

Some properties of the zero-set intersection graph of $C(X)$ and its line graph

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Abstract. Let $C(X)$ be the ring of all continuous real valued functions defined on a completely regular Hausdorff topological space X . The zero-set intersection graph $\Gamma(C(X))$ of $C(X)$ is a simple graph with vertex set all non units of $C(X)$ and two vertices are adjacent if the intersection of the zero sets of the functions is non empty. In this paper, we study the zero-set intersection graph of $C(X)$ and its line graph. We show that if X has more than two points, then these graphs are connected with diameter and radius 2. We show that the girth of the graph is 3 and the graphs are both triangulated and hypertriangulated. We find the domination number of these graphs and finally we prove that $C(X)$ is a von Neumann regular ring if and only if $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$. Finally, we derive some properties of the line graph of $\Gamma(C(X))$.

Keywords: zero-set intersection graph, diameter, girth, cycles, dominating sets, line graph.

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1. Introduction

The characterization of various algebraic structures by means of graph theory is an interesting area of research for mathematicians in the recent years. It was started by Beck in [12] by defining the graph $\Gamma(R)$ of a commutative ring R with vertices as elements of R in which two different vertices a and b are adjacent if and only if $ab = 0$. The author studied finitely colorable rings by associating this graph structure and this study was further continued by Anderson and Naseer in [14]. Sharma and Bhatwadekar first defined the comaximal graph of a

commutative ring in [16] and further investigation was continued in [5], [6], [7], [8], [9], [13], [15], [17], [18], [20], [21], [22].

Throughout this paper, let X be a Tychonoff space and $C(X)$ the ring of all real valued continuous functions on X . For $f \in C(X)$, the set $Z(f) = \{x \in X \mid f(x) = 0\}$ is called the zero set of f . Let $\text{Coz}(f) = X \setminus Z(f)$. The family of all zero sets is denoted by $Z(C(X))$. A space X is said to be a P -space if every zero set of $f \in C(X)$ is an open set. If I is an ideal in $C(X)$, let $Z[I] = \{Z(f) \mid f \in I\}$. An ideal I in $C(X)$ is said to be a z -ideal if $g \in C(X)$ and $Z(g) \in Z[I]$ imply that $g \in I$. A space X is said to be an almost P -space if every non empty zero set in X has a non empty interior. For $x \in X$, let $O^x = \{f \in C(X) \mid x \in \text{int}_{\beta X} Z(f)\}$. O^x is a prime z -ideal in $C(X)$.

In recent years, many mathematicians have associated graphs to the ring of functions on topological spaces and derived graph theoretic characterizations of both the ring and the associated topological space. Badie [4] in the year 2016 studied the comaximal graph of $C(X)$. In the year 2022, Badie [26] associated the annihilating graph to $C(X)$ and derived various characterizations of $C(X)$ and X in terms of this graph. Acharyya et al. [24] studied the zero divisor graph of the rings $C_{\mathcal{P}}(X)$ and $C_{\infty}^{\mathcal{P}}(X)$ in the year 2022. Here, $C_{\mathcal{P}}(X)$ is the family of all functions whose support belong to \mathcal{P} and $C_{\infty}^{\mathcal{P}}(X) = \{f \in C(X) \mid \text{for each } \epsilon > 0, \{x \in X \mid |f(x)| \geq \epsilon\} \in \mathcal{P}\}$. Hejazipour et al. [25] studied the zero divisor graph of the rings of real measurable functions with measures in the year 2022. In the year 2024, Bharati et al. [27] studied the zero divisor graph and comaximal graph of the ring of continuous functions with countable range.

Let R be a commutative ring with unity. The ring R is called an almost regular ring if each non-unit element of R is a zero-divisor element of R . Also, we know that $C(X)$ is an almost regular ring if X is an almost P -space. An ideal I in $C(X)$ is said to be a pure ideal if for each $f \in I$ there exists $g \in I$ such that $f = fg$ [23].

A graph is a pair $G = (V, E)$, where V is a set whose elements are called vertices, and E is a set of pair of vertices, whose elements are called edges. Two elements u and v in V are said to be adjacent if $\{u, v\} \in E$. A graph G is said to be complete if every pair of vertices can be joined by an edge and G is said to be connected if for any pair of vertices $u, v \in V$, there exists a path joining u and v . The distance between two vertices u and v denoted by $d(u, v)$ is the length of the shortest path between them. The diameter is defined as $\text{diam}(G) = \sup \{d(u, v) : u, v \in V(G)\}$, where $d(u, v)$ is the length of a shortest path from u to v . The eccentricity of a vertex $u \in G$ is denoted by $e(u)$ and is defined as $\max\{d(u, v) : v \in G\}$. The $\min\{e(u) : u \in G\}$ is called the radius of G and it is denoted by $\text{Rad}(G)$. The length of the shortest cycle in a graph G is called girth of the graph and is denoted by $\text{gr}(G)$. In a graph G , a dominating set is a set of vertices A such that every vertex outside A is adjacent to at least one vertex in A . The minimum cardinality of a dominating set in a graph G is called the domination number and it is denoted by $dt(G)$. A graph is said to be triangulated(hypertriangulated) if every vertex of the graph is a vertex(edge) of

