Distance measures between picture fuzzy multisets and their application to medical diagnosis

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Abstract. In this paper, the distance measures between picture fuzzy multisets are proposed as a generalisation of the existing distance measures between picture fuzzy sets. The validity of the transformation from distance measures between PFS to PFMS is carried out using numerical example. Also, an application to medical diagnosis of the proposed distance measures between picture fuzzy multisets is carried out using hypothetical medical database.

Keywords: multiset, fuzzy multiset, picture fuzzy set, picture fuzzy medical diagnosis.
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1. Introduction

Zadeh [12] was the pioneer of fuzzy sets (FSs) which was the generalisation of classical sets. The concept of fuzzy set is a vast and sprawling area which has applications in many areas such as Engineering, Economics, etc. Fuzzy set has a membership function which describes the membership degree of an element with respect to a particular class. Atanassov [1] introduced the concept of intuitionistic fuzzy sets (IFSs) as a generalisation of Zadeh's work.

Picture fuzzy sets (PFSs) was introduced and studied by Cuong and Kreinovich [3] to generalise IFSs and FSs. While it is well-known that intuitionistic fuzzy set generalises fuzzy set in dealing with imprecisions and vagueness, the theory still lacks a very crucial parameter which is the degree of neutrality.

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This degree of neutrality is relevant in variety of situations like voting system, medical diagnosis, personal selection etc. For voting system, a voter has four choices, to vote for, to vote against, to abstain from voting and refuse to vote. In medical diagnosis, symptoms like malaria and typhoid have neutral effect on disorders such as chest pain and stomach pain; headache and temperature have neutral effect on stomach problem and chest problem. As a result of these situations, Cuong and Kreinovich [3] in 2013 introduced and studied PFSs as a generalisation of IFSs and FSs. Thus, PFS is made up of degrees of positive, neutral and negative memberships.

Yagar [11] introduced fuzzy multisets (FMs) to extend fuzzy sets. Shinoj and Sunil [9] extended IFSs and FSs by initiating intuitionistic fuzzy multisets (IFMSs). Cao et al [2] generalised IFMS and FMS by introducing picture fuzzy multisets (PFMS).

In fuzzy mathematics, there is as important concept called distance measure between IFSs. The concept is important because of its wide applications in real life problems. Some researchers have proposed some distance measures between IFSs. Dutta [4] introduced some distance measures between PFSs and extended the work of Wang and Xin [11] and established some properties of them.

In this paper, generalised definition of distance measures between PFMS is put forward, and distance measures between PFSs proposed in [4] are adapted. The validity of the transformation from distance measures between PFS to PFMS is carried out using numerical example. Finally, an application to medical diagnosis of proposed distance measures between picture fuzzy multisets is carried out using hypothetical medical database.

2. Preliminaries

In this section, we recall some basic definitions.

Definition 2.1 ([3]). *Given a nonempty set $\mathcal{C}$. A PFS $\mathcal{D}$ of $\mathcal{C}$ is written as*

$$\mathcal{D} = \{\langle \frac{\sigma_{\mathcal{D}}(z), \tau_{\mathcal{D}}(z), \gamma_{\mathcal{D}}(z)}{z} \rangle | z \in \mathcal{C}\},$$

where the functions $\sigma_{\mathcal{D}}(z), \tau_{\mathcal{D}}(z), \gamma_{\mathcal{D}}(z)$ from $\mathcal{C}$ to $[0,1]$ are called the positive, neutral and negative membership degrees of $z \in \mathcal{C}$ to $\mathcal{D}$, respectively, and for all element $z \in \mathcal{C}$, $0 \leq \sigma_{\mathcal{D}}(z) + \tau_{\mathcal{D}}(z) + \gamma_{\mathcal{D}}(z) \leq 1$. For each PFS $\mathcal{D}$ of $\mathcal{C}$, $\pi_{\mathcal{D}}(z) = 1 - (\sigma_{\mathcal{D}}(z) + \tau_{\mathcal{D}}(z) + \gamma_{\mathcal{D}}(z))$ is the refusal membership degrees of $z \in \mathcal{D}$.

