On intuitionistic Q-fuzzy subalgebras/ideals/deductive systems in Hilbert algebras

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Abstract. This study introduces the concepts of intuitionistic Q-fuzzy subalgebras, ideals, and deductive systems in the setting of Hilbert algebras and investigates their fundamental properties and interrelations. The theoretical results are supported by concrete examples and are structured in a way that facilitates understanding of the algebraic-logical framework underlying fuzzy logic extensions. By presenting clear definitions, illustrative cases, and step-by-step reasoning, the paper serves not only as a contribution to the field of abstract algebra but also as a useful learning resource for upper secondary science students beginning to engage with research. The work encourages early exploration of mathematical structures, fosters critical and creative thinking,

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and promotes accessibility of advanced topics through collaboration across academic levels and institutions.

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1. Introduction and preliminaries

The concept of fuzzy sets was proposed by Zadeh [23]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches, such as soft sets and rough sets, has been discussed in [1, 3, 6]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multi-criteria decision-making [12, 13, 14]. The concept of Hilbert algebra was introduced in the early 50-ties by Henkin [15] for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [9] from an algebraic point of view. Diego proved that Hilbert algebras form a locally finite variety. Hilbert algebras were treated by Busneag [4, 5] and Jun [16], and some of their filters forming deductive systems were recognized. Dudek [10] considered the fuzzification of subalgebras and deductive systems in Hilbert algebras. Zhan and Tan [24] studied intuitionistic fuzzifications of the concept of deductive systems in Hilbert algebras. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy deductive systems in Hilbert algebras.

The study of Q-fuzzy sets has attracted considerable interest due to its ability to model graded uncertainty with respect to an index set Q. Researchers have explored this framework in various algebraic structures, including semigroups, rings, and algebras. Notably, Kim [17, 18] investigated intuitionistic Q-fuzzy ideals and semiprime ideals in semigroups, establishing foundational results that extend classical fuzzy ideal theory and contribute to the development of generalized fuzzy algebraic systems. This line of research was further developed in the context of UP-algebras by Tanamoon et al. [21], who explored the general structure of Q-fuzzy sets and their algebraic behavior. In a subsequent study, Sripaeng et al. [20] introduced and analyzed anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras, offering dual perspectives that enriched the understanding of fuzziness in non-classical algebraic systems. Recent studies have extended the concept of intuitionistic Q-fuzzy ideals to various algebraic structures. Derseh et al. [8] examined intuitionistic Q-fuzzy PMS-ideals in PMS-algebras, offering structural properties and characterizations. Wang [22] investigated similar notions within BE-algebras, while Massa'deh [19] contributed to the development of intuitionistic Q-fuzzy KU-ideals. These works demonstrate the broad applicability of the intuitionistic Q-fuzzy framework and support its further exploration in diverse algebraic settings. Collectively, these works contribute to advancing generalized fuzzy structures in algebra and support further applications in logic and uncertainty modeling.

Given a set Q, this paper introduces the notions of intuitionistic Q-fuzzy subalgebras, ideals, and deductive systems within Hilbert algebras, and explores their fundamental properties and interrelationships. The study also examines the behavior of these structures under homomorphisms, with results presented in a clear and illustrative manner to enhance accessibility for learners and support further research development.

Before we begin the study, let's review the definition of Hilbert algebras, which was defined by Diego [9] in 1966.

Definition 1.1. *A Hilbert algebra is a triplet $H = (H, \cdot, 1)$, where H is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of H such that the following axioms hold:*

1. $(\forall x, y \in H)(x \cdot (y \cdot x) = 1)$,
2. $(\forall x, y, z \in H)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
3. $(\forall x, y \in H)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

The following results were proved in [10].

Lemma 1.1. *Let $H = (H, \cdot, 1)$ be a Hilbert algebra. Then*

1. $(\forall x \in H)(x \cdot x = 1)$,
2. $(\forall x \in H)(1 \cdot x = x)$,
3. $(\forall x \in H)(x \cdot 1 = 1)$,
4. $(\forall x, y, z \in H)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $H = (H, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in H)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on H with 1 as the largest element.

Definition 1.2 ([7])**.** *A nonempty subset I of a Hilbert algebra $H = (H, \cdot, 1)$ is called an ideal of H if*

1. $1 \in I$,
2. $(\forall x \in H, \forall y \in I)(x \cdot y \in I)$,
3. $(\forall x \in H, \forall y_1, y_2 \in I)((y_1 \cdot (y_2 \cdot x)) \cdot x \in I)$.

