L-mosaics and orthomodular lattices

Nicolò Cangiotti
Dipartimento di Matematica
Politecnico di Milano
via Bonardi 9, Campus Leonardo, 20133 Milan
Italy
nicolo.cangiotti@polimi.it

Alessandro Linzi
Center for Information Technologies and Applied Mathematics
University of Nova Gorica
Vipavska 13, Rožna Dolina, SI-5000 Nova Gorica
Slovenia
alessandro.linzi@ung.si

Enrico Talotti*
Center for Information Technologies and Applied Mathematics
University of Nova Gorica
Vipavska 13, Rožna Dolina, SI-5000 Nova Gorica
Slovenia
enrico.talotti@ung.si

Abstract. In this paper, we introduce a class of hypercompositional structures called dualizable L-mosaics. We prove that their category is equivalent to that formed by ortholattices and we formulate an algebraic property characterizing orthomodularity, suggesting possible applications to quantum logic. To achieve this, we establish an equivalence between the category of bounded join-semilattices and that of L-mosaics, thereby providing a categorical foundation for our framework.

Keywords: Hypercompositional Structure, Orthomodular Lattice, Mosaic, Polygroup, Effect algebra, Quantum logic.

MSC 2020: 20N20, 18B10, 03G10.

1. Introduction

We introduce the notion of *L-mosaic*, by extending a point of view initially pushed forward by T. Nakano in his studies on modular lattices [12] and by allowing non-associative multivalued operations as introduced in a recent work of S. Nakamura and M. L. Reyes [11]. We find a natural equivalence of categories between the category formed by ortholattices and the category of the multivalued algebraic structures we call *dualizable L-mosaics*. This study is a

*. Corresponding author

follow-up on a recent article by A. Jenčová and G. Jenča suggesting applications of multivalued algebraic structures in quantum logic [7].

The paper is structured as follows. In Section 2, we introduce the notion of mosaics, following recent developments on multivalued algebraic structures. Section 3 is devoted to the definition and analysis of Nakano mosaics, a class of L-mosaics naturally associated with bounded join-semilattices. In Section 4, we review the theory of orthomodular lattices, setting the stage for Section 5, where we establish an equivalence between the categories of ortholattices and dualizable L-mosaics. We also characterize orthomodularity in terms of a structural condition on Nakano mosaics. Finally, in Section 6, we discuss possible connections between these algebraic frameworks and quantum logic, and suggest directions for future research within quantum foundations.

2. Mosaics

Following Nakamura and M. L. Reyes [11], we introduce a multivalued algebraic structure called mosaic and review in detail some of their basic properties.

We consider two sets A and B. By a *multimap* $A \multimap B$ we mean a function $A \to \wp(B)$, with domain A and codomain the power set $\wp(B)$ of B. We call a multimap $f : A \multimap B$

1. *total* if $f(x) \neq \emptyset$, for all $x \in A$;
2. *partial* if it is not total;
3. *deterministic* if $|f(x)| \leq 1$, for all $x \in A$;
4. a *map* if it is total and deterministic.

The composition $g \circ f : A \multimap C$ of two multimaps $f : A \multimap B$ and $g : B \multimap C$ is defined on every $x \in A$ by the following formula:

$$(g \circ f)(x) := \bigcup_{y \in f(x)} g(y).$$

It is easily verified that this composition is associative and that the multimaps defined by the assignment $x \mapsto \{x\}$ serve as identities. Moreover, it is not difficult to verify that the obtained category formed by sets and multimaps as above is isomorphic to the familiar category $\mathtt{Rel}$ formed by sets and binary relations. The isomorphism is provided by the equivalence

$$y \in f(x) \iff (x, y) \in R,$$

where $f : A \multimap B$ is a multimap and $R \subseteq A \times B$ denotes the binary relation corresponding to f. By identifying these categories, we say that a multimap $f : A \multimap B$ *contains* a multimap $g : A \multimap B$ if the relation corresponding to f contains the relation corresponding to g.

A *(binary) multioperation* on a set A is a multimap $\boxdot : A^2 := A \times A \multimap A$. By a *magma* we mean a set A equipped with a multioperation. To any multioperation $\boxdot$ defined on a set A there correspond a multioperation *dual* of $\boxdot$, defined for $x, y \in A$ by the formula

$$x \boxdot^d y := y \boxdot x.$$

We call a magma $(A, \boxdot)$ *total/partial/deterministic* if so is the multimap $\boxdot$. The magma $(A, \boxdot)$ is *commutative* if the multioperations $\boxdot$ and $\boxdot^d$ coincide on A. If $(A, \boxdot)$ is total and deterministic, then we call it *classical*.