Definition 2.2 ([2]). *Given a nonempty set $\mathcal{C}$. The PFMS $\mathcal{D}$ in $\mathcal{C}$ is characterised by three functions, namely, positive membership count function pmc, neutral membership count function $n_e mc$ and negative membership count function nmc such that pmc, $n_e mc$ and nmc are functions from $\mathcal{C}$ to $\mathcal{W}$, where $\mathcal{W}$ refers to collection of crisp multisets taken from $[0,1]$.*

Thus, every element $r \in \mathcal{C}$, pmc is the crisp multiset from $[0,1]$ whose positive membership sequence is defined by $(\sigma^1_{\mathcal{D}}(r), \sigma^2_{\mathcal{D}}(r), \cdots, \sigma^n_{\mathcal{D}}(r))$ such that

$\sigma^1_{\mathcal{D}}(r) \geq \sigma^2_{\mathcal{D}}(r) \geq \cdots \geq \sigma^n_{\mathcal{D}}(r)$, *$n_e mc$ is the crisp multiset from* $[0,1]$ *whose neutral membership sequence is defined by* $(\tau^1_{\mathcal{D}}(r), \tau^2_{\mathcal{D}}(r), \cdots, \tau^n_{\mathcal{D}}(r))$ *and nmc is the crisp multiset from* $[0,1]$ *whose negative membership sequence is defined by* $(\eta^1_{\mathcal{D}}(r), \eta^2_{\mathcal{D}}(r), \cdots, \eta^n_{\mathcal{D}}(r))$*, these can be either decreasing or increasing functions satisfying* $0 \leq \sigma^k_{\mathcal{D}}(r) + \tau^k_{\mathcal{D}}(r) + \eta^k_{\mathcal{D}}(r) \leq 1$, $\forall r \in \mathcal{C}$, $k = 1, 2, \cdots, n$. *Thus,* $\mathcal{D}$ *is represented by* $\mathcal{D} = \{\langle r, \sigma^k_{\mathcal{D}}(r), \tau^k_{\mathcal{D}}(r), \eta^k_{\mathcal{D}}(r)\rangle | r \in \mathcal{C}\}$, *where* $k = 1, 2, \cdots, n$.

Example 2.1. Let $\mathcal{C} = \{a, b, c\}$

$$\begin{aligned}\mathcal{D} = \quad & \{< a, (0.7, 0.2, 0.1), (0.55, 0.25, 0.10), (0.5, 0.3, 0.2) >,\\ & < b, (0.6, 0.2, 0.2), (0.4, 0.3, 0.2), (0.65, 0, 20, 0.15) >,\\ & < c, (0.8, 0.1, 0.1), (0.4, 0.4, 0.2), (0.9, 0.05, 0.05) >\}\end{aligned}$$

Definition 2.3 ([2])**.** *Let* $\mathcal{D} = \{\langle z, \sigma^k_{\mathcal{D}}(z), \tau^k_{\mathcal{D}}(z), \eta^k_{\mathcal{D}}(z)\rangle \mid z \in \mathcal{C}\}$, $k = 1, 2, \cdots, n$ *be a PFMS,* $z \in \mathcal{D}$. *Then, the order* β *of* $z \in \mathcal{D}$ *is defined as the cardinality of* $C_p M_{\mathcal{D}}(z)$ *or* $C_{ne} M_{\mathcal{D}}(z)$ *or* $C_n M_{\mathcal{D}}(z)$ *for* $0 \leq \sigma^k_{\mathcal{D}}(z) + \tau^k_{\mathcal{D}}(z) + \eta^k_{\mathcal{D}}(z) \leq 1$. *That is*

$$\beta = |C_p M_{\mathcal{D}}(z)| = |C_{ne} M_{\mathcal{D}}(z)| = |C_n M_{\mathcal{D}}(z)|.$$

Definition 2.4 ([2])**.** *Given two PFMSs* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *drawn from* $\mathcal{C}$. *Then, the size of* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *is defined as*

$$\beta = \beta(\mathcal{D}_1) \vee (\beta(\mathcal{D}_2).$$

Definition 2.5 ([8])**.** *Given that PFMS* $\mathcal{D}$ *is not empty. Then, a PFMR* $\mathcal{U}$ *on* $\mathcal{D}$*, defined as* $\mathcal{U} = \{\langle (x, y), \sigma^k_{\mathcal{D}}(x, y), \tau^k_{\mathcal{D}}(x, y), \eta^k_{\mathcal{D}}(x, y)\rangle | (x, y) \in \mathcal{D} \times \mathcal{D}\}$ *is also a PFMS, where* $k = 1, 2, \cdots, \beta$*, where* β *is the cardinality of the PFMS* $\mathcal{D}$ *and* $\sigma^k_{\mathcal{D}}, \tau^k_{\mathcal{D}}, \eta^k_{\mathcal{D}}$ *are functions from* $\mathcal{C}$ *to* $\mathcal{W}$*, where* $\mathcal{W}$ *is the set of all crisp multisets drawn from* $[0, 1]$.