Let X and Q be two nonempty sets. An *Q-fuzzy set* in a nonempty set X is defined to be a function $\mu : X \times Q \to [0,1]$, where $[0,1]$ is the unit closed interval of real numbers.

Definition 1.3 ([11])**.** *An Q-fuzzy set μ in a Hilbert algebra $H = (H, \cdot, 1)$ is called a Q-fuzzy ideal of H if the following conditions hold:*

1. $(\forall x \in H, \forall q \in Q)(\mu(1, q) \geq \mu(x, q))$,
2. $(\forall x, y \in H, \forall q \in Q)(\mu(x \cdot y, q) \geq \mu(y, q))$,
3. $(\forall x, y_1, y_2 \in H, \forall q \in Q)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu(y_1, q), \mu(y_2, q)\})$.

Definition 1.4. *Let X and Q be two nonempty sets. An intuitionistic Q-fuzzy set in X is defined to be a structure*

$$A := \{(x, \mu_A(x, q), \gamma_A(x, q)) \mid x \in X, q \in Q\}, \tag{1.1}$$

where the functions $\mu_A : X \times Q \to [0,1]$ is the degree of membership of x and $\gamma_A : X \times Q \to [0,1]$ is the degree of non-membership of x such that

$$(\forall x \in X, \forall q \in Q)(0 \leq \mu_A(x, q) + \gamma_A(x, q) \leq 1),$$

and the intuitionistic Q-fuzzy set in (1.1) *is simply denoted by $A = (\mu_A, \gamma_A)$.*

2. Intuitionistic Q-fuzzy Hilbert algebras

In this section, we introduce the notions of intuitionistic Q-fuzzy subalgebras, intuitionistic Q-fuzzy ideals, and intuitionistic Q-fuzzy deductive systems of Hilbert algebras and investigate some related properties.

Definition 2.1. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra $H = (H, \cdot, 1)$ is called an intuitionistic Q-fuzzy subalgebra of H if*

$$(\forall x, y \in H, \forall q \in Q)\begin{pmatrix} \mu_A(x \cdot y, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\} \\ \gamma_A(x \cdot y, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\} \end{pmatrix}. \tag{2.1}$$

Example 2.1. Let $H = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then H is a Hilbert algebra. We define an intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in H as follows:

$$\mu_A(1, q) = 1, \mu_A(x, q) = 0.8, \mu_A(y, q) = 0.8, \mu_A(z, q) = 0.7, \mu_A(0, q) = 0.4,$$
$$\gamma_A(1, q) = 0.3, \gamma_A(x, q) = 0.5, \gamma_A(y, q) = 0.7, \gamma_A(z, q) = 0.3, \gamma_A(0, q) = 0.6,$$

for all $q \in Q$. Then A is an intuitionistic Q-fuzzy subalgebra of H.

Proposition 2.1. *Every intuitionistic Q-fuzzy subalgebra $A = (\mu_A, \gamma_A)$ of a Hilbert algebra H satisfies $\mu_A(1, q) \geq \mu_A(x, q)$ and $\gamma_A(1, q) \leq \gamma_A(x, q)$ for all $x \in H$ and $q \in Q$.*

Proof. For any $x \in H$ and $q \in Q$, we have

$$\mu_A(1, q) = \mu_A(x \cdot x, q) \geq \min\{\mu_A(x, q), \mu_A(x, q)\} = \mu_A(x, q)$$

and

$$\gamma_A(1, q) = \gamma_A(x \cdot x, q) \leq \max\{\gamma_A(x, q), \gamma_A(x, q)\} = \gamma_A(x, q). \qquad \square$$

Definition 2.2. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is said to be an intuitionistic Q-fuzzy ideal of H if the following conditions hold:*

$$(\forall x \in H, \forall q \in Q)\left(\begin{array}{c} \mu_A(1, q) \geq \mu_A(x, q) \\ \gamma_A(1, q) \leq \gamma_A(x, q) \end{array}\right), \tag{2.2}$$

$$(\forall x, y \in H, \forall q \in Q)\left(\begin{array}{c} \mu_A(x \cdot y, q) \geq \mu_A(y, q) \\ \gamma_A(x \cdot y, q) \leq \gamma_A(y, q) \end{array}\right), \tag{2.3}$$

$$\begin{array}{l}(\forall x, y_1, y_2 \in H, \forall q \in Q) \\ \left(\begin{array}{c} \mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_A(y_2, q)\} \\ \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\} \end{array}\right).\end{array} \tag{2.4}$$

Example 2.2. Let $H = \{1, x, y, z\}$ with the following Cayley table:

$\cdot$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	x	1	z
z	1	1	y	1

Then H is a Hilbert algebra. We define an intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ as follows:

$$\mu_A(1, q) = 0.9, \mu_A(x, q) = 0.3, \mu_A(y, q) = 0.1, \mu_A(z, q) = 0.6,$$
$$\gamma_A(1, q) = 0.1, \gamma_A(x, q) = 0.2, \gamma_A(y, q) = 0.8, \gamma_A(z, q) = 0.3,$$

for all $q \in Q$. Then A is an intuitionistic Q-fuzzy ideal of H.