By a *morphism (of magmata)* from $(A, \boxdot_A)$ to $(B, \boxdot_B)$ we mean a function $f : A \to B$ such that, for all $x, y \in A$, the following inclusion holds:

$$f(x \boxdot_A y) \subseteq f(x) \boxdot_B f(y).$$

By a *strong morphism (of magmata)* from $(A, \boxdot_A)$ to $(B, \boxdot_B)$ we mean a function $f : A \to B$ such that, for all $x, y \in A$, the following inclusion holds:

$$f(x \boxdot_A y) = f(x) \boxdot_B f(y).$$

An injective morphism of magmata from $f : (A, \boxdot_A) \to (B, \boxdot_B)$ is called an *embedding* if

$$f(x \boxdot_A y) = (f(x) \boxdot_B f(y)) \cap f(A).$$

It is easily seen that there we have a category `Mag` formed by magmata and morphisms and that isomorphisms in `Mag` are precisely bijective strong morphisms or, equivalently, surjective embeddings. In order to lay the foundations for the notion of mosaic, we review and formalize a number of basic concepts related to multivalued operations and magmatic structures. The following definitions and lemmas will be essential in characterizing the algebraic behavior of mosaics and their morphisms.

Definition 2.1. *An element e in a magma $(A, \boxdot)$ is called* neutral *if $e \boxdot x = x \boxdot e = \{x\}$ holds, for all $x \in A$. A magma $(A, \boxdot)$ with a neutral element $e \in A$ for $\boxdot$ is called* unital magma*.*

Remark 2.1. If a neutral element e exists in a magma $(A, \boxdot)$, then it is unique.

Definition 2.2. *A morphism $f : A \to A'$ between two unital magmata A and A' with neutral elements e and e', respectively, is called* unitary *if $f(e) = e'$.*

Definition 2.3. *Let $(A, \boxdot)$ be a magma and $\rho : A \to A$ an endofunction. We say that $(A, \boxdot)$ is* reversible with respect to ρ *(briefly, ρ-*reversible*) if*

(RE) *$z \in x \boxdot y$ implies both $x \in z \boxdot \rho(y)$ and $y \in \rho(x) \boxdot z$, for all $x, y, z \in A$.*

Definition 2.4 ([11, Definition 2.3]). *A (commutative) unital magma $(A, \boxdot, e)$ which is ρ-reversible with respect to some endofunction $\rho : A \to A$ is called* (commutative) mosaic. *By a* (strong) morphism of mosaics *we mean a unitary (strong) morphism of the underlying unital magmata.*

The following is also observed in [11]. We make a slightly more precise statement and write down a short proof.

Lemma 2.1 ([11]). *Let $(A, \boxdot, e, \rho)$ be a mosaic. Then ρ is a unitary isomorphism of magmata*

$$\rho : (A, \boxdot, e) \xrightarrow{\sim} (A, \boxdot^d, e)$$

satisfying the following property:

(RINV) $e \in (x \boxdot \rho(x)) \cap (\rho(x) \boxdot x)$, *for all $x \in A$.*

In addition, the equivalences

$$z \in x \boxdot y \iff x \in z \boxdot \rho(y) \iff y \in \rho(x) \boxdot z$$

hold for all $x, y, z \in A$.

Proof. Indeed, $x \in (x \boxdot e) \cap (e \boxdot x)$ and ρ-reversibility imply $e \in (\rho(x) \boxdot x) \cap (x \boxdot \rho(x))$. Conversely, if $e \in (y \boxdot x) \cap (x \boxdot y)$ for some $y \in A$, then by ρ-reversibility we may deduce that $y \in e \boxdot \rho(x) = \{\rho(x)\}$ and hence $y = \rho(x)$. It follows that $\rho(e) = e$ and that ρ is an involution. In addition, the validity of the following equivalences is readily verified, for all $x, y, z \in A$:

$$\begin{aligned} \rho(z) \in \rho(x) \boxdot^d \rho(y) &\iff \rho(z) \in \rho(y) \boxdot \rho(x) \\ &\iff \rho(y) \in \rho(z) \boxdot \rho(\rho(x)) = \rho(z) \boxdot x \\ &\iff x \in \rho(\rho(z)) \boxdot \rho(y) = z \boxdot \rho(y) \\ &\iff z \in x \boxdot \rho(\rho(y)) = x \boxdot y. \end{aligned}$$