a triangle. If two distinct vertices u and v in a graph G are adjacent and there is no vertex $w \in G$ such that w is adjacent to both u and v , then we say that u and v are orthogonal and is denoted by $u \perp v$. A graph G is called complemented if for each vertex $u \in G$, there is a vertex $v \in G$ such that $u \perp v$. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a chord of that cycle. A graph is said to be chordal if every cycle of length greater than three has a chord. In a graph G , the length of the smallest cycle containing two vertices x, y is denoted by $c(x, y)$ and if there is no cycle containing x, y then $c(x, y) = \infty$. The line graph of G , denoted by $L(G)$, is a graph whose vertices are the edges of G and two vertices of $L(G)$ are said to be adjacent wherever the corresponding edges in G share a common vertex [19]. For any undefined term in graph theory, we refer the reader to [10].

Bose et al. [11] in the year 2020 introduced a graph structure called the zero-set intersection graph $\Gamma(C(X))$ on the ring of real valued continuous functions $C(X)$ as a graph whose set of vertices consists of all non-units in the ring $C(X)$ and there is an edge between distinct vertices f and g in $C(X)$ if $Z(f) \cap Z(g) \neq \emptyset$. The authors showed that the graph is connected, studied the cliques and maximal cliques of $\Gamma(C(X))$ and the inter-relationship of cliques of $\Gamma(C(X))$ and ideals in $C(X)$. Further they showed that two graphs are isomorphic if and only if the corresponding rings are isomorphic if and only if the corresponding topological spaces are homeomorphic either for first countable topological spaces or for realcompact topological spaces.

The comaximal graph $\Gamma(R)$ is defined as a graph with set of vertices as R and two vertices a, b are adjacent if and only if $Ra + Rb = R$. Let $\Gamma_2(R)$ denote the subgraph of $\Gamma(R)$ whose vertex set consists of all non-unit elements of R . If $J(R)$ is the Jacobson radical of R , then the graph with vertex set $V(\Gamma_2(R)) \setminus J(R)$ is denoted by $\Gamma'_2(R)$. The comaximal ideal graph of $C(X)$, $\Gamma'_2(C(X))$, was studied in [2] and [4] where they derived the ring properties of $C(X)$ and topological properties of X via $\Gamma'_2(C(X))$. For any two vertices f and g in $\Gamma'_2(C(X))$, f is adjacent to g if and only if $Z(f) \cap Z(g) = \emptyset$ from lemma 1.2 [4]. The complement graph $\overline{\Gamma'_2(C(X))}$ coincides with the zero-set intersection graph of $C(X)$. In this paper, we study the zero-set intersection graph of $C(X)$. Vertex set of $\Gamma(C(X))$ is denoted by $V(\Gamma(C(X)))$ which consists of all the non-units in $C(X)$ and two vertices f and g are adjacent if $Z(f) \cap Z(g) \neq \emptyset$. If X is singleton, then $\Gamma(C(X))$ is empty. Thus we assume that $|X| > 1$.

In Section 2 of this paper, we show that $\Gamma(C(X))$ is connected and find the diameter and radius of the graph. In Section 3, we find the girth of the graph and show that the graph is always triangulated and hypertriangulated. We also find the conditions when the graph is chordal and show that $\Gamma(C(X))$ is never complemented. In Section 4, we find the dominating number of $\Gamma(C(X))$ and show that $C(X)$ is a von Neumann regular ring if and only if $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$. Finally in Section 5, we study the line graph $L(\Gamma(C(X)))$ of $\Gamma(C(X))$ and derive similar properties as

in $\Gamma(C(X))$. For all notations and undefined terms concerning the ring $C(X)$, the reader may consult [1].

2. Diameter, radius of $\Gamma(C(X))$

We first note that $f \in V(\Gamma(C(X)))$ if and only if $Z(f) \neq \emptyset$. We study the condition when $\Gamma(C(X))$ is connected. We also calculate the diameter and radius of $\Gamma(C(X))$.

Theorem 2.1. *Let X be any topological space then $\Gamma(C(X))$ is connected with $\text{diam}(\Gamma(C(X))) = 2$ if and only if $|X| > 1$.*

Proof. If $|X| = 1$, then $C(X) \cong \mathbb{R}$ and so $\Gamma(C(X))$ is the null graph. Suppose $X = \{a, b\}$, then f is a vertex in $\Gamma(C(X))$ if $f(a) = 0$ and $f(b) \neq 0$ or $f(a) \neq 0$ and $f(b) = 0$ or $f = 0$. Thus $\Gamma(C(X))$ is the disjoint union of two complete subgraphs $A = \{f \in C(X) : f(a) = 0 \text{ and } f(b) \neq 0\}$, $B = \{f \in C(X) : f(a) \neq 0 \text{ and } f(b) = 0\}$ and $f = 0$ is adjacent to each vertex in A and also to each vertex in B . Thus $\text{diam}(\Gamma(C(X))) = 2$. We now assume that $|X| > 2$. Let $f, g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$. Let $a \in Z(f)$, $b \in Z(g)$ and $c \notin \{a, b\}$. By regularity of X , there exists an open set U such that $c \in U \subseteq \overline{U} \subseteq X \setminus \{a, b\}$. Since X is completely regular there exists $h_1, h_2 \in C(X)$ such that $h_1(\overline{U}) = 1, h_1(a) = 0, h_2(\overline{U}) = 1$ and $h_2(b) = 0$. Let $h = h_1 h_2$, then $h \in V(\Gamma(C(X))) \setminus \{f, g\}$ and $f - h - g$ is a path of length 2 joining f and g . Thus $\Gamma(C(X))$ is connected with diameter 2. \square