Proposition 2.2. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then*

$$(\forall x, y \in H, \forall q \in Q)\left(\begin{array}{c} \mu_A((y \cdot x) \cdot x, q) \geq \mu_A(y, q) \\ \gamma_A((y \cdot x) \cdot x, q) \leq \gamma_A(y, q) \end{array}\right). \tag{2.5}$$

Proof. Let $x, y \in H$ and $q \in Q$. Putting $y_1 = y$ and $y_2 = 1$ in (2.4), we have

$$\mu_A((y \cdot x) \cdot x, q) \geq \min\{\mu_A(y, q), \mu_A(1, q)\} = \mu_A(y, q)$$

and

$$\gamma_A((y \cdot x) \cdot x, q) \leq \max\{\gamma_A(y, q), \gamma_A(1, q)\} = \gamma_A(y, q).$$

□

Lemma 2.1. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then we have the following*

$$(\forall x, y \in H, \forall q \in Q) \left(x \leq y \ \Rightarrow \begin{cases} \mu_A(x, q) \leq \mu_A(y, q) \\ \gamma_A(x, q) \geq \gamma_A(y, q) \end{cases} \right). \tag{2.6}$$

Proof. Let $x, y \in H$ be such that $x \leq y$ and $q \in Q$. Then $x \cdot y = 1$ and so

$$\begin{aligned} \mu_A(y, q) &= \mu_A(1 \cdot y, q) \\ &= \mu_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \\ &\geq \min\{\mu_A(x \cdot y, q), \mu_A(x, q)\} \\ &= \min\{\mu_A(1, q), \mu_A(x, q)\} \\ &= \mu_A(x, q) \end{aligned}$$

and

$$\begin{aligned} \gamma_A(y, q) &= \gamma_A(1 \cdot y, q) \\ &= \gamma_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \\ &\leq \max\{\gamma_A(x \cdot y, q), \gamma_A(x, q)\} \\ &= \max\{\gamma_A(1, q), \gamma_A(x, q)\} \\ &= \gamma_A(x, q). \end{aligned}$$

□

Theorem 2.1. *Every intuitionistic Q-fuzzy ideal of a Hilbert algebra H is an intuitionistic Q-fuzzy subalgebra of H.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q-fuzzy ideal of H. Let $x, y \in H$ and $q \in Q$. Since $y \leq x \cdot y$, it follows from Lemma 2.1 that

$$\mu_A(y, q) \leq \mu_A(x \cdot y, q) \text{ and } \gamma_A(y, q) \geq \gamma_A(x \cdot y, q).$$

It follows from (2.3) that

$$\mu_A(x \cdot y, q) \geq \mu_A(y, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\}$$

and

$$\gamma_A(x \cdot y, q) \leq \gamma_A(y, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\}.$$

Hence, A is an intuitionistic Q-fuzzy subalgebra of H. □

Proposition 2.3. *If $A_i = \{(\mu_{A_i}, \gamma_{A_i}) : i \in \Delta\}$ is a family of intuitionistic Q-fuzzy ideals of a Hilbert algebra H, then $\bigwedge_{i \in \Delta} A_i$ is an intuitionistic Q-fuzzy ideal of H.*

Proof. Let $A_i = \{(\mu_{A_i}, \gamma_{A_i}) : i \in \Delta\}$ be a family of intuitionistic Q-fuzzy ideals of H. Let $x \in H$ and $q \in Q$. Then

$$(\bigwedge_{i\in\Delta} \mu_{A_i})(1,q) = \inf_{i\in\Delta}\{\mu_{A_i}(1,q)\} \geq \inf_{i\in\Delta}\{\mu_{A_i}(x,q)\} = (\bigwedge_{i\in\Delta} \mu_{A_i})(x,q)$$

and

$$(\bigwedge_{i\in\Delta} \gamma_{A_i})(1,q) = \sup_{i\in\Delta}\{\gamma_{A_i}(1,q)\} \leq \sup_{i\in\Delta}\{\gamma_{A_i}(x,q)\} = (\bigwedge_{i\in\Delta} \gamma_{A_i})(x,q).$$

