Existence and approximate controllability for random functional differential equations with finite delay

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Abstract. This study examines second-order equations with delays, which frequently arise in various scientific and engineering applications. Within Banach spaces, these equations introduce unique challenges and opportunities for analysis and control. By exploring the existence and approximate controllability of solutions, the research enhances

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the understanding of dynamical systems with delayed feedback. Using mathematical tools such as cosine family theory and the Leray-Schauder theorem, it establishes rigorous conditions for solution existence, contributing to both theoretical and practical advancements. Additionally, the study incorporates empirical validation through a practical example, offering insights into the real-world behavior of these equations. This empirical analysis bridges the gap between theory and application, supporting the development of effective control strategies and engineering solutions. Ultimately, this research deepens the understanding of complex dynamical systems with delays and provides valuable contributions to both theoretical progress and practical implementation.

Keywords: differential equation, Lerray-Schauder fixed point, mild solution; finite delay, semigroup theory; approximate controllability.

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1. Introduction

In the area of mathematical analysis and its applications, various theories, methods, and models have been developed to study complex problems in science. This introduction explores key studies and recent research on functional differential equations, impulsive systems, controllability, and the links between mathematics and physics. The study of functional differential equations has been extensively explored, with foundational results established through semigroup theory and evolution operators [22, 1]. These methods have been widely applied to differential systems involving delays, as seen in the works of [19, 13, 27], which analyze the controllability and phase space properties of retarded systems. Further advancements in this field include research on integral inequalities [20] and stochastic fixed-point theorems [10], which provide crucial tools for investigating the existence of solutions in stochastic domains. The development of neutral functional differential equations has led to several significant contributions in the study of existence and controllability. Many studies focus on establishing uniqueness results in Fréchet spaces [3, 4] and on nonlinear impulsive systems [26, 9]. Recent works have extended these results to nonlocal conditions and fuzzy delay systems [12, 25], while others have examined integral equations on time scales [16]. Additionally, the application of fractional differential equations and numerical methods has gained attention, as demonstrated by [23, 14]. Further advancements include the study of abstract second-order neutral functional integrodifferential equations [6] and the investigation of controllability in Banach spaces [18], which have provided deeper insights into stability and control mechanisms in complex dynamical systems.

In recent years, controllability studies have expanded to include approximate controllability results for semilinear and impulsive systems [6, 7, 15]. Research on second-order neutral functional integro-differential equations with delay and random effects has been particularly relevant in this context [11, 17]. Several studies have investigated state-dependent delays and impulsive systems [2, 24], while others have analyzed random differential equations with nonlocal conditions [8, 5]. These studies highlight the increasing complexity of control problems involving randomness, delays, and impulsive effects, providing valuable insights into mathematical modeling and real-world applications. Random integral equations play a crucial role in modeling real-world problems in life sciences and engineering, providing a framework for studying systems influenced by randomness [28]. The concept of mild solutions for second-order semilinear impulsive differential inclusions in Banach spaces has been explored to address the complexities of impulsive systems and their applications in dynamical models [21]. These references help us understand how to control complex systems, especially those with impulses, delays, and nonlocal effects. They offer useful mathematical ideas and methods for studying and creating control strategies, which are important for science and engineering.

In this we demonstrate that the Second order functional differential equation with delay and random effects is of the form.

$$
\begin{aligned}
x''(t,\varpi) =& Ax(t,\varpi) + \Upsilon(t, x_t(.,\varpi),\varpi); \\
& t \in J = (0,\varrho], t \neq t_\xi, \xi = 1,2,3,...,m \\
x(0,\varpi) =& x_0(\varpi), \\
x'(0,\varpi) =& x'_0(\varpi), \\
\Delta x(t_\xi) =& I_\xi(x(t_\xi)), \\
\Delta' x(t_\xi) =& I'_\xi(x(t_\xi)).
\end{aligned} \tag{1}
$$

Consider a Banach space $\mathcal{S}$ equipped with the norm $\|.\|$, where A is the infinitesimal generator of a strongly continuous cosine family $\{T_1(t) : t \in \mathbb{R}\}$ consisting of bounded linear operators on $\mathcal{S}$. Additionally, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω represents the sample space and $\varpi \in \Omega$. Functions $\Upsilon : J \times \mathcal{D} \times \Omega \to \mathcal{S}$ where Ω is a random operator with stochastic domain and are continuous functions, $\mathcal{D}$ is a phase space.The term $x_t(\cdot,\varpi)$ represents the state history from time $-\infty$ to the present time t, with histories assumed to belong to the abstract phase space $\mathcal{S}$.The symbol $\Delta x(t_\xi) = x(t_\xi^+) - x(t_\xi^-)$ and $\Delta x'(t_\xi) = x'(t_\xi^+) - x'(t_\xi^-)$ where $x(t_\xi^+), x(t_\xi^-)$ and $x'(t_\xi^+), x'(t_\xi^-)$ represent right and left hand limit of $\Delta x(t_\xi)$ and $\Delta x'(t_\xi)$ respectively at $t = t_\xi$ and

$$
\begin{aligned}
x''(t,\varpi) =& Ax(t,\varpi) + \Upsilon(t, x_t(.,\varpi),\varpi) + By(t,\varpi); \quad t \in J \\
x(0,\varpi) =& x_0(\varpi), \\
x'(0,\varpi) =& x'_0(\varpi), \\
\Delta x(t_\xi) =& I_\xi(x(t_\xi)), \\
\Delta' x(t_\xi) =& I'_\xi(x(t_\xi)).
\end{aligned} \tag{2}
$$

Here, A is the operator and Υ is the continuous function, both defined previously. The control function $y(\cdot,\varpi)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, where U is a Banach space and $B : U \to \mathcal{S}$ is a bounded linear operator.

2. Preliminaries

In this section, we cover some key concepts and terms needed to understand our main results.

$$x''(t,\varpi) = Ax(t,\varpi)+\Upsilon(t,\varpi), \quad 0 \le t \le \varrho \quad x(0,\varpi) = x_0(\varpi), \quad x'(0,\varpi) = y_0(\varpi).$$

Here, $A : D(A) \subseteq \mathcal{S} \to \mathcal{S}$ is a closed operator that is densely defined, where $t \in J = [0, \varrho]$. Also, $\Upsilon : J \times \Omega \to \mathcal{S}$ is a suitable function. Many studies have explored equations of this type. In most cases, the existence of an evolution operator $T_2(t, j)$ for the homogeneous equation is essential for finding a solution to the problem.

$$x''(t,\varpi) = Ax(t,\varpi), \quad 0 \le j, t \le \varrho.$$

Definition 2.1. Let $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ be a seminormed linear space consisting of functions defined on $(-\delta, 0]$ that take values in a Banach space $\mathcal{S}$. The space $\mathcal{D}$ is complete and satisfies the following axioms:

(A) For any continuous function $x : (-\delta, 0] \to \mathcal{S}$ with $x_0 \in \mathcal{D}$ and $\rho > 0$, the following conditions hold for all $t \in J$:

1. The function x_t belongs to $\mathcal{D}$.
2. There exists a positive constant K such that

$$\|x(t,\varpi)\| \le K\|x_t(\cdot,\varpi)\|_{\mathcal{D}}.$$

Moreover, there exist functions $U, \vartheta, \vartheta' : \mathbb{R}_+ \to \mathbb{R}_+$, where U is continuous and bounded, and ϑ, ϑ' are locally bounded and independent of x, such that

$$\|x_t(\cdot,\varpi)\|_{\mathcal{D}} \le U(t)\sup\{\|x(m,\varpi)\| : -\delta \le m \le 0\} + \vartheta\|x_0(\varpi)\|_{\mathcal{D}} + \vartheta'\|x_0'(\varpi)\|_{\mathcal{D}}.$$

(B) The function x_t is $\mathcal{D}$-valued and remains continuous on J for all functions x satisfying (A).
(C) The space $\mathcal{D}$ is complete.