Corollary 2.1. *If $|X| > 2$, then for any two distinct vertices $f, g \in V(\Gamma(C(X)))$ there exists a vertex $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f and g .*

Proof. If $Z(f) \cap Z(g) = \emptyset$, then by Theorem 2.1 there exists $h \in V(\Gamma(C(X)))$ such that $f - h - g$ is a path in $\Gamma(C(X))$. If $Z(f) \cap Z(g) \neq \emptyset$ then let $h = 2f$, if $g \neq 2f$ and $h = 3f$ otherwise. Then $h \in V(\Gamma(C(X))) \setminus \{f, g\}$ and $f - h - g$ is a path in $\Gamma(C(X))$. \square

Corollary 2.2. *Let $f, g \in \Gamma(C(X))$. Then:*

- (1) $d(f, g) = 1$ if and only if $Z(f) \cap Z(g) \neq \emptyset$.
- (2) $d(f, g) = 2$ if and only if $Z(f) \cap Z(g) = \emptyset$.

Theorem 2.2. *For any space X with $|X| > 2$, $\text{Rad}(\Gamma(C(X))) = 2$.*

Proof. For any vertex $f \in \Gamma(C(X))$, it is evident that $1 \leq e(f) \leq 2$, so $1 \leq \text{Rad}(\Gamma(C(X))) \leq 2$. For any vertex $f \in \Gamma(C(X))$, it is clear that $1 \leq e(f) \leq 2$, and hence $1 \leq \text{Rad}(\Gamma(C(X))) \leq 2$. Let $f \in V(\Gamma(C(X)))$ and choose a point $x \in X \setminus Z(f)$, which is possible because $Z(f) \neq X$. Since X is completely regular, there exists an open set $x \in U \subseteq \text{Cl}_X(U) \subseteq X \setminus Z(f)$. Again, by complete regularity, there exists a function $g \in C(X)$ such that $g(x) = 0, g(Z(f)) = 1$. Then $Z(g) \subseteq U$ and $U \subseteq X \setminus Z(f)$, so $Z(f) \cap Z(g) = \emptyset$. By Corollary 2.1 we have $d(f, g) = 2$. Hence, $e(f) = 2$, for every $f \in V(\Gamma(C(X)))$, and therefore $\text{Rad}(\Gamma(C(X))) = 2$. \square

3. Cycles in $\Gamma(C(X))$

In this section we explore the existence of cycles in $\Gamma(C(X))$.

Theorem 3.1. *For any topological space X with $|X| > 1$, $\Gamma(C(X))$ is both triangulated and hypertriangulated.*

Proof. Let $f \in V(\Gamma(C(X)))$. Then $f - 2f - 3f - f$ is a cycle of length 3 in $\Gamma(C(X))$. Hence, $\Gamma(C(X))$ is triangulated. Suppose f and g are adjacent vertices in $\Gamma(C(X))$. Let $h = 2f$, if $g \neq 2f$, otherwise let $h = 3f$. Then $h \in V(\Gamma(C(X))) \setminus \{f, g\}$ and $f - g - h - f$ is a triangle in $\Gamma(C(X))$. Hence, $f - g$ is an edge in a triangle. Thus, $\Gamma(C(X))$ is hypertriangulated. \square

Corollary 3.1. *If $|X| > 1$, then $gr(\Gamma(C(X))) = 3$.*

Theorem 3.2. *Let f and g be two distinct vertices in $\Gamma(C(X))$. Then:*

- (1) $c(f, g) = 3$ if and only if $Z(f) \cap Z(g) \neq \emptyset$.
- (2) $c(f, g) = 4$ if and only if $Z(f) \cap Z(g) = \emptyset$.

Proof. (1) Let $c(f, g) = 3$. Then f and g are adjacent in $\Gamma(C(X))$. Thus $Z(f) \cap Z(g) \neq \emptyset$.

Conversely, if $Z(f) \cap Z(g) \neq \emptyset$ then f and g are adjacent in $\Gamma(C(X))$ and by Corollary 2.1 there exists a vertex $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f and g . Thus $f - g - h - f$ is a cycle containing f and g . Hence, $c(f, g) = 3$.

(2) Let $c(f, g) = 4$. Then $Z(f) \cap Z(g) = \emptyset$ by (1).