This shows that ρ is a strong morphism of magmata $(A, \boxdot) \to (A, \boxdot^d)$ and thus an isomorphism. The rest of the assertions follow as well. □

Definition 2.5. *For an element x in a unital magma $(A, \boxdot, e)$, we call any $y \in A$ such that $e \in (x \boxdot y) \cap (y \boxdot x)$ an* inverse *of x. If in $(A, \boxdot, e)$ all elements have a unique inverse, then we denote by x^{-1} the inverse of any $x \in A$ and call $(A, \boxdot, e)$ an* invertible magma.

The following is an immediate consequence of Lemma 2.1.

Corollary 2.1. *Let $(A, \boxdot, e, \rho)$ be a mosaic. Then $\rho(x)$ is the unique inverse of x in A, for all $x \in A$.*

Lemma 2.2. *Let $(A, \boxdot, e)$ and $(A', \boxdot', e')$ be invertible magmata and $f : A \to A'$ a unitary morphism. Then $f(x^{-1}) = f(x)^{-1}$, for all $x \in A$.*

Proof. The standard argument as, e.g., in [9, Remark 2.6], applies. $\square$

Definition 2.6 ([4])**.** *A (commutative) polygroup $(P, \boxdot, e)$ is an invertible (commutative) magma, which is reversible with respect to the endofunction defined as $x \mapsto x^{-1}$ and where $\boxdot$ is* associative*, that is, the following property is valid:*

(ASC) $(x \boxdot y) \boxdot z = x \boxdot (y \boxdot z)$*, for all* $x, y, z \in P$

Remark 2.2. It follows from Corollary 2.1 that polygroups are precisely associative mosaics.

Definition 2.7. *By a* (strong) morphism of polygroups *we mean a (strong) morphism of the underlying mosaics.*

Definition 2.8. *Let $(A, \boxdot, e)$ be a mosaic. A subset $B \subseteq A$ is a* submosaic *if the inclusion map $B \to A$ is an embedding. A submosaic B of a mosaic $(A, \boxdot, e)$ is* strong *if the inclusion map $B \to A$ is a strong embedding.*

Examples of polygroups and mosaics can be found e.g. in [3, 11].

Definition 2.9. *A commutative mosaic $(A, \boxplus, 0)$ is called a* L-mosaic *if*

(Lms1) $0, x \in x \boxplus x$*, for all* $x \in A$.

(Lms2) $(x \boxplus x) \boxplus (x \boxplus x) = x \boxplus x$*, for all* $x \in A$.

(Lms3) $(x \boxplus (x \boxplus y)) \cap ((x \boxplus y) \boxplus y) \subseteq x \boxplus y$*, for all* $x, y \in A$.

(Lms4) *For all $x, y \in A$ there is a unique $z \in x \boxplus y$ such that $x, y \in z \boxplus z$.*

Definition 2.10. *Let $(A, \boxplus, 0)$ be an L-mosaic. A submosaic B of A is called* L-submosaic *if $(B, \boxplus_B, 0)$ satisfies properties* (Lms1), (Lms2), (Lms3) *and* (Lms4) *(property* (Lms1) *is automatic).*

Lemma 2.3. *Let $(A, \boxplus, 0)$ be an L-mosaic. Then*

$$y \leq x \overset{\text{def}}{\iff} y \in x \boxplus x.$$

is an order relation on A with respect to which 0 is a bottom element.

Proof. Reflexivity and the assertion on 0 can be deduced immediately from (Lms1). Regarding transitivity, note that if $x \in y \boxplus y$ and $y \in z \boxplus z$, then by (Lms2) implies

$$x \in y \boxplus y \subseteq (z \boxplus z) \boxplus (z \boxplus z) = z \boxplus z.$$

As for antisymmetry, if $y \in x \boxplus x$ and $x \in y \boxplus y$, then by reversibility we have $x, y \in x \boxplus y$. Thus, property (Lms4) now implies $x = y$. $\square$