Definition 2.2. A strongly continuous cosine family in the Banach space $\mathcal{S}$ is a collection of bounded linear operators $\{T_1(t) : t \in \mathcal{J}\}$ that satisfies the following conditions:

1. Identity Property: The operator at $t = 0$ is the identity operator, $T_1(0) = I$.
2. Continuity Condition: The operator $T_1(t)$ depends continuously on t for any fixed element $x \in \mathcal{S}$.
3. Addition Rule: For all $j, t \in \mathcal{J}$, the operators satisfy the equation:

$$T_1(j + t) + T_1(j - t) = 2T_1(j)T_1(t).$$

In this case, a set of bounded linear operators $\{T_1(t) : t \in \mathcal{J}\}$ is defined in the Banach space $\mathcal{S}$, where distances are measured using the norm $\|\cdot\|$. These operations change continuously over time and form what is known as a strongly continuous cosine family. The associated sine function, denoted as $\{T_2(t) : t \in \mathcal{J}\}$, corresponds to this family and is expressed as follows.

$$T_2(t)x = \int_0^t T_1(j)x\, dj \quad \text{for } x \in \mathcal{D} \text{ and } t \in \mathcal{J}.$$

For every $t \in J$, the bounds $\|T_1(t)\| \leq \vartheta$ and $\|T_2(t)\| \leq \vartheta_a$ hold, where ϑ and ϑ_a are positive constants that ensure these limits.

Definition 2.3. A system is considered approximately controllable on the interval $(0, T]$ if, for any initial state $x_0 \in \mathcal{S}$, desired final state $x_1 \in \mathcal{S}$, and $\epsilon > 0$, there exists a control $u \in L^2((0, T]; U)$ such that the solution $x(t)$ to the equation

$$x'(t, \varpi) = Ax(t, \varpi) + \Upsilon(t, x_t(., \varpi), \varpi), \quad t \in J = (0, \varrho],$$

with initial condition $x(0) = x_0$, satisfies the condition

$$\|x(\varrho) - x_1\| < \epsilon.$$

Lemma 2.1 (Leray-Schauder Nonlinear Alternative)**.** *Let $\mathcal{S}$ be a Banach space, and let Z be a closed and convex subset of $\mathcal{S}$. Suppose U is a relatively open subset of Z containing the point* 0*, and let $\Gamma : U \to Z$ be a compact mapping. Under these conditions, one of the following alternatives holds:*

1. *There exists a point $z \in \partial U$ such that $z \in \lambda\Gamma(z)$ for some $\lambda \in (0, 1)$, or*
2. *The mapping Γ has a fixed point in U.*

Lemma 2.2. A set $\mathcal{D} \subset \mathcal{S}$ is relatively compact in $\mathcal{S}$ if and only if, for each $\xi = 0, 1, \ldots, m$, the set $\overline{\mathcal{D}}_\xi$ is relatively compact in $C[(t_\xi, t_{\xi+1}]; \mathcal{S})$.

The analysis considers the case where impulses, delays, or nonlocal conditions are not included while examining the approximate controllability of the equations within their domain. It focuses on the following situation: given any function y from the L^2 space over the interval $(0, \varrho]$ with values in U and any initial point x_0 in the space $\mathcal{S}$, the initial-value problem is analyzed.

$$x'(t, \varpi) = Ax(t, \varpi) + By(t, \varpi), \quad x \in \mathcal{S}, \qquad x(0) = x_0, \tag{3}$$

A unique mild solution is obtained for the given problem, where the control function x is an element of $L^2(0, \varrho; U)$

$$x(t, \varpi) = T(t)x_0(\varpi) + \int_0^t T(t - j)By(j, \varpi)\, dj, \quad t \in (0, \varrho].$$

Definition 2.4. For $t > 0$, the controllability mapping $G : L^2((0,\varrho];U) \to \mathcal{S}$ is defined within the given system

$$Gx = \int_0^t T(t-j)Bx(j)\,dj.$$

The adjoint operator $G^* : \mathcal{S} \to L^2((0,\varrho];\mathcal{S})$ is defined by the following rule

$$(G^*x)(j) = B^*T^*(\varrho - j)x(\varpi) \quad \forall j \in [0,\varrho], \forall z \in \mathcal{S}.$$

Therefore, the Grammian operator $W : \mathcal{S} \to \mathcal{S}$ is given by $W = GG^*$

$$kx = GG^*x = \int_0^\varrho T(\varrho - j)BB^*T^*(\varrho - j)x\,dj.$$

3. Existence results

This section establishes the existence of solutions for the problem described by equations (1). Certain conditions must be considered to achieve this.

Definition 3.1. Let $\varrho > 0$ and define $PC_\delta = PC[(-\delta,\varrho],\mathcal{S}]$. Consider a function $x : J \times \Omega \to \mathcal{S}$ that is continuous and satisfies $x(0,\varpi) = x_0(t,\varpi)$. If x satisfies the integral equation, it is called a random mild solution of equation (1)

$$\begin{aligned} x(t,\varpi) =& T_1(t)x_0(\varpi) + T_2(t)x_0'(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, x_j(.,\varpi),\varpi)dj \\ &+ \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(x(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(x(t_\xi)). \end{aligned}$$

The following section will discuss the hypotheses that have been listed:

(G_1) There exist a continuous function $a_0, b_0 : J \times \Omega \to \mathbb{R}$ such that

$$\|\Upsilon(t,x,\varpi)\| \leq a_0(\varpi)\|x,\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi),$$

for all $x \in \mathcal{S}, \varpi \in \Omega$.

(G_2) The function $\Upsilon(t,.,.) : \mathcal{D} \times \Omega \to \mathcal{S}$ is continuous for all $t, j \in J$. Additionally, for every $(x,\varpi) \in \mathcal{D} \times \Omega$, the function $\Upsilon(.,x,\varpi) : J \to \mathcal{S}$ is strongly measurable.