Let $x, y \in H$ and $q \in Q$. Then

$$(\bigwedge_{i\in\Delta} \mu_{A_i})(x \cdot y,q) = \inf_{i\in\Delta}\{\mu_{A_i}(x \cdot y,q)\} \geq \inf_{i\in\Delta}\{\mu_{A_i}(y,q)\} = (\bigwedge_{i\in\Delta} \mu_{A_i})(y,q)$$

and

$$(\bigwedge_{i\in\Delta} \gamma_{A_i})(x \cdot y,q) = \sup_{i\in\Delta}\{\gamma_{A_i}(x \cdot y,q)\} \leq \sup_{i\in\Delta}\{\gamma_{A_i}(y,q)\} = (\bigwedge_{i\in\Delta} \gamma_{A_i})(y,q).$$

Let $x, y_1, y_2 \in H$ and $q \in Q$. Then

$$\begin{aligned}(\bigwedge_{i\in\Delta} \mu_{A_i})((y_1 \cdot (y_2 \cdot x)) \cdot x,q) &= \inf_{i\in\Delta}\{\mu_{A_i}((y_1 \cdot (y_2 \cdot x)) \cdot x,q)\} \\ &\geq \inf_{i\in\Delta}\{\min\{\mu_{A_i}(y_1,q), \mu_{A_i}(y_2,q)\}\} \\ &= \min\{\inf_{i\in\Delta}\{\mu_{A_i}(y_1,q)\}, \inf_{i\in\Delta}\{\mu_{A_i}(y_2,q)\}\} \\ &= \min\{(\bigwedge_{i\in\Delta} \mu_{A_i})(y_1,q), (\bigwedge_{i\in\Delta} \mu_{A_i})(y_2,q)\}\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in\Delta} \gamma_{A_i})((y_1 \cdot (y_2 \cdot x)) \cdot x,q) &= \sup_{i\in\Delta}\{\gamma_{A_i}((y_1 \cdot (y_2 \cdot x)) \cdot x,q)\} \\ &\leq \sup_{i\in\Delta}\{\max\{\gamma_{A_i}(y_1,q), \gamma_{A_i}(y_2,q)\}\} \\ &= \max\{\sup_{i\in\Delta}\{\gamma_{A_i}(y_1,q)\}, \sup_{i\in\Delta}\{\gamma_{A_i}(y_2,q)\}\} \\ &= \max\{(\bigwedge_{i\in\Delta} \gamma_{A_i})(y_1,q), (\bigwedge_{i\in\Delta} \gamma_{A_i})(y_2,q)\}.\end{aligned}$$

Hence, $\bigwedge_{i\in\Delta} A_i$ is an intuitionistic Q-fuzzy ideal of H. □

Definition 2.3. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is said to be an intuitionistic Q-fuzzy deductive system of H if the following conditions hold:*

$$(\forall x \in H, \forall q \in Q)\begin{pmatrix} \mu_A(1,q) \geq \mu_A(x,q) \\ \gamma_A(1,q) \leq \gamma_A(x,q) \end{pmatrix}, \tag{2.7}$$

$$(\forall x, y \in H, \forall q \in Q)\begin{pmatrix} \mu_A(y,q) \geq \min\{\mu_A(x \cdot y,q), \mu_A(x,q)\} \\ \gamma_A(y,q) \leq \max\{\gamma_A(x \cdot y,q), \mu_A(x,q)\} \end{pmatrix}. \tag{2.8}$$

Proposition 2.4. *Every intuitionistic Q-fuzzy ideal of a Hilbert algebra H is an intuitionistic Q-fuzzy deductive system of H.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q-fuzzy ideal of H. Let $x, y \in H$ and $q \in Q$. If $y_1 = x \cdot y$ and $y_2 = x$, then by (1) and (2) of Lemma 1.1 and (2.4), we have

$$\mu_A(y, q) = \mu_A(1 \cdot y, q) = \mu_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \geq \min\{\mu_A(x \cdot y, q), \mu_A(x, q)\}$$

and

$$\gamma_A(y, q) = \gamma_A(1 \cdot y, q) = \gamma_A(((x \cdot y) \cdot (x \cdot y)) \cdot y, q) \leq \max\{\gamma_A(x \cdot y, q), \gamma_A(x, q)\}.$$

Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy deductive system of H. □

Lemma 2.2. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H if and only if μ_A and $\overline{\gamma}_A$ are Q-fuzzy ideals of H.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H. Then obviously μ_A is a Q-fuzzy ideal of H. Consider for every $x, y \in H$ and $q \in Q$, we have $\overline{\gamma}_A(1, q) = 1 - \gamma_A(1, q) \geq 1 - \gamma_A(x, q) = \overline{\gamma}_A(x, q)$. Let $x, y \in H$ and $q \in Q$. Then $\overline{\gamma_A}(y, q) = 1 - \gamma_A(y, q) \leq 1 - \gamma_A(x \cdot y, q) = \overline{\gamma_A}(x \cdot y, q)$.
Let $x, y_1, y_2 \in H$ and $q \in Q$. Then

$$\begin{aligned}
\overline{\gamma_A}((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= 1 - \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \\
&\geq 1 - \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\} \\
&= \min\{1 - \gamma_A(y_1, q), 1 - \gamma_A(y_2, q)\} \\
&= \min\{\overline{\gamma_A}(y_1, q), \overline{\gamma_A}(y_2, q)\}.
\end{aligned}$$

Hence, $\overline{\gamma_A}$ is a Q-fuzzy ideal of H.

Conversely, let us take μ_A and $\overline{\gamma_A}$ are Q-fuzzy ideals of H. Then obviously for every $x \in H$ and $q \in Q$, we have $\mu_A(1, q) \geq \mu_A(x, q)$ and $1 - \gamma_A(1, q) = \overline{\gamma_A}(1, q) \geq \overline{\gamma_A}(x, q) = 1 - \gamma_A(x, q)$, that is, $\gamma_A(1, q) \leq \gamma_A(x, q)$. Let $x, y \in H$ and $q \in Q$. Then obviously, $\mu_A(x \cdot y, q) \geq \mu_A(y, q)$ and $1 - \gamma_A(x \cdot y, q) = \overline{\gamma_A}(x \cdot y, q) \geq \overline{\gamma_A}(y, q) = 1 - \gamma_A(y, q)$, that is, $\gamma_A(x \cdot y, q) \leq \gamma_A(y, q)$. Let $x, y_1, y_2 \in H$ and $q \in Q$. Then obviously $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_A(y_2, q)\}$ and

$$\begin{aligned}
1 - \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= \overline{\gamma_A}((y_1 \cdot (y_2 \cdot x, q), q \cdot x, q) \\
&\geq \min\{\overline{\gamma_A}(y_1, q), \overline{\gamma_A}(y_2, q)\} \\
&= \min\{1 - \gamma_A(y_1, q), 1 - \gamma_A(y_2, q)\} \\
&= 1 - \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\},
\end{aligned}$$

that is, $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\}$. Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H. □

Theorem 2.2. *An intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is an intuitionistic Q-fuzzy ideal of H if and only if $(\mu_A, \overline{\mu}_A)$ and $(\gamma_A, \overline{\gamma}_A)$ are intuitionistic Q-fuzzy ideals of H.*

Proof. If an intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H, then $\mu_A = \overline{\overline{\mu}}_A$ and γ_A are Q-fuzzy ideals of H from Lemma 2.2, hence $(\mu_A, \overline{\mu}_A)$ and $(\overline{\gamma}_A, \gamma_A)$ are intuitionistic Q-fuzzy ideals of H.

Conversely, if $(\mu_A, \overline{\mu}_A)$ and $(\gamma_A, \overline{\gamma}_A)$ are intuitionistic Q-fuzzy ideals of H, then μ_A and $\overline{\gamma}_A$ are Q-fuzzy ideals of H. Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H. $\square$

Definition 2.4. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic Q-fuzzy set of a Hilbert algebra H and $\alpha \in [0, 1]$. Then we define the subsets $U(\mu_A, \alpha) = \{x \in X : \mu_A(x, q) \geq \alpha \ \forall q \in Q\}$ and $L(\gamma_A, \alpha) = \{x \in X : \gamma_A(x, q) \leq \alpha \ \forall q \in Q\}$ of H.*

Theorem 2.3. *Let A be a nonempty subset of a Hilbert algebra H and (μ_A, γ_A) be an intuitionistic Q-fuzzy set in H defined by $\mu_A(x, q) = \begin{cases} \alpha_0, & \text{if } x \in A \\ \alpha_1, & \text{otherwise} \end{cases}$ $\gamma_A(x, q) = \begin{cases} \beta_0, & \text{if } x \in A \\ \beta_1, & \text{otherwise} \end{cases}$ for all $x \in H$, $q \in Q$ and $\alpha_i, \beta_i \in [0, 1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$. Then (μ_A, γ_A) is an intuitionistic Q-fuzzy ideal of H if and only if A is an ideal of H.*