Conversely, suppose $Z(f) \cap Z(g) = \emptyset$. Then by (1) there is no cycle of length 3 containing f and g . By Corollary 2.1, there exists $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f and g . So, $f - h - g - 2h - f$ is a cycle of length 4 and it is the smallest cycle containing f and g . Hence, $c(f, g) = 4$. \square

Theorem 3.3. *$\Gamma(C(X))$ is chordal if and only if $|X| \leq 3$.*

Proof. Suppose $\Gamma(C(X))$ is chordal and let $|X| \geq 4$. Let $x_1, x_2, x_3, x_4 \in X$. Since X is a Tychonoff space, there exist mutually disjoint open sets U_1, U_2, U_3 and U_4 such that $x_i \in U_i$, where $i \in \{1, 2, 3, 4\}$. Let $h_i \in V(\Gamma(C(X)))$ such that $h_i(x_i) = 0$, and $h_i(X \setminus U_i) = 1$ for each $i \in \{1, 2, 3, 4\}$. Then $Z(h_i) \cap Z(h_j) = \emptyset$, whenever $i \neq j$. Consider the functions $f_1 = h_1h_4$, $f_2 = h_1h_2$, $f_3 = h_2h_3$ and $f_4 = h_3h_4$. Clearly, $f_i \in V(\Gamma(C(X)))$ ($i = 1, 2, 3, 4$) and $f_1 - f_2 - f_3 - f_4 - f_1$ is a chordless cycle since $Z(h_1) \cap Z(h_3) = \emptyset$ and $Z(h_2) \cap Z(h_4) = \emptyset$ which is a contradiction. Hence, if $\Gamma(C(X))$ is chordal then $|X| \leq 3$.

Conversely, if $|X| \leq 3$ then the following two cases arise.

Case I: If $X = \{a, b\}$, then f is a vertex in $\Gamma(C(X))$ if $f(a) = 0$ and $f(b) \neq 0$, $f(a) \neq 0$ and $f(b) = 0$ or $f(a) = 0$ and $f(b) = 0$. Thus, $\Gamma(C(X))$ has two complete subgraphs $A = \{f \in C(X) : f(a) = 0 \text{ and } f(b) \neq 0\}$ and $B = \{f \in C(X) : f(a) \neq 0 \text{ and } f(b) = 0\}$ and $f = 0$ is adjacent to each vertex of A as

well as each vertex of B . Hence, if C is an induced cycle of length greater than 3 in $\Gamma(C(X))$, then it is contained in the complete subgraph induced by A or B and so C has a chord. Therefore, $\Gamma(C(X))$ is chordal.

Case II: Let $X = \{a, b, c\}$ and C be an induced cycle in $\Gamma(C(X))$ of length greater than 3. Consider a path $f_1 - f_2 - f_3 - f_4$ in C such that f_1, f_2, f_3 and f_4 are distinct. If $Z(f_1) \cap Z(f_3) \neq \emptyset$ or $Z(f_2) \cap Z(f_4) \neq \emptyset$, then we have a chord joining f_1 and f_3 or a chord joining f_2 and f_4 . So, we assume that $Z(f_1) \cap Z(f_3) = \emptyset$ and $Z(f_2) \cap Z(f_4) = \emptyset$. Then, we must have $|Z(f_2)| = |Z(f_3)| = 2$. If f_1 and f_4 are adjacent then, there is a chord joining f_2 and f_4 . So, assume that f_1 and f_4 are not adjacent then, there is a vertex f_5 which is adjacent to f_4 . If $|Z(f_5)| = 1$, then f_5 is adjacent to f_3 and if $|Z(f_5)| = 2$, then either f_5 is adjacent to f_2 and f_3 or f_5 is adjacent to f_1 and f_2 . Thus in each case C has a chord and hence, $\Gamma(C(X))$ is chordal. \square

Example 3.1. When $X = \{a, b\}$, we classify the non-zero non-unit functions $f \in C(X)$ into two disjoint classes of functions $[f_1]$ and $[f_2]$ and let the zero function be denoted by f_3 . Let the class $[f_1]$ represent the functions with $Z(f_1) = \{a\}$, the class $[f_2]$ represent the functions having $Z(f_2) = \{b\}$. We note that the classes $[f_1]$ and $[f_2]$ are complete graphs and each function in these classes are adjacent to f_3 . The graph $\Gamma(C(X))$ may be depicted as shown in Figure 1.

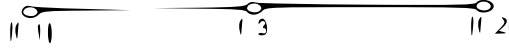


Figure 1: Zero-set intersection graph of $C(X)$ for $|X| = 2$.

Example 3.2. When $|X| = 3$, let $X = \{a, b, c\}$. The non unit functions $f \in C(X)$ can be classified into six classes $[f_i]$ with $1 \leq i \leq 6$ such that $Z(f_1) = \{a\}$, $Z(f_2) = \{b\}$, $Z(f_3) = \{c\}$, $Z(f_4) = \{a, b\}$, $Z(f_5) = \{b, c\}$, $Z(f_6) = \{a, c\}$ and the function $f_7 = 0$. Each function in a class is adjacent to f_7 and if the classes $[f_i]$, $[f_j]$ are adjacent then they form complete bipartite subgraph of the graph. The graph $\Gamma(C(X))$ is shown in Figure 2.

