_3 - 6s_1^2s_2 - 6s_1^2s_3 - 6s_1^2 + 3s_1^3 + 14s_1s_2s_3))}{(5040(s_1 - 1)(s_1 - s_3)(s_1 - s_2))},$$

$$\bar{E}_{12}^{[3]_4} = \frac{(s_1^6(28s_3 - 8s_1 - 8s_1s_3 + 3s_1^2))}{(5040s_2(s_2 - 1)(s_2 - s_3)(s_1 - s_2))},$$

$$\bar{E}_{13}^{[3]_4} = -\frac{(s_1^6(28s_2 - 8s_1 - 8s_1s_2 + 3s_1^2))}{(5040s_3(s_3 - 1)(s_2 - s_3)(s_1 - s_3))},$$

$$\bar{E}_{14}^{[3]_4} = \frac{(s_1^6(28s_2s_3 - 8s_1s_3 - 8s_1s_2 + 3s_1^2))}{(5040(s_3 - 1)(s_2 - 1)(s_1 - 1))},$$

$$\bar{E}_{21}^{[3]_4} = -\frac{(s_2^6(28s_3 - 8s_2 - 8s_2s_3 + 3s_2^2))}{(5040s_1(s_1 - 1)(s_1 - s_3)(s_1 - s_2))},$$

$$\bar{E}_{22}^{[3]_4} = \frac{(s_2^4(42s_1s_3 - 14s_1s_2 - 14s_2s_3 + 6s_1s_2^2 + 6s_2^2s_3 + 6s_2^2 - 3s_2^3 - 14s_1s_2s_3))}{(5040(s_2 - 1)(s_2 - s_3)(s_1 - s_2))},$$

$$\bar{E}_{23}^{[3]_4} = \frac{(s_2^6(8s_2 - 28s_1 + 8s_1s_2 - 3s_2^2))}{(5040s_3(s_3 - 1)(s_2 - s_3)(s_1 - s_3))},$$

$$\bar{E}_{24}^{[3]_4} = -\frac{(s_2^6(8s_1s_2 - 28s_1s_3 + 8s_2s_3 - 3s_2^2))}{(5040(s_3 - 1)(s_1 - 1)(s_2 - 1))},$$

$$\bar{E}_{31}^{[3]_4} = \frac{(s_3^6(8s_3 - 28s_2 + 8s_2s_3 - 3s_3^2))}{(5040s_1(s_1 - 1)(s_1 - s_3)(s_1 - s_2))},$$

$$\bar{E}_{32}^{[3]_4} = -\frac{(h^4s_3^6(8s_3 - 28s_1 + 8s_1s_3 - 3s_3^2))}{(5040s_2(s_2 - 1)(s_2 - s_3)(s_1 - s_2))},$$

$$\bar{E}_{33}^{[3]_4} = \frac{(s_3^4(14s_1s_3 - 42s_1s_2 + 14s_2s_3 - 6s_1s_3^2 - 6s_2s_3^2 - 6s_3^2 + 3s_3^3 + 14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))},$$
$$\bar{E}_{34}^{[3]_4} = \frac{(s_3^6(28s_1s_2 - 8s_1s_3 - 8s_2s_3 + 3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))},$$
$$\bar{E}_{41}^{[3]_4} = -\frac{((28s_2s_3 - 8s_3 - 8s_2 + 3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))},$$
$$\bar{E}_{42}^{[3]_4} = \frac{((28s_1s_3 - 8s_3 - 8s_1 + 3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))},$$
$$\bar{E}_{43}^{[3]_4} = -\frac{((28s_1s_2 - 8s_2 - 8s_1 + 3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))},$$
$$\bar{E}_{44}^{[3]_4} = \frac{((6s_1 + 6s_2 + 6s_3 - 14s_1s_2 - 14s_1s_3 - 14s_2s_3 + 42s_1s_2s_3 - 3))}{(5040(s_3-1)(s_2-1)(s_1-1))}.$$

Equation (5) can also be written as

$$
\begin{aligned}
y_{n+s_1} &= y_n + s_1hy'_n + \frac{s_1^2h^2}{2}y''_n + \frac{s_1^3h^3}{6}y'''_n \\
&\quad - \frac{\begin{aligned}(h^4s_1^4(&28s_1s_2 + 28s_1s_3 - 168s_2s_3 - 8s_1^2s_2 - 8s_1^2s_3 \\ &- 8s_1^2 + 3s_1^3 + 28s_1s_2s_3))\end{aligned}}{(5040s_2s_3)}f_n \\
&\quad + \frac{\begin{aligned}(h^4s_1^4(&14s_1s_2 + 14s_1s_3 - 42s_2s_3 - 6s_1^2s_2 - 6s_1^2s_3 - 6s_1^2 \\ &+ 3s_1^3 + 14s_1s_2s_3))\end{aligned}}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&\quad + \frac{(h^4s_1^6(28s_3 - 8s_1 - 8s_1s_3 + 3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&\quad - \frac{(h^4s_1^6(28s_2 - 8s_1 - 8s_1s_2 + 3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&\quad + \frac{(h^4s_1^6(28s_2s_3 - 8s_1s_3 - 8s_1s_2 + 3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}f_{n+1}, \\
y_{n+s_2} &= y_n + s_2hy'_n + \frac{s_2^2h^2}{2}y''_n + \frac{s_2^3h^3}{6}y'''_n \\
&\quad + \frac{\begin{aligned}(h^4s_2^4(&168s_1s_3 - 28s_1s_2 - 28s_2s_3 + 8s_1s_2^2 + 8s_2^2s_3 \\ &+ 8s_2^2 - 3s_2^3 - 28s_1s_2s_3))\end{aligned}}{(5040s_1s_3)}f_n \\
&\quad - \frac{(h^4s_2^6(28s_3 - 8s_2 - 8s_2s_3 + 3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&\quad + \frac{\begin{aligned}(h^4s_2^4(&42s_1s_3 - 14s_1s_2 - 14s_2s_3 + 6s_1s_2^2 + 6s_2^2s_3 \\ &+ 6s_2^2 - 3s_2^3 - 14s_1s_2s_3))\end{aligned}}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&\quad + \frac{(h^4s_2^6(8s_2 - 28s_1 + 8s_1s_2 - 3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&- \frac{(h^4 s_2^6(8s_1s_2 - 28s_1s_3 + 8s_2s_3 - 3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))} f_{n+1}, \qquad (8)\\
y_{n+s_3} &= y_n - s_3 h y_n' - \frac{s_3^2h^2}{2} y_n'' - \frac{s_3^3h^3}{6} y_n'''\\
&- \frac{\begin{aligned}(h^4 s_3^4(&28s_1s_3 - 168s_1s_2 + 28s_2s_3 - 8s_1s_3^2 - 8s_2s_3^2\\ &- 8s_3^2 + 3s_3^3 + 28s_1s_2s_3))\end{aligned}}{(5040s_1s_2)} f_n\\
&+ \frac{(h^4 s_3^6(8s_3 - 28s_2 + 8s_2s_3 - 3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1}\\
&- \frac{(h^4 s_3^6(8s_3 - 28s_1 + 8s_1s_3 - 3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2}\\
&+ \frac{\begin{aligned}(h^4 s_3^4(&14s_1s_3 - 42s_1s_2 + 14s_2s_3 - 6s_1s_3^2 - 6s_2s_3^2 - 6s_3^2\\ &+ 3s_3^3 + 14s_1s_2s_3))\end{aligned}}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3}\\
&+ \frac{(h^4 s_3^6(28s_1s_2 - 8s_1s_3 - 8s_2s_3 + 3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))} f_{n+1}, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1} &= y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n'''\\
&+ \frac{(h^4(8s_1 + 8s_2 + 8s_3 - 28s_1s_2 - 28s_1s_3 - 28s_2s_3 + 168s_1s_2s_3 - 3))}{(5040s_1s_2s_3)} f_n\\
&- \frac{(h^4(28s_2s_3 - 8s_3 - 8s_2 + 3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1}\\
&+ \frac{(h^4(28s_1s_3 - 8s_3 - 8s_1 + 3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2}\\
&- \frac{(h^4(28s_1s_2 - 8s_2 - 8s_1 + 3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3}\\
&+ \frac{(h^4(6s_1 + 6s_2 + 6s_3 - 14s_1s_2 - 14s_1s_3 - 14s_2s_3 + 42s_1s_2s_3 - 3))}{(5040(s_3-1)(s_2-1)(s_1-1))} f_{n+1} \qquad (10)
\end{aligned}
$$

3. Analysis of the method

3.1 Order of method

Applying the process of finding the order of a linear multistep method proposed by lambert (1973). the order of the method are found by expanding y and f-function in Taylor series

$$\left[\begin{array}{c}
\sum_{j=0}^{\infty} \frac{(s_1)^j h^j}{j!} y_n^j - y_n - s_1 h y_n' - \frac{s_1^2 h^2}{2} y_n'' - \frac{s_1^3 h^3}{6} y_n''' \\
+\frac{(h^4 s_1^4(28s_1s_2+28s_1s_3-168s_2s_3-8s_1^2s_2-8s_1^2s_3-8s_1^2+3s_1^3+28s_1s_2s_3))}{(5040s_2s_3)} y_n^{iv} \\
-\frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+4}}{j!} y_n^{j+4} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4} \\
\sum_{j=0}^{\infty} \frac{(s_2)^j h^j}{j!} y_n^j - y_n - s_2 h y_n' - \frac{s_2^2 h^2}{2} y_n'' - \frac{s_2^3 h^3}{6} y_n''' \\
-\frac{(h^4 s_2^4(168s_1s_3-28s_1s_2-28s_2s_3+8s_1s_2^2+8s_2^2s_3+8s_2^2-3s_2^3-28s_1s_2s_3))}{(5040s_1s_3)} y_n^{iv} \\
+\frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+4}}{j!} y_n^{j+4} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4} \\
\sum_{j=0}^{\infty} \frac{(s_3)^j h^j}{j!} y_n^j - y_n - s_3 h y_n' - \frac{s_3^2 h^2}{2} y_n'' - \frac{s_3^3 h^3}{6} y_n''' \\
+\frac{(h^4 s_3^4(28s_1s_3-168s_1s_2+28s_2s_3-8s_1s_3^2-8s_2s_3^2-8s_3^2+3s_3^3+28s_1s_2s_3))}{(5040s_1s_2)} y_n^{iv} \\
-\frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+4}}{j!} y_n^{j+4} \\
+\frac{(h^4 s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4} \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - h y_n' - \frac{h^2}{2} y_n'' - \frac{h^3}{6} y_n''' \\
-\frac{(h^4(8s_1+8s_2+8s_3-28s_1s_2-28s_1s_3-28s_2s_3+168s_1s_2s_3-3))}{(5040s_1s_2s_3)} y_n^{iv} \\
+\frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_1)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))} \sum_{j=0}^{\infty} \frac{(s_2)^j h^{j+4}}{j!} y_n^{j+4} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))} \sum_{j=0}^{\infty} \frac{(s_3)^j h^{j+4}}{j!} y_n^{j+4} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))} \sum_{j=0}^{\infty} \frac{h^{j+4}}{j!} y_n^{j+4}
\end{array}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Collecting like terms ($\bar{C}_i\ 's$) to h gives $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_8 = 0$, and $\bar{C}_{5+4} \neq 0$. Hence, the new method is of order$[5,5,5,5]^T$ with error constant

$$\bar{C}_9 = \begin{bmatrix} -\dfrac{(s_1^6(24s_1s_2 + 24s_1s_3 - 84s_2s_3 - 9s_1^2s_2 - 9s_1^2s_3 - 9s_1^2 + 4s_1^3 + 24s_1s_2s_3))}{1814400} \\ \dfrac{(s_2^6(84s_1s_3 - 24s_1s_2 - 24s_2s_3 + 9s_1s_2^2 + 9s_2^2s_3 + 9s_2^2 - 4s_2^3 - 24s_1s_2s_3))}{1814400} \\ -\dfrac{(s_3^6(24s_1s_3 - 84s_1s_2 + 24s_2s_3 - 9s_1s_3^2 - 9s_2s_3^2 - 9s_3^2 + 4s_3^3 + 24s_1s_2s_3))}{1814400} \\ \dfrac{(9s_1 + 9s_2 + 9s_3 - 24s_1s_2 - 24s_1s_3 - 24s_2s_3 + 84s_1s_2s_3 - 4)}{1814400} \end{bmatrix}$$

which is true for all

$$\begin{aligned} & s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{9s_1^2s_3 + 9s_1^2 - 4s_1^3 - 24s_1s_3}{24s_1 - 84s_3 - 9s_1^2 + 24s_1s_3}\} \\ & \cup \{s_1 = \frac{24s_2s_3 - 9s_2^2s_3 - 9s_2^2 + 4s_2^3}{84s_3 - 24s_2 + 9s_2^2 - 24s_2s_3}\} \\ & \cup \{s_2 = \frac{-24s_1s_3 + 9s_1s_3^2 + 9s_3^2 - 4s_3^3}{-84s_2 + 24s_3 - 9s_3^2 + 24s_1s_3}\} \\ & \cup \{s_1 = \frac{-9s_2 - 9s_3 + 24s_2s_3 + 4}{9 - 24s_2 - 24s_3 + 84s_2s_3}\}. \end{aligned}$$

3.2 Zero stability

In finding the zero stability of the method, definition in Fatunla (1991) is used. This is

$$\begin{aligned} \Pi(r) &= |r\ I - \bar{B}_1^{[3]_4}| \\ &= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\ &= r^3(r-1), \end{aligned}$$

which gives $r = 0, 0, 0, 1$. According to (Fatunla, 1991), (Lambert, 1973) and (Henrici, 1962) our method is zero stable, consistent and then convergent.

3.3 Region of absolute stability

In this subsection, the locus boundary method is used to confirm the absolute stability interval. By substituting test equation $y'''' = -\lambda^4 y$ in (**??**) where $\bar{h} = \lambda^4 h^4$ and $\lambda = \frac{df}{dy}$. let $r = \cos\theta - i\sin\theta$ and considering real part yields the equation of absolute stability region.

$$\bar{h}(\theta, h) = \frac{60963840000(\cos(\theta) - 1)}{(s_1^3s_2^3s_3^3(20s_1 + 20s_2 + 20s_3 - 10s_1s_2 - 10s_1s_3 - 10s_2s_3 + 4s_1s_2s_3 + s_1s_2s_3\cos(\theta) - 35))}.$$

4. Numerical experimental

In this part, the following linear and non-linear IVPs available in the previous literatures were also solved to specific off step points $x_{n+\frac{1}{4}}$, $x_{n+\frac{2}{4}}$ and $x_{n+\frac{3}{4}}$ in order to compare the performance of the new method with existing ones. Computed solution (COP) , exact solution (EXT) and absolute errors (ERR) were carried out using flexible Matlab code. This is clear in Table 1 and Table 2.

Problem 1:

$$y^{iv}-(y')^2+yy''+4x^2-e^x(1-4x+x^2)=0, y(0)=1, y'(0)=1, y''(0)=1, h=\frac{1}{100}.$$

Exact solution: $y(x)=x^2+e^x$.

Problem 2:

$$y^{iv}-x=0, y(0)=0, y'(0)=1, y''(0)=0, y'''(0)=0, h=\frac{1}{320}.$$

Exact solution: $y(x)=\frac{x^5}{120}+x$.

Table 1: Comparison of the new method with (Olabode et al,2015) for solving problem 1, where $h=\frac{1}{320}$

x		New method, $P=5$	Olabode and Omole (2015), $P=6$
0.0031250	EXT	1.0031396535277390	1.003139653527739149
	CPS	1.0031396535277390	1.003139653526590265
	ERR	$0.000000e^{+00}$	$1.148884e^{-12}$
0.0062500	EXT	1.0063086345037620	1.006308634503762010
	CPS	1.0063086345037617	1.006308634484910542
	ERR	$2.220446e^{-16}$	$1.8851468e^{-11}$
0.0093750	EXT	1.0095069735890709	1.009506973589071086
	CPS	1.0095069735890709	1.009506973491318106
	ERR	$0.000000e^{+00}$	$9.7752980e^{-11}$
0.0125000	EXT	1.0127347015406345	1.012734701540634377
	CPS	1.0127347015406303	1.0127347015406341
	ERR	$4.440892e^{-16}$	$3.15759129e^{-10}$
0.0156250	EXT	1.0159918492116857	1.015991849211685747
	CPS	1.0159918492116851	1.015991848424806972
	Error	$6.661338e^{-16}$	$1.15463e^{-10}$

Table 2: Comparison of the new method with (Kayode et al, 2014) for solving problem 3, where $h = \frac{1}{10}$

x		New method, $P = 5$	(kyode et al,2014), $P = 8$
0.1	EXT	0.100000083333333340	0.100000083333334000
	CPS	0.100000083333333340	0.10000008333351720
	ERR	$0.000000e^{+00}$	$1.832e^{-13}$
0.2	EXT	0.200002666666666690	0.200002666666666900
	CPS	0.200002666666666660	0.20000266667150250
	ERR	$2.775558e^{-17}$	$4.835e^{-12}$
0.3	EXT	0.300020250000000040	0.300020250000000004
	CPS	0.300020249999999990	0.30002025000721480
	ERR	$5.551115e^{-17}$	$7.214e^{-12}$
0.4	EXT	0.400085333333333350	0.400008533333333333
	CPS	0.400085333333333350	0.40000853340160457
	ERR	$0.000000e^{+00}$	$6.832e^{-11}$
0.5	EXT	0.500260416666666650	0.500260416666666665
	CPS	0.500260416666666650	0.50026041674083458
	ERR	$0.000000e^{+00}$	$7.416e^{-11}$

5. Conclusion

A one step hybrid block method with three generalized off step points for solving linear and no-linear fourth order initial value problem has been developed in this article. The numerical properties of the new method are also established. The method competes better than its counterparts in terms of accuracy when solving fourth order initial value problems.

Appendix A:

$$\begin{aligned}
D_{14}^{[3]_4} &= \frac{-(s_1-1)(s_2-1)(s_3-1)}{(5040 s_1 s_2 s_3)}(5s_1^3 s_2 + 5s_1^3 s_3 + 5s_1 - 15 s_1 s_2^2 s_3 + 5s_3^2 \\
&+ 5s_3 - 3s_1^4 + 5s_1^3 + 5s_1^2 s_2^2 - 15 s_1^2 s_2 s_3 - 15 s_1^2 s_2 + 5 s_1^2 s_3^2 - 15 s_1^2 s_3 \\
&+ 5s_1^2 + 5s_1 s_2^3 - 3s_2^4 + 5s_2^3 - 15 s_1 s_2^2 - 15 s_1 s_2 s_3^2 - 15 s_1 s_2 \\
&+ 5 s_1 s_3^3 - 15 s_1 s_3^2 - 15 s_1 s_3 + 5 s_3^3 - 3 s_3^4 + 5 s_2 \\
&+ 105 s_1 s_2 s_3 + 5 s_2^3 s_3 + 5 s_2^2 s_3^2 - 15 s_2^2 s_3 \\
&+ 5 s_2^2 + 5 s_2 s_3^3 - 15 s_2 s_3^2 - 15 s_2 s_3 - 3), \\
D_{24}^{[3]_4} &= \frac{1}{5040h}(3s_1^4 - 5s_1^3 s_2 + 15 s_1 s_2 s_3^2 - 120 s_1 s_2 s_3 - 5 s_1 s_3^3 + 20 s_1 s_3^2 + 3 s_2^4 \\
&- 5 s_2^3 s_3 - 8 s_1^3 - 5 s_1^2 s_2^2 + 15 s_1^2 s_2 s_3 + 20 s_1^2 s_2 - 5 s_1^2 s_3^2 + 20 s_1^2 s_3 \\
&- 5 s_1 s_2^3 + 15 s_1 s_2^2 s_3 + 20 s_1 s_2^2 - 8 s_2^3 - 5 s_1^3 s_3 \\
&- 5 s_2^2 s_3^2 + 3 s_3^4 + 20 s_2^2 s_3 - 5 s_2 s_3^3 + 20 s_2 s_3^2 - 8 s_3^3),
\end{aligned}$$

$$
\begin{aligned}
D_{34}^{[3]_4} &= \frac{-1}{(2520h^2 s_1 s_2 s_3)}(3s_1^5 s_2 + 3s_1^5 s_3 - 5s_1^4 s_2^2 - 10s_1^4 s_2 s_3 - 8s_1^4 s_2 \\
&\quad - 5s_1^4 s_3^2 + 20s_1^3 s_3^2 - 8s_1^4 s_3 - 5s_1^3 s_2^3 + 10s_1^3 s_2^2 s_3 \\
&\quad + 20s_1^3 s_2^2 + 10s_1^3 s_2 s_3^2 + 40s_1^3 s_2 s_3 - 5s_1^3 s_3^3 - 5s_1^2 s_2^4 \\
&\quad - 5s_2^2 s_3^4 + 10s_1^2 s_2 s_3^3 - 100s_1^2 s_2 s_3^2 - 5s_1^2 s_3^4 + 20s_1^2 s_3^3 \\
&\quad + 3s_1 s_2^5 - 10s_1 s_2^4 s_3 - 8s_1 s_2^4 + 10s_1^2 s_2^3 s_3 \\
&\quad + 10s_1 s_2^3 s_3^2 + 40s_1 s_2^3 s_3 + 10s_1 s_2^2 s_3^3 - 100s_1 s_2^2 s_3^2 - 10s_1 s_2 s_3^4 \\
&\quad + 40s_1 s_2 s_3^3 + 20s_1^2 s_2^3 + 3s_1 s_3^5 - 8s_1 s_3^4 + 3s_2^5 s_3 - 5s_2^4 s_3^2 \\
&\quad - 8s_2^4 s_3 - 5s_2^3 s_3^3 + 20s_2^3 s_3^2 + 30s_1^2 s_2^2 s_3^2 - 100s_1^2 s_2^2 s_3 \\
&\quad + 20s_2^2 s_3^3 + 3s_2 s_3^5 - 8s_2 s_3^4), \\
D_{44}^{[3]_4} &= \frac{1}{(840h^3 s_1 s_2 s_3)}(3s_1^5 - 5s_1^4 s_2 - 5s_1^4 s_3 - 8s_1^4 - 5s_1^3 s_2^2 \\
&\quad + 15s_1^3 s_2 s_3 + 20s_2^2 s_3^2 + 20s_1^3 s_2 - 5s_1^3 s_3^2 + 20s_1^3 s_3 \\
&\quad - 5s_1^2 s_2^3 + 15s_1^2 s_2^2 s_3 + 20s_1^2 s_2^2 + 15s_1^2 s_2 s_3^2 - 120s_1^2 s_2 s_3 \\
&\quad - 5s_1^2 s_3^3 + 20s_1^2 s_3^2 - 5s_1 s_2^4 + 15s_1 s_2^3 s_3 + 20s_1 s_2^3 + 15s_1 s_2^2 s_3^2 \\
&\quad - 120s_1 s_2^2 s_3 + 15s_1 s_2 s_3^3 - 5s_2 s_3^4 - 120s_1 s_2 s_3^2 - 5s_1 s_3^4 \\
&\quad + 20s_1 s_3^3 - 5s_2^4 s_3 - 8s_2^4 - 5s_2^3 s_3^2 + 20s_2^3 s_3 - 5s_2^2 s_3^3 \\
&\quad + 20s_2 s_3^3 + 3s_3^5 - 8s_3^4 + 3s_2^5) \\
E_{11}^{[3]_4} &= \frac{-(s_2 - 1)(s_3 - 1)}{5040 s_1 (s_1 - s_2)(s_1 - s_3)}(3s_1^4 - 3s_1^3 s_2 - 3s_1^3 s_3 - 3s_1^3 \\
&\quad - 3s_1^2 s_2^2 + 5s_1^2 s_2 s_3 + 5s_1^2 s_2 - 3s_1^2 s_3^2 + 5s_1^2 s_3 - 3s_1^2 - 3s_1 s_2^3 + 5s_1 s_2^2 s_3 \\
&\quad + 5s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 - 3s_1 s_3^3 + 5s_1 s_3^2 \\
&\quad + 5s_1 s_3 - 3s_1 - 3s_2^4 + 5s_2^3 s_3 + 5s_2^3 + 5s_2^2 s_3^2 - 15s_2^2 s_3 + 5s_2^2 + 5s_2^2 \\
&\quad + 5s_1 s_2 + 5s_2 s_3^3 - 15s_2 s_3^2 - 15s_2 s_3 + 5s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 + 5s_3 - 3), \\
E_{12}^{[3]_4} &= \frac{(s_1 - 1)(s_3 - 1)}{5040 s_2 (s_1 - s_2)(s_2 - s_3)}(5s_1^3 - 3s_1^4 - 3s_1^3 s_2 \\
&\quad + 5s_1^3 s_3 - 3s_1^2 s_2^2 + 5s_1^2 s_2 s_3 + 5s_1^2 s_3^2 - 15s_1^2 s_3 + 5s_1^2 - 3s_1 s_2^3 + 5s_1 s_2^2 s_3 \\
&\quad + 5s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 + 5s_1 s_2 + 5s_1 s_3^3 - 15s_1 s_3^2 - 15s_1 s_3 \\
&\quad + 5s_1 + 3s_2^4 - 3s_2^3 s_3 - 3s_2^3 - 3s_2^2 s_3^2 + 5s_2^2 s_3 - 3s_2^2 + 5s_3 \\
&\quad + 5s_1^2 s_2 - 3s_2 s_3^3 + 5s_2 s_3^2 + 5s_2 s_3 - 3s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 - 3), \\
E_{13}^{[3]_4} &= -\frac{(s_1 - 1)(s_2 - 1)}{(5040 s_3 (s_1 - s_3)(s_2 - s_3)}(-3s_1^4 + 5s_1^3 s_2 - 3s_1^3 s_3 + 5s_1^3 \\
&\quad + 5s_1^2 s_2^2 + 5s_1^2 s_2 s_3 - 15s_1^2 s_2 - 3s_1^2 s_3^2 + 5s_1^2 s_3 + 5s_1^2 + 5s_1 s_2^3 \\
&\quad + 5s_1 s_2^2 s_3 - 15s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 - 15s_1 s_2 - 3s_1 s_3^3 \\
&\quad + 5s_1 s_3^2 + 5s_1 s_3 + 5s_1 - 3s_2^4 - 3s_2^3 s_3 + 5s_2^3 - 3s_2^2 s_3^2 + 5s_2^2 s_3 + 5s_2^2 \\
&\quad - 3s_2 s_3^3 + 5s_2 s_3^2 - 3s_3 + 5s_2 s_3 + 5s_2 + 3s_3^4 - 3s_3^3 - 3s_3^2 - 3),
\end{aligned}
$$

$$
\begin{aligned}
E_{14}^{[3]_4} &= \frac{1}{5040}(-3s_1^4+5s_1^3s_2+5s_1^3s_3-3s_1^3+5s_1^2s_2^2-15s_1^2s_2s_3\\
&+5s_1^2s_2+5s_1^2s_3^2-3s_1^2+5s_1s_2^3-15s_1s_2^2s_3+5s_1s_2^2\\
&-15s_1s_2s_3^2-15s_1s_2s_3+5s_1s_2+5s_1s_3^3+5s_1s_3^2\\
&+5s_1s_3-3s_1-3s_2^4+5s_2^3s_3-3s_2^3+5s_2^2s_3^2+5s_2^2s_3-3s_2^2+5s_2s_3^3+5s_2s_3^2\\
&+5s_2s_3+5s_1^2s_3-3s_2-3s_3^4-3s_3^3-3s_3^2-3s_3+3)\\
E_{21}^{[3]_4} &= -\frac{s_2s_3}{5040h(s_1-s_2)(s_1-s_3)(s_1-1)}(-3s_1^3s_2-3s_1^3s_3-3s_1^2s_2^2\\
&+8s_3^3+5s_2^2s_3^2+5s_1^2s_2s_3+8s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3-3s_1s_2^3\\
&+5s_1s_2^2s_3+8s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3\\
&+3s_1^4-6s_1^3-3s_1s_3^3+8s_1s_3^2-3s_2^4+5s_2^3s_3+8s_2^3-20s_2^2s_3\\
&+5s_2s_3^3-20s_2s_3^2-3s_3^4),\\
E_{22}^{[3]_4} &= \frac{s_1s_3}{5040h(s_1-s_2)(s_2-s_3)(s_2-1)}(5s_1^3s_3-3s_1^3s_2-3s_1^2s_2^2\\
&-3s_3^4+5s_1^2s_2s_3+8s_1^3+8s_1^2s_2+5s_1^2s_3^2-20s_1^2s_3-3s_1s_2^3+5s_1s_2^2s_3\\
&+8s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3-3s_1^4+5s_1s_3^3-20s_1s_3^2\\
&+3s_2^4-3s_2^3s_3-6s_2^3-3s_2^2s_3^2+8s_2^2s_3-3s_2s_3^3+8s_2s_3^2+8s_3^3),\\
E_{23}^{[3]_4} &= -\frac{s_1s_2}{5040h(s_1-s_3)(s_2-s_3)(s_3-1)}(3s_3^4+5s_1^3s_2-3s_1^3s_3\\
&+5s_1^2s_2^2+5s_1^2s_2s_3-20s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3+5s_1s_2^3+5s_1s_2^2s_3\\
&-20s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3-3s_1s_3^3+8s_1^3+8s_1s_3^2\\
&-3s_2^4-3s_2^3s_3+8s_2^3-3s_2^2s_3^2+8s_2^2s_3-3s_2s_3^3+8s_2s_3^2-6s_3^3-3s_1^4),\\
E_{24}^{[3]_4} &= \frac{s_1s_2s_3}{5040h(s_1-1)(s_2-1)(s_3-1)}(5s_2^2s_3^25s_2^3s_3-3s_1^4+5s_1^3s_2\\
&+5s_1^3s_3+5s_1^2s_2^2-15s_1^2s_2s_3+5s_1^2s_3^2+5s_1s_2^3-15s_1s_2^2s_3-15s_1s_2s_3^2\\
&+5s_1s_3^3-3s_2^4+5s_2s_3^3-3s_3^4),\\
E_{31}^{[3]_4} &= \frac{1}{(2520h^2s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^5s_2+3s_1^5s_3\\
&-3s_1^4s_2^2-6s_1^4s_2s_3-6s_1^4s_2-3s_1^3s_2^3+2s_1^3s_2^2s_3+8s_1^3s_2^2\\
&+2s_1^3s_2s_3^2+16s_1^3s_2s_3-3s_1^3s_3^3+8s_1^3s_3^2-3s_1^4s_3^2\\
&-3s_1^2s_2^4+2s_1^2s_2^3s_3+8s_1^2s_2^3+10s_1^2s_2^2s_3^2-12s_1^2s_2^2s_3\\
&+2s_1^2s_2s_3^3-12s_1^2s_2s_3^2-6s_1^4s_3\\
&-3s_1^2s_3^4+8s_1^2s_3^3-3s_1s_2^5+2s_1s_2^4s_3+8s_1s_2^4+10s_1s_2^3s_3^2\\
&-12s_1s_2^3s_3+5s_2^4s_3^2+10s_1s_2^2s_3^3-40s_1s_2^2s_3^2+2s_1s_2s_3^4\\
&-12s_1s_2s_3^3-3s_1s_3^5+8s_1s_3^4-3s_2^5s_3+8s_2^4s_3\\
&+5s_2^3s_3^3-20s_2^3s_3^2+5s_2^2s_3^4-20s_2^2s_3^3-3s_2s_3^5+8s_2s_3^4),
\end{aligned}
$$

$$
\begin{aligned}
E_{32}^{[3]_4} = &-\frac{1}{(2520h^2 s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(-3s_1^5s_2 - 3s_1^5s_3\\
&- 3s_1^4s_2^2 + 2s_1^4s_2s_3 + 8s_1^4s_2 + 5s_1^4s_3^2 + 8s_1^4s_3 - 3s_1^3s_2^3 + 2s_1^3s_2^2s_3\\
&+ 8s_1^3s_2^2 + 10s_1^3s_2s_3^2 + 2s_1s_2s_3^4 + 3s_1s_2^5 - 12s_1^3s_2s_3\\
&+ 5s_1^3s_3^3 - 20s_1^3s_3^2 - 3s_1^2s_2^4 + 2s_1^2s_2^3s_3\\
&+ 8s_1^2s_2^3 - 3s_1s_3^5 + 8s_1s_3^4 + 3s_2^5s_3 + 10s_1^2s_2^2s_3^2 - 12s_1^2s_2^2s_3\\
&+ 10s_1^2s_2s_3^3 - 40s_1^2s_2s_3^2 + 5s_1^2s_3^4 - 20s_1^2s_3^3 - 3s_2^4s_3^2 - 6s_2^4s_3\\
&- 6s_1s_2^4s_3 - 6s_1s_2^4 + 2s_1s_2^3s_3^2 + 16s_1s_2^3s_3 + 2s_1s_2^2s_3^3 - 12s_1s_2^2s_3^2\\
&- 12s_1s_2s_3^3 - 3s_2^3s_3^3 + 8s_2^3s_3^2 - 3s_2^2s_3^4 + 8s_2^2s_3^3\\
&- 3s_2s_3^5 + 8s_2s_3^4),\\
E_{33}^{[3]_4} = &\frac{1}{(2520h^2 s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^5s_2 - 3s_1^5s_3\\
&+ 5s_1^4s_2^2 + 2s_1^4s_2s_3 + 8s_1^4s_2 - 3s_1^4s_3^2 + 8s_1^4s_3 + 5s_1^3s_2^3 + 10s_1^3s_2^2s_3\\
&- 20s_1^3s_2^2 + 2s_1^3s_2s_3^2 + 3s_1s_3^5 - 6s_1s_3^4 - 12s_1^3s_2s_3\\
&- 3s_1^3s_3^3 + 8s_1^3s_3^2 + 5s_1^2s_2^4 + 10s_1^2s_2^3s_3\\
&- 20s_1^2s_2^3 + 10s_1^2s_2^2s_3^2 + 16s_1s_2s_3^3 + 8s_2^4s_3 + 2s_1^2s_2s_3^3\\
&- 12s_1^2s_2s_3^2 - 3s_1^2s_3^4 + 8s_1^2s_3^3 - 3s_1s_2^5\\
&+ 2s_1s_2^4s_3 - 3s_2^3s_3^3 + 8s_2^3s_3^2 + 8s_1s_2^4 + 2s_1s_2^3s_3^2\\
&- 12s_1s_2^3s_3 + 2s_1s_2^2s_3^3 - 12s_1s_2^2s_3^2 - 6s_1s_2s_3^4 - 3s_2^5s_3 - 40s_1^2s_2^2s_3\\
&- 3s_2^4s_3^2 - 3s_2^2s_3^4 + 8s_2^2s_3^3 + 3s_2s_3^5 - 6s_2s_3^4),\\
E_{34}^{[3]_4} = &\frac{1}{h^2(2520s_1-2520)(s_2-1)(s_3-1)}(3s_1^5s_2 + 3s_1^5s_3 - 5s_1^4s_2^2\\
&+ 3s_1s_3^5 + 3s_2^5s_3 - 10s_1^4s_2s_3 - 5s_1^4s_3^2 - 5s_1^3s_2^3\\
&+ 10s_1^3s_2^2s_3 + 10s_1^3s_2s_3^2 - 5s_1^3s_3^3 - 10s_1s_2^4s_3 + 10s_1s_2^3s_3^2\\
&- 5s_1^2s_2^4 + 10s_1^2s_2^3s_3 + 30s_1^2s_2^2s_3^2 + 10s_1^2s_2s_3^3 - 5s_1^2s_3^4 + 3s_1s_2^5\\
&+ 10s_1s_2^2s_3^3 - 10s_1s_2s_3^4 - 5s_2^4s_3^2 - 5s_2^3s_3^3 - 5s_2^2s_3^4 + 3s_2s_3^5),\\
E_{41}^{[3]_4} = &\frac{-1}{840h^3 s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(3s_1^5 - 3s_1^4s_2 - 3s_1^4s_3 - 6s_1^4\\
&- 3s_1^3s_2^2 - 3s_3^5 + 8s_1^3s_2 - 3s_1^3s_3^2 + 8s_1^3s_3 - 3s_1^2s_2^3 + 5s_1^2s_2^2s_3\\
&+ 8s_1^2s_2^2 + 5s_1^2s_2s_3^2 - 20s_1^2s_2s_3 - 20s_2^2s_3^2\\
&- 3s_1^2s_3^3 + 8s_1^2s_3^2 - 3s_1s_2^4 + 5s_1s_2^3s_3 + 8s_1s_2^3 + 5s_1s_2^2s_3^2\\
&- 20s_1s_2^2s_3 + 5s_1s_2s_3^3 + 5s_2s_3^4 - 20s_1s_2s_3^2 - 3s_1s_3^4\\
&+ 8s_1s_3^3 - 3s_2^5 + 5s_2^4s_3 + 8s_2^4 + 5s_2^3s_3^2 - 20s_2^3s_3\\
&+ 5s_2^2s_3^3 - 20s_2s_3^3 + 5s_1^3s_2s_3 + 8s_3^4),
\end{aligned}
$$

$$
\begin{aligned}
E_{42}^{[3]_4} &= \frac{1}{840h^3 s_2(s_1-s_2)(s_2-s_3)(s_2-1)}(8s_3^4 - 3s_1^5 - 3s_1^4 s_2 + 5s_1^4 s_3 \\
&- 3s_1^3 s_2^2 - 3s_3^5 + 8s_1^3 s_2 + 5s_1^3 s_3^2 - 20s_1^3 s_3 - 3s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3 + 8s_1^2 s_2^2 \\
&+ 5s_1^2 s_2 s_3^2 - 20s_1^2 s_2 s_3 + 8s_2^2 s_3^2 + 5s_1^2 s_3^3 - 20s_1^2 s_3^2 - 3s_1 s_2^4 \\
&+ 5s_1 s_2^3 s_3 + 8s_1 s_2^3 + 5s_1 s_2^2 s_3^2 - 20s_1 s_2^2 s_3 + 5s_1 s_2 s_3^3 \\
&- 20s_1 s_2 s_3^2 + 5s_1 s_3^4 - 20s_1 s_3^3 + 3s_2^5 - 3s_2^4 s_3 - 6s_2^4 - 3s_2^3 s_3^2 + 8s_2^3 s_3 - 3s_2^2 s_3^3 \\
&+ 8s_2 s_3^3 + 5s_1^3 s_2 s_3 + 8s_1^4 - 3s_2 s_3^4), \\
E_{43}^{[3]_4} &= \frac{-1}{840h^3 s_3(s_1-s_3)(s_2-s_3)(s_3-1)}(5s_1^4 s_2 - 3s_1^5 - 3s_1^4 s_3 \\
&+ 5s_1^3 s_2^2 + 5s_1^3 s_2 s_3 - 20s_1^3 s_2 - 3s_1^3 s_3^2 + 8s_1^3 s_3 + 5s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3 \\
&- 20s_1^2 s_2^2 + 5s_1^2 s_2 s_3^2 - 20s_1^2 s_2 s_3 - 3s_1^2 s_3^3 + 8s_1^2 s_3^2 + 5s_1 s_2^4 \\
&+ 5s_1 s_2^3 s_3 - 20s_1 s_2^3 + 5s_1 s_2^2 s_3^2 - 20s_1 s_2^2 s_3 + 5s_1 s_2 s_3^3 \\
&- 20s_1 s_2 s_3^2 - 3s_1 s_3^4 + 8s_1 s_3^3 - 3s_2^5 - 3s_2^4 s_3 \\
&+ 8s_2^4 - 3s_2^3 s_3^2 + 8s_2^3 s_3 - 3s_2^2 s_3^3 + 8s_2^2 s_3^2 - 3s_2 s_3^4 \\
&+ 8s_2 s_3^3 + 3s_3^5 - 6s_3^4 + 8s_1^4), \\
E_{44}^{[3]_4} &= \frac{1}{840h^3 (s_1-1)(s_2-1)(s_3-1)}(5s_1^4 s_2 - 3s_1^5 + 5s_1^4 s_3 - 3s_3^5 \\
&+ 5s_1^3 s_3^2 + 5s_1^3 s_2^2 + 5s_1^2 s_2^3 - 15s_1^2 s_2^2 s_3 - 15s_1^2 s_2 s_3^2 + 5s_1^2 s_3^3 + 5s_1 s_2^4 \\
&- 15s_1 s_2^3 s_3 - 15s_1 s_2^2 s_3^2 - 15s_1 s_2 s_3^3 - 15s_1^3 s_2 s_3 + 5s_1 s_3^4 + 5s_2^4 s_3 \\
&+ 5s_2^3 s_3^2 + 5s_2^2 s_3^3 + 5s_2 s_3^4 - 3s_2^5).
\end{aligned}
$$

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Acceptance sampling plans for truncated lifetime tests under two-parameter Pranav distribution

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Abstract. The single acceptance sampling plans (SASP) are one of the main statistical tools in industry and production fields. Both of the customers and producers are interesting in the product, where the customers want a product of good quality with long life time and the producers want to keep the quality of the products with minimum cost and variation. In this study, it is supposed that the lifetime of the products follows the two parameters Pranav distribution (TPPD) and the mean is taken as a quality parameter. The necessary tables of the minimum sample size, operating characteristic (OC) function and the producer's risk values are obtained for various model parameters. Also, for applicability investigation of the suggested SASP based on TPPD, a real data set of failure times of 20 identical components is analyzed and used. It turns out that the new ASP gives minimum sample sizes and it is recommended for practitioners.

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Keywords: Acceptance sampling plan, two-parameter Pranav distribution, lifetime, producer's risk, truncated tests, operating characteristic function.

1. Introduction

The acceptance sampling plans are one of the most commonly used sampling methods in quality control when the product quality depends on its life time. It is used to find the optimal plan parameters as the minimum sample size and its acceptance number to save the time and cost of testing the lots within the experiment. In such life tests the final decision based on the tested units is to accept or reject the lot.

Several authors have suggested various types of acceptance sampling plans using different distributions. In the past few years, much strength is employed in the studying of acceptance sampling plans under a truncated life test. For illustration, Jose and Sivadas [17] suggested ASP for negative binomial Marshall-Olkin Rayleigh distribution; Al-Omari et al. [5,6] for under two-parameter Quasi Shanker distribution and length-biased weighted Lomax distribution, respectively. Al-Omari et al. [6] for the Akash distribution, Al-Omari et al. (2019) for two parameter quasi Lindley distribution, Singh et al. (2020) for generalized Pareto distribution, Gillariose and Tommy (2020) for extended Birnbaum-Saunders distribution, Hamurkaroglu et al. (2020) for single and double ASP for the compound Weibull-Exponential distribution, Al-Nasser et al. (2018) for the Ishita distribution, Al-Omari (2015, 2018) for generalized inverted exponential and Garima distributions respectively, Lio et al. (2010) for Burr type XII percentiles, Kaviyarasu and Fawaz (2017) for percentiles using Weibull-Poisson distribution, Gadde and Durgamamba (2021) for group ASP for size biased Lomax distribution, Chiang et al. (2018) for group ASP based on the Kumaraswamy Burr XII distribution, Aslam et al. (2009) for group ASP for gamma distribution, Rao et al. (2019) for percentiles for Type-II generalized log logistic distribution, Al-Omari and Zamanzade (2017) offered double ASP for transmuted generalized inverse Weibull distribution.Several authors have suggested various types of acceptance sampling plans using different distributions. In the past few years, much strength is employed in the studying of acceptance sampling plans under a truncated life test. For illustration, Aslam et al. (2009) for group ASP for gamma distribution, Lio et al. (2010) for Burr type XII percentiles, Jose and Sivadas [17] suggested ASP for negative binomial Marshall-Olkin Rayleigh distribution; Al-Omari (2015) for generalized inverted exponential distribution, Kaviyarasu and Fawaz (2017) for percentiles using Weibull-Poisson distribution, Al-Omari and Zamanzade (2017) offered double ASP for transmuted generalized inverse Weibull distribution, Al-Omari (2018) for Garima distribution, Al-Nasser et al. (2018) for Quasi Lindley distribution, Al-Nasser et al. (2018) for Ishita distribution, Al-Omari (2018) for Sushila distribution, Chiang et al. (2018) for group ASP based on the Kumaraswamy Burr XII distribution, Al-Nasser et al. (2018) for the Ishita distribution, Al-Omari et al. (2019) for two parameter quasi Lindley distribution, Rao et al. (2019) for percentiles for Type-II generalized log logistic distribution, Al-Omari et al. (2019) for Rama distribution, Al-Omari et al. (2020) for the Akash distribution, Singh et al. (2020) for generalized Pareto distribution, Gillariose and Tommy (2020) for extended Birnbaum-Saunders distribution, Hamurkaroglu et al. (2020) for single and double ASP for the compound Weibull-Exponential distribution, Singh et al. (2020) for generalized Pareto distribution. Al-Omari et al. (2021a, b) for two-parameter Quasi Shanker distribution and length-biased weighted Lomax distribution, respectively, Gadde and Durgamamba (2021) for group ASP for size biased Lomax distribution, Al-Nasser and ul Haq (2021) for Lomax distribution.

To the best of our knowledge this work is the first one considered the SASP based on the two-parameter Pranav distribution. In this article, the two-parameter Pranav distribution is introduced in Section 2. The proposed acceptance sampling plan with its main parameters and illustrations are given in Section 3. Section 4 deals with the tables of minimum sample sizes, OC values and the minimum ratio of true average life time as well as some illustration examples are introduced. Section 5 exhibits an application of a real data set in industry, and some conclusions and recommendations are presented in Section 6.

2. The two-parameter Pranav distribution

Shukla (2018) suggested a one parameter lifetime distribution known as the Pranav distribution (PD) with probability density function (pdf) given by

$$f_{PD}(x) = \frac{\theta^4}{6+\theta^4}(\theta + x^3)e^{-\theta x}, \quad x > 0,\ \theta > 0, \tag{1}$$

and cumulative distribution function (cdf) defined by

$$F_{PD}(x) = 1 - \left(1 + \frac{6\theta x + 3\theta^2\ x^2 + \theta^3 x^3}{6+\theta^4}\right) e^{-\theta x}, \quad x > 0,\ \theta > 0. \tag{2}$$

As a modification of the PD, Umeh and Ibenegbu (2019) proposed a new distribution of two parameters called as a two-parameter Pranav distribution (TPPD) with probability density function (pdf) defined as

$$f_{TPPD}(x) = \frac{\theta^4}{6+\alpha\theta^4}(\alpha\theta + x^3)e^{-\theta x}, \quad x > 0,\ \alpha > 0,\ \theta > 0. \tag{3}$$

Figure 1 shows the pdf of the TPPD plots for some selections of model parameters.

The corresponding cumulative distribution function (cdf) of (3) is

$$F_{TPPD}(x) = 1 - \left(1 + \frac{6\theta x + 3\theta^2\ x^2 + \theta^4 x^4}{6+\alpha\theta^4}\right) e^{-\theta x},\ x > 0,\ \alpha > 0,\ \theta > 0. \tag{4}$$

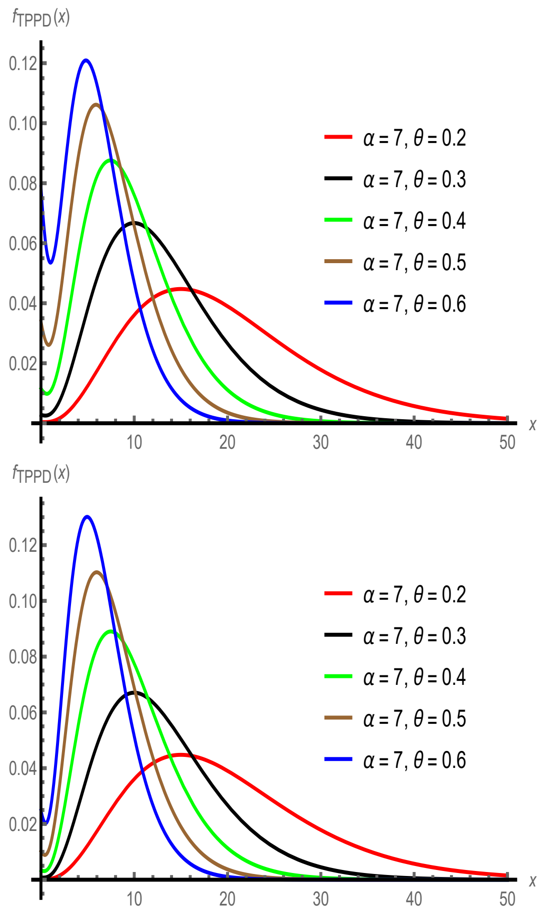

Figure 1: The TPPD pdf plots for some model parameters

The additional parameters to the base PD makes the TPPD more flexible and applicable model more than the Ishita, Akash, Pranav, Shanker, Lindley, Sujatha, and exponential distributions. The flexibility of the TPP distribution is due that is a mixture of two well-known distributions, which are exponential (θ) and gamma $(4,\theta)$ with a mixture factor $A=\frac{\alpha\theta^4}{\alpha\theta^4+6}$. The survival function of the TPPD is given by

$$S_{TPPD}(x)=1-F_{TPPD}(x)=\left(1+\frac{6\theta x+3\theta^2\ x^2+\theta^4x^4}{6+\alpha\theta^4}\right)e^{-\theta x}, \tag{5}$$

$$x>0,\ \alpha>0,\ \theta>0.$$

Figure 2 presents the survival function of the TPPD for some selected parameters. It can be seen that the survival function plots are decreasing for large values of X.

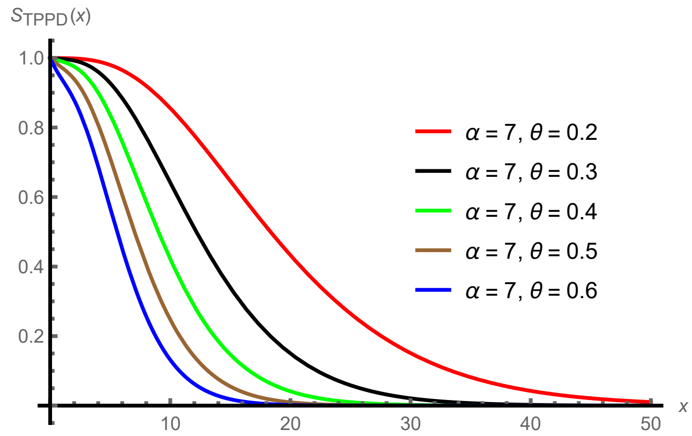

Figure 2: The survival function of the TPPD for some model parameters

The mean, hazard rate and mean residual life functions of the TPPD, respectively, are defined as

$$E(X) = \frac{\alpha\theta^4 + 24}{\theta(\alpha\theta^4 + 6)},$$

$$h_{TPPD}(x) = \frac{f_{TPPD}(x)}{1 - F_{TPPD}(x)} = \frac{\theta^4(\alpha\theta + x^3)}{\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x + \alpha\theta^4 + 6}, \tag{6}$$

and

$$\begin{aligned} m_{TPPD}(x) &= \frac{1}{1 - F_{TPPD}(x)} \int_x^{\infty} [1 - F_{TPPD}(x)]\, dv \\ &= \frac{\theta^3 x^3 + 6\theta^2 x^2 + 18\theta x + \alpha\theta^4 + 24}{\theta(\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x + \alpha\theta^4 + 6)}. \end{aligned} \tag{7}$$

Note that, $f(0) = h(0) = \frac{\alpha\theta^5}{\alpha\theta^4+6}$ and $m(0) = E(X) = \frac{\alpha\theta^4+6}{\alpha\theta^4+24}$. The rth moment and coefficient of variation (C.V) of the TPPD are

$$\mu^r = \frac{r!(\alpha\theta^4 + (r+1)(r+2)(r+3))}{\theta^r(6 + \alpha\theta^4)}, \quad r = 1, 2, ...$$

and

$$CV = \frac{\alpha^2\theta^8 + 84\alpha\theta^4 + 144}{\theta(24 + \alpha\theta^4)(6 + \alpha\theta^4)}.$$

3. Designing the SASP

In this section, a new SASP is developed supposing that the lifetime distribution of the products follows the TPPD. A produced lot is considered good if the true mean life time of items, say μ, is not less than a identified value μ_0. And the lot is not good if $\mu < \mu_0$. The test terminates at a pre-specified time t, while the failures number detected on the time interval given by $[0, t]$ are determined. The decision to accept the determined mean depends on the number of failures at the final of the time t that doesn't exceeds the acceptance number c. It is assumed that the lot is large enough so to that the mathematical theory of the binomial distribution can be employed. The rejection of acceptance of the product are same to the rejection or acceptance of the hypothesis $H_0 : \mu \geq \mu_0$. A SASP $(n, c, t/\mu_0)$ consists of (1). The number of items to be tested, say n, (2) the acceptance number c, and (3) the ratio $t/\mu_0 \longrightarrow t$, where μ_0 is the indicated mean lifetime and t is the pre-identified testing time. The producer's risk which is known as the probability of acceptance lot classified as a bad is fixed to be at most $1-p^*$, where p^* is the confidence level in the direction that the probability of rejecting a lot with a mean $\mu < \mu_0$ is p^* at least. At this stage, the researcher want to obtain the minimum sample size (MSS), n holding the inequality

$$\sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i} \leq 1 - p^*, \tag{8}$$

where $p = F(t, \mu_0)$ is the probability of a failure occurring in time t when the true mean life is μ_0. It depends simply on t/μ_0 and this function is a monotonically increasing in the ratio. Therefore, the experiment requires to determine this ratio. If the number of failures detected is at most equal to c, then from (8) we can assert with probability level p^* that $F(t; \mu) \leq F(t; \mu_0)$, that implies $\mu \geq \mu_0$. Hence, the mean life of the units can be asserted to be at least equal to their determined value with predetermined probability p^*. The minimum values of the sample size favorable (5) are obtained and presented in Table 1 for p^*=0.75, 0.90, 0.95, 0.99, t/μ_0=0.628, 0.942, 1.257, 1.571, 2.356, 3.141, 3.927, 4.712 and $c = 0, 1, 2, ..., 10$ when $\alpha = 83.7$ and $\theta = 0.092$.

The operating characteristic function (OCF) is very important in SASP where it determine the effectiveness of a statistical hypothesis test structured to reject or accept a lot. The OCF of any sampling plan, say $(n, c, t/\mu_0)$ gives the probability of accepting the lot and it is defined as

$$L(P) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \tag{9}$$

where $p = F(t; \mu)$ is a function of lot quality parameter μ. The OCF is an increasing function in μ; a decreasing function of p while p is a decreasing function of μ. Now, for given probability p^* and ratio t/μ_0, the selection of the MSS n with acceptance number c based on the OCF values. The OCF values for the proposed SASP are presented in Table 2 for $\alpha = 83.7$ and $\theta = 0.092$.

The producer's risk (PR) is the probability of rejecting a lot with $\mu > \mu_0$. For the SASP under investigation and a fixed value of the PR η, the researchers are involved in determining the value of μ/μ_0 that will emphasize the PR is less than or equal to η. Therefore, the probability function is found as

$$p = F\left(\frac{t}{\mu_0}\frac{\mu_0}{\mu}\right). \tag{10}$$

Therefore, μ/μ_0 is the lowest positive number for which p fulfills the inequality

$$\sum_{i=0}^{c}\binom{n}{i}p^i(1-p)^{n-i} \leq \eta. \tag{11}$$

For $\alpha = 83.7$ and $\theta = 0.092$ with a given p^*, the smallest values of the ratio μ/μ_0 satisfying the last inequality (9) are provided in Table 3.

4. Description of tables

Assuming that the life distribution follows the TPPD and Table 1 displayed the MSS needed to assert that μ is greater than μ_0 with probability at least p^* with c as an acceptance number. For illustration, when $p^* = 0.95$, $c = 2$ and $t/\mu_0 = 0.942$, the corresponding entry table is $n = 10$. Hence, if out of the 10 items, less than or equal to two fail before time t, then the decision is the lot can be accepted with a probability of 0.95. This means that out of the 10 items, if there are two items fail previous the time t, then a 95% upper confidence interval for μ is $(t/0.942, \infty)$. Table 2 devoted to the OCF values for the suggested ASP, and for the plan $(n = 10, c = 2, t/\mu_0 = 0.942)$ with $p^* = 0.95$ the OCF values are:

μ/μ_0	2	4	6	8	10	12
$L(P)$	0.884749	0.999532	0.999990	0.999999	1	1
PR	0.115251	0.000468	0.0001	0.000001	0	0

From this OCF values, it is found that if the real mean life time is twice the identified mean life, then the producer's risk is approximately 0.115251 and zero for big values of μ/μ_0.
Table 3 includes the values of the minimum ratio of the true average life to the identified mean lifetime (μ/μ_0) for different choices of c and t/μ_0 provided that the producer's risk not than 0.05. Hence, for the $(n = 10, c = 2, t/\mu_0 = 0.942)$, the value of μ/μ_0 is 2.276. This displays that the product must have a mean life of 2.276 times the determined mean life 1000 hours accept the lot with probability of at least 0.90.

5. Application of real data

We take a dataset that is already investigated by Murthy (2004). The dataset represents the failure times of 20 identical components. The observations are:

15.32, 8.29, 8.09, 11.89, 11.03, 10.54, 4.51, 1.79, 7.93, 6.29, 5.46, 2.87, 11.12, 11.23, 3.58, 9.74, 8.45, 2.99, 3.14, 1.80.

Figure 3 displays the fitted density and cdf for the dataset. Figure 4 provides the Total test time (TTT) curve and box plot of the estimates for the data set based on the TPPD, for more details about the TTT see Aarset (1987).

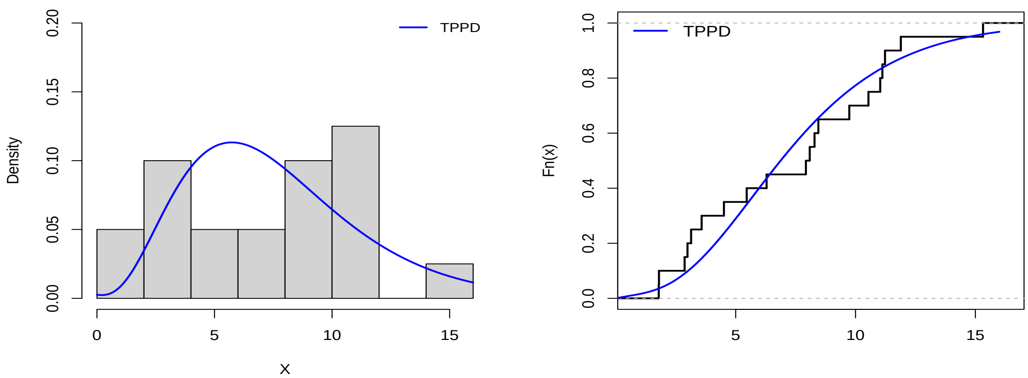

Figure 3: Fitted pdf for failure times of 20 data and the estimated cumulative distribution function for the TPPD

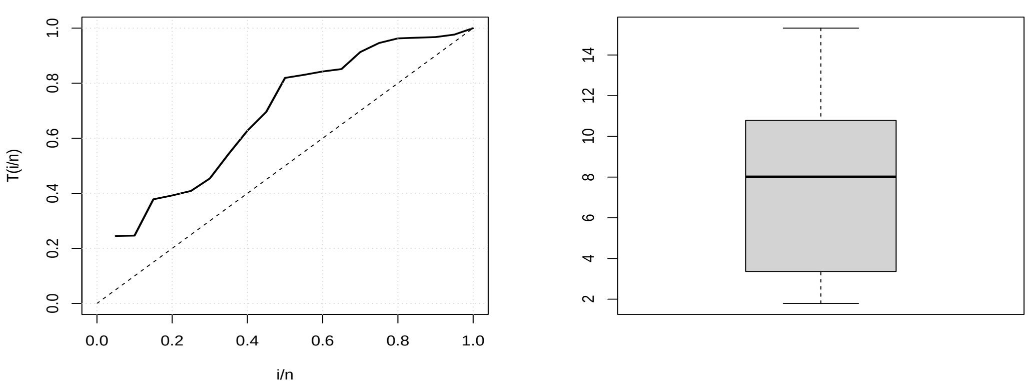

Figure 4: TTT curve and box plot of the estimates for the data set based on the TPPD

First we test whether the TPPD can be used or not. The maximum likelihood estimation method (MLE) is used to estimatethe unknown TPPD parameters. The following criteria consist of the Akaike Information criterion (AIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC), are presented. Also, the Kolmogorov-Smirnov (KS), Anderson-Darling (A) and Cramer-von-Mises (W) are obtained and the results are presented in Table 4.

Table 4: Fitting criteria values for the real dataset

Model	W	A	AIC	CAIC	BIC	HQIC	Statistic	p-value
TPPD	0.09219	0.56157	115.7915	116.4974	117.7829	116.1802	0.15758	0.647

The KS is the distance between the fitted and observed distribution functions is 0.15758 with p-value of 0.647. Thus, the TPPD showed a very good fit. For this data, it is found that the MLEs of the distribution parameters are $\hat{\theta} = 0.5226901$, $\hat{\alpha} = 2.8294078$ and hence $\hat{E}(X) = \frac{(\hat{\alpha}\hat{\theta}^4+24)}{\hat{\theta}(\hat{\alpha}\hat{\theta}^4+6)} = 7.45757$.
Assume that the specified mean lifetime is $\mu_0 = 7.45757$ and the time test is $t_0 = 4.6834$. Then, from Table 6 with $p^* = 0.75$, the acceptance sampling plan is $(n = 20, c = 8, t/\mu_0 = 0.628)$. Thus, for the suggested SASP if more than 8 failures obtained before the time 4.6834 the lot is rejected. Since there are only 7 failures $(4.51, 1.79, 2.87, 3.58, 2.99, 3.14, 1.80)$ before 4.6834, then we accept the lot.

6. Conclusions

In this paper, a new truncated life single acceptance sampling plan has been introduced when the life time of the test units follows the TPP distribution. The required tables are presented for the minimum sample size, operating characteristic function values and the minimum ratio for the suggested sampling plan. An application of a real data for the suggested SASP is presented using the failure times of 20 identical components data and can be employed excellently in analyzing the data. The TTPD might entices various applications in reliability and one can use it for other types of ASP or by using the ranked set sampling methods (Haq et al. 2013, 2014a,b).

Table 1: MSS for a given μ_0 with p^* for c with $\alpha = 83.7, \theta = 0.092$ in the TPPD

		t/μ_0							
p^*	c	0.628	0.942	1.257	1.571	2.356	3.141	3.927	4.712
0.75	0	5	2	2	1	1	1	1	1
	1	11	5	3	2	2	2	2	2
	2	15	7	5	4	3	3	3	3
	3	20	9	6	5	4	4	4	4
	4	25	11	8	6	5	5	5	5
	5	29	13	9	7	6	6	6	6
	6	34	16	11	9	7	7	7	7
	7	39	18	12	10	8	8	8	8
	8	43	20	13	11	9	9	9	9
	9	48	22	15	12	10	10	10	10
	10	52	24	16	13	11	11	11	11
0.90	0	9	4	2	2	1	1	1	1
	1	15	6	4	3	2	2	2	2
	2	20	9	6	4	3	3	3	3
	3	26	11	7	6	4	4	4	4
	4	31	14	9	7	5	5	5	5
	5	36	16	10	8	6	6	6	6
	6	41	18	12	9	8	7	7	7
	7	46	20	13	11	9	8	8	8
	8	51	23	15	12	10	9	9	9
	9	56	25	16	13	11	10	10	10
	10	60	27	18	14	12	11	11	11
0.95	0	11	5	3	2	1	1	1	1
	1	18	8	5	3	2	2	2	2
	2	24	10	6	5	3	3	3	3
	3	30	13	8	6	5	4	4	4
	4	35	15	10	7	6	5	5	5
	5	40	18	11	9	7	6	6	6
	6	46	20	13	10	8	7	7	7
	7	51	22	14	11	9	8	8	8
	8	56	25	16	13	10	9	9	9
	9	61	27	18	14	11	10	10	10
	10	66	29	19	15	12	11	11	11
0.99	0	17	7	4	3	2	1	1	1
	1	25	10	6	4	3	2	2	2
	2	31	13	8	6	4	3	3	3
	3	38	16	10	7	5	4	4	4
	4	44	19	11	9	6	5	5	5
	5	50	21	13	10	7	6	6	6
	6	55	24	15	11	8	8	7	7
	7	61	26	17	13	9	9	8	8
	8	66	29	18	14	11	10	9	9
	9	72	31	20	15	12	11	10	10
	10	77	33	21	17	13	12	11	11

Table 2: OCF values of the sampling plan $(n, c = 2, t/\mu_0)$ with $\alpha = 83.7, \theta = 0.092$ in the TPPD

			μ/μ_0					
p^*	m	t/μ_0	2	4	6	8	10	12
0.75	15	0.628	0.980479	0.999963	0.999999	1	1	1
	7	0.942	0.955535	0.999859	0.999997	1	1	1
	5	1.257	0.900978	0.999419	0.999986	0.999999	1	1
	4	1.571	0.838090	0.998381	0.999955	0.999997	1	1
	3	2.356	0.668703	0.990410	0.999572	0.999966	0.999996	0.999999
	3	3.141	0.336935	0.943213	0.995854	0.999572	0.999939	0.999989
	3	3.927	0.134187	0.831819	0.980719	0.997439	0.999571	0.999911
	3	4.712	0.046533	0.668703	0.943180	0.990410	0.998130	0.999572
0.90	20	0.628	0.957714	0.999909	0.999998	1	1	1
	9	0.942	0.911507	0.999669	0.999993	1	1	1
	6	1.257	0.837326	0.998873	0.999973	0.999998	1	1
	4	1.571	0.838090	0.998381	0.999955	0.999997	1	1
	3	2.356	0.668703	0.990410	0.999572	0.999966	0.999996	0.999999
	3	3.141	0.336935	0.943213	0.995854	0.999572	0.999939	0.999989
	3	3.927	0.134187	0.831819	0.980719	0.997439	0.999571	0.999911
	3	4.712	0.046533	0.668703	0.943180	0.990410	0.998130	0.999572
0.95	24	0.628	0.933093	0.999840	0.999996	1	1	1
	10	0.942	0.884749	0.999532	0.999990	0.999999	1	1
	6	1.257	0.837326	0.998873	0.999973	0.999998	1	1
	5	1.571	0.708825	0.996180	0.999888	0.999993	0.999999	1
	3	2.356	0.668703	0.990410	0.999572	0.999966	0.999996	0.999999
	3	3.141	0.336935	0.943213	0.995854	0.999572	0.999939	0.999989
	3	3.927	0.134187	0.831819	0.980719	0.997439	0.999571	0.999911
	3	4.712	0.046533	0.668703	0.943180	0.990410	0.998130	0.999572
0.99	31	0.628	0.878208	0.999652	0.999991	0.999999	1	1
	13	0.942	0.790899	0.998925	0.999977	0.999998	1	1
	8	1.257	0.689250	0.997026	0.999926	0.999995	0.999999	1
	6	1.571	0.576240	0.992788	0.999781	0.999986	0.999998	1
	4	2.356	0.362531	0.967752	0.998384	0.999866	0.999983	0.999997
	3	3.141	0.336935	0.943213	0.995854	0.999572	0.999939	0.999989
	3	3.927	0.134187	0.831819	0.980719	0.997439	0.999571	0.999911
	3	4.712	0.046533	0.668703	0.943180	0.990410	0.998130	0.999572

Table 3: Minimum ratio of the true mean life to the specified mean lifetime for suitability of a lot with PR of 0.05 and $\alpha = 83.7, \theta = 0.092$

		t/μ_0							
p^*	c	0.628	0.942	1.257	1.571	2.356	3.141	3.927	4.712
0.75	0	3.081	3.469	4.629	4.618	6.925	9.232	11.543	13.85
	1	2.115	2.388	2.572	2.608	3.911	5.215	6.519	7.822
	2	1.767	1.966	2.244	2.487	3.071	4.094	5.118	6.141
	3	1.631	1.764	1.910	2.128	2.659	3.545	4.432	5.318
	4	1.550	1.644	1.853	1.909	2.409	3.211	4.014	4.817
	5	1.476	1.563	1.697	1.758	2.237	2.982	3.728	4.473
6	1.439	1.555	1.684	1.827	2.110	2.813	3.517	4.220	
	7	1.410	1.504	1.587	1.727	2.012	2.682	3.354	4.024
	8	1.375	1.464	1.508	1.647	1.933	2.577	3.222	3.866
	9	1.357	1.432	1.518	1.581	1.868	2.490	3.113	3.736
10	1.332	1.405	1.460	1.525	1.813	2.417	3.022	3.625	
0.90	0	3.694	4.313	4.629	5.785	6.925	9.232	11.543	13.85
	1	2.346	2.559	2.914	3.214	3.911	5.215	6.519	7.822
	2	1.956	2.183	2.449	2.487	3.071	4.094	5.118	6.141
	3	1.795	1.929	2.080	2.387	2.659	3.545	4.432	5.318
	4	1.679	1.837	1.980	2.132	2.409	3.211	4.014	4.817
	5	1.601	1.726	1.810	1.957	2.237	2.982	3.728	4.473
	6	1.545	1.646	1.776	1.827	2.471	2.813	3.517	4.220
	7	1.503	1.585	1.672	1.864	2.343	2.682	3.354	4.024
	8	1.469	1.570	1.660	1.774	2.240	2.577	3.222	3.866
	9	1.442	1.528	1.586	1.700	2.155	2.490	3.113	3.736
	10	1.410	1.493	1.584	1.637	2.084	2.417	3.022	3.625
0.95	0	3.930	4.621	5.261	5.785	6.925	9.232	11.543	13.85
	1	2.490	2.841	3.186	3.214	3.911	5.215	6.519	7.822
	2	2.082	2.276	2.449	2.804	3.071	4.094	5.118	6.141
	3	1.888	2.069	2.225	2.387	3.191	3.545	4.432	5.318
	4	1.754	1.893	2.092	2.132	2.862	3.211	4.014	4.817
5	1.665	1.820	1.911	2.120	2.637	2.982	3.728	4.473	
	6	1.613	1.728	1.860	1.976	2.471	2.813	3.517	4.220
	7	1.562	1.658	1.749	1.864	2.343	2.682	3.354	4.024
	8	1.522	1.634	1.727	1.885	2.240	2.577	3.222	3.866
	9	1.490	1.586	1.708	1.804	2.155	2.490	3.113	3.736
	10	1.463	1.547	1.639	1.736	2.084	2.417	3.022	3.625
0.99	0	4.503	5.127	5.755	6.575	8.676	9.232	11.543	13.85
	1	2.767	3.071	3.414	3.642	4.820	5.215	6.519	7.822
	2	2.268	2.515	2.776	3.060	3.729	4.094	5.118	6.141
	3	2.047	2.248	2.469	2.600	3.191	3.545	4.432	5.318
	4	1.901	2.089	2.193	2.474	2.862	3.211	4.014	4.817
	5	1.803	1.944	2.086	2.262	2.637	2.982	3.728	4.473
	6	1.721	1.872	2.008	2.105	2.471	3.294	3.517	4.220
	7	1.669	1.788	1.949	2.089	2.343	3.123	3.354	4.024
	8	1.618	1.748	1.847	1.984	2.469	2.986	3.222	3.866
	9	1.585	1.691	1.815	1.897	2.370	2.873	3.113	3.736
	10	1.551	1.644	1.741	1.905	2.286	2.778	3.022	3.625

Table 5: MSS of the sampling plans with $\theta = 0.13$ for the real data

		t/μ_0							
p^*	c	0.628	0.942	1.257	1.571	2.356	3.141	3.927	4.712
0.75	0	2	2	2	1	1	1	1	1
	1	5	4	3	3	2	2	2	2
	2	7	6	5	4	4	3	3	3
	3	9	7	6	6	5	4	4	4
	4	12	9	8	7	6	6	5	5
	5	14	11	9	8	7	7	6	6
	6	16	12	11	10	8	8	7	7
	7	18	14	12	11	10	9	8	8
	8	20	16	14	12	11	10	9	9
	9	22	17	15	14	12	11	11	10
	10	24	19	16	15	13	12	12	11
0.90	0	4	3	2	2	2	1	1	1
	1	7	5	4	4	3	3	2	2
	2	9	7	6	5	4	4	3	3
	3	12	9	7	7	5	5	5	4
	4	14	11	9	8	7	6	6	5
	5	16	12	11	9	8	7	7	6
	6	19	14	12	11	9	8	8	7
	7	21	16	14	12	10	9	9	9
	8	23	18	15	14	12	11	10	10
	9	25	20	17	15	13	12	11	11
	10	28	21	18	16	14	13	12	12
0.95	0	5	3	3	2	2	2	1	1
	1	8	6	5	4	3	3	3	2
	2	10	8	6	6	5	4	4	3
	3	13	10	8	7	6	5	5	5
	4	16	12	10	9	7	6	6	6
	5	18	14	11	10	9	8	7	7
	6	20	15	13	12	10	9	8	8
	7	23	17	15	13	11	10	9	9
	8	25	19	16	15	12	11	10	10
	9	27	21	18	16	13	12	11	11
	10	30	23	19	17	15	13	12	12
0.99	0	7	5	4	3	3	2	2	2
	1	10	8	6	5	4	4	3	3
	2	13	10	8	7	6	5	4	4
	3	16	12	10	9	7	6	5	5
	4	19	14	12	10	8	7	7	6
	5	22	16	13	12	10	8	8	7
	6	24	18	15	13	11	10	9	8
	7	27	20	17	15	12	11	10	9
	8	29	22	18	16	14	12	11	10
	9	32	24	20	18	15	13	12	12
	10	34	26	22	19	16	14	13	13

Table 6: OCF values of the sampling plan $(n, c = 2, t/\mu_0)$ with $\theta = 0.13$ for the real data

			μ/μ_0					
p^*	m	t/μ_0	2	4	6	8	10	12
0.75	10	0.628	0.621215	0.894689	0.958488	0.97973	0.988658	0.993033
	7 0.942	0.499465	0.836512	0.931149	0.965168	0.980076	0.987577	
	5	1.257	0.480227	0.821326	0.922917	0.960475	0.977198	0.985698
	4	1.571	0.555292	0.853886	0.937922	0.968412	0.981861	0.988658
	3	2.356	0.330815	0.702311	0.853947	0.919411	0.951274	0.968429
	3	3.141	0.503321	0.802751	0.906758	0.949491	0.969798	0.980575
	3	3.927	0.389740	0.720743	0.856556	0.918268	0.949470	0.966729
	3	4.712	0.298140	0.642183	0.802717	0.882333	0.924960	0.949480
0.90	13	0.628	0.432917	0.805974	0.916977	0.957664	0.975668	0.984780
	8	0.942	0.373656	0.764232	0.894689	0.945013	0.967924	0.979730
	6	1.257	0.324185	0.722659	0.870911	0.931024	0.959169	0.973936
	5	1.571	0.348608	0.733419	0.875575	0.933365	0.960491	0.974748
	4	2.356	0.330815	0.702311	0.853947	0.919411	0.951274	0.968429
	3	3.141	0.193303	0.555466	0.753876	0.853977	0.907408	0.937967
	3	3.927	0.389740	0.720743	0.856556	0.918268	0.949470	0.966729
	3	4.712	0.298140	0.642183	0.802717	0.882333	0.924960	0.94948
0.95	15	0.628	0.353110	0.756532	0.891615	0.943547	0.967125	0.979251
	10	0.942	0.271793	0.687954	0.852489	0.920566	0.952760	0.969751
	7	1.257	0.324185	0.722659	0.870911	0.931024	0.959169	0.973936
	6	1.571	0.204691	0.606875	0.799603	0.887273	0.931049	0.954978
	4	2.356	0.150183	0.518697	0.733514	0.842380	0.900347	0.933397
	3	3.141	0.193303	0.555466	0.753876	0.853977	0.907408	0.937967
	3	3.927	0.111852	0.430568	0.651491	0.779480	0.853923	0.899019
	3	4.712	0.298140	0.642183	0.802717	0.882333	0.924960	0.949480
0.99	19	0.628	0.178801	0.602600	0.801994	0.890316	0.933615	0.956982
	12	0.942	0.134838	0.536994	0.756532	0.860865	0.914086	0.943547
	9	1.257	0.131122	0.522558	0.744357	0.852250	0.908078	0.939277
	7	1.571	0.114501	0.487240	0.716287	0.832715	0.894544	0.929686
	5	2.356	0.062859	0.362634	0.606994	0.751976	0.836375	0.887323
	4	3.141	0.064298	0.348790	0.586764	0.733561	0.821488	0.875656
	3	3.927	0.111852	0.430568	0.651491	0.779480	0.853923	0.899019
	3	4.712	0.063767	0.330815	0.555408	0.702311	0.794793	0.853947

Table 7: Minimum ratio of the true mean life to the specified mean lifetime for suitability of a lot with PR of 0.05 and $\theta = 0.13$ for the real data

		t/μ_0							
p^*	c	0.628	0.942	1.257	1.571	2.356	3.141	3.927	4.712
0.75	0	30.161	45.242	60.37	37.636	56.442	75.248	94.078	112.883
	1	9.679	11.215	10.525	13.154	11.209	14.943	18.683	22.417
	2	5.553	6.886	7.248	6.595	9.890	8.037	10.048	12.056
	3	4.121	4.442	4.748	5.934	6.636	5.634	7.044	8.452
	4	3.808	3.903	4.390	4.448	5.062	6.748	5.537	6.644
	5	3.285	3.560	3.509	3.589	4.140	5.519	4.630	5.556
	6	2.941	2.954	3.445	3.677	3.535	4.713	4.023	4.827
	7	2.696	2.841	2.968	3.183	3.962	4.142	3.587	4.304
	8	2.514	2.749	2.973	2.820	3.528	3.717	3.258	3.909
	9	2.373	2.450	2.661	2.938	3.195	3.387	4.234	3.600
	10	2.260	2.413	2.416	2.674	2.929	3.123	3.904	3.351
0.90	0	60.392	67.915	60.37	75.45	113.151	75.248	94.078	112.883
	1	14.064	14.518	14.966	18.704	19.727	26.299	18.683	22.417
	2	7.466	8.329	9.188	9.058	9.890	13.185	10.048	12.056
	3	5.843	6.182	5.927	7.408	6.636	8.847	11.061	8.452
	4	4.605	5.112	5.208	5.487	6.670	6.748	8.437	6.644
	5	3.887	4.018	4.750	4.386	5.382	5.519	6.900	5.556
	6	3.660	3.686	3.941	4.306	4.547	4.713	5.892	4.827
	7	3.291	3.446	3.790	3.710	3.962	4.142	5.179	6.214
	8	3.019	3.263	3.323	3.715	4.229	4.704	4.647	5.575
	9	2.810	3.118	3.269	3.326	3.811	4.259	4.234	5.080
	10	2.773	2.806	2.955	3.019	3.480	3.905	3.904	4.684
0.95	0	75.508	67.915	90.625	75.45	113.151	150.852	94.078	112.883
	1	16.252	17.810	19.373	18.704	19.727	26.299	32.88	22.417
	2	8.420	9.766	9.188	11.483	13.584	13.185	16.484	12.056
	3	6.414	7.045	7.092	7.408	8.900	8.847	11.061	13.272
	4	5.399	5.712	6.017	6.508	6.670	6.748	8.437	10.123
	5	4.487	4.927	4.750	5.166	6.577	7.175	6.900	8.279
	6	3.899	4.049	4.432	4.925	5.514	6.062	5.892	7.070
	7	3.686	3.746	4.195	4.226	4.774	5.281	5.179	6.214
	8	3.354	3.517	3.669	4.152	4.229	4.704	4.647	5.575
	9	3.101	3.339	3.568	3.708	3.811	4.259	4.234	5.080
	10	3.028	3.196	3.220	3.358	4.010	3.905	3.904	4.684
0.99	0	105.739	113.261	120.88	113.263	169.859	150.852	188.601	226.302
	1	20.625	24.378	23.766	24.212	28.050	37.395	32.880	39.453
	2	11.276	12.63	13.031	13.890	17.221	18.110	16.484	19.78
	3	8.125	8.764	9.400	10.309	11.109	11.865	11.061	13.272
	4	6.587	6.908	7.623	7.520	8.228	8.892	11.117	10.123
	5	5.683	5.831	5.970	6.701	7.748	7.175	8.971	8.279
	6	4.851	5.131	5.403	5.538	6.457	7.351	7.579	7.070
	7	4.472	4.640	4.998	5.243	5.563	6.364	6.603	6.214
	8	4.022	4.277	4.353	4.585	5.572	5.638	5.881	5.575
	9	3.824	3.997	4.161	4.459	4.988	5.081	5.325	6.389
	10	3.537	3.775	4.005	4.024	4.528	4.639	4.882	5.858

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Inequalities of DVT-type-the one-dimensional case continued

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Abstract. In this note, the investigation of particular inequalities of DVT-type in integer numbers is continued.

