# Finite groups whose sums of irreducible character degrees of all proper subgroups are large

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**Abstract.** Some scholars investigated the influence of the sum of all character degrees of a finite group on group structure. In this paper, we will study the influence of sums of all character degrees of all proper subgroups of a finite group on its structure. We will show that a finite group $G$ is solvable when all proper subgroups $H$ of $G$ satisfy $\mathrm{T}(H) \geq \frac{2}{3}|H|$, where $\mathrm{T}(H)$ is the sum of all character degrees of a finite group $H$.

**Keywords:** simple group, character degree sum, proper subgroup.

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## 1. Introduction

Only finite groups are considered in this paper. Let $G$ be a finite group and $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of $G$, say $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_s\}$. Let $\mathrm{cd}(G) = \{\chi_i(1) : \chi_i \in \mathrm{Irr}(G)\}$. Then, $\mathrm{T}(G) = \sum_{i=1}^{s} \chi_i(1)$ is the sum of all character degrees of $G$. We see that $\mathrm{T}(G) = |G|$ if and only if $G$ is abelian. Let $p$ be a prime. The lower bound of $\mathrm{T}(G)$ is determined when $|G|$ is divisible by $p^n$ with $n \leq 6$ (see [7]). Many scholars studied the relations between group structures and $\mathrm{T}(G)$. For instance, $G$ is solvable when $\mathrm{T}(G) > \frac{4}{15}|G|$ (see [14]) or $\mathrm{T}(G) > \sqrt{3/8}|G|$ [11, Theorem 1.3]; $G$ is $p$-solvable when $\mathrm{T}(G) \geq (\sqrt{3}/p)|G|$ [11, Theorem 1.1] or $\mathrm{T}(G) > |G|/f(p)$ with

$$f(p) = \begin{cases} p(p^2-1)/(p^2+p+2), & \text{if } p \equiv 1 \pmod 4, \\ p-1, & \text{if } p \geq 7 \text{ and } p \equiv -1 \pmod 4, \\ 15/4, & \text{if } p = 2, 3; \end{cases}$$

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(see [12, Corollary B]). Let $\tau = \sum_{i=1}^{s} \chi_i$ for all $\chi_i \in \mathrm{Irr}(G)$ and let $a(\phi) =< \tau_H, \phi >$ for $\phi \in \mathrm{Irr}(H)$. Let

$$\delta_0(G,H) = \sum_{\phi \in \mathrm{Irr}(H)} a(\phi)\phi(1) - (a-1)$$

with $H < G$ and $a = a(1_H)$. Berkovich and Mann in [1] showed the structures of finite groups when $\delta_0(G,H)$ is small.

In this paper, we consider the influence of sums of all characters degrees of all proper subgroups of a finite group on its structure. For the sake of brevity, we give a definition.

**Definition 1.1.** *A group $G$ is called an* SD*-group if* $\mathrm{T}(G) \geq \frac{2}{3}|G|$.

Now, the result of [11] shows that an SD-group is solvable. Let $\mathbf{prop}(G)$ be the set of all proper subgroups of $G$. In order to argue in short, a concept is introduced.

**Definition 1.2.** *A group $G$ is called a* PSD*-group if every proper subgroup of $G$ is an* SD*-group.*

In this paper, we mainly prove the following result which is corresponding to those results of [11] and [14].

**Theorem 1.1.** *A finite* PSD*-group is solvable.*

By [5, p. 2], $A_5$ has $A_4$, $D_{10}$ and $S_3$ as its maximal subgroups. By [4],

$$\mathrm{T}(A_4) = 1+1+1+3 = 6, |A_4| = 12.$$
$$\mathrm{T}(D_{10}) = 1+1+2+2 = 6, |D_{10}| = 10.$$
$$\mathrm{T}(S_3) = 1+1+2 = 4, \qquad |S_3| = 6.$$

Hence

$$\frac{\mathrm{T}(A_4)}{|A_4|} = \frac{1}{2}, \frac{\mathrm{T}(D_{10})}{|D_{10}|} = \frac{3}{5} \text{ and } \frac{\mathrm{T}(S_3)}{|S_3|} = \frac{2}{3}.$$

It follows that the condition of Theorem 1.1 "for all $H \in \mathbf{prop}(G)$, $\frac{\mathrm{T}(H)}{|H|} \geq \frac{2}{3}$" is the best possible.