Proof. Assume that (μ_A, γ_A) is an intuitionistic Q-fuzzy ideal of H. Since $\mu_A(1, q) \geq \mu_A(x, q)$ and $\gamma_A(1, q) \leq \gamma_A(x, q)$ for all $x \in H$ and $q \in Q$, we have $\mu_A(1, q) = \alpha_1$ and $\gamma_A(1, q) = \beta_1$ and so $1 \in A$. Let $x \in H$ and $y \in A$. Then $\mu_A(x \cdot y, q) \geq \mu_A(y, q) = \alpha_1$ for all $q \in Q$ and then $\mu_A(x \cdot y, q) = \alpha_1$. Also $\gamma_A(x \cdot y, q) \leq \gamma_A(y, q) = \beta_1$ and then $\gamma_A(x \cdot y, q) = \beta_1$. Hence, $x \cdot y \in A$. For any $y_1, y_2 \in A$ and $x \in H$, we get $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_A(y_2, q)\} = \alpha_1$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\} = \beta_1$ for all $q \in Q$, which implies that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \alpha_1$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \beta_1$. It follows that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in A$. Therefore, A is an ideal of H.

Conversely, assume that A is an ideal of H. Since $1 \in A$, it follows that $\mu_A(1, q) = \alpha_1 \geq \mu_A(x, q)$ for all $x \in H$ and $q \in Q$. Let $x, y \in H$ and $q \in Q$. If $y \in A$, then $x \cdot y \in A$ and so $\mu_A(x \cdot y, q) = \alpha_1 = \mu_A(y, q)$ and $\gamma_A(x \cdot y, q) = \beta_1 = \gamma_A(y, q)$. If $y \in H \backslash A$, then $\mu_A(y, q) = \alpha_2$ and $\gamma_A(y, q) = \beta_2$, and hence $\mu_A(x \cdot y, q) \geq \alpha_2 = \mu_A(y, q)$ and $\gamma_A(x \cdot y, q) \leq \beta_2 = \gamma_A(y, q)$. Finally, let $x, y_1, y_2 \in H$ and $q \in Q$. If $y_1 \in H \backslash A$ or $y_2 \in H \backslash A$, then $\mu_A(y_1, q) = \alpha_2$ or $\mu_A(y_2, q) = \alpha_2$. It follows that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \alpha_2 = \min\{\mu_A(y_1, q), \mu_A(y_2, q)\}$. Also if $y_1 \in H \backslash A$ or $y_2 \in H \backslash A$, then $\gamma_A(y_1, q) = \beta_2$ or $\gamma_A(y_2, q) = \beta_2$. It follows that $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \beta_2 = \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\}$. Assume that $y_1, y_2 \in A$. Then $(y_1 \cdot (y_2 \cdot x)) \cdot x \in A$ and thus $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \alpha_1 = \min\{\mu_A(y_1, q), \mu_A(y_2, q)\}$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) = \beta_1 = \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\}$. Hence, (μ_A, γ_A) is an intuitionistic Q-fuzzy ideal of H. $\square$