Keywords: real numbers, inequality.

1. Introduction

In [2], A. Drápal and V. Valent proved that in a finite quasigroup Q of order n the number of associative triples $a(Q) \geq 2n - i(Q) + (\delta_1 + \delta_2)$, where $i(Q)$ is the number of idempotents in Q, i.e., $i(Q) = |\{\, x \in Q \,|\, xx = x \,\}|$, $\delta_1 = |\{\, z \in Q \,|\, zx \neq x \text{ for all } x \in Q \,\}|$ and $\delta_2 = |\{\, z \in Q \,|\, xz \neq x \text{ for all } x \in Q \,\}|$ (Theorem 2.5). This important result is an easy consequence of the inequality

$$\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) - \sum_{i=1}^{k}(a_i + b_i) \geq 3n - 2k + (r + s),$$

where $n \geq k \geq 0, a_1, \dots, a_n, b_1 \dots, b_n$ are non-negative integers such that $\sum a_i = n = \sum b_i$, $a_i \geq 1$ and $b_i \geq 1$ for $1 \leq i \leq k$, r is the number of i with $a_i = 0$

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and s is the number of i with $b_i = 0$ (Proposition 2.4(ii)). The lengthy and complicated proof of this DVT-inequality (inequality of Drápal-Valent type) in [2] is based on highly semantically involved insight.

In [3], a very short elementary arithmetical proof of a more general inequality of this type was found. This inequality is two-dimensional in the sense that it works with two n-tuples of integers. The approach in [3] opens a road to investigation of similar inequalities of DVT-type which could be useful in further investigations of estimates in non-associative algebra and they are also of independent interest. Hence they deserve a thorough examination, however the research is only at its beginning. In [1], the investigation of the one-dimensional case working with one n-tuple of real numbers was started. This note is an immediate continuation of [1].

2. Second concepts

Let $n \geq 1$ and let $\alpha = (a_1, \dots, a_n)$ be an ordered n-tuple of integers. Let I be any subset (whether empty or non-empty) of the interval $\{1, \dots, n\}$. We put

(1) $z(\alpha, a) = |\{\, i \,|\, 1 \leq i \leq n, a_i = a\}|$ for every $a \in \mathbb{R}$;

(2) $z(\alpha) = z(\alpha, 0)$;

(3) $z(\alpha, +) = \sum_{a>0} z(\alpha, a)$;

(4) $z(\alpha, -) = \sum_{a<0} z(\alpha, a)$;

(5) $s(\alpha) = \sum_{i=1}^{n} a_i$;

(6) $r(\alpha) = \sum_{i=1}^{n} a_i^2$;

(7) $q(\alpha) = r(\alpha) - s(\alpha)$;

(8) $t(\alpha) = q(\alpha) - z(\alpha)$.

(9) $I^{\perp} = \{1, \dots, n\} \setminus I$;

(10) $s(\alpha, I) = -|I| + \sum_{i \in I} a_i \ (= \sum_{i \in I} (a_i - 1)\,)$;

(11) $r(\alpha, I, +) = \sum_{i=1}^{n} a_i^2 + \sum_{i \in I} a_i$;

(12) $r(\alpha, I, -) = \sum_{i=1}^{n} a_i^2 - \sum_{i \in I} a_i$;

(13) $t(\alpha, I) = \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i + |I| - \sum_{i \in I} a_i - z(\alpha)$.

Lemma 2.1. (i) $r(\alpha, I, +) \geq z(\alpha, +)$.

(ii) $r(\alpha, I, +) = z(\alpha, +)$ *if and only if* $a_i \in \{0, -1\}$ *for* $i \in I$ *and* $a_i \in \{0, 1\}$ *for* $i \in I^{\perp}$.

Proof. Put $K_1 = \{ i \,|\, a_i \geq 2 \}$, $K_2 = \{ i \,|\, a_i = 1 \}$, $K_3 = \{ i \,|\, a_i = 0 \}$, $K_4 = \{ i \,|\, a_i = -1 \}$, $K_5 = \{ i \,|\, a_i \leq -2 \}$. Then $r(\alpha) + \sum_{i \in I} a_i \geq 4|K_1 \setminus I| + 6|k_1 \cap I| + |k_2 \setminus I| + 2|K_2 \cap I| + |K_4 \setminus I| + 4|K_5 \setminus I| + 2|K_5 \cap I| \geq |K_1 \setminus I| + |K_1 \cap I| + |K_2 \setminus I| + |K_2 \cap I| = |K_1| + |K_2| = z(\alpha, +)$. If $r(\alpha, I, +) = z(\alpha, +)$ then $K_1 = \emptyset$, $K_\cap I = \emptyset$, $K_4 \subseteq I$, $K_5 = \emptyset$, and therefore $a_i \in \{0, 1, -1\}$ for every i. Moreover, $a_i \neq 1$ for every $i \in I$ and $a_i \neq -1$ for every $i \in I^\perp$. These arguments are reversible. □

Lemma 2.2. (i) *$r(\alpha, I, -) \geq z(\alpha, -)$.*

(ii) *$r(\alpha, I, -) = z(\alpha, -)$ if and only if $a_i \in \{0, 1\}$ for $i \in I$ and $a_i \in \{0, -1\}$ for $i \in I^\perp$.*

Proof. This follows from 2.1 ($a_i \leftrightarrow -a_i$). □

Lemma 2.3. *Let $s(\alpha) \geq 0$. Then:*

(i) *$r(\alpha, I, +) \geq z(\alpha, -) + s(\alpha) \geq z(\alpha, -)$.*

(ii) *$r(\alpha, I, +) = z(\alpha, -) + s(\alpha)$ if and only if $a_i \in \{0, -1\}$ for $i \in I$ and $a_i \in \{0, -1\}$ for $i \notin I$.*

Proof. We have $r(\alpha, I, +) = r(\alpha) + s(a) - \sum_{i \notin I} a_i = r(\alpha, I^\perp, -) + s(\alpha) \geq z(\alpha, -) + s(\alpha) \geq r(\alpha, -)$ by 2.2(i). The rest is clear. □

Lemma 2.4. *Let $s(\alpha) \leq 0$. Then:*

(i) *$r(\alpha, I, -) \geq z(\alpha, +) + s(\alpha) \geq z(\alpha, +)$.*

(ii) *$r(\alpha, I, -) = z(\alpha, +) + s(\alpha)$ if and only if $a_i \in \{0, 1\}$ for $i \in I$ and $a_i \in \{0, -1\}$ for $i \notin I$.*

Proof. This follows from 2.3 ($a_i \leftrightarrow -a_i$). □

Lemma 2.5. (i) *If $s(\alpha) \geq 0$ then $r(\alpha, I, +) \geq \max(z(\alpha, +), z(\alpha, -))$.*

(ii) *If $s(\alpha) \leq 0$ then $r(\alpha, I, -) \geq \max(z(\alpha, +), z(\alpha, -))$.*

Proof. Just combine 2.1(i), 2.3(i), 2.2(i) and 2.4(i). □

Lemma 2.6. (i) *$s(\alpha, I) + s(\alpha, I^\perp) = s(\alpha) - n$.*

(ii) *$r(\alpha, I, +) + r(\alpha, I, -) = 2r(\alpha)$.*

(iii) *$r(\alpha, I, +) + r(\alpha, I^\perp, +) = 2r(\alpha) + s(\alpha)$.*

(iv) *$r(\alpha, I, -) + r(\alpha, I^\perp, -) = 2r(\alpha) - s(\alpha)$.*

(v) *$t(\alpha, I) + t(\alpha, I^\perp) = 2r(\alpha) - 3s(\alpha) + n - 2z(\alpha) = 2t(\alpha) + n - s(\alpha)$.*

Proof. All is obvious. □

Lemma 2.7. (i) *$r(\alpha, I, +) - r(\alpha, I, -) = 2 \sum_{i \in I} a_i$.*

(ii) *$r(\alpha, I, +) - r(\alpha, I^\perp, +) = \sum_{i \in I} a_i - \sum_{i \in I^\perp} a_i$.*

(iii) *$r(\alpha, I, -) - r(\alpha, I^\perp, -) = \sum i \in I^\perp a_i - \sum_{i \in I} a_i$.*

(iv) *$r(\alpha, I, +) - r(\alpha, I^\perp, -) s(\alpha)$.*

(v) *$r(\alpha, I, -) - r(\alpha, I^\perp, +) = -s(\alpha)$.*

(vi) *$t(\alpha, I) - t(\alpha, I^\perp) = |I| - |i^\perp| + \sum_{i \in I^\perp} a_i - \sum_{i \in I} a_i = 2|I| - n + s(\alpha) - 2 \sum_{i \in I} a_i = s(\alpha) - n - 2s(\alpha, I)$.*

Proof. All is obvious. □

Lemma 2.8. (i) $s(\alpha, \emptyset) = 0$.
(ii) $s(\alpha, \{1, \dots, n\}) = s(\alpha) - n$.
(iii) $r(\alpha, \emptyset, +) = r(\alpha)$.
(iv) $r(\alpha, \{1, \dots, n\}, +) = r(a) + s(\alpha)$.
(v) $r(\alpha, \emptyset, -) = r(\alpha)$.
(vi) $r(\alpha, \{1, \dots, n\}, -) = r(\alpha) - s(\alpha)$
(vii) $t(\alpha, \emptyset) = t(\alpha)$.
(viii) $t(\alpha, \{1, \dots, n\}) = t(\alpha) + n - s(\alpha)$.

Proof. It is obvious. □

3. Technical results

Lemma 3.1. *Put* $\beta = \alpha - 1 = (a_1 - 1, \dots, a_n - 1)$*. Then:*
(i) $s(\beta, I) = s(\alpha, I) - |I|$.
(ii) $r(\beta, I, +) = r(\alpha, I, +) - 2s(\alpha) + n - |I|$.
(iii) $r(\beta, I, +) = t(\alpha, I^{\perp}) + z(\alpha)$.
(iv) $r(\beta, I, -) = r(\alpha, I, -) - 2s(\alpha) + n + |I|$.
(v) $t(\beta, I) = t(\alpha, I) - 2s(\alpha) + 2n + |I| + z(\alpha) - z(\alpha, 1)$.
(vi) $t(\alpha, I) = r(\beta) + s(\beta) - \sum_{i \in I} b_i - z(\beta, -1)$.

Proof. (i), (ii), (iii) and (iv) are obvious.

(v) We have $t(\beta, I) = \sum_{i=1}^{n} (a_i - 1)^2 - \sum_{i=1}^{n} (a_i - 1) + |I| - \sum_{i \in I} (a_i - 1) - z(\beta) = \sum_{i=1}^{n} a_i^2 - 2 \sum_{i=1}^{n} a_i + n - \sum_{i=1}^{n} a_i + n + |I| - \sum_{i \in I} a_i + |I| - z(\alpha, 1) = t(\alpha, I) - 2s(\alpha) + 2n + |I| + z(\alpha) - z(\alpha, 1)$.

(vi) We have $r(\beta) + s(\beta) - \sum_{i \in I} b_i - z(\beta, -1) = \sum_{i=1}^{n} a_i^2 - 2 \sum_{i=1}^{n} a_i + n + \sum_{i=1}^{n} a_n - n + |I| - \sum_{i \in I} a_i - z(\alpha) = \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i + |I| - \sum_{i \in I} a_i - z(\alpha) = t(\alpha, I)$. □

Put $\tau(\alpha) = 2r(\alpha) + 2 \sum_{i \in J} a_i - 3z(\alpha, -)$, where $J = \{\, i \mid 1 \le i \le n, a_i < 0 \,\}$.

Lemma 3.2. *Let* $\max(\alpha) = 1$ *and* $\min(\alpha) = -1$*. Then:*
(i) $s(\alpha) = z(\alpha, +) - z(\alpha, -)$.
(ii) $s(\alpha) \ge 0$ *if and only if* $z(\alpha, +) \ge z(\alpha, -)$.
(iii) $r(\alpha) = z(\alpha, +) + z(\alpha, -)$.
(iv) $\tau(\alpha) = 2z(\alpha, +) - 3z(\alpha, -)$.

Proof. All is obvious. □

Lemma 3.3. *Let* $\max(\alpha) = 1$*,* $\min(\alpha) \le -2$*. Then:*
(i) $s(\alpha) \le z(\alpha, +) - z(\alpha, -) + 1 + \min(\alpha)$.
(ii) *If* $s(\alpha) \ge 0$ *then* $z(\alpha, +) \ge z(\alpha, -) + 1$.

Proof. It is obvious. □

Lemma 3.4. *Let* $\max(\alpha) = 1$, $\min(\alpha) \le -2$, $s(\alpha) \ge 0$) *and* $z(\alpha, +) = z(\alpha, -) + 1$. *Then:*

(i) $\min(\alpha) = -2$ *and* $z(\alpha, \min(\alpha)) = 1$.

(ii) $s(\alpha) = 0$.

(iii) $r(\alpha) = 2z(\alpha, -) + 4$.

(iv) $\tau(\alpha) = 6 - z(a, -)$.

(v) $\tau(\alpha) = 0$ *if and only if* $z(\alpha, +) = 7$ *and* $z(\alpha, -) = 6$.

Proof. It follows from 3.3 that $0 \le s(\alpha) \le 2 + \min(\alpha)$, and so $\min(\alpha) = -2$ and $s(\alpha) = 0$. The rest is easy. □

Lemma 3.5. *Let* $\max(\alpha) \ge 2$, $\min(\alpha) = -1$. *Then:*

(i) $s(\alpha) \ge \max(\alpha) + z(\alpha, +) - 1 - z(\alpha, -)$.

(ii) *If* $z(\alpha, -) \le z(\alpha, +) + 1$ *then* $s(\alpha) \ge 0$.

(iii) $r(\alpha) \ge \max(\alpha)^2 + z(a, +) + z(\alpha, -) - 1$.

(iv) $r(\alpha) = \max(\alpha)^2 + z(\alpha, +) + z(\alpha, -) - 1$ *if and only if* $z(\alpha, +) - z(\alpha, 1) = 1$.

(v) $\tau(\alpha) \ge 2\max(\alpha)^2 + 2z(\alpha, +) - 3z(\alpha, -) - 2$.

(vi) $\tau(\alpha) = 2\max(\alpha)^2 + 2z(\alpha, +) - 3z(\alpha, -) - 2$ *if and only if* $z(a, +) - z(\alpha, 1) = 1$.

(vii) *If* $2z(\alpha, +) \ge 3z(\alpha, -)$ *then* $\tau(\alpha) \ge 6$.

Proof. It is easy. □

Now, assume that $n \ge 2$ and choose $j, k \in \{1, \dots, n\}$ such that $j \ne k$ and $a_j = \max(\alpha)$, $a_k = \min(\alpha)$. Consider the n-tuple $\beta = (b_1, \dots, b_n)$ such that $b_i = a_i$ for $i \ne j, k$, $b_j = a_j - 1$ and $b_k = a_k + 1$.

Lemma 3.6. *Let* $a_j \ge 1$ *and* $a_k \le -2$. *Then:*

(i) $z(\alpha, -) = z(\beta, -)$.

(ii) *If* $a_j \ge 2$ *then* $z(\alpha, +) = z(\beta, +)$.

(iii) *If* $a_j = 1$ *then* $z(\beta, +) = z(\alpha, +) - 1$.

(iv) *If* $z(\alpha, \max(\alpha)) \ge 2$ *then* $\max(\beta) = \max(\alpha)$.

(v) *If* $z(\alpha, \max(\alpha)) = 1$ *then* $\max(\beta) = \max(\alpha) - 1$.

(vi) *If* $z(\alpha, \min(\alpha)) = 1$ *then* $\min(\beta) = \min(\alpha) + 1$.

(vii) *If* $z(\alpha, \min(\alpha)) \ge 2$ *then* $\min(\beta) = \min(\alpha)$.

(viii) $\tau(\alpha) - \tau(\beta) = 4a_j - 4a_k - 6 \ge 6$.

Proof. We have $r(\alpha) - r(\beta) = 2(a_j - a_k - 1)$ (see [1, 2.8]) and the rest is easy. □

Lemma 3.7. *Let* $a_j \ge 1$ *and* $a_k = -1$. *Then:*

(i) $z(\beta, -) = z(\alpha, -) - 1$.

(ii) *If* $a_j \ge 2$ *then* $z(\beta, +) = z(\alpha, +)$.

(iii) *If* $a_j = 1$ *then* $z(\beta, +) = z(\alpha, +) - 1$.

(iv) *If* $z(\alpha, \max(\alpha)) \ge 2$ *then* $\max(\beta) = \max(\alpha)$.

(v) *If* $z(\alpha, \max(\alpha)) = 1$ *then* $\max(\beta) = \max(\alpha) - 1$.

(vi) *If* $z(\alpha, -) = z(\alpha, \min(\alpha)) = 1$ *then* $\min(\beta) = 0$.
(vii) *If* $z(\alpha, -) = z(\alpha, \min(\alpha)) \geq 2$ *then* $min(\beta) = -1$.
(viii) $\tau(\alpha) = \tau(\beta) = 4a_j - 5 \geq -1$.

Proof. Similar to that of 3.6. □

4. The inequalities

Theorem 4.1. *Let* $n \geq 1$ *and let* $a_1, \dots, a_n$ *be integers such that* $\sum_{i=1}^n |a_i| \geq n$. *Let* $I \subseteq \{1, \dots, n\}$ *and let* z *be the number of indices* $i \in \{1, \dots, n\}$ *such that* $a_i = 0$. *Then:*

$$(1) \qquad \sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) \geq \sum_{i=1}^n a_i^2 - \sum_{i=1}^n |a_i| - \sum_{i \in I}(|a_i| - 1) \geq z,$$

$$(2) \qquad \sum_{i=1}^n a_i^2 - \sum_{i=1}^n |a_i| - \sum_{i \in I}(|a_i| - 1) = z$$

if and only if $\sum_{i=1}^n |a_i| = n$, $a_i \in \{\pm 1, \pm 2\}$ *for* $i \in I$ *and* $a_i \in \{0, \pm 1\}$ *for* $i \notin I$.

Proof. (1) Since $|a_i| \geq a_i$, we can assume that all the numbers a_i are non-negative. Then $\sum_{i=1}^n a_i \geq n$. Put $b_i = a_i - 1$ for every $i = 1, \dots, n$. Evidently, $\sum_{i=1}^n b_i \geq 0$ and z is now the number of indices i such that $b_i < 0$ ($b_i = -1$ in fact). By [1, 1.4(i)] we have $\sum_{i=1}^n b_i^2 \geq 2z$. Henceforth, $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) - z = \sum_{i=1}^n (a_i - 1)^2 + \sum_{i=1}^n (a_i - 1) - \sum_{i \in I}(a_i - 1) - z = \sum_{i=1}^n b_i^2 + \sum_{i \in I^\perp} b_i - z = (\sum_{i=1}^n b_i^2 - 2z) + (z + \sum_{i \in I^\perp} b_i) \geq z + \sum_{i \in I^\perp} b_i \geq z - z_1 \geq 0$, where $z_1 = |\{\, i \in I^\perp \mid b_i = -1 \,\}|$. Of course, $z \geq z_1 = |\{\, i \in I^\perp \mid a_i = 0 \,\}|$.

(2) If $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) - z = 0$ then $\sum_{i=1}^n b_i^2 = 2z$, $z + \sum_{i \in I^\perp} b_i = 0$ and $z = z_1$. First, it follows from [1, 1.4(ii)] that $\sum_{i=1}^n b_i = 0$ and $b_i \in \{0, 1, -1\}$. It means that $\sum_{i=1}^n a_i = n$ and $a_i \in \{0, 1, 2\}$ for every $i = 1, \dots, n$. Since $z = z_1$, we have $a_i \neq 0$ for every $i \in I$. If $z_2 = |\{\, i \in I^\perp \mid b_i = 1 \,\}| = |\{\, i \in I^\perp \mid a_i = 2 \,\}|$ then $\sum_{i \in I^\perp} b_i = z_2 - z_1 = z_2 - z$. Since $z + \sum_{i \in I^\perp} = 0$, we have $z_2 = 0$. Thus $a_i \neq 2$ for every $i \notin I$. □

Example 4.2. (i) Put $n = 2$, $a_1 = 2$, $a_2 = 0$, $I = \{1\}$. Then $\sum_{i=1}^2 a_i^2 - \sum_{i=1}^2 a_i - (a_1 - 1) = 1 = z$ (cf. [1, 6.1(i)].

(ii) Put $n = 5$, $a_1 = a_2 = a_3 = 2$, $a_4 = a_5 = 0$, $I = \{1, 2, 3\}$. Then $\sum_{i=1}^5 a_i^2 - \sum_{i=1}^5 a_i - \sum_{i=1}^3 (a_i - 1) = 3$ and $z = 2$. We have also $\sum_{i=1}^5 b_i^2 - 2z = 1$ and $z + \sum_{i=1}^5 b_i = 0$.

Remark 4.3. Consider the situation from 4.1 (and the proof). Assume that the numbers a_i are non-negative.

(i) If $\sum_{i \in I^\perp} a_i \geq |I^\perp| = n - |I|$ then $\sum_{i \in I^\perp} b_i \geq 0$ and we conclude that $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) \geq 2z$. For instance, if $|I| = n - 1$ then the latter inequality is true provided that $a_j \neq 0$, where $\{j\} = I^\perp$.

(ii) Assume that $a_i = 0$ for every $i \in I^\perp$. Then $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1) = \sum_{i\in I}(a_i-1)^2 = \sum_{i\in I} b_i = \sum_{i=1}^n b_i^2 - n + |I| \geq 2z - n + |I|$. For instance, if $|I| = n-1$ then $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1) \geq 2z - 1$.

Assume, moreover, that $z_1 = |\{\, i \in I \,|\, a_i = 0\,\}| \geq n - |I|$. We have $z = z_1 + n - |I|$, so that $z \geq 2(n - |I|)$, $4z - 2n + 2|I| \geq 3z$ and $2(\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1)) \geq 3z$.

Proposition 4.4. *Let $n \geq 1$ and let $a_1, \dots, a_n$ be non-negative integers. Denote $z = |\{\, i \,|\, 1 \leq i \leq n, a_i = 0\,\}|$, $w = |\{\, i \,|\, 1 \leq i \leq n, a_i \geq 2\,\}|$ and $a = \max(a_1, \dots, a_n)$. Assume that $a \geq 3$ and $2a^2 - 4a + 2w \geq 3z$. Then, for every subset $I \subseteq \{1, \dots, n\}$,*

$$2(\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1)) \geq 3z.$$

Proof. Put $\alpha = (a_1, \dots, n)$ and $\beta = \alpha - 1 = (a_1 - 1, \dots, a_n - 1)$. Then $2r(\alpha) - 2s(\alpha) - 2\sum_{i\in I}(a_i - 1) - 3z = 2(r(\beta) + \sum_{i\in I^\perp} b_i) - 3z(\beta, -)$ (see the proof of 4.1). Furthermore, $a = \max(\beta) + 1$, $\max(\beta) \geq 2$, $w = z(\beta, +)$ and $2\max(\beta)^2 + 2z(\beta,+) - 2 - 3z(\beta,-) \geq 0$. If $(z(\beta,-) =)\ z \neq 0$ then $\min(\beta) = -1$ and $2(r(\beta) + \sum_{i\in J} b_i) - 3z(\beta,-) \geq 0$ by 3.5(v), where $J = \{\, i \,|\, b_i < 0\,\} = \{\, i \,|\, b_i = -1\,\} = \{\, i \,|\, a_i = 0\,\}$. Of course, $\sum_{i\in I^\perp} b_i \geq \sum_{i\in J} b_i = -|J| = -z$, and hence $2(r(\beta) + \sum_{i\in I^\perp} b_i) - 3z(\beta,-) \geq 0$. On the other hand, if $z = 0$ then $z(\beta,-) = 0$ and $2(r(\beta) + \sum_{i\in I^\perp} b_i) - 3z(\beta,-) \geq 0$ trivially. □

Remark 4.5. Let $n \geq 1$ and let $a_1, \dots, a_n$ be non-negative integers such that $\max(a_1, \dots, a_n) = 2$ and $\alpha = (a_1, \dots, a_n)$. Put $J = \{\, i \,|\, a_i = 2\,\}$. Using 3.2(v) and proceeding similarly as in the proof of 4.4, we show that $2(\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1)) \geq 3z$ if and only if $2|J| = 2z(\alpha, 2) \geq 3z$. Of course, we have $s(\alpha) = 2z(\alpha,2) + z(\alpha,1) = 2z(\alpha,2) + n - z(\alpha,2) - z = z(\alpha,2) + n - z$. Henceforth, if $s(\alpha) \leq n$ then $1 \leq z(\alpha,2) \leq z$.

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Prime-valent one-regular graphs of order $20p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its full automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify connected one-regular graphs of prime valency and order $20p$ for each prime p. As a result there is only one infinite family of such graphs, that is, the cycle C_{20p} with valency two.
Keywords: symmetric graph, arc-transitive graph, one-regular graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts and graph-theoretic terms not defined here we refer the reader to [20, 22] and [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G,s)-*arc-transitive* or (G,s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G,s)-arc-transitive graph is said to be (G,s)-*transitive* if it is not $(G,s+1)$-arc-transitive. In particular, a $(G,1)$-arc-transitive graph is called G-*symmetric.* A graph X

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is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all know that there is only one connected 2-valent graph of order n, that is, the cycle C_n, which is 1-regular with full automorphism group D_{2n}. Let p and q be two primes. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [5] and [7], respectively. Liu [15] characterized prime-valent arc-transitive basic graphs of order $4p$ or $4p^2$. Li [14] and Chen [6] classified prime-valent one-regular graph of order $8p$ and $12p$, respectively. Pan [19] and Huang [13] classified the pentavalent s-transitive graphs of order $4pq$ and $4p^n$ for n a positive integer, respectively. Pan [18] determined heptavalent symmetric graph of order four times an odd square-free integer. Zhou [24] gave a complete classification of cubic one-regular graphs of order twice a square-free integer.

For 2-valent case, s-transitivity always means 1-regularity, and for cubic case, s-transitivity always means s-regularity by Miller [17]. However, for the other prime-valent case, this is not true, see for example [10] for pentavalent case and [11] for heptavalent case. Thus, the characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graph of order $20p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [16, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, and the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [7], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$

and $V' = \{0', 1', \cdots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p,q)$ the unique subgroup of Z_p^* of order q. Define the graph $G(2p,q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p,q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p, q primes. Then X is isomorphic to K_{2p} with $q = 2p-1$, $K_{p,p}$ or $G(2p,q)$ with $q \,\big|\, (p-1)$. Furthermore, if $(p,q) \neq (11,5)$ then $\mathrm{Aut}(G(2p,q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p,q) = (11,5)$ then $\mathrm{Aut}(G(2p,q)) = \mathrm{PGL}(2,11)$.*

Let $p \neq 5$ be an odd prime. Then $20p = 4{\cdot}5{\cdot}p$ is four times a square-free integer. From [18, Theorem 1.1], we have the following characterization about the full automorphism groups of connected heptavalent symmetric graphs of order $20p$ with $p \neq 5$.

Proposition 2.3. *Let p be an odd prime different from 5 and X a connected heptavalent symmetric graphs of order $20p$. Then, the full automorphism group $\mathrm{Aut}(X) \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$ or $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$ with $p \equiv 1(\mathrm{mod}\ 7)$.*

The following proposition is the famous "N/C-Theorem", see for example [12, Chapter I, Theorem 4.5]).

Proposition 2.4. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

From [8, p.12-14] and [21, Theorem 2], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 2.5. *Let G be a non-abelian simple group. If the order $|G|$ has exactly three different prime divisors, then G is called K_3-simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2,3,p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(3,3)$	$2^4 \cdot 3^3 \cdot 13$
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSU}(3,3)$	$2^5 \cdot 3^3 \cdot 7$
$\mathrm{PSL}(2,8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSU}(4,2)$	$2^6 \cdot 3^4 \cdot 5$

The next proposition is proved originally by John Thompson, but now an easy consequence of the Classification of Finite Simple Groups (see for example [23, Chapter 1, Section 2]), that all finite non-abelian simple $3'$-groups (whose order is not divisible by 3) are Suzuki groups. This is well known in the sense that it is mentioned frequently in the literature.

Proposition 2.6. *Any non-abelian finite simple group whose order is not divisible by* 3 *is isomorphic to a Suzuki group* $\mathrm{Sz}(q)$ *with* $q = 2^{2n+1}$ *and* $n \geq 1$. *In particular, the order of* $\mathrm{Sz}(q)$ *is* $q^2(q^2+1)(q-1)$.

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $20p$ for each prime p. Let q be a prime. In what follows, we always denote by X a connected q-valent one-regular graph of order $20p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 20pq$.

Now, we first deal with the special case $q \leq 5$. Clearly, any connected graph of order $20p$ and valency two is isomorphic to the cycle C_{20p}. Thus, for $q = 2$, $X \cong C_{20p}$ and $A \cong D_{40p}$. Let $q = 3$. By [24, Corollary 3.3], there exists no cubic one-regular graph of order $4{\cdot}5{\cdot}p$. Let $q = 5$. By [19, Theorem 3.1], [15, Theorem 1.1] and [13], there exists no pentavalent one-regular graph of order $4{\cdot}5{\cdot}p$. The next lemma is about the case $q = 7$.

Lemma 3.1. *There exists no heptavalent one-regular graph of order* $20p$

Proof. Let X be a heptavalent one-regular graph of order $20p$. Then $q = 7$ and $|A| = 4{\cdot}5{\cdot}7{\cdot}p = 140p$. By Proposition 2.3, $A \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$ or $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$, where $p \equiv 1 (\mathrm{mod}\ 7)$.

Suppose that $A = \mathrm{PSL}(2,p)$. Then $|A| = 140p = \frac{1}{2}p(p^2-1)$. It forces that $p^2 = 281$. Note that p^2 is a prime square. However, 281 is a prime, a contradiction.

Suppose that $A = \mathrm{PGL}(2,p)$ or $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$. Then $|A| = 140p = p(p^2-1)$. This implies that $p^2 = 141 = 3{\cdot}47$, a contradiction.

Suppose that $A = \mathrm{PGL}(2,p)\times\mathbb{Z}_2$. Then $|A| = 140p = 2p(p^2-1)$. An easy calculation implies that $p^2 = 71$, contrary to the fact that p^2 is a prime square. □

To finish the classification, we treat the general case $q > 7$.

Lemma 3.2. *Let* $q > 7$. *Then there exists no* q*-valent one-regular graph of order* $20p$

Proof. Let X be a q-valent one-regular graph of order $20p$. Then $|A| = 20pq$, $|V(X)| = 20p$ and $A_v \cong \mathbb{Z}_q$. If $p = 2$, then $|V(X)| = 40$. By [14, Theorem 3.3], there exists no q-valent one-regular graph of order 40 with $q > 7$. If $p = 3$, then $|V(X)| = 60 = 12{\cdot}5$. By [6, Theorem 3.1], there exists no q-valent one-regular graph of order 60 with $q > 7$. Next, we deal with $p \geq 5$ and separate them into two cases: $p = 5$ and $p \geq 7$.

Case 1. Suppose that $p = 5$. Then $|V(X)| = 20{\cdot}5$ and $|A| = 2^2{\cdot}5^2{\cdot}q$.

Note that $q > 7$. If A is not solvable, then A has a composition factor isomorphic to a K_3-simple group. By Proposition 2.5, every K_3-simple group has divisor 3. It forces that $3 \,\big|\, |A|$, a contradiction. Thus, A is solvable. Let N be a minimal normal subgroup of A. Then N is also solvable and hence isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_5$, $\mathbb{Z}_5^2$ or $\mathbb{Z}_q$. An easy calculation implies that the number of the orbits of N acting on $V(X)$ is at least 4. By Proposition 2.1, N is semiregular on $V(X)$ and so $N \ncong \mathbb{Z}_q$. Since there exists no regular graph of odd order and odd valency, we have $N \ncong \mathbb{Z}_2^2$, and since there is no q-valent graph of order 4 with $q > 7$, we have that $N \ncong \mathbb{Z}_5^2$. It follows that $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_5$.

Assume that $N \cong \mathbb{Z}_2$. Then X_N is a q-valent symmetric graph of order $2{\cdot}5^2$. Recall that A is solvable. Thus, A/N is also solvable. Since $q > 7$, we have that A/N has no normal subgroup of order q by Proposition 2.1, and since there exists no regular graph of odd order and odd valency, we have that A/N has no normal subgroup of order 2. The solvability of A/N implies that A/N has a normal 5-subgroup, say M/N. With an easy calculation, we have that $|M/N| = 5^2$ or 5.

Let $|M/N| = 5^2$. Then by Proposition 2.4, $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_1$ because $N \cong \mathbb{Z}_2$. It forces that $M = C_M(N)$. Let P be a Sylow 5-subgroup of M. Then $|P| = 5^2$ and $M = P{\times}N$. Since P is a normal Sylow 5-subgroup of M, we have that P is characteristic in M. The normality of M in A implies that P is also normal in A. By Proposition 2.1, X_P is a q-valent symmetric graph of order 4. However, any symmetric graphs of order 4 is isomorphic to either the cycle C_4 with valency 2 or the complete graph K_4 with valency 3. This is contrary to the fact that X_P has valency $q > 7$.

Let $|M/N| = 5$. Then M has order 10. By elementary group theory, any group of order 10 is isomorphic to either a cyclic group $\mathbb{Z}_{10}$ or a dihedral group D_{10}. Clearly, the former group has a normal subgroup of order 2 and the latter group has no normal subgroup of order 2. Since M has a normal subgroup $N \cong \mathbb{Z}_2$, we have that $M \cong \mathbb{Z}_{10}$ and M has a normal subgroup $P \cong \mathbb{Z}_5$. Clearly, P is a Sylow 5-subgroup of M. The normality of P in M forces that P is characteristic in M, and since M is normal in A, we have that P is also normal in A. By Proposition 2.1, X_P is a q-valent symmetric graph of order 20 with $q > 7$. Checking the list of symmetric graph of order up to 30 in [9], we have that $X_P \cong K_{20}$ with $q = 19$ and $A/P \lesssim \mathrm{Aut}(K_{20}) \cong \mathrm{S}_{20}$. An easy calculation implies that $|A/P| = 2^2{\cdot}5{\cdot}19$. However, by Magma [3], S_{20} has no subgroup of order $2^2{\cdot}5{\cdot}19$, a contradiction.

Assume that $N \cong \mathbb{Z}_5$. Then by Proposition 2.1, X_N is a q-valent symmetric graph of order 20, and by [9], $X_N \cong K_{20}$ with $q = 19$. A similar argument as the above paragraph we can deduce a contradiction.

Case 2. Suppose that $p \geq 7$. Then $|A| = 2^2{\cdot}5{\cdot}p{\cdot}q$ with $q > 7$.

If A is non-solvable, then A must have a composition factor isomorphic to a non-abelian simple group. Note that the order of A has exactly four different prime divisors. Thus, this non-solvable composition factor is either K_3-simple group or K_4-simple group. Since $p \geq 7$ and $q > 7$, we have that 3 is not a divisor of $|A|$. By Propositions 2.5 and 2.6, the only possibilities are the Suzuki groups $\mathrm{Sz}(2^{2n+1})$ with $n \geq 1$. This forces that

$$|\mathrm{Sz}(2^{2n+1})| = (2^{2n+1})^2((2^{2n+1})^2+1)(2^{2n+1}-1) \mid |A|.$$

This is contrary to the fact that $|A| = 2^2{\cdot}5{\cdot}p{\cdot}q$ with $p \geq 7$ and $q > 7$. Thus, A is solvable. For convenience we still use N to denote a minimal normal subgroup of A. Clearly, N is also solvable and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_5$, $\mathbb{Z}_p$, $\mathbb{Z}_q$ or $\mathbb{Z}_p^2$ with $q = p$. Since $|V(X))| = 20p$, we have that N is not transitive and has at least 20 orbits on $V(X)$. By Proposition 2.1, N is semiregular on $V(X)$, and so $N \ncong \mathbb{Z}_p^2$ with $p = q$. Since there exists no connected regular graph of odd order and odd valency, we have that A has no normal subgroup of order 4 and so $N \ncong \mathbb{Z}_2^2$. Thus, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_5$ or $\mathbb{Z}_p$.

Assume that $N \cong \mathbb{Z}_p$. Then X_N is a q-valent symmetric graph of order 20 and $A/N \lesssim \mathrm{Aut}(X_N)$ by Proposition 2.1. Since $q > 7$, we have that $X_N \cong K_{20}$ with $q = 19$ by [9] and $A/N \lesssim \mathrm{S}_{20}$. Note that $|A/N| = 2^2{\cdot}5{\cdot}19$. However, by Magma [3], S_{20} has no subgroup of order $2^2{\cdot}5{\cdot}19$, a contradiction.

Assume that $N \cong \mathbb{Z}_5$. Then X_N is a q-valent symmetric graph of order $4p$ and $A/N \lesssim \mathrm{Aut}(X_N)$ by Proposition 2.1. The solvability of A forces that A/N is also solvable. Recall that A has no normal subgroup of order 4, q or p^2 with $p = q$. Thus, A/N has normal subgroup $M/N \cong \mathbb{Z}_p$ or $\mathbb{Z}_2$. It follows that M is a normal subgroup of A and has order $5p$ or 10. Again by Proposition 2.1, X_M is a q-valent symmetric graph of order 4 or $2p$. For the former, $X_M \cong C_4$ or K_4. Clearly, this is impossible because X_M has valency $q > 7$. For the latter, $X_M \cong K_{2p}$, $K_{p,p}$ or $G(2p, q)$.

Let $X_M \cong K_{2p}$. Then $q = 2p - 1$ and $A/M \lesssim \mathrm{S}_{2p}$. Clearly, A/M has order $2{\cdot}p{\cdot}q$ and is 2-transitive on $V(X_M)$ because $q = 2p - 1$. By Burnside's Theorem [4, p.192, Theorem IX], any 2-transitive permutation group is either almost simple or affine. The solvability of A forces that A/M is also solvable and hence affine. It follows that A/M has a normal subgroup $K/M \cong \mathbb{Z}_p$. Since $|M| = 10$ and $p \geq 7$, we have that K has a normal Sylow p-subgroup $P \cong \mathbb{Z}_p$ by Sylow Theorem. The normality of the Sylow p-subgroup P in K forces that P is characteristic in K. Thus, P is normal in A. Again by Proposition 2.1, X_P is a q-valent symmetric graph of order 20 and by [9], $X_P \cong K_{20}$. A similar argument as the above, we deduce that A/P has order $2^2{\cdot}5{\cdot}19$ and can not be embedded in S_{20}, a contradiction.

Let $X_M \cong K_{p,p}$. Then $p = q$ and $|A/M| = 2p^2$. Since $q = p > 7$, we have that A/M has a normal Sylow p-subgroup K/M by Sylow Theorem and $|K/M| = p^2$. It follows that $|K| = 10{\cdot}p^2$. Let P be a Sylow p-subgroup of K. Then P has order p^2 and again by Slow Theorem, P is normal and hence characteristic in K. The normality of K in A forces that P is also normal in A. Since $|V(X)| = 20{\cdot}p$, we have that P acting on $V(X)$ has 20 orbits and $P_v \cong \mathbb{Z}_p$. However, by Proposition 2.1, P must be semiregular on $V(X)$, that is, $P_v = 1$, a contradiction.

Let $X_M \cong G(2p,q)$. Then $|A/M| = 2{\cdot}p{\cdot}q$ and by Proposition 2.2, $A/M \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$ with $q \,\big|\, (p-1)$. This implies that A/M has a normal subgroup $K/M \cong \mathbb{Z}_p$. Since M has order 10, we have that K has a normal Sylow p-subgroup $P \cong \mathbb{Z}_p$ by Sylow Theorem. Thus, P is characteristic in K and hence normal in A. By Proposition 2.1, X_P is a q-valent symmetric graph of order 20 and by [9], $X_P \cong K_{20}$. Similarly, A/P has order $2^2{\cdot}5{\cdot}19$ and by Magma [3], S_{20} has no subgroup of order $2^2{\cdot}5{\cdot}19$, a contradiction.

Assume that $N \cong \mathbb{Z}_2$. Then X_N is a q-valent symmetric graph of order $2{\cdot}5{\cdot}p$, $|A/N| = 2{\cdot}5{\cdot}p{\cdot}q$ and $A/N \lesssim \mathrm{Aut}(X_N)$. Since A/N has order twice an odd number, we have that A/N must have a normal subgroup M/N of index two. Thus, $|M/N| = 5{\cdot}p{\cdot}q$ and M also has order twice an odd number. It follows that M is also has a normal subgroup K of index two and so $|K| = 5{\cdot}p{\cdot}q$. This implies that $|A:K| = 4$ and K is also normal in A. Recall that A has no normal subgroup of order 4, q or p^2 with $q = p$. If $p = q$, then $|K| = 5{\cdot}p^2$. Since $q > 7$, we have that K must a normal Sylow p-subgroup P by Sylow Theorem. It forces that P is characteristic in K and hence normal in A. Clearly, P has order p^2, this is impossible. Thus, $p \neq q$. Since K is solvable, we have that K must have a normal subgroup $H \cong \mathbb{Z}_5$, $\mathbb{Z}_p$ or $\mathbb{Z}_q$. Note that $5, p$ and q are different primes. Thus, H is characteristic in K and hence normal in A. Since $A_v \cong \mathbb{Z}_q$, we have that $H \ncong \mathbb{Z}_q$. This implies that A has a normal subgroup $H \cong \mathbb{Z}_5$ or $\mathbb{Z}_p$. Similar arguments as the above, we can deduce that this is impossible. $\square$

Combining the above arguments with the cases $q = 2, 3, 5$, and Lemmas 3.1-3.2, we have the following result.

Theorem 3.1. *Let p, q be two primes and X a connected q-valent one-regular graph of order $20p$. Then X is isomorphic to the cycle C_{20p} with valency 2 and* $\mathrm{Aut}(X) \cong D_{40p}$.

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Units of a class of finite rings of characteristic p^3

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Abstract. Let R be a commutative completely primary finite ring with Jacobson radical $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$ and $R/\mathcal{J} \cong GF(p^r)$, the finite field with p^r elements, for any prime p and any positive integer r. Then, characteristic of R is either p, p^2 or p^3. In this paper, we determine the structure and generators of the group of units of the ring R in the special case when the characteristic of R is p^3. We treat the problem by considering fixed dimensions and bases for the vector spaces $\mathcal{J}^i/\mathcal{J}^{i+1}$ $(i = 1,\ 2)$ over the residue field $R/\mathcal{J}$ and by fixing the order of the ideal $\mathcal{J}^2$. This complements the author's earlier solution to the problem in the case when the characteristic of R is p or p^2 and $\mathcal{J}^2 \subseteq ann(\mathcal{J})$, the annihilator of $\mathcal{J}$.
Keywords: finite commutative rings, unit groups.

1. Introduction

Throughout this paper, all rings are finite and commutative (unless otherwise stated) with identity element $1 \neq 0$, subrings have the same identity, ring homomorphisms preserve 1 and modules are unital. A finite ring R is called *completely primary* if all its zero divisors including the zero element form the unique maximal ideal $\mathcal{J}$. Completely primary finite rings are precisely local rings with unique maximal ideals. For a given completely primary finite ring R, unless otherwise stated, $\mathcal{J}$ will denote the Jacobson radical of R, and we will denote the Galois ring $GR(p^{nr},\ p^n)$ of characteristic p^n and order p^{nr} by R_o, for a prime integer p and positive integers $n,\ r$. We denote the group of units of R by $U(R)$; if g is an element of $U(R)$, then $o(g)$ denotes its order, and $< g >$ denotes the cyclic group generated by g. Similarly, if $f(x) \in R[x]$, we shall denote by $< f(x) >$ the ideal generated by $f(x)$. Further, for a subset A of R or $U(R)$, $|A|$ will denote the number of elements in A. The ring of integers modulo the number n will be denoted by $\mathbb{Z}_n$, and the characteristic of R will be denoted by char$R$. The symbol K will denote the residue field $R/\mathcal{J}$ and K_o will denote the set of coset representatives of the maximal ideal $\mathcal{J}$ in the ring R. We denote a direct product of r cyclic groups of $\mathbb{Z}_m$ by $\mathbb{Z}_m^r$ or by $\underbrace{\mathbb{Z}_m \times \ \cdots\ \times \mathbb{Z}_m}_{r}$.

If I is an ideal of R generated by the elements $a,\ b$, we shall denote this by $I = (a,\ b)$.

Let R be a completely primary finite ring with maximal ideal $\mathcal{J}$. Then, $|R| = p^{nr}$, $\mathcal{J}$ is the Jacobson radical of R, $\mathcal{J}^m = (0)$, where $m \leqslant n$, $|\mathcal{J}| = p^{(n-1)r}$, and the residue field $R/\mathcal{J} \cong GF(p^r)$, the finite field of p^r elements, for some prime p and positive integers $n,\ r$. The characteristic char$R$ of R is equal to $R = p^k$, where $1 \leqslant k \leqslant m$. If $k = n$, then $R = \mathbb{Z}_{p^k}[b]$, where b is an element of R of order $p^r - 1$; $\mathcal{J} = pR$ and $Aut(R) \cong Aut(R/pR)$ (see Proposition 2 in [5]). Such a ring is called a *Galois ring*, denoted by $GR(p^{kr},\ p^k)$, and a concrete model is the quotient $\mathbb{Z}_{p^k}[x]/ < f(x) >$, where $f(x)$ is a monic polynomial of degree r, irreducible modulo p. Any such polynomial will do: the rings are all isomorphic. Trivial cases are $GR(p^n,\ p^n) = \mathbb{Z}_{p^n}$ and $GR(p^n,\ p) = \mathbb{F}_{p^n}$. Furthermore, if $k < n$ and char$R = p^k$, it can be deduced from [4] that R has a coefficient subring R_o of the form $GR(p^{kr},\ p^k)$ which is clearly a maximal Galois subring of R. Moreover, if $R_o^{'}$ is another coefficient subring of R then there exists an invertible element x in R such that $R_o^{'} = xR_ox^{-1}$ (see Theorem 8 in [5]). The maximal ideal of R_o is

$$\mathcal{J}_o = pR_o = \mathcal{J} \cap R_o, \quad \text{and} \quad R_o/\mathcal{J}_o \cong GF(p^r).$$

Let $\psi : R_o \longrightarrow R_o/\mathcal{J}_o$ be the canonical map. Since the element b has order $p^r - 1$ and $\mathcal{J}_o \subset \mathcal{J}$, we have that $\psi(b)$ is a primitive element of $R_o/\mathcal{J}_o$. Let $K_o =< b > \cup\{0\}$ and let $R_o = \mathbb{Z}_{p^k}[b]$ be a coefficient subring of R of order p^{kr}. Then, it is easy to show that every element of R_o can be written uniquely as $\sum_{i=0}^{k-1} \lambda_i p^i$, where $\lambda_i \in K_o$. Also, there exist elements $m_1,\ m_2, \ldots, m_h \in \mathcal{J}$ and automorphisms $\sigma_1, \ldots, \sigma_h \in Aut(R_o)$ such that

$$R = R_o \oplus \sum_{i=1}^{h} R_o m_i \ \ (\text{as } R_o - \text{modules}), \quad m_i r = \sigma_i(r) m_i,$$

for every $r \in R_o$ and any $i = 1, \ldots, h$. Further, $\sigma_1, \ldots, \sigma_h$ are uniquely determined by R and R_o. The maximal ideal of R is

$$\mathcal{J} = pR_o \oplus \sum_{i=1}^{h} R_o m_i.$$

Let R be a completely primary ring (not necessarily commutative) of order p^{nr} with unique maximal ideal $\mathcal{J}$. Then, the set $R - \mathcal{J}$ consisting of invertible elements in R forms a group with respect to the multiplication defined on R, called the group of units of R. The following facts are useful for our purpose (e.g. see [5, §2]): The group of units $U(R)$ of R contains a cyclic subgroup $< b >$ of order $p^r - 1$, and it is a semi-direct product of $1 + \mathcal{J}$ by $< b >$; the group $U(R)$ is solvable; if G is a subgroup of $U(R)$ of order $p^r - 1$, then G is conjugate to $< b >$ in $U(R)$; if $U(R)$ contains a normal subgroup of order $p^r - 1$, then the set $K_o =< b > \cup\{0\}$ is contained in the center of the ring R; and $(1 + \mathcal{J}^i)/(1 + \mathcal{J}^{i+1}) \cong \mathcal{J}^i/\mathcal{J}^{i+1}$ (the left hand side as a multiplicative group and the right hand side as an additive group). It is easy to check that $|U(R)| = p^{(n-1)r}(p^r - 1)$ and that $|1 + \mathcal{J}| = p^{(n-1)r}$, so that $1 + \mathcal{J}$ is a p-group.

In [1], the author studied completely primary finite rings with unique maximal ideals $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$ for all the characteristics. For more details on the structure and construction of these rings, the interested reader may refer to [1].

Let R be a commutative completely primary finite ring with Jacobson radical $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$ and $\mathcal{J}^2 \neq (0)$ (see, for example, [1]). Then, in view of the above results, char$R$ is either $p,\ p^2$ or p^3. The ring R contains a coefficient subring R_o with charR_o =char$R$, and with R_o/pR_o equal to $R/\mathcal{J}$. Moreover, R_o is a Galois ring of the form $GR(p^{kr},\ p^k)$, $k = 1,\ 2$ or 3. Let $ann(\mathcal{J})$ denote the two-sided annihilator of $\mathcal{J}$ in R. Of course $ann(\mathcal{J})$ is an ideal of R. Because $\mathcal{J}^3 = (0)$, it follows easily that $\mathcal{J}^2 \subseteq ann(\mathcal{J})$.

From now on, we assume that the characteristic of the ring R is p^3. Because $\mathcal{J}^3 = (0)$, we have that $p^2 m_i = 0$, for all $m_i \in \mathcal{J}$. Further, $pm_i = 0$ for all $m_i \in ann(\mathcal{J})$. In particular, $pm_i = 0$ for all $m_i \in \mathcal{J}^2$. It is now obvious to see that p lies in $\mathcal{J} - \mathcal{J}^2$, and $p^2 \in \mathcal{J}^2$. Let $B_1 = \{p,\ u_1,\ ...,\ u_s\}$ denote the set of elements of $\mathcal{J}$ whose images modulo $\mathcal{J}^2$ form a $K-$basis for $\mathcal{J}/\mathcal{J}^2$ so that $dim_K(\mathcal{J}/\mathcal{J}^2)$ is $d_1 = 1 + s$, and let $B_2 = \{p^2,\ pu_1,\ ..., pu_d,\ u_1^2,\ u_1u_2, \dots, u_s^2\}$ denote the set of elements of $\mathcal{J}$ whose images modulo $\mathcal{J}^3 (\cong (0))$ form a $K-$basis for $\mathcal{J}^2$, so that $dim_K(\mathcal{J}^2)$ is $d_2 = 1+d+t$, where $t \leqslant s(s+1)/2$, i.e. $d_2 \leqslant (1+s)(2+s)/2$. Then, an arbitrary element in R is of the form

$$a_o + a_1 p + a_2 p^2 + \sum_{i}^{s} b_i u_i + \sum_{l=1}^{d} c_l p u_l + \sum_{i,j=1}^{s} d_{ij} u_i u_j, \quad (a_o,\ a_1,\ b_i,\ c_l,\ d_{ij} \in K_o).$$

Clearly, the products $u_i u_j \in \mathcal{J}^2$. Hence, we conclude that $p^2,\ pu_i$ and $u_i u_j$ $(i,\ j = 1, \dots, s)$ generate $\mathcal{J}^2$. In fact, we can write any $v \in \mathcal{J}^2$ as a linear combination of $p^2,\ pu_i$ and $u_i u_j$ as follows:

$$v = \alpha_0 p^2 + \sum_{i=1}^{d} \alpha_i p u_i + \sum_{i,\ j=1}^{s} \alpha_{ij} u_i u_j,$$

where $\alpha_0,\ \alpha_i,\ \alpha_{ij} \in R_o/pR_o$. Clearly, $|R| = p^{3r} \cdot p^{2dr} \cdot p^{(s-d)r} \cdot p^{tr} = p^{(3+s+d+t)r}$ and $|\mathcal{J}| = p^{(2+s+d+t)r}$. (Notice that $|R_o u_i| = p^{2r}$ if $pu_i \neq 0$, and $|R_o u_i| = p^r$, if otherwise.)

In this paper, we determine explicitly the group of units of all commutative completely primary finite rings R with Jacobson radical $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, and of characteristic p^3. We treat the problem by considering fixed dimensions and bases for the vector spaces $\mathcal{J}^i/\mathcal{J}^{i+1}$ $(i = 1,\ 2)$ over the residue field $K = R/\mathcal{J}$ and by fixing the order of the ideal $\mathcal{J}^2$. First, if a ring R has $\mathcal{J}$ such that d_i =dim$_K\mathcal{J}^i/\mathcal{J}^{i+1}$ $(i = 1,\ 2)$, then as we shall see later, we may further classify R according to the behaviour of a generating set for $\mathcal{J}$. In particular, we determine the group of units of the ring R with $dim_K(\mathcal{J}/\mathcal{J}^2) = 1 + s$ and $dim_K(\mathcal{J}^2) \leqslant (1 + s)(2 + s)/2$ under the given conditions (see Section

2) on the basis elements of $\mathcal{J}^2$ over K. This complements the author's earlier solution of the problem [2] in the case when the characteristic of R is p or p^2 and $\mathcal{J}^2 \subseteq ann(\mathcal{J})$, the annihilator of $\mathcal{J}$.

2. The group of units

Let R be a commutative completely primary finite ring with Jacobson radical $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$, and of characteristic p^3. Suppose that $\mathcal{J} = (p,\ u_1,\ ...,\ u_s)$ so that $dim_K(\mathcal{J}/\mathcal{J}^2) = 1 + s$, for any integer $s \geqslant 0$. As noted above, the non-zero elements $p^2,\ pu_i\ (i = 1, \ldots, s),\ u_iu_j\ (i,\ j = 1, \ldots, s)$ span $\mathcal{J}^2$ over K. If $pu_i = 0$, then $u_i \in ann(\mathcal{J}) \supseteq \mathcal{J}^2$ and as such for every element $x \in \mathcal{J}$, we have $u_ix = xu_i = 0$. In particular, $u_iu_j = 0\ (\forall i,\ j = 1, \ldots, s)$. We also note that $p\mathcal{J} \subseteq \mathcal{J}^2$ is spanned by the non-zero elements p^2 and $pu_i\ (i = 1, \ldots, s)$, since $pu_iu_j = 0\ (\forall i,\ j = 1, \ldots, s)$.

Following the above observations, we determine the structure of the group of units $U(R)$ of the ring R under the conditions listed below:
(i) $\mathcal{J} = (p,\ u_1, \ldots, u_s)$, $pu_i = u_iu_j = 0$, so that $\mathcal{J}^2 = (p^2)$, $dim_K(\mathcal{J}^2) = 1$ and $|\mathcal{J}^2| = p^r$;
(ii) $\mathcal{J} = (p,\ u_1, \ldots, u_s)$, $\mathcal{J}^2 = p\mathcal{J}$, so that $u_iu_j = 0$, $dim_K(\mathcal{J}^2) \leqslant 1 + s$ and $|\mathcal{J}^2| \leqslant p^{(1+s)r}$; and
(iii) $\mathcal{J} = (p,\ u_1, \ldots, u_s)$, $\mathcal{J}^2 = (p^2,\ pu_i, \ldots, pu_s,\ u_iu_j)\ (i,\ j = 1, \ldots, s)$, $dim_K(\mathcal{J}^2) \leqslant (s+1)(s+2)/2$ and $|\mathcal{J}^2| \leqslant p^{[(s+1)(s+2)/2]r}$.

One easily verifies that the above cases are all commutative completely primary finite rings of characteristic p^3 with unique maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$ and $\mathcal{J}^2 \neq (0)$. Also, to distinguish (iii) from the other two cases, we assume that $pu_i \neq 0$ for at least one u_i, and $u_iu_j \neq 0$ for at least one product.

We know that for a commutative completely primary finite ring R,

$$U(R) = <b> \cdot (1 + \mathcal{J}) \cong <b> \times (1 + \mathcal{J});$$

a direct product of the $p-$group $1 + \mathcal{J}$ by the cyclic subgroup $<b>$. Thus, since the structure of $<b>$ is basic, it suffices to determine the structure of the subgroup $1 + \mathcal{J}$ in order to obtain the complete structure of $U(R)$.

There are many important results on the group of units of certain finite rings. For example, it is well known that the multiplicative group of the finite field $GF(p^r)$ is a cyclic group of order $p^r - 1$, and the multiplicative group of the finite ring $\mathbb{Z}/p^k\mathbb{Z}$, the ring of integers modulo p^k, for p a prime number, and k a positive integer, is a cyclic group of order $p^{k-1}(p-1)$ if p is odd, and is a direct product of a cyclic group of order 2 and a cyclic group of order 2^{k-2}, if $p = 2$.

Let $U(R_o)$ denote the group of units of the Galois ring $R_o = GR(p^{nr},\ p^n)$. Then, $U(R_o)$ has the following structure [5]:

Theorem 2.1. *$U(R_o) = <b> \times (1 + pR_o)$, where $<b>$ is the cyclic group of order $p^r - 1$ and $1 + pR_o$ is of order $p^{(n-1)r}$ whose structure is described below.*

(i) If (a) p is odd, or (b) $p=2$ and $n \leqslant 2$, then $1+pR_o$ is the direct product of r cyclic groups each of order $p^{(n-1)}$.

(ii) When $p=2$ and $n \geqslant 3$, the group $1+pR_o$ is the direct product of a cyclic group of order 2, a cyclic group of order $2^{(n-2)}$ and $r-1$ cyclic groups each of order $2^{(n-1)}$.

In Propositions 2.2 and 2.3, we will provide detailed proofs for the two types of rings of this paper, while in Proposition 2.1, we merely state $U(R)$ and their generators for the other type of rings, as the proofs are very similar and may be proved by slight modifications of these.

For the rest of this paper, we shall take r elements $\varepsilon_1, \ldots, \varepsilon_r$ in R_o with $\varepsilon_1 = 1$ such that $\{\overline{\varepsilon_1}, \ldots, \overline{\varepsilon_r}\}$ is a basis for the quotient ring R_o/pR_o regarded as a vector space over its prime subfield $GF(p)$.

2.1 The case when $\mathcal{J}^2 = (p^2)$ and $pu_i = u_iu_j = 0$

Let $dim_K(\mathcal{J}/\mathcal{J}^2) = 1+s$ and suppose that $\mathcal{J} = (p,\ u_1, \ldots, u_s)$, for any integer $s \geqslant 0$. Suppose further that $pu_i = u_iu_j = 0$, for every $i,\ j = 1, \ldots, s$. Then, $\mathcal{J}^2 = (p^2)$, $dim_K(\mathcal{J}^2) = 1$ and $|\mathcal{J}^2| = p^r$. The following result determines the structure of the group of units of R and its generators.

Proposition 2.1. *Let R be a ring of characteristic p^3 with maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = \{0\}$, $\mathcal{J}^2 \neq \{0\}$. Suppose further that there exist elements $u_1, \ldots, u_s$ in $\mathcal{J}$ such that the multiplication in R is defined by $pu_i = 0$, $u_iu_j = 0$, for every $i,\ j = 1,\ \ldots,\ s$. Then,*

$$U(R) \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4^{r-1} \times \underbrace{\mathbb{Z}_2^r \times \ \cdots \ \times \mathbb{Z}_2^r}_{s}, & if\ p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times \underbrace{\mathbb{Z}_p^r \times \ \cdots \ \times \mathbb{Z}_p^r}_{s}, & if\ p\ is\ odd. \end{cases}$$

Moreover, if $p=2$, then $1+\mathcal{J}$ is generated by $(-1+4\varepsilon_1)$, $(1+4\varepsilon_1)$, each of order 2, $(r-1)$ cyclic groups $< 1+2\varepsilon_j >$ $(j = 2, \ldots, r)$, each of order 4, and sr cyclic groups $< 1+\varepsilon_ju_i >$, each of order 2, for $i = 1, \ldots, s$. If p is odd, then $1+\mathcal{J}$ is generated by $1+p\varepsilon_j$ $(j = 1, \ldots, r)$, each of order p^2, and sr cyclic groups $1+\varepsilon_ju_i$ $(j = 1, \ldots, r)$, each of order p, for $i = 1, \ldots, s$.

2.2 The case when $\mathcal{J}^2 = p\mathcal{J}$

Let $dim_K(\mathcal{J}/\mathcal{J}^2) = 1+s$ and suppose that $\mathcal{J} = (p,\ u_1, \ldots, u_s)$, for any integer $s \geqslant 0$. Suppose further that $pu_i \neq 0$ for $i = 1, \ldots, d \leqslant s$ and $pu_j = 0$ for $j = d+1, \ldots, s$. Then, $u_iu_j = 0$, for every $i,\ j = 1, \ldots, s$ and $dim_K(\mathcal{J}^2) \leqslant 1+s$. The following determines the structure and generators of the group of units of the ring R.

Proposition 2.2. *Let R be a ring of characteristic p^3 with maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = \{0\}$, $\mathcal{J}^2 \neq \{0\}$. Suppose further that there exist elements $u_1, \ldots, u_s$*

in $\mathcal{J}$ such that the multiplication in R is defined by $pu_i \neq 0$, for $i = 1, \ldots, d \leqslant s$, $pu_j = 0$, for $j = d+1, \ldots, s$, and $u_i u_j = 0$, for every $i,\ j = 1,\ \ldots,\ s$. Then,

$$U(R) \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4^{r-1} \times \underbrace{\mathbb{Z}_4^r \times \ \cdots \ \times \mathbb{Z}_4^r}_{d} \\ \times \underbrace{\mathbb{Z}_2^r \times \ \cdots \ \times \mathbb{Z}_2^r}_{s-d}, \quad if\ p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times \underbrace{\mathbb{Z}_{p^2}^r \times \ \cdots \ \times \mathbb{Z}_{p^2}^r}_{d} \\ \times \underbrace{\mathbb{Z}_p^r \times \ \cdots \ \times \mathbb{Z}_p^r}_{s-d}, \quad if\ p \neq 2 \end{cases}$$

Moreover, if p is odd, then $1+\mathcal{J}$ is generated by r elements $1+p\varepsilon_k$ $(k = 1, \ldots, r)$, each of order p^2, dr elements $1+\varepsilon_k u_i$ $(k = 1, \ldots, r)$, each of order p^2, for $i = 1, \ldots, d \leqslant s$ and by $(s-d)r$ elements $1+\varepsilon_k u_j$ $(k = 1, \ldots, r)$, each of order p, for $j = d+1, \ldots, s$.

If $p = 2$, then $1+\mathcal{J}$ is a direct product of 2 cyclic groups $< -1+4\varepsilon_1 >$ and $< 1+4\varepsilon_1 >$, each of order 2, $(r-1)$ cyclic groups $< 1+2\varepsilon_k >$ $(k = 2, \ldots, r)$, each of order 4, dr cyclic groups $< 1+\varepsilon_k u_i >$, each of order 4, for $i = 1, \ldots, d \leqslant s$ and by $(s-d)r$ cyclic groups $< 1+\varepsilon_k u_j >$, each of order 2, for $j = d+1, \ldots, s$.

Proof. Suppose $pu_i \neq 0$ for $i = 1,\ \ldots,\ d \leqslant s$, $pu_j = 0$ for $j = d+1, \ldots, s$ and $u_i u_j = 0$ for $1 \leqslant i,\ j \leqslant d \leqslant s$. Let $a = 1+x$ be an element of $1+\mathcal{J}$ with the highest possible order and assume that $x \in \mathcal{J} - \mathcal{J}^2$. Then, $o(a) = p^2$. This is true because, for any ε_k $(k = 1, \ldots, r)$,

$$(1+\varepsilon_k x)^p = 1 + p\varepsilon_k x + \frac{p(p-1)}{2}(\varepsilon_k x)^2 \quad \text{(since } x^3 = 0\text{)}.$$

For every odd prime p, $(1+\varepsilon_k x)^p = 1+p\varepsilon_k x$, since $px^2 = 0$. Now,

$$\begin{aligned} (1+p\varepsilon_k x)^p &= 1 + p^2\varepsilon_k x + \frac{p(p-1)}{2}(p\varepsilon_k x)^2 \\ &= 1, \quad \text{since } p^2 x = 0 \text{ and } p^3 x^2 = 0. \end{aligned}$$

Hence, $(1+\varepsilon_k x)^{p^2} = 1$.

For $p = 2$, $(1+\varepsilon_k x)^2 = 1+2\varepsilon_k x + (\varepsilon_k x)^2$, and $(1+\varepsilon_k x)^4 = 1$, since $4x = 0,\ 6x^2 = 0,\ 4x^3 = 0$ and $x^4 = 0$.

For any prime number p and for each $k = 1,\ \ldots,\ r$, we see that $(1+\varepsilon_k p)^{p^2} = 1$, $(1+\varepsilon_k u_i)^{p^2} = 1$, for $i = 1, \ldots, d \leqslant s$, while $(1+\varepsilon_k u_j)^p = 1$, for $j = d+1, \ldots, s$.

For integers $h_{ki} \leqslant p^2$, $l_{kj} \leqslant p$, we asset that

$$\prod_{k=1}^{r}\prod_{i=1}^{d}(1+\varepsilon_k u_i)^{h_{ki}} \cdot \prod_{k=1}^{r}\prod_{j=d+1}^{s}(1+\varepsilon_k u_j)^{l_{kj}} = 1,$$

will imply $h_{ki} = p^2$, for all $k = 1,\ldots,r$ and $i = 1,\ldots,d \leqslant s$; $l_{kj} = p$, for all $k = 1,\ldots,r$ and $j = d+1,\ \ldots,\ s$.

If we set $D_{ki} = \{(1+\varepsilon_k u_i)^{h_{ki}} : h_{ki} = 1,\ldots,p^2\}, E_{kj} = \{(1+\varepsilon_k u_j)^{l_{kj}} : l_{kj} = 1,\ldots,p\}$, for all $k = 1,\ldots,r$, we see that D_{ki}, E_{kj} are all subgroups of $1+\sum R_0u_i \oplus \sum pR_0u_i$ and that D_{ki} are all of order p^2 and that E_{kj} are all of order p as indicated in their definition. Also, pairwise intersection of these subgroups is trivial.

The argument above will show that the product of the dr subgroups D_{kj}, and $(s-d)r$ subgroups E_{ki} is direct. Thus, their product will exhaust $1+\sum R_0u_i \oplus \sum pR_0u_i$.

It is straightforward to check that if $p = 2$, then $1+\mathcal{J}$ is a direct product of 2 cyclic groups $< -1+4\varepsilon_1 >$ and $< 1+4\varepsilon_1 >$, each of order 2, $(r-1)$ cyclic groups $< 1+2\varepsilon_k >$ $(k = 2,\ldots,r)$, each of order 4, dr cyclic groups $< 1+\varepsilon_k u_i >$, each of order 4, for $i = 1,\ldots,d \leqslant s$ and by $(s-d)r$ cyclic groups $< 1+\varepsilon_k u_j >$, each of order 2, for $j = d+1,\ldots,s$.

Also, if p is odd, then $1+\mathcal{J}$ is generated by r elements $1+p\varepsilon_k$ $(k = 1,\ldots,r)$, each of order p^2, dr elements $1+\varepsilon_k u_i$ $(k = 1,\ldots,r)$, each of order p^2, for $i = 1,\ldots,d \leqslant s$ and by $(s-d)r$ elements $1+\varepsilon_k u_j$ $(k = 1,\ldots,r)$, each of order p, for $j = d+1,\ldots,s$.

This completes the proof. □

2.3 The case when $\mathcal{J}^2 = (p^2,\ pu_i,\ u_iu_j)$

Let $dim_K(\mathcal{J}/\mathcal{J}^2) = 1+s$ and suppose that $dim_K(\mathcal{J}^2) \leqslant (s+1)(s+2)/2$. Then, $\mathcal{J}^2 = (p^2,\ pu_i,\ u_iu_j)$. Suppose further that $pu_i \neq 0$, for $i = 1,\ldots,d \leqslant s$ and that $pu_j = 0$, for $j = d+1,\ldots,s$. Then, $u_iu_j \neq 0$ for all $i,\ j = 1,\ldots,d \leqslant s$ (since in this case u_i, u_j are not in $ann(\mathcal{J})$); and $u_iu_j = 0$ for all $i = 1,\ldots,s$ and $j = d+1,\ldots,s$. Recall that if $pu_j = 0$, then $u_j \in ann(\mathcal{J})$ and as such $u_jx = 0$ for every $x \in \mathcal{J}$. The following describes the structure of $U(R)$ and its possible generators.

Proposition 2.3. *Let R be a ring of characteristic p^3 with maximal ideal $\mathcal{J}$ such that $\mathcal{J}^3 = \{0\}$, $\mathcal{J}^2 \neq \{0\}$. Suppose further that there exist elements $u_1,\ldots,u_s$ in $\mathcal{J}$ such that the multiplication in R is defined by $pu_i \neq 0$, for $i = 1,\ldots,d \leqslant s$, and that $pu_j = 0$, for $j = d+1,\ldots,s$ so that $u_iu_j \neq 0$ for all $i,\ j = 1,\ldots,d \leqslant s$; and $u_iu_j = 0$ for all $i = 1,\ldots,s$ and $j = d+1,\ldots,s$. Then,*

$$U(R) \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4^{r-1} \times \underbrace{\mathbb{Z}_4^r \times \ \cdots \ \mathbb{Z}_4^r}_{d} \times \underbrace{\mathbb{Z}_2^r \times \ \cdots \ \times \mathbb{Z}_2^r}_{s-d} \times \\ \underbrace{\mathbb{Z}_2^r \times \ \cdots \ \times \mathbb{Z}_2^r}_{d(d+1)/2}, \quad if\ p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times \underbrace{\mathbb{Z}_{p^2}^r \times \ \cdots \ \mathbb{Z}_{p^2}^r}_{d} \times \underbrace{\mathbb{Z}_p^r \times \ \cdots \ \times \mathbb{Z}_p^r}_{s-d} \times \\ \underbrace{\mathbb{Z}_p^r \times \ \cdots \ \times \mathbb{Z}_p^r}_{d(d+1)/2}, \quad if\ p \neq 2 \end{cases}$$

Proof. Suppose $pu_i \neq 0$ for all $i = 1, \ \ldots, \ d \leqslant s$ and $u_i u_j \neq 0$ for $1 \leqslant i, \ j \leqslant d \leqslant s$. Let $a = 1 + x$ be an element of $1 + \mathcal{J}$ with the highest possible order and assume that $x \in \mathcal{J} - \mathcal{J}^2$. Then, $o(a) = p^2$. This is true because, for any ε_k $(k = 1, \ldots, r)$,

$$(1 + \varepsilon_k x)^p = 1 + p\varepsilon_k x + \frac{p(p-1)}{2}(\varepsilon_k x)^2 \quad (\text{since } x^3 = 0).$$

If p is odd, then $(1 + \varepsilon_k x)^p = 1 + p\varepsilon_k x$, since $px^2 = 0$. Now,

$$\begin{aligned} (1 + p\varepsilon_k x)^p &= 1 + p^2 \varepsilon_k x + \frac{p(p-1)}{2}(p\varepsilon_k x)^2 \\ &= 1, \quad \text{since } p^2 x = 0 \ \text{ and } p^3 x^2 = 0. \end{aligned}$$

Hence, $(1 + \varepsilon_k x)^{p^2} = 1$.

For any odd prime number p and for each $k = 1, \ \ldots, \ r$, we see that $(1 + \varepsilon_k p)^{p^2} = 1, \ \ (1 + \varepsilon_k u_i)^{p^2} = 1$, for $i = 1, \ldots, d \leqslant s$, while $(1 + \varepsilon_k u_j)^p = 1$, for $j = d + 1, \ldots, s$, and for non-zero elements $u_i^2, \ u_i u_j \ (i \neq j)$, we have $(1 + \varepsilon_k u_i^2)^p = 1$, $(1 + \varepsilon_k u_i u_j)^p = 1$.

For integers $h_{ki} \leqslant p^2$, $l_{kj} \leqslant p$, m_{ki} and $n_{kij} \leqslant p$, we asset that

$$\prod_{k=1}^{r} \prod_{i=1}^{d} (1 + \varepsilon_k u_i)^{h_{ki}} \cdot \prod_{k=1}^{r} \prod_{j=d+1}^{s} (1 + \varepsilon_k u_j)^{l_{kj}} \cdot \prod_{k=1}^{r} \prod_{i=1}^{d} (1 + \varepsilon_k u_i^2)^{m_{ki}} \cdot$$
$$\prod_{k=1}^{r} \prod_{i, \ j=1}^{d} (1 + \varepsilon_k u_i u_j)^{n_{kij}} = 1,$$

will imply $h_{ki} = p^2$, for all $k = 1, \ldots, r$ and $i = 1, \ldots, d$; $l_{kj} = p$, for all $k = 1, \ldots, r$ and $j = d+1, \ \ldots, \ s$; $m_{ki} = p$ for all $k = 1, \ldots, r$ and $i = 1, \ \ldots, \ d$; and $n_{kij} = p$, for all $k = 1, \ldots, r$ and $i, \ j = 1, \ \ldots \ d$.

If we set

$$\begin{aligned} D_{ki} &= \{(1 + \varepsilon_k u_i)^{h_{ki}} : h_{ki} = 1, \ldots, p^2\} \\ E_{kj} &= \{(1 + \varepsilon_k u_j)^{l_{kj}} : l_{kj} = 1, \ldots, p\}, \\ F_{ki} &= \{(1 + \varepsilon_k u_i^2)^{m_{ki}} : m_{ki} = 1, \ldots, p\}, \\ G_{kij} &= \{(1 + \varepsilon_k u_i u_j)^{n_{kij}} : n_{kij} = 1, \ldots, p\}, \end{aligned}$$

for all $k = 1, \ldots, r$, we see that D_{ki}, E_{kj}, F_{ki} and G_{kij} are all subgroups of $1 + \sum R_0 u_i \oplus \sum R_0 u_i u_j$ and that D_{ki} are all of order p^2 and the others are all of order p as indicated in their definition. Also, pairwise intersection of these subgroups is trivial.