2.1 Distance measures between picture fuzzy sets

Dutta [4], defined distance measures between PFSs as follow.

Let

$$\mathcal{D}_1 = \{< x, \sigma_{\mathcal{D}_1}(x), \tau_{\mathcal{D}_1}(x), \gamma_{\mathcal{D}_1}(x) > \ | \ x \in X\}$$

and

$$\mathcal{D}_2 = \{< x, \sigma_{\mathcal{D}_2}(x), \tau_{\mathcal{D}_2}(x), \gamma_{\mathcal{D}_2}(x) > \ | \ x \in X\}$$

defined on X. Then,

(1) the Hamming Distance is defined as

$$\begin{aligned}d(\mathcal{D}_1, \mathcal{D}_2) = \frac{1}{2}\sum_{j=1}^{n}\{&|\sigma_{\mathcal{D}_1}(x_j) - \sigma_{\mathcal{D}_2}(x_j)| + |\tau_{\mathcal{D}_1}(x_j) - \tau_{\mathcal{D}_2}(x_j)|\\ &+ |\eta_{\mathcal{D}_1}(x_j) - \eta_{\mathcal{D}_2}(x_j)| + |\pi_{\mathcal{D}_1}(x_j) - \pi_{\mathcal{D}_2}(x_j)|\};\end{aligned}$$

(2) the Normalised Hamming Distance is defined as

$$l(\mathcal{D}_1,\mathcal{D}_2)=\frac{1}{2n}\sum_{j=1}^{n}\{|\sigma_{\mathcal{D}_1}(x_j)-\sigma_{\mathcal{D}_2}(x_j)|+|\tau_{\mathcal{D}_1}(x_j)-\tau_{\mathcal{D}_2}(x_j)|$$
$$+|\eta_{\mathcal{D}_1}(x_j)-\eta_{\mathcal{D}_2}(x_j)|+|\pi_{\mathcal{D}_1}(x_j)-\pi_{\mathcal{D}_2}(x_j)|\};$$

(3) the Euclidean Distance is defined as

$$e(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2}\sum_{j=1}^{n}[(\sigma_{\mathcal{D}_1}(x_j)-\sigma_{\mathcal{D}_2}(x_j))^2+(\tau_{\mathcal{D}_1}(x_j)-\tau_{\mathcal{D}_2}(x_j))^2$$
$$+(\eta_{\mathcal{D}_1}(x_j)-\eta_{\mathcal{D}_2}(x_j))^2+(\pi_{\mathcal{D}_1}(x_j)-\pi_{\mathcal{D}_2}(x_j))^2]\}^{\frac{1}{2}};$$

(4) the Normalised Euclidean Distance is defined as

$$q(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2n}\sum_{j=1}^{n}[(\sigma_{\mathcal{D}_1}(x_j)-\sigma_{\mathcal{D}_2}(x_j))^2+(\tau_{\mathcal{D}_1}(x_j)-\tau_{\mathcal{D}_2}(x_j))^2$$
$$+(\eta_{\mathcal{D}_1}(x_j)-\eta_{\mathcal{D}_2}(x_j))^2+(\pi_{\mathcal{D}_1}(x_j)-\pi_{\mathcal{D}_2}(x_j))^2)]\}^{\frac{1}{2}}.$$

3. Distance measure between picture fuzzy multisets

This section extends distance measures between PFSs to PFMSs. Throughout this section, $\mathbb{I}$ denotes closed interval $[0,1]$.

Definition 3.1. *Let $\mathcal{C}$ be a nonempty set such that PFMSs $\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3\in\mathcal{C}$. Then, the distance measure is a mapping $d:\mathcal{C}\times\mathcal{C}\to\mathbb{I}$, if it satisfies the following properties;*

(i) $d(\mathcal{D}_1,\mathcal{D}_2)\in\mathbb{I}$.

