A note on hyperrings and hypermodules-Corrigendum

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Abstract. In this corrigendum to the paper, "A note on hyperrings and hypermodules" [5], we present revised Theorem 3.3 and Example 3.3 and correct some typographical errors.

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1. Corrigendum

Unfortunately, we have found some errors in [5, Theorem 3.3] and [5, Example 3.3].

Let $m > 1$. Define the relation "$\equiv$" on $\mathbb{N}$ by for all $x, y \in \mathbb{N}$

$$\text{“}x \equiv y \Longleftrightarrow m|k, \text{ where } \min\{x, y\} + k = \max\{x, y\}\text{”}.$$

It can be seen that "$\equiv$" is an equivalence relation on $\mathbb{N}$. Let $\mathbb{N}_m = \{\overline{x} \,|\, x \in \mathbb{N}\}$, where $\overline{x} = \{0 + x, m + x, 2m + x, ...\} = \{nm + x \,|\, n \in \mathbb{N}\}$. Let $0 \leq x < y < m$. Suppose that $\overline{x} = \overline{y}$. Then, $y \in \overline{x}$ and so $m|k$, $x + k = y$ for some $k \in \mathbb{N}$. This is a contradiction since $0 < k < m$. Hence, the equivalence classes $\overline{0}$, $\overline{1}$, ..., $\overline{m-1}$

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are distinct. Let $\overline{x}$ be any element of $\mathbb{N}_m$. By the division algorithm, $x = mq+r$ for some elements q and r such that $0 \leq r < m$. Since $m|mq$, we obtain that $\overline{r} = \overline{x}$. Hence, $\mathbb{N}_m = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{m-1}\}$.

The definition of the hyperoperation given [5, Theorem 3.3] is incorrect and does not satisfy the condition of the uniqueness of an element, which is among the axioms for being a canonical hypergroup. We can observe this situation in [5, Example 3.3]. In this example, the inverse of the element $\overline{2}$ is both $\overline{2}$ and $\overline{4}$. This violates the definition of a canonical hypergroup. Therefore, the hyperoperation should be defined as we provide below.

The following Theorem are corrected versions of these results.

Theorem 1.1. *Let $m > 1$. Define the hyperoperation "$\oplus_m$" on $\mathbb{N}_m$ by*

$$\overline{x} \oplus_m \overline{y} = \begin{cases} \{\overline{x+y}\}, & \text{if } \overline{x} = \overline{y}; \\ \{\overline{x+y}, \overline{k}\}, & \text{if } \overline{x} \neq \overline{y}, \text{ where } \min\{x, y\} + k = \max\{x, y\}. \end{cases}$$

for all $\overline{x}, \overline{y} \in \mathbb{N}_m$. Then

(1) $(\mathbb{N}_m, \oplus_m)$ *is a canonical hypergroup with scalar identity* $\overline{0}$.

(2) $(\mathbb{N}_m, \oplus_m, .)$ *is a commutative and unitary hyperring, where "." is the usual multiplication.*

(3) $(\mathbb{N}^*_m, .)$ *is a group, where* $\mathbb{N}^*_m = \{\overline{x} \in \mathbb{N}_m \,|\, (x, m) = 1\}$.

(4) $(\mathbb{N}_m, \oplus_m, .)$ *is a hyperfield if and only if m is prime.*

(5) *The canonical hypergroup* $(\mathbb{N}_m, \oplus_m)$ *is a* $\mathbb{N}$*-hypermodule.*

Proof. (1), (2) and (3) are straightforward.

(4) ($\Rightarrow$) Let $m = ab$, where $1 \leq a < b < m$. Then $\overline{a}, \overline{b} \in \mathbb{N}_m$ and so $\overline{a}.\overline{b} = \overline{ab} = \overline{0}$, a contradiction.

($\Leftarrow$) Let $\overline{a} \in \mathbb{N}^*_m$. Then $(a, m) = 1$ and so we get $1 = ax + my$ for some $x, y \in \mathbb{N}$. It follows that $\overline{1} = \overline{ax+my} = \overline{ax} = \overline{a}.\overline{x}$. Hence, $\mathbb{N}^*_m = \mathbb{N}_m \backslash \{\overline{0}\}$. By (3), $(\mathbb{N}_m, \oplus_m, .)$ is a hyperfield.

(5) Define the map $\cdot : \mathbb{N} \times \mathbb{N}_m \longrightarrow \mathbb{N}_m$ via $n \cdot \overline{x} = \overline{nx}$ for all $n \in \mathbb{N}$ and for all $\overline{x} \in \mathbb{N}_m$. According to the map, it can be checked that N_m is a $\mathbb{N}$-hypermodule. □

Note that the condition "prime positive integer" in the [5, Proposition 3.3] is necessary. Let's take the following example to see this.

Example 1.1. Given the the hyperring $\mathbb{N}_6$. Using Theorem 1.1, we obtain the following tables:

$\oplus_6$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$
$\overline{1}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{0}, \overline{4}\}$
$\overline{2}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{3}$	$\{\overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$
$\overline{4}$	$\{\overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{5}$	$\{\overline{5}\}$	$\{\overline{0}, \overline{4}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{4}\}$

and

.	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{0}$	$\overline{2}$	$\overline{4}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{2}$	$\overline{0}$	$\overline{4}$	$\overline{2}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Thus, the only maximal hyperideals of the hyperring $\mathbb{N}_6$ are $I_1 = \{\overline{0}, \overline{3}\}$ and $I_2 = \{\overline{0}, \overline{2}, \overline{4}\}$. Also, we have $\mathbb{N}_6 = I_1 \oplus_6 I_2$. It follows that every hyperideal of $\mathbb{N}_6$ is a direct summand of $\mathbb{N}_6$. Hence, the hyperring $\mathbb{N}_6$ is not local.

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