Lemma 2.4. *Let* $(A, \boxplus, 0)$ *be a mosaic satisfying properties* (Lms1) *and* (Lms2). *If* $x \in A$*, then* $A_x := x \boxplus x$ *is a strong submosaic of* A*. In addition, for any strong submosaic* B *of* A *we have that*

$$B = \bigcup_{x \in B} A_x = \bigcup_{x \in B} B_x.$$

Proof. Let $x \in A$, then by (Lms1) we deduce that $0 \in A_x$. By property (Lms2), for all $y, z \in A_x$ we obtain that

$$y \boxplus z \subseteq (x \boxplus x) \boxplus (x \boxplus x) = A_x.$$

This suffices to prove that A_x is a strong submosaic of A. In addition, we obtain that $B_x = A_x$, for all strong submosaics B of A, whence

$$\bigcup_{x \in B} A_x = \bigcup_{x \in B} B_x \subseteq B$$

For the converse inclusion, note that by property (Lms1) it follows that $x \in B_x$ for all $x \in B$. □

The following is an obvious corollary.

Corollary 2.2. *Let* $(A, \boxplus, 0)$ *be a mosaic satisfying properties* (Lms1) *and* (Lms2) *and* B *a submosaic of* A*. Then*

$$\bar{B} := \bigcup_{x \in B} A_x$$

is a strong submosaic of A*, which is contained in any strong submosaic of* A *containing* B*.*

Definition 2.11. *We call* $\bar{B}$ *the* strong closure *of a submosaic* B *in a mosaic* $(A, \boxplus, 0)$ *satisfying properties* (Lms1) *and* (Lms2).

The following notion will be useful in Section 5.

Definition 2.12. *A commutative mosaic* $(A, \boxplus, 0)$ *will be called* π-dualizable *if* $\pi : A \to A$ *is an involution such that* $(A, \boxplus_\pi, \pi(0))$ *is a commutative mosaic, where for* $x, y \in A$ *we have set*

$$x \boxplus_\pi y := \pi(x) \boxplus \pi(y).$$

Clearly, π becomes an isomorphism between the mosaic $(A, \boxplus, 0)$ and its π-*dual* $(A, \boxplus_\pi, \pi(0))$, whenever A is π-dualizable. Therefore, a π-dualizable mosaic is an L-mosaic if and only if its π-dual is.

3. Nakano mosaics

We introduce in this section the objects we shall call *Nakano mosaics*. These are inspired by the work of Nakano [12], where the modularity of a lattice L is shown to be equivalent to the associativity of some multioperations defined on L.

Let us start by fixing a *bounded join-semilattice* $(L, \vee, 0)$. Thus, $\vee$ is an associative, commutative and idempotent binary operation and $x \vee 0 = x$ for all $x \in L$. We highlight that the results of this section can be easily dualized to bounded-meet semilattices. For all $x, y \in L$ we define the following subset of L:

$$\mathrm{Nak}_{\vee}(x, y) := \{z \in L \mid x \vee y = x \vee z = z \vee y\}.$$

Much of the following proposition was already observed in [11, Example 2.20].

Proposition 3.1. *For all $x, y \in L$ set $x \boxplus y := \mathrm{Nak}_{\vee}(x, y)$. Then $(L, \boxplus, 0)$ is a commutative and total mosaic, where the inverse of each $x \in L$ is x itself. Moreover, for all $x, y \in L$ we have that*

$$x \in y \boxplus y \iff x \le y,$$

where $x \le y :\Leftrightarrow y = x \vee y$ is the canonical partial order associated to the join-semilattice $(L, \vee)$.

Proof. By definition $\mathrm{Nak}_{\vee}(x, 0) = \mathrm{Nak}_{\vee}(0, x) = \{x\}$ holds for all $x \in L$. Moreover, it is easily verified from the definitions that

$$0 \in \mathrm{Nak}_{\vee}(x, y) \implies y = x,$$

for all $x, y \in L$. Thus, $(L, \boxplus, 0)$ is an invertible magma where the inverse of each $x \in L$ is x itself. To show that it is a mosaic it suffices to note that for $x, y, z \in L$ we have that:

$$z \in x \boxplus y \iff x \vee y = z \vee x = z \vee y \iff x \in y \boxplus z.$$

Furthermore, we note that $x \vee y \in x \boxplus y$ holds for all $x, y \in L$, thus $\boxplus$ is total. For the last assertion, $x \in y \boxplus y$ if and only if the displayed equation holds,

$$y = y \vee y = x \vee y,$$

namely, if and only if $x \le y$. □

Definition 3.1. *We call the mosaic $(L, \boxplus, 0)$ associated to a bounded join-semilattice $(L, \vee, 0)$ the* Nakano mosaic *associated to $(L, \vee, 0)$.*

Proposition 3.2. *The Nakano mosaic associated to a subsemilattice*[1] *L' of L such that $0 \in L'$ is a submosaic of the Nakano mosaic associated to L.*

1. Recall that a subset L' of a join-semilattice $(L, \vee)$ is a subsemilattice if it is closed with respect to the operation $\vee$.