The argument above will show that the product of the dr subgroups D_{ki}, $(s - d)r$ subgroups E_{kj}, dr subgroups F_{ki} and the $r[d(d + 1)/2]$ subgroups G_{kij} is direct. Thus, their product will exhaust $1 + \sum R_0 u_i \oplus \sum R_0 u_i u_j$, and we see that the proof for the case when p is odd is complete.

Now, assume that $p = 2$. Then, for each $k = 1, \ldots, r$, we see that $(1 + \varepsilon_k u_i)^2 = 1 + 2\varepsilon_k u_i + \varepsilon_k^2 u_i^2$, $(1 + \varepsilon_k u_i)^4 = 1$, for $i = 1, \ldots, d \leqslant s$; $(1 + \varepsilon_k u_j)^2 = 1$, for $j = d + 1, \ldots, s$; and $(1 + \varepsilon_k u_i u_j)^2 = 1$, for every $i \neq j = 1, \ldots, d$.

For integers $h_{ki} \leqslant 4$, $l_{kj} \leqslant 2$, m_{ki} and $n_{kij} \leqslant 2$, we asset that

$$\prod_{k=1}^{r}\prod_{i=1}^{d}(1+\varepsilon_k u_i)^{h_{ki}} \cdot \prod_{k=1}^{r}\prod_{j=d+1}^{s}(1+\varepsilon_k u_j)^{l_{kj}} \cdot \prod_{k=1}^{r}\prod_{i=1}^{d}(1+\varepsilon_k u_i^2)^{m_{ki}} \cdot$$
$$\prod_{k=1}^{r}\prod_{i,\ j=1}^{d}(1+\varepsilon_k u_i u_j)^{n_{kij}} = 1,$$

will imply $h_{ki} = 4$, for all $k = 1, \ldots, r$; and $i = 1, \ldots, d$; $l_{kj} = 2$, for all $k = 1, \ldots, r$; and $j = d+1, \ \ldots, \ s$; $m_{ki} = 2$ for all $k = 1, \ldots, r$; and $i = 1, \ \ldots, \ d$; and $n_{kij} = 2$, for all $k = 1, \ldots, r$; and $i, \ j = 1, \ \ldots \ d$.

If we set

$$\begin{aligned} D_{ki} &= \{(1+\varepsilon_k u_i)^{h_{ki}} : h_{ki} = 1, \ldots, 4\} \\ E_{kj} &= \{(1+\varepsilon_k u_j)^{l_{kj}} : l_{kj} = 1,\ 2\}, \\ F_{ki} &= \{(1+\varepsilon_k u_i^2)^{m_{ki}} : m_{ki} = 1,\ 2\}, \\ G_{kij} &= \{(1+\varepsilon_k u_i u_j)^{n_{kij}} : n_{kij} = 1,\ 2\}, \end{aligned}$$

for all $k = 1, \ldots, r$, we see that D_{ki}, E_{kj}, F_{ki} and G_{kij} are all subgroups of $1 + \sum R_0 u_i \oplus \sum R_0 u_i u_j$ and that D_{ki} are all of order 4 and the others are all of order 2 as indicated in their definition. Also, pairwise intersection of these subgroups is trivial.

The argument above will show that the product of the dr subgroups D_{ki}, $(s-d)r$ subgroups E_{kj}, dr subgroups F_{ki} and the $r[d(d+1)/2]$ subgroups G_{kij} is direct. Thus, their product will exhaust $1 + \sum R_0 u_i \oplus \sum R_0 u_i u_j$, and we see that the proof for the case when $p = 2$ is complete. □

This completes our investigation of the structure of the group of units of commutative completely primary finite rings of characteristic p^3 with unique maximal ideals $\mathcal{J}$ such that $\mathcal{J}^3 = (0)$, $\mathcal{J}^2 \neq (0)$ with given constraints on the generators for the ideals $\mathcal{J}$ and $\mathcal{J}^2$.

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Solvability in fuzzy multigroup context

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Abstract. The adventure of fuzzy sets has witnessed myriad of group's theoretical concepts being studied as fuzzy algebraic structures. In this present work, the notion of solvable fuzzy multigroups is considered for the first time as an algebraic structure in fuzzy multigroup context. Solvable series for a fuzzy multigroup is defined in such a way that the family of the fuzzy submultigroups of the considered fuzzy multigroup has the same support. Some precursory results in normality and quotient of fuzzy multigroups are considered. It is established that there exists an if and only if condition between the solvability of a fuzzy multigroup and its support. Finally, certain results on solvable fuzzy multigroups are obtained.

Keywords: fuzzy algebra, fuzzy multigroup, normality in fuzzy multigroup, quotient fuzzy multigroup, solvable fuzzy multigroup.

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1. Introduction

The introduction of fuzzy sets by Zadeh [32] was a boost to the solution of uncertainties. With fuzzy sets as foundation, Rosenfeld [27] proposed fuzzy group as an algebraic structure of fuzzy set. Many authors have extended some group's theoretic notions to fuzzy sets as seen in [4, 23, 24]. By a way of generalization, Yager [31] introduced the notion of fuzzy multiset as a special fuzzy set that allows the repetitions of membership functions of elements of a set. Many works have been extensively carried out on fuzzy multisets and applied to many real-life problems [3, 20, 21, 22, 30].

In a continuation of the study of fuzzy algebra, Shinoj et al. [29] proposed fuzzy multigroup as an application of group theory to fuzzy multisets and deduced some related results. As a follow up, the analog of subgroups in fuzzy multigroup context was studied [5]. The notion of commutative fuzzy multigroups has been studied and a number of results were presented [2, 6]. Some group's analog notions have been investigated in fuzzy multigroup context [1, 7, 8, 12, 14, 16, 17, 18, 15, 19]. The concept of direct product of fuzzy multigroups and its generalization have been discussed [9, 13]. To show the connection between fuzzy multigroup and group, the notion of alpha-cuts of fuzzy multigroups was proposed [10, 11].

Though several concepts of group theory have been extended to fuzzy multigroups via fuzzy multisets, the notion of solvable/soluble fuzzy multigroups has not been investigated in fuzzy multigroup context. This paper seeks to introduce solvable fuzzy multigroup. The concept of solvable groups has been studied in fuzzy group setting [28, 26]. The rest of the paper is delineated as follows: Section 2 presents the notions of fuzzy multisets, fuzzy multigroups and certain existing results, Section 3 presents the concept of solvable fuzzy multigroups and discusses certain of its properties, and Section 4 summarizes and gives recommendations for future studies.

2. Preliminaries

We denote a non-empty set as X and a group as G throughout the paper.

Definition 2.1 ([32]). *A fuzzy subset F of X is an object characterized by the form*

$$F = \{\langle x, \mu_F(x)\rangle \mid x \in X\}, \tag{1}$$

where the function $\mu_F\colon X \to [0,1]$ defines the membership grade of x in X.

Definition 2.2 ([31]). *A fuzzy multiset A of X is a structure of a form*

$$A = \{\langle x, CM_A(x)\rangle \mid x \in X\} \tag{2}$$

characterized by a count membership function

$$CM_A\colon X \to N^I \text{ or } CM_A\colon X \to [0,1] \to N, \tag{3}$$

where $I = [0, 1]$, $N = \{0, 1, 2, \cdots\}$ *and* $CM_A(x) = (\mu_A^1(x), \mu_A^2(x), \cdots, \mu_A^n(x))$ *such that* $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^n(x)$.

Definition 2.3 ([31]). *Let A and B be fuzzy multisets of X. Then*

(i) $A = B \iff CM_A(x) = CM_B(x)$, $\forall x \in X$,

(ii) $A \subseteq B \iff CM_A(x) \leq CM_B(x)$, $\forall x \in X$,

(iii) $A \cap B \implies CM_{A\cap B}(x) = \min(CM_A(x), CM_B(x))$, $\forall x \in X$,

(iv) $A \cup B \implies CM_{A\cup B}(x) = \max(CM_A(x), CM_B(x))$, $\forall x \in X$,

(v) $A \oplus B \implies CM_{A\oplus B}(x) = CM_A(x) \oplus CM_B(x)$, $\forall x \in X$.

Definition 2.4 ([29]). *A fuzzy multiset A of G is a fuzzy multigroup if we have (i) $CM_A(xy) \geq \min(CM_A(x), CM_A(y))$, and (ii) $CM_A(x^{-1}) = CM_A(x)$, $\forall x, y \in G$. Because*

$$\begin{aligned} CM_A(e) &= CM_A(xx^{-1}) \geq \min(CM_A(x), CM_A(x)) \\ &= CM_A(x), \forall x \in G, \end{aligned}$$

where e is the identity element of G, then $CM_A(e)$ is the upper bound of A, which is called the tip of A.

Definition 2.5 ([29]). *Let A be a fuzzy multigroup of G. Then, the support of A is the set $supp(A) = \{x \in G \mid CM_A(x) \geq 0\}$.*

Proposition 2.6 ([29]). *The support of a fuzzy multigroup A of G is a subgroup of G.*

Definition 2.7 ([5]). *Let A and B be fuzzy multigroups of G. Then, the product $A \circ B$ is defined to be a fuzzy multiset of G as follows:*

$$CM_{A\circ B}(x) = \begin{cases} \bigvee_{x=yz} \min(CM_A(y), CM_B(z)), & \text{if there exist } y, z \in G \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8 ([6]). *A fuzzy multigroup A of G is said to be commutative if $CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$. Certainly, if G is a commutative group, then a fuzzy multigroup A of G is commutative.*

Definition 2.9 ([5]). *Let A and B be fuzzy multigroups of G. We say A is a fuzzy submultigroup of B if $A \subseteq B$. Again, A is a proper fuzzy submultigroup of B if $A \subseteq B$ and $A \neq B$.*

Definition 2.10 ([7]). *Let A be a fuzzy submultigroup of a fuzzy multigroup B of G. We say A is normal in B if $CM_A(xy) = CM_A(yx) \iff CM_A(y) = CM_A(x^{-1}yx)$, $\forall x, y \in G$, and we write $A \lhd B$.*

Remark 2.11. Certainly, every normal fuzzy submultigroup is self-normal and abelian.

Definition 2.12 ([14])**.** *Suppose A is a fuzzy submultigroup of a fuzzy multigroup B of G. Then, the fuzzy submultiset yA of B for $y \in G$ defined by $CM_{yA}(x) = CM_A(y^{-1}x)$, $\forall x \in G$ is called the left fuzzy comultiset of A. Similarly, the fuzzy submultiset Ay of B for $y \in G$ defined by $CM_{Ay}(x) = CM_A(xy^{-1})$, $\forall x \in G$ is called the right fuzzy comultiset of A.*

Definition 2.13 ([14])**.** *Suppose A and B are fuzzy multigroups of G and $A \triangleleft B$. Then, the union of the set of left/right fuzzy comultisets of A such that $xA \circ yA = xyA$, $\forall x, y \in G$ is called a quotient fuzzy multigroup of B by A, denoted by B/A.*

3. On solvable fuzzy multigroups

Before investigating the notion of solvability in fuzzy multigroup, we first present the following results which are helpful in establishing solvable fuzzy multigroups.

Theorem 3.1. (*i*) *Every abelian fuzzy multigroup is self-normal.* (*ii*) *If A and B are fuzzy multigroups of G such that $A \triangleleft B$, then A is self-normal.*

Proof. Suppose A is an abelian fuzzy multigroup of G. Then

$$CM_A(xy) = CM_A(yx),\ \forall x, y \in G,$$

and so $CM_A(y) = CM_A(x^{-1}yx)$. Hence $A \triangleleft A$, which proves (i). Again, if $A \triangleleft B$ then $CM_A(x) \leq CM_B(x)$ for all $x \in G$ and $CM_A(x) > 0 < CM_A(y) \implies CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$. Thus, (ii) holds from (i). □

Theorem 3.2. *Let A be a fuzzy multigroup of G. Then supp(A) is abelian iff A is abelian.*

Proof. Let $x, y \in \text{supp}(A)$. If $\text{supp}(A)$ is abelian then $xy = yx$, and so $CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$.

Conversely, if A is abelian then $CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$. Thus $\text{supp}(A)$ is an abelian group because $CM_A(xy) > 0 < CM_A(yx) \implies xy = yx$, $\forall x, y \in \text{supp}(A)$. □

Theorem 3.3. *Let A, B and C be fuzzy multigroups of G such that:* (*i*) *B/A and C/A are both in the canonical form,* (*ii*) *$B/A \triangleleft C/A$, and (iii) $(C/A)/(B/A)$ is abelian. Then, $B \triangleleft C$ and C/B is abelian.*

Proof. Let H, H_1 and H_2 be the supports of A, B and C, respectively, and let H' be the zone of A and C. Then B/A and C/A are both fuzzy multigroups of H'/H. If $x \in H_1$, then $CM_B(x) = CM_{B/A}(xH) \leq CM_{C/A}(xH) = CM_C(x)$, and $B \subseteq C$. Thus $CM_C(x) > 0 < CM_C(y) \implies CM_{C/A}(xH) = CM_C(x) > 0 < CM_C(y) = CM_{C/A}(yH) \implies CM_{B/A}(xyH) = CM_{B/A}(yxH) \implies CM_B(xy) = CM_B(yx)$ for all $x, y \in G$, and so $B \triangleleft C$.

Again, we see that, $\text{supp}(C/B) = H_2/H_1$ is equivalent to $(H_2/H)/(H_1/H) = \text{supp}(C/A)/\text{supp}(B/A)$, and it is abelian. By Theorem 3.2, it follows that C/B is abelian. □

Now, we define the notion of solvability of fuzzy multigroup as follows:

Definition 3.4. *If A is a fuzzy multigroup of G, then there must exist a chain of successive fuzzy submultigroups of A:*

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A, \tag{4}$$

such that $supp(A_0) = supp(A_1) = \cdots = supp(A_n) = supp(A)$.

Thus (4) can be rewritten as

$$CM_{A_0}(x) \leq CM_{A_1}(x) \leq \cdots \leq CM_{A_n}(x) = CM_A(x) \text{ for all } x \in G. \tag{5}$$

Albeit, if A is a trivial fuzzy multigroup, we have $A_0 = A$.

Definition 3.5. *A fuzzy multigroup A of G is solvable/soluble if there exists a chain of successive fuzzy submultigroups*

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A, \tag{6}$$

where $A_i \lhd A_{i+1}$ and A_{i+1}/A_i is abelian for all $0 \leq i \leq n-1$.

Thus, such a finite chain of successive fuzzy submultigroups of A is a solvable/soluble series for A denoted by A_i. Without contradiction, the solvable series for A can be written as

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A. \tag{7}$$

Theorem 3.6. *Let A be a fuzzy multigroup of a group G. Then A is solvable iff $supp(A)$ is a solvable group.*

Proof. Let A be a solvable fuzzy multigroup of G. Then there exists a solvable series of A as follows:

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A.$$

Set $H = \text{supp}(A)$, i.e., H is a subgroup of G. Then

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$$

is a solvable series for H since $\text{supp}(A) = \text{supp}(A_i)$. Thus H is a solvable group.

Conversely, let $H = \text{supp}(A)$ be solvable. Then

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$$

is a solvable series for H. Consequently, we have

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A,$$

which is a solvable series for A. Hence A is a solvable fuzzy multigroup of G □

Theorem 3.7. *Let A and B be fuzzy multigroups of G with the same support H such that $A \subseteq B$ and A is self-normal. If A is solvable, then B is a solvable fuzzy multigroup of G.*

Proof. Let $A_0 \lhd A_1 \lhd \cdots \lhd A_n = A$ be a solvable series for A. Because A is self-normal and $\text{supp}(A) = \text{supp}(B) = H$, we get $A \lhd B$. Consequently, we have $\text{supp}(B/A) = H/H = H$ and is abelian. Thus

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A \lhd B$$

is a solvable series for B. Hence B is a solvable fuzzy multigroup of G. □

Theorem 3.8. *Let B be a solvable fuzzy multigroup of G and let A be a self-normal fuzzy submultigroup of B with $A \subseteq B_i$. Then A is solvable.*

Proof. Let $B_0 \lhd B_1 \lhd \cdots \lhd B_n = B$ be a solvable series for B since B is solvable. Because $A \subseteq B_i$, we have

$$B_0 \cap A \subseteq B_1 \cap A \subseteq \cdots \subseteq B_n \cap A = A.$$

Clearly, $CM_{B_1 \cap A}(x) > 0 < CM_{B_1 \cap A}(y) \Longrightarrow CM_{B_1}(x) > 0 < CM_{B_1}(y)$ and $CM_A(x) > 0 < CM_A(y) \Longrightarrow CM_{B_1}(xy) > 0 < CM_{B_1}(yx)$ and $CM_A(xy) > 0 < CM_A(yx) \Longrightarrow CM_{B_1 \cap A}(xy) = CM_{B_1 \cap A}(yx)$ for all $x, y \in G$. Thus

$$B_0 \cap A \lhd B_1 \cap A \lhd \cdots \lhd B_n \cap A = A.$$

Again, let $H_i = \text{supp}(B_i)$ and $H = \text{supp}(A)$. Then we get a quotient $(H_2 \cap H)/(H_1 \cap H)$, which is abelian because H_2/H_1 is abelian. The same logic holds for the other quotients, and thus

$$B_0 \cap A \subseteq B_1 \cap A \subseteq \cdots \subseteq B_n \cap A = A$$

is the solvable series for A. Hence A is solvable. □

Theorem 3.9. *Let A be a normal fuzzy submultigroup of a fuzzy multigroup B of G, and let B be self-normal. If A and B/A are solvable, then B is a solvable fuzzy multigroup of G.*

Proof. Let B'/A' be the canonical form of B/A. Then $A \subseteq A'$, $B \subseteq B'$, $\text{supp}(A') = \text{supp}(A) = H_1$ and $\text{supp}(B') = \text{supp}(B) = H_2$. Thus, there exists a solvable series

$$C_0 \lhd C_1 \lhd \cdots \lhd C_n = B'/A'.$$

Set $A' = A_m$ and $B' = B_n$. Assume there exist fuzzy multigroups B_i of G such that $A' \lhd B_i \lhd B_{i+1}$ and $C_i = B_i/A'$ in the canonical form for $0 \leq i \leq n-1$. Thus

$$B_0/A' \lhd B_1/A' \lhd \cdots \lhd B_n/A' = B'/A'$$

is a solvable series for B'/A'. Since B_0/A' is a trivial fuzzy submultigroup of B'/A', then it is meet to say that $B_0 = A'$. By Theorem 3.3, we have

$$A' = B_0 \lhd B_1 \lhd \cdots \lhd B_n = B', \tag{8}$$

where B_{i+1}/B_i is abelian for $0 \leq i \leq n-1$.

Next, A is self-normal by Theorem 3.1, and A' is solvable by Theorem 3.7. Thus, there exists a solvable series for A' as follows:

$$A_0 \lhd A_1 \lhd \cdots \lhd A_m = A'. \tag{9}$$

By juxtaposing (8) and (9), we have a solvable series for B'. Hence B is solvable by Theorem 3.8. □

4. Conclusion

In this paper, we have defined solvable fuzzy multigroup for the first time as an algebraic structure in fuzzy multigroup context and obtained some results. Solvable series of a fuzzy multigroup was defined in such a way that the family of the fuzzy submultigroups of the considered fuzzy multigroup has the same support. Certain results in normality and quotient of fuzzy multigroups were considered. Some results on solvable fuzzy multigroups could still be exploited and the notion of nilpotency is an interest area to consider in fuzzy multigroup context.

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Total edge irregularity strength and edge irregular reflexive labeling for calendula graph

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Abstract. The calendula graph $Cl_{m,n}$ is a graph constructed from a cycle on m vertices C_m and m copies of C_n which are $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ and pasting the $i-$th edge of C_m to an edge of C_{n_i} for each $i \in \{1, 2, \ldots, m\}$. For a simple graph $G(V, E)$ a labeling of vertices and edges by a mapping $\Phi : V(G) \cup E(G) \rightarrow \{ 1, 2, \ldots, k \}$ providing that the weights of any two pair of edges are distinct is called *an edge irregular total $k-$labeling*, where the weight of an edge is the sum of the label of the edge itself and the labels of its two end vertices. If k is minimum and G admits an edge irregular total $k-$labeling, then k is called the total edge irregularity strength, $tes(G)$. The total $k-$ labeling is called *the reflexive edge strength* of G if the edge labeling $\Phi_e : E(G) \rightarrow \{1, 2, \ldots, k_e\}$ and a vertex labeling $\Phi_v : V(G) \rightarrow \{0, 2, 4, \ldots, 2k_v\}$, where $k = \max\{k_e, , 2k_v\}$. In the current paper, we investigate the existence of edge irregular total $k-$ labeling for the calendula graphs $Cl_{m,n}$ and precise the exact value of total edge irregularity strength of calendula graphs $Cl_{m,n}$. Besides, we explore the presence of edge reflexive irregular $r-$ labeling for calendula graphs and determine the perfect value of reflexive edge strength.

Keywords: irregular labeling, total edge irregularity strength, edge irregular reflexive labeling, reflexive edge strength, calendula graph.

1. Introduction

Graph labeling is one of the fundamental mathematical disciplines in graph theory. There are numerous applications of graph labeling in multiple areas such as coding hypothesis, computer science, physics, and astronomy. For more interesting applications of graph labeling see [1, 2]. A labeling of a graph $G = (V, E)$ is a mapping that carries graph elements (edges or vertices, or both) to

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positive integers subject to certain restrictions. If the domain is the vertex-set or the edge-set, the labeling is called vertex labeling or edge labeling respectively. Similarly if the domain is $V(G) \cup E(G)$, then the labeling is called total labeling. There are many different kinds of graph labeling (see [3, 4, 5, 6, 7]) all that kinds of labeling problem will have the following three common characteristics. A set of numbers from which vertex or edge labels are chosen, a rule that assigns a value to each edge or vertex, and a condition that these values must satisfy. A comprehensive survey of graph labeling is given in [8].

Definition 1.1 ([9]). *Let C_m be a cycle of length m with vertices $u_1, u_2, \cdots, u_m$. Let C_{n_i}, $1 \leq i \leq m$ be m copies of a cycle of length n, and v_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ be the vertices of m copies of C_n. Let $a_i = u_i u_{i+1}$ denote to the edge of the cycle C_m for $1 \leq i \leq m-1$ and $a_m = u_m u_1$. Let $e_{ij} = v_{ij} v_{i(j+1)}$ denote to the edges of m copies of C_n for $1 \leq i \leq m$, $1 \leq j \leq n-1$ and $e_{in} = v_{in} v_{i1}$ for $1 \leq i \leq m$. The calendula graphs, denoted by $Cl_{m,n}$ obtained by pasting each edge a_i of C_m to an edge e_{in} of C_{n_i} for each $1 \leq i \leq m$, i.e., $a_i \equiv e_{in}$, $1 \leq i \leq m$ and $u_i \equiv v_{i1} \equiv v_{(i-1)n}$, $2 \leq i \leq m-1$. It is obvious that the order of $Cl_{m,n}$ is $m(n-1)$ and the size of $Cl_{m,n}$ is mn , see Fig. 1.*

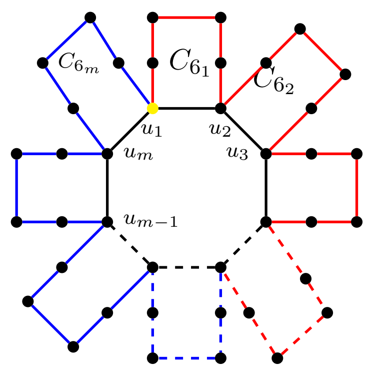

Figure 1: The Calendula graph $Cl_{m,6}$

2. Total edge irregularity strength for calendula graphs

An irregular assignment of G was defined by Chartrand et al. in [10] as a $d-$ labeling of the edges $\Theta : E \longrightarrow \{1, 2, \cdots, d\}$ such that the vertex weights $wt_\Theta(x) = \sum \Theta(xy)$, where the sum is over all vertices y adjacent to x are distinctive for all vertices, i.e., $wt_\Theta(x) \neq wt_\Theta(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. The smallest d for which there is an irregular assignment is the irregularity strength, $S(G)$, this graph parameter $S(G)$ is an invariant for each graph

Bača et al. [11] defined the notion of an edge irregular total $k-$ labeling of a graph $G=(V,E)$ as a labeling of the vertices and edges of G, $\Phi : V\cup E \longrightarrow \{1,2,\cdots,k\}$ such that the edge weights $wt_{\Phi}(xy)=\Phi(x)+\Phi(y)+\Phi(xy)$ are different for all edges, i.e., $wt_{\Phi}(xy)\neq wt_{\Phi}(x^{'}y^{'})$ for all edges $xy, x^{'}y^{'}\in E$ with $xy\neq x^{'}y^{'}$. They also defined the *total edge irregularity strength* of G, $tes(G)$, to be the minimum k for which the graph G has an edge irregular total $k-$labeling. Moreover, in [11], for any graph G a lower bound on the total edge irregularity strength is given by

$$\max\left\{ \left\lceil \frac{\triangle(G)+2}{3} \right\rceil, \ \left\lceil \tfrac{|E(G)|+2}{3} \right\rceil \right\} \leq tes(G) \tag{1}$$

where $\triangle(G)$ is the maximum degree of G.

Since then, many researchers try to find exact values for the total edge irregularity strength of graphs. In [12] Ivanĉo et al. proved that for any tree T $tes(T)$ is equal to its lower bound. Results on the total edge irregularity strength can be found in [13, 14, 15, 16, 17, 18, 19].

before we progress to our main result we discuss the total edge irregularity strength for a small case.

Theorem 2.1. *Let $n\geq 4$ be even positive integer and $Cl_{4,n}$ be the calendula graph. Then*

$$tes(Cl_{4,n}) = \left\lceil \frac{4\ n+2}{3} \right\rceil$$

Proof. The calendula graph $Cl_{4,n}$ has $|V(Cl_{4,n})| = 4(n-1)$, $|E(Cl_{4,n})| = 4n$ and the maximum degree $\triangle(Cl_{4,n})=4$. Thus, the inequality (1) becomes

$$\left\lceil \frac{4\ n+2}{3} \right\rceil \ \leq \ tes(Cl_{4,n}).$$

To prove the equality, we need to show that there exist an edge irregularity total $k-$ labeling, $k=\lceil \frac{4\ n+2}{3} \rceil$, there are three cases

Case (1). When $n\equiv\ 2\ mod\ 3$, $n\geq 4$. Suppose that $k=\lceil \frac{4\ n+2}{3} \rceil$, we construct the total $k-$ labeling function $\Phi : V(Cl_{4,n})\cup E(Cl_{4,n}) \longrightarrow \{1,2,\cdots,k\}$ as follows:

$$\Phi(v_{1j}) = \begin{cases} 1 & \text{if } \ j=\ 1; \\ j-1 & \text{if } \ 2\leq j\leq n-2\ ; \\ 2n-k-2 & \text{if } \ j=\ n-1; \\ k & \text{if } \ j=\ n. \end{cases}$$

$$\Phi(v_{2j}) = \begin{cases} 2n-k+2 & \text{if } \ j=\ 2; \\ n-2 & \begin{array}{l} \text{for } 3\leq j\leq \frac{n}{2},\ \ if\ \ n\ is\ even\ ; \\ \text{or } \ 3\leq j\leq \frac{n-1}{2},\ \ if\ \ n\ is\ odd\ ; \end{array} \\ k & \begin{array}{l} \text{for } \frac{n}{2}+1\leq j\leq n,\ \ if\ \ n\ is\ even\ ; \\ \text{or } \ \frac{n+1}{2}\leq j\leq n,\ \ if\ \ n\ is\ odd\ ; \end{array} \end{cases}$$

$$\Phi(v_{3j}) = \begin{cases} k & \text{for } 1 \le j \le \frac{n}{2},\ \ if\ \ n\ is\ even\ ; \\ & \text{or }\ 1 \le j \le \frac{n+1}{2},\ \ if\ \ n\ is\ odd\ ; \\ n-2 & \text{for } \frac{n}{2}+1 \le j \le n-2,\ \ if\ \ n\ is\ even\ ; \\ & \text{or }\ \frac{n+3}{2} \le j \le n-2,\ \ if\ \ n\ is\ odd\ ; \\ 2n-k+3 & \text{if }\ \ j = n-1; \\ k & \text{if }\ \ j = n. \end{cases}$$

$$\Phi(v_{4j}) = \begin{cases} 2n-k-1 & \text{if }\ \ j = 2; \\ n-j & \text{if }\ \ 3 \le j \le n-1; \\ 1 & \text{if }\ \ j = n. \end{cases}$$

$$\Phi(e_{1j}) = \begin{cases} 1 & \text{if }\ \ j = 1; \\ 2 & \text{if }\ \ 2 \le j \le n-3; \\ k-n+2 & \text{if }\ \ j = n-2; \\ 1 & \text{if }\ \ j = n-1; \\ \dfrac{k}{2}-2 & \text{if }\ \ j = n. \end{cases}$$

$$\Phi(e_{2j}) = \begin{cases} 1 & \text{if }\ \ j = 1; \\ k-n+5 & \text{if }\ \ j = 2; \\ 5+2j & \text{for } 3 \le j \le \frac{n}{2}-1,\ \ if\ \ n\ is\ even\ ; \\ & \text{or }\ 3 \le j \le \frac{n-3}{2},\ \ if\ \ n\ is\ odd\ ; \\ 2k-2n-1 & \text{if }\ \ j = \frac{n}{2}, and\ \ n\ \ even; \\ 2k-2n-2 & \text{if }\ \ j = \frac{n-1}{2}, and\ \ n\ \ odd; \\ k-2n+2j-3 & \text{for } \frac{n}{2}+1 \le j \le n-1,\ \ if\ \ n\ is\ even\ ; \\ & \text{or }\ \frac{n+1}{2} \le j \le n-1,\ \ if\ \ n\ is\ odd\ ; \\ k-3 & \text{if }\ \ j = n. \end{cases}$$

$$\Phi(e_{3j}) = \begin{cases} 2(2n-k-j+1) & \text{for } 1 \le j \le \frac{n}{2}-1,\ \ if\ \ n\ is\ even\ ; \\ & \text{or }\ 1 \le j \le \frac{n-1}{2},\ \ if\ \ n\ is\ odd\ ; \\ 2n-k+4 & \text{if }\ \ j = \frac{n}{2}\ \ and\ \ n\ \ even; \\ 2n-k+3 & \text{if }\ \ j = \frac{n+1}{2}\ \ and\ \ n\ \ odd; \\ 2(n-j+3) & \text{for } \frac{n}{2}+1 \le j \le n-3,\ \ if\ \ n\ is\ even\ ; \\ & \text{or }\ \frac{n+3}{2} \le j \le n-3,\ \ if\ \ n\ is\ odd\ ; \\ k-n+5 & \text{if }\ \ j = n-2; \\ 1 & \text{if }\ \ j = n-1; \\ k-2 & \text{if }\ \ j = n. \end{cases}$$

$$\Phi(e_{4j}) = \begin{cases} 1 & \text{if} \quad j = 1; \\ k-n+2 & \text{if} \quad j = 2; \\ 3 & \text{if} \quad 3 \le j \le n-2; \\ 2 & \text{if} \quad j = n-1; \\ 2n-k+1 & \text{if} \quad j = n. \end{cases}$$

Case (2). When $n \equiv 0 \ mod\ 3$. The labeling function $\Phi : V(Cl_{4,n}) \cup E(Cl_{4,n}) \longrightarrow \{1, 2, \cdots, k\}$ characterized as in case (1) in all vertices but with some modifications in edges given by

$$\Phi(e_{1j}) = \frac{k-3}{2} \qquad if \qquad j = n$$

$$\Phi(e_{2j}) = \begin{cases} 2k-2n & \text{if} \quad j = \frac{n}{2}, and \ \ n \ \ even; \\ 2k-2n-1 & \text{if} \quad j = \frac{n-1}{2}, and \ \ n \ \ odd; \\ k-2n+2j-2 & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \ if \ \ n \ is \ even \ ; \\ & \text{or } \ \frac{n+1}{2} \le j \le n-1, \ \ if \ \ n \ is \ odd \ ; \\ k-2 & \text{if} \quad j = n. \end{cases}$$

$$\Phi(e_{3j}) = k-1 \qquad if \qquad j = n$$

Case (3). When $n \equiv 1 \ mod\ 3$,

• If $n = 4$, the labeling $\Phi : V(Cl_{4,4}) \cup E(Cl_{4,4}) \longrightarrow \{1, 2, \cdots, k = 6\}$ characterized as in Fig. 2.

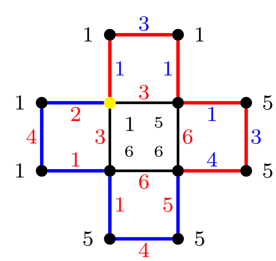

Figure 2: The calendula graph $Cl_{4,4}$ with an edge irregularity total $k = 6$ labeling

• If $n > 4$ the labeling function $\Phi : V(Cl_{4,n}) \cup E(Cl_{4,n}) \longrightarrow \{1, 2, \cdots, k\}$ defined as in case (1) in all vertices but with some modifications in edges given by

$$\Phi(e_{1j}) = \frac{k}{2} - 1 \qquad if \qquad j = n$$

$$\Phi(e_{2j}) = \begin{cases} 2k-2n+1 & \text{if } \ j = \ \frac{n}{2}, and \ \ n \ \ even; \\ 2k-2n & \text{if } \ j = \ \frac{n-1}{2}, and \ \ n \ \ odd; \\ k-2n+2j-1 & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \ if \ \ n \ is \ even \ ; \\ & \text{or } \frac{n+1}{2} \le j \le n-1, \ \ if \ \ n \ is \ odd \ ; \\ k-1 & \text{if } \ j = \ \ n. \end{cases}$$

$\Phi(e_{3j}) = \ k \qquad if \qquad j = \ n$

In all cases, all the vertex and edge labels are at most $k = \lceil \frac{4\ n+2}{3} \rceil$, also under the labeling Φ the weights of the edges are given by

$$\forall \ \ i = \ 1, \ 2 \qquad wt_{\Phi}(e_{ij}) \ = 2n(i-1)+2j+1 \ , \qquad\qquad 1 \le j \le n$$

$$\forall \ \ i = \ 3, 4 \qquad wt_{\Phi}(e_{ij}) = \begin{cases} 2n(5-i)+2-2j & \text{if } \ 1 \le j \le n-1; \\ 2n(\ 5-i)+2 & \text{if } \ \ j = n. \end{cases}$$

We can see that the weights of edges in the first half cycles C_{n_1}, C_{n_2} form an increasing sequence of consecutive odd integers from 3 up to $4n+1$. For the second half cycles C_{n_3}, C_{n_4}, the weights of edges form a decreasing sequence of consecutive even integers from $4n+2$ up to 4. The labeling Φ is the required edge irregular total $k = \lceil \frac{4\ n+2}{3} \rceil$ labeling. This concludes the proof. □

Theorem 2.2. *Let* $n \ge 4$ *be even positive integer and* $Cl_{5,n}$ *be the calendula graph. Then*

$$tes(Cl_{5,n}) = \lceil \ \frac{5\ n+2}{3} \ \rceil \ .$$

Proof. The calendula graph $Cl_{5,n}$ has $|V(Cl_{5,n})| \ = 5(n-1)$, $|E(Cl_{5,n})\,| = 5n$ and the maximum degree $\triangle(Cl_{m,n}) = 4$. Thus, the inequality (1) becomes

$$\lceil \ \frac{5\ n+2}{3} \ \rceil \ \ \le \ tes(Cl_{5,n}).$$

To prove the inverse inequality, we define the function $\Phi : V(Cl_{5,n}) \cup E(Cl_{5,n}) \longrightarrow \{1,2,\cdots,k\}$ to be a total $k = \lceil \frac{5\ n+2}{3} \rceil$ labeling as follows:

$$\Phi(v_{1j}) = \begin{cases} 1 & \text{if } \ j = 1; \\ j-1 & \text{if } \ 2 \le j \le n. \end{cases}$$

$$\Phi(v_{2j}) = \begin{cases} n-1 & \text{for } 1 \le j \le \frac{n}{2}, \ \ if \ \ n \ \ is \ even \ ; \\ & \text{or } \ 1 \le j \le \frac{n+1}{2}, \ \ if \ \ n \ \ is \ \ odd \ ; \\ \lceil \ \dfrac{4\ n+2}{3} \ \rceil - 2 & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \ if \ \ n \ \ is \ \ even \ ; \\ & \text{or } \ \frac{n+3}{2} \le j \le n-1, \ \ if \ \ n \ \ is \ \ odd \ ; \\ k & \text{if } \ j = \ n. \end{cases}$$

$\Phi(v_{3j}) = \ \ k \qquad if \qquad 1 \le j \le n.$

$$\Phi(v_{4j}) = \begin{cases} \lceil \dfrac{4\ n+2}{3} \rceil - 2 & \begin{array}{l}\text{for } 2 \le j \le \frac{n}{2},\quad if \quad n \quad even \ ; \\ \text{or} \quad 2 \le j \le \frac{n-1}{2},\quad if \quad n \quad odd \ ;\end{array} \\ n-1 & \begin{array}{l}\text{for } \frac{n}{2}+1 \le j \le n,\quad if \quad n \quad even \ ; \\ \text{or} \quad \frac{n+1}{2} \le j \le n,\quad if \quad n \quad odd.\end{array} \end{cases}$$

$$\Phi(v_{5j}) = \ n-j \qquad if \qquad 1 \le j \le n-1 \quad \text{and} \qquad \Phi(v_{5n}) = \ 1\ .$$

$$\Phi(e_{1j}) = \begin{cases} 1 & \text{if} \quad j=1; \\ 2 & \text{if} \quad 2 \le j \le n-1; \\ n+1 & \text{if} \quad j = \ n. \end{cases}$$

$$\Phi(e_{2j}) = \begin{cases} 3+2j & \begin{array}{l}\text{for } 1 \le j \le \frac{n}{2}-1,\quad if \quad n \quad even \ ; \\ \text{or} \quad 1 \le j \le \frac{n-1}{2},\quad if \quad n \quad odd \ ;\end{array} \\ 2n+4-\lceil \dfrac{4\ n+2}{3} \rceil & \text{if} \quad j = \ \frac{n}{2}, and \quad n \quad even; \\ 2n+5-\lceil \dfrac{4\ n+2}{3} \rceil & \text{if} \quad j = \ \frac{n+1}{2}, and \quad n \quad odd; \\ 2n+2j+5-2\lceil \dfrac{4\ n+2}{3} \rceil & \begin{array}{l}\text{for } \frac{n}{2}+1 \le j \le n-2,\quad if \quad n \quad even \ ; \\ \text{or} \quad \frac{n+3}{2} \le j \le n-2,\quad if \quad n \quad odd \ ;\end{array} \\ 4n+1-k-\lceil \dfrac{4\ n+2}{3} \rceil & \text{if} \quad j = \ n-1; \\ 3n+2-k & \text{if} \quad j = \ n. \end{cases}$$

$$\Phi(e_{3j}) = \begin{cases} 4n+1+2j-2k & \begin{array}{l}\text{for } 1 \le j \le \frac{n}{2},\quad if \quad n \quad even \ ; \\ \text{or} \quad 1 \le j \le \frac{n+1}{2},\quad if \quad n \quad odd \ ;\end{array} \\ 6n+4-2j-2k & \begin{array}{l}\text{for } \frac{n}{2}+1 \le j \le n,\quad if \quad n \quad even \ ; \\ \text{or} \quad \frac{n+3}{2} \le j \le n,\quad if \quad n \quad odd.\end{array} \end{cases}$$

$$\Phi(e_{4j}) = \begin{cases} 4n+2-\lceil \dfrac{4\ n+2}{3} \rceil - k & \text{if} \quad j = \ 1; \\ 4n+6-2j-2\lceil \dfrac{4\ n+2}{3} \rceil & \begin{array}{l}\text{for } 2 \le j \le \frac{n}{2}-1,\quad if \quad n \quad even \ ; \\ \text{or} \quad 2 \le j \le \frac{n-3}{2},\quad if \quad n \quad odd \ ;\end{array} \\ 2n+5-\lceil \dfrac{4\ n+2}{3} \rceil & \text{if} \quad j = \ \frac{n}{2} \quad and \quad n \quad even; \\ 2n+6-\lceil \dfrac{4\ n+2}{3} \rceil & \text{if} \quad j = \ \frac{n-1}{2} \quad and \quad n \quad odd; \\ 2n+4-2j & \begin{array}{l}\text{for } \frac{n}{2}+1 \le j \le n-1,\quad if \quad n \quad even \ ; \\ \text{or} \quad \frac{n+1}{2} \le j \le n-1,\quad if \quad n \quad odd \ ;\end{array} \\ 3n+3-k & \text{if} \quad j = \ n. \end{cases}$$

$$\Phi(e_{5j}) = \begin{cases} 3 & \text{if} \quad 1 \le j \le n-2; \\ 2 & \text{if} \quad j = \ n-1; \\ n+2 & \text{if} \quad j = \ n. \end{cases}$$

Notice that all the vertex and edge labels are at most $k = \lceil \frac{5\ n+2}{3} \rceil$, moreover under the labeling Φ the weights of the edges are given by

$$\forall \ \ i = \ 1,\ 2 \qquad wt_{\Phi}(e_{ij}) \ = 2n(i-1)+2j+1\ , \qquad\qquad 1 \le j \le n$$

$$wt_\Phi(e_{3j}) = \begin{cases} 4n+2j+1 & \text{for } 1 \le j \le \frac{n}{2}, \quad \textit{and} \ \ n \ \ \textit{is} \ \ \textit{even} \ ; \\ & \text{or } \ 1 \le j \le \frac{n+1}{2}, \quad \textit{and} \ \ n \ \ \textit{is} \ \ \textit{odd} \ ; \\ 6n-2j+4 & \text{for } \frac{n}{2}+1 \le j \le n, \quad \textit{and} \ \ n \ \ \textit{is} \ \ \textit{even} \ ; \\ & \quad \text{or } \ \frac{n+3}{2} \le j \le n, \ \ \textit{and} \ \ n \ \ \textit{is} \ \ \textit{odd} \ ; \end{cases}$$

$$\forall \ \ i = \ 4,5 \qquad wt_\Phi(e_{ij}) = \begin{cases} 2n(6-i)+2-2j & \text{if} \quad 1 \le j \le n-1. \\ 2n(\ 6-i)+2 & \text{if} \quad j = n. \end{cases}$$

The weights of edges in the cycles C_{n_1}, C_{n_2} form an increasing sequence of consecutive odd integers from 3 up to $4n+1$. In the cycle C_{n_3} the first $\frac{n}{2}$ edges form an increasing sequence of consecutive odd integers from $4n+3$ up to $5n+1$, while the last $\frac{n}{2}$ edges form a decreasing sequence of consecutive even integers from $5n+2$ up to $4n+4$. For the cycles C_{n_4}, C_{n_5} , the weights of edges form a decreasing sequence of consecutive even integers from $4n+2$ up to 4 . This complete the proof. $\square$

Illustration: The graph $Cl_{5,10}$ with an edge irregularity total $k = 18$ labeling is shown in Fig. 3.

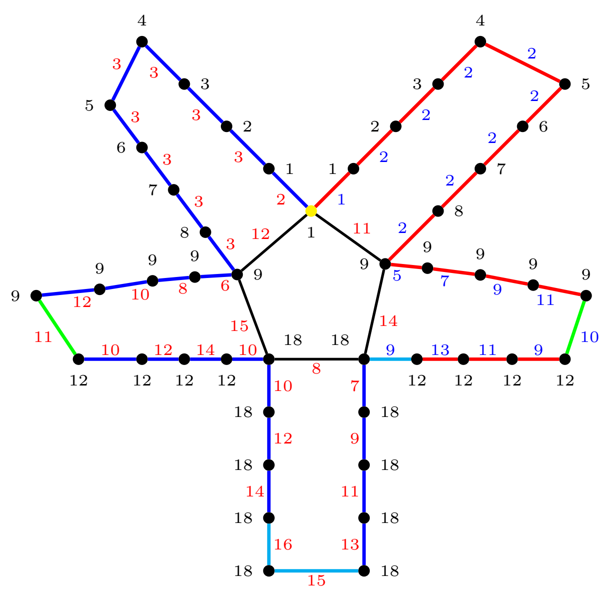

Figure 3: The calendula graph $Cl_{5,10}$ with an edge irregularity total $k = 18$ labeling

Theorem 2.3. *Let $m \ge 6$, $n \ge 4$ be positive integers and $Cl_{m,n}$ be the calendula graph. Then*

$$tes(Cl_{m,n}) = \lceil \ \frac{m\ n+2}{3} \ \rceil$$

Proof. The calendula graph $Cl_{m,n}$ has $|V(Cl_{m,n})| = m(n-1)$, $|E(Cl_{m,n})| = mn$ and the maximum degree $\triangle(Cl_{m,n}) = 4$. Thus, the inequality (1) becomes

$$\lceil \frac{m\,n+2}{3} \rceil \leq tes(Cl_{m,n}).$$

To prove the equality, it suffices to prove the existence of an optimal total $k-$ labeling $\Phi : V(Cl_{m,n}) \cup E(Cl_{m,n}) \longrightarrow \{1, 2, \cdots, k\}$ is a total $k.-$ labeling, $k = \lceil \frac{m\,n+2}{3} \rceil$, we establish the labeling in the following way:

$$\Phi(v_{1j}) = \begin{cases} 1 & \text{if} \quad j = 1; \\ j-1 & \text{if} \quad 2 \leq j \leq n. \end{cases}$$

$$\Phi(v_{2j}) = \begin{cases} n-1 & \text{for } 1 \leq j \leq \frac{n}{2}, \quad if \quad n \quad even\ ; \\ & \text{or } \ 1 \leq j \leq \frac{n+1}{2}, \quad if \quad n \quad odd\ ; \\ \lceil \frac{4\,n+2}{3} \rceil - 2 & \text{for } \frac{n}{2}+1 \leq j \leq n-1, \quad if \quad n \quad even\ ; \\ & \text{or } \quad \frac{n+3}{2} \leq j \leq n-2\ , \quad if \quad n \quad odd\ ; \\ \lceil \frac{6\,n+2}{3} \rceil & \text{if} \quad j = \ n. \end{cases}$$

$\forall\ \ 3 \leq i \leq \frac{m}{2} - 1$ if m is even, or $\forall\ \ 3 \leq i \leq \frac{m-3}{2}$ if m is odd

$$\Phi(v_{ij}) = \begin{cases} \lceil \frac{2ni+2}{3} \rceil & \text{if} \quad 1 \leq j \leq n-1; \\ \lceil \frac{2n(i+1)+2}{3} \rceil & \text{if} \quad j = \ n. \end{cases}$$

$\forall\ \ i = \frac{m-1}{2}$ and m is odd

$$\Phi(v_{(\frac{m-1}{2})j}) = \begin{cases} \lceil \frac{n(m-1)+2}{3} \rceil & \text{if} \quad 1 \leq j \leq n-1; \\ \lceil \frac{nm+2}{3} \rceil & \text{if} \quad j = \ n. \end{cases}$$

$\forall\ \ i = \frac{m}{2}\ ,\ \frac{m}{2}+1$ and m is even, or $\forall\ \ i = \frac{m+1}{2}$ and m is odd

$$\Phi(v_{ij}) = \ k = \lceil \frac{nm+2}{3} \rceil \quad \text{if} \quad 1 \leq j \leq n.$$

if m is odd
$$\Phi(v_{(\frac{m+3}{2})j}) = \begin{cases} \lceil \frac{nm+2}{3} \rceil & \text{if} \quad j = \ 1. \\ \lceil \frac{n(m-1)+2}{3} \rceil & \text{if} \quad 2 \leq j \leq n. \end{cases}$$

$\forall\ \frac{m}{2}+2 \leq i \leq m-2$ and m even, or $\forall\ \frac{m+5}{2} \leq i \leq m-2$ and m is odd

$$\Phi(v_{ij}) = \begin{cases} \lceil \frac{2n[m-(i-2)\ ]+2}{3} \rceil & \text{if} \quad j = \ 1\ ; \\ \lceil \frac{2n[m-(i-1)\ ]+2}{3} \rceil & \text{if} \quad 2 \leq j \leq n. \end{cases}$$

$$\Phi(v_{(m-1)j}) = \begin{cases} \lceil \dfrac{6n+2}{3} \rceil & \text{if} \quad j=1; \\ \lceil \dfrac{4n+2}{3} \rceil - 2 & \begin{array}{l} \text{for } 2 \le j \le \frac{n}{2}, \quad if \quad n \quad even \ ; \\ \text{or} \quad 2 \le j \le \frac{n-1}{2}, \quad if \quad n \quad odd \ ; \end{array} \\ n-1 & \begin{array}{l} \text{for } \frac{n}{2}+1 \le j \le n, \quad if \quad n \quad even \ ; \\ \text{or} \quad \frac{n+1}{2} \le j \le n \ , \quad if \quad n \quad odd \ ; \end{array} \end{cases}$$

$$\Phi(v_{mj}) = \quad n-j \qquad if \qquad 1 \le j \le n-1 \quad \text{and} \qquad \Phi(v_{mn}) = \ 1$$

$$\Phi(e_{1j}) = \begin{cases} 1 & \text{if} \quad j=1; \\ 2 & \text{if} \quad 2 \le j \le n-1. \\ n+1 & \text{if} \quad j = \quad n. \end{cases}$$

$$\Phi(e_{2j}) = \begin{cases} 3+2j & \begin{array}{l} \text{for } 1 \le j \le \frac{n}{2}-1, \quad and \quad n \quad even \ ; \\ \text{or } 1 \le j \le \frac{n-1}{2}, \quad and \quad n \quad odd \ ; \end{array} \\ 2n+4-\lceil \dfrac{4n+2}{3} \rceil & \text{if} \quad j = \ \frac{n}{2} and \quad n \quad is\ even; \\ 2n+5-\lceil \dfrac{4n+2}{3} \rceil & \text{if} \quad j = \ \frac{n+1}{2} and \quad n \quad is\ odd; \\ 2n+2j+5-2\lceil \dfrac{4n+2}{3} \rceil & \begin{array}{l} \text{for } \frac{n}{2}+1 \le j \le n-2, \quad and \quad n \quad even \ ; \\ \text{or } \frac{n+3}{2} \le j \le n-2, \quad and \quad n \quad odd \ ; \end{array} \\ 4n+1-\lceil \dfrac{4n+2}{3} \rceil - \lceil \dfrac{6n+2}{3} \rceil & \text{if} \quad j = \ n-1; \\ 3n+2-\lceil \dfrac{6n+2}{3} \rceil & \text{if} \quad j = \ n; \end{cases}$$

$\forall \ \ 3 \le i \le \frac{m}{2}-1$ and m is even, or $\forall \ \ 3 \le i \le \frac{m-3}{2}$ and m is odd

$$\Phi(e_{ij}) = \begin{cases} 2n(i-1)+2j+1-2\ \lceil \dfrac{2ni+2}{3} \rceil & \text{if} \quad 1 \le j \le n-2; \\ 2ni-1- \ \lceil \dfrac{2ni+2}{3} \rceil - \lceil \dfrac{2n(i+1)+2}{3} \rceil & \text{if} \quad j = \ n-1; \\ 2ni+1- \ \lceil \dfrac{2ni+2}{3} \rceil - \lceil \dfrac{2n(i+1)+2}{3} \rceil & \text{if} \quad j = \ n. \end{cases}$$

$$\Phi(e_{\frac{m-1}{2}j}) = \begin{cases} n(m-3)+2j+1-2\lceil \dfrac{n(m-1)+2}{3} \rceil & \text{if } 1 \le j \le n-2 \text{ and } \ m \ \text{ odd;} \\ n(m-1)-1-\lceil \dfrac{n(m-1)+2}{3} \rceil - k & \text{if} \quad j=n-1 \quad \text{and} \quad m \quad \text{odd} \ ; \\ n(m-1)+1-\lceil \dfrac{n(m-1)+2}{3} \rceil - k & \text{if} \quad j=n \quad \text{and} \quad m \quad \text{odd.} \end{cases}$$

$$\Phi(e_{\frac{m}{2}j}) = \quad n(m-2) \ +2j+1-2k \qquad \text{if} \quad 1 \le j \le n \text{ and } \ m \ \text{ is even.}$$

$$\Phi(e_{\frac{m+1}{2}j}) = \begin{cases} n(m-1)+2j+1-2k & \text{for } 1 \le j \le \frac{n}{2},\ \ and\ \ n\ \ even\ ; \\ & \text{or}\ \ 1 \le j \le \frac{n+1}{2},\ \ and\ \ n\ \ odd\ ; \\ n(m+1)-2j+4-2k & \text{for } \frac{n}{2}+1 \le j \le n,\ \ and\ \ n\ \ even\ ; \\ & \text{or}\ \ \frac{n+3}{2} \le j \le n,\ \ and\ \ n\ \ odd. \end{cases}$$

$$\Phi(e_{\frac{m+3}{2}j}) = \begin{cases} n(m-1) - \lceil \dfrac{n(m-1)+2}{3} \rceil - \lceil \dfrac{nm+2}{3} \rceil & \text{if } j = 1, \text{ and } m \text{ odd;} \\ n(m-1)+2(1-j) - 2\lceil \frac{n(m-1)+2}{3} \rceil & \text{if } 2 \le j \le n-1 \text{ and } m \text{ odd;} \\ n(m-1)+2 - \lceil \dfrac{n(m-1)+2}{3} \rceil - \lceil \dfrac{nm+2}{3} \rceil & \text{if } j = n \text{ and } m \text{ odd.} \end{cases}$$

$$\Phi(e_{(\frac{m}{2}+1)j}) = \begin{cases} n\ m+\ 2-2(k+j) & \text{if } \ 1 \le j \le n-1 \ \text{ and } \ m \ \text{ is even ;} \\ n\ m-2(k-1) & \text{if } \ j = n \ \text{ and } \ m \ \text{ is even.} \end{cases}$$

$\forall\ \frac{m}{2}+2 \le i \le m-2$ and m is even, or $\forall\ \frac{m+5}{2} \le i \le m-2$ and m is odd

$$\Phi(e_{ij}) = \begin{cases} 2n(m+1-i) - \lceil \dfrac{2n(m-i+1)+2}{3} \rceil - \lceil \dfrac{2n(m-i+2)+2}{3} \rceil & \text{if } j = 1; \\ 2n(m+1-i)+2(1-j) - 2\lceil \dfrac{2n(m-i+1)+2}{3} \rceil & 2 \le j \le n-1; \\ 2n(m+1-i)+2 - \lceil \dfrac{2n(m-i+1)+2}{3} \rceil - \lceil \dfrac{2n(m-i+2)+2}{3} \rceil & \text{if } \ j = n. \end{cases}$$

$$\Phi(e_{(m-1)j}) = \begin{cases} 4n+2-\lceil \frac{4n+2}{3} \rceil - \lceil \frac{6n+2}{3} \rceil & \text{if } j = 1; \\ 4n+6-2j-2\lceil \frac{4n+2}{3} \rceil & \text{for } 2 \le j \le \frac{n}{2}-1,\ and\ n\ \ even; \\ & \text{or } 2 \le j \le \frac{n-3}{2},\ and\ \ n\ \ odd; \\ 2n+5-\lceil \frac{4n+2}{3} \rceil & \text{if } j = \frac{n}{2} \text{ and } n \text{ even;} \\ 2n+6-\lceil \frac{4n+2}{3} \rceil & \text{if } \ j = \frac{n-1}{2} \ \text{ and } n \text{ odd;} \\ 2n+4-2j & \text{for } \frac{n}{2}+1 \le j \le n-1,\ and\ \ n\ \ even; \\ & \text{or } \frac{n+1}{2} \le j \le n-1,\ and\ \ n\ \ odd; \\ 3n+3-\lceil \frac{6n+2}{3} \rceil & \text{if } \ j = n. \end{cases}$$

$$\Phi(e_{mj}) = \begin{cases} 3 & \text{if } \ 1 \le j \le n-2; \\ 2 & \text{if } \ j = n-1; \\ n+2 & \text{if } \ j = n. \end{cases}$$

• If $n = m = 5$, the labeling $\Phi : V(Cl_{5,5}) \cup E(Cl_{5,5}) \longrightarrow \{1, 2, \cdots, k = 9\}$ defined as in Fig. 4.

It is evident that all the vertex and edge labels are at most $k = \lceil \frac{m\ n+2}{3} \rceil$. Besides, the weights of edges under the labeling Φ are given by

- If m is even number

$\forall\ 1 \le i \le \frac{m}{2}$, $\qquad wt_\Phi(e_{ij}) = 2n(i-1)+2j+1$, $\qquad 1 \le j \le n$

$\forall\ \frac{m}{2}+1 \le i \le m,$

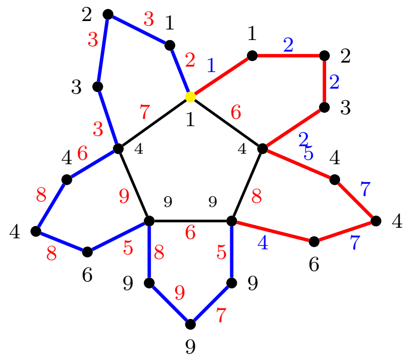

Figure 4: The calendula graph $Cl_{5,5}$ with an edge irregularity total $k = 9$ labeling

$$wt_\Phi(e_{ij}) = \begin{cases} 2n(m+1-i)+2-2j & \text{if} \quad 1 \leq j \leq n-1; \\ 2n(\ m+1-i)+2 & \text{if} \quad j = n. \end{cases}$$

We can see that weights of edges in the first half cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_{\frac{m}{2}}}$ form an increasing sequence of consecutive odd integers from 3 up to $mn+1$. For the second half cycles $C_{n_{\frac{m}{2}+1}}, C_{n_{\frac{m}{2}+2}}, \cdots, C_{n_m}$ weights of edges form a decreasing sequence of consecutive even integers from $mn+2$ up to 4 .

- If m is odd number

$\forall \ 1 \leq i \leq \frac{m-1}{2} \qquad wt_\Phi(e_{ij}) \ = 2n(i-1)+2j+1\ , \qquad\qquad 1 \leq j \leq n$

$$wt_\Phi(e_{(\frac{m+1}{2})j}) = \begin{cases} n(m-1)+2j+1 & \text{for} \ \ 1 \leq j \leq \frac{n}{2}, \quad \textit{and} \ \ n \textit{ is even} \ ; \\ & \text{or} \ \ 1 \leq j \leq \frac{n+1}{2}, \quad \textit{and } n \textit{ is odd} \ ; \\ n(m+1)-2j+4 & \text{for} \ \frac{n}{2}+1 \leq j \leq n, \ \textit{and} \ \ n \textit{ is even} \ ; \\ & \text{or} \ \ \frac{n+3}{2} \leq j \leq n, \ \textit{and } n \textit{ is odd.} \end{cases}$$

$\forall \ 1 \leq i \leq \frac{m+3}{2}, \ \ \cdots, \ m$

$$wt_\Phi(e_{ij}) = \begin{cases} 2n(m+1-i)+2-2j & \text{if} \quad 1 \leq j \leq n-1; \\ 2n(m+1-i)+2 & \text{if} \quad j = n. \end{cases}$$

Since all the edge weights are distinct, the labeling Φ is the required edge irregular total $k = \lceil \frac{m\ n+2}{3} \rceil$ labeling. This concludes the proof. $\square$

3. Edge irregular reflexive for calendula graphs

The concept of the edge irregular reflexive $r-$ labeling was introduced by Ryan et al. [20]. For a graph G, an edge labeling $\Psi_e : E(G) \longrightarrow \{1, 2, \cdots, r_e\}$ and a vertex labeling $\Psi_v : V(G) \longrightarrow \{0, 2, 4, \cdots, 2r_v\}$, then labeling Ψ defined by

$$\Psi(x) = \begin{cases} \Psi_v(x) & \text{if} \quad x \ \in V(G) \\ \Psi_e(x) & \text{if} \quad x \ \in E(G) \end{cases}$$

is a total $r-$ labeling where $r = \max\{r_e, 2r_v\}$. The total $r-$ labeling Ψ is called an edge irregular reflexive $r-$ labeling of the graph G if distinct edges has different weights. The smallest value of r for which such labeling exists is called the reflexive edge strength of the graph G and is denoted by $res(G)$. During the past few years, res(G) has been inspected for distinctive family of graphs (see [21, 22, 23, 24, 25, 26]). In this section, we examine the edge irregular reflexive $r-$ labeling for the calendula graphs.

Let us recall the following lemma proved in [21]

Lemma 3.1. *For every graph G,*

$$res(G) \geq \begin{cases} \lceil \dfrac{E(G)}{3} \rceil & \text{if } \ |E(G)| \not\equiv 2,3 \ (mod\ 6). \\ \lceil \dfrac{E(G)}{3} \rceil \ +1 & \text{if } \ |E(G)| \equiv 2,3 \ (mod\ 6). \end{cases} \tag{2}$$

Also, Baća et al. [22] suggested the following conjecture:

Conjecture 3.1 ([22])**.** *Let G be a simple graph with maximum degree $\triangle = \triangle(G)$. Then*

$$res(G) = \max\{\lfloor \frac{\triangle + 2}{2} \rfloor, \lfloor \frac{E(G)}{3} \rfloor + d\},$$

where $d = 1$ for $|E(G)| \equiv 2,3 \ (mod\ 6)$, and zero otherwise

Theorem 3.1. *Let $n \geq 4$ be positive integer and $Cl_{4,n}$ be the calendula graph. Then*

$$res(Cl_{4,n}) = \begin{cases} \lceil \dfrac{4\ n}{3} \rceil & \text{if } \ |E(G)| \equiv 0,4 \ (mod\ 6). \\ \lceil \dfrac{4\ n}{3} \rceil \ +1 & \text{if } \ |E(G)| \equiv 2 \ (mod\ 6). \end{cases}$$

Proof. Since $|V(Cl_{4,n})| = 4(n-1)$, $|E(Cl_{4,n})| = 4n$ and $\triangle(Cl_{m,n}) = 4$, the inequality (2) becomes

$$res(Cl_{4,n}) \geq \begin{cases} \lceil \dfrac{4\ n}{3} \rceil & \text{if } \ |E(G)| \equiv 0,4 \ (mod\ 6). \\ \lceil \dfrac{4\ n}{3} \rceil \ +1 & \text{if } \ |E(G)| \equiv 2 \ (mod\ 6). \end{cases}$$

To demonstrate the equality, we have to appear that there exists an edge irregularity reflexive $r = \lceil \frac{4\ n}{3} \rceil$ labeling, for the calendula graph $Cl_{4,n}$, there are three cases:

Case (1). When $n \equiv 0 \ mod\ 3$, $n \geq 4$, then $|E(G)| \equiv 0 \ (mod\ 6)$. Suppose that $r = \lceil \frac{4\ n}{3} \rceil$, we build the labeling function $\Psi_v : V(Cl_{4,n})) \longrightarrow \{0, 2, \cdots, 2r_v = r\ \}$ as follows:

$$\Psi_v(v_{1j}) = \begin{cases} j-1 & \begin{array}{r} \text{for } j = \ 1,3,5,\cdots,\ n-3, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or } j = \ 1,3,5,\cdots, n-2, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \\ j-2 & \begin{array}{r} \text{for } j = \ 2,4,6,\cdots,\ n-2, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or } \ j = \ 2,4,6,\cdots, n-3, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \\ 2n-4-r & \text{if} \quad j = \ n-1; \\ r & \text{if} \quad j = \ n. \end{cases}$$

$$\Psi_v(v_{2j}) = \begin{cases} r \qquad for \ \ \text{j=1} \quad \text{and} & \begin{array}{r} \text{for } \frac{n}{2}+1 \le j \le n, \ if \ \ n \ \ is \ \ even; \\ \text{or } \frac{n+3}{2} \le j \le n, \ if \ \ n \ \ is \ \ odd; \end{array} \\ 2n-r \quad \text{if} \quad j = \ 2; \\ n-2 \quad \text{if} \quad 3 \le j \le \frac{n}{2}, \quad n \ \ is \ \ even; \\ n-1 \quad \text{if} \quad 3 \le j \le \frac{n+1}{2}, \quad n \ \ is \ \ odd; \end{cases}$$

$$\Psi_v(v_{3j}) = \begin{cases} r \qquad \begin{array}{r} \text{for } 1 \le j \le \frac{n}{2}, \ \ if \ \ n \ \ is \ \ even; \\ \text{or } \ 1 \le j \le \frac{n-1}{2}, \ \ if \ \ n \ \ is \ \ odd; \end{array} & and \quad \text{j= n ;} \\ n-2 \quad \text{if} \ \ \frac{n}{2}+1 \le j \le n-2, \ \ \ n \ \ is \ \ even \ ; \\ n-1 \quad \text{if} \ \ \frac{n+1}{2} \le j \le n-2, \ \ \ n \ \ is \ \ odd \ ; \\ 2n-r \quad \text{if} \quad j = \ n-1; \end{cases}$$

$$\Psi_v(v_{4j}) = \begin{cases} r & \text{if} \quad j = 1; \\ 2n-4-r & \text{if} \quad j = \ 2; \\ n-j-1 & \begin{array}{r} \text{for } j = \ 3,5,\cdots,\ n-1, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or } \ j = \ 4,6,\cdots, n-1, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \\ n-j & \begin{array}{r} \text{for } j = \ 4,6,\cdots,\ n, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or } \ j = \ 3,5,\cdots, n, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \end{cases}$$

For edges, we construct the labeling function $\Psi_e : E(Cl_{4,n})) \longrightarrow \{1, 2, \cdots, r\}$ as follows:

$$\Psi_e(e_{1j}) = \begin{cases} 1 & \text{if} \quad 1 \le j \le n-3, \ \ and \ \ j = \ n-1; \\ r-n+3 & \text{if} \quad j = \ \ n-2, \quad n \ \ is \ \ even \ ; \\ r-n+2 & \text{if} \quad j = \ n-2, \quad n \ \ is \ \ odd \ ; \\ 2n-r-1 & \text{if} \quad j = \ \ n. \end{cases}$$

$$\Psi_e(e_{2j}) = \begin{cases} 1 & \text{if} \quad j = 1; \\ r-n+5 & \text{if} \quad j = \ 2, \quad n \ \ is \ \ even \ ; \\ r-n+4 & \text{if} \quad j = \ 2, \quad n \ \ is \ \ odd \ ; \\ 3+2j & \text{if} \quad 3 \le j \le \frac{n}{2}-1, \quad n \ \ is \ \ even \ ; \\ 1+2j & \text{if} \quad 3 \le j \le \frac{n-1}{2}, \quad n \ \ is \ \ odd \ ; \\ 2n+1-r & \begin{array}{l} \text{for } j = \ \frac{n}{2}, \quad n \ \ is \ \ even \ \ ; \\ \text{and } \ j = \ \frac{n+1}{2}, \quad n \ \ is \ \ odd \ \ ; \end{array} \\ 2n-1-2r+2j & \begin{array}{l} \text{for } \frac{n}{2}+1 \le j \le n, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or } \ \frac{n+3}{2} \le j \le n, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \end{cases}$$

$$\Psi_e(e_{3j}) = \begin{cases} 4n-2r-2j & \begin{array}{l} \text{for } 1 \le j \le \frac{n}{2}-1, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or } \ 1 \le j \le \frac{n-3}{2}, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \\ 2n+2-r & \begin{array}{l} \text{for } j = \ \frac{n}{2}, \quad n \ \ is \ \ even \ \ ; \\ \text{and } \ j = \ \frac{n-1}{2}, \quad n \ \ is \ \ odd \ \ ; \end{array} \\ 2n+4-2j & \text{if} \quad \frac{n}{2}+1 \le j \le n-3 \ \ and \ \ n \ \ even; \\ 2n+2-2j & \text{if} \quad \frac{n+1}{2} \le j \le n-3 \ \ and \ \ n \ \ odd; \\ r-n+6 & \text{if} \quad j = \ n-2 \ \ and \ \ n \ \ is \ \ even; \\ r-n+5 & \text{if} \quad j = \ n-2 \ \ and \ \ n \ \ is \ \ odd; \\ 2 & \text{if} \quad j = \ n-1; \\ r & \text{if} \quad j = \ \ n. \end{cases}$$

$$\Psi_e(e_{4j}) = \begin{cases} 2 & \text{if} \quad j = 1 \ \ and \ \ 3 \le j \le n-1; \\ r-n+4 & \text{if} \quad j = \ \ 2, \quad n \ \ is \ \ even \ ; \\ r-n+3 & \text{if} \quad j = \ 2, \quad n \ \ is \ \ odd \ ; \\ 2n-r & \text{if} \quad j = \ \ n. \end{cases}$$

Case (2). When $n \equiv \ 1 \, mod \, 3$, then $|E(G)| \equiv 4 \ (mod \ 6)$. Let $r = \lceil \frac{4\ n}{3} \rceil$, the labeling functions Ψ_v and Ψ_e defined as in case (1) in all vertices and edges but with some modifications which are given by

$$\Psi_v(v_{1(n-1)}) = \ 2n-2-r \ , \qquad \Psi_v(v_{1n}) = \ r-2 \ ,$$

$$\Psi_v(v_{21}) = \ r-2 \ , \qquad \Psi_v(v_{22}) = \ 2n-r+2 \ ,$$

$$\Psi_v(v_{3(n-1)}) = \ 2n+2-r \ , \qquad \Psi_v(v_{3n}) = \ r-2 \ ,$$

$$\Psi_v(v_{41}) = \ r-2 \ , \qquad \Psi_v(v_{42}) = \ 2n-r-2 \ ,$$

$$\Psi_e(e_{1j}) = \begin{cases} r-n+1 & \text{if} \quad j = \ \ n-2, \quad n \ \ even \ ; \\ r-n & \text{if} \quad j = \ n-2, \quad n \ \ odd \ ; \\ 2n-r+1 & \text{if} \quad j = \ \ n. \end{cases}$$

$$\Psi_e(e_{2j}) = \begin{cases} r-n+3 & \text{if} \quad j = 2, \quad n \;\; even \; ; \\ r-n+2 & \text{if} \quad j = 2, \quad n \;\; odd \; ; \\ 2n-1-2r+2j & \text{for } \frac{n}{2}+1 \leq j \leq n-1, \quad if \quad n \quad even \; ; \\ & \quad \text{or } \frac{n+3}{2} \leq j \leq n-1, \quad if \quad n \quad odd \; ; \\ 4n-2r+1 & \text{if} \quad j = n. \end{cases}$$

$$\Psi_e(e_{3j}) = \begin{cases} r-n+4 & \text{if} \quad j = n-2 \;\; and \;\; n \;\; even; \\ r-n+3 & \text{if} \quad j = n-2 \;\; and \;\; n \;\; odd; \end{cases}$$

$$\Psi_e(e_{4j}) = \begin{cases} r-n+2 & \text{if} \quad j = 2, \quad n \;\; even \; ; \\ r-n+1 & \text{if} \quad j = 2, \quad n \;\; odd \; ; \\ 2n-r+2 & \text{if} \quad j = n. \end{cases}$$

Case (3). When $n \equiv 2 \; mod \; 3$, then $|E(G)| \equiv 2 \; (mod \; 6)$.suppose that $r = \lceil \frac{4\,n}{3} \rceil + 1$, the labeling functions Ψ_v and Ψ_e defined as in case (1) in all vertices and edges but with some modifications which are given by

$\Psi_v(v_{1(n-1)}) = 2n-r$, $\quad$ $\Psi_v(v_{1n}) = r-4$,

$\Psi_v(v_{21}) = r-4$, $\quad$ $\Psi_v(v_{22}) = 2n-r+4$,

$\Psi_v(v_{3(n-1)}) = 2n+4-r$, $\quad$ $\Psi_v(v_{3n}) = r-4$,

$\Psi_v(v_{41}) = r-4$, $\quad$ $\Psi_v(v_{42}) = 2n-r$,

$$\Psi_e(e_{1j}) = \begin{cases} r-n-1 & \text{if} \quad j = n-2, \quad n \;\; even \; ; \\ r-n-2 & \text{if} \quad j = n-2, \quad n \;\; odd \; ; \\ 2n-r+3 & \text{if} \quad j = n. \end{cases}$$

$$\Psi_e(e_{2j}) = \begin{cases} r-n+1 & \text{if} \quad j = 2, \quad n \;\; even \; ; \\ r-n & \text{if} \quad j = 2, \quad n \;\; odd \; ; \\ 2n-1-2r+2j & \text{for } \frac{n}{2}+1 \leq j \leq n-1, \quad if \quad n \quad even \; ; \\ & \quad \text{or } \frac{n+3}{2} \leq j \leq n-1, \quad if \quad n \quad odd \; ; \\ 4n-2r+3 & \text{if} \quad j = n. \end{cases}$$

$$\Psi_e(e_{3j}) = \begin{cases} r-n+2 & \text{if} \quad j = n-2 \;\; and \;\; n \;\; even; \\ r-n+1 & \text{if} \quad j = n-2 \;\; and \;\; n \;\; odd; \end{cases}$$

$$\Psi_e(e_{4j}) = \begin{cases} r-n & \text{if} \quad j = 2, \quad n \;\; even \; ; \\ r-n-1 & \text{if} \quad j = 2, \quad n \;\; odd \; ; \\ 2n-r+4 & \text{if} \quad j = n. \end{cases}$$

In all cases, notice that all the vertex and edge labels are at most $r = \lceil \frac{4\,n}{3} \rceil$ or $r = \lceil \frac{4\,n}{3} \rceil + 1$. Moreover under the labeling Ψ the weights of the edges are given by

$$\forall \;\; i = 1, \; 2 \qquad wt_\Psi(e_{ij}) = 2n(i-1)+2j-1 \; , \qquad\qquad 1 \leq j \leq n.$$

$$\forall \;\; i = \; 3,4 \qquad wt_{\Psi}(e_{ij}) = \begin{cases} 2n(5-i) - 2j & \text{if} \quad 1 \leq j \leq n-1; \\ 2n(5-i) & \text{if} \quad j = n. \end{cases}$$

We can see that weights of edges in the first half cycles C_{n_1}, C_{n_2} form an increasing sequence of consecutive odd integers from 1 up to $4n-1$. For the second half cycles C_{n_3}, C_{n_4} weights of edges form a decreasing sequence of consecutive even integers from $4n$ up to 2. In this way the labeling Ψ is the required edge irregularity reflexive $r-$ labeling. This completes the proof. $\square$

Illustration: The calendula graph $Cl_{4,12}$ with an edge irregularity total $k = 17$ labeling and an edge irregularity reflexive $r = 16$ labeling are shown in Fig. 5.

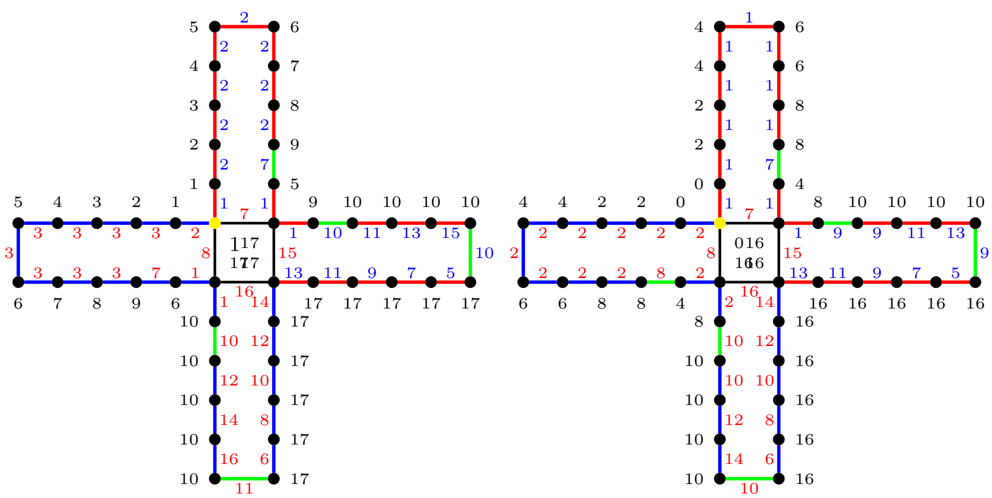

Figure 5: (a) $Cl_{4,12}$ with $k = 17$ (b) $Cl_{4,12}$ with $r = 16$

Theorem 3.2. *Let $n \geq 6$ and m be an even positive integer, $m \geq 6$. Then the calendula graph $Cl_{m,n}$ have .*

$$res(Cl_{m,n}) = \begin{cases} \lceil \dfrac{mn}{3} \rceil & \textit{for } n \equiv 0 \textit{ mod } 3; \\ & \textit{or} \;\; n \equiv \;1\;, 2 \textit{ mod } 3 \textit{ and } |m\; n| \equiv 0, 4 \textit{ mod } 6; \\ \lceil \dfrac{m\;n}{3} \rceil + 1 & \textit{if } n \equiv 1, 2 \textit{ mod } 3 \textit{ and } |mn| \equiv 2 (\textit{mod } 6). \end{cases}$$

Proof. Since $|V(Cl_{m,n})| = m(n-1)$, $|E(Cl_{m,n})| = mn$ and $\triangle(Cl_{m,n}) = 4$. To illustrate the equality in (2), we have to show up that there exists an edge edge irregularity reflexive $r-$ labeling, $r = \lceil \frac{m\;n}{3} \rceil$, or $r = \lceil \frac{m\;n}{3} \rceil + 1$, for the calendula graph $Cl_{m,n}$, there are three cases

Case (1). When $n \equiv \; 0 \; mod \; 3$ or $n \equiv \; 1 \; mod \; 3$ and $|m\; n| \not\equiv 2 \; (mod \; 6)$, we take $r = \lceil \frac{m\;n}{3} \rceil$.