(G_3) Let $I_\xi, I_\xi' \in C(\mathcal{S},\mathcal{S})$ be compact operators for each $\xi = 1,2,3,\ldots,m$

$$\begin{aligned} \|I_\xi(t,x)\| &\leq a_\xi\|x\|_{\mathbb{R}}^{\alpha_\xi}, \\ \|I_\xi'(t,x)\| &\leq a_\xi'\|x\|_{\mathbb{R}}^{\alpha_\xi}. \end{aligned}$$

(G_4) The function $\Upsilon : J \times \mathcal{D} \times \Omega \to \mathcal{S}$ is continuous, and there exists a constant $\mathcal{L}$ such that the following condition holds

$$\|\Upsilon(t,x_1,\varpi) - \Upsilon(t,x_2,\varpi)\| \leq \mathcal{L}(\varpi)(\|(x_1,\varpi) - (x_2,\varpi)\|_{\mathcal{D}}^{\alpha_0}).$$

(G_5) A random function $R : \Omega \to \mathbb{R}^+$ exists such that the following condition holds

$$\vartheta\|z_0\| + \vartheta_a\|z_0'\| + \vartheta_a b_0(\varpi) + \vartheta \sum_{0<t_\xi<t} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}$$
$$+ \vartheta_a \sum_{0<t_\xi<t} a_\xi' \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \int_0^t \|[\sup_{j\in(0,\varrho]} \|z_j(.,\varpi)]\|dj \le R(\varpi).$$

Theorem 3.1. If conditions (G_1)-(G_5) are satisfied, then the problem given in (1) has a mild random solution on the interval $(-\delta, \varrho]$.

Proof of Theorem 3.1. Consider a random operator $\Gamma(\varpi)$ defined as $\Gamma(\varpi) : \Omega \times PC_\delta \to PC_\delta$, where $PC_\delta = [(-\delta, \varrho), \mathcal{S}]$. This operator is given by $(\Gamma(\varpi))z(t)$ for $t \in (\delta, \varrho)$.

$$(\Gamma(\varpi))z(t,\varpi) = \begin{cases} \Upsilon(t,\varpi), & t \in (-\delta, \varrho], \\ T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) & \\ + \int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)dj & \\ + \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) & \\ + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)), & t, j \in J \end{cases} \tag{4}$$

The proof is divided into several steps.

Step 1. The operator $(\Gamma(\varpi))$ maps bounded sets into bounded sets.

To prove this, it is enough to find a positive constant $r(\varpi)$ such that for every z in the bounded set $\mathcal{B}_r(\delta)$, where δ is defined as follows:

$$\mathcal{B}_r(\delta) := \{z \in PC_\delta : \sup_{\delta\le t\le\varrho} \|z(t,\varpi)\| \le r(\varpi)\}$$

one has $\|(\Gamma(\varpi))z\|_{PC} \le R(\varpi)$

$$\begin{aligned} \|(\Gamma(\varpi))z(t)\| \le & \|T_1(t)z_0(\varpi)\| + \|T_2(t)z_0'(\varpi)\| \\ & + \|\int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)\|dj \\ & + \|\sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi))\| + \|\sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi))\|dj \\ & \le \vartheta\|z_0(\varpi)\| + \vartheta_a\|z_0'(\varpi)\| + \vartheta_a \int_0^t \|\Upsilon(j, z_j(.,\varpi),\varpi)\|dj \\ & + \vartheta \sum_{0<t_\xi<t} \|I_\xi(z(t_\xi))\| + \vartheta_a \sum_{0<t_\xi<t} \|I_\xi'(z(t_\xi))\| \\ & \le \vartheta\|z_0\| + \vartheta_a\|z_0'\| + \vartheta_a \int_0^t [\sup_{j\in(0.\varrho]} \|z_j(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj \end{aligned}$$

$$
\begin{aligned}
&+ \vartheta \sum_{0<t_\xi<t} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\le \vartheta \|z_0\| + \vartheta_a \|z'_0\| + \vartheta_a b_0(\varpi) + \vartheta \sum_{0<t_\xi<t} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&+ \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \int_0^t \|[\sup_{j\in(0.\varrho]} \|z_j(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} dj \\
&\le R(\varpi).
\end{aligned}
$$

Hence, $(\Gamma(\varpi))$ is bounded set in PC_δ.

Step 2. We now show that $(\Gamma(\varpi))$ is continuous on $\mathcal{B}_r(\delta)$.

Let us consider that for $z_1, z_2 \in \mathcal{B}_r(\delta)$, $t \in J$,

$$
\begin{aligned}
&\|(\Gamma(\varpi))z_1(t) - (\Gamma(\varpi))z_2(t)\| \\
&\quad \le \| \int_0^t T_2(t-j)\Upsilon(j,(z_{1,j}(.,\varpi)),\varpi) - \Upsilon(j,(z_{2,j}(.,\varpi)),\varpi)\| dj \\
&\quad + \| \sum_{0<t_\xi<t} T_1(t-t_\xi)[I_\xi(z_1(t_\xi)) - I_\xi(z_2(t_\xi))]\| \\
&\quad + \| \sum_{0<t_\xi<t} T_2(t-t_\xi)[I'_\xi(z_1(t_\xi)) - I'_\xi(z_2(t_\xi))]\| dj \\
&\quad \le \vartheta_a \int_0^t \|(\Upsilon(j,z_{1,j}(.,\varpi),\varpi) - \Upsilon(j,z_{2,j}(.,\varpi),\varpi)\| dj \\
&\quad + \vartheta \sum_{0<t_\xi<t} \|I_\xi(z_1(t_\xi)) - I_\xi(z_2(t_\xi))\| \\
&\quad + \vartheta_a \sum_{0<t_\xi<t} \|I'_\xi(z_1(t_\xi)) - I'_\xi(z_2(t_\xi))\| \\
&\quad \le \vartheta_a \int_0^t \sup_{j\in(0.\varrho]} \|(z_{1,j}(.,\varpi),\varpi) - (z_{2,j}(.,\varpi),\varpi)\|_{\mathcal{D}}^{\alpha_0} dj \\
&\quad + \vartheta \sum_{0<t_\xi<t} a_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}, \\
&\|(\Gamma(\varpi))z_1(t) - (\Gamma(\varpi))z_2(t)\| \\
&\quad \le \vartheta \sum_{0<t_\xi<t} a_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \sum_{0<t_\xi<t} a'_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \int_0^t \sup_{j\in(0.\varrho]} \|(z_{1,j}(.,\varpi),\varpi) - (z_{2,j}(.,\varpi),\varpi)\|_{\mathcal{D}}^{\alpha_0} dj.
\end{aligned}
$$

The operators are compact for $t > 0$, ensuring continuity in the uniform operator topology. For all $t \in (-\delta, \varrho]$, the right-hand side of the inequalities is independent when $z_1, z_2 \in \mathcal{B}_r(\delta)$.