(ii) $d(\mathcal{D}_1,\mathcal{D})_2=0$ *if and only if* $\mathcal{D}_1=\mathcal{D}_2$ *(faithful condition).*

(iii) $d(\mathcal{D}_1,\mathcal{D}_2)=d(\mathcal{D}_2,\mathcal{D}_1)$ *(Symmetric property).*

(iv) $d(\mathcal{D}_1,\mathcal{D}_3)\le d(\mathcal{D}_1,\mathcal{D}_2)+d(\mathcal{D}_2,\mathcal{D}_3)$ *(Triangular inequality).*

(v) If $\mathcal{D}_1\subseteq\mathcal{D}_2\subseteq\mathcal{D}_3$, *then* $d(\mathcal{D}_1,\mathcal{D}_3)\ge d(\mathcal{D}_1,\mathcal{D}_2)$ *and* $d(\mathcal{D}_1,\mathcal{D}_3)\ge d(\mathcal{D}_2,\mathcal{D}_3)$.

Then, $d(\mathcal{D}_1,\mathcal{D}_2)$ measures the distance between PFMSs $\mathcal{D}_1$ and $\mathcal{D}_2$.

Distance measure can be considered as a dual concept of similarity measure. Distance measures between PFSs have been introduced and studied recently by Dutta [4]. These distance measures are extended to PFMSs since they satisfied conditions of metric distance.

Definition 3.2. *Let $\mathcal{D}_1=\{<z,\sigma^k_{\mathcal{D}_1}(z),\tau^k_{\mathcal{D}_1}(z),\eta^k_{\mathcal{D}_1}(z)>\ |\ z\in\mathcal{C}\}$ and $\mathcal{D}_2=\{<z,\sigma^k_{\mathcal{D}_2}(z),\tau^k_{\mathcal{D}_2}(z),\eta^k_{\mathcal{D}_2}(z)>\ |\ z\in\mathcal{C}\}$ defined on $\mathcal{C}$. Then,*

(1) the Hamming Distance is defined as

$$hd(\mathcal{D}_1,\mathcal{D}_2)=\frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j)|+|\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j)| + |\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j)|+|\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j)|\};$$

(2) the Normalised Hamming Distance is defined as

$$nhd(\mathcal{D}_1,\mathcal{D}_2)=\frac{1}{2n}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j)|+|\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j)| + |\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j)|+|\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j)|\};$$

(3) the Euclidean Distance is defined as

$$ed(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j))^2+(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j))^2 + (\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j))^2+(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}};$$

(4) the Normalised Euclidean Distance is defined as

$$ned(\mathcal{D}_1,\mathcal{D}_2)=\{\frac{1}{2n}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j))^2+(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j))^2 + (\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j))^2+(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}}.$$

Theorem 3.1. *Given*

$$\mathcal{D}_1=\{<z,\sigma^k_{\mathcal{D}_1}(z),\tau^k_{\mathcal{D}_1}(z),\eta^k_{\mathcal{D}_1}(z)> \ | \ z\in\mathcal{C}\},$$
$$\mathcal{D}_2=\{<z,\sigma^k_{\mathcal{D}_2}(z),\tau^k_{\mathcal{D}_2}(z),\eta^k_{\mathcal{D}_2}(z)> \ | \ z\in\mathcal{C}\}$$

and

$$\mathcal{D}_3=\{<z,\sigma^k_{\mathcal{D}_3}(z),\tau^k_{\mathcal{D}_3}(z),\eta^k_{\mathcal{D}_3}(z)> \ | \ z\in\mathcal{C}\}$$

defined on $\mathcal{C}$. Then, the distance measures defined in Definition 3.2 are metric.

Proof. (1) Hamming Distance

Non-negativity: $hd(\mathcal{D}_1,\mathcal{D}_2)\geq 0$, because by definition, $hd(\mathcal{D}_1,\mathcal{D}_2)$ is always either 0 or 1, since the Hamming distance is the sum of non-negative terms. Faithful condition: Suppose that $\mathcal{D}_1=\mathcal{D}_2$, then every corresponding element is the same. That is

$$\sigma^k_{\mathcal{D}_1}(z_j)=\sigma^k_{\mathcal{D}_2}(z_j),\ \tau^k_{\mathcal{D}_1}(z_j)=\tau^k_{\mathcal{D}_2}(z_j),\ \eta^k_{\mathcal{D}_1}(z_j)=\eta^k_{\mathcal{D}_2}(z_j)$$

and

$$\pi^k_{\mathcal{D}_1}(z_j)=\pi^k_{\mathcal{D}_2}(z_j).$$

Hence, $hd(\mathcal{D}_1, \mathcal{D}_2) = 0$.

Conversely, suppose that $hd(\mathcal{D}_1, \mathcal{D}_2) = 0$, then all the terms in the summation must be zero, which means that $|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)| = 0$, $|\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)| = 0$, $|\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)| = 0$ and $|\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)| = 0$.