When $n \equiv \; 1 \; mod \; 3$ and $|m\; n| \equiv 2 \; (mod \; 6)$, we put $r = \lceil \frac{m\;n}{3} \rceil + 1$.

We construct the labeling function $\Psi_v : V(Cl_{m,n})) \longrightarrow \{0, 2, \cdots, r\}$ as follows:

$$\Psi_v(v_{1j}) = \begin{cases} j-1 & \text{for } j = 1,3,5,\cdots, n-1, \quad if \quad n \quad even\ ; \\ & \text{or } j = 1,3,5,\cdots,n, \quad if \quad n \quad odd\ ; \\ j-2 & \text{for } j = 2,4,6,\cdots, n, \quad if \quad n \quad even\ ; \\ & j = 2,4,6,\cdots,n-1, \quad if \quad n \quad odd. \end{cases} \tag{3}$$

$$\Psi_v(v_{2j}) = \begin{cases} n-2 & \text{if} \quad 1 \le j \le \frac{n}{2}, \quad n \quad even\ ; \\ n-1 & \text{if} \quad 1 \le j \le \frac{n+1}{2}, \quad n \quad odd\ ; \\ \lceil \dfrac{4\ n}{3} \rceil & \text{for } \frac{n}{2}+1 \le j \le n-1, \quad if \quad n \quad even\ ; \\ & \text{or } \frac{n+3}{2} \le j \le n-1, \quad if \quad n \quad odd\ ; \\ \lceil \dfrac{6\ n}{3} \rceil & \text{if} \quad j = n. \end{cases} \tag{4}$$

$\forall \quad 3 \le i \le \frac{m}{2} - 1$

If $n \equiv 0\ mod\ 3$, or $n \equiv 1\ mod\ 3, \quad and \quad i \equiv 2\ mod\ 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\ n\ i}{3} \rceil & \text{if} \quad 1 \le j \le n-1; \\ \lceil \dfrac{2\ n\ (i+1)}{3} \rceil & \text{if} \quad j = n, \end{cases} \tag{5}$$

If $n \equiv 1\ mod\ 3, \quad and \quad i \equiv 0\ mod\ 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\ n\ i}{3} \rceil & \text{if} \quad 1 \le j \le n-1; \\ \lceil \dfrac{2\ n\ (i+1)}{3} \rceil + 1 & \text{if} \quad j = n, \end{cases} \tag{6}$$

If $n \equiv 1\ mod\ 3, \quad and \quad i \equiv 1\ mod\ 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\ n\ i}{3} \rceil + 1 & \text{if} \quad 1 \le j \le n-1; \\ \lceil \dfrac{2\ n\ (i+1)}{3} \rceil & \text{if} \quad j = n, \end{cases} \tag{7}$$

$$\forall \quad i = \frac{m}{2}, \frac{m}{2}+1, \qquad \Psi_v(v_{ij}) = r \quad for \quad 1 \le j \le n \tag{8}$$

$\forall \quad \frac{m}{2} + 2 \le i \le m-2$

If $n \equiv 0 \; mod\; 3$ or $n \equiv 1 \; mod\; 3, \quad and \quad i \equiv 1 \; mod\; 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\; n\; [m-(i-2)]}{3} \rceil & \text{if} \quad j = 1; \\ \lceil \dfrac{2\; n\; [m-(i-1)]}{3} \rceil & \text{if} \quad 2 \le j \le n, \end{cases} \tag{9}$$

If $n \equiv 1 \; mod\; 3, \quad and \quad i \equiv 2 \; mod\; 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\; n\; [m-(i-2)]}{3} \rceil & \text{if} \quad j = 1; \\ \lceil \dfrac{2\; n\; [m-(i-1)]}{3} \rceil + 1 & \text{if} \quad 2 \le j \le n, \end{cases} \tag{10}$$

If $n \equiv 1 \; mod\; 3, \quad and \quad i \equiv 0 \; mod\; 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\; n\; [m-(i-2)]}{3} \rceil + 1 & \text{if} \quad j = 1; \\ \lceil \dfrac{2\; n\; [m-(i-1)]}{3} \rceil & \text{if} \quad 2 \le j \le n, \end{cases} \tag{11}$$

$$\Psi_v(v_{(m-1)j}) = \begin{cases} \lceil \dfrac{6\; n}{3} \rceil & \text{if} \quad j = 1; \\ \lceil \dfrac{4\; n}{3} \rceil & \begin{array}{l} \text{for } 2 \le j \le \frac{n}{2}, \;\; if \;\; n \;\; even \;; \\ \text{or } \;\; 2 \le j \le \frac{n-1}{2}, \;\; if \;\; n \;\; odd \;; \end{array} \\ n-2 & \text{if} \quad \frac{n}{2}+1 \le j \le n, \quad n \;\; even \;; \\ n-1 & \text{if} \quad \frac{n+1}{2} \le j \le n, \quad n \;\; odd, \end{cases} \tag{12}$$

$$\Psi_v(v_{mj}) = \begin{cases} n-j-1 & \begin{array}{l} \text{for } j = 1, 3, \cdots, \; n-1, \;\; if \;\; n \;\; even \;; \\ \text{or } \;\; j = 2, 4, \cdots, n-1, \;\; if \;\; n \;\; odd \;; \end{array} \\ n-j & \begin{array}{l} \text{for } j = 2, 4, \cdots, \; n, \;\; if \;\; n \;\; even \;; \\ \quad j = 1, 3, \cdots, n, \;\; if \;\; n \;\; odd, \end{array} \end{cases} \tag{13}$$

For the edges, we define the labeling function $\Psi_e : E(Cl_{m,n})) \longrightarrow \{1, 2, \cdots, r\}$ as follows:

$$\Psi_e(e_{1j}) = \begin{cases} 1 & \text{if} \quad 1 \le j \le n-1; \\ n+1 & \text{if} \quad j = n \;, \quad n \;\; even \;; \\ n & \text{if} \quad j = n, \quad n \;\; odd, \end{cases} \tag{14}$$

$$\Psi_e(e_{2j}) = \begin{cases} 2j+3 & \text{if } 1 \le j \le \frac{n}{2}-1, \ n \ even; \\ 2j+1 & \text{if } 1 \le j \le \frac{n-1}{2}, \ n \ odd \ ; \\ 2n+1-\lceil \frac{4n}{3} \rceil & \text{for } j=\frac{n}{2} \ if \ n \ even; \\ & \text{or } \ j=\frac{n+1}{2}, \ if \ n \ odd; \\ 2n-1+2j-2\lceil \frac{4n}{3} \rceil & \text{for } \frac{n}{2}+1 \le j \le n-2 \ if \ n \ even; \\ & \text{or } \frac{n+3}{2} \le j \le n-2 \ if \ n \ odd; \\ 4n-3-\lceil \frac{4n}{3} \rceil - \lceil \frac{6n}{3} \rceil & \text{if } \ j=n-1. \\ 3n+1-\lceil \frac{6n}{3} \rceil & \text{if } \ j=n, \ n \ \ is \ \ even. \\ 3n-\lceil \frac{6n}{3} \rceil & \text{if } \ j=n, \ n \ \ is \ \ odd, \end{cases} \tag{15}$$

$\forall \quad 3 \le i \le \frac{m}{2}-1$

If $n \equiv 0 \ mod \ 3$ or $n \equiv 1 \ mod \ 3, \quad and \ \ i \equiv 2 \ mod \ 3,$

$$\Psi_e(e_{ij}) = \begin{cases} 2n(i-1)+2j-1-2\lceil \frac{2ni}{3} \rceil & \text{if } \ 1 \le j \le n-2; \\ 2ni-3-\lceil \frac{2ni}{3} \rceil - \lceil \frac{2n(i+1)}{3} \rceil & \text{if } \ j=n-1, \\ 2ni-1-\lceil \frac{2ni}{3} \rceil - \lceil \frac{2n(i+1)}{3} \rceil & \text{if } \ j=n, \end{cases} \tag{16}$$

If $n \equiv 1 \ mod \ 3, \quad and \quad i \equiv 0 \ mod \ 3,$

$$\Psi_e(e_{ij}) = \begin{cases} 2n(i-1)+2j-1-2\lceil \frac{2ni}{3} \rceil & \text{if } \ 1 \le j \le n-2; \\ 2ni-4-\lceil \frac{2ni}{3} \rceil - \lceil \frac{2n(i+1)}{3} \rceil & \text{if } \ j=n-1, \\ 2ni-2-\lceil \frac{2ni}{3} \rceil - \lceil \frac{2n(i+1)}{3} \rceil & \text{if } \ j=n, \end{cases} \tag{17}$$

If $n \equiv 1 \ mod \ 3, \quad and \quad i \equiv 1 \ mod \ 3,$

$$\Psi_e(e_{ij}) = \begin{cases} 2n(i-1)+2j-3-2\lceil \frac{2ni}{3} \rceil & \text{if } \ 1 \le j \le n-2; \\ 2ni-4-\lceil \frac{2ni}{3} \rceil - \lceil \frac{2n(i+1)}{3} \rceil & \text{if } \ j=n-1, \\ 2ni-2-\lceil \frac{2ni}{3} \rceil - \lceil \frac{2n(i+1)}{3} \rceil & \text{if } \ j=n, \end{cases} \tag{18}$$

$$\Psi_e(e_{\frac{m}{2}j}) = n(m-2) + 2j - 1 - 2r, \qquad for \quad 1 \leq j \leq n \tag{19}$$

$$\Psi_e(e_{(\frac{m}{2}+1)j}) = \begin{cases} nm - 2(r+j) & \text{if} \quad 1 \leq j \leq n-1 \ ; \\ nm - 2r & \text{if} \quad j = \quad n \ . \end{cases} \tag{20}$$

$\forall \quad \frac{m}{2} + 2 \ \leq \ i \ \leq \ m-2$

If $\quad n \equiv \ 0 \ mod\ 3 \quad$ or $\quad n \equiv \ 1 \ mod\ 3, \quad and \ \ i \equiv \ 1 \ mod\ 3,$

(21)

$$\Psi_e(e_{ij}) = \begin{cases} 2n(m+1-i) - 2 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = 1; \\ 2n(m+1-i) - 2j - 2\lceil \frac{2n[m-i+1]}{3} \rceil & \text{if} \ \ 2 \leq j \leq n-1, \\ 2n(m+1-i) - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = n, \end{cases}$$

If $\quad n \equiv \ 1 \ mod\ 3, \quad and \quad i \equiv \ 2 \ mod\ 3,$

(22)

$$\Psi_e(e_{ij}) = \begin{cases} 2n(m+1-i) - 3 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if } j = 1; \\ 2n(m+1-i) - 2 - 2j - 2\lceil \frac{2n[m-i+1]}{3} \rceil & \text{if} \ \ 2 \leq j \leq n-1, \\ 2n(m+1-i) - 1 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = \ n, \end{cases}$$

If $\quad n \equiv \ 1 \ mod\ 3, \quad and \quad i \equiv \ 0 \ mod\ 3,$

(23)

$$\Psi_e(e_{ij}) = \begin{cases} 2n(m+1-i) - 3 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = 1; \\ 2n(m+1-i) - 2j - 2\lceil \frac{2n[m-i+1]}{3} \rceil & \text{if } 2 \leq j \leq n-1; \\ 2n(m+1-i) - 1 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = n, \end{cases}$$

$$\Psi_e(e_{(m-1)j}) = \begin{cases} 4n-2-\lceil\frac{4n}{3}\rceil-\lceil\frac{6n}{3}\rceil & \text{if } j=1; \\ 4n-2j-2\lceil\frac{4n}{3}\rceil & \text{for } 2\le j\le \frac{n}{2}-1, if\ n\ even; \\ & \text{or } 2\le j\le \frac{n-3}{2},\ if\ n\ odd; \\ 2n+2-\lceil\frac{4n}{3}\rceil & \text{for } j=\frac{n}{2},\ if\ n\ even; \\ & \text{or } j=\frac{n-1}{2},\ if\ n\ odd. \\ 2n+4-2j & \text{if } \frac{n}{2}+1\le j\le n-1,\ n\ is\ even; \\ 2n+2-2j & \text{if } \frac{n+1}{2}\le j\le n-1,\ n\ is\ odd; \\ 3n+2-\lceil\frac{6n}{3}\rceil & \text{if } j=n,\ n\ is\ even; \\ 3n+1-\lceil\frac{6\,n}{3}\rceil & \text{if } j=n,\ n\ is\ odd, \end{cases} \tag{24}$$

$$\Psi_e(e_{mj}) = \begin{cases} 2 & \text{if } 1\le j\le n-1; \\ n+2 & \text{if } j=n,\quad n\ is\ even\ ; \\ n+1 & \text{if } j=n,\quad n\ is\ odd \end{cases} \tag{25}$$

Case (2). When $n\equiv 2\,mod\,3$ and $|m\,n|\equiv 0\,,4\ (mod\,6)$, we choose $r=\lceil\frac{m\,n}{3}\rceil$, when $n\equiv 2\ mod\ 3$ and $|m\ n|\equiv 2\ (mod\ 6)$, we choose $r=\lceil\frac{m\ n}{3}\rceil+1$.

The labeling function $\Psi: V(Cl_{m,n})\cup E(Cl_{m,n})\ \longrightarrow\{1,2,\cdots,r\}$ defined as in case (1) but with some modifications which are given by

$$\Psi_v(v_{2j}) = \lceil\frac{4\ n}{3}\rceil+1 \qquad \begin{array}{l}\text{for } \frac{n}{2}+1\le j\le n-1,\ \ if\ \ n\ \ is\ \ even\ ; \\ \text{or } \frac{n+3}{2}\le j\le n-1,\ \ if\ \ n\ \ is\ \ odd,\end{array}$$

$\forall\quad 3\ \le\ i\ \le\ \frac{m}{2}-1$,

If $i\equiv\ 0\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(5).

If $i\equiv\ 1\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(6).

If $i\equiv\ 2\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq. (7).

$\forall\quad \frac{m}{2}+2\ \le\ i\ \le\ m-2$,

If $i\equiv\ 0\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(9).

If $i\equiv\ 1\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(10).

If $i\equiv\ 2\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(11).

$$\Psi_v(v_{(m-1)j}) = \lceil \frac{4\,n}{3} \rceil + 1 \qquad \text{for } 2 \le j \le \tfrac{n}{2},\ \ if\ \ n\ \ is\ \ even\ ; \ \text{or}\ \ 2 \le j \le \tfrac{n-1}{2},\ \ if\ \ n\ \ is\ \ odd,$$

$$\Psi_e(e_{2j}) = \begin{cases} 2n - \lceil \dfrac{4n}{3} \rceil & \text{for } j = \frac{n}{2},\ \ if\ \ n\ \ is\ \ even; \ \text{or}\ \ j = \frac{n+1}{2},\ \ if\ \ n\ \ is\ \ odd; \\ 2n - 3 + 2j - 2\lceil \dfrac{4n}{3} \rceil & \text{for } \frac{n}{2}+1 \le j \le n-2,\ if\ \ n\ \ is\ \ even; \ \text{or}\ \frac{n+3}{2} \le j \le n-2,\ \ if\ \ n\ \ is\ \ odd; \\ 4n - 4 - \lceil \dfrac{4n}{3} \rceil - \lceil \dfrac{6n}{3} \rceil & \text{if}\ \ \ j = \ \ n-1. \end{cases}$$

$\forall$, $3 \le i \le \frac{m}{2} - 1$

If $i \equiv 0 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(16).

If $i \equiv 1 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within the Eq. (17).

If $i \equiv 2 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(18).

$\forall$, $\frac{m}{2} + 2 \le i \le m - 2$

If $i \equiv 0 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(21).

If $i \equiv 1 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(22).

If $i \equiv 2 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(23).

$$\Psi_e(e_{(m-1)j}) = \begin{cases} 4n - 3 - \lceil \dfrac{4n}{3} \rceil - \lceil \dfrac{6\,n}{3} \rceil & \text{if}\ \ \ j = \ 1; \\ 4n - 2j - 2\lceil \dfrac{4n}{3} \rceil - 2 & \text{for } 2 \le j \le \frac{n}{2} - 1,\ \ if\ \ n\ \ is\ \ even; \ \text{or}\ 2 \le j \le \frac{n-3}{2},\ \ if\ \ n\ \ is\ \ odd; \\ 2n + 1 - \lceil \dfrac{4n}{3} \rceil & \text{for } j = \frac{n}{2},\ \ if\ \ n\ \ is\ \ even\ ; \ \text{or}\ j = \frac{n-1}{2},\ \ if\ \ n\ \ is\ \ odd, \end{cases}$$

In all cases, we can see that all the vertex and edge labels are at most $r = \lceil \frac{m\,n}{3} \rceil$ or $r = \lceil \frac{m\,n}{3} \rceil + 1$. Besides, under the labeling Ψ the weights of the edges are given by

- If m is even number

$\forall\ \ 1 \le i \le \frac{m}{2} \qquad wt_\Psi(e_{ij}) \ = 2n(i-1) + 2j - 1\ , \qquad 1 \le j \le n$

$\forall\ \ \frac{m}{2} + 1 \le i \le m$

$$wt_\Psi(e_{ij}) = \begin{cases} 2n(m+1-i) - 2j & \text{if}\ \ 1 \le j \le n-1; \\ 2n(\ m+1-i) & \text{if}\ \ j = n. \end{cases}$$

We can see that weights of edges in the first half cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_{\frac{m}{2}}}$ form an increasing sequence of consecutive odd integers from 1 up to $mn-1$. For the second half cycles $C_{n_{\frac{m}{2}+1}}, C_{n_{\frac{m}{2}+2}}, \cdots, C_{n_m}$ weights of edges form a decreasing sequence of consecutive even integers from mn up to 2. This concludes the proof. $\square$

Theorem 3.3. *Let $n \geq 6$ and m be an odd positive integer, $m \geq 5$. Then the calendula graph $Cl_{m,n}$ have .*

$$res(Cl_{m,n}) = \begin{cases} & \textit{for } n \equiv 0 \textit{ mod } 6; \\ \lceil \frac{m\ n}{3} \rceil & \textit{or } n \equiv 2, 4 \textit{ mod } 6 \textit{ and } |mn| \equiv 0, 4 \textit{ mod } 6; \\ & \textit{or } n \equiv 1, 5 \textit{ mod } 6 \textit{ and } |mn| \equiv 1, 5 \textit{ mod } 6; \\ & \textit{for } n \equiv 3 \textit{ mod } 6 \ ; \\ \lceil \frac{mn}{3} \rceil + 1 & \textit{or } n \equiv 2, 4 \textit{ mod } 6 \textit{ and } |mn| \equiv 2 \textit{ mod } 6; \\ & \textit{or } n \equiv 1, 5 \textit{ mod } 6 \textit{ and } |mn| \equiv 3 \textit{ mod } 6. \end{cases}$$

Proof. Since $|V(Cl_{m,n})| = m(n-1)$, $|E(Cl_{m,n})| = mn$ and the maximum degree $\triangle(Cl_{m,n}) = 4$. Thus, the inequality (2) becomes

$$res(Cl_{m,n}) \geq \begin{cases} \lceil \frac{m\ n}{3} \rceil & \text{if } |m\ n| \not\equiv 2, 3 \ (mod\ 6). \ ; \\ \lceil \frac{m\ n}{3} \rceil + 1 & \text{if } |m\ n| \equiv 2, 3 \ (mod\ 6). \end{cases}$$

To prove the inverse inequality, we need to show that there exist an edge irregularity reflexive $r-$ labeling, $r = \lceil \frac{m\ n}{3} \rceil$, or $r = \lceil \frac{m\ n}{3} \rceil + 1$, for the calendula graph $Cl_{m,n}$. The labeling in the case for m odd is the same as the case in Theorem 3.2 when m is even, but with some modifications, we list it in the following:

For vertices and edges when $3 \leq i \leq \frac{m}{2} - 1$, m is even, the labeling is the same when $3 \leq i \leq \frac{m-3}{2}$, m is odd.

$\forall \quad 1 \leq j \leq n-1$

$$\Psi_v(v_{(\frac{m-1}{2})j}) = \begin{cases} & \text{for } n \equiv 0 \ mod\ 3; \\ \lceil \frac{(m-1)n}{3} \rceil & \text{or } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 1, 5 \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 3 \ \ and\ m \equiv 1, 3 \ mod\ 6; \\ \lceil \frac{(m-1)n}{3} \rceil + 1 & \text{for } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 3 \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 3 \ \ and\ m \equiv 5 \ mod 6, \end{cases}$$

$$\Psi_v(v_{(\frac{m-1}{2})n}) = \begin{cases} & \text{for } n \equiv 0 \ mod\ 3, \ \ or\ \ n \equiv 2, \ 4 \ mod\ 6; \\ r & \text{or } n \equiv 1 \ mod\ 6 \ \ and\ m \equiv 3, 5 \ mod\ 6; \\ & \text{or } n \equiv 5 \ mod\ 6 \ \ and\ m \equiv 1, 3 \ mod\ 6; \\ r-1 & \text{for } n \equiv 1 \ mod\ 6 \ \ and\ m \equiv 1 \ mod\ 6; \\ & \text{or } n \equiv 5 \ mod\ 6 \ and\ m \equiv 5 \ mod\ 6, \end{cases}$$

$\forall \quad 1 \leq j \leq n$

$$\Psi_v(v_{(\frac{m+1}{2})j}) = \begin{cases} r & \text{for } n \equiv 0,2 \ mod\ 3, \quad or \quad n \equiv 4 \ mod\ 6; \\ & \text{or } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 3,5 \ mod\ 6; \\ & \text{or } n \equiv 5 \ mod\ 6 \ \ and\ m \equiv 1,3 \ mod\ 6; \\ r-1 & \text{for } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 1 \ \ mod\ 6; \\ & \text{or } n \equiv 5 \ mod\ 6 \ \ and\ m \equiv 5 \ mod\ 6, \end{cases}$$

$$\Psi_v(v_{(\frac{m+3}{2})1}) = \begin{cases} r & \text{for } n \equiv 0 \ mod\ 3, \quad or \quad n \equiv 2,\ 4 \ mod\ 6; \\ & \text{or } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 3,5 \ mod\ 6; \\ & \text{or } n \equiv 5 \ mod\ 6 \ \ and\ m \equiv 1,3 \ mod\ 6; \\ r-1 & \text{for } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 1 \ mod\ 6; \\ & \text{or } n \equiv 5 \ mod\ 6 \ \ and\ m \equiv 5 \ mod\ 6, \end{cases}$$

$\forall \quad 2 \leq j \leq n$

$$\Psi_v(v_{(\frac{m+3}{2})j}) = \begin{cases} \lceil \frac{(m-1)n}{3} \rceil & \text{for } n \equiv 0 \ mod\ 3 \ ; \\ & \text{or } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 1,5 \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 3 \ \ and\ m \equiv 1,3 \ mod\ 6; \\ \lceil \frac{(m-1)\ n}{3} \rceil + 1 & \text{for } n \equiv 1 \ mod\ 3 \ \ and\ m \equiv 3 \ \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 3 \ \ and\ m \equiv 5 \ mod\ 6, \end{cases}$$

$\forall \quad \frac{m+5}{2} \leq i \leq m-2$, there are three cases

Case (1).

If $n \equiv 0 \ mod\ 3$, or $n \equiv 1 \ mod\ 3$, $m \equiv 1 \ mod\ 6$, and $i \equiv 0 \ mod\ 3$,
or $n \equiv 1 \ mod\ 3$, $m \equiv 5 \ mod\ 6$, and $i \equiv 1 \ mod\ 3$,
or $n \equiv 1 \ mod\ 3$, $m \equiv 3 \ mod\ 6$, and $i \equiv 2 \ mod\ 3$,
or $n \equiv 2 \ mod\ 3$, $m \equiv 1 \ mod\ 6$, and $i \equiv 2 \ mod\ 3$,
or $n \equiv 2 \ mod\ 3$, $m \equiv 5 \ mod\ 6$, and $i \equiv 0 \ mod\ 3$,
or $n \equiv 2 \ mod\ 3$, $m \equiv 3 \ mod\ 6$, and $i \equiv 1 \ mod\ 3$.

In this case, the labeling of vertices $\Psi_v(v_{ij})$ will match as within Eq.(9). Also, the labeling of edges $\Psi_e(e_{ij})$ will match as within Eq.(21).

Case (2).

If $n \equiv 1 \ mod\ 3$, $m \equiv 1 \ mod\ 6$, and $i \equiv 1 \ mod\ 3$,
or $n \equiv 1 \ mod\ 3$, $m \equiv 5 \ mod\ 6$, and $i \equiv 2 \ mod\ 3$,
or $n \equiv 1 \ mod\ 3$, $m \equiv 3 \ mod\ 6$, and $i \equiv 0 \ mod\ 3$,
or $n \equiv 2 \ mod\ 3$, $m \equiv 1 \ mod\ 6$, and $i \equiv 0 \ mod\ 3$,
or $n \equiv 2 \ mod\ 3$, $m \equiv 5 \ mod\ 6$, and $i \equiv 1 \ mod\ 3$,
or $n \equiv 2 \ mod\ 3$, $m \equiv 3 \ mod\ 6$, and $i \equiv 2 \ mod\ 3$.

In this case, the labeling of vertices $\Psi_v(v_{ij})$ will match as within Eq.(10), and the labeling of edges $\Psi_e(e_{ij})$ will match as within Eq.(22).

Case (3).

If $n \equiv 1 \ mod\ 3$, $m \equiv 1 \ mod\ 6$, and $i \equiv 2 \ mod\ 3$,
or $n \equiv 1 \ mod\ 3$, $m \equiv 5 \ mod\ 6$, and $i \equiv 0 \ mod\ 3$,

or $\quad n \equiv 1\ mod\ 3\ , \quad m \equiv 3\ mod\ 6\ , \quad$ and $\quad i \equiv 1\ mod\ 3,$

or $\quad n \equiv 2\ mod\ 3\ , \quad m \equiv 1\ mod\ 6\ , \quad$ and $\quad i \equiv 1\ mod\ 3,$

or $\quad n \equiv 2\ mod\ 3\ , \quad m \equiv 5\ mod\ 6\ , \quad$ and $\quad i \equiv 2\ mod\ 3,$

or $\quad n \equiv 2\ mod\ 3\ , \quad m \equiv 3\ mod\ 6\ , \quad$ and $\quad i \equiv 0\ mod\ 3.$

In this case, the labeling of vertices $\Psi_v(v_{ij})$ will match as within Eq.(11) and the labeling of edges $\Psi_e(e_{ij})$ will match as within Eq.(23).

$$\forall \qquad 1 \leq j \leq n-2$$

$$\Psi_e(e_{(\frac{m-1}{2})j}) = \begin{cases} n(m-3)+2j-1-2\lceil\frac{(m-1)n}{3}\rceil & \begin{array}{r}\text{for } n \equiv 0\ mod\ 3;\\ \text{or } n \equiv 1\ mod\ 3\ and\ m \equiv 1,5\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 3\ and\ m \equiv 1,3\ mod\ 6;\end{array} \\ n(m-3)+2j-3-2\lceil\frac{(m-1)\ n}{3}\rceil & \begin{array}{r}\text{for } n \equiv 1\ mod\ 3\ and\ m \equiv 3\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 3\ and\ m \equiv 5\ mod\ 6,\end{array} \end{cases}$$

$$\Psi_e(e_{(\frac{m-1}{2})(n-1)}) = \begin{cases} n(m-1)-3-\lceil\frac{(m-1)n}{3}\rceil - r & \begin{array}{r}\text{for } n \equiv 0\ mod\ 3,\ or\ n \equiv 5\ mod\ 6;\\ \text{or } n \equiv 1\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 1,3\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 1,5\ mod\ 6;\end{array} \\ n(m-1)-4-\lceil\ \frac{(m-1)n}{3}\rceil - r & \begin{array}{r}\text{for } n \equiv 1\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\end{array} \\ n(m-1)-2-\lceil\frac{(m-1)n}{3}\rceil - r & \text{if } \ n,m \equiv 1\ mod\ 6 \end{cases}$$

$$\Psi_e(e_{(\frac{m-1}{2})n}) = \begin{cases} n(m-1)-1-\lceil\frac{(m-1)n}{3}\rceil - r & \begin{array}{r}\text{for } n \equiv 0\ mod3,\ or\ n \equiv 5\ mod\ 6;\\ \text{or } n \equiv 1\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 1,3\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 1,5\ mod\ 6;\end{array} \\ n(m-1)-2-\lceil\frac{(m-1)n}{3}\rceil - r & \begin{array}{r}\text{for } n \equiv 1\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\end{array} \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil - r & \text{if } n,m \equiv 1\ mod\ 6, \end{cases}$$

$$\forall \qquad 1 \leq j \leq \tfrac{n+1}{2} \ \text{ and } n \ \text{ is odd,} \quad \text{ or } \quad 1 \leq j \leq \tfrac{n}{2} \quad \text{ and } \quad n \quad \text{ is even}$$

$$\Psi_e(e_{(\frac{m+1}{2})j}) = \begin{cases} n(m-1)+2j-1-2r & \begin{array}{l}\text{for } n \equiv 0\ mod\ 3,\ or\ n \equiv 2,4\ mod\ 6\ \ ;\\ \text{or } n \equiv 1\ mod\ 6\ and\ m \equiv 3,5\ mod\ 6\ ;\\ \text{or } n \equiv 5\ mod\ 6\ and\ m \equiv 1,3\ mod\ 6\ ;\end{array} \\ n(m-1)+2j+1-2r & \begin{array}{l}\text{for } n \equiv 1\ mod\ 6\ and\ m \equiv 1\ mod\ 6;\\ \text{or } n \equiv 5\ mod\ 6\ and\ m \equiv 5\ mod\ 6,\end{array} \end{cases}$$

$$\forall \qquad \tfrac{n+3}{2} \leq j \leq n \ \text{ and } n \ \text{ is odd, or } \ \tfrac{n}{2}+1 \leq j \leq n \ \text{ and } n \ \text{ is even}$$

$$\Psi_e(e_{(\frac{m+1}{2})j}) = \begin{cases} & \text{for } n \equiv 0 \; mod \; 3\,; \\ n(m+1)-2j+2-2r & \text{for } n \equiv 2,4 \; mod \; 6\,; \\ & \text{or } n \equiv 1 \; mod \; 6 \; and \; m \equiv 3,5 \; mod \; 6; \\ & \text{or } n \equiv 5 \; mod \; 6 \; and \; m \equiv 1,3 \; mod \; 6; \\ n(m+1)-2j+4-2r & \text{for } n \equiv 1 \; mod \; 6 \; and \; m \equiv 1 \; mod \; 6; \\ & \text{or } n \equiv 5 \; mod \; 6 \; and \; m \equiv 5 \; mod \; 6, \end{cases}$$

$$\Psi_e(e_{(\frac{m+3}{2})1}) = \begin{cases} & \text{for } n \equiv 0 \; mod \; 3, \; or \; n \equiv 5 \; mod \; 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-2-r & \text{or } n \equiv 1 \; mod \; 6 \; and \; m \equiv 5 \; mod \; 6; \\ & \text{or } n \equiv 2 \; mod \; 6 \; and \; m \equiv 1,3 \; mod \; 6; \\ & \text{or } n \equiv 4 \; mod \; 6 \; and \; m \equiv 1,5 \; mod \; 6; \\ & for n \equiv 1 \; mod \; 6 \; and \; m \equiv 3 \; mod \; 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-3-r & \text{or } n \equiv 2 \; mod \; 6 \; and \; m \equiv 5 \; mod \; 6; \\ & \text{or } n \equiv 4 \; mod \; 6 \; and \; m \equiv 3 \; mod \; 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-1-r & \text{if } \; n,m \equiv 1 \; mod \; 6, \end{cases}$$

$\forall \quad 2 \leq j \leq n-1$

$$\Psi_e(e_{(\frac{m+3}{2})j}) = \begin{cases} & \text{for } n \equiv 0 \; mod \; 3; \\ n(m-1)-2j-2\lceil\frac{(m-1)n}{3}\rceil & \text{or } n \equiv 1 \; mod \; 3 \; and \; m \equiv 1,5 \; mod \; 6; \\ & \text{or } n \equiv 2 \; mod \; 3 \; and \; m \equiv 1,3 \; mod \; 6; \\ n(m-1)-2j-2\lceil\frac{(m-1)n}{3}\rceil-2 & \text{for } n \equiv 1 \; mod \; 3 \; and \; m \equiv 3 \; mod \; 6; \\ & \text{or } n \equiv 2 \; mod \; 3 \; and \; m \equiv 5 \; mod \; 6, \end{cases}$$

$$\Psi_e(e_{(\frac{m+3}{2})n}) = \begin{cases} & \text{for } n \equiv 0 \; mod \; 3, \; or \; n \equiv 5 \; mod \; 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-r & \text{or } n \equiv 1 \; mod \; 6 \; and \; m \equiv 5 \; mod \; 6; \\ & \text{or } n \equiv 2 \; mod \; 6 \; and \; m \equiv 1,3 \; mod \; 6; \\ & \text{or } n \equiv 4 \; mod \; 6 \; and \; m \equiv 1,5 \; mod \; 6; \\ & \text{for } n \equiv 1 \; mod \; 6 \; and \; m \equiv 3 \; mod \; 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-1-r & \text{or } n \equiv 2 \; mod \; 6 \; and \; m \equiv 5 \; mod \; 6; \\ & \text{or } n \equiv 4 \; mod \; 6 \; and \; m \equiv 3 \; mod \; 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil+1-r & \text{if } \; n,m \equiv 1 \; mod \; 6. \end{cases}$$

In all cases, we are able see that all the vertex and edge names are at most $r = \lceil \frac{m\,n}{3} \rceil$ or $r = \lceil \frac{m\,n}{3} \rceil + 1$.

When m is odd number, the weight of the edges of the calendula graphs $Cl_{m,n}$ are given by

$$\forall \;\; 1 \leq i \leq \tfrac{m-1}{2} \qquad wt_\Phi(e_{ij}) \; = 2n(i-1)+2j-1\,, \qquad\qquad 1 \leq j \leq n$$

$$wt_\Phi(e_{(\frac{m+1}{2})j}) = \begin{cases} n(m-1)+2j-1 & \text{for } 1 \leq j \leq \frac{n}{2}, \; n \;\; is \;\; even; \\ & \text{or } 1 \leq j \leq \frac{n+1}{2}, \;\; n \;\; is \;\; odd, \\ n(m+1)-2j+2 & \text{for } \; \frac{n}{2}+1 \leq j \leq n, \;\; n \;\; is \;\; even; \\ & \text{or } \; \frac{n+3}{2} \leq j \leq n, \;\; n \;\; is \;\; odd, \end{cases}$$

$$\forall \;\; \tfrac{m+3}{2} \leq i \leq m \qquad wt_\Phi(e_{ij}) = \begin{cases} 2n(m+1-i)-2j & \text{if } \;\; 1 \leq j \leq n-1; \\ 2n(m+1-i) & \text{if } \;\; j = n. \end{cases}$$

We can see that weights of edges in all cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_m}$ are all different, this concludes the proof. □

Table 1 illustrates the different cases of an edge irregularity reflexive labeling for the calendula $Cl_{m,n}$ graph.

Table 1: The different cases of an edge irregularity reflexive labeling for the calendula $Cl_{m,n}$

Graph	n	m	$E(Cl_{m,n})$	r
$Cl_{m,n}$	$n \equiv 0 \ mod\ 6$	$\forall m$	$E \equiv 0 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 4 \ mod\ 6$	$m \equiv 0, 3 \ mod\ 6$	$E \equiv 0 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 4 \ mod\ 6$	$m \equiv 1, 4 \ mod\ 6$	$E \equiv 4 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 4 \ mod\ 6$	$m \equiv 2, 5 \ mod\ 6$	$E \equiv 2 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 2 \ mod\ 6$	$m \equiv 0, 3 \ mod\ 6$	$E \equiv 0 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 2 \ mod\ 6$	$m \equiv 2, 5 \ mod\ 6$	$E \equiv 4 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 2 \ mod\ 6$	$m \equiv 1, 4 \ mod\ 6$	$E \equiv 2 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 3 \ mod\ 6$	$m \equiv 0, 2, 4 \ mod\ 6$	$E \equiv 0 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 3 \ mod\ 6$	$m \equiv 1, 3, 5 \ mod\ 6$	$E \equiv 3 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 1 \ mod\ 6$	$m \equiv 0 \ mod\ 6$	$E \equiv 0 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \ mod\ 6$	$m \equiv 1 \ mod\ 6$	$E \equiv 1 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \ mod\ 6$	$m \equiv 4 \ mod\ 6$	$E \equiv 4 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \ mod\ 6$	$m \equiv 5 \ mod\ 6$	$E \equiv 5 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \ mod\ 6$	$m \equiv 2 \ mod\ 6$	$E \equiv 2 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 1 \ mod\ 6$	$m \equiv 3 \ mod\ 6$	$E \equiv 3 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 5 \ mod\ 6$	$m \equiv 1 \ mod\ 6$	$E \equiv 5 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \ mod\ 6$	$m \equiv 2 \ mod\ 6$	$E \equiv 0 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \ mod\ 6$	$m \equiv 4 \ mod\ 6$	$E \equiv 4 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \ mod\ 6$	$m \equiv 5 \ mod\ 6$	$E \equiv 1 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \ mod\ 6$	$m \equiv 0 \ mod\ 6$	$E \equiv 2 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 5 \ mod\ 6$	$m \equiv 3 \ mod\ 6$	$E \equiv 3 \ mod\ 6$	$\lceil \frac{m\ n}{3} \rceil + 1$

Illustration: The calendula graphs $Cl_{10,12}$ with an edge irregularity total $k = 41$ labeling and $Cl_{10,12}$ with an edge irregularity reflexive $r = 40$ labeling are shown in Fig. 6.

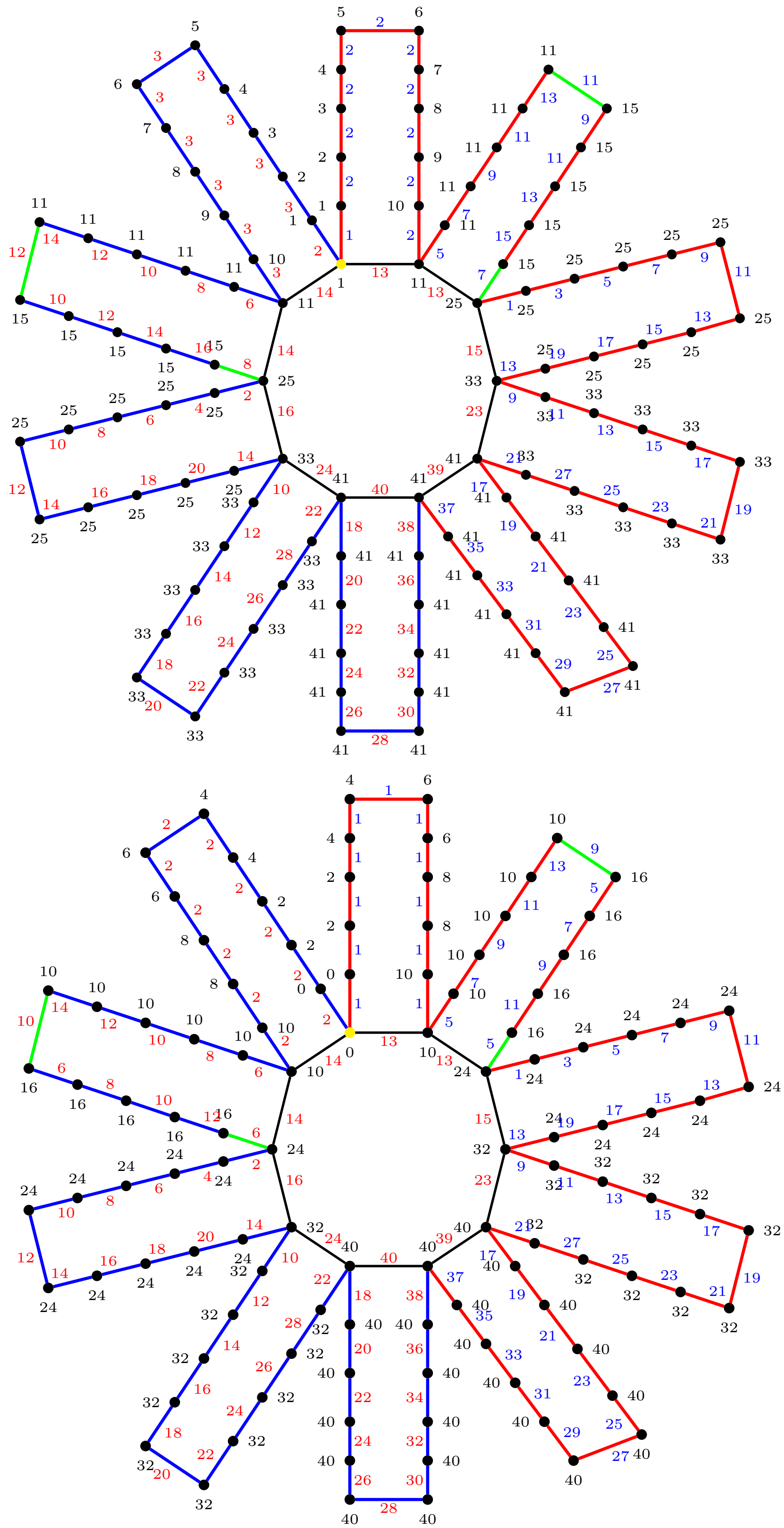

Figure 6: The calendula graphs $Cl_{10,12}$ with an edge irregularity total $k = 41$ labeling and $Cl_{10,12}$ with an edge irregularity reflexive $r = 40$ labeling

Illustration: The calendula graphs $Cl_{9,11}$ with an edge irregularity total $k = 34$ labeling and $Cl_{9,11}$ with an edge irregularity reflexive $r = 34$ labeling are shown in Fig. 7.

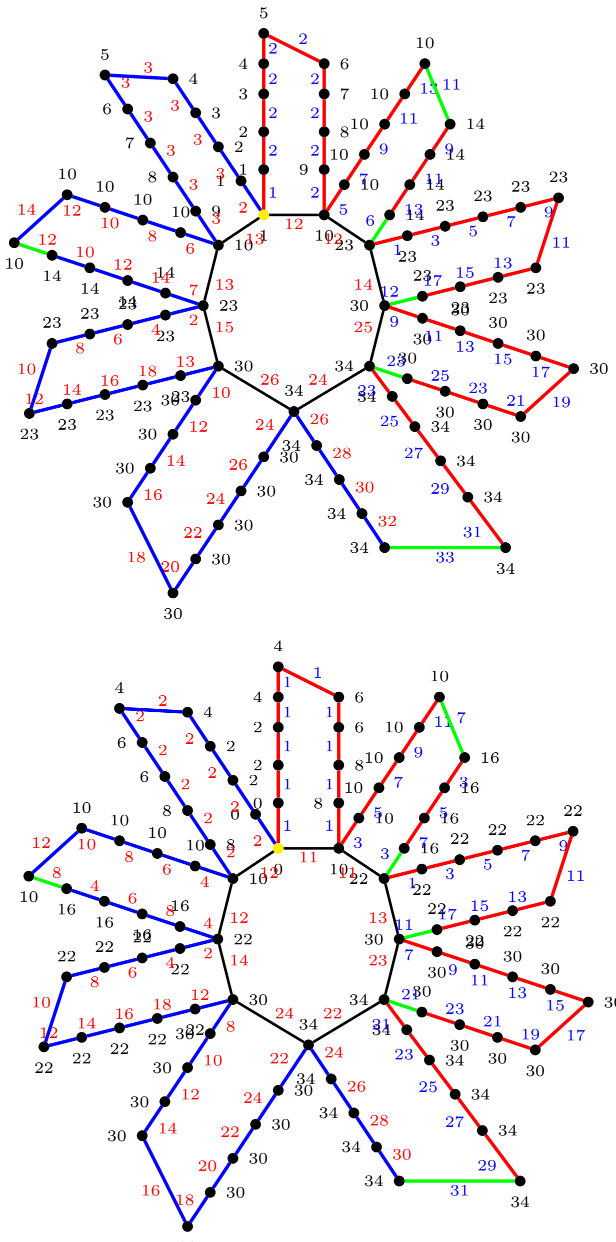

Figure 7: The calendula graphs $Cl_{9,11}$ with an edge irregularity total $k = 34$ labeling and $Cl_{9,11}$ with an edge irregularity reflexive $r = 34$ labeling

4. Conclusion

lately, graph labeling has become a fruitful branch of several research studies in graph theory. It has massive applications in many disciplines, like coding theory, X-rays, radar, communication networks, and astronomy. In this work, we have determined the total edge irregular strength $k-$ and the edge irregularity reflexive $r-$ for calendula $Cl_{4,n}$, $n \geq 4$, $Cl_{5,n}$, $n \geq 4$. Furthermore, the exact value of total edge irregular strength $k-$ and the edge irregularity reflexive $r-$ for a generalized calendula $Cl_{m,n}$ was defined.

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An efficient optimal fourth-order iterative method for scalar equations

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Abstract. In the present paper, using linear combination technique, we introduce an optimal three-step iterative scheme for solving nonlinear equations. We prove the convergence of the proposed method. In order to demonstrate the performance of newly developed method, we consider some commonly used nonlinear equations for numerical as well as graphical comparisons. We also explore polynomiographs in the context of some complex polynomials.

Keywords: iterative methods, nonlinear equations, order of convergence, linear combination.

1. Introduction

Nonlinear equations and their solutions have been a scorching topic for many researchers. In this regard, vast literature is available, for examples see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein. A fundamental technique for solving nonlinear equations is the well-known Newton's method, which converges quadratically:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \ldots. \tag{1}$$

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According to Kung and Turab [5] conjecture, an iterative method is called optimal if it needs $(n+1)$ functional evaluations per iteration and possesses convergence order 2^n. S. Abbasbandy [6] using modified Adomian decomposition method, proposed a fourth-order method which needs three evaluations per iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} - \frac{f^3(x_n)f''^2(x_n)}{2f'^5(x_n)}, \tag{2}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

Cordero et al. [7], developed the following fourth-order method:

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} - \left[\frac{f(y_n)}{f'(x_n)}\right]^2 \left[\frac{2f(x_n) + f(y_n)}{f'(x_n)}\right], \tag{3}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

A second derivative free optimal fourth-order method has been introduced by Chun et al. [8].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left[1 + \frac{3}{4}\frac{f'(x_n) - f'(y_n)}{f'(x_n)} + \frac{9}{8}\left(\frac{f'(x_n) - f'(y_n)}{f'(x_n)}\right)^2\right], \tag{4}$$

where $y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

In 2015, Sherma and Behl [9], also proposed a second derivative free optimal fourth-order method:

$$x_{n+1} = x_n - \left[-\frac{1}{2} + \frac{9f'(x_n)}{8f'(y_n)} + \frac{3f'(y_n)}{8f'(x_n)}\right] \frac{f(x_n)}{f'(x_n)}, \tag{5}$$

where $y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

In this paper, having motivation from the above study, we propose a more effective second derivative free optimal fourth-order iterative method. The effectiveness of our method is explored by its numerical as well as graphical comparisons with some existing methods of the same class. We also investigate the dynamical behavior of newly constructed method for visualization of the roots of complex polynomials.

2. Construction of iterative method

Consider the nonlinear equation

$$f(x) = 0. \tag{6}$$

Using Taylor's expansion about γ (initial guess), equation (6) can be written in the form of the following coupled system:

$$f(x) \approx f(\gamma) + (x - \gamma)f'(\gamma) + g(x) \approx 0, \tag{7}$$

$$g(x) \approx \frac{\lambda f(x)}{f'(\gamma)} - f(\gamma) - (x - \gamma)f'(\gamma), \tag{8}$$

where $\lambda \in R$ is an auxiliary parameter.

From equation (7), we get

$$x \approx \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{g(x)}{f'(\gamma)}, \\ = c + N(x), \tag{9}$$

where

$$c = \gamma - \frac{f(\gamma)}{f'(\gamma)} \quad \text{and} \tag{10}$$

$$N(x) = -\frac{g(x)}{f'(\gamma)}. \tag{11}$$

Here, $N(x)$ is a nonlinear operator and can be approximated by using Taylor's series expansion about x_0 as follows:

$$N(x) = N(x_0) + \sum_{k=1}^{\infty} \frac{(x_i - x_0)^k}{k!} N^{(k)}(x_0). \tag{12}$$

Our aim is to find the series solution of equation (6):

$$x = \sum_{i=0}^{\infty} x_i. \tag{13}$$

Which can alternatively be expressed as

$$x = \lim_{m\to\infty} X_m, \qquad \text{where } X_m = x_0 + x_1 + \ldots + x_m. \tag{14}$$

From equations (9), (12) and (13), we get

$$x = \sum_{i=0}^{\infty} x_i = \sum_{i=0}^{\infty} (c + N(x_0) + \sum_{k=1}^{\infty} \frac{(x_i - x_0)^k}{k!} N^{(k)}(x_0), \text{ which implies}$$

$$x = c + N(x_0) + \sum_{k=1}^{\infty} \frac{(\sum_{i=0}^{k} x_i - x_0)^k}{k!} N^{(k)}(x_0). \tag{15}$$

From the last relation, we have the following scheme:

$$\left\{ \begin{array}{l} x_0 = c, \\ x_1 = N(x_0), \\ x_2 = (x_0 + x_1 - x_0)N'(x_0), \\ x_3 = \frac{(x_0+x_1+x_2-x_0)^2}{2!} N''(x_0), \\ \vdots \\ x_{m+1} = \frac{(x_0+x_1+\ldots+x_m-x_0)^m}{m!} N^{(m)}(x_0), \ m = 0, 1, 2, \ldots. \end{array} \right\} \tag{16}$$

Thus,

$$x_1 + x_2 + \ldots + x_{m+1} = N(x_0) + (x_0 + x_1 - x_0)N'(x_0) + \ldots + \frac{(x_0 + x_1 + \ldots + x_m - x_0)^m}{m!} N^{(m)}(x_0), \tag{17}$$

where $m = 1, 2, \ldots$.

Since $x_0 = c$, therefore, equation (15) gives

$$x = c + \sum_{i=1}^{\infty} x_i. \tag{18}$$

From equation (10) and the first equation of (16), we have

$$x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}. \tag{19}$$

Using equation (14) with $m = 0$ and equation (19), we have

$$x \approx X_0 = x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}. \tag{20}$$

This formulation allows us to propose the following iterative method for solving nonlinear equation (6).

Algorithm. 2.1. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \qquad n = 0, 1, 2, \ldots, \tag{21}$$

which is the well-known Newton's method.

Now, from equation (19), we have

$$x_0 - \gamma = -\frac{f(\gamma)}{f'(\gamma)}. \tag{22}$$

Using equation (11) and the second equation of (16), we get

$$x_1 = N(x_0) = -\frac{g(x_0)}{f'(\gamma)}. \tag{23}$$

Thus, using equation (8), we have

$$g(x_0) = \frac{\lambda f(x_0)}{f'(\gamma)} - f(\gamma) - (x_0 - \gamma)f'(\gamma). \tag{24}$$

From equations (22), (23) and (24), we obtain

$$x_1 = N(x_0) = -\frac{\lambda f(x_0)}{(f'(\gamma))^2} = -\frac{\lambda f(\gamma - \frac{f(\gamma)}{f'(\gamma)})}{(f'(\gamma))^2}. \tag{25}$$

Using equation (14) with $m = 1$ along with equations (19) and (25), we obtain

$$x \approx X_1 = x_0 + x_1 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{\lambda f(\gamma - \frac{f(\gamma)}{f'(\gamma)})}{(f'(\gamma))^2}. \tag{26}$$

This formulation allows us to propose the following iterative method for solving nonlinear equation (6).

Algorithm 2.2. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = y_n - \frac{\lambda f(y_n)}{(f'(x_n))^2}, \tag{27}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

The last algorithm converges cubically for $\lambda = 1$ and requires 3 function evaluations per iteration.

From the third equation of (16), we get

$$x_2 = (x_1)N'(x_0) = -x_1 \frac{g'(x_0)}{f'(\gamma)}. \tag{28}$$

From equation (8), we have

$$g'(x_0) = \frac{\lambda f'(x_0)}{f'(\gamma)} - f'(\gamma). \tag{29}$$

Thus from equations (25), (28) and (29), we have

$$x_2 = \lambda^2 \frac{f(x_0) f'(x_0)}{(f'(\gamma))^4} - \frac{\lambda f(x_0)}{(f'(\gamma))^2}. \tag{30}$$

Using equation (14) with $m = 2$ along with equations (19), (25) and (30), we obtain

$$\begin{aligned} x &\approx X_3 = x_0 + x_1 + x_2 \\ &= c - 2\frac{\lambda f(x_0)}{(f'(\gamma))^2} + \lambda^2 \frac{f(x_0) f'(x_0)}{(f'(\gamma))^4} \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - 2\frac{\lambda f(x_0)}{(f'(\gamma))^2} + \lambda^2 \frac{f(x_0) f'(x_0)}{(f'(\gamma))^4}. \end{aligned} \tag{31}$$

This formulation allows us to propose the following iterative method for solving nonlinear equation (6).

Algorithm 2.3. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = 2z_n - y_n - \lambda(z_n - y_n) \frac{f'(y_n)}{(f'(x_n))^2}, \tag{32}$$

where $z_n = y_n - \frac{\lambda f(y_n)}{(f'(x_n))^2}$, and $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$. The above algorithm has convergence order 3 and needs 4 function evaluations per iteration. In order to reduce the number of function evaluations by one, we make the following approximation:

$$f'(y_n) \approx \frac{f(y_n) - f(x_n)}{(y_n - x_n)}. \tag{33}$$

Algorithm 2.4. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = 2z_n - y_n - \lambda(z_n - y_n)\frac{f(y_n) - f(x_n)}{(y_n - x_n)(f'(x_n))^2}, \tag{34}$$

where

$$z_n = y_n - \frac{\lambda f(y_n)}{(f'(x_n))^2},$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \qquad n = 0, 1, 2, \ldots.$$

Which is a three-step iterative method having convergence order three and needs three function evaluations per iteration.

On the basis of the linear combination of Algorithms 2.2 and 2.4, we suggest the following new optimal fourth-order iterative scheme:

$$x_{n+1} = y_n + (1 + \frac{4\theta}{3})\frac{\lambda f(y_n)}{(f'(x_n))^2} + \frac{4\theta}{3}\lambda(z_n - y_n)\frac{f(y_n) - f(x_n)}{(y_n - x_n)(f'(x_n))^2}, \tag{35}$$

where $\theta \in R$ is the adjusting parameter. Clearly, for $\theta = 0$ and $\lambda = -1$, equation (35) reduces to the method given in equation (27) and for $\theta = 3/4$ and $\lambda = -1$, it reduces to the method defined in equation (34). The performance of newly suggested method depends upon the appropriate choice of θ.

Taking $\theta = 1$ and $\lambda = -1$, the above formulation allows us to suggest the following optimal fourth-order iterative method:

Algorithm 2.5. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = y_n - (\frac{7}{3})\frac{f(y_n)}{(f'(x_n))^2} - \frac{4}{3}(z_n - y_n)\frac{f(y_n) - f(x_n)}{(y_n - x_n)(f'(x_n))^2}, \tag{36}$$

where

$$z_n = y_n - \frac{f(y_n)}{(f'(x_n))^2},$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \qquad n = 0, 1, 2, \ldots.$$

To the best of our knowledge, algorithm 2.5 is a new one to solve the nonlinear equation (6).

3. Convergence analysis

In this section, convergence criteria of newly proposed method is studied in the form of the following theorem.

Theorem 3.1. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ (where I is an open interval) has a simple root $\alpha \in I$ and x_0 is sufficiently close to α. Let $f(x)$ be sufficiently differentiable in the neighborhood of α, then the algorithm 2.5 has the convergence order 4.*

Proof. Let α be a simple zero of $f(x)$. Since f is sufficiently differentiable, therefore, the Taylor's series expansions of $f(x_n)$ and $f'(x_n)$ about α are given by

$$f(x_n) = f'(\alpha)\{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O\left(e_n^6\right)\}, \tag{37}$$

and

$$f'(x_n) = f'(\alpha)\{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_4^3 + 5c_5e_n^4 + 6c_6e_n^5 + O\left(e_n^5\right)\}, \tag{38}$$

where $e_n = x_n - \alpha$ and $c_j = (\frac{1}{j!})\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 1, 2, 3, \ldots$.

From equations (37) and (38), we get

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ &\quad +(8c_2^4 - 20c_2^2c_3 + 10c_2c_4 + 6c_3^2 - 4c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{39}$$

Using equation (39), we find

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad +(-8c_2^4 + 20c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{40}$$

Using equation (40), the Taylor's series of $f(y_n)$ is given as

$$\begin{aligned} f(y_n) &= c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad +(-12c_2^4 + 24c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{41}$$

Using equations (40) and (41), we get

$$\begin{aligned} z_n &= \alpha + 4c_2^2e_n^3 + (-21c_2^3 + 14c_2c_3)e_n^4 \\ &\quad + (80c_2^4 - 104c_2^2c_3 + 20c_2c_4 + c_3^2)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{42}$$

From equations (38), (40) and (41), we obtain

$$\begin{aligned} y_n - \frac{7f(y_n)}{3(f'(x_n))^2} &= \alpha - \frac{4}{3}c_2e_n^2 + (12c_2^2 - \frac{8}{3}c_3)e_n^3 + (-\frac{163}{3}c_2^3 + 42c_2c_3 - 4c_4)e_n^4 \\ &\quad + (\frac{592}{3}c_2^4 - \frac{808}{3}c_2^2c_3 + 60c_2c_4 + 36c_3^2 - \frac{16}{3}c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{43}$$

Using equations (40) and (42), we have

$$
\begin{aligned}
4(z_n - y_n)\frac{f(y_n) - f(x_n)}{3(y_n - x_n)(f'(x_n))^2} &= -\frac{4}{3}c_2e_n^2 + (12c_2^2 - \frac{8}{3}c_3)e_n^3 + (-\frac{208}{3}c_2^3 \\
&\quad + \frac{128}{3}c_2c_3 - 4c_4)e_n^4 + (324c_2^4 - 352c_2^2c_3 \\
&\quad + \frac{184}{3}c_2c_4 + \frac{112}{3}c_3^3 - \frac{16}{3}c_5)e_n^5 + O\left(e_n^6\right). \qquad (44)
\end{aligned}
$$

Using equations (43) and (44), the error term for algorithm 2.5 is given as

$$
e_{n+1} = (15c_2^3 - \frac{2}{3}c_2c_3)e_n^4 + (-\frac{380}{3}c_2^4 + \frac{248}{3}c_2^2c_3 - \frac{4}{3}c_2c_4 - \frac{4}{3}c_3^2)e_n^5 + O(e_n^6). \qquad (45)
$$

This completes the proof. □

4. Numerical examples

In this section, we reveal the validity and efficiency of our proposed iterative method (AN1) given in algorithm 2.5 by considering the nonlinear equations from the fields of mathematical sciences. Taking x_0 as initial guess, we compare AN1 with the standard Newton's method (equation 1) (NM), Abbasbundy's method (equation 2) (AM), Cordero et al. method (equation 3) (DM), Chun's method (equation 4) (CM), and recently developed method (equation 5) (RM) by Sharma and Behl. The numerical comparison is presented in the following table and the graphical behavior is studied in Fig. 1 to Fig. 10 to demonstrate the performance of the methods. We use Maple 18 and Matlab software for comparisons, taking $|x_{n+1} - x_n|$ as stopping criteria, where $\epsilon = 10^{-15}$ represents the tolerance. Both the comparative studies show that the newly developed method i.e. algorithm 2.5 performs better.

In the following table NFE denotes the total number of functional evaluations required to reach the desired result.

Table

$f(x)$	$x_\circ$	$Method$	n	x_n	$f(x_n)$	$(\lvert x_{n+1}-x_n \rvert)$	NFE
x^3+x^2+2	1.4	NM	18	−1.6956207695598621	$2.228890e^{-16}$	$3.137243e^{-09}$	36
		AM	13	−1.6956207695598621	$2.228890e^{-16}$	$2.024970e^{-13}$	39
		DM	50	6889.4069728314899	$3.270458e^{11}$	$6.870399e^{+03}$	150
		CM	28	−1.6956207695598621	$2.228890e^{-16}$	$1.897249e^{-14}$	84
		RM	13	−1.6956207695598621	$2.228890e^{-16}$	$5.034529e^{-05}$	39
		$AN1$	6	−1.6956207695598621	$2.228890e^{-16}$	$1.071447e^{-08}$	18
$e^{sinx}+\sin(3x)-1$	−0.4	NM	6	0.0000000000000000	$1.914154e^{-27}$	$6.187332e^{-14}$	12
		AM	5	−1.8736010948989818	$6.376116e^{-17}$	$4.639035e^{-08}$	15
		DM	4	0.0000000000000000	$3.782606e^{-29}$	$8.934023e^{-08}$	12
		CM	4	3.1415926535897932	$1.538506e^{-16}$	$7.110410e^{-07}$	12
		RM	4	0.0000000000000000	$8.231301e^{-50}$	$6.173081e^{-13}$	12
		$AN1$	3	0.0000000000000000	$2.659091e^{-15}$	$1.031329e^{-07}$	9
$x^3-e^{-\sin 4x}-1$	1.03	NM	7	1.4115624181268047	$1.491129e^{-16}$	$3.531439e^{-09}$	14
		AM	12	1.4115624181268047	$1.491129e^{-16}$	$2.139755e^{-09}$	36
		DM	8	1.4115624181268047	$1.491129e^{-16}$	$8.432900e^{-15}$	24
		CM	9	1.4115624181268047	$1.491129e^{-16}$	$4.229577e^{-14}$	27
		RM	5	1.4115624181268047	$1.491129e^{-16}$	$6.641495e^{-06}$	15
		$AN1$	4	1.4115624181268047	$1.491129e^{-16}$	$7.080969e^{-09}$	12
$ln(e^{-x}+x^2)$	0.28	NM	6	−0.0000000000000000	$1.409068e^{+23}$	$3.753755e^{-12}$	12
		AM	3	0.0000000000000000	$3.426071e^{+35}$	$3.247871e^{-12}$	9
		DM	3	0.0000000000000000	$0.000000e^{+00}$	$8.553213e^{+03}$	9
		CM	5	0.7145563847430097	$1.504888e^{-18}$	$6.332712e^{-11}$	15
		RM	4	−0.0000000000000000	$4.431519e^{-28}$	$1.073788e^{-07}$	12
		$AN1$	2	0.0000000000000000	$0.000000e^{+00}$	$3.185195e^{+02}$	6
$sin^{-1}x$	0.3	NM	3	0.0000000000000000	$6.689713e^{-21}$	$2.717542e^{-07}$	6
		AM	3	−0.0000000000000000	$1.781999e^{-20}$	$3.767121e^{-07}$	9
		DM	2	0.0000000000000000	$1.673591e^{-21}$	$1.246764e^{-04}$	6
		CM	2	0.0000000000000000	$1.797242e^{-23}$	$5.576293e^{-05}$	6
		RM	2	0.0000000000000000	$1.001759e^{-22}$	$7.862795e^{-05}$	6
		$AN1$	2	0.0000000000000000	$1.362037e^{-26}$	$1.297510e^{-05}$	6
$sin(5x)$	−0.22	NM	5	0.0000000000000000	$1.384135e^{-20}$	$6.925692e^{-08}$	10
		AM	5	−1.2566370614359173	$2.307471e^{-17}$	$8.228151e^{-08}$	15
		DM	4	−0.0000000000000000	$4.183607e^{-33}$	$1.192325e^{-07}$	12
		CM	7	324.21236185046666	$1.104672e^{-14}$	$1.908201e^{-05}$	21
		RM	4	−0.0000000000000000	$1.550586e^{-69}$	$6.224901e^{-15}$	12
		$AN1$	3	−0.0000000000000013	$6.732192e^{-15}$	$6.506171e^{-06}$	9
$ln(x^2+e^X)+x$	−0.5	NM	5	0.0000000000000000	$2.761873e^{-17}$	$5.255352e^{-09}$	10
		AM	9	−0.0000000000000000	$9.183366e^{-31}$	$6.739488e^{-11}$	27
		DM	8	0.0000000000000000	$2.076011e^{-21}$	$6.099432e^{-06}$	24
		CM	5	0.0000000000000011	$2.171942e^{-15}$	$1.876283e^{-04}$	15
		RM	4	0.0000000000000000	$6.932928e^{-32}$	$1.590518e^{-08}$	12
		$AN1$	3	0.0000000000000000	$7.299088e^{-26}$	$6.617743e^{-13}$	9
$x+2x^3sin(x)-1$	1.38	NM	6	1.5523226989842709	$2.951026e^{-16}$	$2.008739e^{-11}$	12
		AM	18	4.7079470507119627	$9.955880e^{-15}$	$4.485377e^{-12}$	54
		DM	7	−32504614.839471103	$7.392443e^{+13}$	$3.474241e^{-19}$	21
		CM	4	7.0707301317015739	$2.403279e^{-14}$	$2.078366e^{-07}$	12
		RM	9	7.0707301317015739	$2.403279e^{-14}$	$2.407384e^{-14}$	27
		$AN1$	4	1.5523226989842709	$2.951026e^{-16}$	$2.345465e^{-10}$	12
$sin(5x)-sin(x)$	0.2	NM	5	−0.0000000000000000	$6.387890e^{-15}$	$5.366433e^{-06}$	10
		AM	20	−67.5442420404298480	$1.656949e^{-15}$	$1.958451e^{-08}$	60
		DM	4	0.0000000000000000	$1.387071e^{-20}$	$3.651889e^{-05}$	12
		CM	4	−0.5235987755982989	$1.598912e^{-17}$	$7.380621e^{-14}$	12
		RM	4	0.0000000000000000	$1.897089e^{-57}$	$1.559232e^{-12}$	12
		$AN1$	3	−0.0000000000000000	$2.652270e^{-18}$	$5.044085e^{-07}$	9
x^3-x^2-8	−2.3	NM	14	2.3948586738660659	$5.353746e^{-16}$	$1.947641e^{-15}$	28
		AM	14	2.3948586738660659	$5.353746e^{-16}$	$1.580128e^{-08}$	42
		DM	50	−1.7488390182525641	$1.640715e^{+01}$	$2.566501e^{+00}$	150
		CM	31	2.3948586738660659	$5.353746e^{-16}$	$4.634751e^{-12}$	93
		RM	17	2.3948586738660659	$5.353746e^{-16}$	$8.595530e^{-08}$	51
		$AN1$	7	2.3948586738660659	$5.353746e^{-16}$	$4.022589e^{-15}$	21

From the above results, it is clear that each method converges for the considered test problems but the computational cost of the proposed method is the least.

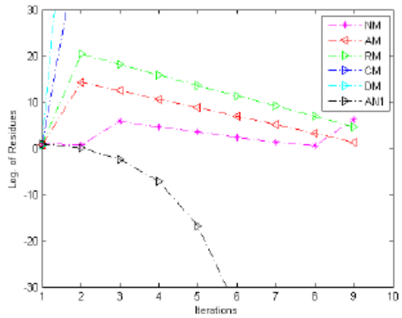

Fig. 1 ($f(x) = x^3 + x^2 + 2$)

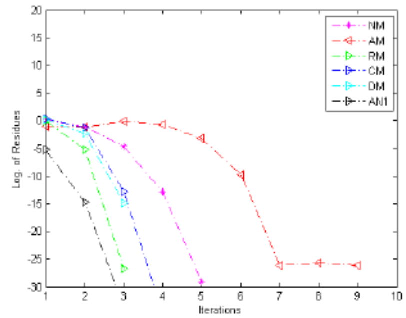

Fig. 2 ($f(x) = e^{\sin x} + \sin(3x) - 1$)

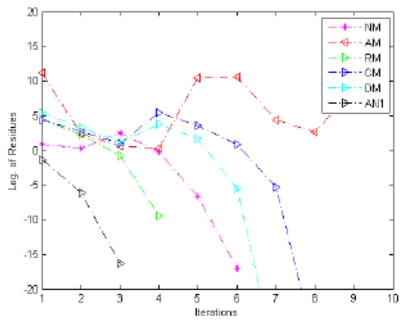

Fig. 3 ($f(x) = x^3 - e^{-\sin 4x} - 1$)

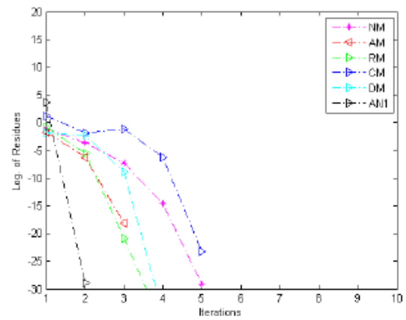

Fig. 4 ($f(x) = ln(e^{-x} + x^2)$)

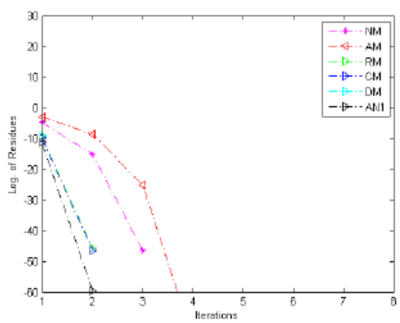

Fig. 5 ($f(x) = \sin^{-1} x$)

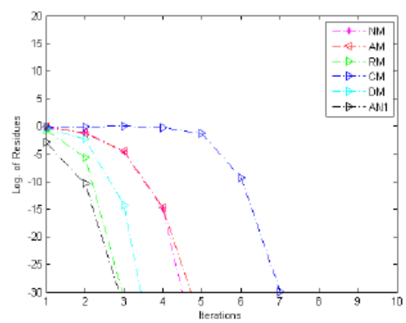

Fig. 6 ($f(x) = \sin(5x)$)

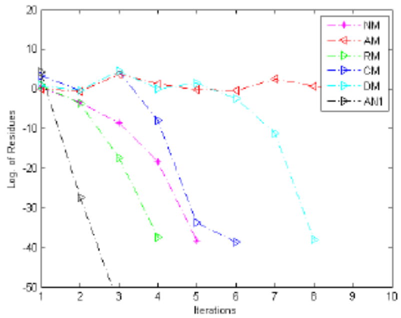

Fig. 7 ($f(x) = ln(x^2 + e^x) + x$)

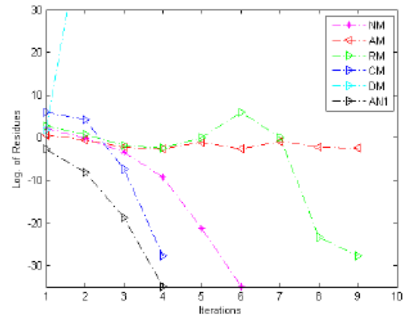

Fig. 8 ($f(x) = x + 2x^3 \sin(x) - 1$)

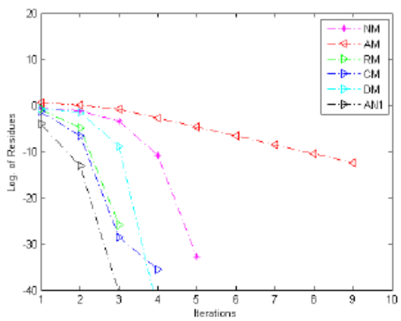

Fig. 9 ($f(x) = \sin(5x) - \sin(x)$)

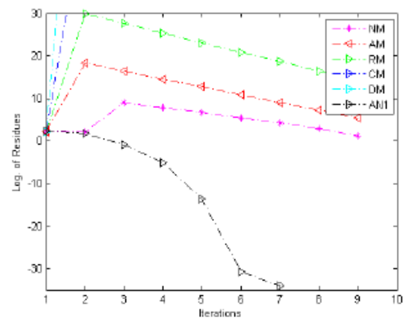

Fig. 10 ($f(x) = x^3 - x^2 - 8$)

5. Dynamical study

Polynomiography is an art and science of visualization of the zeroes of complex polynomials [12]. It has diverse applications in science, engineering, industries etc. Particularly, this art is being applied in textile industry for designing and printing.

In this section, we present some interesting polynomiographs, i.e. Fig. 11 to Fig. 18 of certain complex polynomials in the context of the newly constructed optimal fourth-order iterative method. It is obvious from these figures that we can easily identify the zeros of complex polynomials with remarkable basins of attraction.