As $(z_1 - z_2) \to 0$, the norm $\|((\Gamma(\varpi))z_1)(t) - ((\Gamma(\varpi))z_2)(t)\|$ also tends to zero. Thus, $(\Gamma(\varpi))$ is continuous.

Step 3. The operator $(\Gamma(\varpi))$ is compact. The operator $(\Gamma(\varpi))$ is expressed as the sum $(\Gamma_1(\varpi)) + (\Gamma_2(\varpi))$, where both are defined on $\mathcal{B}_r(\delta)$ as follows:

$$(\Gamma_1(\varpi))z(t) = T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, z_j(., \varpi), \varpi)dj,$$

$$(\Gamma_2(\varpi))z(t) = \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)),$$

for all $t \in [\delta, \varrho]$.

Next, we establish that $(\Gamma_1(\varpi))$ is a compact operator.

(i) We show that $(\Gamma_1(\varpi))(\mathcal{B}_r(\delta))$ is equicontinuous.

Consider $\delta \le t_1 < t_2 \le \varrho$ and let $\epsilon > 0$ be small. Then:

$$\begin{aligned}
&\|(\Gamma_1(\varpi))z(t_2) - (\Gamma_1(\varpi))z(t_1)\| \\
&\quad \le \|[T_1(t_2) - T_1(t_2)]z_0(\varpi)\| + \|[T_2(t_2) - T_2(t_1)]z_0'(\varpi)\| \\
&\quad + \|\int_0^{t_1} T_2(t_2-j)\Upsilon(j, z_j(., \varpi), \varpi) - \int_0^{t_2} T_2(t_1-j)\Upsilon(j, z_j(., \varpi), \varpi)\|dj \\
&\quad \le \|[T_1(t_2) - T_1(t_2)]z_0(\varpi)\| + \|[T_2(t_2) - T_2(t_1)]z_0'(\varpi)\| \\
&\quad + \|\int_0^{t_1-\epsilon} (T_2(t_2-j) - T_2(t_1-j))\Upsilon(j, z_j(., \varpi), \varpi)dj\| \\
&\quad + \|\int_{t_1-\epsilon}^{t_1} T_2(t_2-j) - T_2(t_1-j))\Upsilon(j, z_j(., \varpi), \varpi)dj\| \\
&\quad + \|\int_{t_1}^{t_2} T_2(t_2-j) - T_2(t_1-j))\Upsilon(j, z_j(., \varpi), \varpi)\|dj \\
&\quad \le \|[T_1(t_2) - T_1(t_2)]z_0\| + \|[T_2(t_2) - T_2(t_1)]z_0'\| \\
&\quad + \int_0^{t_1-\epsilon} \|T_2(t_2-j) - T_2(t_1-j)\|[\sup_{j\in(0.\varrho]} \|z_j(., \varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj \\
&\quad + \int_{t_1-\epsilon}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)\|[\sup_{j\in(0.\varrho]} \|z_j(., \varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj \\
&\quad + \int_{t_1}^{t_2} \|T_2(t_2-j) - T_2(t_1-j)\|[\sup_{j\in(0.\varrho]} \|z_j(., \varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj.
\end{aligned}$$

As $t_2 - t_1$ approaches zero,

$$\left\|(\Gamma_1(\varpi))z(t_2) - (\Gamma_1(\varpi))z(t_1)\right\|.$$

Since $T_2(t)$ is compact for $t > 0$, the operator $(\Gamma_1(\varpi))$ maps $\mathcal{B}_r(\delta)$ into a set of uniformly continuous functions. For any $z \in \mathcal{B}_r(\delta)$, the above equation tends to zero.

Next, we prove that $(\Gamma_1(\varpi))(\mathcal{B}_r(\delta))(t)$ is precompact in $\mathcal{S}$.

Let $\delta < t \leq j \leq \varrho$ be fixed, and take a real number ϵ such that $0 < \epsilon < t$. For $z \in \mathcal{B}_r(\delta)$, we define $((\Gamma_1(\varpi)), \epsilon)(t)$ as follows:

$$T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^{t-\epsilon} T_2(t-j)\Upsilon(j, z_j(., \varpi), \varpi)dj.$$

For each $z \in B_r(\delta)$, we establish that the set $\{((\Gamma(\varpi))_{1,\epsilon}z)(t) : z \in B_r(\delta)\}$ is precompact for $0 < \epsilon < t$. This follows from the compactness property of $T_2(t)$ for $t > 0$. Additionally, we ensure that

$$\begin{aligned}\|((\Gamma_1(\varpi))z)(t) - &((\Gamma_{1,\epsilon}(\varpi))z)(t)\| \\ &\leq \int_{t-\epsilon}^{t} \|T_2(t-j)\Upsilon(j, z_j(., \varpi), \varpi)\| \\ &\leq \vartheta_a \int_{t-\epsilon}^{t} [\sup_{j\in(0.\varrho]} \|z_j(., \varpi), \varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]dj.\end{aligned}$$

Thus, there exist precompact sets that are arbitrarily close to

$$\{(\Gamma_1(\varpi))\, z : z \in B_r(\delta)\}.$$

Since $(\Gamma_1(\varpi))(B_r(\delta))$ is uniformly bounded and consists of equicontinuous functions, the Arzelà–Ascoli theorem implies that proving $\Gamma_1(\varpi)$ maps $B_r(\delta)$ into a precompact set in $\mathcal{S}$ is sufficient. As a result, the set $\{\Gamma_1(\varpi)\, z : z \in B_r(\delta)\}$ is precompact in $\mathcal{S}$.

Next, we establish the compactness of $\Gamma_2(\varpi)$. By applying Lemma 2.1, we show that $\Gamma_2(\varpi)$ is completely continuous. Its continuity follows from the phase space axioms. Furthermore, for $r > 0$, $t \in (t_z, t_{z+1}] \cap (0, \varrho]$, and $z \geq 1$, where $z \in \mathcal{B}_r = \mathcal{B}_r(0, \mathcal{B}_r(\delta))$, we observe that

$$(5)\quad (\Gamma(\varpi))z(t) \in \begin{cases} \sum_{j=1}^{z} T(t-t_j)I_j(\mathcal{B}_{r^*}(0, \mathcal{S})), & t \in (t_z, t_{z+1}), \\ \sum_{j=0}^{z} T(t_{z+1}-t_j)I_j(\mathcal{B}_{r^*}(0, \mathcal{S})), & t = t_{z+1}, \\ \sum_{j=0}^{z} T(t_z-t_j)I_j(\mathcal{B}_{r^*}(0, \mathcal{S})) + I_z(\mathcal{B}_{r^*}(0, \mathcal{S})), & t = t_z. \end{cases}$$

Since the mappings I_j are completely continuous, the set $[(\Gamma(\varpi))_2(B_r)]_\xi(t)$ is relatively compact in $\mathcal{S}$ for every $t \in [t_\xi, t_{\xi+1}]$. Furthermore, the strong continuity of $(T(t))_{t_0}$ and the compactness of the operators I_ξ guarantee that $[(\Gamma(\varpi))_2(B_r)]_\xi$ remains equicontinuous at t for all $t \in [t_\xi, t_{\xi+1}]$ and for each

$\xi = 1, 2, \dots, n$. Consequently, by Lemma 2.2, it follows that $(\Gamma(\varpi))_2$ is completely continuous.