Proof. Let L' be a sublattice of L such that $0 \in L'$. Consider the additive Nakano mosaic $(L', \boxplus', 0)$ associated to L'. We have to show that $x \boxplus' y = (x \boxplus y) \cap L'$, for all $x, y \in L'$. Fix $x, y \in L'$. If $z \in x \boxplus' y$, then $z \in L'$ and $x \vee y = x \vee z = y \vee z$ holds in L'. Thus, the same holds in L since L' is a subsemilattice of L and we obtain that $z \in (x \boxplus y) \cap L'$. Conversely, if $z \in (x \boxplus y) \cap L'$ then $x \vee y = x \vee z = y \vee z$ holds in L, but since L' is a subsemilattice and $x, y, z \in L'$ we obtain that $z \in x \boxplus' y$. □

Lemma 3.1. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$. The following equivalence*

$$x, y \in x \boxplus y \iff x = y.$$

holds, for all $x, y \in L$

Proof. We have $x, y \in x \boxplus y$ if and only if both $x \in y \boxplus y$ and $y \in x \boxplus x$ hold. The assertion thus follows by the antisymmetry of the canonical order $\leq$ associated to the join-semilattice L. □

Lemma 3.2. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$, then*

$$(x \boxplus x) \boxplus (x \boxplus x) = x \boxplus x$$

holds, for all $x \in L$.

Proof. If $y, z \leq x$ holds in L, then $t \leq t \vee z = t \vee y = z \vee y \leq x$ follows, for all $t \in y \boxplus z$, i.e., $t \in x \boxplus x$. The converse inclusion is immediately verified after noticing that $x \in x \boxplus x$ holds for all $x \in L$. □

Fact 3.1 ([5, Lemma 24, p. 132])**.** *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$ and take $x, y, z \in L$. Then*

$$x \boxplus (y \boxplus z) \subseteq \{t \in L \mid t \vee x \vee y = t \vee x \vee z = t \vee y \vee z = x \vee y \vee z\}.$$

Remark 3.1. Reference [5] actually contains the result for lattices, but the proof uses only the join-semilattice structure.

Corollary 3.1. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$, then*

$$(x \boxplus (x \boxplus y)) \cap ((x \boxplus y) \boxplus y) \subseteq x \boxplus y$$

holds, for all $x, y \in L$.

Proof. By Fact 3.1, for $z \in (x \boxplus (x \boxplus y)) \cap ((x \boxplus y) \boxplus y)$ we deduce

$$z \vee x = z \vee x \vee x = z \vee x \vee y = x \vee x \vee y = x \vee y$$

and

$$z \vee x \vee y = z \vee y \vee y = z \vee y.$$

It follows, that

$$x \vee y = z \vee x = z \vee x \vee y = z \vee y,$$

whence $z \in x \boxplus y$. $\square$

Lemma 3.3. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$. For all $x, y, z \in L$, we have $z \in x \boxplus y$ implies $z \leq x \vee y$. It follows that $x \boxplus y \subseteq (x \vee y) \boxplus (x \vee y)$.*

Proof. If $z \in x \boxplus y$, then

$$z \vee (x \vee y) = (z \vee x) \vee y = (x \vee y) \vee y = x \vee (y \vee y) = x \vee y,$$

hence $z \leq x \vee y$. $\square$

Lemma 3.4. *Let $(L, \boxplus, 0)$ be the Nakano mosaic associated to the bounded join-semilattice $(L, \vee, 0)$. For $x, y, z \in L$ we have that $z = x \vee y$ if and only if*

$$x, y \in z \boxplus z \text{ and } z \in x \boxplus y \tag{1}$$

hold in $(L, \boxplus, 0)$.

Proof. If $z = x \vee y$, then $x, y \leq z$ and consequently $x \vee y = z = x \vee z = y \vee z$, which shows (1). Conversely, if (1) holds, then (e.g.) $x \leq z$ follows and thus $x \vee y = x \vee z = z$. $\square$

Corollary 3.2. *The Nakano mosaic $(L, \boxplus, 0)$ associated to a bounded join-semilattice $(L, \vee, 0)$ is an L-mosaic.*

Proof. Property (Lms1) is an immediate consequence of Proposition 3.1. Property (Lms2) follows from Lemma 3.2, while Property (Lms3) follows from Lemma 3.1.