Hence, $\mathcal{D}_1 = \mathcal{D}_2$.

Symmetric:

$$\begin{aligned} hd(\mathcal{D}_1, \mathcal{D}_2) &= \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)| + |\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)| \\ &\quad + |\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)| + |\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)|\} \\ &= \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_1}(z_j)| + |\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_1}(z_j)| \\ &\quad + |\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_1}(z_j)| + |\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_1}(z_j)|\} \\ &= hd(\mathcal{D}_2, \mathcal{D}_1). \end{aligned}$$

Triangle inequality:

$$\begin{aligned} hd(\mathcal{D}_1, \mathcal{D}_3) &= \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j)| + |\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_3}(z_j)| \\ &\quad + |\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_3}(z_j)| + |\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_3}(z_j)|\} \\ &= \frac{1}{2}\sum_{k,j=1}^{n}\{|(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)) + (\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j))| \\ &\quad + |(\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)) + (\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_3}(z_j))| \\ &\quad + |(\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)) + (\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_3}(z_j))| \\ &\quad + |(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)) + (\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_3}(z_j))|\} \\ &\leq \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j)| + |\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j)| \\ &\quad + |\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j)| + |\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j)|\} \\ &\quad + \frac{1}{2}\sum_{k,j=1}^{n}\{|\sigma^k_{\mathcal{D}_2}(z_j) - \sigma^k_{\mathcal{D}_3}(z_j)| + |\tau^k_{\mathcal{D}_2}(z_j) - \tau^k_{\mathcal{D}_3}(z_j)| \\ &\quad + |\eta^k_{\mathcal{D}_2}(z_j) - \eta^k_{\mathcal{D}_3}(z_j)| + |\pi^k_{\mathcal{D}_2}(z_j) - \pi^k_{\mathcal{D}_3}(z_j)|\} \\ &= hd(\mathcal{D}_1, \mathcal{D}_2) + hd(\mathcal{D}_2, \mathcal{D}_3). \end{aligned}$$

(2) Normalised Hamming Distance can be proved in a similar way.

(3) Euclidean Distance

Non-negativity: $ed(\mathcal{D}_1, \mathcal{D}_2) \geq 0$ since each squared sum and difference are always non-negative.

Faithful condition: Suppose that $\mathcal{D}_1 = \mathcal{D}_2$, then

$\sigma^k_{\mathcal{D}_1}(z_j) = \sigma^k_{\mathcal{D}_2}(z_j)$, $\tau^k_{\mathcal{D}_1}(z_j) = \tau^k_{\mathcal{D}_2}(z_j)$, $\eta^k_{\mathcal{D}_1}(z_j) = \eta^k_{\mathcal{D}_2}(z_j)$ and $\pi^k_{\mathcal{D}_1}(z_j) = \pi^k_{\mathcal{D}_2}(z_j)$.

Thus $ed(\mathcal{D}_1, \mathcal{D}_2) = \sqrt{0} = 0$.

Conversely, suppose that $ed(\mathcal{D}_1, \mathcal{D}_2) = 0$, this implies that

$$\begin{aligned}
(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\
(\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\
(\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 &= 0
\end{aligned}$$

and

$$(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2 = 0,$$

i.e.

$$\begin{aligned}
&[\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2\\
&+ (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 + (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2]]^{\frac{1}{2}} = 0.
\end{aligned}$$

Since the square root function is only zero when its argument is zero, thus

$$\begin{aligned}
&\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 + (\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2\\
&+ (\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 + (\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2] = 0.
\end{aligned}$$

Since each of these

$$\begin{aligned}
&(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2,\\
&(\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2,\\
&(\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2
\end{aligned}$$

and

$$(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2$$

is non-negative, their sum can only be zero if

$$\begin{aligned}
(\sigma^k_{\mathcal{D}_1}(z_j) - \sigma^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\
(\tau^k_{\mathcal{D}_1}(z_j) - \tau^k_{\mathcal{D}_2}(z_j))^2 &= 0,\\
(\eta^k_{\mathcal{D}_1}(z_j) - \eta^k_{\mathcal{D}_2}(z_j))^2 &= 0
\end{aligned}$$

and

$$(\pi^k_{\mathcal{D}_1}(z_j) - \pi^k_{\mathcal{D}_2}(z_j))^2 = 0.$$

Thus $\sigma^k_{\mathcal{D}_1}(z_j)=\sigma^k_{\mathcal{D}_2}(z_j)$, $\tau^k_{\mathcal{D}_1}(z_j)=\tau^k_{\mathcal{D}_2}(z_j)$, $\eta^k_{\mathcal{D}_1}(z_j)=\eta^k_{\mathcal{D}_2}(z_j)$ and $\pi^k_{\mathcal{D}_1}(z_j)=\pi^k_{\mathcal{D}_2}(z_j)$.

