On algebraically closed Krasner hyperfields

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Abstract. In this short paper, we prove two negative results: the first order theory of algebraically closed Krasner hyperfields is neither complete nor substructure complete, the latter meaning that the theory does not admit quantifier elimination.

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1. Introduction

It is a classical result by A. Robinson ([19]) that the first order theory ACF_p of algebraically closed fields of fixed characteristic p (prime or 0) is complete. In addition, it is known that the theory ACF of algebraically closed fields of unspecified characteristic admits quantifier elimination.

Krasner hyperfields may be quickly described as field-like structures, where the additive operation is multivalued, that is, the result of an addition in a hyperfield F is allowed to be an arbitrary subset of F. Hyperfields have been first described in [8] and subsequently attract the interest of several mathematicians [16, 15, 5, 6, 1, 11, 3, 2, 10, 7, 13] because of a variety of reasons.

It is tempting to define and consider algebraically closed Krasner hyperfields (this has been done for instance in [4]) and to analyse if model theoretic results analogous to the case of algebraically closed fields hold for the theories of algebraically closed Krasner hyperfields (of fixed or unspecified characteristic, respectively).

By considering some simple but fundamental examples we prove in this paper that the first order theory (over a natural first order language) of algebraically closed Krasner hyperfields of fixed characteristic 0 is not complete. We prove in addition that the first order theory of algebraically closed Krasner hyperfields of unspecified characteristic does not admit quantifier elimination.

We also compare the notion of algebraically closed hyperfields with the notion of algebraically closed structures over the language we have selected. In the case of fields these two notions coincide, while we note that in the case of hyperfields they do not.

2. Hyperfields

Definition 2.1. *A* multioperation *on a set F is a map $\boxplus : F \times F \to \mathcal{P}(F)$, where $\mathcal{P}(F)$ denotes the power set of F.*

If $X, Y \subseteq F$, then we set

$$X \boxplus Y := \bigcup_{(x,y)\in X\times Y} x \boxplus y.$$

If $X = \{x\}$ (resp. $Y = \{y\}$), then we write $x \boxplus Y$ (resp. $X \boxplus y$) in place of $X \boxplus Y$.

Definition 2.2. *A* (Krasner) hyperfield *is a tuple $(F, \boxplus, \cdot, 0, 1)$, where $\boxplus$ is a multioperation on F and $\cdot : F \times F \to F$ is an operation on F while 0 and 1 are elements of F, subject to the following conditions:*

(KH1) The multioperation $\boxplus$ is associative on F, i.e., $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$ (as subsets of F), for all $x, y, z \in F$.

(KH2) The multioperation $\boxplus$ is commutative on F, i.e., $x \boxplus y = y \boxplus x$ (as subsets of F), for all $x, y \in F$.

(KH3) For all $x \in F$, there exists a unique $x^- \in F$ such that $0 \in x \boxplus x^-$.

(KH4) $(F \setminus \{0\}, \cdot, 1)$ is an abelian group and $0 \cdot x = x \cdot 0 = 0$, for all $x \in F$.

(KH5) The operation $\cdot$ is distributive over $\boxplus$, i.e.,

$$xy \boxplus xz = x(y \boxplus z) := \{xt \mid t \in y \boxplus z\}$$

(as subsets of F).

Remark 2.1. The expert reader may have noticed that the *reversibility* axiom is absent in our definition, while it explicitly appears in the original definition of hyperfields given by Krasner (cf. [8] or [9]):

- The multioperation $\boxplus$ is reversible on F, i.e., $z \in x \boxplus y$ implies $x \in z \boxplus y^-$, for all $x, y, z \in F$.

Let us briefly explain the reason for this absence. By subsequent applications of the axiom one notices that the multioperation $\boxplus$ of a hyperfield F is reversible if and only if

$$z^- \in x^- \boxplus y^-, \text{ for all } x, y, z \in F. \tag{1}$$

On the other hand, by axioms (KH4) and (KH5) we obtain, for all $x \in F$ that

$$0 \in x \cdot (1 \boxplus 1^-) = x \boxplus x \cdot 1^-.$$

By the uniqueness postulated in axiom (KH3), it follows that $x \cdot 1^- = x^-$, for all $x \in F$. Therefore, (1) and hence the reversibility axiom follow logically from the other axioms.[1]

1. The author is in debt with Ch. Massouros for presenting this observation.

Remark 2.2. By axiom (KH1), summations of finite length are well-defined in any hyperfield, with no need to specify where the parentheses lie.