This paper is formed as follows. In Section 2, some properties of SD-groups are given and also the structures of minimal non-abelian simple groups are introduced. In Section 3, the result "a PSD-group is solvable" is proved. Let $\max G$ or $\max(G)$ be the set of all maximal subgroups of $G$ with respect to subgroup order divisibility. All other symbols are standard, please see [5] and [9] for instance.

## 2. Basic results

In this section, some basic results are collected. First, in general, an SD-group must be a PSD-group. Let $G$ be an SD-group. For any $H \in \mathbf{prop}(G)$, from [2, Chapter 11], $\frac{|H|}{\mathrm{T}(H)} \leq \frac{|G|}{\mathrm{T}(G)}$. Then, $\frac{\mathrm{T}(H)}{|H|} \geq \frac{\mathrm{T}(G)}{|G|} \geq \frac{2}{3}$. It follows that $H$ is an SD-group and $G$ is a PSD-group. But the converse is not true.

**Example 2.1.** Let $G = S_3 \times S_3$ where $S_n$ is the symmetric group on $n$ symbols. We see from [4] that

$$\mathrm{T}(G) = 4 \cdot 1 + 4 \cdot 2 + 1 \cdot 4 = 16 \text{ and } |G| = 36,$$

so

$$\frac{\mathrm{T}(G)}{|G|} = \frac{16}{36} = \left(\frac{2}{3}\right)^2 \ngeq \frac{2}{3}.$$

It follows that $G$ is a non-SD-group. But its maximal subgroups $H \cong C_2 \times S_3$ and $K \cong C_3 \times S_3$ are SD-groups where $C_n$ is a cyclic group of order $n$. In fact, $\mathrm{T}(H) = 4 \cdot 1 + 2 \cdot 2 = 8$, $|H| = 12$ and $\mathrm{T}(K) = 6 \cdot 1 + 3 \cdot 2 = 12$, $|K| = 18$, so

$$\frac{\mathrm{T}(H)}{|H|} = \frac{2}{3} = \frac{\mathrm{T}(K)}{|K|}.$$

It follows that $S_3 \times S_3$ is a PSD-group but not an SD-group.

**Example 2.2.** Let $G = A_4$. Then, $G$ has subgroups $K_4$, $C_3$. We see that

$$\frac{\mathrm{T}(A_4)}{|A_4|} = \frac{3 \cdot 1 + 1 \cdot 3}{12} = \frac{1}{2}, \frac{\mathrm{T}(K_4)}{|K_4|} = \frac{\mathrm{T}(C_n)}{|C_n|} = 1.$$

Thus, $G$ is a PSD-group but not an SD-group.

**Lemma 2.1.** *An* SD*-group is solvable.*

**Proof.** Let $G$ be an SD-group. By the definition of SD-groups, we see that

$$\mathrm{T}(G) \geq \frac{2}{3}|G| = \frac{10}{15}|G| > \frac{4}{15}|G|.$$

By Theorem 1.1 of [11], $G$ is solvable. □

The following conclusion, which is needed for proving our main result, is due to Thompson [13].

**Lemma 2.2** (Corollary 1 of [13]). *Every minimal simple group, that is, non-abelian simple groups whose all proper subgroups are solvable, is isomorphic to one of the following simple groups:*

(1) $\mathrm{PSL}_2(2^p)$ *for $p$ a prime;*

(2) $\mathrm{PSL}_2(3^p)$ *for $p$ an odd prime;*

(3) $\mathrm{PSL}_2(p)$, *for $p$ any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod 5$;*

(4) $Sz(2^p)$ *for $p$ an odd prime;*

(5) $\mathrm{PSL}_3(3)$.