Hence, $\mathcal{D}_1=\mathcal{D}_2$.

Symmetric:

$$\begin{aligned}
ed(\mathcal{D}_1,\mathcal{D}_2) &= \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j))^2+(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j))^2\\
&\quad +(\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j))^2+(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}}\\
&= \{\frac{1}{2}\sum_{j=1}^{n}[(\sigma^k_{\mathcal{D}_2}(z_j)-\sigma^k_{\mathcal{D}_1}(z_j))^2+(\tau^k_{\mathcal{D}_2}(z_j)-\tau^k_{\mathcal{D}_1}(z_j))^2\\
&\quad +(\eta^k_{\mathcal{D}_2}(z_j)-\eta^k_{\mathcal{D}_1}(z_j))^2+(\pi^k_{\mathcal{D}_2}(z_j)-\pi^k_{\mathcal{D}_1}(z_j))^2]\}^{\frac{1}{2}}\\
&= ed(\mathcal{D}_2,\mathcal{D}_1).
\end{aligned}$$

Triangle inequality:

$$\begin{aligned}
ed(\mathcal{D}_1,\mathcal{D}_3) &= \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_3}(z_j))^2+(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_3}(z_j))^2\\
&\quad +(\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_3}(z_j))^2+(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_3}(z_j))^2]\}^{\frac{1}{2}}\\
&= \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j)+\sigma^k_{\mathcal{D}_2}(z_j)-\sigma^k_{\mathcal{D}_3}(z_j))^2\\
&\quad +(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j)+\tau^k_{\mathcal{D}_2}(z_j)-\tau^k_{\mathcal{D}_3}(z_j))^2\\
&\quad +(\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j)+\eta^k_{\mathcal{D}_2}(z_j)-\eta^k_{\mathcal{D}_3}(z_j))^2\\
&\quad +(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j)+\pi^k_{\mathcal{D}_2}(z_j)-\pi^k_{\mathcal{D}_3}(z_j))^2]\}^{\frac{1}{2}}\\
&\leq \{\frac{1}{2}\sum_{k,j=1}^{n}[(\sigma^k_{\mathcal{D}_1}(z_j)-\sigma^k_{\mathcal{D}_2}(z_j))^2+(\tau^k_{\mathcal{D}_1}(z_j)-\tau^k_{\mathcal{D}_2}(z_j))^2\\
&\quad +(\eta^k_{\mathcal{D}_1}(z_j)-\eta^k_{\mathcal{D}_2}(z_j))^2+(\pi^k_{\mathcal{D}_1}(z_j)-\pi^k_{\mathcal{D}_2}(z_j))^2]\}^{\frac{1}{2}}\\
&+ \{\frac{1}{2}\sum_{j=1}^{n}[(\sigma^k_{\mathcal{D}_2}(z_j)-\sigma^k_{\mathcal{D}_3}(z_j))^2+(\tau^k_{\mathcal{D}_2}(z_j)-\tau^k_{\mathcal{D}_3}(z_j))^2\\
&\quad +(\eta^k_{\mathcal{D}_2}(z_j)-\eta^k_{\mathcal{D}_3}(z_j))^2+(\pi^k_{\mathcal{D}_2}(z_j)-\pi^k_{\mathcal{D}_3}(z_j))^2]\}^{\frac{1}{2}}\\
&\leq ed(\mathcal{D}_1,\mathcal{D}_2)+ed(\mathcal{D}_2,\mathcal{D}_3).
\end{aligned}$$

(4) Normalised Euclidean distance can be proved in a similar way. □

4. Application of picture fuzzy multisets in medical diagnosis via distance measures

This section presents an application of PFMSs to medical diagnosis via distance measures. Given that, the following parameters M, L, and K represent sets of symptoms, diseases and patients. Define the PFMR, F from M to L, $M \times L$ which reveals the degrees of association, non-association and nuetrality between symptoms and diseases. Also, define another PFMR, G from K to M, $K \times M$ which reveals the association degree, non-association degree and neutrality degree between patients and symptoms.