Remark 2.3. Note that for a hyperfield $(F, \boxplus, \cdot, 0, 1)$ the multioperation $\boxplus$ only has non-empty values. Moreover, it follows from the uniqueness postulated in axiom (KH3) and Remark 2.1 that for all $x \in F$ one has $x \boxplus 0 = \{x\}$.

Definition 2.3. *A hyperfield F has* characteristic $n \in \mathbb{N}$ *if*

$$0 \in \underbrace{1 \boxplus \ldots \boxplus 1}_{n \ times}$$

and n is the minimal such positive integer. If no such n exists, then F is said to have characteristic 0.

Example 2.1 (Sign hyperfield)**.** Consider the set $\mathbb{S} = \{1^-, 0, 1\}$ endowed with the multioperation $\boxplus$ defined by $1 \boxplus 0 = 1 \boxplus 1 := \{1\}$, $1^- \boxplus 0 = 1^- \boxplus 1^- := \{1^-\}$ and $1 \boxplus 1^- := \mathbb{S}$. With the obvious multiplication $(\mathbb{S}, \boxplus, \cdot, 0, 1)$ is a hyperfield. This hypefield has characteristic 0.

Example 2.2 (Phase hyperfield)**.** Consider the set of complex numbers $\mathbb{P} := S^1 \cup \{0\}$, where S^1 denotes the circle of units. Endow $\mathbb{P}$ with a commutative multioperation defined by $z \boxplus 0 := \{z\}$ and for $0 \le \varphi_1 \le \varphi_2 < 2\pi$

$$e^{i\varphi_1} \boxplus e^{i\varphi_2} := \begin{cases} \{e^{i\varphi_1}\}, & \text{if } \varphi_1 = \varphi_2 \\ \{e^{i\varphi} \mid \varphi_1 < \varphi < \varphi_2\}, & \text{if } \varphi_1 < \varphi_2 \text{ and } \varphi_2 - \varphi_1 < \pi \\ \{e^{i\varphi} \mid \varphi_2 < \varphi < \varphi_1 + 2\pi\}, & \text{if } \varphi_1 < \varphi_2 \text{ and } \varphi_2 - \varphi_1 > \pi \\ \{e^{i\varphi_1}, 0, e^{i\varphi_2}\}, & \text{if } \varphi_2 - \varphi_1 = \pi \end{cases}$$

With the multiplication of complex numbers $(\mathbb{P}, \boxplus, \cdot, 0, 1)$ is a hyperfield (see e.g. [17, p. 6] and also [16]). This hyperfield is nothing but the quotient hyperfield $\mathbb{C}/\mathbb{R}^+$ (cf. [9]), where $\mathbb{R}^+$ denotes the group of positive real numbers. The phase hyperfield has characteristic 0.

Definition 2.4 ([16])**.** *A hyperfield F is* closed *if $x, y \in x \boxplus y$ for all $x, y \in F$.*

Lemma 2.1 ([16, Construction II])**.** *Let $(F, \boxplus, \cdot, 0, 1)$ be a hyperfield. If we define on F a new multioperation $\dot{\boxplus}$ as follows: $x \dot{\boxplus} 0 = 0 \dot{\boxplus} x = \{x\}$ for all $x \in F$ and, for $x, y \neq 0$,*

$$x \dot{\boxplus} y := \begin{cases} x \boxplus y \cup \{x, y\}, & \text{if } y \neq x^- \\ F, & \text{if } y = x^-, \end{cases}$$

then $(F, \dot{\boxplus}, \cdot, 0, 1)$ is a hyperfield.

Definition 2.5. *We call the hyperfield $\dot{F} := (F, \dot{\boxplus}, \cdot, 0, 1)$ constructed as in the previous lemma the* closure *of the given hyperfield $(F, \boxplus, \cdot, 0, 1)$.*

We leave to the reader the straightforward verification of the following lemma.