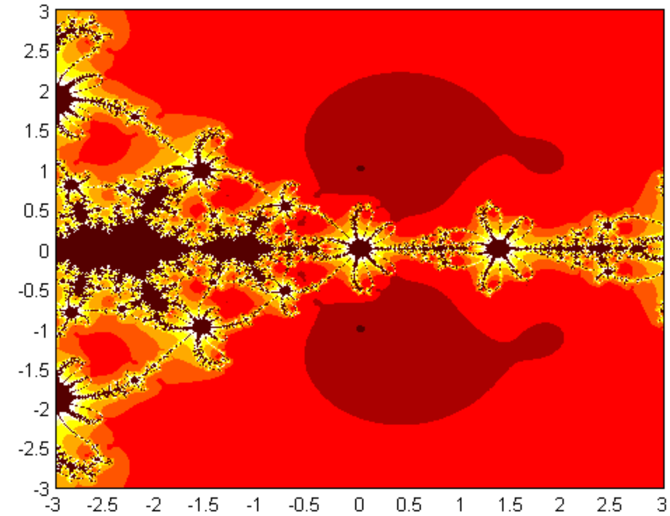

Fig. 11 Polynomiograph of $z^2 + 1$

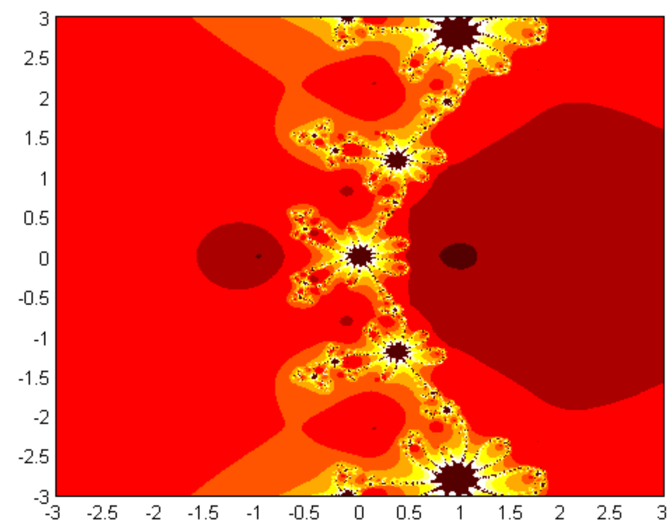

Fig. 12 Polynomiograph of $z^2 - 1$

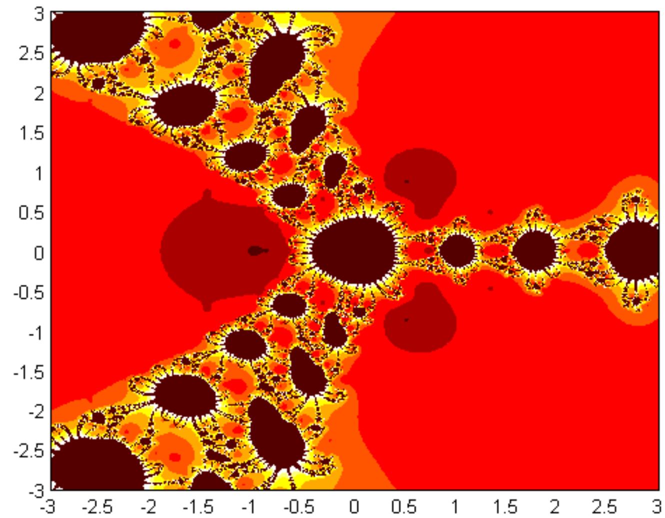

Fig. 13 Polynomiograph of $z^3 + 1$

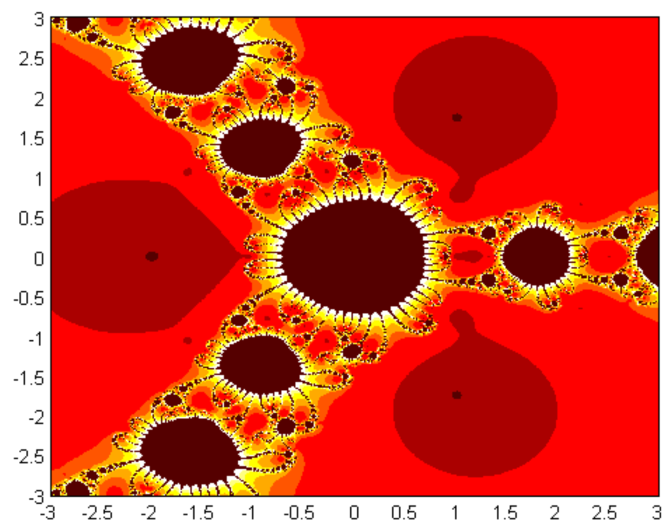

Fig. 14 Polynomiograph of $z^3 + 8$

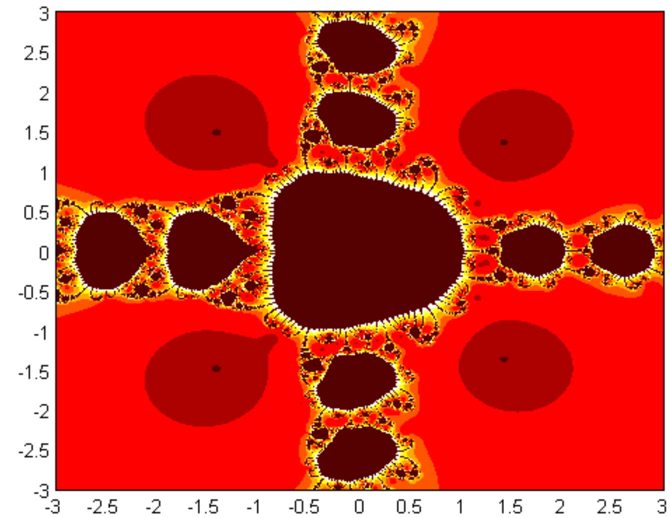

Fig. 15 Polynomiograph of $z^4 - z + 16$

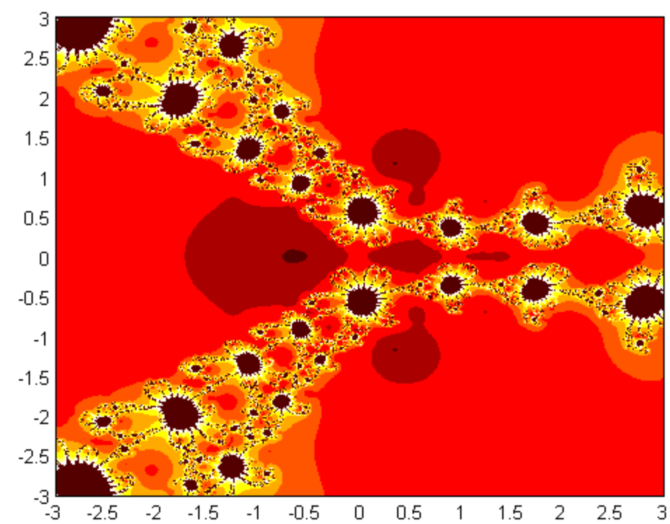

Fig. 16 Polynomiograph of $z^3 + z + 1$

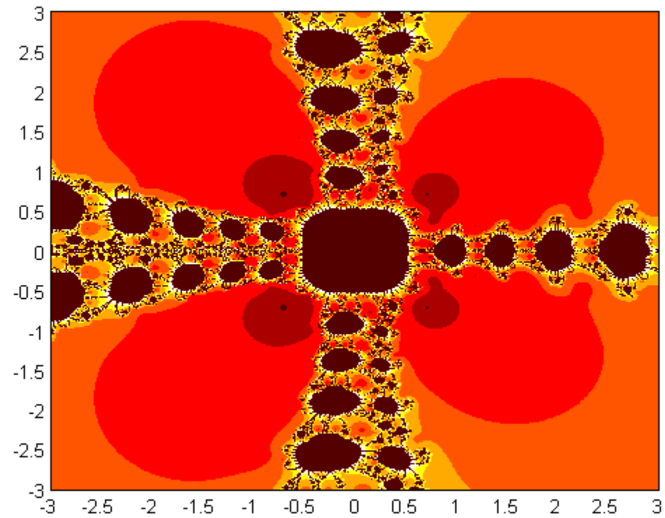

Fig. 17 Polynomiograph of $z^4 + 1$

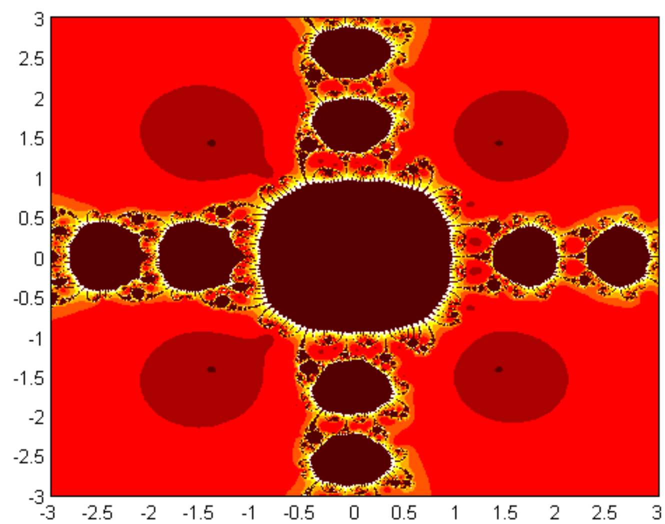

Fig. 18 Polynomiograph of $z^4 - 4$

Conclusions

A new three-step optimal fourth-order second derivative free iterative method based on the technique of linear combination has been introduced in this article. The efficiency of the newly developed method has been demonstrated both numerically and graphically by comparing the same with standard Newton's method and various other methods of the same class. In the context of the suggested method, the visualization process of the roots of certain complex polynomials has exhibited some interesting polynomiographs.

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A study on Riesz I-convergence in intuitionistic fuzzy normed spaces

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Abstract. The primary objective of this study is to introduce the concept of ideal convergent sequences as a domain of regular Riesz triangular matrix in the settings of intuitionistic fuzzy normed spaces(IFNS). Some properties of this notion with respect to the intuitionistic fuzzy norm are also presented in this study. We demonstrate the Riesz ideal Cauchy criterion in intuitionistic fuzzy normed spaces and later on, we show that in an arbitrary IFNS, a sequence is Riesz ideal convergent if and only if it satisfies Riesz ideal Cauchy criterion. We also present major counterexamples for the converse part of some results. Lastly, we define the notion of Riesz I^*–convergence in intuitionistic fuzzy normed spaces and establish the relationship with Riesz I–convergence in IFNS with a certain counterexample.

Keywords: intuitionistic fuzzy normed space, Riesz matrix, I–convergence, I–Cauchy.

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1. Introduction and preliminaries

The field of fuzzy topology plays a pivotal role to obtain significant applications in the theory of quantum particle physics [40]. Zadeh [47] gave a major breakthrough in this field by introducing the idea of fuzzy set, afterward many authors came forward to establish fuzzy analogs of classical theories. Atanassov ([2], [3], [4], [5]) defined intuitionistic fuzzy sets (IFS) and the characteristics of these IFS are given by Deschrijver and Kerre [13]. Inspired by these notions, Coker ([9], [10]) introduced intuitionistic fuzzy topological spaces. Saadati and Park [42, 43] studied these spaces and their generalization which helped them to obtain the concept of intuitionistic fuzzy normed space(IFNS). Mursaleen [36] presented the notion of statistical convergence with respect to the intuitionistic fuzzy norm and proved some fundamental results.

Definition 1.1 ([42])**.** *The five-tuple $(X, f_1, f_2, *, \diamond)$ is said to be an intuitionistic fuzzy normed space if X is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and f_1 & f_2 are the fuzzy sets on $X \times (0, \infty)$ satisfy the following conditions $\forall\ y, z \in X$ and $s, t\ > 0$:*

(a) $f_1(y,t)\ +\ f_2(y,t) \leq 1$;

(b) $f_1(y,t)\ > 0$;

(c) $f_1(y,t)\ =\ 1$ *iff* $y = 0$;

(d) $f_1(cy,t)\ =\ f_1(y, \frac{t}{|c|})$, $\forall c \neq 0,\ c \in F$;

(e) $f_1(y,t) * f_1(z,s) \leq f_1(y+z, t+s)$;

(f) $f_1(y,t) : (0,\infty) \to [0,1]$ *is continuous in* t;

(g) $\lim_{t\to\infty} f_1(y,t)\ =\ 1$ *and* $\lim_{t\to 0} f_1(y,t)\ =\ 0$;

(i) $f_2(y,t)\ > 0$;

(j) $f_2(y,t)\ =\ 0$ *iff* $y = 0$;

(l) $f_2(cy,t)\ =\ f_2(y, \frac{t}{|c|}$, $\forall c \neq 0,\ c \in F$;

(m) $f_2(y,t) \diamond f_2(z,s) \geq f_2(y+z, t+s)$;

(n) $f_2(y,t) : (0,\infty) \to [0,1]$ *is continuous in* t;

(o) $\lim_{t\to\infty} f_2(y,t)\ =\ 0$ *and* $\lim_{t\to 0} f_2(y,t)\ =\ 1$.

In this case, (f_1, f_2) is called intuitionistc fuzzy norm.

Remark 1.1. Hosseini et al. [22] and many others have defined intuitionistic fuzzy normed space by a more complete definition. This study can be studied as a more extended case in this novel setting. The current definition 1.1, however, simplifies the computational aspects.

Example 1.1. If $(X, ||\bullet||)$ forms a normed linear space, let for all $a, b \in [0,1]$, t-norm is defined as $a*b = ab$ and t-co-norm is defined as $a\diamond b = \min\{a+b, 1\}$, then for any $y \in X$ and $\forall\ t > 0$, consider

$$\phi(y,t) = \frac{t}{t+||y||} \quad \text{and} \quad \psi(y,t) = \frac{||y||}{t+||y||}$$

Then, $(X, \phi, \psi, *, \diamond)$ forms an intuitionistic fuzzy normed space.

Lemma 1.1 ([42]). *If $m_i \in (0,1)$, $i = 1$ to 7. $*$ and $\diamond$ are continuous $t-$norm and continuous $t-$conorm, respectively. Then:*

(1) *If $m_1 > m_2$, $\exists\ m_3, m_4 \in (0,1)$ s.t. $m_1 * m_3 \geq m_2$ and $m_1 \geq m_2 \diamond m_4$.*

(2) *If $m_5 \in (0,1)$, $\exists\ m_6, m_7 \in (0,1)$ s.t. $m_6 * m_6 \geq m_5$ and $m_5 \geq m_7 \diamond m_7$.*

Definition 1.2 ([42]). *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be convergent to ζ if for a given $\epsilon > 0$ and $t > 0$, $\exists\ k_0 \in \mathbb{N}$ such that*

$$f_1(y_k - \zeta, t) > 1 - \epsilon \quad \textit{and} \quad f_2(y_k - \zeta, t) < \epsilon, \ \forall\ k \geq k_0.$$

Definition 1.3 ([42]). *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Cauchy if for a given $\epsilon > 0$ and $t > 0$, $\exists\ k_0 \in \mathbb{N}$ such that*

$$f_1(y_k - y_j, t) > 1 - \epsilon \quad \textit{and} \quad f_2(y_k - y_j, t) < \epsilon, \ \forall\ j, k \geq k_0.$$

The notion of convergence of sequence has become a useful notion in the fundamental theory of functional analysis and plays a key role, especially in sequence space.

In 1951, Fast [14] gave the concept of statistical convergence which is a very important extension of usual convergence and is being used widely in different areas of science and technology. Due to the versatility of this concept, various forms of statistical convergence are introduced and these forms are further defined in different settings, e.g. [35, 36]. In the race of inventing the most extended form of statistical convergence, Kostyrko et al. [33] introduced the idea of I-convergence using the notion of ideal defined on $\mathbb{N}$. Nowadays it has become a more important form than many other forms of convergence (see [33]). If $I \subset P(X)$ of any set X with a) $\phi \in I$, b) $A \cup B \in I$ for all $A, B \in I$ and c) $\forall\ A \in I$ and $B \subset A$ then $B \in I$. Then I is called an ideal. If $I \neq 2^X$ then I is called non trivial ideal. If $\{\{x\} : x \in X\} \subset I$ then I is called admissible ideal. If $\mathcal{F} \subset P(X)$ of a set X then $\mathcal{F}$ is called filter if a) $\phi \notin \mathcal{F}$, b) $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$, c) $\forall\ A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$. Ŝalát et al. [44, 45] studied the characterization of ideal convergence and also defined the ideal convergence field. Later, many others (e.g. [15, 24, 25, 39]) further investigated the notion of I-convergence from the sequence space point of view and linked it with the summability theory. In addition, numerous researchers are working on the various extended versions of ideal convergence of sequence and further introduced them in several important spaces as Hazarika [18, 19] introduced the ideal convergence for sequence and double sequence in fuzzy normed space and gave the salient features of this notion in fuzzy norm-setting. Mursaleen and Mohiuddine [37, 38] further studied this concept for multiple sequences with respect to the intuitionistic fuzzy norm as well as probabilistic norm. Moreover, Hazarika [17, 20, 21] also extended several forms of ideal convergence in different spaces like random $2-$normed space, probabilistic normed space, intuitionistic fuzzy normed space. Debnath [11, 12] defined ideal convergence via lacunary and lacunary difference mean in intuitionistic fuzzy normed space and established key results with respect to IFN. Recently, Khan et al. [26, 27] also studied the notion of ideal convergence as a domain of the Nörlund matrix and generalized difference matrix, respectively in intuitionistic fuzzy normed space. Khan et al. also defined their respective intuitionistic fuzzy ideal convergent sequence spaces and proved their topological properties. Other important notions with respect to the intuitionistic fuzzy norm, one can refer to [28, 29, 30, 31, 32].

Proposition 1.1 ([33]). *Class $\mathcal{F}(I) = \{A \subset X : A = X \backslash B,$ for some $B \in I\}$ is a filter on X, where $I \subset P(\mathbb{N})$ is a non trivial ideal.*

$\mathcal{F}(I)$ is known as the filter associated with the ideal I.

Definition 1.4 ([33]). *Let $I \subset P(\mathbb{N})$ is a non-trivial ideal, a sequence $(y_k) \in \omega$ is called to be $I-$convergent to $\zeta \in \mathbb{R}$ if $\forall\ \epsilon > 0$,*

$$\{k \in \mathbb{N} :\mid y_k - \zeta \mid \geq \epsilon\} \in I.$$

We denote it as $I - \lim(y_k) = \zeta$.

Definition 1.5 ([33]). *Let $I \subset P(\mathbb{N})$ is a non-trivial ideal, a sequence $(y_k) \in \omega$ is called to be $I-$Cauchy if $\forall\ \epsilon > 0$, there exists a $K = K(\epsilon)$ such that*

$$\{k \in \mathbb{N} :\mid y_k - y_K \mid \geq \epsilon\} \in I.$$

Definition 1.6 ([33]). *Let $I \subset P(\mathbb{N})$ is a non-trivial ideal, I^*-convergence of a sequence $(y_k) \in \omega$ to number $\zeta \in \mathbb{R}$(i.e. $I^* - \lim y = \zeta$) is defined as if there exists a set $M \in I$, s.t. for $A = \mathbb{N} \backslash M = \{k_i \in \mathbb{N} : k_i < k_{i+1},$ for all $i \in \mathbb{N}\}$ we have, $\lim_{k\to\infty} y_{i_k} = \zeta$.*

Definition 1.7 ([33]). *An admissible ideal $I \subset P(\mathbb{N})$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to I, there exists a countable family $\{B_1, B_2, ...\}$ in I such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i \in I$; where Δ is the symmetric difference.*

As a bounded linear operator on the space of all p-summable sequence l_p, several authors used the riesz matrix $\mathcal{R}_n^b$ in the different disciplines of sequence space. Recall in [1, 8, 7, 16, 23, 34, 46] for a sequence of positive numbers (b_k), the entries of infinite riesz matrix $\mathcal{R}_j^b = (r_{jk}^b)$ is defined as

$$r_{jk}^b = \begin{cases} \dfrac{b_k}{\mathcal{B}_j}, & 0 \leq k \leq j \\ 0, & k \geq j. \end{cases} \tag{1}$$

where $\mathcal{B}_j = \sum_{k=0}^{j} b_k$. By (1) clearly, Riesz matrix is lower triangular and is regular if $\mathcal{B}_j \to \infty$ as $j \to \infty$, (see [1, 6, 23, 41]). Recently by using the notion of $I-$convergence and domain of Riesz matrix $\mathcal{R}_j^b$, Khan et al. [24] introduced Riesz $I-$convergent sequence space

$$c^I(\mathcal{R}_j^b) := \{y = (y_k) \in \omega : \{j \in \mathbb{N} : |\mathcal{R}_j^b(y) - L| \geq \varepsilon \text{ for some } L \in \mathbb{R}\} \in I\}, \tag{2}$$

$$c_0^I(\mathcal{R}_j^b) := \left\{y = (y_k) \in \omega : \{j \in \mathbb{N} : |\mathcal{R}_j^b(y)| \geq \varepsilon\} \in I\right\}, \tag{3}$$

where

$$\mathcal{R}_j^b(y) := \frac{1}{\mathcal{B}_j} \sum_{k=0}^{j} b_k y_k \quad \text{ for all } j \in \mathbb{N}. \tag{4}$$

Clearly, Riesz I–convergent sequence is a more generalized form of ideal convergent sequence and each I–convergent sequence is Riesz I–convergent sequence but converse is not true.

Example 1.2. Let I is an ideal defined on the set of natural numbers such that it contains the subsets of natural numbers whose natural density is zero. If we take sequence (b_k) as $b_k = k$ for all k and sequence (y_k) as

$$\mathcal{R}_j^b(y) = \begin{cases} 1, & \text{if } j = m^2,\ (m \in \mathbb{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is obvious that sequence (y_k) is Riesz I–convergent to 0 but sequence (y_k) is not I–convergent.

Since Riesz ideal convergence is a novel and a more extended variant of ideal convergence and IFNS is a unified and generalized space of various important spaces so these facts motivate us to define Riesz ideal convergence in intuitionistic fuzzy norm-setting.

2. Main results

In this particular section, we define Riesz I-convergence in intuitionistic fuzzy normed space and try to give some theorems about it. Throughout the paper, we assume that sequence $y = (y_k)$ and $\mathcal{R}_j^b(y)$ are related as (4) and I is an admissible ideal of subset of $\mathbb{N}$.

Definition 2.1. *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Riesz I-convergent to ζ if for a given $\epsilon > 0$ and $s > 0$, the following set*

$$\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}_j^b(y) - \zeta, s) \geq \epsilon\} \in I.$$

In this case, we say Riesz $I_{(f_1,f_2)}-$ limit of sequence (y_k) is ζ and denote it as $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta$.

Proposition 2.1. *Let $(X, f_1, f_2, *, \diamond)$ is an IFNS and $y = (y_k)$ is a sequence in X. Then for every $\epsilon > 0$ and $s > 0$, these following are equivalent:*

(a) $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta$.

(b) $\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y)-\zeta, s) \leq 1-\epsilon\} \in I$ *and* $\{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y)-\zeta, s) \geq \epsilon\} \in I$.

(c) $\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) > 1 - \epsilon$ *and* $f_2(\mathcal{R}_j^b(y) - \zeta, s) < \epsilon\} \in \mathcal{F}(I)$.

(d) $\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) > 1 - \epsilon\} \in \mathcal{F}(I)$ *and* $\{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta, s) < \epsilon\} \in \mathcal{F}(I)$.

(e) $I - \lim_{j\to\infty} f_1(\mathcal{R}_j^b(y) - \zeta, s) = 1$ *and* $I - \lim_{j\to\infty} f_2(\mathcal{R}_j^b(y) - \zeta, s) = 0$.

Theorem 2.1. *Let $y = (y_k)$ be a sequence in IFNS $(X, f_1, f_2, *, \diamond)$. If sequence (y_k) is Riesz $I-$convergent in X, then Riesz $I_{(f_1,f_2)}-$ limit of (y_k) is unique.*

Proof of Theorem 2.1. Let on contrary that ζ_1 and ζ_2 are two different elements such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta_1$ and $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta_2$. For a given $\epsilon > 0$, choose $r > 0$ such that $(1 - r) * (1 - r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. For $s > 0$, we define

$$\begin{aligned}
A_1 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta_1, s) \leq 1 - r\},\\
A_2 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta_1, s) \geq r\},\\
A_3 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta_2, s) \leq 1 - r\},\\
A_4 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta_2, s) \geq r\}
\end{aligned}$$

and $A = (A_1 \cup A_3) \cap (A_2 \cup A_4)$.

Sets A_1, A_2, A_3, A_4 and A must belong to I as (y_k) has two different Riesz $I-$limit with respect to intuitionistic fuzzy norm (f_1, f_2) i.e. ζ_1 and ζ_2. Hence, $A^c \in \mathcal{F}(I)$ then A^c is non empty. Let us say some $j_0 \in A^c$ then either $j_0 \in {A_1}^c \cap {A_3}^c$ or $j_0 \in {A_2}^c \cap {A_4}^c$.

If $j_0 \in {A_1}^c \cap {A_3}^c$ which implies that

$$f_1\left(\mathcal{R}_{j_0}^b(y) - \zeta_1, \frac{s}{2}\right) > 1 - r \text{ and } f_1\left(\mathcal{R}_{j_0}^b(y) - \zeta_2, \frac{s}{2}\right) > 1 - r.$$

Hence,

$$\begin{aligned} f_1(\zeta_1-\zeta_2,s) &\geq f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right)*f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_2,\frac{s}{2}\right)\\ &> (1-r)*(1-r)>1-\epsilon. \end{aligned}$$

As $\epsilon>0$ was arbitrary hence $f_1(\zeta_1-\zeta_2,s)=1$ for all $s>0$. So, we have $\zeta_1=\zeta_2$, which is a contradiction.

If $j_0\in {A_2}^c\cap {A_4}^c$ which implies that

$$f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right)<r \text{ and } f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_2,\frac{s}{2}\right)<r.$$

Hence,

$$\begin{aligned} f_2(\zeta_1-\zeta_2,s) &\leq f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right)\diamond f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_2,\frac{s}{2}\right)\\ &< r\diamond r<\epsilon. \end{aligned}$$

As $\epsilon>0$ was arbitrary hence $f_2(\zeta_1-\zeta_2,s)=0$ for all $s>0$. So, we have $\zeta_1=\zeta_2$, which is a contradiction. Hence, $y=(y_k)$ has unique Riesz $I_{(f_1,f_2)}-$limit.

Theorem 2.2. *Let $y=(y_k)$ be any sequence in IFNS $(X,f_1,f_2,*,\diamond)$ such that ordinary Riesz limit w.r.t. IFN (f_1,f_2) of (y_k) is ζ, then $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim y=\zeta$.*

Proof of Theorem 2.2. As we are given that $\mathcal{R}^b_{(f_1,f_2)}-\lim y=\zeta$, hence for any $\epsilon>0$, and $s>0$, we can find a natural number $j_0\in\mathbb{N}$ in such a way that

$$f_1(\mathcal{R}^b_j(y)-\zeta,s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(y)-\zeta,s)<\epsilon,$$

for all $j\geq j_0$.

Now, let

$$K = \{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y)-\zeta,s)\leq 1-\epsilon \text{ or } f_2(\mathcal{R}^b_j(y)-\zeta,s)\geq\epsilon\}.$$

As $K\subset\{1,2,....,j_0-1\}$ and I is an admissible ideal so $K\in I$. Hence, $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim y=\zeta$.

Remark 2.1. *Converse of Theorem 2.2 may not be hold in general.*

Example 2.1. In the Example 1.1, let X=$\mathbb{R}$ with usual norm, then $(\mathbb{R},f_1,f_2,*,\diamond)$ forms an IFNS.

Suppose $I=\{A\subset\mathbb{N}:\delta(A)=0\}$, where $\delta(A)$ is the natural density of the set A in $\mathbb{N}$ which is defined as $\delta(A)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\chi_A(i)$, where χ_A is the characteristic function on A, hence I is a non-trivial admissible ideal.

Now, for a positive sequence of real numbers $b=(b_k)$, let us define a sequnece $y=(y_k)$ such that

$$\mathcal{R}^b_j(y) = \begin{cases} 1, & \text{if } j=m^2,\ (m\in\mathbb{N})\\ 0, & \text{otherwise.}\end{cases}$$

Now, for any $\epsilon > 0$ and $s > 0$, we define

$$\begin{aligned} K(\epsilon, s) &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y), s) \leq 1-\epsilon \text{ or } f_2(\mathcal{R}_j^b(y), s) \geq \epsilon\} \\ \text{then,} & \\ K(\epsilon, s) &= \left\{j \in \mathbb{N} : \frac{s}{s + ||\mathcal{R}_j^b(y)||} \leq 1-\epsilon \text{ or } \frac{||\mathcal{R}_j^b(y)||}{s + ||\mathcal{R}_j^b(y)||} \geq \epsilon\right\} \\ &= \left\{j \in \mathbb{N} : |\mathcal{R}_j^b(y)| \geq \frac{\epsilon s}{1-\epsilon} > 0\right\} \\ &\subseteq \{j \in \mathbb{N} : j = m^2, (m \in \mathbb{N})\}. \end{aligned}$$

Hence, $\delta(K(\epsilon, s)) = 0$, which implies that $K(\epsilon, s) \in I$. Hence, $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = 0$. On the other hand, $\mathcal{R}_j^b(y)$ is not convergent with respect to the intuitionistic fuzzy norm (f_1, f_2) as $\mathcal{R}_j^b(y)$ is not convergent in $(\mathbb{R}, ||.||)$.

Theorem 2.3. *Let $y = (y_k)$ and $z = (z_k)$ be any two sequences in IFNS $(X, f_1, f_2, *, \diamond)$ such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta_1$ and $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(z) = \zeta_2$ then:*

(1) $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y+z) = \zeta_1 + \zeta_2$.

(2) *For any real number α,* $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(\alpha y) = \alpha\zeta_1$.

Proof of Theorem 2.3. (1) For any $\epsilon > 0$, we may find $r > 0$ such that $(1-r) * (1-r) > 1-\epsilon$ and $r \diamond r < \epsilon$.

For $s > 0$, we define

$$\begin{aligned} A_1 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta_1, s) \leq 1-r\}, \\ A_2 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta_1, s) \geq r\}, \\ A_3 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(z) - \zeta_2, s) \leq 1-r\}, \\ A_4 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(z) - \zeta_2, s) \geq r\} \\ \text{and } A &= (A_1 \cup A_3) \cap (A_2 \cup A_4). \end{aligned}$$

Sets A_1, A_2, A_3, A_4 and A must belong to I as $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta_1$ and $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(z) = \zeta_2$. Hence, $A^c \in \mathcal{F}(I)$ then A^c is non empty. Now, we show that

$$A^c \subset \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y+z) - (\zeta_1+\zeta_2), s) > 1-\epsilon \text{ and } f_2(\mathcal{R}_j^b(y+z) - (\zeta_1+\zeta_2), s) < \epsilon\}.$$

To show this we let $j_0 \in A^c$. So, we have

$$\begin{aligned} f_1\left(\mathcal{R}_{j_0}^b(y) - \zeta_1, \frac{s}{2}\right) &> 1-r, \ f_1\left(\mathcal{R}_{j_0}^b(z) - \zeta_2, \frac{s}{2}\right) > 1-r, \\ f_2\left(\mathcal{R}_{j_0}^b(y) - \zeta_1, \frac{s}{2}\right) &< r \text{ and } f_2\left(\mathcal{R}_{j_0}^b(z) - \zeta_2, \frac{s}{2}\right) < r. \end{aligned}$$

Hence, we have

$$\begin{aligned} f_1(\mathcal{R}^b_{j_0}(y+z)-(\zeta_1+\zeta_2),s) &\geq f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right) * f_1\left(\mathcal{R}^b_{j_0}(z)-\zeta_2,\frac{s}{2}\right)\\ &> (1-r)*(1-r)\\ &> 1-\epsilon \end{aligned}$$

and

$$\begin{aligned} f_2(\mathcal{R}^b_{j_0}(y+z)-(\zeta_1+\zeta_2),s) &\leq f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right) \diamond f_2\left(\mathcal{R}^b_{j_0}(z)-\zeta_2,\frac{s}{2}\right)\\ &< r\diamond r\\ &< \epsilon \end{aligned}$$

which implies that

$$A^c\subset\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y+z)-(\zeta_1+\zeta_2),s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(y+z)-(\zeta_1+\zeta_2),s)<\epsilon\}.$$

As $A^c\in\mathcal{F}(I)$, hence the later set belongs to $\mathcal{F}(I)$, which implies that $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(y+z)=\zeta_1+\zeta_2$.

(2) If $\alpha=0$ then for any $\epsilon>0$ and $s>0$,

$$\begin{aligned} f_1(\mathcal{R}^b_j(0y)-(0\zeta_1),s)=f_1(0,s)=1>1-\epsilon\\ \text{and } f_2(\mathcal{R}^b_j(0y)-(0\zeta_1),s)=f_2(0,s)=0<\epsilon \end{aligned}$$

which implies that $\mathcal{R}^b_{(f_1,f_2)}-\lim(0y)=\theta$. Hence, by Theorem 2.3, $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(0y)=\theta$.

If $\alpha(\neq 0)\in\mathbb{R}$. To prove the result, we will show that for any $\epsilon>0$ and $s>0$, the set

$$\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)<\epsilon\}\in\mathcal{F}(I),$$

for any $\alpha(\neq 0)\in\mathbb{R}$.

As we have given that $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(y)=\zeta_1$ so we have for any $\epsilon>0$ and $s>0$, the set

$$K=\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y)-\zeta_1,s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(y)-\zeta_1,s)<\epsilon\}\in\mathcal{F}(I).$$

Choose any $j_0\in K$, hence we have $f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)>1-\epsilon$ and $f_2(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)<\epsilon$. Now,

$$\begin{aligned} f_1(\mathcal{R}^b_{j_0}(\alpha y)-(\alpha\zeta_1),s) &= f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{|\alpha|}\right)\\ &\geq f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)*f_1\left(0,\frac{s}{|\alpha|}-s\right)\\ &= f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)*1\\ &= f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)>1-\epsilon \end{aligned}$$

and

$$\begin{aligned} f_2(\mathcal{R}^b_{j_0}(\alpha y)-(\alpha\zeta_1),s) &= f_2\bigg(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{|\alpha|}\bigg)\\ &\leq f_2\bigg(\mathcal{R}^b_{j_0}(y)-\zeta_1,s\bigg)\diamond f_2\bigg(0,\frac{s}{|\alpha|}-s\bigg)\\ &= f_2\bigg(\mathcal{R}^b_{j_0}(y)-\zeta_1,s\bigg)\diamond 0\\ &= f_2\bigg(\mathcal{R}^b_{j_0}(y)-\zeta_1,s\bigg)<\epsilon \end{aligned}$$

which implies that

$j_0\in\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)>1-\epsilon$ and $f_2(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)<\epsilon\}$.

Hence,

$K\subset\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)>1-\epsilon$ and $f_2(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)<\epsilon\}$.

Since $K\in\mathcal{F}(I)$, hence the set

$\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)>1-\epsilon$ and $f_2(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)<\epsilon\}\in\mathcal{F}(I)$

which implies that $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(\alpha y)=\alpha\zeta_1$.

Theorem 2.4. *Let $y=(y_k)$ be any sequence in IFNS $(X,f_1,f_2,*,\diamond)$ and let I be a non-trivial ideal in $\mathbb{N}$. If $z=(z_k)$ is Riesz I-convergent sequence in X w.r.t. IFN (f_1,f_2) such that the set $\{j\in\mathbb{N}:\mathcal{R}^b_j(y)\neq\mathcal{R}^b_j(z)\}\in I$, then the sequence $y=(y_k)$ is Riesz I-convergent sequence in X w.r.t. IFN (f_1,f_2).*

Proof of Theorem 2.4. Let

$$\{j\in\mathbb{N}:\mathcal{R}^b_j(y)\neq\mathcal{R}^b_j(z)\}\in I$$

and let $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(z)=\zeta$. Then, for any given $\epsilon>0$ and $s>0$, we have

$$A=\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(z)-\zeta,s)\leq 1-\epsilon \text{ or } f_2(\mathcal{R}^b_j(z)-\zeta,s)\geq\epsilon\}\in I.$$

Thus, for any $\epsilon>0$,

$$\begin{aligned} &\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y)-\zeta,s)\leq 1-\epsilon \text{ or } f_2(\mathcal{R}^b_j(y)-\zeta,s)\geq\epsilon\}\\ &\qquad\subseteq\{j\in\mathbb{N}:\mathcal{R}^b_j(y)\neq\mathcal{R}^b_j(z)\}\cup A \end{aligned}$$

which implies that

$$\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y)-\zeta,s)\leq 1-\epsilon \text{ or } f_2(\mathcal{R}^b_j(y)-\zeta,s)\geq\epsilon\}\in I.$$

Hence, $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(y)=\zeta$ that is $y=(y_k)$ is Riesz I-convergent sequence in X w.r.t. IFN (f_1,f_2).

Now, we define Riesz I-Cauchy sequence in IFNS and establish results about relathionship with Riesz I-convergence in IFNS.

Definition 2.2. *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Riesz Cauchy w.r.t. IFN (f_1, f_2) if for all $\epsilon > 0$ and $s > 0$, $\exists K \in \mathbb{N}$ such that*

$$f_1(\mathcal{R}_j^b(y) - \mathcal{R}_K^b(y), s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}_j^b(y) - \mathcal{R}_K^b(y), s) < \epsilon \text{ for all } j \geq K.$$

Definition 2.3. *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Riesz I-Cauchy w.r.t. IFN (f_1, f_2) if for a given $\epsilon > 0$ and $s > 0$, $\exists K \in \mathbb{N}$ such that the following set*

$$\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \mathcal{R}_K^b(y), s) \geq 1 - \epsilon \text{ or } f_2(\mathcal{R}_j^b(y) - \mathcal{R}_K^b(y), s) \leq \epsilon\} \in I.$$

Theorem 2.5. *Let $y = (y_k)$ be any sequence in IFNS $(X, f_1, f_2, *, \diamond)$ such that (y_k) is Riesz $I-$convergent with respect to intuitiontic fuzzy norm (f_1, f_2) if and only if (y_k) is Riesz $I-$Cauchy with respect to intuitiontic fuzzy norm (f_1, f_2).*

Proof of Theorem 2.5. In X, let $y = (y_k)$ is such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$, then for any given $\epsilon > 0$, we can choose $0 < r < 1$ in such a way that $(1-r) * (1-r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. Then, for any $s > 0$, we define

$$\begin{aligned} A_1 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) \leq 1 - r\},\ A_2 = \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta, s) \geq r\} \\ \text{and } A &= (A_1 \cup A_2). \end{aligned}$$

Sets A_1, A_2 and A must belong to I as $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$. Hence, $A^c \in \mathcal{F}(I)$ then A^c is non empty. Let if $m \in A^c$, choose a fixed $l \in A^c$. So, we have,

$$\begin{aligned} f_1(\mathcal{R}_m^b(y) - \zeta, \frac{s}{2}) &> 1 - r,\ f_1(\mathcal{R}_l^b(y) - \zeta, \frac{s}{2}) > 1 - r, \\ f_2(\mathcal{R}_m^b(y) - \zeta, \frac{s}{2}) &< r \text{ and } f_2(\mathcal{R}_l^b(y) - \zeta, \frac{s}{2}) < r. \end{aligned}$$

Hence, we have

$$\begin{aligned} f_1(\mathcal{R}_m^b(y) - \mathcal{R}_l^b(y), s) &\geq f_1\left(\mathcal{R}_m^b(y) - \zeta, \frac{s}{2}\right) * f_1\left(\mathcal{R}_l^b(y) - \zeta, \frac{s}{2}\right) \\ &> (1-r) * (1-r) > 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} f_2(\mathcal{R}_m^b - \mathcal{R}_l^b(y), s) &\leq f_2\left(\mathcal{R}_m^b - \zeta, \frac{s}{2}\right) \diamond f_2\left(\mathcal{R}_l^b(y) - \zeta, \frac{s}{2}\right) \\ &< r \diamond r < \epsilon \end{aligned}$$

which implies that

$$m \in \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \mathcal{R}_l^b(y), s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}_j^b(y) - \mathcal{R}_l^b(y), s) < \epsilon\}.$$

Hence,

$$A^c \subset \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \mathcal{R}^b_l(y), s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}^b_j(y) - \mathcal{R}^b_l(y), s) < \epsilon\}.$$

As $A^c \in \mathcal{F}(I)$, hence the later set belongs to $\mathcal{F}(I)$, which implies that sequence $y = (y_k)$ is Riesz I-Cauchy sequence with respect to intuitionistic fuzzy norm (f_1, f_2).

Conversely, let on contrary, $y = (y_k)$ is a sequence in X which is Riesz I-Cauchy but not Riesz I-convergent with respect to IF norm (f_1, f_2) then

$$R = \left\{ j \in \mathbb{N} : f_1\left(\mathcal{R}^b_j(y) - \zeta, \frac{s}{2}\right) > 1 - \epsilon \text{ and } f_2\left(\mathcal{R}^b_j(y) - \zeta, \frac{s}{2}\right) < \epsilon \right\} \in I$$

which implies that, $R^c \in \mathcal{F}(I)$.

Since $y = (y_k)$ is Riesz I-Cauchy with respect to IF norm (f_1, f_2), then there exists $M = M(y, \epsilon, s)$ s.t. the set

$$S = \left\{ j \in \mathbb{N} : f_1\left(\mathcal{R}^b_j(y) - \mathcal{R}^b_M(y), \frac{s}{2}\right) \leq 1 - \epsilon \textit{ or } f_2\left(\mathcal{R}^b_j(y) - \mathcal{R}^b_M(y), \frac{s}{2}\right) \geq \epsilon \right\} \in I.$$

As

$$\begin{aligned} f_1(\mathcal{R}^b_j(y) - \mathcal{R}^b_M(y), s) &\geq 2f_1\left(\mathcal{R}^b_j(y) - \zeta, \frac{s}{2}\right) > 1 - \epsilon \text{ and} \\ f_2(\mathcal{R}^b_j(y) - \mathcal{R}^b_M(y), s) &\leq 2f_2\left(\mathcal{R}^b_j(y) - \zeta, \frac{s}{2}\right) < \epsilon, \end{aligned}$$

if $f_1(\mathcal{R}^b_j(y) - \zeta, \frac{s}{2}) > \frac{(1-\epsilon)}{2}$ and $f_2(\mathcal{R}^b_j(y) - \zeta, \frac{s}{2}) > \frac{\epsilon}{2}$, respectively.

Hence, we have $S^c \in I$. Equivalently, $S \in \mathcal{F}(I)$, which is a contradiction, as $y = (y_k)$ is Riesz I-Cauchy with respect to IF norm (f_1, f_2).

The proof of following Theorems are straight-forward.

Theorem 2.6. *Let $y = (y_k)$ be a sequence in an IFNS $(X, f_1, f_2, *, \diamond)$ is Riesz Cauchy w.r.t. IFN (f_1, f_2) and $\mathcal{R}^b_j(y)$ clusters to ζ in X then (y_k) is Riesz I-convergent to ζ w.r.t. same IFN.*

Theorem 2.7. *Let $y = (y_k)$ be a sequence in an IFNS $(X, f_1, f_2, *, \diamond)$ is Riesz Cauchy w.r.t. IFN (f_1, f_2), then it is Riesz I-Cauchy w.r.t. IFN (f_1, f_2).*

Theorem 2.8. *Let $y = (y_k)$ be a sequence in an IFNS $(X, f_1, f_2, *, \diamond)$ is Riesz Cauchy w.r.t. IFN (f_1, f_2), then $\exists$ a subsequence (y_{k_n}) of (y_k) such that (y_{k_n}) is a Riesz Cauchy sequence w.r.t. IFN (f_1, f_2).*

Now, we define Riesz I^*-convergence in intuitionistic fuzzzy normed space and and prove some theorems about its relathionship with Riesz I-convergence in IFNS.

Definition 2.4. *A sequence $y = (y_k)$ in IFNS $(X, f_1, f_2, *, \diamond)$ is said to be Riesz I^*-convergent to $\zeta \in X$ with respect to the intuitionistic fuzzy norm (f_1, f_2) if there exists a set $A = \{j_i \in \mathbb{N} : j_i < j_{i+1},$ for all $i \in \mathbb{N}\}$ such that $A \in \mathcal{F}(I)$ and $\lim \mathcal{R}^b_{j_i}(y) = \zeta$ w.r.t. IFN (f_1, f_2). In this case, we say $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$*

Theorem 2.9. *Let I be an admissible ideal and a sequence $y = (y_k)$ in IFNS $(X, f_1, f_2, *, \diamond)$ is such that $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$ then $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$.*

Proof of Theorem 2.6. Since $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$ so there exists a subset $A = \{j_i \in \mathbb{N} : j_i < j_{i+1},$ for all $i \in \mathbb{N}\}$ such that $A \in \mathcal{F}(I)$ and $\lim \mathcal{R}^b_{j_i}(y) = \zeta$ w.r.t. IFN (f_1, f_2). Hence, for each $\epsilon > 0$ and $s > 0$, there exists $m \in \mathbb{N}$ in such a way that

$$f_1(\mathcal{R}^b_{j_i}(y) - \zeta, s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}^b_{j_i}(y) - \zeta, t) < \epsilon \text{ for all } i \geq m.$$

As the set

$$\{j_i \in A : f_1(\mathcal{R}^b_{j_i}(y), s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_{j_i}(y) - \zeta, s) \geq \epsilon\}$$

is contained in $\{j_1, j_2, ..., j_{m-1}\}$. Hence,

$$\{j_i \in A : f_1(\mathcal{R}^b_{j_i}(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_{j_i}(y) - \zeta, s) \geq \epsilon\} \in I,$$

as I is an admissible ideal. Also $A \in \mathcal{F}(I)$, then by the definition of $\mathcal{F}(I)$ there exists a set $B \in I$ such that $A = \mathbb{N} \backslash B$. So,

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \epsilon\} \subset B \cup \{j_1, j_2, ..., j_{m-1}\}$$

Hence,

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, t) \leq 1 - \epsilon \text{ or } f_1(\mathcal{R}^b_j(y) - \zeta, t) \geq \epsilon\} \in I$$

which implies that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$.

Remark 2.2. Converse of Theorem 2.9 may not be hold in general.

Example 2.2. In the Example 1.1, let X=$\mathbb{R}$ with usual norm, then $(\mathbb{R}, f_1, f_2, *, \diamond)$ forms an IFNS.

Now, we take a decomposition of $\mathbb{N}$ as $\mathbb{N} = \cup A_i$, where every A_i is an infinite set and $A_i \cap A_l = \emptyset$, for $i \neq l$. Suppose $I = \{N \subset \mathbb{N} : N \subset \cup_{i=1}^r A_i$, for some finite natural number $r\}$ then I is a non-trivial admissible ideal.

Now, for a positive sequence of real numbers $b = (b_k)$, take a sequence (y_k) in such a way that

$$\mathcal{R}^b_j(y) = \frac{1}{i}, \text{ if } j \in A_i$$

Hence, we have for $s > 0$

$$\begin{aligned} f_1(\mathcal{R}^b_j(y), s) &= \frac{s}{s + ||\mathcal{R}^b_j(y)||} \to 1 \text{ as } j \to \infty \\ \text{and } f_2(\mathcal{R}^b_j(y), s) &= \frac{||\mathcal{R}^b_j(y)||}{s + ||\mathcal{R}^b_j(y)||} \to 0, \text{ as } j \to \infty. \end{aligned}$$

Hence, by Proposition 2.1, $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = 0$.

Let on contrary that $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = 0$ then $\exists$ a set $A = \{j_i \in \mathbb{N} : j_i < j_{i+1}, \text{ for all } i \in \mathbb{N}\}$ such that $A \in \mathcal{F}(I)$ and $\lim \mathcal{R}^b_{j_i}(y) = \zeta$ w.r.t. IFN (f_1, f_2). As $A \in \mathcal{F}(I)$, $\exists$ $K = \mathbb{N}\backslash A$ and $K \in I$. Then, there exists a natural numer r such that $K \subset \cup_{i=1}^r N_i$. Then $N_{r+1} \subset A$. Hence,

$$\mathcal{R}^b_{j_i}(y) = \frac{1}{r+1}, \text{ for infinitely many values of } j_i \text{ in } A,$$

which is a contradiction. Hence, $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) \neq 0$.

Theorem 2.10. *Let $y = (y_k)$ be a sequence in IFNS $(X, f_1, f_2, *, \diamond)$ such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$ and ideal I satisfies condition (AP). Then $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.*

Proof of Theorem 2.7. As $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$. Then $\forall \epsilon > 0$ and $s > 0$, we have

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \epsilon\} \in I,$$

For $r \in \mathbb{N}$ and $s > 0$, we define

$$A_r = \{j \in \mathbb{N} : 1 - \frac{1}{r} \leq f_1(\mathcal{R}^b_j(y) - \zeta, s) < 1 - \frac{1}{r+1} \text{ or } \frac{1}{r+1} < f_2(\mathcal{R}^b_j(y) - \zeta, s) \leq \frac{1}{r}\}.$$

Now, it is clear that $\{A_1, A_2, ...\}$ is a countable family of mutually disjoint sets belonging to I and therefore by the condition (AP) there is a countable family of sets $\{B_1, B_2, ...\}$ in I such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^\infty B_i$. Since $B \in I$ so by definition of associate filter $\mathcal{F}(I)$ there is set $K \in \mathcal{F}(I)$ such that $K = \mathbb{N} - B$. Now, to prove the result it is sufficient to prove that the subsequence $(y_k)_k \in K$ is ordinary Riesz convergent to with respect to the intuitionistic fuzzy norm (ϕ, ψ). For this, let $\eta > 0$ and $s > 0$. Choose a positive integer q such that $\frac{1}{q} < \eta$. Hence, we have

$$\begin{aligned} &\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \eta \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \eta\} \\ &\subset \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \frac{1}{q} \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \frac{1}{q}\} \\ &\subset \cup_{i=1}^{q+1} A_i. \end{aligned}$$

Since $A_i \Delta B_i$ be a finite set for each $i = 1, 2, ..q + 1$, $\exists$ j_0 such that

$$(\cup_{i=1}^{q+1} B_i) \cap \{j \in \mathbb{N} : j \geq j_0\} = (\cup_{i=1}^{q+1} A_i) \cap \{j \in \mathbb{N} : j \geq j_0\}.$$

If $j \geq j_0$ and $j \in K$, then $j \notin B$. This implies that $j \in \cup_{i=1}^{q+1} B_i$ and therefore $j \notin \cup_{i=1}^{q+1} A_i$. Hence, for every $j \geq j_0$ and $j \in K$, we have

$$f_1(\mathcal{R}_j^b(y) - \zeta, s) > 1 - \eta \text{ and } f_2(\mathcal{R}_j^b(y) - \zeta, s) < \eta$$

As this holds for every $\eta > 0$ and $s > 0$, so $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.

Theorem 2.11. *For any sequence $y = (y_k)$ in IFNS $(X, f_1, f_2, *, \diamond)$, the following statements are equivalent.*

(1) $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.

(2) $\exists$ *sequences $p = (p_k)$ and $q = (q_k)$ in X such that $\mathcal{R}_j^b(y) = \mathcal{R}_j^b(p) + \mathcal{R}_j^b(q)$, $\lim_{j\to\infty} \mathcal{R}_j^b(p) = \zeta$ w.r.t. IFN (f_1, f_2) and the set $\{j \in \mathbb{N} : \mathcal{R}_j^b(q) \neq \theta\} \in I$ where zero elements of X are denoted by θ.*

Proof of Theorem 2.8. Let statement (1) holds. Then we have a subset $A = \{i_1, i_2, i_3, ... : i_1 < i_2 < ..\}$ of $\mathbb{N}$ in such a way that $A \in \mathcal{F}(I)$ and $\lim_{j\to\infty} \mathcal{R}_j^b(y) = \zeta$ w.r.t. IFN (f_1, f_2).

We define (p_k) and (q_k) such that,

$$\mathcal{R}_j^b(p) \quad = \quad \begin{cases} \mathcal{R}_j^b(y), & \text{if } j \in A \\ \zeta, & \text{otherwise} \end{cases}$$

and $\mathcal{R}_j^b(q) = \mathcal{R}_j^b(y) - \mathcal{R}_j^b(p)$ for $j \in \mathbb{N}$. For $j \in A^c$, for all $\epsilon > 0$ and $s > 0$,

$$f_1(\mathcal{R}_j^b(p) - \zeta, s) = 1 > 1 - \epsilon \text{ and } f_2(\mathcal{R}_j^b(p) - \zeta, s) = 0 < \epsilon$$

which implies that $\lim_{j\to\infty} \mathcal{R}_j^b(p) = \zeta$ w.r.t. IFN (f_1, f_2). As $\{j \in \mathbb{N} : \mathcal{R}_j^b(q) \neq \theta\} \subset A^c$, which implies that $\{j \in \mathbb{N} : \mathcal{R}_j^b(q) \neq \theta\} \in I$.

Now, let statement (2) holds, $A = \{j \in \mathbb{N} : \mathcal{R}_j^b(q) = 0\}$, hence $A \in \mathcal{F}(I)$ so it is an infinite set. Now, suppose $A = \{i_1, i_2, .. : i_1 < i_2 < ...\}$. As $\mathcal{R}_{i_j}^b(y) = \mathcal{R}_{i_j}^b(p)$ and $\lim_{j\to\infty} \mathcal{R}_j^b(p) = \zeta$ w.r.t. IFN (f_1, f_2), hence $\lim_{j\to\infty} \mathcal{R}_{i_j}^b(y) = \zeta$ w.r.t. IFN (f_1, f_2), which implies that $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.

Conclusion

In this article, we have defined Riesz ideal convergent and Riesz ideal Cauchy sequences in intuitionistic fuzzy normed space. We have also discussed the behavior of this notion by proving various fundamental results with counterexamples. Usage of fuzzy logic nowadays increased massively in the various fields of science and technology to tackle real-world problems. Since ideal convergence is a unified and generalized notion of various well-known notions of convergence and Riesz ideal convergence is a more generalized variant of ideal convergence. On the other hand, intuitionistic fuzzy normed space is an extension of various famous spaces and it also handles complicated situations very easily because of its inexactness of the norm, hence our study is in a more general setting than other existing studies. Therefore, these new results will further give a superior tool to tackle complex problems and also help the researchers expand their work in the area of sequence spaces in view of fuzzy theory.

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On lacunary statistical convergence of difference sequences

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Abstract. In this paper, the $S_\theta(\Delta)$ and $N_\theta(\Delta)$ summabilities are used along with the notion of weakly unconditionally Cauchy series (in brief wuC series) to characterize a Banach space. We examine these two kinds of summabilities which are regular methods and we recall some features. Furthermore, we investigate the spaces $S_{N_\theta}(\sum_p \Delta w_p)$ and $S_{S_\theta}(\sum_p \Delta w_p)$ which will be thought to characterize the completeness of a space.
Keywords: completeness, unconditionally Cauchy series, lacunary convergence, difference sequence.

1. Introduction and background

The notion of statistical convergence was introduced under the name almost convergence by Zygmund [1]. It was formally presented by Fast [2]. Later the idea was associated with summability theory by Fridy [3] and many others (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]).

By a lacunary sequence we mean an increasing integer sequence $\theta = \{n_r\}$ such that $n_0 = 0$ and $h_r = n_r - n_{r-1} \to \infty$ as $r \to \infty$ and ratio $\frac{n_r}{n_{r-1}}$ will be abbreviated by q_r. Throughout this paper the intervals determined by θ will be denoted by $I_r = (n_{r-1}, n_r]$. Utilizing lacunary sequence, Fridy and Orhan [15] presented the notion of lacunary statistical convergence. Some works in lacunary statistical convergence can be found in [16, 17, 18, 19, 20].

Let us define the forward difference matrix $\Delta^F = (c_{nk})$ and the backward difference matrix $\Delta^B = (d_{nk})$ by

$$c_{nk} = \begin{cases} (-1)^{n-k}, & n \le k \le n+1, \\ 0, & 0 \le k < n \text{ or } k > n+1, \end{cases}$$

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$$d_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n, \\ 0, & 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

for all $k, n \in \mathbb{N} = \{0, 1, 2, ...\}$. Then, the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced by Kızmaz [21], can be seen as the domain of forward difference matrix Δ^F in the classical spaces l_∞, c and c_0 of bounded, convergent and null sequences, respectively. Quite recently, the difference space bv_p was introduced as the domain of the backward difference matrix Δ^B in the classical space l_p of absolutely p-summable sequences for $0 < p < 1$ by Altay and Başar [22], and for $1 \leq p \leq \infty$ by Başar and Altay [23].

Later on the notion was generalized by Et and Çolak [24]. Başarır [25] investigated the Δ-statistical convergence of sequences. Also, the generalized difference sequence spaces were worked by various authors [26, 27, 28, 29, 30].

The characterization of a Banach space through various types of convergence has been examined by authors such as Kolk [31], Connor et al. [32].

The purpose of this study originates in the PhD thesis of the second author [33] who identified a relationship between features of a normed space Y and some sequence spaces which are named convergence spaces associated to a wuC series. These sequence spaces associated to a wuC series were examined [33] in terms of the norm topology and the usual weak topology of the space. These types of consequences have been researched in various convergence spaces connected with a wuC series utilizing different types of convergence [34, 35, 36, 37, 38, 40, 39]. The readers can refer to the recent papers [41, 42, 43, 44], and references therein on the wuC series in a normed space and the examples of multiplier convergent series that characterizes the uc and wuC series, and related topics.

Y be a normed space and $\sum w_i$ also be a series in Y. In [33], the authors defined the space of convergence $S(\sum w_i)$ connected with the series $\sum w_i$, which is introduced as the space of sequence (β_i) in l_∞ such that $\sum \beta_i w_i$ converges. They demonstrated that the necessary and sufficient condition for Y to be a complete space is that for every wuC series $\sum w_i$, the space $S(\sum w_i)$ is complete. Diestel [45] showed that $\sum w_i$ is wuC iff $\sum |f(w_i)| < \infty$ for all $f \in Y^*$. In [46, 47], a Banach space is characterized by means of the strong p-Cesàro summability and ideal-convergence.

In this paper, we examine the completeness of a normed space through the lacunary statistical convergence and lacunary strongly convergence of series for difference sequences. We also describe the summability spaces associated with these summabilities with strongly (p, Δ)-Cesàro summability spaces for difference sequences.

2. Main results

We identify the notion of lacunary Δ-statistically convergent sequence for Banach spaces.

Let $A \subset \mathbb{N}$ and $r \in \mathbb{N}$. $d_\theta^r(A)$ is named the rth partial lacunary density of A, if

$$d_\theta^r(A) = \frac{|A \cap I_r|}{h_r},$$

where $I_r = (k_{r-1}, k_r]$.

The number $d_\theta(A)$ is indicated the lacunary density (θ-density) of A if

$$d_\theta(A) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : k \in A\}|, \text{ (i.e., } d_\theta(A) = \lim_{r\to\infty} d_\theta^r(A))$$

exists. Also, $\Lambda = \{A \subset \mathbb{N} : d_\theta(A) = 0\}$ is called to be zero density set.

It is easy to demonstrate that this density is a finitely additive measure and we can introduce the notion of lacunary statistically convergent difference sequences for Banach spaces.

Definition 2.1. *Let Y be a Banach space and $\theta = \{n_r\}$ a lacunary sequence. A sequence $w = (w_p)$ is a lacunary Δ-statistically convergent or sequence to $\xi \in Y$ if given $\zeta > 0$,*

$$d_\theta (\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}) = 0,$$

or equivalently,

$$d_\theta (\{p \in I_r : \|\Delta w_p - \xi\| < \zeta\}) = 1,$$

we say that (w_p) is $S_\theta(\Delta)$-convergent and is written as S_θ-$\lim \Delta w_p = \xi$.

Definition 2.2. *A sequence $w = (w_p)$ in Y is lacunary strongly Δ-convergent or $N_\theta(\Delta)$-summable to $\xi \in Y$ if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{p \in I_r} \|\Delta w_p - \xi\| = 0,$$

and we write N_θ-$\lim \Delta w_p = \xi$.

Theorem 2.1. *Let Y be a Banach space and (w_p) a sequence in Y. Note that $S_\theta(\Delta)$ and $N_\theta(\Delta)$ are regular methods.*

Proof. First, we prove that $S_\theta(\Delta)$ is a regular method. If $(\Delta w_p) \to \xi$, then N_θ-$\lim \Delta w_p = \xi$. Let $\zeta > 0$, then there is p_0 such that if $p \geq p_0$, then

$$\|\Delta w_p - \xi\| < \zeta.$$

Therefore, there is $r_0 \in \mathbb{N}$ with $r_0 \geq p_0$ such that if $r \geq r_0$ we obtain

$$\frac{1}{h_r} \sum_{p \in I_r} \|\Delta w_p - \xi\| < \frac{1}{h_r} \sum_{p \in I_r} \zeta = \frac{h_r}{h_r} \zeta = \zeta,$$

which gives that $\lim_{r\to\infty} \frac{1}{h_r} \sum_{p \in I_r} \|\Delta w_p - \xi\| = 0$.

Now, we show that $N_\theta(\Delta)$ is a regular method. If $(\Delta w_p) \to \xi$, then $S_\theta-\lim \Delta w_p = \xi$. One can easily observe that $(\Delta w_p) \to \xi$, given $\zeta > 0$ there is p_0 such that for every $p > p_0$ we obtain

$$card\left(\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right) = 0,$$

which gives

$$d_\theta\left(\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right) = 0$$

for every $p > p_0$. □

The reverse is not true, as was shown in Example 2.1, in which we introduce an unbounded sequence that is $N_\theta(\Delta)$-summable and Example 2.2 where an unbounded $S_\theta(\Delta)$-convergent sequence is given.

Example 2.1. There exist unbounded sequences which are $N_\theta(\Delta)$-summable. Let $\theta = \{n_r\}$ be a lacunary sequence with $n_0 = 0$ and $n_r = 2^r$. Think that

$$h_1 = n_1 - n_0 = 2 \text{ and } h_r = 2^{r-1}, \text{ for every } r \geq 2,$$

$$I_1 = (n_0, n_1] = (0, 2] \text{ and } I_r = \left(2^{r-1}, 2^r\right], \text{ for every } r \geq 2.$$

Think the sequence determined by

$$\Delta w_p = \begin{cases} 0, & \text{if } p \neq 2^j \text{ for all } j, \\ j-1, & \text{if } p = 2^j \text{ for all } j. \end{cases}$$

Notice that, (w_p) is unbounded and observe that

$$\frac{\sum_{p\in I_r} |\Delta w_p - 0|}{h_r} = \left\{ \begin{array}{cl} 0, & \text{if } r = 1 \\ \frac{r-1}{2^{r-1}}, & \text{if } r \geq 2 \end{array} \right\} \to 0, \text{ as } r \to \infty$$

which gives that N_θ-$\lim \Delta w_p = 0$.

Theorem 2.2. *Let Y be a Banach space and $\theta = \{n_r\}$ be a lacunary sequence. Then, we have the followings:*

(*i*) N_θ-$\lim \Delta w_p = \xi$ *implies* S_θ-$\lim \Delta w_p = \xi$,

(*ii*) (w_p) *is bounded and* S_θ-$\lim \Delta w_p = \xi$ *imply* N_θ-$\lim \Delta w_p = \xi$.

Proof. (*i*) If N_θ-$\lim \Delta w_p = \xi$ then, for every $\zeta > 0$,

$$\sum_{p\in I_r} \|\Delta w_p - \xi\| \geq \sum_{\substack{p\in I_r \\ \|\Delta w_p - \xi\| \geq \zeta}} \|\Delta w_p - \xi\| \geq \zeta \left|\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right|,$$

which gives that S_θ-$\lim \Delta w_p = \xi$.

(ii) Let us assume that (w_p) is bounded and S_θ-$\lim \Delta w_p = \xi$. Since (w_p) is bounded, there exists $H > 0$ such that $\|\Delta w_p - \xi\| < H$ for every $p \in \mathbb{N}$. Given $\zeta > 0$,

$$\frac{1}{h_r}\sum_{p\in I_r} \|\Delta w_p - \xi\| = \frac{1}{h_r} \sum_{\substack{p\in I_r \\ \|\Delta w_p - \xi\| \geq \zeta}} \|\Delta w_p - \xi\| + \frac{1}{h_r} \sum_{\substack{p\in I_r \\ \|\Delta w_p - \xi\| < \zeta}} \|\Delta w_p - \xi\|$$
$$\leq \frac{H}{h_r} |\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}| + \zeta,$$

so, we obtain N_θ-$\lim \Delta w_p = \xi$. □

Next, we give an example to demonstrate that the assumption over the sequence to be bounded is necessary and cannot be removed.

Example 2.2. There exist unbounded $S_\theta(\Delta)$-convergent sequences to ξ which are not $N_\theta(\Delta)$-summable to ξ. Let $\theta = \{n_r\}$ be a lacunary sequence with $n_0 = 0$ and $n_r = 2^r$. Consider that

$$h_1 = n_1 - n_0 = 2 \text{ and } h_r = 2^{r-1} \text{ for every } r \geq 2,$$
$$I_1 = (n_0, n_1] = (0, 2] \text{ and } I_r = \left(2^{r-1}, 2^r\right] \text{ for every } r \geq 2.$$

Think the sequence determined by

$$\Delta w_p = \begin{cases} 0, & \text{if } p \neq 2^j \text{ for all } j, \\ 2^j, & \text{if } p = 2^j \text{ for all } j. \end{cases}$$

Given $\zeta > 0$, it is simply to denote that

$$\frac{|\{p \in I_r : \|\Delta w_p - 0\| \geq \zeta\}|}{h_r} \to 0 \text{ as } r \to \infty,$$

which gives that S_θ-$\lim \Delta w_p = 0$. Also, note that (w_p) is an unbounded sequence. However,

$$\frac{\sum_{p\in I_r} |\Delta w_p - 0|}{h_r} = \left\{ \begin{array}{ll} \frac{2}{2} = 1, & \text{if } r = 1, \\ \frac{2^r}{2^{r-1}} = 2, & \text{if } r \geq 2 \end{array} \right\} \to 2, \text{ as } r \to \infty$$

which gives that N_θ-$\lim \Delta w_p \neq 0$.

Definition 2.3. *Take Y as a Banach space. A sequence $w = (w_p)$ is named to be lacunary Δ-statistically Cauchy sequence if there exists a subsequence $(w_{p'(r)})$ of (w_p) sucht that $p'_r \in I_r$, for every $r \in \mathbb{N}$, $\lim_{r\to\infty} \Delta w_{p'(r)} = \xi$, for some $\xi \in Y$ and for each $\zeta > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{p \in I_r : \left\|\Delta w_p - \Delta w_{p'(r)}\right\| \geq \zeta\right\}\right| = 0,$$

or equivalently,

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{p \in I_r : \left\|\Delta w_p - \Delta w_{p'(r)}\right\| < \zeta\right\}\right| = 1.$$

In this case, we say that (w_p) is $S_\theta(\Delta)$-Cauchy.

The following consequence is acquired for sequences in Banach spaces, and we involve the proof for the sake of completeness.

Theorem 2.3. *Take Y as a Banach space. A sequence $w = (w_p)$ is $S_\theta(\Delta)$-convergent iff it is $S_\theta(\Delta)$-Cauchy.*

Proof. Let $w = (w_p)$ be an $S_\theta(\Delta)$-convergent sequence in Y and for each $p \in \mathbb{N}$, we determine

$$K_q = \left\{ p \in \mathbb{N} : \|\Delta w_p - \xi\| < \frac{1}{q} \right\}.$$

Observe that $K_q \supseteq K_{q+1}$ and $\frac{\operatorname{card}(K_q \cap I_r)}{h_r} \to 1$ as $r \to \infty$.

Establish m_1 such that $r \leq m_1$ then $\operatorname{card}(K_1 \cap I_r)/h_r > 0$, i.e., $K_1 \cap I_r \neq \emptyset$. Next, select $m_2 > m_1$ such that if $r \geq m_2$, then $K_2 \cap I_r \neq \emptyset$. Now, for each $m_1 \leq r \leq m_2$, we select $p'_r \in I_r$ such that $p'_r \in I_r \cap K_1$, i.e., $\left\|\Delta w_{p'(r)} - \xi\right\| < 1$. Technically, we select $m_{k+1} > m_k$, such that if $r > m_{k+1}$, then $I_r \cap K_{k+1} \neq \emptyset$. So, for all r such that $m_k \leq r < m_{k+1}$, we select $p'_r \in I_r \cap K_k$, and we obtain $\left\|\Delta w_{p'(r)} - \xi\right\| < \frac{1}{k}$.

Therefore, we get a sequence (p'_r) such that $p'_r \in I_r$ for every $r \in \mathbb{N}$ and $\lim_{r\to\infty} \Delta w_{p'(r)} = \xi$. As a result, we acquire

$$\begin{aligned}\frac{1}{h_r}\left|\left\{p \in I_r : \left\|\Delta w_p - \Delta w_{p'(r)}\right\| \geq \zeta\right\}\right| \leq{}& \frac{1}{h_r}\left|\left\{p \in I_r : \|\Delta w_p - \xi\| \geq \frac{\zeta}{2}\right\}\right| \\ &+ \frac{1}{h_r}\left|\left\{p \in I_r : \left\|\Delta w_{p'(r)} - \xi\right\| \geq \frac{\zeta}{2}\right\}\right|.\end{aligned}$$

Since S_θ-$\lim \Delta w_p = \xi$ and $\lim_{r\to\infty} \Delta w_{p'(r)} = \xi$ we conclude that (w_p) is $S_\theta(\Delta)$-Cauchy.

Conversely, if (w_p) is $S_\theta(\Delta)$-Cauchy sequence, for every $\zeta > 0$,

$$\begin{aligned}\left|\left\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\right\}\right| \leq{}& \left|\left\{p \in I_r : \left\|\Delta w_p - \Delta w_{p'(r)}\right\| \geq \tfrac{\zeta}{2}\right\}\right| \\ &+ \left|\left\{p \in I_r : \left\|\Delta w_{p'(r)} - \xi\right\| \geq \tfrac{\zeta}{2}\right\}\right|.\end{aligned}$$

Since (w_p) is $S_\theta(\Delta)$-Cauchy and $\lim_{r\to\infty} \Delta w_{p'(r)} = \xi$, we conclude that S_θ-$\lim \Delta w_p = \xi$. □

Now, we examine some features of the statistical lacunary summability spaces for Banach spaces.

Let us think Y a real Banach space, $\sum_j \Delta w_j$ a series in Y and $\theta = (n_r)$ a lacunary sequence. We identify

$$S_{S_\theta}\Big(\sum_j \Delta w_j\Big) = \left\{ (a_j)_j \in l_\infty : \sum_j a_j \Delta w_j \text{ is } S_\theta\text{-summable} \right\}$$

endowed with the supremum norm. The space will be called as the space of $S_\theta(\Delta)$-summability connected with $\sum_j \Delta w_j$. We will describe the completeness of the space $S_{S_\theta}(\sum_j \Delta w_j)$ in Theorem 2.4, but first we have to give the following Lemma.

Lemma 2.1. *Let Y be a Banach space and presume that the series $\sum_j \Delta w_j$ is not wuC. Then, there is $f \in Y^*$ and a null sequence $(a_j)_j \in c_0$ such that*

$$\sum_j a_j f(\Delta w_j) = +\infty$$

and

$$a_j f(\Delta w_j) \geq 0.$$

Proof. Since $\sum_{j=1}^{\infty} |f(\Delta w_j)| = +\infty$, there exists t_1 such that $\sum_{j=1}^{t_1} |f(\Delta w_j)| > 2 \cdot 2$. We itendify $a_j = \frac{1}{2}$ if $f(\Delta w_j) \geq 0$ and $a_j = -\frac{1}{2}$ if $f(\Delta w_j) < 0$ for $j = \{1, 2, ..., t_1\}$. This gives that $\sum_{j=1}^{t_1} a_j f(\Delta w_j) > 2$ and $a_j f(\Delta w_j) \geq 0$ if $j = \{1, 2, ..., t_1\}$. Let $t_2 > t_1$ be such that $\sum_{j=t_1+1}^{t_2} |f(\Delta w_j)| > 2^2 \cdot 2^2$. We determine $a_j = \frac{1}{2^2}$ if $f(\Delta w_j) \geq 0$ and $a_j = -\frac{1}{2^2}$ if $f(\Delta w_j) < 0$ for $j = \{t_1 + 1, ..., t_2\}$. Hence, $\sum_{j=t_1+1}^{t_2} a_j f(\Delta w_j) > 2^2$ and $a_j f(\Delta w_j) \geq 0$ if $j = \{t_1 + 1, ..., t_2\}$. So, we have acquired a sequence $(a_j)_j \in c_0$ with the above features. $\square$

Theorem 2.4. *Let Y be a Banach space and $\theta = \{n_r\}$ a lacunary sequence. The subsequent are equivalent:*

(*i*) *The series $\sum_j \Delta w_j$ is wuC.*

(*ii*) *The space $S_{S_\theta}(\sum_j \Delta w_j)$ is complete.*

(*iii*) *The space c_0 of all null sequences is included in $S_{S_\theta}(\sum_j \Delta w_j)$.*

Proof. $(i) \Rightarrow (ii)$: Since $\sum_j \Delta w_j$ is wuC, the subsequent supremum is finite:

$$Q = \sup \left\{ \left\| \sum_{j=1}^{n} \beta_j \Delta w_j \right\| : |\beta_j| \leq 1,\ 1 \leq j \leq n,\ n \in \mathbb{N} \right\} < +\infty.$$

Let $(\beta^s)_s \subset S_{S_\theta}(\sum_j \Delta w_j)$ such that $\lim_s \left\| \beta^s - \beta^0 \right\|_\infty = 0$, with $\beta^0 \in l_\infty$. We will denote that $\beta^0 \in S_{S_\theta}(\sum_j \Delta w_j)$. Let us assume without any loss of generality that $\left\| \beta^0 \right\|_\infty \leq 1$. Then, the partial sums $S_p^0 = \sum_{j=1}^{p} \beta_j^0 \Delta w_j$ satisfy $\left\| S_p^0 \right\| \leq Q$ for every $p \in \mathbb{N}$, i.e., the sequence $\left(S_p^0\right)$ is bounded. Then, $\beta^0 \in S_{S_\theta}(\sum_j \Delta w_j)$ iff $\left(S_p^0\right)$ is $S_\theta(\Delta)$-summable to some $\xi \in Y$. In accordance with Theorem 2.3, $\left(S_p^0\right)$ is lacunary Δ-statistically convergent to $\xi \in Y$ iff $\left(S_p^0\right)$ is lacunary Δ-statistically Cauchy sequence.

Let $\zeta > 0$ and $n \in \mathbb{N}$. Then, we acquire statement (ii) if we indicate that there is a subsequence $\left(S_{p'(r)}\right)$ such that $p'_r \in I_r$ for every $r \in \mathbb{N}$, $\lim_{r \to \infty} S_{p'(r)} = \xi$ and

$$d_\theta \left(\left\{ p \in I_r : \left\| S_p^0 - S_{p'(r)}^0 \right\| < \zeta \right\} \right) = 1.$$

Since $\beta^s \to \beta^0$ in l_∞, there is $s_0 > n$ such that $\left\|\beta^s - \beta^0\right\|_\infty < \frac{\zeta}{4Q}$ for all $s > s_0$, and since $S_p^{s_0}$ is $S_\theta(\Delta)$-Cauchy, there is $p'_r \in I_r$ such that $\lim_{r\to\infty} S_{p'(r)}^{s_0} = \xi$ for some ξ and

$$d_\theta\left(\left\{p \in I_r : \left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| < \frac{\zeta}{2}\right\}\right) = 1.$$

Think $r \in \mathbb{N}$ and fix $p \in I_r$ such that

$$\left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| < \frac{\zeta}{2}. \tag{1}$$

We will signify that $\left\|S_p^0 - S_{p'(r)}^0\right\| < \zeta$, and this will evidence that

$$\left\{p \in I_r : \left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| < \frac{\zeta}{2}\right\} \subset \left\{p \in I_r : \left\|S_p^0 - S_{p'(r)}^0\right\| < \zeta\right\}.$$

Since the first set has density 1, the second will also have density 1 and we will be done.

Let us observe first that for each $i \in \mathbb{N}$,

$$\left\|\sum_{j=1}^{i} \frac{4Q}{\zeta}\left(\beta_j^s - \beta_j^{s_0}\right)\Delta w_j\right\| \leq Q,$$

for every $s > s_0$, therefore

$$\left\|S_i^0 - S_i^{s_0}\right\| = \left\|\sum_{j=1}^{i}\left(\beta_j^0 - \beta_j^{s_0}\right)\Delta w_j\right\| \leq \frac{\zeta}{4}. \tag{2}$$

Then, by using the triangular inequality,

$$\begin{aligned}\left\|S_p^0 - S_{p'(r)}^0\right\| &\leq \left\|S_p^0 - S_p^{s_0}\right\| + \left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| + \left\|S_{p'(r)}^{s_0} - S_{p'(r)}^0\right\| \\ &< \tfrac{\zeta}{4} + \tfrac{\zeta}{2} + \tfrac{\zeta}{4} = \zeta.\end{aligned}$$

Therefore, by applying (1) and (2), the last inequality yields the desired result.

(ii) $\Rightarrow$ *(iii)*: Let us observe that if $S_{S_\theta}(\sum_j \Delta w_j)$ is complete, then it includes the space of ultimately zero sequences c_{00} and therefore the thesis comes, since the supremum norm completion of c_{00} is c_0.

(iii) $\Rightarrow$ *(i)*: By utilizing the contradiction, presume that the series $\sum \Delta w_j$ is not wuC. So, there is $f \in Y^*$ such that $\sum_{j=1}^\infty |f(\Delta w_j)| = +\infty$. By Lemma 2.1, we can create technically a sequence $(\beta_j)_j \in c_0$ such that

$$\sum_j \beta_j f(\Delta w_j) = +\infty$$

and

$$\beta_j f(\Delta w_j) \geq 0.$$

Now, we will examine that the sequence $(S_p) = (\sum_{j=1}^{p} \beta_j f(\Delta w_j))$ is not $S_\theta(\Delta)$-summable to any $\xi \in \mathbb{R}$. By utilizing the contradiction, assume that it is $S_\theta(\Delta)$-summable to $\xi \in \mathbb{R}$, then we obtain

$$\frac{1}{h_r} |\{p \in I_r : |S_p - \xi| \geq \zeta\}| = \frac{1}{h_r} \sum_{\substack{p=n_r-1 \\ |S_p-\xi|\geq\zeta}}^{n_r} 1 \to 0 \text{ as } r \to \infty.$$

Since S_p is an inreasing sequence and $S_p \to \infty$, there is n_0 such that $|S_p - \xi| \geq \zeta$ for every $p \geq n_0$. Let us presume that $n_r > n_0$ for every r. Consequently,

$$\frac{1}{h_r} \sum_{\substack{n=n_r-1 \\ |S_p-\xi|\geq\zeta}}^{n_r} 1 = \frac{h_r}{h_r} = 1 \nrightarrow 0 \text{ as } r \to \infty,$$

a contradiction. This gives that (S_p) is not $S_\theta(\Delta)$-convergent and is a contradiction with (iii). □

Now, we examine some features of the lacunary strongly Δ-summability space for Banach spaces.

Let Y a real Banach space, $\sum_j \Delta w_j$ a series in Y and $\theta = (n_r)$ a lacunary sequence. We itendify

$$S_{N_\theta}(\sum_j \Delta w_j) = \left\{ (a_j)_j \in l_\infty : \sum_j a_j \Delta w_j \text{ is } N_\theta\text{-summable} \right\}$$

endowed with the supremum norm. This will be characterised as the space of $N_\theta(\Delta)$-summability connected with the series $\sum_j \Delta w_j$. We can now give a theorem very same as that of Theorem 2.4 but for the case of $N_\theta(\Delta)$-summability. Actually Theorem 2.5 describes the completeness of the space $S_{N_\theta}(\sum_j \Delta w_j)$.

Theorem 2.5. *Let Y be a real Banach space and $\theta = (n_r)$ a lacunary sequence. The subsequent are equivalent:*

(i) The series $\sum_j \Delta w_j$ is wuC.

(ii) The space $S_{N_\theta}(\sum_j \Delta w_j)$ is complete.

(iii) The space c_0 of all null sequences is included in $S_{N_\theta}(\sum_j \Delta w_j)$.

Proof. $(i) \Rightarrow (ii)$: Since $\sum_j \Delta w_j$ is wuC, the subsequent supremum is finite:

$$Q = \sup \left\{ \left\| \sum_{j=1}^{k} \beta_j \Delta w_j \right\| : |\beta_j| \leq 1, 1 \leq j \leq k, k \in \mathbb{N} \right\} < \infty.$$

Let $(\beta^s)_s \subset S_{N_\theta}(\sum_j \Delta w_j)$ such that $\lim_s \left\|\beta^s - \beta^0\right\|_\infty = 0$, with $\beta^0 \in l_\infty$. We will denote that $\beta^0 \in S_{N_\theta}(\sum_j \Delta w_j)$. With no loss of generality, we can presume that $\left\|\beta^0\right\|_\infty \leq 1$. So, the partial sums $S_p^0 = \sum_{j=1}^p \beta_j^0 \Delta w_j$ satisfy $\left\|S_p^0\right\| \leq Q$ for every $p \in \mathbb{N}$, i.e., the sequence $\left(S_p^0\right)$ is bounded. Then, $\beta^0 \in S_{N_\theta}(\sum_j \Delta w_j)$ iff $\left(S_p^0\right)$ is $N_\theta(\Delta)$-summable to some $\xi \in Y$. Since $\left(S_p^0\right)$ is bounded, it is sufficient to show that (S_p) is $S_\theta(\Delta)$-convergent, as a consequence of Theorem 2.1 due to Fridy and Orhan [15]. The results follows similarly as in Theorem 2.4.