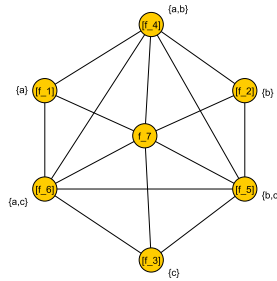


Figure 2: Zero-set intersection graph of $C(X)$ for $|X| = 3$.

Remark 3.1. We notice that the graphs in Figure 1 and Figure 2 are chordal graphs.

Theorem 3.4. For $|X| > 1$, $\Gamma(C(X))$ is not complemented.

Proof. Suppose $\Gamma(C(X))$ is complemented. Then, for every $f \in \overline{\Gamma'_2(C(X))}$ there exist g such that $f \perp g$, thus $c(f, g) = 4$. By Theorem 3.2, $Z(f) \cap Z(g) = \emptyset$, which is a contradiction. \square

4. Dominating sets in $\Gamma(C(X))$

Theorem 4.1. If $|X| > 1$, then $dt(\Gamma(C(X))) = 2$.

Proof. It is evident that $dt(\Gamma(C(X))) \neq 1$. Let $f \in V(\Gamma(C(X)))$ and $a \in \text{Int}_X Z(f)$, then $f \in O^a$. Since O^a is a pure ideal, there exists $g \in O^a \subseteq Z(C(X))$ such that $f = fg$, that is $g = 1$ on $\text{Supp}(f)$ [23]. Hence, $g^2 \in V(\Gamma(C(X)))$ and $\{g, g^2\}$ dominates $V(\Gamma(C(X)))$. Therefore, $dt(\Gamma(C(X))) = 2$. \square

Theorem 4.2. If $|X| > 1$, then for any $f, g \in V(\Gamma(C(X)))$, $\{f, g\}$ dominates $\Gamma(C(X))$ if and only if $Z(f) \cup Z(g) = X$.

Proof. Suppose $\{f, g\}$ dominates $\Gamma(C(X))$ and assume that $y \in X \setminus (Z(f) \cup Z(g))$. Let $h \in C(X)$ such that $h(y) = 0$ and $h(Z(f) \cup Z(g)) = 1$. Then, $h \in V(\Gamma(C(X)))$, $h \notin \{f, g\}$, $Z(f) \cap Z(h) = \emptyset$ and $Z(g) \cap Z(h) = \emptyset$. This is a contradiction since $\{f, g\}$ dominates $\Gamma(C(X))$.

Conversely, suppose $Z(f) \cup Z(g) = X$. Let $h \in C(X)$ such that $Z(f) \cap Z(h) = \emptyset$ and $Z(g) \cap Z(h) = \emptyset$. Then, $Z(h) \cap (Z(f) \cup Z(g)) = \emptyset$ which implies that $Z(h) = \emptyset$, i.e., $h \notin V(\Gamma(C(X)))$. Thus, $\{f, g\}$ dominates $\Gamma(C(X))$. \square

Theorem 4.3. $C(X)$ is a von Neumann regular ring if and only if $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$.

Proof. Suppose $C(X)$ is a von Neumann regular ring. Then, it is clear that $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$, $Z(f)$ is open. Consider the function $g \in C(X)$ such that

$$g(x) = \begin{cases} 1, & x \in Z(f) \\ 0, & x \notin Z(f). \end{cases}$$

Then, $g \in V(\Gamma(C(X)))$ and $Z(f) \cup Z(g) = Z(f) \cup (X \setminus Z(f)) = X$. Hence, by Theorem 4.2 $\{f, g\}$ dominates $\Gamma(C(X))$.

Conversely, suppose $C(X)$ is an almost regular ring and for all $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$. Since X is an almost P -space, every non-unit element in $C(X)$ has a zero-set with non-empty interior. That is every element in $C(X)$ is either a unit or a zero-divisor. If $f \in C(X)$ is a unit then, $f = f^2 f^{-1}$. Suppose now

$f \in V(\Gamma(C(X)))$. By hypothesis $Z(f) \cap Z(g) = \emptyset$, so $\text{Coz}(f) \cup \text{Coz}(g) = X$ which implies that $Z(f)$ is open since $Z(f) = \text{Coz}(g)$. Therefore, $C(X)$ is a von Neumann regular ring. \square

Example 4.1. Consider $X = \{a, b, c\}$ equipped with discrete topology. Then, every non-empty zero set $Z(f)$ is open. Hence, X is an almost P -space. Also, $C(X)$ is a von Neumann regular ring. Consider the functions

$$f_1(x) = \begin{cases} 0, & \text{when } x = a \\ 1, & \text{when } x = b \text{ or } x = c, \end{cases}, f_2(x) = \begin{cases} 1, & \text{when } x = a \text{ or } x = c \\ 0, & \text{when } x = b, \end{cases},$$

$$f_3(x) = \begin{cases} 1, & \text{when } x = a \text{ or } x = b \\ 0, & \text{when } x = c. \end{cases}$$

Let $g = f_2 f_3$. Then, $g \in V(\Gamma(C(X)))$ such that $Z(f_1) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$.

The following theorem is a direct consequence of Theorem 5.3 [4] and Theorem 4.2.