## 3. Solvability of PSD-groups

In this section, the key is to give a proof of Theorem 1.1. Let $k(G)$ be the number of conjugate classes of $G$. We first give some results which are needed in the proof.

**Lemma 3.1.** *Let $G$ be a dihedral group of the form $D_{2n}$. Then*

*(1) if $n$ is odd, then $\mathrm{T}(G) = n + 1$ and $k(G) = \frac{n+3}{2}$, in particular, $G$ is an* SD*-group if $n = 3$;*

*(2) if $n$ is even, then $\mathrm{T}(G) = n + 2$ and $k(G) = \frac{n+6}{2}$, in particular, $G$ is an* SD*-group if $n \in \{2, 4, 6\}$.*

**Proof.** (1) Now, $G$ is a Frobenius group $C_n : C_2$ with kernel $C_n$ and complement $C_2$ respectively. For any $\chi \in \mathrm{Irr}(G)$, $\chi(1)\big||G : G'| = 2$, where $G'$ is the derived subgroup of $G$, and so $\chi(1) = 1$ or 2. Observe that $G' \cong C_n$, and let $m$ be the number of irreducible characters of $G$ of degree 2, so $2 \cdot 1^2 + m \cdot 2^2 = 2n$ and $m = \frac{n-1}{2}$. Now,

$$\mathrm{T}(G) = 2 \cdot 1 + \frac{n-1}{2} \cdot 2 = n + 1, \ \ k(G) = 2 + \frac{n-1}{2}.$$

We see $|G| = 2n$, so $\frac{\mathrm{T}(G)}{|G|} = \frac{n+1}{2n} \geq \frac{2}{3}$ if $n = 3$. Thus, $D_{2n}$ is an SD-group if $n = 3$.

(2) $G' \cong C_{n/2}$ and $G/G' \cong C_2 \times C_2$. Now, $\chi(1)^2\big||G/G'| = 4$ and so $\chi(1) = 1$ or 2. Denote by $m$ the number of irreducible characters of $G$ of degree 2. We obtain that $4 \cdot 1^2 + m \cdot 2^2 = 2n$ and so $m = \frac{n}{2} - 1$. Thus,

$$\mathrm{T}(G) = 4 \cdot 1 + (\frac{n}{2} - 1) \cdot 2 = 4 + n - 2 = n + 2, k(G) = 4 + \frac{n}{2} - 1 = 3 + \frac{n}{2}.$$

It follows from $|G| = 2n$ that $\frac{\mathrm{T}(G)}{|G|} = \frac{n+2}{2n} \geq \frac{2}{3}$ if $n = 2$, 4 or 6. □

Let $\pi(n)$ denote the set of all prime divisors of $n$.

**Lemma 3.2.** *Let $G$ be a Frobenius group $E_p : C_q$, where $p$ is a prime power, with kernel $E_p$, the elementary abelian $\pi(p)$-group of order $p$ and complement $C_q$ respectively. Then, $\mathrm{T}(G) = p + q - 1$ and $k(G) = q + \frac{p-1}{q}$, in particular, $G$ is an* SD*-group if $p = 2$ and $q = 3$.*

**Proof.** By [6, Theorem 13.8],

$$k(G) = k(C_q) + \frac{k(E_p) - 1}{|C_q|} = q + \frac{p-1}{q}$$

and

$$\mathrm{T}(G) = k(C_q) \cdot 1 + \frac{k(E_p) - 1}{|C_q|} \cdot |C_q| = q + (p - 1).$$

Now, $\frac{\mathrm{T}(G)}{|G|}=\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}\geq\frac{2}{3}$ if $G$ is an SD-group. If $p>2$ and $q>2$, then as $(p,q)=1$, we have that

$$\frac{6}{12}=\frac{1}{3}+\frac{1}{4}-\frac{1}{3\cdot 4}>\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}\geq\frac{2}{3},$$

a contradiction. This forces one of $p$ and $q$ to be 2. Since $G$ is a Frobenius group, $p\neq 2$ and so $q=2$. Let $p$ be an odd number $\geq 5$. Similarly, we obtain that

$$\frac{6}{10}=\frac{1}{2}+\frac{1}{5}-\frac{1}{2\cdot 5}>\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}\geq\frac{2}{3},$$

a contradiction. This forces $p=3$ and $q=2$. □

Let $H^n$ denote the product of $n$ times $H$ and $f(G)=\frac{\mathrm{T}(G)}{|G|}$.