Step 4. The goal is to determine an open set $U \subseteq PC_\delta$ such that for every $t \in (0, \varrho]$, any point z on its boundary is not part of $\lambda(\Gamma(\varpi))(z)$ for any $\lambda \in (0, 1)$

$$\begin{aligned}(\Gamma(\varpi))z(t) =& \lambda x(t,\varpi)\\ &= \lambda T_1(t)z_0(\varpi) + \lambda T_2(t)z_0'(\varpi) + \lambda \int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)dj\\ &+ \lambda \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \lambda \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)),\end{aligned}$$

for each $t \in (0, \varrho]$, we have $\|x(t,\varpi)\| \leq \|(\Gamma(\varpi))z(t)\|$ and

$$\begin{aligned}\|(\Gamma(\varpi))z(t)\| \leq& \|T_1(t)z_0(\varpi)\| + \|T_2(t)z_0'(\varpi)\|\\ &+ \int_0^t \|T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)\|dj\\ &+ \|\sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi))\| + \|\sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi))\|.\end{aligned}$$

By step 1, $\|(\Gamma(\varpi))z(t)\| \leq R(\varpi)$. We can find a constant $R(\varpi)$ such that $\|z\|_{PC} \neq R(\varpi)$. Set

$$U = \{z \in PC([\delta,\varrho],\mathcal{S}) \mid \sup_{\delta\leq t\leq\varrho} \|z(t,\varpi)\| < R(\varpi)\}.$$

From Steps 1-3 in Theorem 3.1, it is sufficient to show that $(\Gamma(\varpi)) : U \to PC_\delta$ is a compact map. Since no $x \in \partial U$ satisfies $z \in \lambda(\Gamma(\varpi))(z)$ for any $\lambda \in (0, 1)$, Lemma 2.1 implies that $(\Gamma(\varpi))$ has a fixed point $z^* \in U$. Thus, we obtain the result

$$\begin{aligned}z^*(t,\varpi) =& T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, z_j^*(.,\varpi),\varpi)dj\\ &+ \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z^*(t_\xi)).\end{aligned} \tag{6}$$

This shows that $z^*(t,\varpi)$ has a fixed point and satisfies the conditions of a mild solution to problem (1). Thus, the proof of the theorem is complete.

4. Approximate controllability of random functional differential equation

Definition 4.1. The problem (2) is considered controllable on the interval $(0, \varrho]$ if, for any given final state $z^1(\varpi)$, there exists a control function $y(t,\varpi)$ in $L^2(J, U)$ such that the solution $z(t,\varpi)$ satisfies $z(\varrho,\varpi) = z^1(\varpi)$.

Before presenting the main existence result for problem (2), we introduce the definition of a mild random solution

Definition 4.2. Let $\varrho > 0$ and define $PC_\delta = PC[(-\delta, \varrho], \mathcal{S}]$. Consider a function $z : J \times \Omega \to \mathcal{S}$ that is continuous and satisfies $z(0, \varpi) = z_0(\varpi)$. If z fulfills the given integral equation, it is referred to as a random mild solution of equation (2)

$$z(t,\varpi) = T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)[\Upsilon(j, z_j(.,\varpi), \varpi) + By(t,\varpi)]dj + \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)).$$

The next section will cover additional hypotheses, which have been listed here.

Let (G_6)The linear operator $k : L^2(J, U) \to \mathcal{S}$, defined by

$$ky = \int_0^\varrho T_2(\varrho - j)By(j,\varpi)dj,$$

admits a pseudo-inverse operator k^{-1} in the quotient space $L^2(J, U)/\ker k$.

(G_7) A random function $Q : \mathcal{S} \to \mathbb{R}_+$ exists, satisfying the following condition.

$$\vartheta_a Bk^{-1} \int_0^t [\|z^1(\varpi)\| + \vartheta\|z_0(\varpi)\| + \vartheta_a\|z_0'(\varpi)\| + \vartheta_a \int_0^\varrho [\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta + \vartheta \sum_{0<t_\xi<\varrho} a_\xi\|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \sum_{0<t_\xi<\varrho} a_\xi'\|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}]dj \leq Q(\varpi).$$

Theorem 4.1. If (G_1) - (G_7) are satisfied, then the problem (2) is approximately controllable on J.

Proof of Theorem 4.1. Let us define the control:

$$\begin{aligned} z(t,\varpi) =& k^{-1}(z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\ &- \int_0^\varrho T_2(\varrho - j)\Upsilon(j, z_j(.,\varpi),\varpi)dj \\ &- \sum_{0<t_\xi<\varrho} T_1(\varrho - t_\xi)I_\xi(z(t_\xi)) - \sum_{0<t_\xi<\varrho} T_2(\varrho - t_\xi)I_\xi'(z(t_\xi))). \end{aligned}$$

The random operator $(\Gamma(\varpi))'$ is defined as $(\Gamma(\varpi))' : \Omega \times PC_\delta \to PC_\delta$

$$\begin{aligned} ((\Gamma(\varpi))'z)(t) =& T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) + \int_0^t T_2(t-j)[\Upsilon(j, z_j(.,\varpi),\varpi)dj \\ &+ \int_0^t T_2(t-j)Bk^{-1}[(z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\ &- \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta - \sum_{0<t_\xi<\varrho} T_1(\varrho - t_\xi)I_\xi(z(t_\xi)))] \\ &- \sum_{0<t_\xi<\varrho} T_2(\varrho - t_\xi)I_\xi'(z(t_\xi))]dj + \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) \end{aligned} \tag{7}$$

$$+\sum_{0<t_\xi<t} T_2(t-t_\xi)I'_\xi(z(t_\xi)) \quad t\in(-\delta,\varrho].$$

$(\Gamma(\varpi))' = (\Gamma(\varpi))'_1 + (\Gamma(\varpi))'_2$

$$\begin{aligned}(\Gamma(\varpi))'_1 z(t) &= T_1(t)z_0(\varpi) + T_2(t)z'_0(\varpi) + \int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)dj \\ &+ \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \sum_{0<t_\xi<t} T_2(t-t_\xi)I'_\xi(z(t_\xi)),\end{aligned} \tag{8}$$

$$\begin{aligned}(\Gamma(\varpi))'_2 z(t) &= \int_0^t T_2(t-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z'_0(\varpi) \\ &- \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta \\ &- \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)) - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))]dj.\end{aligned} \tag{9}$$

In Theorem (3.1), we have already considered four cases for $(\Gamma_1(\varpi))$. Therefore, it remains to verify the result for $(\Gamma(\varpi))_2$.