Theorem 4.4. *The graph $\Gamma(C(X))$ is complemented if and only if for every $f \in V(\Gamma(C(X)))$ there exists $g \in V(\Gamma(C(X)))$ such that $Z(f) \cap Z(g) = \emptyset$ and $\{f, g\}$ dominates $\Gamma(C(X))$.*

5. Line graph of $\Gamma(C(X))$

In this section, we study the properties of the line graph $L(\Gamma(C(X)))$ of $\Gamma(C(X))$. The study of the line graph of $\Gamma(C(X))$ is interesting as the results on some of the properties of $\Gamma(C(X))$ and $L(\Gamma(C(X)))$ differ from each other.

Suppose $f, g \in V(\Gamma(C(X)))$, then $[f, g]$ is a vertex in $L(\Gamma(C(X)))$ if $Z(f) \cap Z(g) \neq \emptyset$. In $L(\Gamma(C(X)))$, $[f, g] = [g, f]$ as $\Gamma(C(X))$ is an undirected graph and for two distinct vertices $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\Gamma(C(X)))$, $[f_1, f_2]$ is adjacent to $[g_1, g_2]$ if $f_i = g_j$, for some $i, j \in \{1, 2\}$.

We first investigate when is $L(\Gamma(C(X)))$ connected and then compute diameter and radius of $L(\Gamma(C(X)))$.

Lemma 5.1. *Let $[f_1, f_2]$ and $[g_1, g_2]$ be distinct vertices in $L(\Gamma(C(X)))$. Then, there is a vertex $[h_1, h_2]$ which is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\Gamma(C(X)))$ if and only if $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i, j \in \{1, 2\}$.*

Proof. Suppose there exists a vertex $[h_1, h_2]$ which is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in $L(\Gamma(C(X)))$. If $f_i = g_j$ for some $i, j \in \{1, 2\}$, then $Z(f_i) \cap Z(g_j) = Z(f_i) \neq \emptyset$. So, assume that $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. Consider a path $[f_1, f_2] - [h_1, h_2] - [g_1, g_2]$ in $L(\Gamma(C(X)))$, then $h_1 = f_i$ for some $i \in \{1, 2\}$ and $h_2 = g_j$ for some $j \in \{1, 2\}$. Thus $Z(f_i) \cap Z(g_j) = Z(h_1) \cap Z(h_2) \neq \emptyset$.

Conversely, suppose $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i, j \in \{1, 2\}$. Without loss of generality let $f_1 \neq g_1$, then $[f_1, g_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ in

$L(\Gamma(C(X)))$. If $f_1 = g_1$ then there exists $r \in \mathbb{R} \setminus \{0, 1\}$ such that $g_2 \neq rg_1$ and $f_2 \neq rf_1$. Thus $[f_1, rf_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$. \square

Theorem 5.1. *Let $|X| > 2$ and $[f_1, f_2], [g_1, g_2]$ be distinct vertices in $L(\Gamma(C(X)))$. Then:*

- (1) $d([f_1, f_2], [g_1, g_2]) = 1$ if and only if $f_i = g_j$ for some $i, j \in \{1, 2\}$.
- (2) $d([f_1, f_2], [g_1, g_2]) = 2$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i, j \in \{1, 2\}$.
- (3) $d([f_1, f_2], [g_1, g_2]) = 3$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$.

Proof. (1) Clearly holds.

(2) Suppose $d([f_1, f_2], [g_1, g_2]) = 2$. Then, $[f_1, f_2] - [f_i, g_j] - [g_1, g_2]$ is a path of length 2 for any $i, j \in \{1, 2\}$ implies that $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i, j \in \{1, 2\}$.

The converse follows clearly.

(3) From (1) and (2) we have $d([f_1, f_2], [g_1, g_2]) > 2$. By Corollary 2.1, there exists $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f_1 and g_1 in $\Gamma(C(X))$. Clearly, we can choose h such that $h \notin \{f_2, g_2\}$. Thus $[f_1, f_2] - [f_1, h] - [h, g_1] - [g_1, g_2]$ is a path in $L(\Gamma(C(X)))$ and hence $d([f_1, f_2], [g_1, g_2]) = 3$ \square

The following corollary is immediate from Theorem 2.1 and Theorem 5.1.

Corollary 5.1. *If $|X| > 2$, then $L(\Gamma(C(X)))$ is a connected graph with*

$$\text{diam}(L(\Gamma(C(X)))) \leq 3.$$

Theorem 5.2. *Let $|X| > 2$ and $[f_1, f_2]$ be a vertex in $L(\Gamma(C(X)))$. Then*

$$e([f_1, f_2]) = \begin{cases} 2, & \text{if } Z(f_1) \cup Z(f_2) = X \\ 3, & \text{otherwise.} \end{cases}$$