It remains to verify Property (Lms4). For this observe that for $x, y \in L$ the element $z := x \vee y \in L$ satisfies $x, y \in z \boxplus z$ and $z \in x \boxplus y$ by Lemma 3.4. This proves the existence part. For uniqueness, note that if $z' \in L$ satisfies $x, y \in z' \boxplus z'$ and $z' \in x \boxplus y$, then by Lemma 3.4, we obtain that $z' = x \vee y = z$. $\square$

We now turn to a structural result that places the theory of Nakano mosaics within a categorical framework. By identifying the correspondence between bounded join-semilattices and L-mosaics, we are able to establish a categorical equivalence that highlights the algebraic robustness of these multivalued structures.

Theorem 3.1. *Consider the category* $\mathtt{JSem}^0$ *formed by bounded join-semilattices and their morphisms*[2] *and the category* $\mathtt{LMsc}$ *formed by L-mosaics and morphisms of mosaics. Then there is an equivalence of categories* $\mathtt{JSem}^0 \simeq \mathtt{LMsc}$*.*

Proof. We employ Corollary 3.2 to define the object assignment of a functor $\mathcal{E} : \mathtt{JSem}^0 \to \mathtt{LMsc}$ as

$$\mathcal{E}(L, \vee, 0) := (L, \boxplus, 0).$$

We start by claiming that $f : L \to L'$ is a morphism of join-semilattices if and only if the same map is a morphism of the corresponding mosaics. Indeed, if $z \in x \boxplus y$ holds in the mosaic $(L, \boxplus, 0)$ defined above, then we deduce

$$f(x) \vee f(y) = f(x \vee y) = f(x \vee z) = f(x) \vee f(z), \text{ and}$$
$$f(x) \vee f(y) = f(x \vee y) = f(z \vee y) = f(z) \vee f(y),$$

that is, $f(z) \in f(x) \boxplus f(y)$.

Conversely, if f is a morphism of mosaics, then since $x \vee y \in x \boxplus y$ holds for any $x, y \in L$, we obtain that $f(x \vee y) \in f(x) \boxplus' f(y)$, whence $f(x \vee y) \vee f(x) = f(x) \vee f(y)$. On the other hand, we have that $x \vee y \geq x$, i.e., $x \vee y \in x \boxplus x$. Therefore, $f(x \vee y) \in f(x) \boxplus' f(x)$, meaning that $f(x \vee y) \geq f(x)$ holds in L'. We conclude that $f(x \vee y) \vee f(x) = f(x \vee y) = f(x) \vee f(y)$.

Thus, $\mathcal{E}$ is fully faithful. It remains to prove that it is essentially surjective, i.e., if $(A, \boxplus, 0)$ is an L-mosaic, then we have to define a unique (up to isomorphism) join-semilattice structure on A such that $(A, \boxplus, 0)$ is the associated Nakano mosaic. Since $(A, \boxplus, 0)$ is an L-mosaic, by Lemma 2.3 we obtain the poset $(A, \leq)$ with bottom element 0.

We claim that for all $x, y \in A$ the unique z, whose existence is guaranteed by property (Lms4), is the least upper bound of x and y in $(A, \leq)$. Indeed, it is an upper bound since $x, y \in z \boxplus z$. If $z' \in A$ is any upper bound, i.e, $x, y \in z' \boxplus z'$, then $z \leq z'$ because

$$z \in x \boxplus y \subseteq (z' \boxplus z') \boxplus (z' \boxplus z') = z' \boxplus z'.$$

Now, we set $x \vee y := z$ and by the arbitrary choice of $x, y \in A$ we obtain a bounded join-semilattice $(A, \vee, 0)$. It remains to verify that $a \boxplus b = \mathrm{Nak}_\vee(a, b)$ for all a, b in the bounded join-semilattice $(A, \vee, 0)$.