Step 1. The operator $(\Gamma(\varpi))_2$ maps bounded sets into bounded sets. To show this, we need to find a positive constant $q(\varpi)$ such that for every $z\in\mathcal{B}_q(\delta)$, the following holds:

$$\mathcal{B}_r(\delta) := \{z\in PC_\delta : \sup_{\delta\le t\le\varrho} \|z(t,\varpi)\| \le q(\varpi)\}$$

one has $\|(\Gamma(\varpi))_2 z\|_{PC} \le Q(\varpi)$.

$$\begin{aligned}&\|(\Gamma(\varpi))'_2 z(t)\| \\ &\le \int_0^t \|T_2(t-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z'_0(\varpi) \\ &- \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta \\ &- \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)) - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))]\|dj \\ &\le \int_0^t \|T_2(t-j)\|Bk^{-1}[\|z^1(\varpi)\| + \|T_1(\varrho)z_0(\varpi)\| + \|T_2(\varrho)z'_0(\varpi)\| \\ &+ \|\int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta\| \\ &+ \sum_{0<t_\xi<\varrho} \|T_1(\varrho-t_\xi)I_\xi(z(t_\xi))\| + \|\sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))\|]dj \\ &\le \vartheta_a Bk^{-1}\int_0^t [\|z^1(\varpi)\| + \vartheta\|z_0(\varpi)\| + \vartheta_a\|z'_0(\varpi)\| \\ &+ \vartheta_a \int_0^\varrho [\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|^{\alpha_0}_{\mathcal{D}} + b_0(\varpi)]d\eta\end{aligned}$$

$$+ \vartheta \sum_{0<t_\xi<\varrho} a_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \sum_{0<t_\xi<\varrho} a'_\xi \|z(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}]dj$$
$$\leq Q(\varpi).$$

Hence, $(\Gamma(\varpi))'_2$ is bounded in PC_δ.

Step 2. We show that $(\Gamma(\varpi))'_2$ is continuous on $\mathcal{B}_r(\delta)$.

Let $z_1, z_2 \in \mathcal{B}_r(\delta)$ and $t \in J$.

$$\begin{aligned}
&\|(\Gamma(\varpi))'_2 z_1(t) - (\Gamma(\varpi))'_2 z_2(t)\| \\
&\quad \leq \int_0^t \|T_2(t-j)Bk^{-1}[\int_0^\varrho T_2(\varrho-\eta)[\Upsilon(\eta,(z_{1,\eta}(.,\varpi),\varpi),\varpi) \\
&\quad - \Upsilon(\eta,(z_{2,\eta}(.,\varpi),\varpi),\varpi)]d\eta - \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)[I_\xi(z_1(t_\xi)) - I_\xi(z_2(t_\xi))] \\
&\quad - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)[I'_\xi(z_1(t_\xi)) - I'_\xi(z_2(t_\xi))]\|]dj \\
&\quad \leq \int_0^t \vartheta_a Bk^{-1}[\int_0^\varrho \vartheta_a \|[\sup_{\eta\in(0.\varrho]} \|(z_{1,\eta}(.,\varpi),\varpi) - (z_{2,\eta}(.,\varpi),\varpi)\|_{\mathbb{R}}^{\alpha_0} d\eta \\
&\quad + \vartheta \sum_{0<t_\xi<\varrho} a_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi} \\
&\quad + \vartheta_a \sum_{0<t_\xi<\varrho} a'_\xi \|z_1(t_\xi) - z_2(t_\xi)\|_{\mathbb{R}}^{\alpha_\xi}]dj.
\end{aligned}$$

For all $t \in (-\delta, \varrho]$, the uniform operator topology remains continuous due to compactness for $t > 0$. Since $z_1, z_2 \in \mathcal{B}_r(\delta)$, the right-hand side of the inequalities is independent. As $(z_1 - z_2) \to 0$, it follows that

$$\|((\Gamma(\varpi))'_2 z_1)(t) - ((\Gamma(\varpi))'_2 z_2)(t)\| \to 0.$$

Thus, $(\Gamma(\varpi))$ is continuous.

Step 3. $(\Gamma(\varpi))'_2$ is a compact operator. To show this, we write $(\Gamma(\varpi))'_2$ as

$$(\Gamma(\varpi))'_2 = (\Gamma_a(\varpi))'_2 + (\Gamma_b(\varpi))'_2.$$

Here, $(\Gamma_1(\varpi))$ and $(\Gamma(\varpi))_2$ are operators on $\mathcal{B}_r(\delta)$ and are defined as follows.

$$\begin{aligned}
(\Gamma_a(\varpi))'_2 &= \int_0^t \|T_2(t-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z'_0(\varpi) \\
&\quad - \int_0^\varrho T_2(\varrho-\eta)[\Upsilon(\eta, z_\eta(.,\varpi),\varpi)]]d\eta dj, \\
(\Gamma_b(\varpi))'_2 &= \int_0^t \|T_2(t-j)Bk^{-1}[\sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)) \\
&\quad - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I'_\xi(z(t_\xi))]dj.
\end{aligned}$$

First, we show that $(\Gamma(\varpi))_{2,a}(\mathcal{B}_r(\delta))$ is equicontinuous.

Let $\delta \leq t_1 < t_2 \leq \varrho$ and take a small $\epsilon > 0$. Then,

$$\begin{aligned}
&\|(\Gamma_a(\varpi))'_2 z(t_2) - (\Gamma_a(\varpi))'_2 z(t_1)\| \\
&\quad \leq \int_0^{t_1} \|T_2(t_2-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\
&\quad - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]dj \\
&\quad - \int_0^{t_2} T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\
&\quad - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\quad \leq \int_0^{t_1-\theta} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\quad - T_2(\varrho)z_0'(\varpi) - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\quad + \| \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\quad - T_2(\varrho)z_0'(\varpi) - \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\quad + \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\quad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta]\|dj \\
&\quad \leq \int_0^{t_1-\theta} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\quad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta]\|dj \\
&\quad + \| \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\quad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta]\|dj \\
&\quad + \int_{t_1-\theta}^{t_1} \|T_2(t_2-j) - T_2(t_1-j)Bk^{-1}[z^1(\varpi) - T_1(\varrho)z_0(\varpi) \\
&\quad - T_2(\varrho)z_0'(\varpi) + \int_0^{\varrho} T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} \|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)]d\eta]\|dj.
\end{aligned}$$

As $t_2 - t_1$ approaches zero, the expression $\|(\Gamma_a(\varpi))'_2 z(t_2) - (\Gamma_a(\varpi))'_2 z(t_1)\|$. Additionally, for any $z \in \mathcal{B}_r(\delta)$, the value tends to zero. This occurs because the operator $T_2(t)$ is compact for $t > 0$, ensuring continuity in the uniform operator topology. Consequently, $(\Gamma_1(\varpi))$ maps $\mathcal{B}_r(\delta)$ into an equicontinuous family of functions.