Theorem 2.4. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then the subsets $U(\mu_A, \alpha)$ and $L(\gamma_A, \alpha)$ of H are ideals of H for every $\alpha \in Im(f_A) \cap Im(g_A) \subset [0, 1]$.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H and let $\alpha \in Im(f_A) \cap Im(g_A) \subset [0,1]$. Let $x \in U(\mu_A, \alpha)$. Then $\mu_A(x,q) \geq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\mu_A(1,q) \geq \mu_A(x,q) \geq \alpha$ for all $q \in Q$. Hence, $1 \in U(\mu_A, \alpha)$. Let $x \in L(\gamma_A, \alpha)$. Then $\gamma_A(x,q) \leq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\gamma_A(1,q) \leq \gamma_A(x,q) \leq \alpha$ for all $q \in Q$. Hence, $1 \in L(\gamma_A, \alpha)$. Let $x \in H$ and $y \in U(\mu_A, \alpha)$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\mu_A(x \cdot y, q) \geq \mu_A(y,q) \geq \alpha$ for all $q \in Q$. Hence, $x \cdot y \in U(\mu_A, \alpha)$. Let $x_1 \in H$ and $y_1 \in L(\gamma_A, \alpha)$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\gamma_A(x_1 \cdot y_1, q) \leq \gamma_A(y_1, q) \leq \alpha$ for all $q \in Q$. Hence, $x_1 \cdot y_1 \in L(\gamma_A, \alpha)$. Let $x \in H$ and $y_1, y_2 \in U(\mu_A, \alpha)$. Then $\mu_A(y_1, q) \geq \alpha$ and $\mu_A(y_2, q) \geq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\mu_A(y_1 \cdot (y_2 \cdot x, q), q) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_A(y_2, q)\} \geq \alpha$ for all $q \in Q$. Hence, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\mu_A, \alpha)$. Let $x' \in H$ and $y_1', y_2' \in L(\gamma_A, \alpha)$. Then $\gamma_A(y_1', q) \leq \alpha$ and $\gamma_A(y_2', q) \leq \alpha$ for all $q \in Q$. Since A is an intuitionistic Q-fuzzy ideal of H, we have $\gamma_A((y_1' \cdot (y_2' \cdot x', q), q) \cdot x', q) \leq \max\{\gamma_A(y_1', q), \gamma_A(y_2', q)\} \leq \alpha$ for all $q \in Q$. Hence, $(y_1' \cdot (y_2' \cdot x')) \cdot x' \in L(\gamma_A, \alpha)$. Therefore, $U(\mu_A, \alpha)$ and $L(\gamma_A, \alpha)$ are ideals of H. $\square$

Theorem 2.5. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of a Hilbert algebra H, then for all $s, t \in [0,1]$, the sets $U(\mu_A, t)$ and $L(\gamma_A, s)$ are either empty or ideals of H.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic Q-fuzzy ideal of H and let $s, t \in [0,1]$ be such that $U(\mu_A, t)$ and $L(\gamma_A, s)$ are nonempty subsets of H. It is clear that $1 \in U(\mu_A, t) \cap L(\gamma_A, s)$ since $\mu_A(1,q) \geq t$ and $\gamma_A(1,q) \leq s$ for all $q \in Q$. Let $x \in H$ and $y \in U(\mu_A, t)$. Then $\mu_A(y,q) \geq t$ for all $q \in Q$. It follows that $\mu_A(x \cdot y, q) \geq \mu_A(y,q) \geq t$ so that $x \cdot y \in U(\mu_A, t)$. Let $x \in H$ and $y_1, y_2 \in U(\mu_A, t)$. Then $\mu_A(y_1, q) \geq t$ and $\mu_A(y_2, q) \geq t$ for all $q \in Q$. Hence, $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \geq \min\{\mu_A(y_1, q), \mu_a(y_2, q)\} \geq t$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\mu_A, t)$. Hence, $U(\mu_A, t)$ is an ideal of H. Let $x \in H$ and $y \in L(\gamma_A, s)$. Then $\gamma_A(y,q) \leq s$ for all $q \in Q$. It follows that $\gamma_A(x \cdot y, q) \leq \gamma_A(y,q) \leq s$ so that $x \cdot y \in L(\gamma_A, s)$. Let $x \in H$ and $y_1, y_2 \in L(\gamma_A, s)$. Then $\gamma_A(y_1, q) \leq s$ and $\gamma_A(y_2, q) \leq s$ for all $q \in Q$. Hence, $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x, q) \leq \max\{\gamma_A(y_1, q), \gamma_A(y_2, q)\} \leq s$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(\gamma_A, s)$. Hence, $L(\gamma_A, s)$ is an ideal of H. $\square$

A mapping $f : X \to Y$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras. For any intuitionistic Q-fuzzy set $A = (\mu_A, \gamma_A)$ in Y, we define a new intuitionistic Q-fuzzy set $f^{-1}(A) = (\mu_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$ in X by

$$(\forall x \in X)(\mu_{f^{-1}(A)}(x,q) = \mu_A(f(x), q))$$

and

$$(\forall x \in X)(\gamma_{f^{-1}(A)}(x,q) = \gamma_A(f(x), q)).$$

Proposition 2.5. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and $A = (\mu_A, \gamma_A)$ an intuitionistic Q-fuzzy subalgebra of Y. Then the inverse image $f^{-1}(A)$ of A is an intuitionistic Q-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $q \in Q$. Then

$$\begin{aligned} \mu_{f^{-1}(A)}(x \cdot y, q) &= \mu_A(f(x \cdot y), q) \\ &= \mu_A(f(x) \cdot f(y), q) \\ &\geq \min\{\mu_A(f(x), q), \mu_A(f(y), q)\} \\ &= \min\{\mu_{f^{-1}(A)}(x, q), \mu_{f^{-1}(A)}(y, q)\} \end{aligned}$$