If $c \in a \boxplus b$ holds in $(A, \boxplus, 0)$ and we set $x := a \vee b$, $y := a \vee c$ as well as $z := b \vee c$, then we have that $a, b \in x \boxplus x$ and

$$c \in a \boxplus b \subseteq (x \boxplus x) \boxplus (x \boxplus x) = x \boxplus x.$$

Thus, we deduce $a, c \leq x$ and $y = a \vee c \leq x$. On the other hand,

$$b \in a \boxplus c \subseteq (y \boxplus y) \boxplus (y \boxplus y) = y \boxplus y.$$

2. Recall that a *morphism* of join-semilattices $(L, \vee, 0)$ and $(L', \vee', 0')$ is a function $f : L \to L'$ such that $f(0) = 0'$ and $f(x \vee y) = f(x) \vee' f(y)$, for all $x, y \in L$.

Thus, we deduce $a, b \leq y$ and $x = a \vee b \leq y$. Therefore, $x = y$ follows by antisymmetry and similarly one proves that $z = x = y$. This proves that $a \boxplus b \subseteq \mathrm{Nak}_{\vee}(a, b)$.

For the converse inclusion, take $c \in \mathrm{Nak}_{\vee}(a, b)$, i.e., $a \vee b = a \vee c = b \vee c =: x$. By definition of $\vee$, we obtain that

$$x \in (a \boxplus b) \cap (a \boxplus c) \cap (b \boxplus c),$$

therefore, using reversibility,

$$c \in a \boxplus x \subseteq a \boxplus (a \boxplus b)$$

and

$$c \in x \boxplus b \subseteq (a \boxplus b) \boxplus b.$$

Now, $c \in a \boxplus b$ follows by property (Lms3) of L-mosaics. □

Since fully faithful functors are conservative, we obtain a useful corollary.

Corollary 3.3. *Let $(L, \vee, 0), (L', \vee', 0')$ be bounded join-semilattices. The associated Nakano mosaics $(L, \boxplus, 0)$ and $(L', \boxplus', 0')$ are isomorphic if and only if $(L, \vee, 0)$ and $(L', \vee', 0')$ are.*

4. Orthomodular lattices

We now review the main concepts around the orthomodularity property in the framework of lattices. Despite their introduction in the 1930s is strictly linked with the arising of quantum mechanics and the studies of J. V. Neumann [13], orthomodular lattices have become an independent purely algebraic dimension beyond their physical significance. The reader interested in general details on orthomodular lattices is referred to [8].

Two elements x, y in a bounded lattice $(L, \wedge, \vee, 0, 1)$ are *complements*, written $x \bowtie y$, if and only if

$$x \vee y = 1 \quad \text{and} \quad x \wedge y = 0.$$

We define the multimap $\omega : L \multimap L$ by setting

$$\omega(x) := \{y \in L \mid x \bowtie y\}.$$

By the symmetry of the relation $\bowtie$, we obtain the following properties:

(O1) $\omega(x) \neq \emptyset \implies x \in (\omega \circ \omega)(x) =: \omega^2(x)$.

(O2) $\omega^3(x) = \omega(x)$, for all $x \in L$.

Definition 4.1. *A bounded lattice L is called* complemented *if ω is total. A lattice L is called an* ortholattice *if ω contains a map (called* orthocomplementation*).*

It follows from properties (O1) and (O2) above that all orthocomplementations on a bounded lattice are involutions. We shall denote by $\leq$ the canonical partial order associated to a lattice L.

Fact 4.1 ([8, Chpt. 1, Sect. 2])**.** *Let L be an ortholattice and $\pi : L \to L$ an orthocomplementation. Then the following statements hold:*

(OC1) *$x \leq y$ if and only if $\pi(y) \leq \pi(x)$, for all $x, y \in L$.*

(OC2) *$\pi(x \vee y) = \pi(x) \wedge \pi(y)$, for all $x, y \in L$.*

(OC3) *$\pi(x \wedge y) = \pi(x) \vee \pi(y)$, for all $x, y \in L$.*

Definition 4.2. *Let $(L, \wedge, \vee, 0, 1)$ be an ortholattice. If $\pi : L \to L$ is an orthocomplementation of L such that for all $x, y \in L$ we have that*

$$x \leq y \implies x \vee (\pi(x) \wedge y) = y.$$

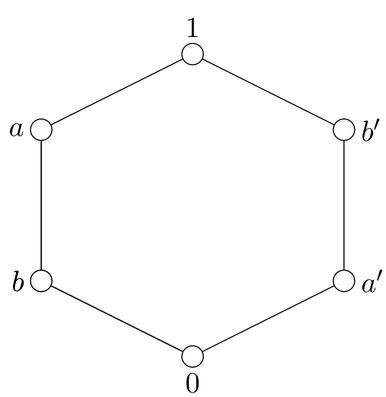
Then the lattice L is called orthomodular with respect to π *(*π-orthomodular*).*