Lemma 2.2. *The closure $\dot{\mathbb{P}}$ of the phase hyperfield has characteristic* 0.

Next, we define algebraically closed hyperfields by considering polynomials over hyperfields. Polynomials over hyperfields have been considered also e.g. in [2] from where we took the terminology analogous to that standardly adopted for fields.

Definition 2.6. *A hyperfield F is called* algebraically closed *if every non-constant polynomial with coefficients in F has a root in F, i.e., for all $a_0, \ldots, a_n \in F$ with $n > 0$, there exists $z \in F$ such that*

$$0 \in a_0 \boxplus a_1 z \ldots \boxplus a_n z^n.$$

Lemma 2.3. *Let $(F, \boxplus, \cdot, 0, 1)$ denote the phase hyperfield $\mathbb{P}$ or its closure $\dot{\mathbb{P}}$ and take $a, b, c \in F$. Then the following inclusion*

$$a \boxplus b \subseteq a \boxplus b \boxplus c$$

holds.

Proof. By definition, we have that

$$a \boxplus b \boxplus c = \bigcup_{d \in a \boxplus b} d \boxplus c.$$

There are two possibilities:

1. If $c \in a \boxplus b$, then the right hand side is equal to the arc $a \boxplus b$.
2. If $c \notin a \boxplus b$, then the right hand side is an arc connecting c and either a or b, which by the assumption contains the shortest arc connecting a and b, i.e., $a \boxplus b$. □

Theorem 2.1. *Let $(F, \boxplus, \cdot, 0, 1)$ denote the phase hyperfield $\mathbb{P}$ or its closure $\dot{\mathbb{P}}$ and take a non-constant polynomial $a_0 \boxplus a_1 z \boxplus \ldots \boxplus a_n z^n$ such that $a_0 \neq 0$. Consider the minimal $m > 0$ such that $a_m \neq 0$, then*

$$a_0 \boxplus a_m z^m \subseteq a_0 \boxplus a_1 z \boxplus \ldots \boxplus a_n z^n.$$

In particular, F is algebraically closed.

Proof. The inclusion is straightforward to verify by induction on n, using Lemma 2.3. The last assertion follows since the multiplicative group of F is divisible. □

3. Model theory of hyperfields

We consider hyperfields as structures over the language $\mathcal{L} = \{\boxplus, (\_)^{-}, \cdot, 0, 1\}$, where $\boxplus$ is a ternary relation symbol, which is interpreted as $z \in x \boxplus y$, $(\_)^{-}$ is a unary function symbol, $\cdot$ is a binary function symbol and $0, 1$ are constant symbols. This language is a natural choice and is the one employed in [12] in relation to the model theory of valued fields.

The following proposition is quickly verified by writing down the (countably many!) existential sentences needed.

Proposition 3.1. *Over the language $\mathcal{L}$ the theory of algebraically closed hyperfields and the theory of algebraically closed hyperfields of fixed characteristic are first order theories.*

Over the language $\mathcal{L}$, a *substructure* of a hyperfield $(F, \boxplus, \cdot, 0, 1)$ is formed by a tuple $(F', \boxplus', \cdot, 0, 1)$, where $F' \subseteq F$ is multiplicative closed, contains 0 and 1, and

$$x \boxplus' y := (x \boxplus y) \cap F' \quad (x, y \in F').$$

Definition 3.1. *Two hyperfields F_1, F_2 are* elementarily equivalent *(we write $F_1 \equiv F_2$) if F_1 and F_2 satisfy the same first order sentences over the language $\mathcal{L}$.*

If F is a common substructure of F_1 and F_2, then we say that F_1 and F_2 are elementarily equivalent over F *(and write $F_1 \equiv_F F_2$) if F_1 and F_2 satisfy the same first order sentences, with parameters from F, over the language $\mathcal{L}$.*

Definition 3.2. *A first order theory is* substructure complete *if every two models with a common substructure are elementarily equivalent over that substructure.*

It is well-known that a first order theory is *complete* if and only if any two of its models are elementarily equivalent. For more details we refer the reader to [18]. Perhaps the following fact is less known.