and

$$\begin{aligned} \gamma_{f^{-1}(A)}(x \cdot y, q) &= \gamma_A(f(x \cdot y), q) \\ &= \gamma_A(f(x) \cdot f(y), q) \\ &\leq \max\{\gamma_A(f(x), q), \gamma_A(f(y), q)\} \\ &= \max\{\gamma_{f^{-1}(A)}(x, q), \gamma_{f^{-1}(A)}(y, q)\}. \end{aligned}$$

Hence, $f^{-1}(A)$ is an intuitionistic Q-fuzzy subalgebra of X. □

Theorem 2.6. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y and $A = (\mu_A, \gamma_A)$ an intuitionistic Q-fuzzy ideal of Y. Then the inverse image $f^{-1}(A)$ of A is an intuitionistic Q-fuzzy ideal of X.*

Proof. Since f is a homomorphism of X into Y, we have $f(1) = 1 \in Y$ and, by the assumption, $\mu_A(f(1), q) = \mu_A(1, q) \geq \mu_A(y, q)$ for all $y \in Y$ and $q \in Q$. In particular, $\mu_A(f(1), q) \geq \mu_A(f(x), q)$ for all $x \in X$ and $q \in Q$. Hence, $\mu_{f^{-1}(A)}(1, q) \geq \mu_{f^{-1}(A)}(x, q)$ for all $x \in X$ and $q \in Q$. Also $\gamma_A(f(1), q) = \gamma_A(1, q) \leq \gamma_A(y, q)$ for all $y \in Y$ and $q \in Q$. In particular, $\gamma_A(f(1), q) \leq \gamma_A(f(x), q)$ for all $x \in X$ and $q \in Q$. Hence, $\gamma_{f^{-1}(A)}(1, q) \leq \gamma_{f^{-1}(A)}(x, q)$ for all $x \in X$ and $q \in Q$, which proves (2.2). Now, let $x, y \in X$ and $q \in Q$. Then, by the assumption, we have

$$\mu_{f^{-1}(A)}(x \cdot y, q) = \mu_A(f(x \cdot y), q) = \mu_A(f(x) \cdot f(y), q) \geq \mu_A(f(y), q) = \mu_{f^{-1}(A)}(y, q)$$

and

$$\gamma_{f^{-1}(A)}(x \cdot y, q) = \gamma_A(f(x \cdot y), q) = \gamma_A(f(x) \cdot f(y), q) \leq \gamma_A(f(y), q) = \gamma_{f^{-1}(A)}(y, q),$$

which proves (2.3). Let $x, y_1, y_2 \in X$ and $q \in Q$. Then, by the assumption, we have

$$\begin{aligned} \mu_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= \mu_A(f((y_1 \cdot (y_2 \cdot x)) \cdot x), q) \\ &= \mu_A((f(y_1) \cdot (f(y_2) \cdot f(x))) \cdot f(x), q) \\ &\geq \min\{\mu_A(f(y_1), q), \mu_A(f(y_2), q)\} \\ &= \min\{\mu_{f^{-1}(A)}(y_1, q), \mu_{f^{-1}(A)}(y_2, q)\} \end{aligned}$$

and

$$\begin{aligned}\gamma_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x, q) &= \gamma_A(f((y_1 \cdot (y_2 \cdot x)) \cdot x), q) \\ &= \gamma_A((f(y_1) \cdot (f(y_2) \cdot f(x))) \cdot f(x), q) \\ &\leq \max\{\gamma_A(f(y_1), q), \gamma_A(f(y_2), q)\} \\ &= \max\{\gamma_{f^{-1}(A)}(y_1, q), \gamma_{f^{-1}(A)}(y_2, q)\},\end{aligned}$$

which proves (2.4). Hence, $f^{-1}(A)$ is an intuitionistic Q-fuzzy ideal of X. □

3. Conclusions

This work presented a detailed study of intuitionistic Q-fuzzy subalgebras, ideals, and deductive systems in Hilbert algebras. We established key relationships among these structures, including that every intuitionistic Q-fuzzy ideal is both a subalgebra and a deductive system. In addition, we examined how these structures behave under homomorphisms by analyzing their inverse images. The results not only contribute to the theoretical development of fuzzy algebraic systems but also provide a foundation that is accessible to new researchers and adaptable for educational purposes.

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