We now show that the set $(\Gamma_a(\varpi))_2'(\mathcal{B}_r(\delta))(t)$ is precompact in $\mathcal{S}$. Consider $\delta < t \leq j \leq \varrho$ and a real number ϵ satisfying $0 < \epsilon < t$. For $z \in \mathcal{B}_r(\delta)$, we define $((\Gamma_{a,\epsilon}(\varpi))_2' z)(t)$ as $\int_0^{t-\epsilon} T_2(t_2-j)Bk^{-1}[z^1(\varpi)-T_1(\varrho)z_0(\varpi)-T_2(\varrho)z_0'(\varpi)+\int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)]d\eta$.

Since $T_2(t)$ is compact for $t > 0$, it follows that the set $\{((\Gamma_{a,\epsilon}(\varpi))_2' z)(t) : z \in \mathcal{B}_r(\delta)\}$ is precompact for $z \in \mathcal{B}_r(\delta)$ and $0 < \epsilon < t$.

Additionally, for every $z \in \mathcal{B}_r(\delta)$, we state that

$$\begin{aligned}
&\|((\Gamma_a(\varpi))_2 z)(t) - ((\Gamma_{a,\epsilon}(\varpi))_2 z)(t)\| \\
&\qquad \leq \int_{t-\epsilon}^{t} \|T_2(t_2-j)Bk^{-1}\int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)\|d\eta\, dj \\
&\qquad \leq \int_{t-\epsilon}^{t} \vartheta_a Bk^{-1}\int_0^\varrho \vartheta_a[\sup_{\eta\in(0.\varrho]}\|z_\eta(.,\varpi),\varpi\|_{\mathcal{D}}^{\alpha_0} + b_0(\varpi)d\eta]dj.
\end{aligned}$$

Thus, we can find sets that are precompact and close to $\{(\Gamma_a(\varpi))_2' z : z \in B_r(\delta)\}$. This shows that the set $\{(\Gamma_a(\varpi))_2' z : z \in B_r(\delta)\}$ is precompact in $\mathcal{S}$. The set $(\Gamma_a(\varpi))_2'(B_r(\delta))$ is uniformly bounded. Since the functions in this set are equicontinuous, the Arzelà-Ascoli theorem implies that it is enough to show that $(\Gamma_a(\varpi))_2'$ maps $B_r(\delta)$ into a precompact set in $\mathcal{S}$.

Next, we need to confirm that $(\Gamma_b(\varpi))_2'$ is also a compact operator. Using Step 3 of Theorem 3.1, we establish that $(\Gamma_b(\varpi))_2'$ is compact.

Step 4. Next, we find an open set $U \subseteq PC_\delta$ where any z on its boundary satisfies $z \notin \lambda(\Gamma(\varpi))'(z)$ for $\lambda \in (0,1)$.

Let $\lambda \in (0,1)$ and assume $z \in PC_\delta$ is a solution of $z = \lambda(\Gamma(\varpi))'(z)$ for some $0 < \lambda < 1$. This means that for every $t \in (0,\varrho]$, the condition holds

$$\begin{aligned}
z(t,\varpi) =& \lambda T_1(t)z_0(\varpi) + \lambda T_2(t)z_0'(\varpi) + \lambda\int_0^t T_2(t-j)\Upsilon(j, z_j(.,\varpi),\varpi)dj \\
&+ \lambda\int_0^t T_2(t-j)Bk^{-1}[(z^1(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\
&- \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta(.,\varpi),\varpi)d\eta \\
&- \sum_{0<t_\xi<\varrho} T_1(\varrho-t_\xi)I_\xi(z(t_\xi)))] - \sum_{0<t_\xi<\varrho} T_2(\varrho-t_\xi)I_\xi'(z(t_\xi))]dj \\
&+ \lambda\sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z(t_\xi)) + \lambda\sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z(t_\xi)).
\end{aligned}$$

By step 1 of Theorem 3.1 and 3.2, $\|(\Gamma(\varpi))'z(t)\| \leq R(\varpi) + Q(\varpi)$. We can find a constant $R(\varpi) + Q(\varpi)$ such that $\|z\|_{PC} \neq R(\varpi) + Q(\varpi)$. Set

$$U = \{z \in PC([\delta,\varrho],\mathcal{S}) \mid \sup_{\delta\leq t\leq\varrho}\|z(t)\| < R(\varpi) + Q(\varpi)\}.$$

From Steps 1-3 in Theorem 3.2, it is enough to show that $(\Gamma(\varpi))' : U \to PC_\delta$ is a compact operator.

With the chosen set U, no x on its boundary satisfies $z \in \lambda(\Gamma(\varpi))(z)$ for $\lambda \in (0,1)$.

Using Lemma 2.1, we conclude that the operator $(\Gamma(\varpi))$ has a fixed point $z^* \in U$. Hence, we obtain the result

$$
\begin{aligned}
z^*(t,\varpi) &= T_1(t)z_0(\varpi) + T_2(t)z_0'(\varpi) \\
&\quad + \lambda \int_0^t T_2(t-j)[\Upsilon(j, z_j^*(.,\varpi),\varpi)dj \\
&\quad + \int_0^t T_2(t-j)Bk^{-1}[(z^{*,1}(\varpi) - T_1(\varrho)z_0(\varpi) - T_2(\varrho)z_0'(\varpi) \\
&\quad - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta, z_\eta^*(.,\varpi),\varpi)d\eta - \sum_{0<t_\xi<\varrho} T_1(\varrho - t_\xi)I_\xi(z^*(t_\xi)))] \\
&\quad - \sum_{0<t_\xi<\varrho} T_2(k-t_\xi)I_\xi'(z^*(t_\xi))]dj + \lambda \sum_{0<t_\xi<t} T_1(t-t_\xi)I_\xi(z^*(t_\xi)) \\
&\quad + \lambda \sum_{0<t_\xi<t} T_2(t-t_\xi)I_\xi'(z^*(t_\xi))].
\end{aligned} \tag{10}
$$

Thus, $z^*(t,\varpi)$ is a mild solution of problem (2) and has a fixed point. This concludes the proof of the theorem.

5. Example

Before using our abstract results, we first establish some necessary details. This section provides an example to illustrate the findings.

Let $\mathcal{S} = L^2([0,\pi])$ and define $D(A)$ as the set of functions $x \in \mathcal{S}$ for which $x'' \in \mathcal{S}$ and $x(0) = x(\pi) = 0$. The linear operator $A : D(A) \subseteq \mathcal{S} \to \mathcal{S}$ is defined by $Ax = x''$. It is well known that A generates a strongly continuous cosine family $(T_1(t))_{t\in\mathbb{R}}$ on $\mathcal{S}$.