$(ii) \Rightarrow (iii)$: It is adequate to observe that $S_{N_\theta}(\sum_j \Delta w_j)$ is a complete space and it includes the space of ultimately zero sequences c_{00}, so it involves the completion of c_{00} with regards to the supremum norm, hence it includes c_0.

$(iii) \Rightarrow (i)$: By utilizing the contradiction, presume that the series $\sum \Delta w_j$ is not wuC. So, there is $f \in Y^*$ such that $\sum_{j=1}^\infty |f(\Delta w_j)| = +\infty$. By Lemma 2.1, we can create technically a sequence $(\beta_j)_j \in c_0$ such that $\sum_j \beta_j f(\Delta w_j) = +\infty$ and $\beta_j f(\Delta w_j) \geq 0$.

The sequence $S_p = \sum_{j=1}^p \beta_j f(\Delta w_j)$ is not $N_\theta(\Delta)$-summable to any $\xi \in \mathbb{R}$. Since $S_p \to \infty$, for every $H > 0$, there is p_0 such that $|S_p| > H$ if $p > p_0$. Then, we acquire

$$\frac{1}{h_r}\sum_{p \in I_r} |S_p| > \frac{h_r Q}{h_r} = Q.$$

Hence S_p is not $N_\theta(\Delta)$-summable to any $\xi \in \mathbb{R}$, on the other hand

$$\infty \leftarrow \frac{1}{h_r}\sum_{p \in I_r} |S_p| \leq |\xi| + \sum_{p \in I_r} |S_p - \xi| \to |\xi|$$

We can deduce that S_p is not $N_\theta(\Delta)$-convergent, a contradiction with (iii). $\square$

A Banach space Y can be characterized by the completeness of the space $S_{N_\theta}(\sum_p \Delta w_p)$ for every wuC series $\sum_p \Delta w_p$, as we will show, nextly.

Theorem 2.6. *Take Y as a normed real vector space. Then, Y is complete iff $S_{N_\theta}(\sum_p \Delta w_p)$ is a complete space for every wuC series $\sum_p \Delta w_p$.*

Proof. The necessary condition is obvious from Theorem 2.4. Now, suppose that Y is not complete, hence there is a series $\sum_p \Delta w_p$ in Y such that $\|\Delta w_p\| \leq \frac{1}{p 2^p}$ and $\sum \Delta w_p = w^{**} \in Y^{**} \setminus Y$. We will provide a wuC series $\sum_p \Delta y_p$ such that $S_{N_\theta}(\sum_p \Delta y_p)$ is not complete, a contradiction. Set $S_M = \sum_{p=1}^M \Delta w_p$. As Y^{**} is a Banach space endowed with the dual topology, $\sup\limits_{\|y^*\| \leq 1} |y^*(S_M) - w^{**}(y^*)|$ tends to 0 as $M \to \infty$, i.e.,

$$\lim_{M \to +\infty} y^*(S_M) = \lim_{M \to +\infty} \sum_{p=1}^M y^*(\Delta w_p) = w^{**}(y^*), \tag{3}$$

for every $\|y^*\| \leq 1$. Put $\Delta y_p = p \Delta w_p$ and let us observe that $\|\Delta y_p\| < \frac{1}{2^p}$. Therefore, $\sum_p \Delta y_p$ is absolutely convergent, so it is unconditionally convergent and weakly unconditionally Cauchy. We claim that the series $\sum_p \frac{1}{p}\Delta y_p$ is not N_θ-summable in Y. Using contradiction assume that $S_M = \sum_{p=1}^{M} \frac{1}{p}\Delta y_p$ is N_θ-summable in Y, i.e., there exists ξ in Y such that $\lim_{r\to\infty} \frac{1}{h_r}\sum_{p\in I_r} \|S_p - \xi\| = 0$. This gives that

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{p\in I_r} y^*(S_p) = y^*(\xi), \tag{4}$$

for every $\|y^*\| \leq 1$. By the relations (3) and (4), the uniqueness of the limit and since $N_\theta(\Delta)$ is a regular method, we get $w^{**}(y^*) = y^*(\xi)$ for every $\|y^*\| \leq 1$, so we acquire $w^{**} = \xi \in Y$, a contradiction. Hence, $S_M = \sum_{p=1}^{M} \frac{1}{p}\Delta y_p$ is not N_θ-summable to any $\xi \in Y$.

Finally, let us observe that since $\sum_p \Delta y_p$ is a weakly unconditionally Cauchy series and $S_M = \sum_{p=1}^{M} \frac{1}{p}\Delta y_p$ is not N_θ-summable, we get $\left(\frac{1}{p}\right) \notin S_{N_\theta}(\sum_p \Delta y_p)$ and this means that $c_0 \nsubseteq S_{N_\theta}(\sum_p \Delta y_p)$ which contradicts Part (iii) of Theorem 2.5. This completes the proof. □

Definition 2.4. *Let $0 < p < \infty$, the sequence $w = (w_n)$ is named to be strongly (p,Δ)-Cesàro or $|\sigma_p|(\Delta)$-summable if there is $\xi \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} \|\Delta w_i - \xi\|^p = 0$$

and is writen as $(\Delta w_n) \underset{|\sigma_p|}{\rightarrow} \xi$ or $|\sigma_p| - \lim_{n\to\infty} \Delta w_n = \xi$.

Let $\sum \Delta w_i$ be a series in a real Banach space Y,

$$S_{|\sigma_p|}(\sum_i \Delta w_i) = \left\{ (a_i)_i \in l_\infty : \sum_i a_i \Delta w_i \text{ is } |\sigma_p|\text{-summable} \right\}$$

endowed with the supremum norm.

Corollary 2.1. *Take Y as a normed real vector space and $p \geq 1$. The subsequent are equivalent:*

- (i) *Y is complete.*
- (ii) *$S_{N_\theta}(\sum_p \Delta w_p)$ is complete for every wuC series $\sum_p \Delta w_p$.*
- (iii) *$S_{S_\theta}(\sum_p \Delta w_p$ is complete for every wuC series $\sum_p \Delta w_p$.*
- (iv) *$S_{|\sigma_p|}(\sum_p \Delta w_p)$ is complete, for every wuC series $\sum_p \Delta w_p$.*

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On finite groups with s-weakly normal subgroups

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Abstract. A subgroup H of a group G is weakly normal in G if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ for any element $g \in G$. A subgroup H of a group G is s-weakly normal in G if there exists a normal subgroup T such that $G = HT$ and $H \cap T$ is weakly normal in G. Clearly a weakly normal subgroup of G is an s-weakly normal subgroup of G. In this paper, we investigate the influence of s-weakly normal subgroups on the structure of a finite group, especially some criteria for supersolvability, nilpotency, formation and hypercenter of a finite group are proved. Based on our results, some recent results can be generalized easily.

Keywords: finite group, weakly normal subgroup, s-weakly normal subgroup, supersolvable group, nilpotent group.

1. Introduction

The groups which appear throughout this paper are assumed to be finite groups and G always denotes a finite group. Let's first introduce some frequently used notations and terminologies, any unexplained terms can be found in [9, 11, 12, 20].

Let $|G|$ be the order of G and $\pi(G)$ be the set of all prime divisors of $|G|$. For a p-group P, where p is a prime, we write $\Omega_1(P) = \langle x \in P | x^p = 1\rangle$ and $\Omega_2(P) = \langle x \in P | x^{p^2} = 1\rangle$; we say a p'-group H means a group H satisfying $p \nmid |H|$.

Let $\mathfrak{F}$ be a class of groups. Recall that $\mathfrak{F}$ is said to be a *formation* if $\mathfrak{F}$ is closed under taking homomorphic image and finite subdirect product, that is, for each group G and a normal subgroup N of G, $G \in \mathfrak{F}$ implies that $G/N \in \mathfrak{F}$, moreover, if $M \trianglelefteq G$, then $G/N \in \mathfrak{F}$ and $G/M \in \mathfrak{F}$ imply $G/(N \cap M) \in \mathfrak{F}$. A

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formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$, where $\Phi(G)$ is the intersection of all maximal subgroups of G.

We denote by $\mathfrak{U}$ the class of all supersolvable groups and by $\mathfrak{N}$ the class of all nilpotent groups. It is known that $\mathfrak{U}$ and $\mathfrak{N}$ are both saturated formations (see [12]). We denote by $Z_{\mathfrak{F}}(G)$ the product of all $\mathfrak{F}$-hypercentral subgroups of G. In particular, $Z_{\mathfrak{U}}(G)$ denotes the product of all normal subgroups N of G such that each chief factor of G below N has prime order. It is known that for the formation $\mathfrak{N}$, $Z_{\mathfrak{N}}(G) = Z_{\infty}(G)$ is the hypercenter of G.

It is known that a subgroup H of a group G is *pronormal* in G if the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$ for each element g of G. This concept was introduced by P. Hall [13] and the first general results about pronormality appeared in a paper by J. S. Rose [21]. A subgroup H of G is *c-normal* in G if there is a normal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G = Core_G(H)$, see for example [22]. In [7], the authors introduced the concept of $\mathcal{H}$-subgroup of a group and proved a number of interesting results about such subgroups. A subgroup H of a group G is called an *$\mathcal{H}$-subgroup* provided that $H^g \cap N_G(H) \leq H$ for all $g \in G$. It is easy to see that the Sylow p-subgroups, normal subgroups and self-normalizing subgroups of an arbitrary group are $\mathcal{H}$-subgroups. Following Müller [18], a subgroup H of a group G is *weakly normal* in G if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ for any element $g \in G$. It is known that every pronormal subgroup and $\mathcal{H}$-subgroup of G are weakly normal in G, but the converse is not true (see [1, p.28] and [6]) for more details. In [17], the authors investigated the behaviour of weakly normal subgroups, and obtained some characterizations about the supersolvability and nilpotency of G by assuming that some subgroups of prime power order of G are weakly normal in G. Recently, the authors in [24] gave some results about formation under the condition that some subgroups of prime square order are weakly normal in G.

It is known that there is no obvious general relationship between the concepts of c-normal subgroup and $\mathcal{H}$-subgroup. For a generalization of both $\mathcal{H}$-subgroup and c-normal subgroup, Assad, et al.[2] introduced the concept of weakly $\mathcal{H}$-subgroup, which describes subgroup embedding properties of a finite group. A subgroup H of a group G is called a weakly $\mathcal{H}$-subgroup in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is a $\mathcal{H}$-subgroup of G. The authors in [2] determined the structure of a finite group G when all maximal subgroups of every Sylow subgroup of certain subgroups of G are weakly $\mathcal{H}$-subgroups in G.

Inspired by the above works, we consider the following question:

How is the structure of a finite group G determined by its subgroup H with the property that there exists a normal subgroup T of G such that $HT = G$ and $H \cap T$ is weakly normal in G?

We first introduce a new notion of s-weakly normal subgroup which is a generalization of c-normal subgroup, $\mathcal{H}$-subgroup, and weakly normal subgroup.

Definition 1.1. *A subgroup H of a group G is an s-weakly normal subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G.*

Clearly a c-normal subgroup, $\mathcal{H}$-subgroup or weakly normal subgroup of G is an s-weakly normal subgroup of G, but the converse is not true. The following two examples will show this.

Example 1.1. Let $G = S_4$ be the symmetric group of degree 4. Suppose that $H = \langle(12)\rangle$ and A_4 is the alternating group of degree 4. Since A_4 is a normal subgroup of G such that $G = HA_4$ and $H \cap A_4 = \{(1)\}$ is weakly normal in G, H is s-weakly normal in G. It is easy to see that $N_G(H) = \{(1), (12), (34), (12)(34)\}$. Let $g = (13)(24) \in G$. Since $H^g = \{(1), (34)\}$ and $H^g \cap N_G(H) = \{(1), (34)\} \nleq H$, H is not an $\mathcal{H}$-subgroup of G. Also note that $H^g \leq N_G(H)$, but $g = (13)(24) \notin N_G(H)$. It follows that H is not weakly normal in G.

Example 1.2. Let $G = S_4$, $H = \{(1), (12), (13), (23), (123), (132)\}$. It is easy to check that $N_G(H) = H$, this means that H is an $\mathcal{H}$-subgroup of G and so it is weakly normal in G. Hence H is s-weakly normal in G. But H is not c-normal in G. In fact, since $H_G = \{(1)\}$, there is no such a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$.

The aim of this paper is to obtain some new characterizations of the nilpotency and supersolvability of finite groups by studying the s-weakly normality properties of some certain primary subgroups. In Section 2, some necessary lemmas are given. In Section 3, some criteria for supersolvability, nilpotency, formation and hypercenter of a finite group are proved, based on these criteria, some recent results can be improved and extended easily. These results show that the concept of s-weakly normal subgroup provides us a useful tool to investigate the structure of finite groups.

2. Preliminaries

In this section we list some basic facts which are needed in this paper.

Lemma 2.1 ([17], Lemma 2.1, Lemma 2.2)**.** *Let N, H and K be subgroups of a finite group G. Then:*

(1) *If H is weakly normal in G, and $H \leq K \leq G$, then H is weakly normal in K.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is weakly normal in G if and only if H/N is weakly normal in G/N.*

(3) *If H is weakly normal in G and $H \trianglelefteq\trianglelefteq K \leq G$, then $H \trianglelefteq K$.*

(4) *If $N \trianglelefteq G$, P is a weakly normal p-subgroup of G such that $(|N|, p) = 1$, then PN is weakly normal in G and $PN/N$is weakly normal in G/N.*

Lemma 2.2. *Let G be a finite group and N, H, K be subgroups of group G.*

(1) *If H is s-weakly normal in G, and $H \leq K \leq G$, then H is s-weakly normal in K.*

(2) *Let $N \leq H$ and $N \trianglelefteq G$. Then H is s-weakly normal in G if and only if H/N is s-weakly normal in G/N.*

(3) *Let N be a normal subgroup of G. If H is a p-subgroup of G such that $(|N|, |H|) = 1$ and H is s-weakly normal in G, then HN/N is s-weakly normal in G/N.*

Proof. (1) Since H is an s-weakly normal subgroup of G, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. It is easy to see that $K = K \cap HT = H(K \cap T)$ and $H \cap (K \cap T) = H \cap T$ is weakly normal in G. It follows by Lemma 2.1(1) that $H \cap (K \cap T) = H \cap T$ is weakly normal in K. Hence H is s-weakly normal in K.

(2) Assume that H is s-weakly normal in G, that is, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Since $N \trianglelefteq G$, it is clear that $G/N = (H/N)(TN/N)$ and $TN/N \trianglelefteq G/N$. Then it follows from Lemma 2.1(2) that $(H/N) \cap (TN/N) = N(H \cap T)/N$ is weakly normal in G/N. Hence H/N is s-weakly normal in G/N.

Conversely, suppose that H/N is s-weakly normal in G/N. Then there exists a normal subgroup T/N of G/N such that $G/N = (H/N)(T/N)$ and $(H/N) \cap (T/N) = (H \cap T)/N$ is weakly normal in G/N. It is easy to see that $G = HT$ and $H \cap T$ is weakly normal in G by Lemma 2.1(2), that is, H is s-weakly normal in G.

(3) Suppose that H is s-weakly normal in G. Then there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Note that $(|N|, |H|) = 1$, we have $N \leq T$, and hence $HN \cap T = N(H \cap T)$. It is easy to see that $HN \cap T = N(H \cap T)$ is weakly normal in G. By Lemma 2.1(4), $N(H \cap T)/N$ is weakly normal in G/N. Note that $G/N = (HN/N)(T/N)$ and $(HN/N) \cap (T/N) = (HN \cap T)/N = N(H \cap T)/N$ is weakly normal in G/N. Hence HN/N is s-weakly normal in G/N. □

Lemma 2.3 ([14], Satz 5.4, p.434)**.** *Let G be a group and $p \in \pi(G)$, If G is a minimal non-p-nilpotent group, that is, G is not nilpotent but all of its proper subgroups are p-nilpotent, then*

(*i*) *$G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G.*

(*ii*) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(*iii*) *If $p > 2$, then $exp(P)$ is p, and when $p = 2$, $exp(P)$ is at most 4, where $exp(P)$ is the exponent of group P.*

Lemma 2.4 ([23], Theorem 6.3, p.221 and Corollary 7.8, p.33)**.** *Let P be a normal p-subgroup of a group G such that $|G/C_G(P)|$ is a power of prime p. Then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.5 ([9], Theorem 6.10, p.390)**.** *If a class of groups $\mathfrak{F}$ is a saturated formation, then $[G^{\mathfrak{F}}, Z_{\mathfrak{F}}(G)] = 1$.*

Lemma 2.6. *Let H be an s-weakly normal subgroup of G and K be a subgroup of G such that $H \leq K$. If $K/\Phi(K)$ is a chief factor of G, then H is weakly normal in G.*

Proof. Since H is s-weakly normal in G, there is a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Note that $K/\Phi(K)$ is a chief factor of G. Thus either $(K \cap T)\Phi(K)/\Phi(K) = 1$ or $(K \cap T)\Phi(K)/\Phi(K) = K/\Phi(K)$. In the former case, since $K \cap T \leq \Phi(K)$, we have $K = K \cap HT = H(K \cap T) = H$. This implies that $H \trianglelefteq G$, and clearly, H is weakly normal in G. In the latter case, we have $(K \cap T)\Phi(K) = K$, and hence $K \cap T = K$ and $G = T$. This also implies that H is weakly normal in G. □

Let G be a finite group. It is known that the Fitting subgroup $F(G)$ of G is the unique maximal normal nilpotent subgroup of G, and the generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. The following results about $F^*(G)$ and $F(G)$are useful in our paper.

Lemma 2.7 ([15], Chapter X 13)**.** *Let G be a group.*

(1) *Suppose that $F^*(G)$ is solvable. Then $F^*(G) = F(G)$;*
(2) *$C(F^*(G)) \leq F(G)$;*
(3) *If N is a normal subgroup of G, then $F^*(N) = N \cap F^*(G)$.*

Lemma 2.8 ([6], Lemma 2)**.** *Let H be a p-subgroup of G. Then the following properties are equivalent:*

(1) *H is a pronormal subgroup of G;*
(2) *H is a weakly normal subgroup of G.*

Lemma 2.9 ([4], Theorem 4.1)**.** *Let p be the smallest prime of $\pi(G)$ and P a Sylow p-subgroup of G. If every subgroup of P of order p or 4 (when $p = 2$) is pronormal in G, then G is p-nilpotent.*

From Lemma 2.8 and Lemma 2.9, we immediately get the following result.

Lemma 2.10. *Let G be a group and p be the smallest prime of $\pi(G)$. If P is a Sylow p-subgroup of G and every subgroup of $P$of order p or 4 (when $p = 2$) is weakly normal in G, then G is p-nilpotent.*

Lemma 2.11 ([16], Lemma 2.8)**.** *Let P be a normal p-subgroup of G contained in $Z_\infty(G)$. Then $O^p(G) \leq C_G(P)$.*

Lemma 2.12 ([10], Lemma 2.4)**.** *Let P be a p-group. If α is a p'-automorphism of P which centralizes $\Omega_1(P)$, then $\alpha = 1$ unless P is a non-abelian 2-group. If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$ without restriction.*

3. Main results

In the sequel, we discuss the influence of s-weakly normal subgroups on the structure of a group.

Theorem 3.1. *Let G be a group with a Sylow p-subgroup P, where p is the smallest prime in $\pi(G)$. Suppose that every subgroup of P of order p or 4 (when $p = 2$) is s-weakly normal in G. Then G is p-nilpotent.*

Proof. Suppose that the required result is not true and let G be a counterexample of minimal order.

Firstly, suppose that p is an odd prime. If every subgroup of P of order p is weakly normal in G, then by Lemma 2.10, G is p-nilpotent, a contradiction. Hence there exists a subgroup P_1 of P such that $|P_1| = p$ and P_1 is not weakly normal in G. By the hypotheses of the theorem, P_1 is s-weakly normal in G, i.e. there is a normal subgroup T of G such that $G = P_1T$ and $P_1 \cap T$ is weakly normal in G. If $P_1 \cap T \neq 1$, then $P_1 \cap T = P_1$ is weakly normal in G, a contradiction. Thus $P_1 \cap T = 1$, and therefore T is a proper subgroup of G. By Lemma 2.2(1), T satisfies the hypotheses of the theorem. Then T is p-nilpotent by the minimal choice of G. Let $T_{p'}$ be a normal p-complement of G. Clearly, $T_{p'}$ *char* G. Note that T is normal in G, and therefore $T_{p'} \trianglelefteq G$. This means that G is p-nilpotent, which is a contradiction.

Now suppose that $p = 2$. Since G is not 2-nilpotent, it follows that G contains a minimal non-2-nilpotent subgroup K. Then K is a minimal non-nilpotent subgroup of G and $K = K_2 \rtimes K_q$, where K_2 is a normal Sylow 2-subgroup of K and K_q is a non-normal Sylow q-subgroup of K, where $q > 2$, and $exp(K_2)$ is at most 4. By using Lemma 2.1(1), we can easily see that the hypothesis is inherited by K. Then by Lemma 2.10, K_2 contains a subgroup L of order 2 or 4 such that L is not weakly normal in K. By the hypotheses of the theorem, L is s-weakly normal in G and thereby L is s-weakly normal in K by Lemma 2.2(1), that is, there is a normal subgroup T of K such that $K = LT$ and $L \cap T$ is weakly normal in K. If $T = K$, then $L \cap T = L$ is weakly normal in K, a contradiction. Thus T is a proper subgroup of K. If $|L| = 2$, then $L \cap T = 1$. Since T is a normal nilpotent subgroup of K, K_q *char* $T \trianglelefteq K$, and hence $K_q \trianglelefteq K$, a contradiction. If $|L| = 4$, then we can always conclude that $K_q \trianglelefteq K$ when $|L \cap T| = 1$ or $|L \cap T| = 2$ since T is a normal nilpotent subgroup of K, a contradiction. This completes the proof of the theorem. □

Theorem 3.2. *Let G be a group with a normal p-subgroup P, where $p \in \pi(G)$. Suppose that every subgroup of P of order p or of order 4 (when $p = 2$) is s-weakly normal in G. Then we have $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. We proceed the proof of the theorem by induction on $|G| + |P|$ and distinguish the following two cases.

Case (1): p is an odd prime.

If every subgroup of P of order p is normal in G, then it is easy to see from [5, Theorem 1.1] that $P \leq Z_{\mathfrak{U}}(G)$. Now we assume that there exists a subgroup H of order p in P such that H is not normal in G. By the hypothesis of the theorem, H is s-weakly normal in G, that is, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Assume that $H \cap T \neq 1$. Then $H \cap T = H$ is weakly normal in G. It is easy to know that H is subnormal in G, then we have that $H \trianglelefteq G$ by Lemma 2.1(3), which is a contradiction. Therefore $H \cap T = 1$. Note that $P \cap T \trianglelefteq G$. By hypothesis of the theorem, every subgroup of $P \cap T$ of order p is s-weakly normal in G. Hence, by induction on $|G| + |P|$, we have that $P \cap T \leq Z_{\mathfrak{U}}(G)$. Since $P = H(P \cap T)$, we have $P/(P \cap T) = H(P \cap T)/(P \cap T)$ is a normal subgroup of $G/(P \cap T)$ of order p. Consequently, $P/(P \cap T) \leq Z_{\mathfrak{U}}(G/(P \cap T))$. Note that $P \cap T \leq Z_{\mathfrak{U}}(G)$, and then by [23, Theorem 7.7, p.32], we have

$$Z_{\mathfrak{U}}(G/(P \cap T)) = Z_{\mathfrak{U}}(G)/(P \cap T).$$

Therefore, $P \leq Z_{\mathfrak{U}}(G)$.

Case (2): $p = 2$.

Let Q be any Sylow q-subgroup of G, where $q \neq 2$. Then it is clear that PQ is a subgroup of G. By Lemma 2.2(1) and Theorem 3.1, PQ is 2-nilpotent, this implies that $PQ = P \times Q$. And then $|G/C_G(P)|$ is a power of 2. By Lemma 2.4, we have $P \leq Z_{\mathfrak{U}}(G)$. □

Theorem 3.3. *Let G be a group and $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$. Then G lies in $\mathfrak{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathfrak{F}$, and every subgroup of H of prime order or of order 4 is s-weakly normal in G.*

Proof. The necessity is obvious, and we only need to prove the sufficiency part. We use induction on the order of group G. By Lemma 2.2(1) and using repeated applications of Theorem 3.1, H has a Sylow tower of supersolvable type. Without loss of generality, let p be the largest prime of $\pi(H)$ and P be the Sylow p-subgroup of H. Clearly P is a characteristic subgroup of H, and note $H \trianglelefteq G$, we have $P \trianglelefteq G$. This implies that $H/P \trianglelefteq G/P$ and $(G/P)/(H/P) \cong G/H \in \mathfrak{F}$. It follows from Lemma 2.2(3) that every subgroup of H/P of prime order or of order 4 is s-weakly normal in G/P. By induction on $|G|$, we have $G/P \in \mathfrak{F}$. It is easy to see from Theorem 3.2 that $P \leq Z_{\mathfrak{U}}(G)$. And by Lemma[9, Propositin 3.11, p.362], $Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{F}}(G)$, consequently, we have $P \leq Z_{\mathfrak{F}}(G)$. Therefore, $G \in \mathfrak{F}$. □

As an immediate consequence of Theorem 3.3, we have the following two corollaries.

Corollary 3.1. *Let G be a group with a normal subgroup E. If G/E is supersolvable and every subgroup of E of prime order or of order 4 is s-weakly normal in G, then G is supersolvable.*

Corollary 3.2. *Suppose that every subgroup of prime order or of order* 4 *is* s*-weakly normal in a group* G*. Then* G *is supersolvable.*

Theorem 3.4. *Let* G *be a group and* $\mathfrak{F}$ *be a saturated formation containing the class of supersolvable groups* $\mathfrak{U}$*. Then* G *lies in* $\mathfrak{F}$ *if and only if there is a normal subgroup* E *of* G *such that* $G/E \in \mathfrak{F}$*, and every subgroup of* $F^*(E)$ *of prime order or of order* 4 *is* s*-weakly normal in* G*.*

Proof. We only need to prove the sufficiency part. We use induction on the order of group G. By Lemma 2.2(1), every subgroup of $F^*(E)$ of prime order or of order 4 is s-weakly normal in $F^*(E)$. It follows from Corollary 3.2 that $F^*(E)$ is supersolvable. By Lemma 2.7(1), we have $F^*(E) = F(E)$, and then $F(E) \leq Z_{\mathfrak{U}}(G)$ by Theorem 3.2. Since $Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{F}}(G)$, we have $F(E) \leq Z_{\mathfrak{F}}(G)$. Hence by Lemma 2.5, we have $G/C_G(F(E)) \in \mathfrak{F}$. This implies that $G/(E \cap C_G(F(E))) = G/C_E(F(E)) \in \mathfrak{F}$. Since $C_E(F(E)) \leq F(E)$ by Lemma 2.7(2) and $F^*(E) = F(E)$, we have

$$G/F(E) \cong (G/C_E(F(E)))/(F(E)/C_E(F(E))) \in \mathfrak{F}.$$

And then it is easy to see that $G \in \mathfrak{F}$ by Theorem 3.3. □

Corollary 3.3. *Let* G *be a group and* $\mathfrak{F}$ *be a saturated formation containing the class of supersolvable groups* $\mathfrak{U}$*. Then* G *lies in* $\mathfrak{F}$ *if and only if there is a solvable normal subgroup* E *such that* $G/E \in \mathfrak{F}$ *and every subgroup of* $F(E)$ *of prime order or of order* 4 *is* s*-weakly normal in* G*.*

In the following part, we characterize the nilpotency of finite groups by the s-weakly normality of some subgroups of prime power order in G.

Theorem 3.5. *Let* G *be a group with a normal subgroup* E *such that* G/E *is nilpotent. If every minimal subgroup of* E *is contained in* $Z_\infty(G)$*, and every cyclic subgroup of* E *of order* 4 *is* s*-weakly normal in* G*, then* G *is nilpotent.*

Proof. Assume that the result is false and let (G, E) be a counterexample such that $|G| + |E|$ is minimal. Then we prove the theorem via the following steps.

(1) G is a minimal non-nilpotent group, that is, $G = P \rtimes Q$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G for some prime $q \neq p$; $P/\Phi(P)$ is a chief factor of G; $exp(P) = p$ when $p > 2$ and $exp(P)$ is at most 4 when $p = 2$.

Let K be any proper subgroup of G. Then $K/(E \cap K) \cong EK/E \leq G/E$ is nilpotent, and every minimal subgroup of $E \cap K$ is contained in $Z_\infty(G) \cap K \leq Z_\infty(K)$. By hypothesis, every cyclic subgroup of $E \cap K$ of order 4 is s-weakly normal in G. Thus by Lemma 2.2(1), every cyclic subgroup of $E \cap K$ of order 4 is s-weakly normal in K. Hence $(K, E \cap K)$ satisfies the hypothesis of the theorem. Then the choice of (G, E) implies that K is nilpotent. Hence G is a minimal non-nilpotent group, and so (1) holds by [14, Chapter III, Satz 5.2].

(2) $P \leq E$.

If $P \nleq E$, then clearly $P \cap E < P$, and so $(P \cap E)Q < G$. By (1), $(P \cap E)Q$ is nilpotent. This implies that $Q \trianglelefteq (P \cap E)Q$. Since $G/(P \cap E) \lesssim G/P \times G/E$ is nilpotent, $(P \cap E)Q \trianglelefteq G$, and thus $Q \trianglelefteq G$, which contradicts to (1).

(3) Final contradiction.

If $exp(P) = p$, then $P \leq Z_\infty(G)$, and so G is nilpotent, which is impossible. Hence we may assume that $p = 2$ and $exp(P) = 4$. Then by Lemma 2.6, every cyclic subgroup of P of order 4 is weakly normal in G, and so every cyclic subgroup of P of order 4 is normal in G by Lemma 2.1(3). Take an element $x \in P \backslash \Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $P = \langle x \rangle^G \Phi(P) = \langle x \rangle^G$. If x is of order 2, then $P = \langle x \rangle^G \leq Z_\infty(G)$, also we have G is nilpotent. a contradiction. Now assume that x is of order 4. Then $\langle x \rangle \trianglelefteq G$, and so $P = \langle x \rangle$ is cyclic. By [20, (10.1.9)], G is 2-nilpotent, and so $Q \trianglelefteq G$, a contradiction. This completes the proof of the theorem. □

Theorem 3.6. *Let G be a group with a normal subgroup E such that G/E is nilpotent. If every minimal subgroup of $F^*(E)$ is contained in $Z_\infty(G)$ and every cyclic subgroup of $F^*(E)$ of order 4 is s-weakly normal in G, then G is nilpotent.*

Proof. Assume that the result is false and let (G, E) be a counterexample such that $|G| + |E|$ is minimal. Then we prove the theorem via the following steps.

(1) Every proper normal subgroup of G is nilpotent.

Let K be any proper normal subgroup of G. Then $K/(E \cap K) \cong EK/E \leq G/E$ is nilpotent. By Lemma 2.7(3), $F^*(E \cap K) = F^*(E) \cap K$. Hence by Lemma 2.2(1), $(K, E \cap K)$ satisfies the hypothesis of the theorem. The choice of (G, E) implies that K is nilpotent.

(2) $E = G = \gamma_\infty(G)$ and $F^*(G) = F(G) < G$, where $\gamma_\infty(G)$ is the nilpotent residual of G.

If E is a proper subgroup of G, then E is nilpotent by (1), and so $F^*(E) = F(E) = E$. By Theorem 3.5, G is nilpotent, a contradiction. Thus $E = G$. Now suppose that $F^*(G) = G$. Then by Theorem 3.5 again, G is nilpotent, which is impossible. Hence $F^*(G) < G$, and $F^*(G) = F(G)$ by (1). If $\gamma_\infty(G) < G$, then by (1), $\gamma_\infty(G) \leq F(G)$, and so $G/F(G)$ is nilpotent. It follows that G is nilpotent, a contradiction. Thus $\gamma_\infty(G) = G$.

(3) Every cyclic subgroup of $F(G)$ of order 4 is contained in $Z(G)$.

By hypothesis and (2), every cyclic subgroup H of $F(G)$ of order 4 is s-weakly normal in G. Then there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. If $T < G$, then $T \leq F(G)$ by (1), and thereby $F(G) = G$, a contradiction. Hence $T = G$, and so H is weakly normal in G. By Lemma 2.1(3), $H \trianglelefteq G$. This implies that $G/C_G(H)$ is abelian. Then by (2), $C_G(H) = \gamma_\infty(G) = G$, and so $H \leq Z(G)$. Thus (3) holds.

(4) Final contradiction.

Let p be any prime divisor of $|F(G)|$ and P be the Sylow p-subgroup of $F(G)$. Then $P \trianglelefteq G$. If p is odd, then by hypothesis, $\Omega_1(P) \leq Z_\infty(G)$. It follows from Lemma 2.11 that $O^p(G) \leq C_G(\Omega_1(P))$, and so $O^p(G) \leq C_G(P)$ by Lemma 2.12. Then by (2), $C_G(P) = \gamma_\infty(G) = G$. Now consider that $p = 2$. Then by hypothesis and (3), $\Omega_2(P) \leq Z_\infty(G)$. A similar discussion as above also shows that $C_G(P) = G$. Therefore, we have $C_G(F(G)) = G$, which contradicts the fact that $C_G(F(G)) \leq F(G)$ by (2) and Lemma 2.7(2). This completes the proof of the theorem. □

Remark 3.1. Note that a c-normal subgroup, $\mathcal{H}$-subgroup and weakly normal subgroup of G is an s-weakly normal subgroup of G, thus some recent results can be generalized and improved by applications of the results given in this paper. For example,[22, Theorem 4.2] and [3, Theorem 3.6] are immediate results of Theorem 3.3; It is easy to obtain [8, Theorem 11]) and [17, Theorem 3.1] by Corollary 3.1; [17, Theorem 3.2] and [17, Corollary 3.4] are immediate result of Theorem 3.4; [17, Theorem 3.5] and [17, Theorem 3.6] are immediate results of Theorem 3.5 and Theorem 3.6, respectively.

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On a maximal subgroup $\overline{G} = 5^4{:}((3{\times}2L_2(25)){:}2_2)$ of the Monster $\mathbb{M}$

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Abstract. The split extension $\overline{G} = 5^4{:}((3 \times 2L_2(25)){:}2_2)$ is a maximal subgroup of the sporadic Monster group $\mathbb{M}$ of order $58500000 = 2^5.3^2.5^6.13$. The technique of Fischer-Clifford matrices has been applied to numerous examples of split and non-split extensions where the kernels are either elementary abelian 2 or 3-groups but very few examples exist where the kernel is an elementary abelian 5-group. In this paper, the Fischer-Clifford matrices technique is applied to the group $\overline{G} = 5^4{:}((3 \times 2L_2(25)){:}2_2)$, where the kernel 5^4 of the extension is an elementary abelian 5-group.

Keywords: coset analysis, Fischer-Clifford matrices, split extension, inertia factor, character table, fusion map, restriction of characters.

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1. Introduction

The sporadic Monster group $\mathbb{M}$ has a conjugacy class of maximal 5-local subgroups of the form $5^4{:}((3 \times 2L_2(25)){:}2_2)$ [6]. Obtaining a permutation representation on 625 points for $\overline{G}{=}5^4{:}((3{\times}2L_2(25)){:}2_2)$ from the online ATLAS [23], the group $\overline{G}$ is generated by using the algebra computational system MAGMA [5]. The normal subgroup $N = 5^4$ and subgroup $G = (3 \times 2L_2(25)){:}2_2 \cong SL_2(25){:}S_3$ of $\overline{G}$ are constructed by MAGMA as permutation groups on 625 points. Using the MAGMA commands, "M:=GModule($\overline{G}, N$); and "M:Maximal;", the group $G =< g_1, g_2 >$ is constructed as a matrix group of degree 4 over $GF(5)$ with generators g_1 and g_2 such that $o(g_1) = 2$, $o(g_2) = 39$ and $o(g_1g_2) = 8$ (see, Figure 1).

$$g_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}, \; g_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 4 & 2 & 0 \end{pmatrix}$$

Figure 1: Generators of G

Considering $N = V_4(5)$ as the vector space of dimension 4 over $GF(5)$, on which the matrix group $G =< g_1, g_2 >$ acts absolutely irreducibly, it was found with aid of GAP [8] that G has two orbits on N of lengths 1 and 624 with corresponding point stabilizers $P_1 = G$ and $P_2 = 5^2{:}S_3$. By Brauer's theorem (see Theorem 5.1.5 in [12]), the action of G on Irr(N) also has two orbits of lengths 1 and 624 with corresponding inertia factor groups $H_1 = G$ and $H_2 = 5^2{:}S_3$. It is worth noting that the vector space N and its dual space $N^* =$Irr(N) are isomorphic as 4-dimensional modules over $GF(5)$ for G. Having obtained G as a 4-dimensional matrix group over the finite field $GF(5)$ and treating N as the vector space $V_4(5)$ we can apply Fischer-Clifford theory (see, for example, [7] and [14]) to the split extension $\overline{G}$ to construct its ordinary character table. The Fischer-Clifford matrices technique is powerful if the kernel of a suitable split extension group is elementary abelian as it is the case with the group $\overline{G}$. A GAP routine found in [21] which is based on coset analysis technique found in [11] and [14] is used to compute the conjugacy classes of $\overline{G}$. This method is very efficient when the kernel of a split extension is an elementary-abelian p-group. The importance of computing conjugacy classes of $\overline{G}$ from a coset Ng is that the centralizer orders of these classes play a role in the computation of the entries of a Fischer-Clifford matrix $M(g)$, where g is a conjugacy class representative of G. In the paper [10], Fischer-Clifford technique was applied to a non-split extension $\overline{G_1} = 5^{3\cdot}L_3(5)$, which is a maximal subgroup of the Lyons sporadic simple group $\mathbb{L}y$. Besides our group $\overline{G}$, $\overline{G_1}$ is one of the few extension groups in the literature with the kernel being an elementary abelian 5-group, where the method of Fischer-Clifford matrices has been applied to.

In the sections that follow, an outline of the Fischer-Clifford matrices technique is going to be given. The conjugacy classes and Fischer-Clifford matrices of $\overline{G}$ are also computed using appropriate GAP routines. In addition, the ordinary character table of $\overline{G}$ is constructed and the fusion of conjugacy classes of $\overline{G}$ into those of the Monster $\mathbb{M}$ is determined. For an update on recent developments around Fischer-Clifford matrices, interested readers are referred to the papers [1], [2], [15] [16], [17], [18] and [19]. Most of the computations in this paper are carried out with computer algebra systems MAGMA and GAP. Notation from the ATLAS [6] is mostly followed.

2. Theory of Fischer-Clifford matrices

Since the ordinary character table of $\overline{G} = 5^4{:}((3 \times 2L_2(25)){:}2_2)$ will be constructed by the technique of Fischer-Clifford matrices, an outline of this technique is given for a split extension $\overline{G} = N{:}G$, where N is an elementary abelian p-group, see for example, [14] or [22].

Let $\overline{G} = N{:}G$ be a split extension of N by G, where N is an elementary abelian p-group. The subgroup $\overline{H} = N{:}H = \left\{x \in \overline{G} | \theta^x = \theta\right\}$ of $\overline{G}$ is defined as the inertia group of $\theta \in \mathrm{Irr}(N)$ in $\overline{G}$, with inertia factor $H = \overline{H}/N$. Note that a lifting $\overline{g} \in \overline{G}$ of $g \in G$ into $\overline{G}$ under the natural homomorphism $\eta{:}\overline{G} \longrightarrow G$ is just g itself, since $G \leq \overline{G}$. Let $X(g) = \{x_1, x_2, \cdots, x_{c(g)}\}$ be a set of representatives of the conjugacy classes of $\overline{G}$ from the coset Ng whose images under the natural homomorphism η are in the conjugacy class $[g]$ of G where $x_1 = g$. Now let $\theta_1 = 1_N, \theta_2, \cdots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $\mathrm{Irr}(N)$. Since N is elementary abelian, we have by Mackey's Theorem (see Theorem 5.1.15 in [12]) that each θ_i, $1 \leq i \leq t$, extends to a $\psi_i \in \mathrm{Irr}(\overline{H_i})$, i.e. $\psi_i \downarrow_N = \theta_i$. By Theorem 5.1.7, Remark 5.1.8 and Theorem 5.1.19 in [12], an ordinary irreducible character $\chi = (\psi_i\overline{\beta})^{\overline{G}}$ of $\overline{G}$ consists of $\psi_i\overline{\beta} \in \mathrm{Irr}(\overline{H_i})$ which is induced to $\overline{G}$, where N is contained in the kernel $\ker(\overline{\beta})$ of a lifting $\overline{\beta} \in \mathrm{Irr}(\overline{H_i})$ of $\beta \in \mathrm{Irr}(H_i)$ into $\overline{H_i}$. Therefore,

$$\mathrm{Irr}(\overline{G}) = \bigcup_{i=1}^{t}\{(\psi_i\overline{\beta})^{\overline{G}} | \overline{\beta} \in \mathrm{Irr}(\overline{H_i}), N \subseteq \ker(\overline{\beta})\} = \bigcup_{i=1}^{t}\{(\psi_i\overline{\beta})^{\overline{G}} | \beta \in \mathrm{Irr}(H_i)\}.$$

Hence, the set $\mathrm{Irr}(\overline{G})$ are partitioned into t blocks B_i with each block B_i corresponding to an inertia subgroup $\overline{H_i}$ of $\overline{G}$. Observe that $|\mathrm{Irr}(\overline{G})| = |\mathrm{Irr}(H_1)| + ... + |\mathrm{Irr}(H_t)|$.

We take $\overline{H_1} = \overline{G}$ and $H_1 = G$. Choose $y_1, y_2, .., y_r$ to be representatives of the conjugacy classes $[y_k]$, $k = 1, ..., r$, of H_i that fuse to $[g]$ in G. We define $R(g) = \{(i, y_k)\,|\, 1 \leq i \leq t, H_i \cap [g] \neq \emptyset, 1 \leq k \leq r\}$ and we observe that y_k runs over representatives of the conjugacy classes $[y_k]$ of H_i which fuse into $[g]$ of G. We define $y_{l_k} \in \overline{H_i}$ such that y_{l_k} ranges over all representatives of the conjugacy classes of $\overline{H}_i$ which map to y_k under the homomorphism $\overline{H_i} \longrightarrow H_i$ whose kernel is N.

Lemma 2.1. *With notation as above,*

$$(\psi_i\overline{\beta})^{\overline{G}}(x_j) = \sum_{y_k:(i,y_k)\in R(g)} \left[{\sum_l}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k}) \right] \beta(y_k).$$

Proof. See [22]. $\square$

Then, the Fischer-Clifford matrix $M(g) = (a^j_{(i,y_k)})$ is defined as $(a^j_{(i,y_k)}) = ({\sum_l}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k}))$, with columns indexed by $X(g)$ and rows indexed by $R(g)$ and where ${\sum_l}'$ is the summation over all l for which $y_{l_k} \sim x_j$ in $\overline{G}$. So, we can write Lemma 2.1 as

$$(\psi_i\overline{\beta})^{\overline{G}}(x_j) = \sum_{y_k:(i,y_k)\in R(g)} a^j_{(i,y_k)}\beta(y_k).$$

The Fischer-Clifford $M(g)$ (see, Figure 2) is partitioned row-wise into blocks $M_i(g)$, where each block corresponds to an inertia group $\overline{H}_i$. We write $|C_{\overline{G}}(x_j)|$, for each $x_j \in X(g)$, at the top of the columns of $M(g)$ and at the bottom we write $m_j \in \mathbb{N}$, where we define $m_j = |N|\frac{|C_G(g)|}{|C_{\overline{G}}(x_j)|}$. On the left of each row we write $|C_{H_i}(y_k)|$, where the conjugacy classes $[y_k]$, $k = 1, 2, ..., r$, of an inertia factor H_i fuse into the conjugacy class $[g]$ of G.

$$M(g)=\begin{array}{c|cccc|}
 & |C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \cdots & |C_{\overline{G}}(x_{c(g)})| \\
\hline
|C_G(g)| & a^1_{(1,g)} & a^2_{(1,g)} & \cdots & a^{c(g)}_{(1,g)} \\
\hline
|C_{H_2}(y_1)| & a^1_{(2,y_1)} & a^2_{(2,y_1)} & \cdots & a^{c(g)}_{(2,y_1)} \\
|C_{H_2}(y_2)| & a^1_{(2,y_2)} & a^2_{(2,y_2)} & \cdots & a^{c(g)}_{(2,y_2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
|C_{H_i}(y_1)| & a^1_{(i,y_1)} & a^2_{(i,y_1)} & \cdots & a^{c(g)}_{(i,y_1)} \\
|C_{H_i}(y_2)| & a^1_{(i,y_2)} & a^2_{(i,y_2)} & \cdots & a^{c(g)}_{(i,y_2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
|C_{H_t}(y_1)| & a^1_{(t,y_1)} & a^2_{(t,y_1)} & \cdots & a^{c(g)}_{(t,y_1)} \\
|C_{H_t}(y_2)| & a^1_{(t,y_2)} & a^2_{(t,y_2)} & \cdots & a^{c(g)}_{(t,y_2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
 & m_1 & m_2 & \cdots & m_{c(g)}
\end{array}$$

Figure 2: The Fischer-Clifford Matrix $M(g)$

In practice it is difficult to compute the elements y_{l_k} or the ordinary irreducible character tables of the inertia groups $\overline{H}_i$, since the sets $\mathrm{Irr}(\overline{H}_i)$ of ordinary irreducible characters of the $\overline{H}_i$'s are in general much larger and more complicated to compute than the one for $\overline{G}$. Instead of using the above formal definition of a Fischer-Clifford matrix $M(g)$, the arithmetical properties of $M(g)$ found in [14] are used to compute the entries of $M(g)$. The matrix $M(g)$ is square where the number of rows is equal to the number of conjugacy classes of the inertia factors H_i's, $1 \leq i \leq t$, which fuse into the class $[g]$ in G and the number of columns is equal to the number $c(g)$ of conjugacy classes of $\overline{G}$ which is obtained from the coset $N\overline{g}$. Then, the partial character table of $\overline{G}$ on the classes $\{x_1, x_2, \cdots, x_{c(g)}\}$ is given by

$$\begin{bmatrix} C_1(g)\, M_1(g) \\ C_2(g)\, M_2(g) \\ \vdots \\ C_t(g)\, M_t(g) \end{bmatrix}$$

with each block $M_i(g)$ of $M(g)$ (see Figure 2) corresponding to an inertia group $\overline{H}_i$ and $C_i(g)$ consists of the columns of the ordinary character table of H_i which correspond to the conjugacy classes of H_i that fuse into the class $[g]$ of G. We obtain the characters of $\overline{G}$ by multiplying the relevant columns of the ordinary irreducible characters of H_i by the rows of $M(g)$.

3. The conjugacy classes of $\overline{G}$

In this section, a GAP routine (labelled as Programme A in [21]), which is based on the method of coset analysis (see [11], [13] or [14]), is used to compute the conjugacy classes of $\overline{G}$. This GAP routine is written for a split extension $S = p^n{:}Q$ of an elementary abelian p-group p^n by a linear matrix group Q of dimension n over the field $GF(p)$. The group p^n (regarded as a vector space $V_n(p)$ of degree n over the finite field $GF(p)$ (p is a prime)) is a Q-module where upon the matrix group Q acts naturally. A coset p^nq is considered for each conjugacy class $[q]$ representative q in Q and then consider the action of the stabilizer $C_g = p^n{:}C_Q(q) = \{x \in S | x(p^nq)x^{-1} = p^nq\}$ of the coset p^nq in S by conjugation on the elements of p^nq. Since C_g is split extension we will first act p^n on p^nq to form k orbits $Q_1, Q_2, ..., Q_k$, with each orbit Q_i containing $|p^n|/k$ elements. Under the action of the centralizer $C_Q(q)$ of q in Q, f_j of the k orbits Q_i fuse together to form an orbit O_j. The orbit O_j contains the elements from the coset p^nq which belong to a conjugacy class $[x_j]$ of S with class representative x_j. Note that $\sum f_j = k$. The order of the centralizer $|C_S(x_j)|$ of the class representative x_j is then computed by $|C_S(x_j)| = \frac{k|C_Q(q)|}{f_j}$. In this manner, the conjugacy classes of S, with class representatives $X(q) = \{x_1, x_2, ..., x_{c(q)}\}$ (see Section 2) coming from the coset p^nq, are obtained.

Using similar techniques as in [14], the permutation character $\chi(G|5^4)$ of $G = (3 \times 2L_2(25)){:}2_2$ on the conjugacy classes of $N = 5^4$ is computed as

$$\chi(G|5^4) = \sum_{i=1}^{2} I_{P_i}^{G} = 1aa + 13cd + 25b + 26dd + 52bbdejklmno.$$

Note that $\chi(G|5^4)$ is the sum of the identity characters $I_{P_i}^{G}$, $i = 1, 2$, of the point stabilizers P_i of the orbits of G on N, which are induced to G. Also, $\chi(G|5^4)$ is written in terms of the ordinary irreducible characters of G. For an element g in a conjugacy class $[g]$ of G, it is required that $\chi(G|5^4)(g) = 5^n$, for some $n \in \{0, 1, 2, 3, 4\}$. The value $\chi(G|5^4)(g)$ gives the number of elements of N which is fixed by an element $g \in G$ and it is also the number of orbits of N on a coset Ng.

In Section 1, the group $G = (3 \times 2L_2(25)){:}2_2 =< g_1, g_2 >$ was computed as a 4-dimensional matrix group over the field $GF(5)$ and with $N = 5^4$ represented as a vector space $V_4(5)$ of dimension 4 over $GF(5)$, we now proceed to compute the conjugacy classes for $\overline{G}$ as described above. The permutation character $\chi(G|5^4)$ is evaluated on each class representative $g \in G$ to determine the number $k = \chi(G|5^4)(g)$ of orbits of N on Ng. Programme A in [21] written in GAP is then used to calculate the number f_j of these k orbits which come together as an orbit O_j under the action of $C_G(g)$. With the values of k and the f_j's obtained, the order of the centralizer $|C_{\overline{G}}(d_j g)| = \frac{k|C_G(g)|}{f_j}$ of a class representative $d_j g \in O_j$, where $d_j \in N$ and $g \in G$, is computed (see Table 1). Altogether 70 conjugacy classes are obtained for $\overline{G}$. Using the GAP routine, Programme B in [21], which is based on Theorem 2.7 and Remark 2.8 in [14], the order $o(d_j g)$ of a representative $d_j g$ in the orbit O_j, is computed. Let $(d_j g)^{o(g)} = w \in N$. If $w = 1_N$, then $o(d_j g) = o(g)$. Otherwise for $w \neq 1_N$ we have $o(d_j g) = 5o(g)$, since N is an elementary abelian 5-group. Hence the order for each class representative $d_j g$ in a conjugacy class $[d_j g]$ of $\overline{G}$ coming from a coset Ng is determined and is found in Table 1. From Programme A and Programme B in [21] the p-power maps, p a prime, are computed for the elements in each conjugacy class $[d_j g]$ of $\overline{G}$ and are listed in Table 1. The values of the parameter, $m_j = \frac{f_j|N|}{k}$, which are useful in determining the entries of a Fischer-Clifford matrix $M(g)$ are also listed in Table 1. We identify $d_j g$ with x_j used in Section 2 and in the beginning of Section 3.

Table 1: The Conjugacy Classes of $\bar{G}$

$[g]_G$	k	f_j	m_j	d_j	w	$[d_jg]_{\overline{G}}$	$\lvert C_{\overline{G}}(d_jg)\rvert$	2	3	5	13	$\mapsto \mathbb{M}$
$1A$	625	1	1	$(0,0,0,0)$	$(0,0,0,0)$	$1A$	58500000					$1A$
		624	624	$(0,0,0,1)$	$(0,0,0,1)$	$5A$	93750			$1A$		$5B$
$2A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$2A$	93600	$1A$				$2B$
$2B$	25	1	25	$(0,0,0,0)$	$(0,0,0,0)$	$2B$	6000	$1A$				$2B$
		24	600	$(0,0,0,1)$	$(1,2,3,2)$	$10A$	250	$5A$		$2B$		$10E$
$3A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$3A$	46800		$1A$			$3B$
$3B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$3B$	144		$1A$			$3C$
$3C$	25	1	25	$(0,0,0,0)$	$(0,0,0,0)$	$3C$	1800		$1A$			$3C$
		24	600	$(0,1,2,2)$	$(1,2,4,4)$	$15A$	75		$5A$	$3C$		$15D$
$4A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$4A$	144	$2A$				$4D$
$4B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$4B$	240	$2A$				$4D$
$5A$	25	1	25	$(0,0,0,0)$	$(0,0,0,0)$	$5B$	7500			$1A$		$5B$
		12	300	$(0,0,2,4)$	$(0,0,0,0)$	$5C$	625			$1A$		$5B$
		12	300	$(0,1,0,0)$	$(0,0,0,0)$	$5D$	625			$1A$		$5B$
$5B$	25	1	25	$(0,0,0,0)$	$(0,0,0,0)$	$5E$	7500			$1A$		$5A$
		6	150	$(0,0,0,4)$	$(0,0,0,0)$	$5F$	1250			$1A$		$5B$
		6	150	$(0,0,0,2)$	$(0,0,0,0)$	$5G$	1250			$1A$		$5B$
		6	150	$(0,1,0,0)$	$(0,0,0,0)$	$5H$	1250			$1A$		$5B$
		6	150	$(0,1,0,3)$	$(0,0,0,0)$	$5I$	1250			$1A$		$5B$
$6A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$6A$	46800	$3A$	$2A$			$6B$
$6B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$6B$	72	$3B$	$2A$			$6F$
$6C$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$6C$	144	$3C$	$2A$			$6F$
$6D$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$6D$	12	$3B$	$2B$			$6F$
$8A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$8A$	72	$4B$				$8F$
$8B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$8B$	8	$4B$				$8F$
$10A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$10B$	300	$5B$		$2A$		$10D$
$10B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$10C$	300	$5E$		$2A$		$10B$
$10C$	5	1	125	$(0,0,0,0)$	$(0,0,0,0)$	$10D$	100	$5E$		$2B$		$10B$
		2	250	$(0,0,0,1)$	$(0,0,0,0)$	$10E$	50	$5H$		$2B$		$10E$
		2	250	$(0,0,0,2)$	$(0,0,0,0)$	$10F$	50	$5I$		$2B$		$10E$
$10D$	5	1	125	$(0,0,0,0)$	$(0,0,0,0)$	$10G$	100	$5E$		$2B$		$10B$
		2	250	$(0,0,2,2)$	$(0,0,0,0)$	$10H$	50	$5G$		$2B$		$10E$
		2	250	$(0,0,0,3)$	$(0,0,0,0)$	$10I$	50	$5F$		$2B$		$10E$
$12A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$12A$	72	$6A$	$4A$			$12J$
$12B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$12B$	72	$6A$	$4B$			$12F$
$12C$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$12C$	72	$6C$	$4B$			$12J$
$12D$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$12D$	72	$6C$	$4B$			$12J$
$12E$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$12E$	12	$6B$	$4B$			$12J$
$13A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$13A$	78				$1A$	$13B$
$13B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$13B$	78				$1A$	$13B$
$13C$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$13C$	78				$1A$	$13B$
$15A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$15B$	150		$5B$	$3A$		$15C$
$15B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$15C$	150		$5E$	$3A$		$15B$
$20A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$20A$	20	$10B$		$4A$		$20E$
$20B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$20B$	20	$10B$		$4A$		$20E$

Table 1. The Conjugacy Classes of $\bar{G}$ (continued)

$[g]_G$	k	f_j	m_j	d_j	w	$[d_jg]_{\overline{G}}$	$\lvert C_{\overline{G}}(d_jg)\rvert$	2	3	5	13	$\mapsto \mathbb{M}$
$24A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24A$	72	$12B$	$8B$			$12F$
$24B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24B$	72	$12B$	$8B$			$24J$
$24C$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24C$	72	$12D$	$8B$			$24J$
$24D$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24D$	72	$12D$	$8B$			$24J$
$24E$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24E$	72	$12C$	$8B$			$24J$
$24F$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24F$	72	$12C$	$8B$			$24J$
$24G$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24G$	72	$12E$	$8B$			$24J$
$24H$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$24H$	72	$12E$	$8B$			$24J$
$26A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$26A$	78	$13C$			$2A$	$26B$
$26B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$26B$	78	$13A$			$2A$	$26B$
$26C$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$26C$	78	$13B$			$2A$	$26B$
$30A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$30A$	150	$15A$	$10B$	$6A$		$30A$
$30B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$30B$	150	$15B$	$10C$	$6A$		$30D$
$39A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$39A$	78		$13C$		$3A$	$39C$
$39B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$39B$	78		$13C$		$3A$	$39D$
$39C$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$39C$	78		$13A$		$3A$	$39C$
$39D$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$39D$	78		$13A$		$3A$	$39D$
$39E$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$39E$	78		$13B$		$3A$	$39C$
$39F$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$39F$	78		$13B$		$3A$	$39D$
$78A$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$78A$	78	$39E$	$26C$		$6A$	$78B$
$78B$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$78B$	78	$39F$	$26C$		$6A$	$78C$
$78C$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$78C$	78	$39A$	$26A$		$6A$	$78B$
$78D$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$78D$	78	$39B$	$26A$		$6A$	$78C$
$78E$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$78E$	78	$39C$	$26B$		$6A$	$78B$
$78F$	1	1	625	$(0,0,0,0)$	$(0,0,0,0)$	$78F$	78	$39D$	$26B$		$6A$	$78C$

4. Inertia factor groups of $\overline{G}$

We have already seen in the Introduction of this paper, that the orbit stabilizers (the so-called inertia factors) of the action of G on $\mathrm{Irr}(N)$ are two groups of the form $H_1 = G$ and $H_2 = 5^2{:}S_3$. The inertia factor $H_2 =<\alpha_1, \alpha_2>$ is generated from elements $\alpha_1 \in 2B$ and $\alpha_2 \in 10C$ (see Figure 3) in the conjugacy classes $2B$ and $10C$ of G.

The fusion maps of the conjugacy classes of H_2 into G are shown in Table 2 and will be used in the construction of the Fischer-Clifford matrices and ordinary character table of $\overline{G}$.

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 2 & 4 \\ 2 & 0 & 4 & 0 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

Figure 3: Generators of H_2

Table 2: The fusion of H_2 into G

$[h]_{H_2}$	$\longrightarrow [g]_{(3\times 2L_2(25)):2_2}$	$[h]_{H_2}$	$\longrightarrow [g]_{(3\times 2L_2(25)):2_2}$
$1A$	$1A$	$5E$	$5B$
$2A$	$2B$	$5F$	$5B$
$3A$	$3C$	$10A$	$10C$
$5A$	$5A$	$10B$	$10D$
$5B$	$5A$	$10C$	$10C$
$5C$	$5B$	$10D$	$10D$
$5D$	$5B$		

5. The Fischer-matrices of $\overline{G}$

In this section, the Fischer-Clifford matrices of the group $\overline{G}$ are going to be obtained by using a GAP routine, Programme D in [3] and [4]. This routine gives a possible candidate for a Fischer-Clifford matrix $M(g)$ and then the properties of Fischer-Clifford matrices (see [2], [14]) are used to rearrange the rows and columns in order to get the unique matrix $M(g)$ corresponding to a class representative $g \in G$. A brief outline of the theory behind the development of Programme D, as found in [9] and [14], is given first.

We restrict our discussion to a split extension $S = p^n{:}Q$, with p^n an elementary abelian p-group. For a class representative $q \in Q$, it can be shown that the map $\phi_q{:}p^n \longrightarrow p^n$, defined by $\phi_q(\overline{n}) = \overline{n}q\overline{n}^{-1}q^{-1}$, is an endomorphism of p^n. The image $\mathbb{I} = \mathrm{Im}(\phi_q)$ and kernel $\ker(\phi_q)$ are C_q-sub-modules of p^n, where $C_q = p^n{:}C_Q(q)$ is the stabilizer of the coset $p^n q$. The actions of p^n by conjugation on $p^n q$ and that of $\mathbb{I}$ by left multiplication result in the same number k of orbits. It follows that the action of C_q on the k orbits of p^n on $p^n q$ is the same as the action of C_q on the module $p^n/\mathbb{I} \cong ker(\phi_q)$. Therefore, we can identify the k orbits of the action of $\mathbb{I}$ on $p^n q$ with the k elements of $p^n/\mathbb{I}$. Since p^n is an elementary abelian p-group, $\mathbb{I}$ and $\ker(\phi_q)$ are also elementary abelian p-groups and it follows that the index of $\mathbb{I}$ in p^n is $[p^n{:}\mathbb{I}] = k$. Instead of acting C_q on the k orbits, the centralizer $C_Q(q)$ of q in Q is used. With the above discussion and notation and more details in [9], the following theorem is formulated.

Theorem 5.1. *A Fischer-Clifford matrix $M(q)$ of a split extension $S = p^n{:}Q$, corresponding to a class representative $q \in Q$, is a matrix of orbit sums of*

C_q acting on the rows of the ordinary character table of $p^n/\mathbb{I}$ with duplicating columns discarded.

Corollary 5.1. *If $q = 1_Q$, then $\mathbb{I} = Im(\phi_q) = 1_{p^n}$ and the Fischer-Clifford matrix $M(1_Q)$ is the matrix of orbit sums of $C_q = S$ acting on the rows of the ordinary character table of $p^n/\mathbb{I} = p^n$ with duplicating columns discarded.*

The following GAP routine, which is based on the above theoretical discussion, is taken from Programme D in [3] and can compute a candidate FM for a Fischer-Clifford matrix $M(q)$ of $S = p^n{:}Q$.

```
C:=List(ConjugacyClasses(G),Representative);; M:=[];;
g:=C[i];; for n in N do
Add(M, n*g*Inverse(n)*Inverse(g));; od;
M:=AsGroup(M);; cent:=Centralizer(G, g);
I:=Irr(N);; IM:=[];; for i in [1..Size(I)] do
if IsSubgroup(Kernel(I[i]), M) then Add(IM,I[i]);
fi; od; oo:=Orbits(cent,IM);; FM:=[];;
for i in [1..Size(oo)] do
Append(FM,[AsList(Sum(oo[i]))]);od;
M1:=TransposedMat(FM);;
M2:=AsDuplicateFreeList(M1);;
FM:=TransposedMat(M2);; Display(FM)
```

As an example, consider the conjugacy class $5B$ of G. By making use of Theorem 5.2.4 and property (e) in [12], $M(5B)$ has the following form with corresponding weights attached to the rows and columns,

$$
\begin{array}{ll}
 & \\
 & \\
|C_{H_1}(5B)| = & 300 \\
|C_{H_2}(5C)| = & 50 \\
|C_{H_2}(5D)| = & 50 \\
|C_{H_2}(5E)| = & 50 \\
|C_{H_2}(5F)| = & 50 \\
 & m_j
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
|C_{\overline{G}}(5E)| & |C_{\overline{G}}(5F)| & |C_{\overline{G}}(5G)| & |C_{\overline{G}}(5H)| & |C_{\overline{G}}(5I)| \\
7500 & 1250 & 1250 & 1250 & 1250
\end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
6 & g & h & i & j \\
6 & l & m & n & o \\
6 & q & r & s & t \\
6 & v & w & x & y
\end{pmatrix} \\
\begin{array}{ccccc}
25 & 150 & 150 & 150 & 150
\end{array}
\end{array}.
$$

To determine the unknown entries $M(5B)$, the above GAP routine gives the candidate FM,

$$M(5B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & A^* & A & B^* & B \\ 6 & A & A^* & B & B^* \\ 6 & B^* & B & A & A^* \\ 6 & B & B^* & A^* & A \end{pmatrix}$$
where $A = 1 - \sqrt{5}$ and $B = (-3 - \sqrt{5})/2$.

From the p-power maps of $\overline{G}$ in Table 1, we have that $(10I)^2 = 5F$, $(10H)^2 = 5G$, $(10E)^2 = 5H$ and $(10F)^2 = 5I$. Thus, for any $\chi \in \mathrm{Irr}(\overline{G})$, the congruent relations $\chi(5F) \equiv \chi(10I) \pmod 2$, $\chi(5G) \equiv \chi(10H) \pmod 2$, $\chi(5H) \equiv \chi(10E) \pmod 2$ and $\chi(5I) \equiv \chi(10F) \pmod 2$ must be satisfied. Checking the validity of these relations for the parts of the ordinary character tables of $\overline{G}$ corresponding to $M(10C)$, $M(10D)$ and the candidate FM for $M(5B)$, the rows of FM are rearranged to find the desired Fischer-Clifford matrix $M(5B)$ of $\overline{G}$ (see Figure 4).

$$M(5B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & A^* & A & B^* & B \\ 6 & B & B^* & A^* & A \\ 6 & A & A^* & B & B^* \\ 6 & B^* & B & A & A^* \end{pmatrix}$$

Figure 4: Fischer-Clifford matrix $M(5B)$

Only the Fischer-matrices $M(5B)$, $M(10C)$ and $M(10D)$ were computed with the aid of the above GAP routine. The rest of the Fischer-Clifford matrices of $\overline{G}$ were computed manually. The above GAP routine comes in very handy when some entries of the Fischer-Clifford matrices are algebraic integers which are not integers. If there are considerately many inertia factors H_i for the action of a split extension $S = p^n{:}Q$ on $\mathrm{Irr}(p^n)$, the Fischer-Clifford matrices can become very large. Consequently, to compute the desired Fischer-Clifford matrices of S, it is necessary also to use other techniques such as restriction of ordinary characters of the parent group of S to the ordinary irreducible characters of S together with the GAP routine. However, when the group S becomes too large, the computational power to use the GAP routine becomes difficult. We have then to resort to other methods, if possible, to compute the Fischer-Clifford matrices. The Fischer-Clifford matrices of $\overline{G}$ have sizes ranging from 1 to 5 and are contained in Table 3.

Table 3: The Fischer-Clifford Matrices of $\overline{G}$

$M(g)$	$M(g)$
$M(1A) = \begin{pmatrix} 1 & 1 \\ 624 & -1 \end{pmatrix}$	$M(2B) = \begin{pmatrix} 1 & 1 \\ 24 & -1 \end{pmatrix}$
$M(3C) = \begin{pmatrix} 1 & 1 \\ 24 & -1 \end{pmatrix}$	$M(5A) = \begin{pmatrix} 1 & 1 & 1 \\ 12 & -3 & 2 \\ 12 & 2 & -3 \end{pmatrix}$
$M(5B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & A^* & A & B^* & B \\ 6 & B & B^* & A^* & A \\ 6 & A & A^* & B & B^* \\ 6 & B^* & B & A & A^* \end{pmatrix}$	$M(10C) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & C & C^* \\ 2 & C^* & C \end{pmatrix}$
$M(10D) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & C & C^* \\ 2 & C^* & C \end{pmatrix}$	$M(g_i) = (1), \forall g_i \notin \{1A, 2B, 3C, 5A, 5B, 10C, 10D\}$

where $A = 1 - \sqrt{5}$, $B = (-3 - \sqrt{5})/2$, $C = (-1 - \sqrt{5})/2$

6. The character table of $\overline{G}$ and fusion into the Monster $\mathbb{M}$

With all the necessary information obtained in the previous sections, the ordinary character table of $\overline{G}$ can now be constructed by the technique of Fischer-Clifford matrices as discussed in Section 2. The character table (see Table 4) is a 70×70 $\mathbb{C}-$valued matrix partitioned row-wise into two blocks $\triangle_1 = \{\chi_i | 1 \leq i \leq 57\}$ and $\triangle_2 = \{\chi_i | 58 \leq i \leq 70\}$, where $\chi_i \in \mathrm{Irr}(\overline{G}) = \cup_{i=1}^{2} \triangle_i$. Note that each block corresponds to an inertia group $\overline{H}_i = 5^4{:}H_i$. Checks for consistency and accuracy of the character table obtained have been carried out with the GAP routine, Programme C [20].

Unique p-power maps for the elements of $\overline{G}$ are obtained for our Table 4 using Programme C, which coincide with the p-power maps in Table 1. Using the power maps of $\overline{G}$ and $\mathbb{M}$, the permutation character $\chi(\mathbb{M}|\overline{G})$ of $\mathbb{M}$ on the classes of $\overline{G}$ which was computed directly by GAP, we obtained partial fusion from the classes of $\overline{G}$ into $\mathbb{M}$. To complete the fusion map from $\overline{G}$ to $\mathbb{M}$, the technique of set intersections [14] was used to restrict ordinary irreducible characters of $\mathbb{M}$ of small degrees to $\overline{G}$. For example, the character $196883a \in \mathrm{Irr}(\mathbb{M})$ will restrict to $\overline{G}$ as $(196883a)_{\overline{G}} = 13c + 24a + 26cef + 52acjk + 624a + 4(624b) + 5(1248a) + 5(1872a) + 7(1872b) + 5(1872c) + 7(1872d) + 5(1872e) + 7(1872f) + 5(1872g) + 7(1872h) + 13(3744a) + 13(3744b)$. The fusion map of the classes of $\overline{G}$ into the classes of $\mathbb{M}$ is found in the last column of Table 1.