The steps to determine which disease explains patience's symptoms and signs better is what is called medical decision making. We employ the same methodology for the picture fuzzy medical diagnosis stated in [4] for PFMR. These are stated below:

i. Determine $M_i,\ i = 1, 2, 3, 4$.

ii. Formulate medical knowledge PFMR, F, $M \times L$ and PFMR, G, $K \times M$.

iii. Evaluate π in the data set.

iv. Computation of the distance between K_i and L_i using Definition 3.2.

v. Identification of the shortest distance between K_i and L_i reflects that the patient K_i is suffering from disease $L_i,\ i = 1, 2, 3, 4$.

4.1 Case study

Let

$$K = \{\text{Rabat, Roy, Ranic, Raja}\}$$

be a set of patients,

$$L = \{\text{Viral fever, malaria, Typhoid, Stomach problem}\}$$

be a set of diseases and

$$M = \{\text{Temperature, Headache, Stomach pain, Cough, Chest pain}\}$$

be a set of symptoms. Here, each patient was examined more than once because there is a possibility that a particular patient has symptoms of different diseases. This will enable the doctor to draw a conclusion that a particular patient is suffering from a particular disease based on the different examinations carried out on each patient.

In Table 1, each symptom M_i is described by four parameters: positive membership degree $\sigma(x)$, neutral membership degree $\tau(x)$, negative membership degree $\eta(x)$ and refusal degree $\pi(x)$.

In Table 2, the data were recorded at different intervals, and the distance of each patient K_i from the set of symptoms M_i for each diagnosis $L_i,\ i = 1, 2, 3, 4$. The first set represents the positive membership values, the second represents the neutral membership values, the third represents the negative membership values and the fourth represents the refusal membership values. Thus, using Definition 2.2 Table 2 is constructed.

In Table 3, conversion of PFMS to PFS was carried out by finding the mean values for each set in Table 2.

Table 1: SYMPTOMS VS DISEASES

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Temperature	(0.6,0.2,0.1,0.1)	(0.1,0.5,0.1,0.3)	(0.3,0.5,0.1,0.1)	(0.3,0.1,0.5,0.1)
Cough	(0.1,0.5,0.1,0.3)	(0.7,0.1,0.0,0.2)	(0.2,0.4,0.2,0.2)	(0.4,0.2,0.0,0.4)
Chest pain	(0.2,0.4,0.2,0.2)	(0.5,0.1,0.3,0.1)	(0.1,0.5,0.1,0.3)	(0.6,0.0,0.1,0.3)
Headache	(0.3,0.5,0.1,0.1)	(0.6,0.0,0.1,0.3)	(0.1,0.4,0.2,0.3)	(0.4,0.4,0.1,0.1)
Stomach pain	(0.4,0.3,0.2,0.1)	(0.6,0.0,0.1,0.3)	(0.3,0.3,0.2,0.2)	(0.5,0.2,0.0,0.3)

Table 2: PATIENTS VS SYMPTOMS

	Temperature	Cough	Chest pain	Headache	Stomach pain
Rabat	(0.5,0.2,0.1,0.2)	(0.3,0.4,0.2,0.1)	(0.4,0.3,0.1,0.2)	(0.2,0.3,0.2,0.3)	(0.6,0.0,0.3,0.0)
	(0.6,0.2,0.1,0.1)	(0.3,0.1,0.3,0.3)	(0.2,0.3,0.2,0.2)	(0.1,0.0,0.2,0.7)	(0.0,0.4,0.2,0.4)
	(0.4,0.3,0.2,0.1)	(0.4,0.4,0.1,0.1)	(0.3,0.2,0.3,0.2)	(0.0,0.4,0.3,0.3)	(0.2,0.3,0.2,0.3)
Roy	(0.4,0.3,0.1,0.2)	(0.6,0.0,0.2,0.2)	(0.5,0.1,0.1,0.3)	(0.1,0.3,0.3,0.3)	(0.4,0.1,0.2,0.3)
	(0.3,0.3,0.2,0.2)	(0.4,0.1,0.2,0.3)	(0.4,0.2,0.0,0.4)	(0.3,0.2,0.1,0.4)	(0.3,0.5,0.1,0.1)
	(0.4,0.2,0.1,0.3)	(0.3,0.3,0.0,0.4)	(0.1,0.3,0.1,0.5)	(0.6,0.1,0.2,0.1)	(0.2,0.3,0.3,0.2)
Ranic	(0.6,0.1,0.1,0.2)	(0.2,0.3,0.1,0.4)	(0.3,0.2,0.2,0.3)	(0.3,0.4,0.1,0.2)	(0.5,0.2,0.1,0.2)
	(0.2,0.4,0.2,0.2)	(0.1,0.4,0.0,0.5)	(0.2,0.5,0.1,0.2)	(0.4,0.2,0.1,0.3)	(0.4,0.3,0.2,0.1)
	(0.3,0.3,0.1,0.3)	(0.6,0.0,0.2,0.2)	(0.1,0.3,0.5,0.1)	(0.2,0.3,0.2,0.3)	(0.0,0.3,0.4,0.3)
Raja	(0.3,0.0,0.3,0.4)	(0.5,0.2,0.1,0.2)	(0.4,0.3,0.1,0.2)	(0.6,0.2,0.1,0.1)	(0.2,0.4,0.2,0.2)
	(0.4,0.2,0.2,0.2)	(0.3,0.3,0.2,0.2)	(0.2,0.5,0.2,0.1)	(0.3,0.2,0.3,0.2)	(0.3,0.1,0.3,0.3)
	(0.1,0.5,0.1,0.3)	(0.2,0.4,0.1,0.3)	(0.3,0.4,0.1,0.2)	(0.1,0.5,0.1,0.3)	(0.6,0.2,0.1,0.1)