Proof. It is clear that $1 \leq e([f_1, f_2]) \leq 3$. Let $[g_1, g_2]$ be a vertex in $L(\Gamma(C(X)))$ such that $[g_1, g_2]$ is not adjacent to $[f_1, f_2]$. Then, for all $i, j \in \{1, 2\}$, $g_i \neq f_j$. But $\emptyset \neq Z(g_1) = Z(g_1) \cap X = Z(g_1) \cap (Z(f_1) \cup Z(f_2))$. Suppose, $Z(f_1) \cap Z(g_1) \neq \emptyset$ then, $[f_1, f_2] - [f_1, g_1] - [g_1, g_2]$ is a path in $L(\Gamma(C(X)))$ and so $d([f_1, f_2], [g_1, g_2]) = 2$. Hence, $e([f_1, f_2]) = 2$. Now, suppose $y \in X \setminus Z(f_1) \cup Z(f_2)$ and V be an open set in X such that $y \in V \subseteq Cl_X V \subseteq X \setminus Z(f_1) \cup Z(f_2)$. Consider $g_1, g_2 \in C(X)$, $g_1 \neq g_2$ such that $g_i(y) = 0$ and $g_i(Z(f_1) \cup Z(f_2)) = 1$ for $i \in \{1, 2\}$. Then, $f_i \neq g_j$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$. So, by Theorem 2.1, $d([f_1, f_2], [g_1, g_2]) = 3$ and hence $e([f_1, f_2]) = 3$. \square

An immediate conclusion from Corollary 2.1 and Theorem 2.2 is the following corollary.

Corollary 5.2. *If $|X| > 2$, then $2 \leq \text{Rad}(L(\Gamma(C(X)))) \leq 3$.*

We now find the girth of $L(\Gamma(C(X)))$ and show that $L(\Gamma(C(X)))$ is always triangulated and hypertriangulated. We also show that $L(\Gamma(C(X)))$ is never chordal.

Theorem 5.3. *If $|X| > 1$, then $gr(L(\Gamma(C(X)))) = 3$.*

Proof. Let $[f, g]$ be a vertex in $L(\Gamma(C(X)))$, then $[f, g] - [g, 2f] - [2f, f] - [f, g]$ is a cycle of length 3. Hence, $gr(L(\Gamma(C(X)))) = 3$. \square

Theorem 5.4. *For any space X with $|X| > 1$, $L(\Gamma(C(X)))$ is both triangulated and hypertriangulated.*

Proof. By Theorem 5.3, it follows that $L(\Gamma(C(X)))$ is triangulated. Let $[f_1, f_2] - [f_1, g] - [f_1, 2f_1] - [f_1, f_2]$ be an edge in $L(\Gamma(C(X)))$. Then, $f_1 \neq 2f_1$ and $f_2 \neq 2f_1$ and $[f_1, f_2] - [f_1, g] - [f_1, 2f_1] - [f_1, f_2]$ is a cycle in $L(\Gamma(C(X)))$. Hence, $L(\Gamma(C(X)))$ is hypertriangulated. \square

Theorem 5.5. *If $[f_1, f_2]$ and $[g_1, g_2]$ are distinct vertices in $L(\Gamma(C(X)))$. Then:*

- (1) $c([f_1, f_2], [g_1, g_2]) = 3$ if and only if $f_i = g_j$ for some $i, j \in \{1, 2\}$.
- (2) $c([f_1, f_2], [g_1, g_2]) = 4$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) \neq \emptyset$ for some $i \in \{1, 2\}$ and for all $j \in \{1, 2\}$ or $(Z(f_1) \cap Z(g_i) \neq \emptyset$ and $Z(f_2) \cap Z(g_j) \neq \emptyset$, where $i, j \in \{1, 2\}$).
- (3) $c([f_1, f_2], [g_1, g_2]) = 5$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$ and only one $j \in \{1, 2\}$, $Z(f_i) \cap Z(g_j) \neq \emptyset$.
- (4) $c([f_1, f_2], [g_1, g_2]) = 6$ if and only if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$.

Proof. (1) If $c([f_1, f_2], [g_1, g_2]) = 3$, then it is clear that $f_i = g_j$ for some $i, j \in \{1, 2\}$.

Conversely, suppose $f_1 = g_1$. Then, there exists $r \in \mathbb{R} \setminus \{0\}$ such that $rg_1 \notin \{f_1, f_2, g_1\}$ and $[f_1, f_2] - [f_1, g_1] - [f_1, rg_1] - [f_1, f_2]$ is a cycle of length 3 in $L(\Gamma(C(X)))$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Hence, $c([f_1, f_2], [g_1, g_2]) = 3$.

(2) Suppose $c([f_1, f_2], [g_1, g_2]) = 4$ then by (1), $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. So, we have the cycle $[f_1, f_2] - [a, b] - [g_1, g_2] - [c', d] - [f_1, f_2]$, where $a, c' \in \{f_1, f_2\}$ and $b, d \in \{g_1, g_2\}$.

Case I. If $a = f_1 = c'$, $b = g_1$ and $d = g_2$ then $Z(f_1) \cap Z(g_1) \neq \emptyset$ and $Z(f_1) \cap Z(g_2) \neq \emptyset$.

Case II. If $a = f_1$, $c' = f_2$, $b = g_1$ and $d = g_2$, then $Z(f_1) \cap Z(g_1) \neq \emptyset$ and $Z(f_2) \cap Z(g_2) \neq \emptyset$.

Case III. If $a = f_1 = c'$ and $b = g_2 = d$, then $[a, b] = [c, d]$, which is a contradiction.