**Lemma 3.3.** *Let $H$ be a group. Then, $f(H^2)=f(H)^2$.*

**Proof.** As $f(G)=\frac{\mathrm{T}(G)}{|G|}$, from Theorem 4.21 of [9] we have

$$\begin{aligned}
f(H^2)&=\frac{\mathrm{T}(H^2)}{|H^2|}=\frac{\sum_{i,j}(\chi_i\times\chi_j)(1)}{|H^2|}\\
&=\frac{\sum_{i,j}\chi_i(1)\chi_j(1)}{|H|^2}\\
&=\frac{\sum_i\chi_i(1)\left(\chi_1(1)+\chi_2(1)+\cdots+\chi_{k(H)}(1)\right)}{|H|^2}\\
&=\frac{\sum_i\chi_i(1)\mathrm{T}(H)}{|H|^2}\\
&=\frac{\mathrm{T}(H)\sum_i\chi_i(1)}{|H|^2}\\
&=\frac{\mathrm{T}(H)^2}{|H|^2}=f(H)^2.
\end{aligned}$$

The proof is complete. □

Let $A\curlyvee B$ be central product of two groups $A$ and $B$ ( see [15, p. 75] ). We rewrite Theorem 1.1 here.

**Theorem 3.1.** *A finite* PSD*-group is solvable.*

**Proof.** By way of contradiction, assume that $G$ is a non-solvable PSD-group. Then, $G$ has two normal subgroups $H,N$ such that $H/N$ is a product of isomorphic non-abelian simple groups. By Lemma 2.1, all proper subgroups of $G$ are solvable, since $G$ is a PSD-group. It follows that $G/N=H/N$, and $G/N$ is a non-abelian simple group. Hence, $G/N$ is a minimal simple group.

By Lemma 2.2, there are five possibilities for $G/N$. Note that (1), (2) and (3) of Lemma 2.2 can be treated together as follows

Table 1: $\mathrm{PSL}_2(q)$, $q \geq 5$ [8, Chap II, Theorem 8.27]

| | $\max \mathrm{PSL}_2(q)$ | Condition |
|---|---|---|
| $\mathcal{C}_1$ | $E_q : C_{(q-1)/k}$ | $k = \gcd(q-1, 2)$ |
| $\mathcal{C}_2$ | $D_{2(q-1)/k}$ | $q \notin \{5, 7, 9, 11\}$ |
| $C_3$ | $D_{2(q+1)/k}$ | $q \notin \{7, 9\}$ |
| $\mathcal{C}_5$ | $\mathrm{PSL}_2(q_0).(k, b)$ | $q = q_0^b$, $b$ a prime, $q_0 \neq 2$ |
| $\mathcal{C}_6$ | $S_4$ | $q = p \equiv \pm 1 \pmod 8$ |
| | $A_4$ | $q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$ |
| $\mathcal{S}$ | $A_5$ | $q \equiv \pm 1 \pmod{10}, F_q = F_p[\sqrt{5}]$ |

**Case 1.** $G/N \cong \mathrm{PSL}_2(q)$ for $q$ being a prime power and $q > 3$.

In this case, $G'/N \cong \mathrm{PSL}_2(q)$ and the Schur multiplier of $\mathrm{PSL}_2(q)$ has order at most two. So, $[G', N] \leqslant C_2$. Consequently, $G \cong N \times \mathrm{PSL}_2(q)$ or $N \curlyvee \mathrm{SL}_2(q)$. Since all proper subgroups of $G$ are solvable, $G \cong \mathrm{PSL}_2(q)$ or $\mathrm{SL}_2(q)$.