Next, we convert PFMS to PFS by finding the mean values for each set in Table 2 to get Table 3.

Table 3: PATIENTS VS SYMPTOMS

	Temperature	Cough	Chest pain	Headache	Stomach pain
Rabat	(0.50,0.23, 0.13,0.13)	(0.33,0.30, 0.20,0.17)	(0.30,0.27, 0.20,0.23)	(0.10,0.23, 0.23,0.43)	(0.27,0.23, 0.23,0.27)
Roy	(0.37,0.27, 0.13,0.23)	(0.43,0.13, 0.13,0.30)	(0.33,0.20, 0.07,0.40)	(0.33,0.20, 0.20,0.27)	(0.30,0.30, 0.20,0.20)
Ranic	(0.37,0.27, 0.13,0.23)	(0.30,0.23, 0.10,0.37)	(0.20,0.33, 0.27,0.20)	(0.30,0.30, 0.13,0.27)	(0.30,0.27, 0.23,0.20)
Raja	(0.27,0.23, 0.20,0.30)	(0.33,0.30, 0.13,0.23)	(0.30,0.40, 0.13,0.17)	(0.33,0.30, 0.17,0.20)	(0.37,0.23, 0.20,0.20)

Table 4: HAMMING DISTANCE BETWEEN PATIENTS AND DISEASES

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Rabat	1.220	2.060	**0.945**	1.790
Roy	1.325	1.765	**1.125**	1.505
Ranic	**0.865**	1.860	0.980	1.570
Raja	0.995	1.705	**0.975**	1.395

Table 5: N-HAMMING DISTANCE BETWEEN PATIENTS AND DISEASES

	Viral ever	Tuberculosis	Typhoid	Stomach problem
Rabat	0.244	0.412	**0.189**	0.358
Roy	0.265	0.353	**0.225**	0.301
Ranic	**0.173**	0.372	0.196	0.314
Raja	0.199	0.341	**0.195**	0.279

Table 6: Euclidean distance between patients and diseases

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Rabat	0.475	0.755	**0.393**	0.645
Roy	0.541	0.581	**0.482**	0.514
Ranic	**0.362**	0.635	0.377	0.581
Raja	0.416	0.808	**0.390**	0.541

Table 7: N-Euclidean distance between patients and diseases

	Viral fever	Tuberculosis	Typhoid	Stomach problem
Rabat	0.212	0.338	**0.176**	0.289
Roy	0.242	0.260	**0.215**	0.230
Ranic	**0.162**	0.284	0.169	0.260
Raja	0.186	0.361	**0.174**	0.242

4.2 Results and discussion

With respect to Tables 4 - 7, the decision making is presented. Decisions are made based on the smallest distance between the patients and diseases. From Tables 4 - 7, it was established that Rabat is suffering from Typhoid, Roy is suffering from Typhoid, Rank is suffering from Typhoid and Ranic is suffering from Viral fever.

5. Conclusion

In this paper, it has been established that the distance measures; Hamming Distance, Euclidean Distance together with their normalised versions transformed from PFSs to PFMSs are valid simply because the properties of the definition of distance measures proposed in [4] are satisfied. Also, an application to medical diagnosis of distance measures between picture fuzzy multisets is carried out using hypothetical medical database.

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