Conversely, if $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ then there is no triangle containing $[f_1, f_2]$ and $[g_1, g_2]$. Suppose $Z(f_2) \cap Z(g_j) \neq \emptyset$ for all $j \in \{1, 2\}$ then $[f_1, f_2] - [f_2, g_2] - [g_1, g_2] - [g_1, f_2] - [f_1, f_2]$ is a cycle of length 4 in $L(\Gamma(C(X)))$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Suppose $Z(f_1) \cap Z(g_2) \neq \emptyset$ and $Z(f_2) \cap Z(g_2) \neq \emptyset$

then we get a cycle $[f_1, f_2] - [f_1, g_2] - [g_1, g_2] - [g_2, f_2] - [f_1, f_2]$ of length 4 in $L(\Gamma(C(X)))$. Hence, $c([f_1, f_2], [g_1, g_2]) = 4$.

(3) Suppose $c([f_1, f_2], [g_1, g_2]) = 5$. Then, by (1), $f_i \neq g_j$ for all $i, j \in \{1, 2\}$. Suppose $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$ then, $[f_1, f_2] - [k_1, l_1] - [g_1, g_2] - [k_2, l_2] - [k_3, l_3] - [f_1, f_2]$ is a cycle of length 5 in $L(\Gamma(C(X)))$, where $k_1 \in \{f_1, f_2\}$ and $l_1 \in \{g_1, g_2\}$. But $Z(k_1) \cap Z(l_1) \neq \emptyset$ which is a contradiction to our assumption. Similarly, if we consider the cycle $[f_1, f_2] - [k_1, l_1] - [k_2, l_2] - [g_1, g_2] - [k_3, l_3] - [f_1, f_2]$, we get a contradiction. Hence, for only one $i \in \{1, 2\}$ and only one $j \in \{1, 2\}$, $Z(f_i) \cap Z(g_j) \neq \emptyset$.

Conversely, suppose $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$ and only one $j \in \{1, 2\}$, $Z(f_i) \cap Z(g_j) \neq \emptyset$. Let $Z(f_2) \cap Z(g_2) \neq \emptyset$. Then, by (1) and (2) it follows that there is no cycle of length 3 or 4 containing $[f_1, f_2]$ and $[g_1, g_2]$. Now there exists $r \in \mathbb{R} \setminus \{0\}$ such that $rg_2 \notin \{f_1, g_1, g_2\}$ and so $[f_1, f_2] - [f_2, g_2] - [g_1, g_2] - [g_2, rg_2] - [rg_2, f_2] - [f_1, f_2]$ is a cycle of length 5 in $L(\Gamma(C(X)))$ containing $[f_1, f_2]$ and $[g_1, g_2]$. Hence, $c([f_1, f_2], [g_1, g_2]) = 5$.

(4) Suppose $c([f_1, f_2], [g_1, g_2]) = 6$. Then, by (1), (2) and (3), $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$.

Conversely, suppose $f_i \neq g_j$ for all $i, j \in \{1, 2\}$ and $Z(f_i) \cap Z(g_j) = \emptyset$ for all $i, j \in \{1, 2\}$, then by (1), (2) and (3), $c([f_1, f_2], [g_1, g_2]) > 5$. By Corollary 2.1, there exists a vertex $h \in V(\Gamma(C(X)))$ such that h is adjacent to both f_1 and g_1 in $\Gamma(C(X))$. Consider, $r \in \mathbb{R} \setminus \{0\}$ then we get a cycle $[f_1, f_2] - [f_1, h] - [h, g_1] - [g_1, g_2] - [g_1, rh] - [rh, f_1] - [f_1, f_2]$ of length 6 containing $[f_1, f_2]$ and $[g_1, g_2]$. Therefore, $c([f_1, f_2], [g_1, g_2]) = 6$. \square

Theorem 5.6. *The graph $L(\Gamma(C(X)))$ is never chordal.*

Proof. Let $f \in V(\Gamma(C(X)))$. Then, $[f, 3f] - [3f, 5f] - [5f, 7f] - [7f, f] - [f, 3f]$ is a chordless cycle of length 4 in $L(\Gamma(C(X)))$. Hence, $L(\Gamma(C(X)))$ is never chordal. \square

Example 5.1. Consider the graph in Figure 2. In the line graph of this graph, the cycle $[f, f_1] - [f, f_2] - [f_2, f_4] - [f_4, f_1] - [f_1, f]$ is not chordal.

6. Conclusion

Bose et al. [11] studied cliques and maximal cliques of $\Gamma(C(X))$ and graph isomorphism of the graph $\Gamma(C(X))$. In this paper, we determined diameter, girth, radius, domination number of $\Gamma(C(X))$. We connected the von Neumann regularity property of the ring $C(X)$ with the domination property of $\Gamma(C(X))$. We have studied the line graph $L(\Gamma(C(X)))$ of $\Gamma(C(X))$. The independence number of the graph $\Gamma(C(X))$, the vertex cover, matching and many other properties of the graph are yet to be determined. The problem of finding the genus of the graph $\Gamma(C(X))$ is still an open problem.

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