Table 4: The Character Table of $\overline{G}$

$[g]_G$	1A		2A	2B		3A	3B	3C		4A	4B	5A			5B				
$[x]_{\overline{G}}$	1A	5A	2A	2B	10A	3A	3B	3C	15A	4A	4B	5B	5C	5D	5E	5F	5G	5H	5I
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	1	1	1	1	-1	1	1	1	1	1	1	1	1	1
χ_3	2	2	2	0	0	-1	2	-1	-1	0	2	2	2	2	2	2	2	2	2
χ_4	12	12	-12	0	0	12	0	0	0	0	0	-3	-3	-3	2	2	2	2	2
χ_5	12	12	-12	0	0	12	0	0	0	0	0	-3	-3	-3	2	2	2	2	2
χ_6	12	12	-12	0	0	12	0	0	0	0	0	2	2	2	-3	-3	-3	-3	-3
χ_7	12	12	-12	0	0	12	0	0	0	0	0	2	2	2	-3	-3	-3	-3	-3
χ_8	13	13	13	-1	-1	13	1	1	1	-5	1	-2	-2	-2	3	3	3	3	3
χ_9	13	13	13	-5	-5	13	1	1	1	-1	1	3	3	3	-2	-2	-2	-2	-2
χ_{10}	13	13	13	5	5	13	1	1	1	1	1	3	3	3	-2	-2	-2	-2	-2
χ_{11}	13	13	13	1	1	13	1	1	1	5	1	-2	-2	-2	3	3	3	3	3
χ_{12}	24	24	-24	0	0	-12	0	0	0	0	0	-6	-6	-6	4	4	4	4	4
χ_{13}	24	24	-24	0	0	-12	0	0	0	0	0	4	4	4	-6	-6	-6	-6	-6
χ_{14}	25	25	25	-5	-5	25	1	1	1	-5	1	0	0	0	0	0	0	0	0
χ_{15}	25	25	25	5	5	25	1	1	1	5	1	0	0	0	0	0	0	0	0
χ_{16}	26	26	26	0	0	-13	2	-1	-1	0	2	6	6	6	-4	-4	-4	-4	-4
χ_{17}	26	26	26	0	0	-13	2	-1	-1	0	2	-4	-4	-4	6	6	6	6	6
χ_{18}	26	26	26	6	6	26	2	2	2	-6	-2	1	1	1	1	1	1	1	1
χ_{19}	26	26	26	-6	-6	26	2	2	2	6	-2	1	1	1	1	1	1	1	1
χ_{20}	26	26	26	-4	-4	26	-1	-1	-1	-4	2	1	1	1	1	1	1	1	1
χ_{21}	26	26	26	4	4	26	-1	-1	-1	-4	2	1	1	1	1	1	1	1	1
χ_{22}	26	26	26	-4	-4	26	-1	-1	-1	4	2	1	1	1	1	1	1	1	1
χ_{23}	26	26	26	4	4	26	-1	-1	-1	4	2	1	1	1	1	1	1	1	1
χ_{24}	48	48	48	0	0	48	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{25}	48	48	48	0	0	48	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{26}	48	48	48	0	0	48	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{27}	48	48	-48	0	0	48	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{28}	48	48	-48	0	0	48	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{29}	48	48	-48	0	0	48	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{30}	48	48	-48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{31}	48	48	-48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{32}	48	48	-48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{33}	48	48	-48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{34}	48	48	-48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{35}	48	48	-48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{36}	48	48	48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{37}	48	48	48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{38}	48	48	48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{39}	48	48	48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{40}	48	48	48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{41}	48	48	48	0	0	-24	0	0	0	0	0	-2	-2	-2	-2	-2	-2	-2	-2
χ_{42}	50	50	50	0	0	-25	2	-1	-1	0	2	0	0	0	0	0	0	0	0
χ_{43}	52	52	52	0	0	-26	4	-2	-2	0	-4	2	2	2	2	2	2	2	2
χ_{44}	52	52	-52	0	0	52	4	4	4	0	0	2	2	2	2	2	2	2	2
χ_{45}	52	52	52	0	0	52	-2	-2	-2	0	-4	2	2	2	2	2	2	2	2
χ_{46}	52	52	52	0	0	-26	-2	1	1	0	4	2	2	2	2	2	2	2	2
χ_{47}	52	52	52	0	0	-26	-2	1	1	0	4	2	2	2	2	2	2	2	2
χ_{48}	52	52	-52	0	0	-26	4	-2	-2	0	0	2	2	2	2	2	2	2	2
χ_{49}	52	52	-52	0	0	-26	4	-2	-2	0	0	2	2	2	2	2	2	2	2
χ_{50}	52	52	-52	0	0	52	-2	-2	-2	0	0	2	2	2	2	2	2	2	2
χ_{51}	52	52	-52	0	0	52	-2	-2	-2	0	0	2	2	2	2	2	2	2	2
χ_{52}	52	52	52	0	0	-26	-2	1	1	0	-4	2	2	2	2	2	2	2	2
χ_{53}	52	52	52	0	0	-26	-2	1	1	0	-4	2	2	2	2	2	2	2	2
χ_{54}	52	52	-52	0	0	-26	-2	1	1	0	0	2	2	2	2	2	2	2	2
χ_{55}	52	52	-52	0	0	-26	-2	1	1	0	0	2	2	2	2	2	2	2	2
χ_{56}	52	52	-52	0	0	-26	-2	1	1	0	0	2	2	2	2	2	2	2	2
χ_{57}	52	52	-52	0	0	-26	-2	1	1	0	0	2	2	2	2	2	2	2	2
χ_{58}	624	-1	0	24	-1	0	0	24	-1	0	0	24	-1	-1	24	-1	-1	-1	-1
χ_{59}	624	-1	0	-24	1	0	0	24	-1	0	0	24	-1	-1	24	-1	-1	-1	-1
χ_{60}	1248	-2	0	0	0	0	0	-24	1	0	0	48	-2	-2	48	-2	-2	-2	-2
χ_{61}	1872	-3	0	-24	1	0	0	0	0	0	0	12	A	*A	-18	C	$\overline{C}$	$\overline{D}$	D
χ_{62}	1872	-3	0	-24	1	0	0	0	0	0	0	12	*A	A	-18	D	$\overline{D}$	C	$\overline{C}$
χ_{63}	1872	-3	0	-24	1	0	0	0	0	0	0	12	A	*A	-18	$\overline{C}$	C	D	$\overline{D}$
χ_{64}	1872	-3	0	-24	1	0	0	0	0	0	0	12	*A	A	-18	$\overline{D}$	D	$\overline{C}$	C
χ_{65}	1872	-3	0	24	-1	0	0	0	0	0	0	12	A	*A	-18	C	$\overline{C}$	$\overline{D}$	D
χ_{66}	1872	-3	0	24	-1	0	0	0	0	0	0	12	*A	A	-18	D	$\overline{D}$	C	$\overline{C}$
χ_{67}	1872	-3	0	24	-1	0	0	0	0	0	0	12	A	*A	-18	$\overline{C}$	C	D	$\overline{D}$
χ_{68}	1872	-3	0	24	-1	0	0	0	0	0	0	12	*A	A	-18	$\overline{D}$	D	$\overline{C}$	C
χ_{69}	3744	-6	0	0	0	0	0	0	0	0	0	-36	B	*B	24	E	E	*E	*E
χ_{70}	3744	-6	0	0	0	0	0	0	0	0	0	-36	-*B	B	24	*E	*E	E	E

where A$=\frac{-1-5\sqrt{5}}{2}$, B$=\frac{3+5\sqrt{5}}{2}$, C$=-7E(5)-2E(5)^2+3E(5)^3+3E(5)^4$,
D$=3E(5)-7E(5)^2+3E(5)^3-2E(5)^4$, E$=-1-5\sqrt{5}$

Table 4: The Character Table of $\overline{G}$ (continued)

$[g]_G$	6A	6B	6C	6D	8A	8B	10A	10B	10C			10D			12A	12B	12C	12D	12E
$[x]_{\overline{G}}$	6A	6B	6C	6D	8A	8B	10B	10C	10D	10E	10F	10G	10H	10I	12A	12B	12C	12D	12E
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
χ_3	-1	2	-1	0	0	2	2	2	0	0	0	0	0	0	0	-1	-1	-1	2
χ_4	-12	0	0	0	0	0	3	-2	0	0	0	0	0	0	0	0	0	0	0
χ_5	-12	0	0	0	0	0	3	-2	0	0	0	0	0	0	0	0	0	0	0
χ_6	-12	0	0	0	0	0	-2	3	F	F	F	-F	-F	-F	0	0	0	0	0
χ_7	-12	0	0	0	0	0	-2	3	-F	-F	-F	F	F	F	0	0	0	0	0
χ_8	13	1	1	-1	1	-1	-2	3	-1	-1	-1	-1	-1	-1	1	1	1	1	1
χ_9	13	1	1	1	1	-1	3	-2	0	0	0	0	0	0	-1	1	1	1	1
χ_{10}	13	1	1	-1	-1	-1	3	-2	0	0	0	0	0	0	1	1	1	1	1
χ_{11}	13	1	1	1	-1	-1	-2	3	1	1	1	1	1	1	-1	1	1	1	1
χ_{12}	12	0	0	0	0	0	6	-4	0	0	0	0	0	0	0	0	0	0	0
χ_{13}	12	0	0	0	0	0	-4	6	0	0	0	0	0	0	0	0	0	0	0
χ_{14}	25	1	1	1	-1	1	0	0	0	0	0	0	0	0	1	1	1	1	1
χ_{15}	25	1	1	-1	1	1	0	0	0	0	0	0	0	0	-1	1	1	1	1
χ_{16}	-13	2	-1	0	0	-2	6	-4	0	0	0	0	0	0	0	-1	-1	-1	2
χ_{17}	-13	2	-1	0	0	-2	-4	6	0	0	0	0	0	0	0	-1	-1	-1	2
χ_{18}	26	2	2	0	0	0	1	1	1	1	1	1	1	1	0	-2	-2	-2	-2
χ_{19}	26	2	2	0	0	0	1	1	-1	-1	-1	-1	-1	-1	0	-2	-2	-2	-2
χ_{20}	26	-1	-1	-1	0	2	1	1	1	1	1	1	1	1	-1	2	-1	-1	-1
χ_{21}	26	-1	-1	1	0	-2	1	1	-1	-1	-1	-1	-1	-1	-1	2	-1	-1	-1
χ_{22}	26	-1	-1	-1	0	-2	1	1	1	1	1	1	1	1	1	2	-1	-1	-1
χ_{23}	26	-1	-1	1	0	2	1	1	-1	-1	-1	-1	-1	-1	1	2	-1	-1	-1
χ_{24}	48	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{25}	48	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{26}	48	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	-48	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{28}	-48	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	-48	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{30}	24	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	24	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{32}	24	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	24	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	24	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	24	0	0	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0
χ_{36}	-24	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{37}	-24	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{38}	-24	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{39}	-24	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{40}	-24	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{41}	-24	0	0	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{42}	-25	2	-1	0	0	2	0	0	0	0	0	0	0	0	0	-1	-1	-1	2
χ_{43}	-26	4	-2	0	0	0	2	2	0	0	0	0	0	0	0	2	2	2	-4
χ_{44}	-52	-4	-4	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{45}	52	-2	-2	0	0	0	2	2	0	0	0	0	0	0	0	-4	2	2	2
χ_{46}	-26	-2	1	0	0	-4	2	2	0	0	0	0	0	0	0	-2	1	1	-2
χ_{47}	-26	-2	1	0	0	4	2	2	0	0	0	0	0	0	0	-2	1	1	-2
χ_{48}	26	-4	2	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{49}	26	-4	2	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{50}	-52	2	2	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{51}	-52	2	2	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0
χ_{52}	26	-2	1	0	0	0	2	2	0	0	0	0	0	0	0	2	-1	-1	2
χ_{53}	-26	-2	1	0	0	0	2	2	0	0	0	0	0	0	0	2	-1	-1	2
χ_{54}	26	2	-1	0	0	0	-2	-2	0	0	0	0	0	0	0	0	J	-J	0
χ_{55}	26	2	-1	0	0	0	-2	-2	0	0	0	0	0	0	0	0	J	-J	0
χ_{56}	26	2	-1	0	0	0	-2	-2	0	0	0	0	0	0	0	0	-J	J	0
χ_{57}	26	2	-1	0	0	0	-2	-2	0	0	0	0	0	0	0	0	-J	J	0
χ_{58}	0	0	0	0	0	0	0	0	4	-1	-1	4	-1	-1	0	0	0	0	0
χ_{59}	0	0	0	0	0	0	0	0	-4	1	1	-4	1	1	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{61}	0	0	0	0	0	0	0	0	G	H	$\bar{H}$	*G	$\bar{I}$	I	0	0	0	0	0
χ_{62}	0	0	0	0	0	0	0	0	*G	I	$\bar{I}$	G	H	$\bar{H}$	0	0	0	0	0
χ_{63}	0	0	0	0	0	0	0	0	G	$\bar{H}$	H	*G	I	$\bar{I}$	0	0	0	0	0
χ_{64}	0	0	0	0	0	0	0	0	*G	$\bar{I}$	I	G	$\bar{H}$	H	0	0	0	0	0
χ_{65}	0	0	0	0	0	0	0	0	-G	-H	-$\bar{H}$	-*G	-$\bar{I}$	-I	0	0	0	0	0
χ_{66}	0	0	0	0	0	0	0	0	-*G	-I	-$\bar{I}$	-G	-H	-$\bar{H}$	0	0	0	0	0
χ_{67}	0	0	0	0	0	0	0	0	-G	-$\bar{H}$	-H	-*G	-I	-$\bar{I}$	0	0	0	0	0
χ_{68}	0	0	0	0	0	0	0	0	-*G	-$\bar{I}$	-I	-G	-$\bar{H}$	-H	0	0	0	0	0
χ_{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{70}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

where F$=-\sqrt{5}$, G$=1+\sqrt{5}$, H$=-E(5)+E(5)^2+E(5)^4$,
I$=-E(5)^2+E(5)^3+E(5)^4$, J $=-3E(4)$

Table 4: The Character Table of $\overline{G}$ (continued)

$[g]_G$	13A	13B	13C	15A	15B	20A	20B	24A	24B	24C	24D	24E	24F	24G	24H
$[x]_{\overline{G}}$	13A	13B	13C	15A	15B	20A	20B	24A	24B	24C	24D	24E	24F	24G	24H
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	1	1	1	1	1	1	1	1
χ_3	2	2	2	-1	-1	0	0	-1	-1	-1	-1	-1	-1	2	2
χ_4	-1	-1	-1	-3	2	N	-N	0	0	0	0	0	0	0	0
χ_5	-1	-1	-1	-3	2	-N	N	0	0	0	0	0	0	0	0
χ_6	-1	-1	-1	2	-3	0	0	0	0	0	0	0	0	0	0
χ_7	-1	-1	-1	2	-3	0	0	0	0	0	0	0	0	0	0
χ_8	0	0	0	-2	3	0	0	-1	-1	-1	-1	-1	-1	-1	-1
χ_9	0	0	0	3	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_{10}	0	0	0	3	-2	1	1	-1	-1	-1	-1	-1	-1	-1	-1
χ_{11}	0	0	0	-2	3	0	0	-1	-1	-1	-1	-1	-1	-1	-1
χ_{12}	-2	-2	-2	3	-2	0	0	0	0	0	0	0	0	0	0
χ_{13}	-2	-2	-2	-2	3	0	0	0	0	0	0	0	0	0	0
χ_{14}	-1	-1	-1	0	0	0	0	1	1	1	1	1	1	1	1
χ_{15}	-1	-1	-1	0	0	0	0	1	1	1	1	1	1	1	1
χ_{16}	0	0	0	-3	2	0	0	1	1	1	1	1	1	-2	-2
χ_{17}	0	0	0	2	-3	0	0	1	1	1	1	1	1	-2	-2
χ_{18}	0	0	0	1	1	-1	-1	0	0	0	0	0	0	0	0
χ_{19}	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
χ_{20}	0	0	0	1	1	1	1	2	2	-1	-1	-1	-1	-1	-1
χ_{21}	0	0	0	1	1	1	1	-2	-2	1	1	1	1	1	1
χ_{22}	0	0	0	1	1	-1	-1	-2	-2	1	1	1	1	1	1
χ_{23}	0	0	0	1	1	-1	-1	2	2	-1	-1	-1	-1	-1	-1
χ_{24}	K	M	L	-2	-2	0	0	0	0	0	0	0	0	0	0
χ_{25}	L	K	M	-2	-2	0	0	0	0	0	0	0	0	0	0
χ_{26}	M	L	K	-2	-2	0	0	0	0	0	0	0	0	0	0
χ_{27}	K	M	L	-2	-2	0	0	0	0	0	0	0	0	0	0
χ_{28}	L	K	M	-2	-2	0	0	0	0	0	0	0	0	0	0
χ_{29}	M	L	K	-2	-2	0	0	0	0	0	0	0	0	0	0
χ_{30}	K	M	L	1	1	0	0	0	0	0	0	0	0	0	0
χ_{31}	L	K	M	1	1	0	0	0	0	0	0	0	0	0	0
χ_{32}	M	L	K	1	1	0	0	0	0	0	0	0	0	0	0
χ_{33}	M	L	K	1	1	0	0	0	0	0	0	0	0	0	0
χ_{34}	K	M	L	1	1	0	0	0	0	0	0	0	0	0	0
χ_{35}	L	K	M	1	1	0	0	0	0	0	0	0	0	0	0
χ_{36}	K	M	L	1	1	0	0	0	0	0	0	0	0	0	0
χ_{37}	L	K	M	1	1	0	0	0	0	0	0	0	0	0	0
χ_{38}	M	L	K	1	1	0	0	0	0	0	0	0	0	0	0
χ_{39}	M	L	K	1	1	0	0	0	0	0	0	0	0	0	0
χ_{40}	K	M	L	1	1	0	0	0	0	0	0	0	0	0	0
χ_{41}	L	K	M	1	1	0	0	0	0	0	0	0	0	0	0
χ_{42}	-2	-2	-2	0	0	0	0	-1	-1	-1	-1	-1	-1	2	2
χ_{43}	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0
χ_{45}	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0
χ_{46}	0	0	0	-1	-1	0	0	2	2	-1	-1	-1	-1	2	2
χ_{47}	0	0	0	-1	-1	0	0	-2	-2	1	1	1	1	-2	-2
χ_{48}	0	0	0	-1	-1	0	0	R	-R	R	-R	R	-R	0	0
χ_{49}	0	0	0	-1	-1	0	0	-R	R	-R	R	-R	R	0	0
χ_{50}	0	0	0	2	2	0	0	0	0	S	-S	S	-S	S	-S
χ_{51}	0	0	0	2	2	0	0	0	0	-S	S	-S	S	-S	S
χ_{52}	0	0	0	-1	-1	0	0	0	0	J	J	-J	-J	0	0
χ_{53}	0	0	0	-1	-1	0	0	0	0	-J	-J	J	J	0	0
χ_{54}	0	0	0	-1	-1	0	0	R	-R	T	-T	$\overline{\text{T}}$	-$\overline{\text{T}}$	S	-S
χ_{55}	0	0	0	-1	-1	0	0	-R	R	-T	T	-$\overline{\text{T}}$	$\overline{\text{T}}$	-S	S
χ_{56}	0	0	0	-1	-1	0	0	R	-R	-$\overline{\text{T}}$	$\overline{\text{T}}$	-T	T	-S	S
χ_{57}	0	0	0	-1	-1	0	0	-R	R	$\overline{\text{T}}$	-$\overline{\text{T}}$	T	-T	S	-S
χ_{58}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{59}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{61}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{62}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{63}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{65}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{66}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{67}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{68}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{70}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

where K$=-E(13)^4-E(13)^6-E(13)^7-E(13)^9$, L$=-E(13)-E(13)^5-E(13)^8-E(13)^{12}$, M$=-E(13)^2-E(13)^3-E(13)^{10}-E(13)^{11}$, R$=-\sqrt{6}i$, S$=-\sqrt{6}$, T$=E(24)-E(24)^{17}$

Table 4: The Character Table of $\overline{G}$ (continued)

$[g]_G$	26A	26B	26C	30A	30B	39A	39B	39C	39D	39E	39F	78A	78B	78C	78D	78E	78F
$[x]_{\overline{G}}$	26A	26B	26C	30A	30B	39A	39B	39C	39D	39E	39F	78A	78B	78C	78D	78E	78F
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_3	2	2	2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_4	1	1	1	3	-2	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_5	1	1	1	3	-2	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_6	1	1	1	-2	3	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_7	1	1	1	-2	3	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1
χ_8	0	0	0	-2	3	0	0	0	0	0	0	0	0	0	0	0	0
χ_9	0	0	0	3	-2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	0	0	0	3	-2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{11}	0	0	0	-2	3	0	0	0	0	0	0	0	0	0	0	0	0
χ_{12}	2	2	2	-3	2	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_{13}	2	2	2	2	-3	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_{14}	-1	-1	-1	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_{15}	-1	-1	-1	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_{16}	0	0	0	-3	2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{17}	0	0	0	2	-3	0	0	0	0	0	0	0	0	0	0	0	0
χ_{18}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{19}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{20}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{21}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{22}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{23}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	K	M	L	-2	-2	K	K	M	M	L	L	K	K	M	M	L	L
χ_{25}	L	K	M	-2	-2	L	L	K	K	M	M	L	L	K	K	M	M
χ_{26}	M	L	K	-2	-2	M	M	L	L	K	K	M	M	L	L	K	K
χ_{27}	-K	-M	-L	2	2	K	K	M	M	L	L	-K	-K	-M	-M	-L	-L
χ_{28}	-L	-K	-M	2	2	L	L	K	K	M	M	-L	-L	-K	-K	-M	-M
χ_{29}	-M	-L	-K	2	2	M	M	L	L	K	K	-M	-M	-L	-L	-K	-K
χ_{30}	-K	-M	-L	-1	-1	O	$\overline{\text{O}}$	Q	$\overline{\text{Q}}$	P	$\overline{\text{P}}$	-O	-$\overline{\text{O}}$	-Q	-$\overline{\text{Q}}$	-P	-$\overline{\text{P}}$
χ_{31}	-L	-K	-M	-1	-1	P	$\overline{\text{P}}$	O	$\overline{\text{O}}$	Q	$\overline{\text{Q}}$	-P	-$\overline{\text{P}}$	-O	-$\overline{\text{O}}$	-Q	-$\overline{\text{Q}}$
χ_{32}	-M	-L	-K	-1	-1	Q	$\overline{\text{Q}}$	P	$\overline{\text{P}}$	O	$\overline{\text{O}}$	-Q	-$\overline{\text{Q}}$	-P	-$\overline{\text{P}}$	-O	-$\overline{\text{O}}$
χ_{33}	-M	-L	-K	-1	-1	$\overline{\text{Q}}$	Q	$\overline{\text{P}}$	P	$\overline{\text{O}}$	O	-$\overline{\text{Q}}$	-Q	-$\overline{\text{P}}$	-P	-$\overline{\text{O}}$	-O
χ_{34}	-K	-M	-L	-1	-1	$\overline{\text{O}}$	O	$\overline{\text{Q}}$	Q	$\overline{\text{P}}$	P	-$\overline{\text{O}}$	-O	-$\overline{\text{Q}}$	-Q	-$\overline{\text{P}}$	-P
χ_{35}	-L	-K	-M	-1	-1	$\overline{\text{P}}$	P	$\overline{\text{O}}$	O	$\overline{\text{Q}}$	Q	-$\overline{\text{P}}$	-P	-$\overline{\text{O}}$	-O	-$\overline{\text{Q}}$	-Q
χ_{36}	K	M	L	1	1	O	$\overline{\text{O}}$	Q	$\overline{\text{Q}}$	P	$\overline{\text{P}}$	O	$\overline{\text{O}}$	Q	$\overline{\text{Q}}$	P	$\overline{\text{P}}$
χ_{37}	L	K	M	1	1	P	$\overline{\text{P}}$	O	$\overline{\text{O}}$	Q	$\overline{\text{Q}}$	P	$\overline{\text{P}}$	O	$\overline{\text{O}}$	Q	$\overline{\text{Q}}$
χ_{38}	M	L	K	1	1	Q	$\overline{\text{Q}}$	P	$\overline{\text{P}}$	O	$\overline{\text{O}}$	Q	$\overline{\text{Q}}$	P	$\overline{\text{P}}$	O	$\overline{\text{O}}$
χ_{39}	M	L	K	1	1	$\overline{\text{Q}}$	Q	$\overline{\text{P}}$	P	$\overline{\text{O}}$	O	$\overline{\text{Q}}$	Q	$\overline{\text{P}}$	P	$\overline{\text{O}}$	O
χ_{40}	K	M	L	1	1	$\overline{\text{O}}$	O	$\overline{\text{Q}}$	Q	$\overline{\text{P}}$	P	$\overline{\text{O}}$	O	$\overline{\text{Q}}$	Q	$\overline{\text{P}}$	P
χ_{41}	L	K	M	1	1	$\overline{\text{P}}$	P	$\overline{\text{O}}$	O	$\overline{\text{Q}}$	Q	$\overline{\text{P}}$	P	$\overline{\text{O}}$	O	$\overline{\text{Q}}$	Q
χ_{42}	-2	-2	-2	0	0	1	1	1	1	1	1	1	1	1	1	1	1
χ_{43}	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{45}	0	0	0	2	2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{46}	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{47}	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{48}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{49}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{50}	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{51}	0	0	0	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{52}	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{53}	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{54}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{56}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{57}	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{58}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{59}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{61}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{62}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{63}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{65}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{66}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{67}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{68}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{70}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

where K$=-E(13)^4 - E(13)^6 - E(13)^7 - E(13)^9$, L$=-E(13) - E(13)^5 - E(13)^8 - E(13)^{12}$, M$=-E(13)^2 - E(13)^3 - E(13)^{10} - E(13)^{11}$, O$=-E(39) - E(39)^5 - E(39)^8 - E(39)^{25}$, P$=-E(39)^2 - E(39)^{10} - E(39)^{11} - E(39)^{16}$,Q$=-E(39)^4 - E(39)^{20} - E(39)^{22} - E(39)^{32}$

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A note on (I, J)-e-continuous and (I, J)-e^*-continuous functions

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Abstract. In this paper the notions of I-e-open set and I-e^*-open set are introduced and used to define a large number of modifications of the concept of continuous function, such as (I, J)-e-continuous functions, (I, J)-e^*-continuous functions, contra (I, J)-e-continuous functions, contra (I, J)-e^*-continuous functions, almost weakly (I, J)-e-continuous functions, almost weakly (I, J)-e^*-continuous functions, almost (I, J)-

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e-continuous functions, almost (I, J)-e^*-continuous functions, almost contra (I, J)-e-continuous functions and almost contra (I, J)-e^*-continuous functions. Also, several characterizations of these new classes of functions are given and finally relations between them are investigated.

Keywords: topological ideal, (I, J)-e-continuous functions, (I, J)-e^*-continuous functions, contra (I, J)-e-continuous functions, almost contra (I, J)-e^*-continuous functions.

1. Introduction

In 2008, E. Ekici [10] introduced a new class of generalized open sets in a topological space called e-open sets and, in 2009, [11] introduced a new generalization of open sets called e^*-open sets. Also, [5], [6], [7], [8], [9], [17] studied another generalized forms of open sets using e-open sets and e^*-open sets. Currently using the notion of topological ideal, different types of continuous functions have been introduced and studied. The concept of ideal topological spaces has been introduced and studied by Kuratowski [13] and the local function of a subset A of a topological space (X, τ) was introduced by Vaidyanathaswamy [16] as follows: given a topological space (X, τ) with an ideal I on X and $P(X)$ the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, defined for each $A \subseteq X$, as $A^*(\tau, I) = \{x \in X \ / \ U \cap A \notin I \text{ for every } U \in \tau_x\}$, where $\tau_x = \{U \in \tau : x \in U\}$, is called the local function of A with respect to τ and I. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(\tau, I)$ called the $*$-topology, finer than τ, is defined by $cl^*(A) = A \cup A^*(\tau, I)$. We will denote $A^*(\tau, I)$ by A^* and $\tau^*(\tau, I)$ by τ^*. Note that when $I = \{\emptyset\}$, (respectively, $I = \mathcal{P}(X)$) $A^* = cl(A)$ (respectively, $A^* = \emptyset$). In 1990, Jankovic and Hamlett [12] introduced the notion of I-open set in a topological space (X, τ) with an ideal I on X. In 2018, Rosas et al. [15] introduced, studied and investigated the (I, J)-continuous functions and its relations with another functions and, in this same direction, Al-Omeri and Noiri (see [1], [2] [3]) introduced several modifications of continuity that have served as inspiration for other researchers to focus their attention on this topic. Motivated by this, we introduce the notions of I-e-open set and I-e^*-open set in an ideal topological spaces, and using these notions, we define and study a large number of modifications of the concept of continuous function, such as (I, J)-e-continuous functions, (I, J)-e^*-continuous functions, contra (I, J)-e-continuous functions, contra (I, J)-e^*-continuous functions, almost weakly (I, J)-e-continuous functions, almost weakly (I, J)-e^*-continuous functions, almost (I, J)-e-continuous functions, almost (I, J)-e^*-continuous functions, almost contra (I, J)-e-continuous functions and almost contra (I, J)-e^*-continuous functions. Finally, we give several characterizations of these new classes of functions and we investigate some relations between them.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated. If I is an ideal on X, (X, τ, I) mean an ideal topological space. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $int(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of (X, τ, I) is an I-open [12], if $S \subseteq int(S^*)$. The complement of an I-open set is called I-closed set. In the case that $I = \{\emptyset\}$ (respectively, $I = P(X)$,) the I-open sets form the collection of all preopen sets of X (respectively, the only I-open set is $\emptyset$). The I-closure and the I-interior of a subset A of X, denoted by $I\text{-}\mathrm{Cl}(A)$ and $I\text{-}int(A)$, respectively, can be defined in the same way as $\mathrm{Cl}(A)$ and $int(A)$, respectively. The family of all I-open (resp. I-closed) subsets of a (X, τ, I), denoted by $IO(X)$(resp. $IC(X)$). We set $IO(X, x) = \{A : A \in IO(X) \text{ and } x \in A\}$. It is well known that in an ideal topological space (X, τ, I), the $I\text{-}\mathrm{Cl}(A)$ is an I-closed set and $I\text{-}int(A)$ is an I-open set and then, the following two results are immediate, using the notions of I-closure and I-interior.

Theorem 2.1. *Let (X, τ, I) be an ideal topological space, $A \subseteq X$ and $x \in X$. $x \in I\text{-}\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for all $U \in IO(X, x)$.*

Theorem 2.2. *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then, the following conditions hold:*

1. $I\text{-}int(X \setminus A) = X \setminus I\text{-}\mathrm{Cl}(A)$.
2. $I\text{-}\mathrm{Cl}(X \setminus A) = X \setminus I\text{-}int(A)$.
3. $A \subseteq B \Rightarrow I\text{-}\mathrm{Cl}(A) \subseteq I\text{-}\mathrm{Cl}(B)$.
4. $A \subseteq B \Rightarrow I\text{-}int(A) \subseteq I\text{-}int(B)$.
5. $I\text{-}\mathrm{Cl}(A) \cup I\text{-}\mathrm{Cl}(B) \subseteq I\text{-}\mathrm{Cl}(A \cup B)$.
6. $I\text{-}int(A \cap B \subseteq I\text{-}int(A \cap B)$.

Definition 2.3 ([14])**.** *Let (X, τ, I) be an ideal topological space. $A \subseteq X$ is said to be:*

1. *I-regular open if $A = I\text{-}int(I\text{-}\mathrm{Cl}(A))$;*
2. *I-semiopen if $A \subseteq I\text{-}\mathrm{Cl}(I\text{-}int(A))$;*
3. *I-preopen if $A \subseteq I\text{-}int(I\text{-}\mathrm{Cl}(A))$.*

The class of I-regular open (resp.,I-semiopen, I-preopen) sets in X, is denoted by $IRO(X)$ (resp., $ISO(X)$, $IPO(X)$). The complement of an I-regular open set is called an I-regular closed set and the class of this sets is denoted by $IRC(X)$. The complement of an I-semiopen set is called an I-semiclosed set and the family of this sets is denoted by $ISC(X)$. Similarly, the complement of an I-preopen set is called an I-preclosed set and the class of this sets is denoted by $IPC(X)$.

Definition 2.4 ([18])**.** *Let (X, τ, I) be an ideal topological space and $A \subseteq X$:*

1. *A subset A of X is said to be I-δ-open if for each $x \in A$, there exists an I-regular open set G such that $x \in G \subseteq A$.*
2. *A point $x \in X$ is called an I-δ-cluster point of A if I-$int(I\,\mathrm{Cl}(U)) \cap A \neq \emptyset$ for every I-open set U of X containing x.*

The set of all I-δ-cluster point of A is called the I-δ-closure of A and is denoted by I-δ-$\mathrm{Cl}(A)$. If I-δ-$\mathrm{Cl}(A) = A$, A is said to be I-δ-closed. The set $\{x \in X : x \in U \subseteq A$ for some I-regular open set $U \subseteq X\}$ is called I-δ-interior of A and is denoted by I-δ-$int(A)$. A is I-δ-open if I-δ-$int(A) = A$.

Theorem 2.5 ([18])**.** *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then, the following conditions hold:*

1. *if $A \subseteq B$ then I-δ-$int(A) \subseteq I$-δ-$int(B)$.*
2. *if $A \subseteq B$ then I-δ-$\mathrm{Cl}(A) \subseteq I$-$\delta$-$\mathrm{Cl}(B)$.*
3. *I-δ-$int(X \setminus A) = X \setminus I$-$\delta$-$\mathrm{Cl}(A)$.*
4. *I-δ-$\mathrm{Cl}(X \setminus A) = X \setminus I$-$\delta$-$int(A)$.*
5. *I-δ-$int(A) \subseteq I$-$int(A) \subseteq I$-$\mathrm{Cl}(A) \subseteq I$-$\delta$-$\mathrm{Cl}(A)$.*

3. Contra (I, J)-e-continuous functions and contra (I, J)-e^*-continuous functions

In this section, we introduced and defined a notion of I-e-open sets and I-e^*-open sets in order to define and characterize a new notions of continuous functions.

Definition 3.1. *Let (X, τ, I) be an ideal topological space. $A \subseteq X$ is said to be:*

1. *I-e-open set if $A \subseteq I$-$\mathrm{Cl}(I$-δ-$int(A)) \cup I$-$int(I$-δ-$\mathrm{Cl}(A))$.*
2. *I-e*-open set if $A \subseteq I$-$\mathrm{Cl}(I$-$int(I$-δ-$\mathrm{Cl}(A)))$.*

The complement of an I-e-open (respectively, I-e^*-open) set is called an I-e-closed (respectively, I-e^*-closed) set. The family of all I-e-open (respectively, I-e^*-open) sets is denoted by I-$eO(X)$ (respectively, I-$e^*O(X)$).

Theorem 3.2. *Let (X, τ, I) be an ideal topological space. The following conditions hold:*

1. *the union of any collection of I-e-open sets is an I-e-open set.*
2. *the union of any collection of I-e*-open sets is an I-e*-open set.*

We define the I-e-closure (respectively, I-e-closure) of $A \subseteq X$ denoted by I-e-$\mathrm{Cl}(A)$ (respectively, I-e^*-$\mathrm{Cl}(A)$) as the intersection of all I-e-closed (respectively, I-e^*-closed) sets containing A. From the above A is I-e-closed (respectively, I-e^*-closed) if $A = I$-e-$\mathrm{Cl}(A)$ (respectively, $A = I$-e^*-$\mathrm{Cl}(A)$).

Theorem 3.3. *Let (X, τ, I) be an ideal topological space. The following conditions hold:*

1. *every $I - \delta$-open is an I-e-open set, but not conversely.*
2. *every I-e-open set is an I-e^*-open set, but not conversely.*

Proof. The proof follows directly from the definition. □

Example 3.4. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = \{\emptyset\}$. Then, we obtain that:
$IO(X) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$,
$IRO(X) = \{\emptyset, X, \{c\}, \{a, b\}\}$,
I-δ-open set $= \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$,
I-$e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Now, it is easy to see that: $\{a, c\}$ is an I-e-open but is not I-δ-open. In the same form $\{a, d\}$ is an I-e^*-open but is not I-e-open.

In a natural form, we define the (I, J)-e-continuous and (I, J)-e^*-continuous functions.

Definition 3.5. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *(I, J)-e-continuous if $f^{-1}(U)$ is I-e-open for every J-open set U in Y;*
2. *(I, J)-e^*-continuous if $f^{-1}(U)$ is I-e^*-open for every J-open set U in Y.*

The characterization of (I, J)-e-continuous are very similar as the characterization of (I, J)-continuous functions due in [15].

Definition 3.6. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *contra (I, J)-e-continuous if $f^{-1}(U)$ is I-e-closed for every J-open set U in Y;*
2. *contra (I, J)-e^*-continuous if $f^{-1}(U)$ is I-e^*-closed for every J-open set U in Y.*

Example 3.7. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = J = \{\emptyset\}$. Then, we obtain that:

$$IO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}.$$

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = c$. It is easy to see that f is contra (I, J)-e-continuous.

Example 3.8. Let $X = Y = \{a,b,c,d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$ and $I = J = \{\emptyset\}$. Then, we obtain that:
$IO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$, I-$e^*C(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = c$. It is easy to see that f is contra (I,J)-e^*-continuous.

Definition 3.9. *Let (X,τ,I) be an ideal topological space and $A \subseteq X$. The I-kernel of A, denoted by I-$ker(A)$ is defined as the intersection of all I-open sets that contains A, that is I-$ker(A) = \cap\{U : U \in IO(X), A \subseteq U\}$.*

In a natural form as in topological spaces, we have the following result.

Theorem 3.10. *Let (X,τ,I) be an ideal topological space, $A \subseteq X$ and $x \in X$, then:*

1. *$x \in I$-$ker(A)$ if and only if $A \cap F \neq \emptyset$ for every I-closed set F containing x;*
2. *$A \subseteq I$-$ker(A)$ and $A = I$-$ker(A)$ if A is an I-open set.*

Using the above notion, we obtain the following characterizations of contra (I,J)-e-continuous and contra (I,J)-e^*-continuous functions.

Theorem 3.11. *For a function $f : (X,\tau,I) \to (Y,\sigma,J)$, the following conditions are equivalent:*

1. *f is contra (I,J)-e-continuous;*
2. *for each $x \in X$ and each J-closed set F of Y containing $f(x)$ there exists $U \in I$-$eO(X)$ such that $f(U) \subseteq F$;*
3. *for each J-closed subset F of Y, $f^{-1}(F)$ is an I-e-open set;*
4. *$f(I$-e-$\mathrm{Cl}(A)) \subseteq J$-$ker(f(A))$ for all $A \subseteq X$;*
5. *I-e-$\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(J$-$ker(B))$ for all $B \subseteq Y$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and F any J-closed set of Y containing $f(x)$. Then, $Y \setminus F$ is a J-open set in Y and by hypothesis $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is I-e-closed in X, in consequence, $f^{-1}(F)$ is an I-e-open. Taking $U = f^{-1}(F)$, $x \in U$ and $f(U) \subseteq F$.

(2) $\Rightarrow$ (3). Let F be any J-closed subset of Y. Consider $x \in f^{-1}(F)$, then $f(x) \in F$. By hypothesis, there exists $U \in I$-$eO(X)$ such that $f(U) \subseteq F$. In consequence, $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(F)$. It follows that $f^{-1}(F) = \cup_{x \in f^{-1}(F)} U$ and then by Theorem 3.2, $f^{-1}(F)$ is an I-e-open set.

(3) $\Rightarrow$ (4). Consider that for some subset A of X, $y \in f(I$-e-$\mathrm{Cl}(A))$ but $y \notin J$-$ker(f(A))$. This implies that there exists a J-closed set F such that $y \in F$ and $f(A) \cap F = \emptyset$. It follows that $A \cap f^{-1}(F) = \emptyset$. Since F is a J-closed

set, by hypothesis, $f^{-1}(F)$ is an I-e-open set and then, I-e-$\text{Cl}(A) \cap f^{-1}(F) = \emptyset$. Since $y \in f(I\text{-}e\text{-Cl}(A)$, then $y = f(x)$ for some $x \in I$-e-$\text{Cl}(A)$, since $f(x) \in F$, then $x \in f^{-1}(F)$ and hence $x \in I$-e-$\text{Cl}(A) \cap f^{-1}(F)$, which is a contradiction.

(4) $\Rightarrow$ (5). Let B be any subset of Y. By hypothesis, $f(I\text{-}e\text{-Cl}(f^{-1}(B))) \subseteq J\text{-}ker(B)$. Thus $I\text{-}e\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(J\text{-}ker(B))$ for all $B \subseteq Y$.

(5) $\Rightarrow$ (1). Let V any J-open set of Y. By hypothesis, $I\text{-}e\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(J\text{-}ker(V)) = f^{-1}(V)$. Follows that $I\text{-}e\text{-Cl}(f^{-1}(V)) = f^{-1}(V)$. Hence $f^{-1}(V)$ is an I-e-closed set in X. □

Theorem 3.12. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following conditions are equivalent:*

1. *f is contra (I, J)-e^*-continuous;*
2. *for each J-closed subset F of Y, $f^{-1}(F)$ is an I-e^*-open set;*
3. *for each $x \in X$ and each J-closed set F of Y containing $f(x)$ there exists $U \in I\text{-}e^*O(X)$ such that $f(U) \subseteq F$;*
4. *$f(I\text{-}e^*\text{-Cl}(A)) \subseteq J\text{-}ker(f(A))$ for all $A \subseteq X$;*
5. *$I\text{-}e^*\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(J\text{-}ker(B))$ for all $B \subseteq Y$.*

Proof. The proof is similar to that of Theorem 3.11. □

Definition 3.13. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *almost weakly (I, J)-e-continuous at a point $x \in X$, if for each J-open set V of Y containing $f(x)$, there exists an I-e-open set U containing x such that $f(U) \subseteq J\text{-Cl}(V)$;*
2. *almost weakly (I, J)-e^*-continuous at a point $x \in X$, if for each J-open set V of Y containing $f(x)$, there exists an I-e^*-open set U containing x such that $f(U) \subseteq J\text{-Cl}(V)$.*

If $f : (X, \tau, I) \to (Y, \sigma, J)$ is almost weakly (I, J)-e-continuous (respectively, almost weakly (I, J)-e^*-continuous) at each point $x \in X$, then f is said to be almost weakly (I, J)-e-continuous (respectively, almost weakly (I, J)-e^*-continuous).

Theorem 3.14. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following conditions are satisfied:*

1. *If f is (I, J)-e-continuous function then f is almost weakly (I, J)-e-continuous;*
2. *If f is (I, J)-e^*-continuous function then f is almost weakly (I, J)-e^*-continuous.*

Proof. The proof is a consequence of the Definition 3.13 and the notion of J-closure of a set. □

The following examples shows that the converse of Theorem 3.14 are not necessarily true.

Example 3.15. As in Example 3.7. Then, we obtain that:
$IO(X) = JO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = d$, $f(b) = a$, $f(c) = b$ and $f(d) = c$. It is easy to see that f is almost weakly (I,J)-e-continuous but is not (I,J)-e-continuous.

Example 3.16. Let $X = Y = \{a,b,c\}$, $\tau = \sigma = \{\emptyset, X, \{b,c\}\}$, $I = J = \{\emptyset, \{b\}\}$. Then, we obtain that:
$JO(Y) = \{\emptyset, Y, \{c\}, \{a,c\}, \{b,c\}\}$,
I-$e^*O(X) = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$.

Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = c$, $f(b) = a$ and $f(c) = b$. It is easy to see that f is almost weakly (I,J)-e^*-continuous but is not (I,J)-e^*-continuous.

Definition 3.17. *A function $f : (X,\tau,I) \to (Y,\sigma,J)$ is said to be:*

1. *almost (I,J)-e-continuous if and only if for each $x \in X$ and each J-regular open set V of Y containing $f(x)$, there exists an I-e-open set U containing x such that $f(U) \subseteq V$.*

2. *almost (I,J)-e*-continuous if and only if for each $x \in X$ and each J-regular open set V of Y containing $f(x)$, there exists an I-e*-open set U containing x such that $f(U) \subseteq V$.*

Example 3.18. As in Example 3.7. Then, we obtain that:
$IO(Y) = \{\emptyset, Y, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
$IRO(Y) = \{\emptyset, Y, \{c\}, \{a,b\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$,
I-$e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = c$. It is easy to see that f is not almost (I,J)-e-continuous.

Example 3.19. As in Example 3.7. Then, we obtain that:
$IO(Y) = \{\emptyset, Y, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
$IRO(Y) = \{\emptyset, Y, \{c\}, \{a,b\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$,

I-$e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = d$, $f(b) = c$, $f(c) = b$ and $f(d) = a$. It is easy to see that f is almost (I, J)-e-continuous but is not (I, J)-e-continuous function.

Example 3.20. As in Example 3.15, f is almost weakly (I, J)-e-continuous but is not almost (I, J)-e-continuous.

4. Almost contra (I, J)-e-continuous functions and almost contra (I, J)-e^*-continuous functions

In this section, we introduced and defined the notions of almost contra (I, J)-e-continuous and almost contra (I, J)-e^*-continuous functions in order to characterize it and find its relations with another notions of continuous functions.

Definition 4.1. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *almost contra (I, J)-e-continuous if $f^{-1}(V)$ is I-e-closed for every J-regular open set V of Y,*
2. *almost contra (I, J)-e^*-continuous if $f^{-1}(V)$ is I-e^*-closed for every J-open set V in Y.*

Example 4.2. As in Example 3.7. Then, we obtain that:
$IO(X) = JO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
$IRO(X) = \{\emptyset, X, \{c\}, \{a,b\}\}$,
I-$eC(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = d$, $f(b) = a$, $f(c) = b$ and $f(d) = c$. It is easy to see that f is almost contra (I, J)-e-continuous.

Theorem 4.3. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following conditions are equivalent:*

1. *f is almost contra (I, J)-e-continuous;*
2. *$f^{-1}(F)$ is I-e-open for every J-regular closed set F of Y;*
3. *for each $x \in X$ and each J-regular open set F of Y containing $f(x)$ there exists $U \in I$-$eO(X, x)$ such that $f(U) \subseteq F$;*
4. *for each $x \in X$ and each J-regular open set V of Y containing $f(x)$ there exists a I-e-closed set K containing x such that $f^{-1}(V) \subseteq K$.*

Proof. (1) $\Rightarrow$ (2). Let F be any J-regular closed set of Y. Then, $Y \setminus F$ is a J-regular open set of Y, and $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \in I$-$eC(X)$. Therefore, $f^{-1}(F) \in I$-$eO(X)$.

(2) $\Rightarrow$ (3). Let F be any J-regular closed set of Y and $x \in X$ such that $f(x) \in F$. Then, $f^{-1}(F) \in I\text{-}eO(X, x)$, $x \in f^{-1}(F)$. Therefore, take $U = f^{-1}(F)$, $f(U) \subseteq F$.

(3) $\Rightarrow$ (4) Let V any J-regular open set of Y such that $f(x) \notin V$, then $f(x) \in Y \setminus V$ and $Y \setminus V$ is a J-regular closed set of Y. By hypothesis, there exists $U \in I\text{-}eO(X, x)$ such that $f(U) \subseteq Y \setminus V$, therefore $U \subseteq f^{-1}(Y \setminus V) \subseteq X \setminus f^{-1}(V)$. Follows $f^{-1}(U) \subseteq X \setminus U$, $X \setminus U$ is an I-e-closed set and $x \notin X \setminus U$.

(4) $\Rightarrow$ (1). Straighforward. □

Theorem 4.4. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following conditions are equivalent:*

1. *f is almost contra (I, J)-e^*-continuous;*
2. *$f^{-1}(F)$ is I-e^*-open for every J-regular closed set F of Y;*
3. *for each $x \in X$ and each J-regular open set F of Y containing $f(x)$ there exists $U \in I\text{-}e^*O(X, x)$ such that $f(U) \subseteq F$;*
4. *for each $x \in X$ and each J-regular open set F of Y containing $f(x)$ there exists a I-e^*-closed set K containing x such that $f^{-1}(V) \subseteq K$.*

Proof. The proof is similar to that of Theorem 4.3. □

Theorem 4.5. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is almost contra (I, J)-e-continuous then f is almost contra (I, J)-e^*-continuous.*

Proof. The proof follows from the fact that every I-e-open set is an I-e^*-open set. □

The following example, shows that there exists an almost contra (I, J)-e^*-continuous that is not almost contra (I, J)-e-continuous.

Example 4.6. As in Example 4.2. Then, we obtain that:
$IO(X) = JO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$,
$IRC(Y) = \{\emptyset, Y, \{c, d\}, \{a, b, d\}\}$,
$I\text{-}eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$,
$I\text{-}e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = c, f(b) = a, f(c) = b, f(d) = d$. It is easy to see that f is almost contra (I, J)-e^*-continuous but is not almost contra (I, J)-e-continuous.

Definition 4.7. *An ideal topological space (X, τ, I) is said to be I-extremally disconnected if $I\text{-}\mathrm{Cl}(U) \in IO(X)$ for each $U \in IO(X)$.*

Example 4.8. Let $X = \{a, b, c, d\}$, $\tau = \mathcal{P}(X)$, $I = \{\emptyset\}$. Follows that: $IO(X) = \mathcal{P}(X)$, the I-Cl$(U) = U$ for all $U \in IO(X)$. It follows that (X, τ, I) is extremally disconnected.

Theorem 4.9. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function, where Y is J-extremally disconnected. The following conditions hold:*

1. *f is almost contra (I, J)-e-continuous if and only if f is almost (I, J)-e-continuous,*
2. *f is almost contra (I, J)-e^*-continuous if and only if f is almost (I, J)-e^*-continuous.*

Proof. (1). Suppose that $x \in X$ and V is any J-regular open set of Y containing $f(x)$. Since Y is J-extremally disconnected, J-Cl$(V) \in JO(Y)$ and then V is J-regular closed, follows that V is J-clopen. Using Theorem 4.3, there exists $U \in IO(X, x)$ such that $f(U) \subseteq V$ and then f is almost (I, J)-e-continuous. Conversely, Let f be almost (I, J)-e-continuous function and W be any J-regular closed set of Y. By hypothesis, Y is J-extremally disconnected, then W is J-regular open, therefore $f^{-1}(W)$ is an I-e-open set of X. Take $U = f^{-1}(W)$ and obtain that $f(U) \subseteq W$.

(2). The proof is similar to that of part (1). □

Definition 4.10. *An ideal topological space (X, τ, I) is said to be I-e^*-$T_{1/2}$ if each I-e^*-closed set is I-δ-closed.*

Example 4.11. Let $X = \{a, b, c, d\}$, $\tau = \mathcal{P}(X)$, $I = \{\emptyset\}$. Then $IO(X) = \mathcal{P}(X)$, $IRO(X) = \mathcal{P}(X)$, I-δ-open sets$=\mathcal{P}(X)$, I-$e^*O(X) = \mathcal{P}(X)$.

It follows that (X, τ, I) is an I-e^*-$T_{1/2}$.

Theorem 4.12. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function and (X, τ, I) an I-e^*-$T_{1/2}$. Then, f is almost (I, J)-e-continuous if and only if f is almost (I, J)-e^*-continuous.*

Proof. By Theorem 3.10, each I-e-open set is I-e^*-open and then each almost (I, J)-e-continuous f is almost (I, J)-e^*-continuous. Conversely, since (X, τ, I) is an I-e^*-$T_{1/2}$-space, each I-e^*-open set is I-δ-open. By Theorem 3.10, each I-δ-open set is I-e-open and, then each almost (I, J)-e^*-continuous f is almost (I, J)-e-continuous. □

Theorem 4.13. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function and (X, τ, I) an I-e^*-$T_{1/2}$-space. Then, f is almost contra (I, J)-e-continuous if and only if f is almost contra (I, J)-e^*-continuous.*

Proof. The proof is similar to that of Theorem 4.12. □

Definition 4.14. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *(I, J)-e-irresolute if $f^{-1}(V)$ is I-e-open for every J-e-open set V of Y;*

2. *(I, J)-e^*-irresolute if $f^{-1}(V)$ is I-e^*-open for every J-e^*-open set V of Y.*

Theorem 4.15. 1. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is an (I, J)-e-irresolute function and $g : (Y, \sigma, J) \to (Z, \beta, L)$ is an (J, L)-e-irresolute function, then $g \circ f : (X, \tau, I) \to (Z, \beta, L)$ is an (I, L)-e-irresolute function;*

2. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is an (I, J)-e^*-irresolute function and $g : (Y, \sigma, J) \to (Z, \beta, L)$ is an (J, L)-e^*-irresolute function, then $g \circ f : (X, \tau, I) \to (Z, \beta, L)$ is an (I, L)-e^*-irresolute function;*

3. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is an (I, J)-e-irresolute function and $g : (Y, \sigma, J) \to (Z, \beta, L)$ is an (J, L)-e-continuous function, then $g \circ f : (X, \tau, I) \to (Z, \beta, L)$ is an (I, L)-e- continuous function;*

4. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is an (I, J)-e^*-irresolute function and $g : (Y, \sigma, J) \to (Z, \beta, L)$ is an (J, L)-e^*-continuous function, then $g \circ f : (X, \tau, I) \to (Z, \beta, L)$ is an (I, L)-e^*- continuous function.*

5. Conclusion

In this work, in the theoretical framework of an ideal topological space, we have introduced the notions of I-e-open set and I-e^*-open set. By using these notions we have defined the (I, J)-e-continuous functions, (I, J)-e^*-continuous functions, contra (I, J)-e-continuous functions, contra (I, J)-e^*-continuous functions, almost weakly (I, J)-e-continuous functions, almost weakly (I, J)-e^*-continuous functions, almost (I, J)-e-continuous functions, almost (I, J)-e^*-continuous functions, almost contra (I, J)-e-continuous functions and almost contra (I, J)-e^*-continuous functions. Also, we gave various characterizations of these new classes of functions and have obtained some relationships between them. For future research, notions similar to those studied in the reference [4] can be investigated, such as defining an I-semi*-open (respectively, I-pre *-open) set A to the one that satisfies the inclusion $A \subseteq I\text{-Cl}(I\text{-}\delta\text{-}int(A))$ (respectively, $A \subseteq I\text{-}int(I\text{-}\delta\text{-Cl}(A)))$.

In consequence, these notions can be applied in the study of new modifications of continuous functions that are similar to those presented in [4].

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Extended GE-filters in weak eGE-algebras

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Abstract. A broader concept than eGE algebra, called weak eGE algebra, is introduced, and related properties are studied. The concept of transitive and tightly (weak) eGE algebra is also considered and some properties are discussed. A weak eGE-algebra with additional conditions is used to give a way to create a GE-algebra. Extended GE filters are described in the last section. The concept of eGE-filters and upper sets is introduced and associated properties are investigated. Conditions for a superset of E in a weak eGE-algebra $(X, *, E)$ to be an eGE-filter are provided. Also, conditions for the upper set to become an eGE-filter are discussed. The characterization of the eGE-filter is established.

Keywords: (weak) eGE-algebra, transitive (weak) eGE-algebra, tightly (weak) eGE-algebra, eGE-filter.

1. Introduction

The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego [7] from algebraic point of view. Hilbert algebras are a valuable tool for some algebraic logic investigations as they can be regarded as fragments of any propositional logic that contains a logical connective implication $(\rightarrow)$

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and the constant 1 that is assumed to be the logical meaning "true". Many researchers have done a significant amount of work on Hilbert algebras [4, 5, 6, 8, 9, 10, 12, 13, 14]. As a generalization of Hilbert algebras, R.K. Bandaru et al. [1] introduced the notion of GE-algebras. They studied the various properties and filter theory of GE-algebras [2, 11, 15]. Bandaru et al. [3] introduced the notion of eGE-algebra as a generalization of GE-algebra and investigated its properties. We observed that there is a condition that do not play a remarkable role in that paper [3]. Algebraic structures with conditions that play no many role will inevitably narrow their objects, so they can weaken the value of their use. Everyone knows that the wider the object for a new algebraic structure, the wider the application. Therefore, it is necessary to increase the value of use by expanding the object of algebraic structures except for the conditions in which the role is insignificant. From this point of view, we would like to introduce a more generalized concept by deleting conditions that do not play an important role.

In this manuscript, we introduce more general version than eGE-algebras, so called weak eGE-algebra, and investigate its properties. This can generalize the several results of paper [3], and allows some of the results of the paper [3] to be classified as corollaries. We provide a condition for a weak eGE-algebra to be an eGE-algebra. We consider the concepts of a transitive and tightly (weak) eGE-algebra, and discuss some properties. Using a weak eGE-algebra with additional conditions, we provide a way to create a GE-algebra. The last section describes the expanded GE filters. We introduce the concepts of eGE-filters and upper sets and investigate their associated properties. We provide conditions for a superset of E in a weak eGE-algebra $(X, *, E)$ to be an eGE-filter. We provide conditions for the upper set to become an eGE-filter. We establish the characterization of eGE filters.

2. Preliminaries

Definition 2.1 ([1]). *By a GE-algebra we mean a nonempty set X with a constant 1 and a binary operation "$*$" satisfying the following axioms:*

$(GE1)$ $u * u = 1,$

$(GE2)$ $1 * u = u,$

$(GE3)$ $u * (v * w) = u * (v * (u * w)),$

for all $u, v, w \in X$.

Definition 2.2 ([3]). *Let E be a nonempty subset of a set X. By a extended GE-algebra (briefly, eGE-algebra) we mean a structure $(X, *, E)$ in which $*$ is a binary operation on X satisfying the condition (GE3) and*

$(eGE1)$ $(\forall x \in X)$ $(x * x \in E),$

(*eGE*2) $(\forall x \in X)\ (x * E \subseteq E)$,

(*eGE*3) $(\forall x \in X)\ (E * x = \{x\})$,

where $E * x := \{a * x \mid a \in E\}$ *and* $x * E := \{x * a \mid a \in E\}$.

In an eGE-algebra $(X, *, E)$, we define a binary operation "$\leq_e$" as follows:

$$(\forall x, y \in X)(x \leq_e y \ \Leftrightarrow \ x * y \in E). \tag{1}$$

It could be noted that the binary operation "$\leq_e$" is reflexive, but it is neither antisymmetric nor transitive.

Proposition 2.1 ([3]). *Every eGE-algebra* $(X, *, E)$ *satisfies:*

$$(\forall x, y \in X)(x * (x * y) = x * y), \tag{2}$$

$$(\forall x, y, z \in X)(y * z \in E \ \Rightarrow \ x * (y * z) \in E), \tag{3}$$

$$(\forall x, y, z \in X)(x \leq_e y * z \ \Rightarrow \ y \leq_e x * z). \tag{4}$$

Definition 2.3 ([3]). *Let* $(X, *, E)$ *be a (weak) eGE-algebra. A subset* F *of* X *is called an extended GE-filter (briefly, eGE-filter) of* $(X, *, E)$ *if* F *is a superset of* E *which satisfies the next condition*

$$(\forall x, y \in X)(x * y \in F,\ x \in F \ \Rightarrow \ y \in F). \tag{5}$$

Lemma 2.1 ([3]). *Every eGE-filter* F *of an eGE-algebra* $(X, *, E)$ *satisfies:*

$$(\forall x, y \in X)(x \in F,\ x \leq_e y \ \Rightarrow \ y \in F). \tag{6}$$

3. Weak extended GE-algebras

Definition 3.1. *Let* E *be a nonempty subset of a set* X *and let "$*$" be a binary operation on* X*. A structure* $(X, *, E)$ *is called a weak extended GE-algebra (briefly, weak eGE-algebra) if it satisfies the following three conditions (GE3), (eGE1) and (eGE3).*

It is obvious that every eGE-algebra is a weak eGE-algebra, but the converse is not true in general as shown in the following example.

Example 3.2. Let $X = \{a, b, c, d\}$ be a set with the Cayley table which is given in Table 1.

Then, $(X, *, E)$ with $E = \{b, c\}$ is a weak eGE-algebra. But it is not an eGE-algebra since $d * E = \{a, c\} \nsubseteq E$.

It is clear that if $E = \{1\}$, then the weak eGE-algebra $(X, *, E)$ is only a GE-algebra, and vice versa. If $|E| \geq 2$, then the weak eGE-algebra $(X, *, E)$ may not be a GE-algebra as seen in the following example. Hence, we know that the weak eGE-algebra is an extension of a GE-algebra.

Table 1: Cayley table for the binary operation "$*$"

$*$	a	b	c	d
a	b	b	c	c
b	a	b	c	d
c	a	b	c	d
d	a	a	c	c

Example 3.3. Consider the weak eGE-algebra $(X, *, E)$ which is given in Example 3.2. We can see that there is no element to play a constant role and we can check that (GE1) and (GE2) are not true. Hence, $(X, *, E)$ is not a GE-algebra.

Proposition 3.1. *Every weak eGE-algebra $(X, *, E)$ satisfies:*

$$(\forall x, y \in X)(x * (x * y) = x * y). \tag{7}$$

Proof. For every $x, y \in X$, we have

$$x * (x * y) = x * ((x * x) * (x * y)) = x * ((x * x) * y) = x * y$$

by (GE3), (eGE1) and (eGE3). $\square$

Definition 3.4. *If $(X, *, E)$ is a (weak) eGE-algebra in which $(X, *, 1)$ is a GE-algebra, we say that $(X, *, E)$ is a tightly (weak) eGE-algebra.*

It is clear that every (weak) eGE-algebra $(X, *, E)$ is a tightly (weak) eGE-algebra if and only if $E = \{1\}$.

In the example below, we can see that if $(X, *, E)$ is a (weak) eGE-algebra satisfying $1 \in E$ and $|E| \geq 2$, $(X, *, E)$ may not be a tightly (weak) eGE-algebra.

Example 3.5. 1. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 2. Then $(X, *, E)$ with $E = \{0, 1\}$ is an eGE-algebra. But it is not a tightly eGE-algebra since $(X, *, 1)$ fails to satisfy (GE1), i.e., $0 * 0 = 0 \neq 1$.

Table 2: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	1	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0

2. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 3.

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0

Then $(X, *, E)$ with $E = \{0, 1\}$ is a weak eGE-algebra. But it is not a tightly weak eGE-algebra since $(X, *, 1)$ fails to satisfy (GE1), i.e., $0 * 0 = 0 \neq 1$.

Proposition 3.2. *If $(X, *, E)$ is a weak eGE-algebra, then E is closed under the binary operation "$*$".*

Proof. Let $x, y \in E$. Then $x * y = y \in E$ and $y * x = x \in E$ by (eGE3). Hence, E is closed under "$*$" □

Question 3.1. *Let B be a subset of X such that $E \subseteq B$.*
1. If $(X, *, E)$ is a weak eGE-algebra, then is $(X, *, B)$ a weak eGE-algebra.
2. If $(X, *, B)$ is a weak eGE-algebra, then is $(X, *, E)$ a weak eGE-algebra.

The next example give a negative answer to Question 3.1.

Example 3.6. 1. Let $X = \{a, b, c, d, e\}$ be a set with the Cayley table which is given in Table 4.

Table 4: Cayley table for the binary operation "$*$"

$*$	a	b	c	d	e
a	c	b	c	c	b
b	d	d	d	d	d
c	a	b	c	d	e
d	a	b	c	d	e
e	a	c	c	d	e

Then $(X, *, E)$ with $E = \{c, d\}$ is a weak eGE-algebra. But $(X, *, B)$ with $B = \{c, d, e\}$ is not a weak eGE-algebra since $B * b = \{b, c\} \neq \{b\}$. Also $(X, *, B)$ with $B = \{c, d\}$ is a weak eGE-algebra. But $(X, *, E)$ with $E = \{c\}$ is not a weak eGE-algebra since $b * b = d \notin E$.

Remark 3.7. Let $\{E, B\}$ be a partition of X. If $(X, *, E)$ is a (weak) eGE-algebra, then $(X, *, B)$ can never be a (weak) eGE-algebra.

The following example describes Remark 3.7.

Example 3.8. In Example 3.6, if we take $E = \{c, d\}$ and $B = \{a, b, e\}$, then $\{E, B\}$ is a partition of X. We can observe that $(X, *, E)$ is a weak eGE-algebra, but $(X, *, B)$ is not a weak eGE-algebra since $B * b = \{b, c, d\} \neq \{b\}$.

By Remark 3.7, we know that if $(X, *, E)$ is a weak eGE-algebra, then $(X, *, X \setminus E)$ is not a weak eGE-algebra.

Theorem 3.2. *Every weak eGE-algebra $(X, *, E)$ with $E = \{1\}$ satisfies the condition (eGE2).*

Proof. It is straightforward. □

Question 3.3. *Let $(X, *, E)$ be a weak eGE-algebra. If E contains the constant 1, then does (eGE2) hold?*

The next example give a negative answer to Question 3.3.

Example 3.9. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 5.

Table 5: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	1	1	3	3	1
1	0	1	2	3	4
2	0	1	2	3	4
3	0	1	0	1	1
4	0	1	0	1	1

Then $(X, *, E)$ with $E = \{1, 2\}$ is a weak eGE-algebra. But it does not satisfy (eGE2) since $2 * E = \{0, 2\} \nsubseteq E$.

Note that two facts below are equivalent in an eGE-algebra $(X, *, E)$ (see [3]).

(8) $$(\forall x, y, z \in X)(x * y \leq_e (z * x) * (z * y)).$$

(9) $$(\forall x, y, z \in X)(x * y \leq_e (y * z) * (x * z)).$$

In the next example, we can verify that (8) and (9) are not equivalent in a weak eGE-algebra.

Example 3.10. Let $X = \{0, a, b, c, d, e, f\}$ be a set with the Cayley table which is given in Table 6.

Then $(X, *, E)$ with $E = \{a, d\}$ is a weak eGE-algebra. But (8) and (9) are not equivalent. In fact, $(0 * b) * ((c * 0) * (c * b)) = a * (0 * b) = a * a = a \in E$, that is, $(0 * b) \leq_e (c * 0) * (c * b)$. But $(0 * b) * ((b * c) * (0 * c)) = a * (f * a) = a * e = e \notin E$, i.e., $(0 * b) \leq_e (b * c) * (0 * c)$ does not hold.

Table 6: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d	e	f
0	a	a	a	a	a	a	a
a	0	a	b	c	d	e	f
b	0	0	d	f	d	0	f
c	0	0	b	d	d	0	d
d	0	a	b	c	d	e	f
e	0	a	0	0	0	a	0
f	0	e	b	d	d	e	d

Definition 3.11. *A (weak) eGE-algebra $(X, *, E)$ is said to be transitive if it satisfies:*

$$(\forall x, y, z \in X)(x * y \leq_e (z * x) * (z * y)). \tag{10}$$

It is clear that every transitive eGE-algebra is a transitive weak eGE-algebra.

Example 3.12. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 7.

Table 7: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	0
b	0	a	b	c	d
c	0	a	0	0	a
d	b	b	b	c	b

Then $(X, *, E)$ with $E = \{0, b\}$ is a transitive (weak) eGE-algebra.

Lemma 3.1. *Every transitive weak eGE-algebra $(X, *, E)$ satisfies:*

$$(\forall x, y, z \in X)(y \leq_e z \ \Rightarrow \ x * y \leq_e x * z,\ z * x \leq_e y * x). \tag{11}$$

$$(\forall x, y, z \in X)(x \leq_e y,\ y \leq_e z \ \Rightarrow \ x \leq_e z). \tag{12}$$

Proof. Let $x, y, z \in X$ be such that $y \leq_e z$. Then $y * z \in E$, which implies from (eGE3) and (10) that

$$(x * y) * (x * z) = (y * z) * ((x * y) * (x * z)) \in E,$$

that is, $x * y \leq_e x * z$. The combination of (GE3), (eGE3) and (10) induces

$$\begin{aligned}(z * x) * (y * x) &= (y * z) * ((z * x) * (y * x)) \\ &= (y * z) * ((z * x) * ((y * z) * (y * x))) \\ &= (z * x) * ((y * z) * (y * x)) \in E,\end{aligned}$$

and so $z * x \leq_e y * x$. Hence, (11) is valid. Let $x, y, z \in X$ be such that $x \leq_e y$ and $y \leq_e z$. Then $x * y \in E$ and $y * z \in E$. Using (eGE3) and (10), we have

$$x * z = (y * z) * ((x * y) * (x * z)) \in E,$$

and thus $x \leq_e z$. □

Corollary 3.1. *Every transitive eGE-algebra* $(X, *, E)$ *satisfies* (11) *and* (12).

The following example shows that any weak eGE-algebra $(X, *, E)$ does not satisfy the following assertion.

$$(\forall x, y, z \in X)(y * z \in E \ \Rightarrow x * (y * z) \in E). \tag{13}$$

Example 3.13. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 8.

Table 8: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	0	a	b	c	d
a	b	b	b	c	c
b	0	a	b	c	d
c	a	a	b	b	a
d	0	0	b	b	0

Then, $(X, *, E)$ with $E = \{0, b\}$ is a weak eGE-algebra. But it doesn't satisfy (13). In fact, $d * 0 = 0 \in E$ but $c * (d * 0) = c * 0 = a \notin E$.

Proposition 3.3. *For any weak eGE-algebra* $(X, *, E)$ *satisfying the condition* (13), *we have*

$$(\forall x, y, z \in X)\left(x \leq_e y * z \ \Rightarrow \begin{cases} y * (x * z) \in E \\ y * (x * (y * z)) \in E \end{cases} \right). \tag{14}$$

Proof. Let $(X, *, E)$ be a weak eGE-algebra that satisfies the condition (13). Let $x, y, z \in X$ be such that $x \leq_e y * z$. Then $x * (y * z) \in E$ and hence

$$y * (x * z) = y * (x * (y * z)) \in E$$

by (GE3) and (13). □

Since every eGE-algebra $(X, *, E)$ satisfies the condition (13) (see [3]), we have the following corollary.

Corollary 3.2. *Every eGE-algebra* $(X, *, E)$ *satisfies the condition* (14).

Theorem 3.4. *Let* $(X, *, E)$ *be a weak eGE-algebra satisfying the condition* (13) *where* E *contains the constant* 1, *and let* $Y := \{1\} \cup (X \setminus E)$. *Define a binary operation "*$\circledast$*" on* Y *as follows:*

$$\circledast : Y \times Y \to Y,\ (x, y) \mapsto \begin{cases} x * y, & \text{if } x \neq 1 \neq y,\, x * y \notin E, \\ 1, & \text{if } x \neq 1 \neq y,\, x * y \in E, \\ y, & \text{if } x = 1, \\ 1, & \text{if } y = 1. \end{cases} \tag{15}$$

Then $(Y, \circledast, 1)$ *is a GE-algebra.*

Proof. (GE1) and (GE2) are directly identified by the definition of $\circledast$. Let $x, y, z \in X$. It is clear that if $x = 1$, $y = 1$ or $z = 1$, then

$$x \circledast (y \circledast z) = x \circledast (y \circledast (x \circledast z)).$$

Assume that $x \neq 1$, $y \neq 1$ and $z \neq 1$. If $y * z \in E$, then $y \circledast z = 1$ and so $x \circledast (y \circledast z) = x \circledast 1 = 1$.

On the other hand, if $x * z \in E$, then $x \circledast z = 1$. Hence,

$$x \circledast (y \circledast (x \circledast z)) = x \circledast (y \circledast 1) = x \circledast 1 = 1.$$

If $x * z \notin E$, then $x \circledast z = x * z$. Since $y * z \in E$, we have $x * (y * z) \in E$, that is, $x \leq_e y * z$ by (13), and thus $y * (x * z) \in E$ by Proposition 3.3. Hence, $y \circledast (x \circledast z) = y \circledast (x * z) = 1$, and so $x \circledast (y \circledast (x \circledast z)) = x \circledast 1 = 1$. This shows that $x \circledast (y \circledast z) = x \circledast (y \circledast (x \circledast z))$ when $y * z \in E$. If $y * z \notin E$, then $y \circledast z = y * z$, and either $x * (y * z) \in E$ or $x * (y * z) \notin E$. For the case $x * (y * z) \in E$, we get $x \circledast (y \circledast z) = x \circledast (y * z) = 1$, and $y * (x * z) \in E$ by Proposition 3.3. Thus $y \circledast (x \circledast z) = y \circledast (x * z) = 1$ when $x * z \notin E$. If $x * z \in E$, then $x \circledast z = 1$ and so $y \circledast (x \circledast z) = y \circledast 1 = 1$. Hence,

$$x \circledast (y \circledast (x \circledast z)) = x \circledast 1 = 1 = x \circledast (y \circledast z).$$

For the case $x * (y * z) \notin E$, we get $y * z \notin E$ by (13), and $y * (x * z) \notin E$ by Proposition 3.3. Then $x * z \notin E$ by (13). Since $1 \in E$ and

$$x * (y * (x * z)) = x * (y * z) \notin E$$

by (GE3), it follows that

$$\begin{aligned} x \circledast (y \circledast z) &= x \circledast (y * z) = x * (y * z) \\ &= x * (y * (x * z)) \\ &= x \circledast (y * (x * z)) \\ &= x \circledast (y \circledast (x \circledast z)). \end{aligned}$$

Therefore, $(Y, \circledast, 1)$ is a GE-algebra. □

Note that, every eGE-algebra $(X, *, E)$ satisfies the condition (13) and it is a weak eGE-algebra. Hence, we have the next corollary.

Corollary 3.3 ([3]). *Let $(X, *, E)$ be an eGE-algebra where E contains the constant 1 and consider $Y := \{1\} \cup (X \setminus E)$. If we give a binary operation "$\circledast$" on Y by (15), then $(Y, \circledast, 1)$ is a GE-algebra.*

The following example illustrates Theorem 3.4.

Example 3.14. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 9.

Table 9: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	0	0	0	4
3	0	0	0	0	4
4	1	1	1	3	1

Then $(X, *, E)$ with $E = \{0, 1\}$ is a weak eGE-algebra satisfying the condition (13), and $Y = \{1\} \cup (X \setminus E) = \{1, 2, 3, 4\}$. The operation $\circledast$ on Y is given by Table 10, and $(Y, \circledast, 1)$ is a GE-algebra.

Table 10: Cayley table for the binary operation "$\circledast$"

$\circledast$	1	2	3	4
1	1	2	3	4
2	1	1	1	4
3	1	1	1	4
4	1	1	3	1

4. Extended GE-filters

Given a superset F of E in a weak eGE-algebra $(X, *, E)$, we consider the next arguments:

$$(\forall a \in E)(\forall x, y \in X)(x * (a * y) \in F \;\Rightarrow\; a * y \in F), \tag{16}$$

$$(\forall a \in E^c)(\forall x, y \in X)(x * (a * y) \in F \;\Rightarrow\; a * y \in F). \tag{17}$$

The following example shows that there exists a weak eGE-algebra $(X, *, E)$ in which any supserset F of E does not satisfy the assertion (16) or (17).

Table 11: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	a	b	c	d
b	c	a	a	c	c
c	0	0	0	0	0
d	0	0	0	0	0

Example 4.1. 1. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 11.

Then, $(X, *, E)$ with $E = \{0, a\}$ is a weak eGE-algebra. If we take a superset $F = \{0, a, b\}$ of E, then $d * (a * c) = d * c = 0 \in F$ and $a \in E$ but $a * c = c \notin F$. Hence, F does not satisfy the assertion (16).

2. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 12.

Table 12: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	b	a	b	a	a
a	0	c	0	c	c
b	0	a	b	c	d
c	0	a	b	c	d
d	0	a	0	c	c

Then, $(X, *, E)$ with $E = \{b, c\}$ is a weak eGE-algebra, and the set $F = \{b, c, d\}$ does not satisfy (17) since $0 * (a * 0) = 0 * 0 = b \in F$ but $a * 0 = 0 \notin F$.

We provide a condition for a superset of E in a weak eGE-algebra $(X, *, E)$ to be an eGE-filter.

Theorem 4.1. *Let F be a superset of E in a weak eGE-algebra $(X, *, E)$. If F satisfies* (16)*, then F is an eGE-filter of $(X, *, E)$.*

Proof. Let $x, y \in X$ be such that $x \in F$ and $x * y \in F$. Then $x * (E * y) = x * \{y\} \subseteq F$, and so $x * (a * y) \in F$, for all $a \in E$. It follows from (16) that $a * y \in F$, for all $a \in E$. Hence, $\{y\} = E * y \subseteq F$, and thus $y \in F$. Therefore, F is an eGE-filter of $(X, *, E)$. □

Question 4.2. *If a superset F of E in a weak eGE-algebra $(X, *, E)$ satisfies* (17)*, then is F an eGE-filter of $(X, *, E)$?*

The answer to the Question 4.2 is negative as seen in the next example.

Table 13: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	b	b	b	d	d
a	b	b	b	b	b
b	0	a	b	c	d
c	0	a	b	c	d
d	b	b	b	b	b

Example 4.2. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 13. Then $(X, *, E)$ with $E = \{b, c\}$ is a weak eGE-algebra, and the set $F = \{b, c, d\}$ satisfies (17). But F is not eGE-filter of $(X, *, E)$ since $d * 0 = b \in F$ and $d \in F$ but $0 \notin F$.

Theorem 4.3. *Let F be a superset of E in a weak eGE-algebra $(X, *, E)$. Then F is an eGE-filter of $(X, *, E)$ if and only if it satisfies:*

$$(\forall x \in E)(\forall y, z \in X)(x * (y * z) \in F,\ x * y \in F \ \Rightarrow\ x * z \in F). \tag{18}$$

Proof. Assume that F is an eGE-filter of $(X, *, E)$. Let $x, y, z \in X$ be such that $x \in E$, $x * (y * z) \in F$ and $x * y \in F$. Then $y * z = x * (y * z) \in F$ and $y = x * y \in F$ by (eGE3). It follows from (eGE3) and (5) that $x * z = z \in F$.

Conversely, suppose that F satisfies (18). Assume that $x \in F$ and $x * y \in F$, for all $x, y \in X$. Then $E * x = \{x\} \subseteq F$ and $E * (x * y) = \{x * y\} \subseteq F$. It follows that $a * x \in F$ and $a * (x * y) \in F$, for all $a \in E$. Hence, $a * y \in F$, for all $a \in E$ by (18), and so $\{y\} = E * y \subseteq F$, that is $y \in F$. Therefore, F is an eGE-filter of $(X, *, E)$. □

Given an eGE-algebra $(X, *, E)$ and any element $a, b \in X$, consider the following set.

$$E_a := \{x \in X \mid a * x \in E\}, \tag{19}$$

$$E(a, b) := \{x \in X \mid a * (b * x) \in E\}. \tag{20}$$

The set E_a(resp. $E(a, b)$) is called an upper set of a(resp. of a and b).

Proposition 4.1. *Let $(X, *, E)$ be an eGE-algebra and $a, b \in X$. Then*

(i) $a \in E_a$ *and* $a, b \in E(a, b)$.

(ii) $E_a \subseteq E(a, x)$, *for all* $x \in X$.

(iii) $E(a, b) = E(b, a)$.

(iv) $a \leq_e b \ \Rightarrow\ b \in E(a, c)$, *for all* $c \in X$.

(v) $b \in E \ \Rightarrow\ E(a, b) \subseteq E_a$.

(vi) $E_a = \bigcap_{x\in X} E(a,x)$.

Proof. (i) It is straightforward.

(ii) If $z \in E_a$ and $x \in X$, then $a * z \in E$ and so $x * (a * z) \in x * E \subseteq E$ by (eGE2). It follows from (14) that $a * (x * z) \in E$. Hence, $z \in E(a,x)$, and thus $E_a \subseteq E(a,x)$, for all $x \in X$.

(iii) it is straightforward by (14).

(iv) Assume $a \leq_e b$ and let $c \in X$. Then $a*b \in E$, and so $c*(a*b) \in c*E \subseteq E$. Hence, $b \in E(c,a) = E(a,c)$.

(v) Let $b \in E$. Then $b * x \in E * x = \{x\}$ by (eGE3), and so $b * x = x$, for all $x \in X$. If $y \in E(a,b)$, then $a * y = a * (b * y) \in E$, i.e., $y \in E_a$. Hence, $E(a,b) \subseteq E_a$.

(vi) We have $E_a \subseteq \bigcap_{x\in X} E(a,x)$ by (ii). If $y \in \bigcap_{x\in X} E(a,x)$, then $y \in E(a,x)$, i.e., $a * (x * y) \in E$, for all $x \in X$ and so $a * (b * y) \in E$ for $b \in E$. It follows from (eGE3) that $a * y = a * (b * y) \in E$, that is, $y \in E_a$. Hence, $\bigcap_{x\in X} E(a,x) \subseteq E_a$, and therefore (vi) is valid. □

The following example shows that the set E_a may not be an eGE-filter of a weak eGE-algebra $(X,*,E)$.

Example 4.3. Let $X = \{0,a,b,c,d\}$ be a set with the Cayley table which is given in Table 14. Then $(X,*,E)$ with $E = \{a,b\}$ is a weak eGE-algebra which

Table 14: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	a	a	a	a	a
a	0	a	b	c	d
b	0	a	b	c	d
c	0	a	0	a	d
d	b	a	b	c	b

is not eGE-algebra. We can observe that $E_d = \{0,a,b,d\} \subseteq E$. But E_d is not an eGE-filter of $(X,*,E)$. In fact, $0 * c = a \in E_d$ and $0 \in E_d$ but $c \notin E_d$.

We provide conditions for the set E_a to be an eGE-filter.

Theorem 4.4. *If a weak eGE-algebra $(X,*,E)$ satisfies:*

$$(\forall x,y,z \in X)(x * (y * z) = (x * y) * (x * z)), \tag{21}$$

then E_a is an eGE-filter of $(X,*,E)$, for all $a \in X$.

Proof. It is clear that E_a is a superset of E. Let $x,y \in X$ be such that $x \in E_a$ and $x * y \in E_a$. Then $a * x \in E$ and $(a * x) * (a * y) = a * (x * y) \in E$ by (21). Since E is an eGE-filter of X, it follows from (5) that $a * y \in E$, that is, $y \in E_a$. Therefore, E_a is an eGE-filter of X. □

Corollary 4.1. *If an eGE-algebra* $(X, *, E)$ *satisfies* (21)*, then* E_a *is an eGE-filter of* $(X, *, E)$*, for all* $a \in X$*.*

The following example shows that there exist $a, b \in X$ such that the set $E(a, b)$ may not be an eGE-filter of a weak eGE-algebra $(X, *, E)$.

Example 4.4. Let $X = \{0, a, b, c, d, e\}$ be a set with the Cayley table which is given in Table 15. Then $(X, *, E)$ with $E = \{a, b\}$ is a weak eGE-algebra

Table 15: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d	e
0	a	a	a	a	a	a
a	0	a	b	c	d	e
b	0	a	b	c	d	e
c	0	a	b	b	a	e
d	0	b	b	c	b	b
e	b	d	b	c	d	b

which is not an eGE-algebra. Let $c, d \in X$. Then we can observe that $E(d, c) = \{a, b, d, e\}$ and $E \subseteq E(d, c)$. But $E(d, c)$ is not an eGE-filter of $(X, *, E)$ since $e * 0 = b \in E(d, c)$ and $e \in E(d, c)$ but $0 \notin E(d, c)$.

We provide a condition for the set $E(a, b)$ to be an eGE-filter, for all $a, b \in X$.

Theorem 4.5. *If a weak eGE-algebra* $(X, *, E)$ *satisfies* (21)*, then* $E(a, b)$ *is an eGE-filter of* $(X, *, E)$*, for all* $a, b \in X$*.*

Proof. Let $a, b \in X$. It is clear that $E(a, b)$ is a superset of E. Let $x, y \in X$ be such that $x \in E(a, b)$ and $x * y \in E(a, b)$. Then $a * (b * x) \in E$ and $a * (b * (x * y)) \in E$. Using (21), we have

$$(a * (b * x)) * (a * (b * y)) = a * ((b * x) * (b * y)) = a * (b * (x * y)) \in E.$$

Since E is an eGE-filter of $(X, *, E)$, it follows from (5) that $a * (b * y) \in E$, i.e., $y \in E(a, b)$. Therefore, $E(a, b)$ is an eGE-filter of $(X, *, E)$, for all $a, b \in X$. $\square$

Corollary 4.2. *If an eGE-algebra* $(X, *, E)$ *satisfies* (21)*, then* $E(a, b)$ *is an eGE-filter of* $(X, *, E)$*, for all* $a, b \in X$*.*

Theorem 4.6. *Let* F *be a nonempty subset of* X *in a weak eGE-algebra* $(X, *, E)$*. Then* F *is an eGE-filter of* $(X, *, E)$ *if and only if it satisfies:*

$$(\forall a, b \in F)(E(a, b) \subseteq F). \tag{22}$$

Proof. Assume that F is an eGE-filter of $(X, *, E)$ and let $x \in E(a, b)$, for all $a, b \in F$. Then $a * (b * x) \in E \subseteq F$, and so $x \in F$ by (5). Hence, $E(a, b) \subseteq F$, for all $a, b \in F$.