The operator A has a discrete spectrum, with eigenvalues $-n^2$ for $n \in \vartheta$. Each eigenvalue corresponds to an eigenvector given by $z_n(\varrho) = (\frac{2}{\pi})^{1/2}$. Now, we consider the following impulsive partial functional integro-differential equation.

$$
\frac{\partial^2}{\partial t^2} z(t,x,\varpi) = \frac{\partial^2}{\partial x^2} z(t,x,\varpi) + \Upsilon(t, z(\sin t, x, \varpi), \varpi), \quad \varpi \in (-\infty, 0], \tag{11}
$$

$$
\begin{aligned}
\Delta z(t_\xi, x, \varpi) &= \int_0^\pi q_\xi(x,y) z(t_\xi, y, \varpi)dy \quad \text{and} \\
\Delta' z(t_\xi, x, \varpi) &= \int_0^\pi q_\xi'(x,y) z(t_\xi, y, \varpi)dy, \quad \xi = 1, \ldots, m,
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
z(t,0,\varpi) =& z(t,\pi,\varpi) = 0; \quad z(0,x,\varpi) = z_0(x,\varpi), \\
z_t(0,x,\varpi) =& z_1(x,\varpi), \quad t \in J = [0,1], \quad 0 \le x \le \pi,
\end{aligned} \tag{13}
$$

(14) $\quad z(0,x,\varpi)=z_0(x,\varpi), \quad \text{and} \quad z_t(0,x,\varpi)=z_1(x,\varpi), \quad 0\le x\le \pi,$

where we assume the following conditions:

The functions $\Upsilon(\cdot,\varpi)$ and are continuous on $[0,1]$ with

$$n=\sup_{0\le j\le 1}|\Upsilon(j,\varpi)|<1.$$

The functions $q_\xi,\ q'_\xi:[0,\pi]\times[0,\pi]\to\mathbb{R},\ k=1,1,\dots,m$, are continuously differentiable, and

$$\psi_\xi=\left(\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial x}q_\xi(x,y)\right)^2dxdy\right)^{\frac{1}{2}}<\infty,$$
$$\psi'_\xi=\left(\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial x}q'_\xi(x,y)\right)^2dxdy\right)^{\frac{1}{2}}<\infty,$$

for every $\xi=1,2,\dots,m$.

Now, we define the operators respectively $\Upsilon:J\times\mathcal{D}\times\Omega\to\mathcal{S}$,

$$\Upsilon(t,z_t(.,\varpi),\varpi)(x)=\Upsilon(t,z(\sin t,x,\varpi),\varpi),$$
$$I_\xi(z,\varpi)(x)=\int_0^\pi q_\xi(x,y)z(t_\xi,y,\varpi)dy,\quad \xi=1,2,...,m,$$
$$I'_\xi(z,\varpi)(x)=\int_0^\pi q'_\xi(x,y)z(t_\xi,y,\varpi)dy,\quad \xi=1,2,...,m.$$

The equations (5.13-5.16) can be rewritten in a more general form, represented as (1). Using the previously defined functions, we satisfy the conditions of Theorem 3.1. Therefore, by applying Theorem 3.1, we conclude that the nonlocal impulsive Cauchy problem (5.13-5.16) has a mild solution on the interval J.

The graphical illustration of the solution surface for the second-order impulsive random functional differential equation with finite delay, given in equation (1). This illustration provides a more comprehensive understanding of the spatio-temporal behavior of the system in question. The solution surface $z(t,x,\varpi)$ is shown over the spatial domain $x\in[0,\pi]$ and the temporal domain $t\in[0,1]$. The graph gives a visual representation of the dynamic behavior of the system, which includes the interaction between diffusion effects, delay factors, and impulsive influences. The exponential decay term, in conjunction with the sinusoidal components, captures the oscillatory character of the solution, which is heavily dominated by the boundary and initial conditions imposed. The plot also shows the influence of the impulses at the discrete time instances t_k. These impulses cause discontinuities in the system, which then manifest as jumps in the solution trajectory. As for the influence of the kernel functions $q_k(x,y)$ and $q'_k(x,y)$ of the integral formulas, it will be reflected through the solution's amplitudes in the space domain.

Inspecting the above graph, it is possible to make several useful observations. As the solution depends on the sines, one can see its oscillations along with the periodic variation within the above domain. In other words, the exponential decay term alone causes an amplitude to fall off gradually due to time going on, showing system stability under specific conditions. Impulsive effects are captured by the solution surface at certain instances of time, and these effects appear as sudden deviations or peaks in the surface. The boundary condition $z(t, 0, \varpi) = z(t, \pi, \varpi) = 0$ is used to enforce that the solution is discarded at the spatial boundaries, and this causes characteristic nodal structures. The solution surface is represented in below figure. The plot of the time development of the system with respect to the spatial variable is depicted here, and color gradient represents the magnitude of $z(t, x, \varpi)$. Light regions indicate high values, while the dark regions signify low values. The graphical interpretation of the solution

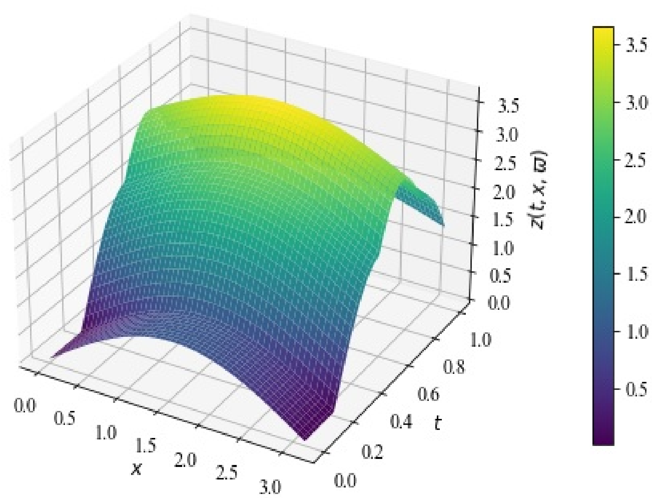

Figure 1: Graphical representation of the solution $z(y, t, \varpi)$ for the second-order impulsive random functional differential equation.

surface would carry an important behavior of the system with respect to different parametric conditions. The solutions profile could be discussed by adjusting the parameters such as kernel functions q_k and initial conditions $z_0(x, \varpi)$. The visualization more strongly supports the verification process of theoretical results through which it ensures the provided solutions are in accordance with expected physical and mathematical properties. In general, graphical representation is essential for the deeper understanding of such a complex dynamics system and helps in further research and analysis.

6. Conclusion

This study explores a class of second-order equations with delays, which are widely applicable in scientific and engineering fields. By analyzing these equations in Banach spaces, it reveals unique challenges and opportunities for control and analysis. The research rigorously examines the existence and approximate controllability of solutions, contributing to a deeper understanding of dynamical systems with delayed feedback. Using mathematical tools such as cosine family theory and the Leray-Schauder theorem, it establishes precise conditions for solution existence. A practical example provides empirical validation, offering valuable insights into real-world behavior. This investigation not only advances theoretical knowledge but also aids in developing effective control strategies and engineering solutions across various domains.

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