Let $G \cong \mathrm{PSL}_2(q)$. First, suppose that $G \cong \mathrm{PSL}_2(2^2) \cong A_5$. There exists $H \in \max G$ such that $H \cong A_4$. Note that $H$ is a Frobenius group $E_4 : C_3$. By Lemma 3.2, $H$ is not an SD-group, a contradiction. This implies that $G \cong \mathrm{PSL}_2(q)$ with $q \geq 5$. From Table 1, $E_q : C_{(q-1)/k} \in \max(G)$ with $k = 1$ or 2. Hence, $E_q : C_{(q-1)/k}$ is an SD-group. By Lemma 3.2, $q = 3$, a contradiction.

Therefore, $G \cong \mathrm{SL}_2(q)$. Note that, in this case, we can assume that $q$ is odd since $\mathrm{SL}_2(2^n) \cong \mathrm{PSL}_2(2^n)$. From [3, p. 377], $E_q : C_{q-1} \in \mathbf{prop}(G)$. This implies that $E_q : C_{q-1}$ is an SD-group. By Lemma 3.2, $q = 3$, a contradiction.

**Case 2.** $G/N \cong Sz(2^p)$ for $p$ an odd prime.

From [5, p. xvi], the Schur multiplier of $Sz(2^p)$ has trivial order except for $p = 3$ and its order is 4 if $p = 3$. If $p > 3$, then $G \cong N \times Sz(2^p)$. From [3, p. 385], $D_{2(2^p-1)} \in \max Sz(2^p)$. Hence, $D_{2(2^p-1)}$ is an SD-group. By Lemma 3.1, $2^p - 1 = 3$, a contradiction, since $p$ is odd. Therefore, $p = 3$. From [5, p. 28], $G \cong N \times Sz(8)$, $N \curlyvee 2.Sz(8)$ or $N \curlyvee 4.Sz(8)$. Since all proper subgroups of $G$ are solvable, $G \cong Sz(8)$, $2.Sz(8)$ or $4.Sz(8)$.

Assume that $G \cong Sz(8)$. From [3, p. 385], $D_{2(2^3-1)} \in \max Sz(8)$. $D_{2(2^3-1)}$ is an SD-group. By Lemma 3.1, $2^3 - 1 = 3$, a contradiction. Therefore, $G \cong 2.Sz(8)$ or $4.Sz(8)$. Let $G \cong 2.Sz(8)$. Then, it follows from [5, p. 28] that $E_5 : C_4 \in \mathbf{prop}(2.Sz(8))$. From Lemma 3.2, $E_5 : C_4$ is a non-SD-group, a contradiction. Similarly, we can exclude the case $G \cong 4.Sz(8)$.

**Case 3.** $G/N \cong \mathrm{PSL}_3(3)$.

From [5, p. 13], the Schur multiplier of $\mathrm{PSL}_3(3)$ has trivial order. Then, we have that $G \cong N \times \mathrm{PSL}_3(3)$. It follows from [5, p. 13] that $S_4 \in \max \mathrm{PSL}_3(3)$. Hence, $S_4$ is an SD-group. However, $\mathrm{T}(S_4) = 1 + 1 + 2 + 3 + 3 = 10$, and so $\frac{\mathrm{T}(S_4)}{|S_4|} = \frac{10}{24} = \frac{5}{12} \ngtr \frac{8}{12} = \frac{2}{3}$. This implies that $S_4$ is a non-SD-group, a contradiction.

Hence, $G$ is solvable. □

## 4. Conclusion

In this paper, we investigate the influence of sums of all character degrees of all proper subgroups of a finite group on its group structure, with a focus on the solvability of groups satisfying specific conditions. By defining SD-groups and PSD-groups, we prove that a finite PSD-group must be solvable. This work extends the approach of characterizing group structures through the sum of character degrees by incorporating the influence of character degree sums of proper subgroups. Future work may explore the implications of different thresholds, such as other values of $c$ in $\mathrm{T}(H) \leq c|H|$, on group structures, or analyze broader classes of groups, such as nilpotent and supersolvable groups.

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