Conversely, suppose F satisfies (22). Then F is a superset of E since $E \subseteq E(a,b) \subseteq F$, for all $a,b \in F$. Let $x,y \in X$ be such that $x \in F$ and $x * y \in F$. Since $(x*y)*(x*y) \in E$ by (eGE1), we have $y \in E(x*y,x) \subseteq F$. Consequently, F is an eGE-filter of $(X,*,E)$. □

Corollary 4.3. *Let F be a nonempty subset of X in an eGE-algebra $(X,*,E)$. Then F is an eGE-filter of $(X,*,E)$ if and only if it satisfies (22).*

Proposition 4.2. *If F is an eGE-filter of a weak eGE-algebra $(X,*,E)$, then $F = \bigcup_{a,b\in F} E(a,b)$.*

Proof. Let $x \in F$. The combination of (eGE1) and (eGE3) induces $x*(y*x) \in E$, for all $y \in E$. Hence, $x \in E(x,y)$, and so

$$F \subseteq \bigcup_{x\in F, y\in E} E(x,y) \bigcup_{a,b\in F} E(a,b).$$

If $x \in \bigcup_{a,b\in F} E(a,b)$, then $x \in E(y,z)$ for some $y,z \in F$ and thus $x \in F$ by Theorem 4.6. This shows that $\bigcup_{a,b\in F} E(a,b) \subseteq F$, and we conclude that $F = \bigcup_{a,b\in F} E(a,b)$. □

Corollary 4.4. *If F is an eGE-filter of an eGE-algebra $(X,*,E)$, then $F = \bigcup_{a,b\in F} E(a,b)$.*

5. Conclusion

We have introduced a broader concept than eGE algebra, called weak eGE algebra and its properties are investigated. We have also considered the concept of transitive and tightly (weak) eGE algebra and some properties are discussed. We have provided a way to create a GE-algebra using a weak eGE-algebra with additional conditions. We have introduced the notions of eGE-filter and upper set and associated properties are investigated. Conditions for a superset of E in a weak eGE-algebra $(X,*,E)$ to be an eGE-filter are provided. We have established the characterization of the eGE-filter.

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A study on intuitionistic fuzzy topological operators

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Abstract. In this paper, new intuitionistic fuzzy topological operators are introduced by considering Marinov and Atanassov's last operators. We show that these operators are also pair of conjugate preinterior-preclosure operators. In addition, some properties of these operators are examined.
Keywords: intuitionistic fuzzy sets, intuitionistic fuzzy pretopological operators, intuitionistic fuzzy topological operators.

1. Introduction

Fuzzy set theory was introduced by Zadeh [18] as an object whose elements have memberships degrees in the $[0,1]$ interval. In following years, many researchers studied on the generalization of the fuzzy set concept. Atanassov introduced the concept of Intuitionistic Fuzzy Sets, form an extension of fuzzy sets by expanding the truth value set to the lattice $[0,1]\times[0,1]$ is defined as following:

Definition 1.1. *Let* $L=[0,1]$ *then* $L^*=\{(x_1,x_2)\in[0,1]^2 : x_1+x_2\leq 1\}$ *is a lattice with* $(x_1,x_2)\leq(y_1,y_2) :\Longleftrightarrow$ "$x_1\leq y_1$ *and* $x_2\geq y_2$"*. For* $(x_1,y_1),(x_2,y_2)\in L^*$*,the operators* $\wedge$ *and* $\vee$ *on* $(L^*,\leq)$ *are defined as following;*

$$(x_1,y_1)\wedge(x_2,y_2)=(\min(x_1,x_2),\max(y_1,y_2)),$$
$$(x_1,y_1)\vee(x_2,y_2)=(\max(x_1,x_2),\min(y_1,y_2)).$$

For each $J\subseteq L^*$ $\sup J=(\sup\{x:(x,y\in[0,1]),((x,y)\in J)\},\inf\{y:(x,y\in[0,1])((x,y)\in J)\})$ *and* $\inf J=(\inf\{x:(x,y\in[0,1])((x,y)\in J)\},\sup\{y:(x,y\in[0,1])((x,y)\in J)\})$.

Topology concept is widely used by mathematicians and other scientists in modeling real-world structures and problems. This approach is based on identifying and using the common points of different shapes. The Intuitionistic Fuzzy Topology was defined by Çoker in 1997 ([6]). Expanding the topology theory on intuitionistic fuzzy sets has attracted the attention of many researchers.Various studies were done based on the application and theoretical fields [8, 9, 10, 13].

The application of intuitionistic fuzzy topology on spatial objects was examined firstly by M.R. Malek [11].

Operator theory has an important role in modeling real-world problems. The concept of intuitionistic fuzzy modal operators was defined by K. Atanassov in 1999 and then modal operators were studied extensively in various fields (see [2, 4, 7, 15]). The first intuitionistic fuzzy topological operators were defined by K. Atanassov and in subsequent studies new intuitionistic fuzzy topological operators were introduced (see [4, 8, 15]). Fuzzy and intuitionistic fuzzy pre-topological /topological operators are applied in computing the values of fuzzy relations of spatial objects with uncertainty in determining the boundaries such as forest area, lake, sea, etc. ([5, 13, 14, 16, 17]). Therefore, the defining of new topological operators is important for approaching spatial problems.

In this paper, new intuitionistic fuzzy topological operators are introduced and some properties are examined.

2. Preliminaries

Definition 2.1 ([2])**.** *An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A", $\nu_A(x), (\nu_A : X \to [0,1])$ is called the "degree of non- membership of x in A", and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \textit{ for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$. The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Definition 2.2 ([2])**.** *An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.*

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 2.3 ([2])**.** *Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the above set is callede the complement of A*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\},$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}.$$

Some special Intuitionistic Fuzzy Sets on X are defined as following;

$$O^* = \{\langle x, 0, 1\rangle : x \in X\},$$

$$X^{*} = \{< x, 1, 0 >: x \in X\}.$$

Atanassov introduced topological operators, and the extensions of these operators was defined by same author in 2001 as:

Definition 2.4 ([2]). *Let X be a set and $A \in IFS(X)$*

$$C(A) = \{\langle x, K, L\rangle : x \in X\},$$

where $K = \sup_{y \in X} \mu_A(y)$, $L = \inf_{y \in X} \nu_A(y)$ and

$$I(A) = \{\langle x, k, l\rangle : x \in X\},$$

where $k = \inf_{y \in X} \mu_A(y), l = \sup_{y \in X} \nu_A(y)$.

Definition 2.5 ([3, 4]). *Let X be a set and $A \in IFS(X)$. Let K, L, k and l be as above forms:*

1. $C_\mu(A) = \{\langle x, K, \min(1 - K, \nu_A(x))\rangle : x \in X\}$;
2. $C_\nu(A) = \{\langle x, \mu_A(x), L\rangle : x \in X\}$;
3. $I_\mu(A) = \{\langle x, k, \nu_A(x)\rangle : x \in X\}$;
4. $I_\nu(A) = \{\langle x, \min(1 - l, \mu_A(x)), l\rangle : x \in X\}$.

The characteristic of these operators were investigated in the same study. We will now define some topological concepts that we refer to in this study.

Definition 2.6 ([1]). *An pre$-$closure operator $\mathbf{c} : \mathbf{X} \to \mathbf{X}$ is a map which associates to each set $A \in \mathbf{X}$ a set $\mathbf{c}(A)$ such that:*

1. $\mathbf{c}(\emptyset) = \emptyset$;
2. $A \subseteq \mathbf{c}(A)$;
3. $\mathbf{c}(A \cup B) = \mathbf{c}(A) \cup \mathbf{c}(B)$, *for all $A, B \subset X$.*

If in addition to above axioms the operator $\mathbf{c}$ is idempotent, that is $\mathbf{c}(A) = \mathbf{c}(\mathbf{c}(A))$ then $\mathbf{c}$ is called closure operator in $\mathbf{X}$. $\mathbf{X}$ can be $\wp(X), FS(X)$ or $IFS(X)$.

Definition 2.7 ([1]). *For the pre$-$closure operator $\mathbf{c}$ defined on $\mathbf{X}$ we say that a set $A \in \mathbf{X}$ is closed iff $\mathbf{c}(A) = A$. Also,*

$$\tau^{\mathbf{c}} = \{A : A \in \mathbf{X} \ \& \ \mathbf{c}(A) = A\}$$

is the topology generated by the pre$-$closure operator $\mathbf{c}$. If $\mathbf{X}$ is $\wp(X), FS(X)$ or $IFS(X)$ then τ is called crisp topology, fuzzy topology or intuitionistic fuzzy topology, respectively.

Definition 2.8 ([1]). *An pre−interior operator* $\mathbf{i} : \mathbf{X} \to \mathbf{X}$ *is a map which associates to each set* $A \in \mathbf{X}$ *a set* $\mathbf{i}(A)$ *such that:*

1. $\mathbf{i}(\mathbf{X}) = \mathbf{X}$;
2. $\mathbf{i}(A) \subseteq A$;
3. $\mathbf{i}(A \cap B) = \mathbf{i}(A) \cap \mathbf{i}(B)$, *for all* $A, B \subset X$.

If in addition to above axioms the operator $\mathbf{i}$ is idempotent, that is $\mathbf{i}(A) = \mathbf{i}(\mathbf{i}(A))$ then $\mathbf{i}$ is called interior operator in $\mathbf{X}$. $\mathbf{X}$ can be $\wp(X), FS(X)$ or $IFS(X)$.

Definition 2.9 ([1]). *For the pre−interior operator* $\mathbf{i}$ *defined on* $\mathbf{X}$ *we say that a set* $A \in \mathbf{X}$ *is open iff* $\mathbf{i}(A) = A$. *Also,*

$$\tau_{\mathbf{i}} = \{A : A \in \mathbf{X} \ \& \ \mathbf{i}(A) = A\}$$

is the topology generated by the pre−interior operator $\mathbf{i}$. *If* $\mathbf{X}$ *is* $\wp(X), FS(X)$ *or* $IFS(X)$ *then* τ *is called crisp topology, fuzzy topology or intuitionistic fuzzy topology, respectively.*

Remark 2.1. If $\mathbf{i}$ is (pre)interior operator then $\mathbf{c}(A) = \neg\mathbf{i}(\neg A)$ is its corresponding $pre-$closure. That is $(\mathbf{c}(A), \neg\mathbf{i}(\neg A))$ is a pair of conjugate preclosure-preinterior operators.

Proposition 2.1 ([1]). *If* $\mathbf{i}$ *and* $\mathbf{c}$ *is a conjugate pair of preinterior and preclosure operators in* $\mathbf{X}$, *then*

$$\tau^{\mathbf{c}} = \{\neg A : A \in \tau_{\mathbf{i}}\} \ \text{ and } \tau_{\mathbf{i}} = \{\neg B : B \in \tau^{\mathbf{c}}\}.$$

In [14], Marinov and Atanassov generalized the $pre-$interior and $pre-$closure operators to intuitionistic fuzzy sets and introduced new intuitionistic fuzzy topological operators. In the same paper, they examined topological properties of these operators in detail.

Definition 2.10 ([14]). *Let us denote* $\overline{\alpha} = (\alpha_0, \alpha_1)$ *and* $\overline{\beta} = (\beta_0, \beta_1)$, *where* $\alpha_i, \beta_i \in [0,1]$ *for* $i \in \{0,1\}$ *and* $\alpha_0 \leq \alpha_1, \beta_0 \leq \beta_1$. *For every* $\gamma_{\overline{\alpha}}, \gamma_{\overline{\beta}} \in [0,1]$ *and based on an arbitrary* $A \in IFS(X)$. *The topological operators*

$$I^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\mu;\overline{\alpha},\overline{\beta}}, C^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\nu;\overline{\alpha},\overline{\beta}} : IFS(X) \to IFS(X)$$

are defined as follow;

$$\mu_{I^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\mu;\overline{\alpha},\overline{\beta}}(A)}(x) = \begin{cases} \mu_A(x), & 0 \leq \mu_A(x) < \alpha_0 \\ \alpha_0, & \alpha_0 \leq \mu_A(x) < \alpha_0 + \gamma_{\overline{\alpha}}(\alpha_1 - \alpha_0) \\ \frac{1}{1-\gamma_{\overline{\alpha}}}(\mu_A(x) - \alpha_1) + \alpha_1, & \alpha_0 + \gamma_{\overline{\alpha}}(\alpha_1 - \alpha_0) \leq \mu_A(x) < \alpha_1 \\ \mu_A(x), & \alpha_1 \leq \mu_A(x) \leq 1 \end{cases},$$

$$\nu_{I^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\mu;\overline{\alpha},\overline{\beta}}(A)}(x)=\begin{cases}\nu_A(x), & 0\leq \nu_A(x)<\beta_0\\ \min\left\{\begin{array}{c}\left(1-\gamma_{\overline{\beta}}\right)\nu_A(x)+\beta_1\gamma_{\overline{\beta}},\\ 1-\mu_{I^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\mu;\overline{\alpha},\overline{\beta}}(A)}(x)\end{array}\right\}, & \beta_0\leq \nu_A(x)<\beta_1\\ \nu_A(x), & \beta_1\leq \nu_A(x)\leq 1\end{cases}$$

and

$$\mu_{C^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\nu;\overline{\alpha},\overline{\beta}}(A)}(x)=\begin{cases}\mu_A(x), & 0\leq \mu_A(x)<\beta_0\\ \min\left\{\begin{array}{c}\left(1-\gamma_{\overline{\beta}}\right)\mu_A(x)+\beta_1\gamma_{\overline{\beta}},\\ 1-\nu_{C^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\nu;\overline{\alpha},\overline{\beta}}(A)}(x)\end{array}\right\}, & \beta_0\leq \mu_A(x)<\beta_1\ ,\\ \mu_A(x), & \beta_1\leq \mu_A(x)\leq 1\end{cases}$$

$$\nu_{C^{\gamma_{\overline{\alpha}},\gamma_{\overline{\beta}}}_{\nu;\overline{\alpha},\overline{\beta}}(A)}(x)=\begin{cases}\nu_A(x), & 0\leq \nu_A(x)<\alpha_0\\ \alpha_0, & \alpha_0\leq \nu_A(x)<\alpha_0+\gamma_{\overline{\alpha}}(\alpha_1-\alpha_0)\\ \frac{1}{1-\gamma_{\overline{\alpha}}}(\nu_A(x)-\alpha_1)+\alpha_1, & \alpha_0+\gamma_{\overline{\alpha}}(\alpha_1-\alpha_0)\leq \nu_A(x)<\alpha_1\\ \nu_A(x), & \alpha_1\leq \nu_A(x)\leq 1\end{cases}$$

3. Main results

In this study, new intuitionistic fuzzy topological operators are defined by considering the operators defined by Marinov and Atanassov in [14]. The variation of *pre*−closure operator, *pre*−interior operator and boundary value according to varying α,β,γ and ω values is examined with an example.

Definition 3.1. *Let X be a set and $A\in IFS(X)$. For $\alpha,\beta,\gamma,\omega\in[0,1]$, the topological operator $I^{\gamma,\omega}_{\alpha,\beta}$ is defined as follow;*

$$I^{\gamma,\omega}_{\alpha,\beta}: IFS(X)\to IFS(X)$$

such that

$$\mu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x)=\begin{cases}\inf \mu_A(x), & 0\leq \mu_A(x)<\alpha\gamma(1-\beta)\\ (1-\beta)\mu_A(x), & \alpha\gamma(1-\beta)\leq \mu_A(x)<\alpha\gamma\\ \frac{1}{1-\gamma}(\mu_A(x)-\alpha)+\alpha, & \alpha\gamma\leq \mu_A(x)<\alpha\\ \mu_A(x), & \alpha\leq \mu_A(x)\leq 1\end{cases}$$

and

$$\nu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x)=\begin{cases}\nu_A(x), & 0\leq \nu_A(x)<\beta\omega\\ \min\left\{(1-\omega)\nu_A(x)+\beta\omega, 1-\mu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x)\right\}, & \beta\omega\leq \nu_A(x)<\beta\\ \nu_A(x), & \beta\leq \nu_A(x)\leq 1\end{cases}$$

Proposition 3.1. *Let X be a set and $A \in IFS(X)$. For $\alpha, \beta, \gamma, \omega \in [0,1]$, the topological operator $I_{\alpha,\beta}^{\gamma,\omega}(A)$ is an intuitionistic fuzzy set.*

Proof. Suppose that $0 \leq \nu_A(x) < \beta\omega$ or $\beta \leq \nu_A(x) \leq 1$ then $\nu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = \nu_A(x)$ and it is clear that $\mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) \leq \mu_A(x)$, we obtain that $\mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) + \nu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) \leq \mu_A(x) + \nu_A(x) \leq 1$.

On the other hand, if $\beta\omega \leq \nu_A(x) < \beta$ then

$$\begin{aligned}
&\mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) + \min\{(1-\omega)\nu_A(x) + \beta\omega, 1 - \mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x)\}\\
&\leq \mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) + 1 - \mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = 1\\
&\Rightarrow \mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) + \nu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) \leq 1.
\end{aligned}$$ □

Proposition 3.2. *Let X be a set and $A \in IFS(X)$. For $\alpha, \beta, \gamma, \omega \in [0,1]$, the operator $I_{\alpha,\beta}^{\gamma,\omega}(A)$ is a pre$-$interior operator in $IFS(X)$.*

Proof. (i) Let $A = X^*$ then $\mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = \mu_{X^*(A)}(x)$ and $\nu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = \nu_{X^*(A)}(x)$ for all $x \in X^*$. $I_{\alpha,\beta}^{\gamma,\omega}(X^*) = X^*$

(ii) Let's examine the $I_{\alpha,\beta}^{\gamma,\omega}(A)$ under the given conditions.

First, $\inf \mu_A(x) \leq \mu_A(x)$ and $(1-\beta)\mu_A(x) \leq \mu_A(x), \beta \in [0,1]$ for all $x \in X$. Now, let $\mu_A(x) < \alpha$, $x \in X$ then for $\gamma \in [0,1]$,

$$\begin{aligned}
\gamma\mu_A(x) &< \gamma\alpha \Rightarrow \mu_A(x) - \gamma\alpha < \mu_A(x) - \gamma\mu_A(x)\\
&\Rightarrow \mu_A(x) > \frac{\mu_A(x) - \alpha}{1-\gamma} + \alpha\\
&\Rightarrow \mu_A(x) > \frac{1}{1-\gamma}(\mu_A(x) - \alpha) + \alpha
\end{aligned}$$

also, let $\nu_A(x) < \beta$, $x \in X$ then for $\omega \in [0,1]$,$\omega\nu_A(x) < \omega\beta \Rightarrow \nu_A(x) < (1-\omega)\nu_A(x) + \omega\beta$ and $\nu_A(x) < 1 - \mu_{I_{\alpha,\beta}^{\gamma,\omega}(A)}(x)$ for all $x \in X$. $I_{\alpha,\beta}^{\gamma,\omega}(A) \sqsubseteq A$.

(iii) Let $A, B \in IFS(X)$.

$$\mu_{I_{\alpha,\beta}^{\gamma,\omega}(A \sqcap B)}(x)$$
$$= \begin{cases}
\inf(\mu_A(x) \wedge \mu_B(x)), & 0 \leq \mu_A(x) \wedge \mu_B(x) < \alpha\gamma(1-\beta)\\
(1-\beta)(\mu_A(x) \wedge \mu_B(x)), & \alpha\gamma(1-\beta) \leq \mu_A(x) \wedge \mu_B(x) < \alpha\gamma\\
\frac{1}{1-\gamma}((\mu_A(x) \wedge \mu_B(x)) - \alpha) + \alpha, & \alpha\gamma \leq \mu_A(x) \wedge \mu_B(x) < \alpha\\
(\mu_A(x) \wedge \mu_B(x)), & \alpha \leq \mu_A(x) \wedge \mu_B(x) \leq 1
\end{cases}$$

$$= \begin{cases} \inf \mu_A(x) \wedge \inf \mu_B(x), & 0 \le \mu_A(x) \wedge \mu_B(x) < \alpha\gamma(1-\beta) \\ (1-\beta)\mu_A(x) \wedge (1-\beta)\mu_B(x), & \alpha\gamma(1-\beta) \le \mu_A(x) \wedge \mu_B(x) < \alpha\gamma \\ \begin{array}{l} \left(\frac{1}{1-\gamma}(\mu_A(x)-\alpha)+\alpha\right) \wedge \\ \left(\frac{1}{1-\gamma}(\mu_B(x)-\alpha)+\alpha\right), \end{array} & \alpha\gamma \le \mu_A(x) \wedge \mu_B(x) < \alpha \\ (\mu_A(x) \wedge \mu_B(x)), & \alpha \le \mu_A(x) \wedge \mu_B(x) \le 1 \end{cases}$$

$$= \begin{cases} \inf \mu_A(x), & 0 \le \mu_A(x) < \alpha\gamma(1-\beta) \\ (1-\beta)\mu_A(x), & \alpha\gamma(1-\beta) \le \mu_A(x) < \alpha\gamma \\ \frac{1}{1-\gamma}(\mu_A(x)-\alpha)+\alpha, & \alpha\gamma \le \mu_A(x) < \alpha \\ \mu_A(x), & \alpha \le \mu_A(x) \le 1 \end{cases}$$

$$\wedge \begin{cases} \inf \mu_B(x), & 0 \le \mu_A(x) < \alpha\gamma(1-\beta) \\ (1-\beta)\mu_B(x), & \alpha\gamma(1-\beta) \le \mu_A(x) < \alpha\gamma \\ \frac{1}{1-\gamma}(\mu_B(x)-\alpha)+\alpha, & \alpha\gamma \le \mu_A(x) < \alpha \\ \mu_B(x), & \alpha \le \mu_A(x) \le 1 \end{cases}$$

$$= \mu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x) \wedge \mu_{I^{\gamma,\omega}_{\alpha,\beta}(B)}(x)$$

and

$$\nu_{I^{\gamma,\omega}_{\alpha,\beta}(A \sqcap B)}(x)$$

$$= \begin{cases} \nu_A(x) \vee \nu_B(x), & 0 \le \nu_A(x) \vee \nu_B(x) < \beta\omega \\ \min\left\{ \begin{array}{c} (1-\omega)(\nu_A(x) \vee \nu_B(x)) + \beta\omega, \\ 1-(\mu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x) \wedge \mu_{I^{\gamma,\omega}_{\alpha,\beta}(B)}(x)) \end{array} \right\}, & \beta\omega \le \nu_A(x) \vee \nu_B(x) < \beta \\ \nu_A(x) \vee \nu_B(x), & \beta \le \nu_A(x) \vee \nu_B(x) \le 1 \end{cases}$$

$$= \begin{cases} \nu_A(x) \vee \nu_B(x), & 0 \le \nu_A(x) \vee \nu_B(x) < \beta\omega \\ \min\left\{ \begin{array}{c} ((1-\omega)\nu_A(x)+\beta\omega) \vee \\ ((1-\omega)\nu_B(x)+\beta\omega), \\ (1-\mu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x)) \vee (1-\mu_{I^{\gamma,\omega}_{\alpha,\beta}(B)}(x)) \end{array} \right\}, & \beta\omega \le \nu_A(x) \vee \nu_B(x) < \beta \\ \nu_A(x) \vee \nu_B(x), & \beta \le \nu_A(x) \vee \nu_B(x) \le 1 \end{cases}$$

$$= \begin{cases} \nu_A(x), & 0 \le \nu_A(x) < \beta\omega \\ \min\{(1-\omega)\nu_A(x)+\beta\omega, 1-\mu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x)\}, & \beta\omega \le \nu_A(x) < \beta \\ \nu_A(x), & \beta \le \nu_A(x) \le 1 \end{cases}$$

$$\vee \begin{cases} \nu_B(x), & 0 \le \nu_A(x) < \beta\omega \\ \min\{(1-\omega)\nu_B(x)+\beta\omega, 1-\mu_{I^{\gamma,\omega}_{\alpha,\beta}(B)}(x)\}, & \beta\omega \le \nu_A(x) < \beta \\ \nu_B(x), & \beta \le \nu_A(x) \le 1 \end{cases}$$

$$= \nu_{I^{\gamma,\omega}_{\alpha,\beta}(A)}(x) \vee \nu_{I^{\gamma,\omega}_{\alpha,\beta}(B)}(x)$$

This completes the proof. □

Definition 3.2. *Let X be a set and $A \in IFS(X)$. For $\alpha, \beta, \gamma, \omega \in [0,1]$, the topological operator $C_{\nu;\alpha,\beta}^{\gamma,\omega}$ is defined as follow;*

$$C_{\alpha,\beta}^{\gamma,\omega} : IFS(X) \to IFS(X)$$

such that

$$\mu_{C_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = \begin{cases} \mu_A(x), & 0 \le \mu_A(x) < \beta\omega \\ \min\left\{(1-\omega)\mu_A(x) + \beta\omega, 1 - \nu_{C_{\alpha,\beta}^{\gamma,\omega}(A)}(x)\right\}, & \beta\omega \le \mu_A(x) < \beta \\ \mu_A(x), & \beta \le \mu_A(x) \le 1 \end{cases}$$

and

$$\nu_{C_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = \begin{cases} \inf \nu_A(x), & 0 \le \nu_A(x) < \alpha\gamma(1-\beta) \\ (1-\beta)\nu_A(x), & \alpha\gamma(1-\beta) \le \nu_A(x) < \alpha\gamma \\ \frac{1}{1-\gamma}(\nu_A(x) - \alpha) + \alpha, & \alpha\gamma \le \nu_A(x) < \alpha \\ \nu_A(x), & \alpha \le \nu_A(x) \le 1 \end{cases}$$

Proposition 3.3. *Let X be a set and $A \in IFS(X)$. For $\alpha, \beta, \gamma, \omega \in [0,1]$, the topological operator $C_{\alpha,\beta}^{\gamma,\omega}(A)$ is an intuitionistic fuzzy set.*

Proof. It can be proved similarly to Proposition 3.1. □

Proposition 3.4. *Let X be a set and $A \in IFS(X)$. For $\alpha, \beta, \gamma, \omega \in [0,1]$, the operator $C_{\alpha,\beta}^{\gamma,\omega}(A)$ is a pre$-$closure operator in $IFS(X)$.*

Proof. (i) Let $A = O^*$ then $\mu_{C_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = \mu_{O^*(A)}(x)$ and $\nu_{C_{\alpha,\beta}^{\gamma,\omega}(A)}(x) = \nu_{O^*(A)}(x)$ for all $x \in O^*$. $C_{\alpha,\beta}^{\gamma,\omega}(O^*) = O^*$

(ii) It is clear that $\inf \nu_A(x) \le \nu_A(x)$ and $(1-\beta)\nu_A(x) \le \nu_A(x), \beta \in [0,1]$ for all $x \in X$.

Now, let $\nu_A(x) < \alpha$, $x \in X$ then for $\gamma \in [0,1]$,

$$\begin{aligned} \gamma\nu_A(x) &< \gamma\alpha \Rightarrow \nu_A(x) - \gamma\alpha < \nu_A(x) - \gamma\nu_A(x) \\ &\Rightarrow \nu_A(x) > \frac{\nu_A(x) - \alpha}{1-\gamma} + \alpha \\ &\Rightarrow \nu_A(x) > \frac{1}{1-\gamma}(\nu_A(x) - \alpha) + \alpha \end{aligned}$$

and also, let $\mu_A(x) < \beta$, $x \in X$ then for $\omega \in [0,1]$,

$$\omega\mu_A(x) < \omega\beta \Rightarrow \mu_A(x) < (1-\omega)\mu_A(x) + \omega\beta$$

and $\mu_A(x) < 1 - \nu_{C_{\alpha,\beta}^{\gamma,\omega}(A)}(x)$ for all $x \in X$. So, $A \sqsubseteq C_{\alpha,\beta}^{\gamma,\omega}(A)$.

(iii) Let $A, B \in IFS(X)$.

$$
\begin{aligned}
&\nu_{C^{\gamma,\omega}_{\alpha,\beta}(A \sqcup B)}(x) \\
&= \begin{cases}
\inf\left(\nu_A(x) \wedge \nu_B(x)\right), & 0 \le \nu_A(x) \wedge \nu_B(x) < \alpha\gamma(1-\beta) \\
(1-\beta)\left(\nu_A(x) \wedge \nu_B(x)\right), & \alpha\gamma(1-\beta) \le \nu_A(x) \wedge \nu_B(x) < \alpha\gamma \\
\frac{1}{1-\gamma}\left(\left(\nu_A(x) \wedge \nu_B(x)\right) - \alpha\right) + \alpha, & \alpha\gamma \le \nu_A(x) \wedge \nu_B(x) < \alpha \\
\left(\nu_A(x) \wedge \nu_B(x)\right), & \alpha \le \nu_A(x) \wedge \nu_B(x) \le 1
\end{cases} \\
&= \begin{cases}
\inf \nu_A(x) \wedge \inf \nu_B(x), & 0 \le \nu_A(x) \wedge \nu_B(x) < \alpha\gamma(1-\beta) \\
(1-\beta)\nu_A(x) \wedge (1-\beta)\nu_B(x), & \alpha\gamma(1-\beta) \le \nu_A(x) \wedge \nu_B(x) < \alpha\gamma \\
\left(\frac{1}{1-\gamma}\left(\nu_A(x) - \alpha\right) + \alpha\right) \wedge \\ \quad \left(\frac{1}{1-\gamma}\left(\nu_B(x) - \alpha\right) + \alpha\right), & \alpha\gamma \le \nu_A(x) \wedge \nu_B(x) < \alpha \\
\left(\nu_A(x) \wedge \nu_B(x)\right), & \alpha \le \nu_A(x) \wedge \nu_B(x) \le 1
\end{cases} \\
&= \begin{cases}
\inf \nu_A(x), & 0 \le \nu_A(x) < \alpha\gamma(1-\beta) \\
(1-\beta)\nu_A(x), & \alpha\gamma(1-\beta) \le \nu_A(x) < \alpha\gamma \\
\frac{1}{1-\gamma}\left(\nu_A(x) - \alpha\right) + \alpha, & \alpha\gamma \le \nu_A(x) < \alpha \\
\nu_A(x), & \alpha \le \nu_A(x) \le 1
\end{cases} \\
&\wedge \begin{cases}
\inf \nu_B(x), & 0 \le \nu_B(x) < \alpha\gamma(1-\beta) \\
(1-\beta)\nu_B(x), & \alpha\gamma(1-\beta) \le \nu_B(x) < \alpha\gamma \\
\frac{1}{1-\gamma}\left(\nu_B(x) - \alpha\right) + \alpha, & \alpha\gamma \le \nu_B(x) < \alpha \\
\nu_B(x), & \alpha \le \nu_B(x) \le 1
\end{cases} \\
&= \nu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}(x) \wedge \nu_{C^{\gamma,\omega}_{\alpha,\beta}(B)}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mu_{C^{\gamma,\omega}_{\alpha,\beta}(A \sqcup B)}(x) \\
&= \begin{cases}
\mu_A(x) \vee \mu_B(x), & 0 \le \mu_A(x) \vee \mu_B(x) < \beta\omega \\
\min\left\{ \begin{array}{c} (1-\omega)(\mu_A(x) \vee \mu_B(x)) + \beta\omega, \\ 1 - (\nu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}(x) \wedge \nu_{C^{\gamma,\omega}_{\alpha,\beta}(B)}(x)) \end{array} \right\}, & \beta\omega \le \mu_A(x) \vee \mu_B(x) < \beta \\
\nu_A(x) \vee \nu_B(x), & \beta \le \mu_A(x) \vee \mu_B(x) \le 1
\end{cases} \\
&= \begin{cases}
\mu_A(x) \vee \mu_B(x), & 0 \le \mu_A(x) \vee \mu_B(x) < \beta\omega \\
\min\left\{ \begin{array}{c} ((1-\omega)\mu_A(x) + \beta\omega) \vee \\ ((1-\omega)\mu_B(x) + \beta\omega), \\ (1 - \nu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}(x)) \vee (1 - \nu_{C^{\gamma,\omega}_{\alpha,\beta}(B)}(x)) \end{array} \right\}, & \beta\omega \le \mu_A(x) \vee \mu_B(x) < \beta \\
\mu_A(x) \vee \mu_B(x), & \beta \le \mu_A(x) \vee \mu_B(x) \le 1
\end{cases}
\end{aligned}
$$

$$= \begin{cases} \mu_A(x), & 0 \le \mu_A(x) < \beta\omega \\ \min\{(1-\omega)\mu_A(x)+\beta\omega, 1-\nu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}(x)\}, & \beta\omega \le \mu_A(x) < \beta \\ \mu_A(x), & \beta \le \mu_A(x) \le 1 \end{cases}$$

$$\vee \begin{cases} \mu_B(x), & 0 \le \mu_A(x) < \beta\omega \\ \min\{(1-\omega)\mu_B(x)+\beta\omega, 1-\nu_{C^{\gamma,\omega}_{\alpha,\beta}(B)}(x)\}, & \beta\omega \le \mu_A(x) < \beta \\ \mu_B(x), & \beta \le \mu_A(x) \le 1 \end{cases}$$

$$= \mu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}(x) \vee \mu_{C^{\gamma,\omega}_{\alpha,\beta}(B)}(x)$$

Hence, proof completed. □

Proposition 3.5. *The operator $I^{\gamma,\omega}_{\alpha,\beta}$ is generalization of the operator I_μ and the operator $C^{\gamma,\omega}_{\alpha,\beta}$ is generalization of the operator C_ν.*

Proof. Let X be a set and $A \in IFS(X)$. It is clear that to take $\gamma = \alpha = 1, \omega = 0$ and $\beta = 1 - \inf \mu_A(x)$, i.e. $I_\mu = I^{1,0}_{1,(1-k)}$ is provides the definition. On the other hand, if $\gamma{=}\alpha{=}1, \omega{=}0$ and $\beta{=}1 - \inf \nu_A(x)$ then $C_\nu{=}C^{1,0}_{1,(1-L)}$. □

Theorem 1. *Let X be a set and $A \in IFS(X)$ then $C^{\gamma,\omega}_{\alpha,\beta}(A) = \neg\, I^{\gamma,\omega}_{\alpha,\beta}(\neg A)$, i.e $I^{\gamma,\omega}_{\alpha,\beta}$ and $C^{\gamma,\omega}_{\alpha,\beta}$ is a conjugate pair of pre−interior and pre−closure operators. They define the same topology $\tau_{I^{\gamma,\omega}_{\alpha,\beta}} = \{\neg B : B \in \tau^{C^{\gamma,\omega}_{\alpha,\beta}}\}$.*

Proof. Let X be a set and $A \in IFS(X)$.

$$\mu_{I^{\gamma,\omega}_{\alpha,\beta}(\neg A)}(x) = \begin{cases} \inf \nu_A(x), & 0 \le \nu_A(x) < \alpha\gamma(1-\beta) \\ (1-\beta)\nu_A(x), & \alpha\gamma(1-\beta) \le \nu_A(x) < \alpha\gamma \\ \frac{1}{1-\gamma}(\nu_A(x)-\alpha)+\alpha, & \alpha\gamma \le \nu_A(x) < \alpha \\ \nu_A(x), & \alpha \le \nu_A(x) \le 1 \end{cases} = \nu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}(x)$$

and

$$\nu_{I^{\gamma,\omega}_{\alpha,\beta}(\neg A)}(x) = \begin{cases} \mu_A(x), & 0 \le \mu_A(x) < \beta\omega \\ \min\left\{(1-\omega)\mu_A(x)+\beta\omega, 1-\nu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}\right\}, & \beta\omega \le \mu_A(x) < \beta \\ \mu_A(x), & \beta \le \mu_A(x) \le 1 \end{cases}$$

$$= \mu_{C^{\gamma,\omega}_{\alpha,\beta}(A)}(x)$$

So, $C^{\gamma,\omega}_{\alpha,\beta}(A) = \neg\, I^{\gamma,\omega}_{\alpha,\beta}(\neg A)$. From Proposition 2.1, we obtain that $\tau_{I^{\gamma,\omega}_{\alpha,\beta}} = \{\neg B : B \in \tau^{C^{\gamma,\omega}_{\alpha,\beta}}\}$. □

Definition 3.3. *The boundary of set A in the intuitionistic fuzzy topology defined by these pre−interior and pre−closure operators is $\partial A = C^{\gamma,\omega}_{\alpha,\beta}(A) \cap (\neg\, I^{\gamma,\omega}_{\alpha,\beta}(A))$ according to the IF boundary definition given by Malek [11].*

In the following example, $I_{\alpha,\beta}^{\gamma,\omega}$, $C_{\alpha,\beta}^{\gamma,\omega}$ and boundary value of an intuitionistic fuzzy set A are examined, for different α, β, γ and ω.

Example 3.1. Let the universal X and $A \in IFS(X)$ be given in the table below.

X	α	β	γ	ω	(μ_A, ν_A)	$I_{\alpha,\beta}^{\gamma,\omega}(A)$	$C_{\alpha,\beta}^{\gamma,\omega}(A)$	$\partial(A)$
a_0	0.4	0.2	0.7	0.6	(1, 0)	(1, 0)	(1, 0)	(0, 1)
a_1	0.4	0.2	0.7	0.6	(0.65, 0.15)	(0.65, 0.18)	(0.65, 0.01)	(0.18, 0.65)
a_2	0.4	0.2	0.7	0.6	(0.3, 0.4)	(0.066667, 0.4)	(0.3, 0.4)	(0.3, 0.4)
a_3	0.4	0.2	0.7	0.6	(0.25, 0.45)	(0.2, 0.45)	(0.25, 0.45)	(0.25, 0.45)
a_4	0.4	0.2	0.7	0.6	(0.8, 0.07)	(0.8, 0.07)	(0.8, 0.01)	(0.07, 0.8)
a_5	0.4	0.2	0.7	0.6	(0.1, 0.2)	(0.08, 0.2)	(0.1, 0.01)	(0.1, 0.08)
a_6	0.4	0.2	0.7	0.6	(0.2, 0.7)	(0.08, 0.7)	(0.2, 0.7)	(0.2, 0.7)
a_7	0.4	0.2	0.7	0.6	(0.72, 0.24)	(0.72, 0.24)	(0.72, 0.192)	(0.24, 0.72)
a_8	0.4	0.2	0.7	0.6	(0.2, 0.7)	(0.08, 0.7)	(0.2, 0.7)	(0.2, 0.7)
a_9	0.4	0.2	0.7	0.6	(0.35, 0.53)	(0.233333, 0.53)	(0.35, 0.53)	(0.35, 0.53)
a_{10}	0.4	0.2	0.7	0.6	(0.82, 0.1)	(0.82, 0.1)	(0.82, 0.01)	(0.1, 0.82)
a_{11}	0.4	0.2	0.7	0.6	(0.97, 0.01)	(0.97, 0.01)	(0.97, 0.01)	(0.01, 0.97)
a_{12}	0.4	0.2	0.7	0.6	(0.48, 0.32)	(0.48, 0.32)	(0.48, 0. 133333)	(0.32, 0.48)
a_{13}	0.4	0.2	0.7	0.6	(0.15, 0.6)	(0.08, 0.6)	(0.18, 0.6)	(0.18, 0.6)
a_{14}	0.4	0.2	0.7	0.6	(0.55, 0.24)	(0.55, 0.24)	(0.55, 0.192)	(0.24, 0.55)
a_{15}	0.4	0.2	0.7	0.6	(0.74, 0.2)	(0.74, 0.2)	(0.74, 0.01)	(0.2, 0.74)
a_0	0,3	0,8	0,4	0,5	(1, 0)	(1, 0)	(1, 0)	(0, 1)
a_1	0,3	0,8	0,4	0,5	(0.65, 0.15)	(0.65, 0.15)	(0.725, 0.05)	(0.15, 0.65)
a_2	0,3	0,8	0,4	0,5	(0.3, 0.4)	(0.3, 0.6)	(0.3, 0.4)	(0.3, 0.4)
a_3	0,3	0,8	0,4	0,5	(0.25, 0.45)	(0.216667, 0.625)	(0.25, 0.45)	(0.25, 0.45)
a_4	0,3	0,8	0,4	0,5	(0.8, 0.07)	(0.8, 0.07)	(0.8, 0.014)	(0.07, 0.8)
a_5	0,3	0,8	0,4	0,5	(0.1, 0.2)	(0.02, 0.2)	(0.1, 0.13333)	(0.1, 0.1333)
a_6	0,3	0,8	0,4	0,5	(0.2, 0.7)	(0.133333, 0.75)	(0.2, 0.7)	(0.2, 0.7)
a_7	0,3	0,8	0,4	0,5	(0.72, 0.24)	(0.72, 0.24)	(0.76, 0.2)	(0.24, 0.72)
a_8	0,3	0,8	0,4	0,5	(0.2, 0.7)	(0.133333, 0.75)	(0.2, 0.7)	(0.2, 0.7)
a_9	0,3	0,8	0,4	0,5	(0.35, 0.53)	(0.35, 0.65)	(0.35, 0.53)	(0.35, 0.53)
a_{10}	0,3	0,8	0,4	0,5	(0.82, 0.1)	(0.82, 0.1)	(0.82, 0.02)	(0.1, 0.82)
a_{11}	0,3	0,8	0,4	0,5	(0.97, 0.01)	(0.97, 0.01)	(0.97, 0.01)	(0.01, 0.97)
a_{12}	0,3	0,8	0,4	0,5	(0.48, 0.32)	(0.48, 0.32)	(0.64, 0.32)	(0.32, 0.48)
a_{13}	0,3	0,8	0,4	0,5	(0.15, 0.6)	(0.05, 0.7)	(0.15, 0.6)	(0.15, 0.6)
a_{14}	0,3	0,8	0,4	0,5	(0.55, 0.24)	(0.55, 0.24)	(0.675, 0.2)	(0.24, 0.55)
a_{15}	0,3	0,8	0,4	0,5	(0.74, 0.2)	(0.74, 0.2)	(0.77, 0.13333)	(0.2, 0.74)
a_0	0,6	0,7	0,8	0,65	(1, 0)	(1, 0)	(1, 0)	(0, 1)
a_1	0,6	0,7	0,8	0,65	(0.65, 0.15)	(0.65, 0.15)	(0.6825, 0.045)	(0.15, 0.65)
a_2	0,6	0,7	0,8	0,65	(0.3, 0.4)	(0.09, 0.4)	(0.3, 0.12)	(0.3, 0.12)
a_3	0,6	0,7	0,8	0,65	(0.25, 0.45)	(0.075, 0.45)	(0.25, 0.135)	(0.25, 0.135)
a_4	0,6	0,7	0,8	0,65	(0.8, 0.07)	(0.8, 0.07)	(0.8, 0.01)	(0.07, 0.8)
a_5	0,6	0,7	0,8	0,65	(0.1, 0.2)	(0.1, 0.2)	(0.1, 0.06)	(0.1, 0.1)
a_6	0,6	0,7	0,8	0,65	(0.2, 0.7)	(0.06, 0.7)	(0.2, 0.7)	(0.2, 0.7)
a_7	0,6	0,7	0,8	0,65	(0.72, 0.24)	(0.72, 0.24)	(0.72, 0.072)	(0.24, 0.72)
a_8	0,6	0,7	0,8	0,65	(0.2, 0.7)	(0.06, 0.7)	(0.2, 0.7)	(0.2, 0.7)
a_9	0,6	0,7	0,8	0,65	(0.35, 0.53)	(0.105, 0.6405)	(0.35, 0.25)	(0.35, 0.25)
a_{10}	0,6	0,7	0,8	0,65	(0.82, 0.1)	(0.82, 0.1)	(0.82, 0.01)	(0.1, 0.82)
a_{11}	0,6	0,7	0,8	0,65	(0.97, 0.01)	(0.97, 0.01)	(0.97, 0.01)	(0.01, 0.97)
a_{12}	0,6	0,7	0,8	0,65	(0.48, 0.32)	(0, 0.32)	(0.623, 0.096)	(0.32, 0.096)
a_{13}	0,6	0,7	0,8	0,65	(0.15, 0.6)	(0.045, 0.665)	(0.15, 0.6)	(0.15, 0.6)
a_{14}	0,6	0,7	0,8	0,65	(0.55, 0.24)	(0.35, 0.24)	(0.6475, 0.072)	(0.24, 0.35)
a_{15}	0,6	0,7	0,8	0,65	(0.74, 0.2)	(0.74, 0.2)	(0.74, 0.06)	(0.2, 0.74)

Figure 1: Table

As can be seen from the tables, since there are no conditions limiting α, β, γ and ω values, this diversity provides wide application for problems studied using topological operators.

4. Conclusion

In this study, new topological operators are defined on intuitionistic fuzzy sets and their theoretical properties are examined. It is obvious that these defined operators will contribute to the modeling of real-world problems with uncertainty in determining the boundaries.

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Normal structure and the modulus of weak uniform rotundity in Banach spaces

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Abstract. In this paper, we present some sufficient conditions for which a Banach space X has normal structure in term of the modulus of weak uniform rotundity $\delta_X(\epsilon, f)$, the Domínguez-Benavides coefficient $R(1, X)$ and the coefficient of weak orthogonality $\omega(X)$. Some known results are improved and strengthened.

Keywords: the modulus of weak uniform rotundity, Domínguez-Benavides coefficient, coefficient of weak orthogonality, normal structure.

1. Introduction

Let X be a Banach space, and $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere and the unit ball of the Banach space X, respectively. For $x \in S_X$, let $\nabla_x \subset S_{X^*}$ be the set of norm 1 supporting functionals of S_X at x, that is $f \in \nabla_x \Leftrightarrow \langle f, x\rangle = 1$, where X^* stands for the dual space of X.

Definition 1.1. *The bounded convex subset C of a Banach space X is said to have normal structure, if for every convex subset H of C that contains more than one point, there exists a point $x_0 \in H$ such that*

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

The Banach space X is said to have weak normal structure, if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both weak normal structure and normal structure coincide. A Banach space X is said to have uniform normal structure, if there

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exists $0 < c < 1$ such that for any closed bounded convex subset H of X that contains more than one point, there exists $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < c \sup\{\|x - y\| : x, y \in H\}.$$

Let C be a nonempty bounded closed convex subset of a Banach space X, a mapping $T : C \to C$ is said to be nonexpansive provided the inequality

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for every $x, y \in C$. A Banach space X is said to have the fixed point property if every nonexpansive mapping $T : C \to C$ has a fixed point.

Weak normal structure, normal structure and uniform normal structure are important in the metric fixed point theory for nonexpansive mapping. It was proved by Kirk [7] that if X has normal structure, then Banach space X has fixed point property. Since then, many mathematicians have investigated many various geometrical properties of Banach spaces implying weak normal structure, normal structure or uniform normal structure. A possible approach to look for some geometric properties in term of some geometric constants which imply weak normal structure, normal structure or uniform normal structure. Among the geometric constants, the modulus of weak uniform rotundity $\delta_X(\epsilon, f)$ plays an important role in the description of various geometric structures.

Definition 1.2. *The modulus of weak uniform rotundity is the function* $\delta_X(\epsilon, f) : [0, 2] \times S_{X^*} \to [0, 1]$ *defined in the following way ([10]):*

$$\delta_X(\epsilon, f) = \inf\left\{\{1\} \cup \{1 - \frac{\|x + y\|}{2} : x, y \in S_X, |\langle f, x - y\rangle| \geq \epsilon\}\right\},$$

where $0 \leq \epsilon \leq 2$ *and* $f \in S_{X^*}$. *The space* X *is weakly uniformly rotund if* $\delta_X(\epsilon, f) > 0$, *whenever* $0 < \epsilon \leq 2$ *and* $f \in S_{X^*}$. *For any* $f \in S_{X^*}$, $\delta_X(\epsilon, f)$ *is a continuous function in* $0 \leq \epsilon < 2$ *and* $\frac{\delta_X(\epsilon, f)}{\epsilon}$ *is increasing in* $(0, 2]$.

Rencently, Gao [3] studies the modulus of weak uniform rotundity extensively, and get some various geometrical properties and some sufficient conditions for normal structure as follows:

(i) If $\delta_X(\epsilon, f) > \frac{1}{2} - \frac{\epsilon}{4}, 0 \leq \epsilon < 2$ for all $f \in S_{X^*}$, then X is uniform nonsquare.

(ii) If $\delta_X(1, f) > 0$ for all $f \in S_{X^*}$, then X has uniform normal structure.

(iii) If $\delta_X(\epsilon, f) > \frac{1}{2} - \frac{\epsilon}{4}, 0 \leq \epsilon < 2$ for all $f \in S_{X^*}$, then X has uniform normal structure.

The purpose of this paper is to obtain some classes of Banach spaces with normal structure, which involves the modulus of weak uniform rotundity $\delta_X(\epsilon, f)$, the Domínguez-Benavides coefficient $R(1, X)$ and the coefficient of weak orthogonality $\omega(X)$. Moreover, these results are strictly wider than the previous Gao's results.

2. Preliminaries

Firstly, let us recall some basic facts about ultrapowers. A filter $\mathcal{F}$ on the set $\mathbb{N}$ of natural numbers is called to be an ultrafilter if it is maximal with respect to set inclusion. The ultrafilter is called trivial if it is of the form $A : A \subset \mathbb{N}, i_0 \in A$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial. The sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$ if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. Let $l_\infty(X)$ denote the subspace of the product space $\Pi_{n\in\mathbb{N}} X$ equipped with the norm

$$\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and

$$N_{\mathcal{U}} = \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}.$$

The ultrapower of X, denoted by $\widetilde{X}$ is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm. $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\|(x_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|.$$

If $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically ([6]).

In what follows, some coefficients are introduced, which will be used in the following sections.

Definition 2.1. *The following Domínguez-Benavides coefficient was introduced in [2]:*

$$R(1, X) = \sup \big\{ \liminf_{n\to\infty} \{\|x_n + x\|\} \big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\| \le 1.$$

It is clear that $1 \le R(1, X) \le 2$. Some geometric conditions sufficient for normal structure in term of Domínguez-Benavides coefficient have been studied in [11], [12], [13].

Definition 2.2. *The coefficient of weak orthogonality of X was introduced by Sims in [9]:*

$$\omega(X) = \sup\{\lambda > 0 : \lambda \cdot \liminf_{n\to\infty} \|x_n + x\| \le \liminf_{n\to\infty} \|x_n - x\|\},$$

where the supremum is taken over all the weakly null sequence (x_n) in X and all elements x of X. It is known that $\frac{1}{3} \le \omega(X) \le 1$ and $\omega(X) = \omega(X^*)$ in the reflexive Banach spaces (see [5], [8]).

3. Main results

Lemma 3.1 ([4]). *Let X be a Banach space without weak normal structure, then there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty} \subseteq S_X$ such that*

$$\lim_n \|x_n - x\| = 1 \quad for\ all \quad x \in co\{x_n\}_{n=1}^{\infty}.$$

Theorem 3.2. *Let X be a Banach space with $\delta_X(1+\epsilon, f) > g(\epsilon)$ for all $f \in S_{X^*}$ and $0 \le \epsilon \le 1$, then X has weak normal structure, where the function $g(\epsilon)$ is defined as*

$$g(\epsilon) := \begin{cases} \frac{(R(1,X)-1)\epsilon}{2}, & 0 \le \epsilon \le \frac{1}{R(1,X)}, \\ \frac{1}{2}\left(1 - \frac{1-\epsilon}{R(1,X)-1}\right), & \frac{1}{R(1,X)} < \epsilon \le 1. \end{cases}$$

Proof. Suppose that X does not have weak normal structure, by the Lemma 3.1, there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty}$ in S_X such that

$$\lim_n \|x_n - x\| = 1 \quad for\ all \quad x \in co\{x_n\}_{n=1}^{\infty}.$$

Take $\{f_n\} \subset S_{X^*}$ such that $f_n \in \nabla_{x_n}$ for all $n \in \mathbb{N}$. By the reflexivity of X^*, without loss of generality, we can assume that $f_n \xrightarrow{w^*} f$ for some $f \in B_{X^*}$. If necessary by passing to a subsequence, we can choose a subsequence of $\{x_n\}_{n=1}^{\infty}$, denoted again by $\{x_n\}_{n=1}^{\infty}$, such that

$$(1) \qquad \lim_n \|x_{n+1} - x_n\| = 1, \quad |(f_{n+1} - f)(x_n)| < \frac{1}{n}, \quad f_n(x_{n+1}) < \frac{1}{n},$$

for all $n \in \mathbb{N}$ and it follows that

$$\lim_n f_{n+1}(x_n) = \lim_n (f_{n+1} - f)(x_n) + f(x_n) = 0.$$

Note that the sequence $\{x_n\}$ is weakly null and verifies $D[\{x_n\}] = 1$. It follows from the definition of $R(1, X)$, then

$$\liminf_n \|x_{n+1} + x_n\| \le R(1, X).$$

Therefore, we can choose a subsequence $\{x_n\}$ such that

$$(2) \qquad \|x_{n+1} + x_n\| \le R(1, X).$$

Denote that $R := R(1, X)$ and consider two cases for $\epsilon \in [0, 1]$.

Firstly, if $\epsilon \in [0, \frac{1}{R}]$, take

$$\tilde{x} = (x_{n+1} - x_n)_{\mathcal{U}}, \tilde{y} = \{[1 - (R-1)\epsilon]x_{n+1} + \epsilon x_n\}_{\mathcal{U}} \text{ and } \tilde{f} = (-f_n)_{\mathcal{U}}.$$

By the 1 and 2, then

$$\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \|\tilde{x}\| = 1.$$

and

$$\begin{aligned}
\|\tilde{y}\| &= \Big\|[1-(R-1)\epsilon]x_{n+1}+\epsilon x_n\Big\| \\
&= \Big\|\epsilon(x_n+x_{n+1})+(1-R\epsilon)x_{n+1}\Big\| \\
&\leq R\epsilon+(1-R\epsilon)=1.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\tilde{f}(\tilde{x}-\tilde{y}) &= \lim_{\mathcal{U}}(-f_n)\Big((R-1)\epsilon x_{n+1}-(1+\epsilon)x_n\Big) \\
&= 1+\epsilon,
\end{aligned}$$

$$\begin{aligned}
\|\tilde{x}+\tilde{y}\| &= \lim_{\mathcal{U}}\Big\|[2-(R-1)\epsilon]x_{n+1}-(1-\epsilon)x_n\Big\| \\
&\geq \lim_{\mathcal{U}}(f_{n+1})\Big([2-(R-1)\epsilon]x_{n+1}-(1-\epsilon)x_n\Big) \\
&= 2-(R-1)\epsilon.
\end{aligned}$$

From the definition of $\delta_X(\epsilon,f)$, then

$$\delta_X(1+\epsilon,f)=\delta_{\widetilde{X}}(1+\epsilon,f)\leq\frac{(R-1)\epsilon}{2},$$

which is a contradiction.

Secondly, if $\epsilon\in(\frac{1}{R},1]$, in this case $R>1$, other $\epsilon>1$. Let

$$\tilde{x}=(x_{n+1}-x_n)_{\mathcal{U}},\tilde{y}=\{[1-(R-1)\epsilon']x_n+\epsilon' x_{n+1}\}_{\mathcal{U}},\ \text{and}\ \tilde{f}=(-f_n)_{\mathcal{U}},$$

where $\epsilon'=\frac{1-\epsilon}{R-1}\in[0,\frac{1}{R})$. It follows from the first case, then

$$\|\tilde{f}\|=\|\tilde{x}\|=1\text{ and }\|\tilde{y}\|\leq 1,$$

$$\begin{aligned}
\tilde{f}(\tilde{x}-\tilde{y}) &= \lim_{\mathcal{U}}(-f_n)\Big((1-\epsilon')x_{n+1}-[2-(R-1)\epsilon']x_n\Big) \\
&= 2-(R-1)\epsilon',
\end{aligned}$$

$$\begin{aligned}
\|\tilde{x}+\tilde{y}\| &= \lim_{\mathcal{U}}\|(1+\epsilon')x_{n+1}-(R-1)\epsilon' x_n\| \\
&\geq \lim_{\mathcal{U}}(f_{n+1})\Big((1+\epsilon')x_{n+1}-(R-1)\epsilon' x_n\Big) \\
&= 1+\epsilon'.
\end{aligned}$$

From the definition of $\delta_X(\epsilon, f)$, then

$$\delta_X(2-(R-1)\epsilon', f) = \delta_{\widetilde{X}}(2-(R-1)\epsilon', f) \le \frac{1-\epsilon'}{2},$$

which is equivalent to

$$\delta_X(1+\epsilon, f) = \delta_{\widetilde{X}}(1+\epsilon, f) \le \frac{1}{2}\left(1 - \frac{1-\epsilon}{R-1}\right).$$

This is a contradiction. □

In fact, take $\epsilon = 0$ in Theorem 3.2, we can easily get the following result in [3].

Corollary 3.3. *Let X be a Banach space with*

$$\delta_X(1, f) > 0 \quad for \quad all \quad f \in S_{X^*},$$

then X has weak normal structure.

Remark 3.4. *(i) Theorem 3.2 strengthens the result of Gao: $\delta_X(1, f) > 0$ for all $f \in S_{X^*} \Longrightarrow X$ has normal structure, which gives the precise sufficient condition for the normal structure, whenever $1 \le 1 + \epsilon \le 2$.*

(ii) It is note that Corollary 3.3 is sharp in the sense that there is a Banach space X such that $\delta_X(1, f) = 0$, X fails to have normal structure. Indeed, we consider the Bynum space $\ell_{p,\infty}$, which is the space ℓ_p $(1 < p < +\infty)$ with the norm

$$\|x\|_{p,\infty} = \max\{\|x^+\|, \|x^-\|\},$$

where x^+ is the positive part of x, defined as $x^+(i) = \max\{x(i), 0\}$ and $x^- = x^+ - x$. It is known that $\ell_{p,\infty}$ is a super-reflexive space that fails normal structure(see [1]), therefore $\delta_X(1, f) = 0$. This example shows that the condition in Corollary 3.3 is the best possible.

In the proof of following Theorem 3.5, we will get a property $\mathcal{P}$ that implying the uniform normal structure of a Banach space and also implying uniform normal structure of its dual. The proof is in the following fashion, suppose X^* fails to have uniform normal structure, then $\tilde{X}^*$ fails to have normal structure [7]. If X is super-reflexive, applying Lemma 3.1 yields vectors in $(\tilde{\tilde{X}}^*)^* = \tilde{\tilde{X}}$ that are used to show $\tilde{\tilde{X}} \notin \mathcal{P}$, which in turn implies $X \notin \mathcal{P}$ (The notation $X \notin \mathcal{P}$ will mean that a Banach space X does not satisfy the property $\mathcal{P}$.) Thus, in order to prove the property $\mathcal{P}$ implying X^* has uniform normal structure, we only need to show that if $X = X^{**}$ fails to have uniform normal structure, then $(\tilde{X})^* = (\tilde{\tilde{X}})^*$ fails to satisfy the property $\mathcal{P}$ by Lemma 3.1.

Theorem 3.5. *Let X be a Banach space such that*

$$\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2}$$

for all $f \in S_{X^*}$, then X and X^* have uniform normal structure.

Proof. Since $\frac{1}{3} \leq \omega(X) \leq 1$, it is easy to check that

$$\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2} \geq \frac{1}{2} - \frac{1+\omega(X)}{4} = \frac{1-\omega(X)}{4},$$

then X is uniformly nonsquare by the result (i) of Gao, it suffices to prove that X has weak normal structure whenever

$$\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2}.$$

Firstly, repeating the arguments in the proof of Theorem 3.2, take $\tilde{x} = (x_{n+1} - x_n)_{\mathcal{U}}$, $\tilde{y} = [\omega(X)(x_{n+1}+x_n)]_{\mathcal{U}}$, and $\tilde{f} = (-f_n)_{\mathcal{U}}$. By the definition of $\omega(X)$ and Lemma 3.1, then

$$\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \|\tilde{x}\| = 1,$$

$$\|\tilde{y}\| = \|[\omega(X)(x_{n+1}+x_n)]_{\mathcal{U}}\| \leq \|(x_{n+1}-x_n)_{\mathcal{U}}\| = 1,$$

$$\begin{aligned}\tilde{f}(\tilde{x}-\tilde{y}) &= \lim_{\mathcal{U}}(-f_n)\Big((1-\omega(X))x_{n+1} - (1+\omega(X))x_n\Big) \\ &= 1+\omega(X),\end{aligned}$$

$$\begin{aligned}\|\tilde{x}+\tilde{y}\| &= \lim_{\mathcal{U}} \|(1+\omega(X))x_{n+1} - (1-\omega(X))x_n\| \\ &\geq \lim_{\mathcal{U}}(f_{n+1})\Big((1+\omega(X))x_{n+1} - (1-\omega(X))x_n\Big) \\ &= 1+\omega(X).\end{aligned}$$

The definition of $\delta_X(\epsilon, f)$ implies that

$$\delta_X(1+\omega(X), f) = \delta_{\tilde{X}}(1+\omega(X), f) \leq \frac{1-\omega(X)}{2}.$$

This is a contradiction. Consequently, if $\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2}$, then X has weak normal structure.

On the other hand, suppose that X^{**} does not have weak normal structure,

$$\tilde{x^*} = (f_n)_{\mathcal{U}}, \tilde{y^*} = [\omega(X^*)(f_{n+1} - f_{n+2})]_{\mathcal{U}} \ \text{ and } \ \tilde{x^{**}} = (x_n - x_{n+1})_{\mathcal{U}}.$$

By the definition of $\omega(X^*)$ and Lemma 3.1, then

$$\|\tilde{x^*}\| = \|(f_n)_{\mathcal{U}}\| = 1 \ \text{ and } \ \langle \tilde{x^*}, \tilde{x^{**}} \rangle = \langle f_n, x_n - x_{n+1} \rangle = 1,$$

and

$$\|\tilde{y^*}\| = \omega(X^*)\|(f_{n+1} - f_{n+2})_{\mathcal{U}}\| \leq 1.$$

Moreover, we have

$$\begin{aligned} \langle \tilde{x^*} - \tilde{y^*}, \tilde{x^{**}} \rangle &= \lim_{\mathcal{U}} \langle f_n - \omega(X^*) f_{n+1} + \omega(X^*) f_{n+2}, x_n - x_{n+1} \rangle \\ &= 1 + \omega(X^*), \end{aligned}$$

$$\begin{aligned} \|\tilde{x^*} + \tilde{y^*}\| &= \lim_{\mathcal{U}} \|f_n + \omega(X^*) f_{n+1} - \omega(X^*) f_{n+2}\| \\ &\geq \lim_{\mathcal{U}} \langle f_n + \omega(X^*) f_{n+1} - \omega(X^*) f_{n+2}, x_n - x_{n+2} \rangle \\ &= 1 + \omega(X^*). \end{aligned}$$

From the definition of $\delta_{X^*}(\epsilon, f)$ and $\omega(X) = \omega(X^*)$, then

$$\delta_{X^*}(1 + \omega(X^*), f) = \delta_{(X^*)_{\mathcal{U}}}(1 + \omega(X), f) = \delta_{(X_{\mathcal{U}})^*}(1 + \omega(X), f) > \frac{1 - \omega(X)}{2}.$$

□

Remark 3.6.

(i) It is well known that $\frac{1}{3} \leq \omega(X) \leq 1$, then $\frac{4}{3} \leq 1 + \omega(X) \leq 2$, therefore the condition in Theorem 3.5 implies the uniform normal structure of Banach spaces are shown to imply uniform normal structure of their dual spaces as well, which are complementary to the Gao's results, whenever $\frac{4}{3} \leq \epsilon \leq 2$.

(ii) Consider the Hilbert space H, it is well known that $\delta_H(\epsilon, f) = 1 - \frac{\sqrt{2(2-\epsilon)}}{2}$ for all $f \in S_{X^*}$ and $0 \leq \epsilon \leq 2$, $R(1, X) = 1$ and $\omega(X) = 1$, it is easy to check that $\delta_H(\epsilon, f) > 0 = \frac{(R(1,X)-1)\epsilon}{2} = \frac{1-\omega(X)}{2}$, then X has weak normal structure from Theorem 3.2 or Theorem 3.5.

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A note on the p-length of a p-soluble group

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Abstract. Suppose that the finite group $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B. By use of several invariant parameters of A and B, we present some bounds of the p-length of G. Some known results are improved.

Keywords: p-soluble, p-length, lower p-series, mutually permutable product.

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1. Introduction

All groups considered are finite. Let G be a group, we denote by $\pi(G)$ the set of all prime divisors of $|G|$. Let $p \in \pi(G)$, by G_p, we mean a Sylow p-subgroup of G. The other notations and terminologies used in this note are standard, as in [1, 2].

The p-length of a p-soluble group is an important invariant parameter. Many scholars have investigated on this invariant parameter, the readers can refer to [3]-[6] for instances. Therefore, the celebrated Hall-Higman theorem has established basic theorem on the p-length of a p-soluble group G, showing that the p-length of G is bounded above by the nilpotent class and the minimal number of generators of G_p and the p-rank of G [3].

In general, a product of two p-soluble subgroups need not be p-soluble. However, if the group G is a mutually permutable product of two p-soluble subgroups, then G is still a p-soluble group [7]. Recall that the product $G = AB$ of the subgroups A and B of a group G is called a mutually permutable product of A and B if $AU = UA$ for any subgroup U of B and $BV = VB$ for any subgroup V of B [7]. Cossey and Li in [6] investigated the p-length of a mutually permutable product of two p-soluble groups and obtained the following result:

Theorem 1.1 ([6, Theorem 1.1])**.** *Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. If $l_p(A) \leq k$ and $l_p(B) \leq k$, then $l_p(G) \leq k + 1$.*

In the note, we continue the study on the p-length of a mutually permutable product of two p-soluble groups. By use of several invariant parameters of A and B, we will improve the above results as follows.

Theorem 1.2. *Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. Then*

(1) $l_p(G) \leq \max\{c(A_p), c(B_p)\}$;

(2) $l_p(G) \leq \max\{d(A_p), d(B_p)\}$;

(3) $l_p(G) \leq \max\{l_p(A), l_p(B)\} + 1$;

(4) $l_p(G) \leq \max\{r_p(A), r_p(B)\} + 1$.

Note that, $\max\{l_p(A), l_p(B)\} \leq l_p(G)$, we get the following corollary:

Corollary 1.1. *Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. Then, either $l_p(G) = \max\{l_p(A), l_p(B)\}$ or $l_p(G) = \max\{l_p(A), l_p(B)\} + 1$.*

2. Preliminaries

Let π be a set of primes and let G be a group. As well-known, $O^{\pi}(G)$ is defined to be the intersection of all normal subgroups N of G such that G/N is a π-group. Hence, $G/O^{\pi}(G)$ is the maximal π-factor group of G ([8, IX, 1.1]). Following [6], we invoke the following definition way of p-length of a p-soluble group.

If p is a prime, the lower p-series of G is

$$G \geq O^{p'}(G) \geq O^{p',p}(G) \geq O^{p',p,p'}(G) \geq \cdots .$$

If G is p-soluble, the last term of the lower p-series is 1 and if the lower p-series of G is

$$G = G_0 \geq G_1 \geq \cdots \geq G_s = 1,$$

then the p-length of G is the number of non-trivial p-groups in the set

$$\{G/G_1, G_1/G_2, \ldots, G_{s-1}/G_s\}.$$

Lemma 2.1 ([7, Theorem 4.1.15])**.** *Let the group G be the product of the mutually permutable subgroups A and B. If A and B are p-soluble, then G is p-soluble.*

Lemma 2.2 ([7, Lemma 4.1.10])**.** *Let the group G be the product of the mutually permutable subgroups A and B. If N is a normal subgroup of G, then G/N is a mutually permutable product of AN/N and BN/N.*

Lemma 2.3 ([7, Theorem 4.3.11])**.** *Let the non-trivial group G be the product of mutually permutable subgroups A and B. Then $A_G B_G$ is not trivial.*

Lemma 2.4 ([7, Lemma 4.3.3])**.** *Let the group G be the product of the mutually permutable subgroups A and B. Then:*

(1) *If N is a minimal normal subgroup of G, then $\{A \cap N, B \cap N\} \subseteq \{N, 1\}$.*

(2) *If N is a minimal normal subgroup of G contained in A and $B \cap N = 1$, then $N \leq C_G(A)$ or $N \leq C_G(B)$. If furthermore N is not cyclic, then $N \leq C_G(B)$.*

Lemma 2.5 ([7, Corollary 4.1.25])**.** *Let the group G be the product of the mutually permutable subgroups A and B. Then A' and B' are subnormal in G.*

3. Proof of Theorem 1.2

Proof. It is clear that (3) implies (4) by Hall-Higman theorem on the p-length of p-soluble groups. Hence, we only need to prove (1), (2) and (3).

Let G be a counter-example of minimal order. We proceed in steps.

Step 1. G is p-soluble.

This follows from Lemma 2.1.

Step 2. $N = O_p(G)$ is unique minimal normal and complemented in G and $N = C_G(N)$.

Let N be a minimal normal subgroup of G. We consider $\overline{G} = G/N$ together with $\overline{A} = AN/N$ and $\overline{B} = BN/N$. It is clear that $\overline{A_p} = A_pN/N$ and $\overline{B_p} = B_pN/N$ is respectively a Sylow p-subgroup of $\overline{A}$ and $\overline{B}$. By Lemma 2.2, $\overline{G}$ is the mutually product of two p-soluble subgroups $\overline{A}$ and $\overline{B}$, hence $\overline{G}$ satisfies the hypotheses of the theorem. For (1), the choice of G implies that

$$l_p(\overline{G}) \leq \max\{c(\overline{A}_p), c(\overline{B}_p)\} \leq \max\{c(A_p), c(B_p)\}.$$

If N_1 is minimal normal in G with $N_1 \neq N$, then we also have

$$l_p(G/N_1) \leq \max\{c(A_p), c(B_p)\}.$$

It follows that

$$l_p(G) \leq \max\{l_p(G/N), l_p(G/N_1)\} \leq \max\{c(A_p), c(B_p)\},$$

a contradiction. Therefore N is the unique minimal normal subgroup of G. Moreover, if $N \leq O_{p'}(G)$ or $N \leq \Phi(G)$, then

$$l_p(G) = l_p(\overline{G}) \leq \max\{c(A_p), c(B_p)\},$$

contradicting to the choice of G. Hence, $O_{p'}(G) = \Phi(G) = 1$ and $N = O_p(G)$, Step 1 follows. Similarly, we can prove Step 1 for (2) and (3).

Step 3. $N \leq A \cap B$.

Since $A_G B_G \neq 1$ by Lemma 2.3, we may assume $N \leq A$ by Step 1. If $N \not\leq B$, then $N \cap B = 1$ by Lemma 2.4(1). If N is cyclic, then $N = C_G(N) \in \mathrm{Syl}_p(G)$, hence $l_p(G) = 1$, a contradiction. Thus, N is not cyclic and $N \leq C_G(B)$ by Lemma 2.4(2). Furthermore, $B \leq C_G(N) = N \leq A$ and so $G = AB = A$, Theorem 1.2 holds by Hall-Higman theorem. This shows $N \leq A \cap B$.

Step 4. If $N \leq M \leq G$, then $O_{p'}(M) = 1$.

Since $O_{p'}(M) \leq C_M(N) \leq C_G(N) = N$, we have $O_{p'}(M) = 1$.

Step 5. Finishing the proof.

For convenience, write $\overline{G} = G/N$, $\overline{A} = A/N$ and $\overline{B} = B/N$. We know that $\overline{G}$ satisfies the hypotheses of the theorem. Now, we prove by distinguishing three invariant parameters.

(1) By Step 2 and 3, $Z(A_p) \leq C_A(N) = N$, hence $c(\overline{A}_p) \leq c(A_p) - 1$. Similarly, $c(\overline{B}_p) \leq c(B_p) - 1$. The minimality of G implies that

$$l_p(\overline{G}) \leq \max\{c(\overline{A}_p), c(\overline{B}_p)\} \leq \max\{c(A_p) - 1, c(B_p) - 1\}.$$

Thus, $l_p(G) \leq \max\{c(A_p), c(B_p)\}$. This is the final contradiction.

(2) Since N is complemented in G, $N \not\leq \Phi(A_p)$, i.e., $N \cap \Phi(A_p) < N$. Now, that A_p is a p-group, we have $\Phi(\overline{A}_p) = \Phi(A_p)N/N$ and so

$$\overline{A}_p/\Phi(\overline{A}_p) = (A_p/N)/(\Phi(A_p)N/N) \cong A_p/(\Phi(A_p)N).$$

Furthermore,

$$|\overline{A}_p/\Phi(\overline{A}_p)| = |A_p/(\Phi(A_p)N)| = |A_p/\Phi(A_p)|/|N/(N \cap \Phi(A_p))| < |A_p/\Phi(A_p)|.$$

This implies that $d(\overline{A}_p) \leq d(A_p) - 1$. Similarly, $d(\overline{B}_p) \leq d(B_p) - 1$.

The choice of G implies that

$$l_p(\overline{G}) \leq \max\{d(\overline{A}_p), d(\overline{B}_p)\} \leq \max\{d(A_p) - 1, d(B_p) - 1\}.$$

Thus, $l_p(G) \leq \max\{d(A_p), d(B_p)\}$. This is the final contradiction.

(3) Firstly, we have

Claim 1. $\max\{l_p(A), l_p(B)\} > 1$.

Suppose otherwise, $\max\{l_p(A), l_p(B)\} = 1$. Then

(i) $\overline{A} = \overline{A}_p \times \overline{A}_{p'}$ and $\overline{B} = \overline{B}_p \times \overline{B}_{p'}$.

Since $l_p(A) \leq 1$, by Step 4, we can write $A = [A_p]A_{p'}$. Since $[A_p, A_{p'}] \trianglelefteq \langle A_p, A_{p'}\rangle = A$ and $[A_p, A_{p'}] \leq [A, A] = A'$, we have $[A_p, A_{p'}] \trianglelefteq A'$. Noticing that A' is subnormal in G by Lemma 2.5, $[A_p, A_{p'}]$ is a subnormal p-subgroup of G. Hence, $[A_p, A_{p'}] \leq O_p(G) = N$ and consequently, $\overline{A} = \overline{A}_p \times \overline{A}_{p'}$.

Similarly, $\overline{B} = \overline{B}_p \times \overline{B}_{p'}$.

(ii) Both $\overline{A}_p$ and $\overline{B}_p$ are abelian groups.

By (i), $(\overline{A})' = (\overline{A}_p)' \times (\overline{B}_p)'$. Note that $(\overline{A})'$ is subnormal in $\overline{G}$ by Lemma 2.2 and 2.5, $(\overline{A}_p)'$ is a subnormal p-subgroup of $\overline{G}$. Hence, $(\overline{A}_p)' \leq O_p(\overline{G}) = 1$, that is, $\overline{A}_p$ is abelian.

Similarly, $\overline{B}_p$ is also abelian.

(iii) Finishing the proof of Claim 1.

In view of (ii) and the result of (1), $l_p(\overline{G}) \leq \max\{c(\overline{A}_p), c(\overline{B}_p)\} \leq 1$. Hence, $l_p(G) \leq 2$, a contradiction.

Now, we may assume that $\max\{l_p(A), l_p(B)\} = l_p(A) > 1$. Furthermore, we have

Claim 2. $1 < O_p(A') \leq N$.

Since $l_p(A) > 1$, $A' \neq 1$. But $O_{p'}(A') \leq O_{p'}(A)$, hence $O_{p'}(A') = 1$ by Step 4. Because A' is subnormal in G, $O_p(A')$ is subnormal in G. Thereby $1 < O_p(A') \leq N$.

Claim 3. Let O be the last non-trivial term of the lower p-series of A. Then $O \leq O_p(A')$.

Since $O_{p'}(A) = 1$, O is a p-group and $O \leq A_p$. On the other hand, since $l_p(A) > 1$,

$$O \leq O^{p',p}(A) = O^p(O^{p'}(A)) \leq O^p(A).$$

Consequently, $O \leq A_pA' \cap O^p(A)A'$. Noticing that

$$A/A' = A_pA'/A' \times O^p(A/A') = A_pA'/A' \times O^p(A)A'/A',$$

we have $A_pA' \cap O^p(A)A' = A'$. Hence, $O \leq A'$ and $O \leq O_p(A')$.

Claim 4. $\max\{l_p(\overline{A}), l_p(\overline{B})\} \leq l_p(A) - 1$.

By Claim 2 and 3, we have $O \leq N$. Clearly, $l_p(\overline{A}) \leq l_p(A/O) \leq l_p(A) - 1$. Similarly, $l_p(\overline{B}) \leq l_p(A) - 1$ if $l_p(B) = l_p(A)$. Of course, $l_p(\overline{B}) \leq l_p(B) \leq l_p(A) - 1$ if $l_p(B) < l_p(A)$. Thus, Claim 4 follows.

Finally, since $l_p(\overline{G}) \leq \max\{l_p(\overline{A}), l_p(\overline{B})\} + 1 \leq l_p(A)$, we obtain

$$l_p(G) \leq l_p(A) + 1 = \max\{l_p(A), l_p(B)\} + 1.$$

This is the final contradiction and the proof is complete. □

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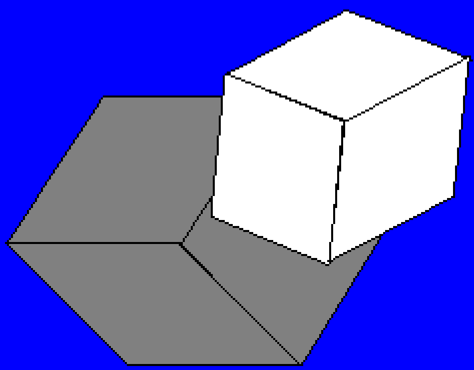

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