Fact 3.1 ([12, Theorem 1.3.1])**.** *A first order theory admits quantifier elimination if and only if it is substructure complete.*

4. Main results

Theorem 4.1. *The theory of algebraically closed hyperfields with characteristic 0 is not complete.*

Proof. In fact, both the phase hyperfield $\mathbb{P}$ and its closure $\dot{\mathbb{P}}$ are models of the theory of algebraically closed hyperfields by Theorem 2.1. We have already verified that they both have characteristic 0. On the other hand, the first order sentence $\forall x \forall y (x \in x \boxplus y)$ does not hold in $\mathbb{P}$ while it does hold in $\dot{\mathbb{P}}$. $\square$

Theorem 4.2. *The theory of algebraically closed hyperfields is not substructure complete.*

Proof. It is a quick verification that the sign hyperfield $\mathbb{S}$ is a common substructure of both $\mathbb{P}$ and $\dot{\mathbb{P}}$. The result follows by considering the first order sentence $\forall x \forall y (x \in x \boxplus y)$ as in the proof of Theorem 4.1. $\square$

5. Algebraic formulæ

An *algebraically closed structure* in model theory is an abstraction of the usual notion of an algebraically closed field, defined purely in terms of definability and finiteness. For more details we refer to e.g. [14, Chapter 5].

Definition 5.1. *Let $\mathcal{L}$ be any first-order language and $\mathfrak{A}$ an $\mathcal{L}$-structure.*

A formula $\varphi(x, a)$ with parameters $a \in \mathfrak{A}$ is called algebraic over $\mathfrak{A}$ *if, in every $\mathcal{L}$-structure $\mathfrak{B} \supseteq \mathfrak{A}$, the solution set*

$$\varphi(\mathfrak{B}, a) = \{b \in \mathfrak{B} : \mathfrak{B} \models \varphi(b, a)\}$$

is finite.

Example 5.1. The formula $0 \in x \boxplus 1$ is algebraic over any hyperfield F. Indeed, the polynomial $x \boxplus 1$ has only one root in any hyperfield H which extends F, that is, the unique additive inverse of 1.

Example 5.2. The formula $0 \in x^2 \boxplus x \boxplus 1$ is not algebraic over $\mathbb{P}$ (and neither over $\mathbb{S}$!). Indeed, it is a simple excercise to check that the polynomial $x^2 \boxplus x \boxplus 1$ has infinitely many roots in $\mathbb{P}$. These roots are indeed

$$\left\{ e^{i\varphi} \mid \frac{\pi}{2} < \varphi < \pi \right\}.$$

Definition 5.2. *An $\mathcal{L}$-structure $\mathfrak{A}$ is* algebraically closed *if whenever $\varphi(x, a)$ is algebraic over $\mathfrak{A}$ and there is some extension $\mathfrak{B} \supseteq \mathfrak{A}$ with*

$$\mathfrak{B} \models \exists x\, \varphi(x, a),$$

then already

$$\mathfrak{A} \models \exists x\, \varphi(x, a).$$

Example 5.2 shows that the notion of algebraically closed hyperfields that we have considered so far does not coincide with the notion of algebraically closed structure over the language of hyperfields we have selected.

6. Further research

It is known that hyperfields are allowed to have any $n \in \mathbb{N}$ as characteristic: not only prime numbers, as it happens in the case of fields (cf. [7]).

On the other hand, the structures associated to non-archimedean local fields originally considered by Krasner and which motivated the introduction of hyperfields in [8], all have finite prime characteristic.

It remains an open problem whether the theory of algebraically closed hyperfields with fixed characteristic $n > 0$ is complete or admits quantifier elimination.

The validity of model-completeness is an entirely open question.

Furthermore, the hyperfields associated to valued fields as in [12, 13] are themselves valued in a way which makes their structure considerably simpler than the general case. By an application of [13, Theorem 4.21] one quickly checks that the hyperfields considered in this paper are not of that form, namely, they are not Krasner valued hyperfields ([13, Definition 4.6]).

We shall consider the case of algebraically closed Krasner valued hyperfields in a subsequent research paper.

In future research, also the notion of algebraically closed structures over our language of hyperfields could unravel intersting facts on the peculiar behavior of hyperfields (in comparison with that of fields).

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