Definition 4.3. *Two elements x, y in a π-orthomodular lattice are called* orthogonal*, written $x \perp y$, if $x \leq \pi(y)$.*

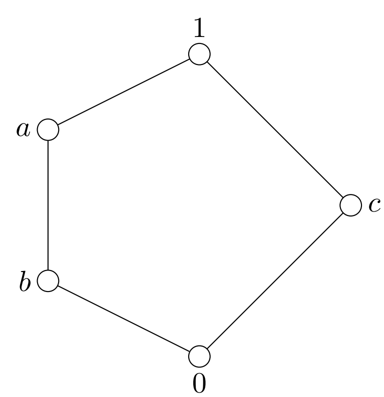
Definition 4.4. *A lattice $(L, \wedge, \vee)$ is called* modular *if for all $x, y, z \in L$*

$$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z).$$

Fact 4.2 ([8, Chpt. 1, Sect. 2])**.** *A lattice L is modular if and only if the pentagon lattice (Figure 1b) is not a sublattice of L.*



(a) The hexagon lattice H.



(b) The pentagon lattice P.

Example 4.1. The ortholattice $H = \{0, a, a', b, b', 1\}$ in Figure 1a is not orthomodular. Indeed $b \leq a$ but

$$b \vee (b' \wedge a) = b \vee 0 = b \neq a.$$

The following characterization of orthomodularity is well-known.

Fact 4.3 ([8], Theorem 2)**.** *Let L be a π-ortholattice. The following statements are equivalent:*

(OM1) *L is orthomodular.*

(OM2) *If $x \leq y$ and $\pi(x) \vee y = 1$, then $x = y$, for all $x, y \in L$.*

(OM3) *H is not a sublattice of L.*

(OM4) *If $x \leq y$, then the smallest sublattice L' of L containing x, y and closed under π is distributive.*

Since the pentagon lattice P of Figure 1b is a sublattice of the hexagon lattice H of Figure 1a, it follows from Fact 4.2 and Fact 4.3 (OM3) that modularity is a property stronger than orthomodularity. For an example of an orthomodular lattice which is not modular see [8, Section 3, p. 33].

The following is the main theorem of Nakano mentioned before.

Fact 4.4 ([12, Theorem 1])**.** *For a bounded lattice L, the following are equivalent statements:*

1. The Nakano mosaic $(L, \boxplus, 0)$ associated to $(L, \vee, 0)$ is a polygroup.

2. L is a modular lattice.

The abstract theory of orthomodularity admits concrete illustration through well-known lattice examples. We now examine two finite lattices, the pentagon and the hexagon, to show how modularity, associativity, and orthomodularity (or their failure) are reflected in the structure of their associated Nakano mosaics.

Example 4.2. The Pentagon lattice $P = \{0, a, b, c, 1\}$ in Figure 1b is not modular. Hence, the associated Nakano mosaic $\mathbf{P}_\vee$ is not associative. Table 1 fully describes the multioperation $\boxplus$ of $\mathbf{P}_\vee$. Note that $\mathbf{P}_\vee$ is not a polygroup as $a \boxplus (b \boxplus c) = \{c, 1\} \neq \{1\} = (a \boxplus b) \boxplus c$.

$\boxplus$	0	a	b	c	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a\}$	$\{1\}$	$\{c, 1\}$
b	$\{b\}$	$\{a\}$	$\{0, b\}$	$\{1\}$	$\{c, 1\}$
c	$\{c\}$	$\{1\}$	$\{1\}$	$\{0, c\}$	$\{a, b, 1\}$
1	$\{1\}$	$\{c, 1\}$	$\{c, 1\}$	$\{a, b, 1\}$	$\{0, a, b, c, 1\}$

Table 1: $\mathbf{P}_\vee$: The Nakano mosaic associated to the non-modular Pentagon lattice.

Corollary 4.1. *Let L be a bounded lattice. If $\mathbf{P}_\vee$ is not a submosaic of $(L, \boxplus, 0)$, then the Nakano mosaic $(L, \boxplus, 0)$ associated to $(L, \vee, 0)$ is a polygroup.*

Proof. If $(L, \boxplus, 0)$ is not a polygroup, then L is not modular by Fact 4.4, thus the Pentagon lattice is a sublattice of it. By Proposition 3.2, it follows that $\mathbf{P}_\vee$ is a submosaic of $(L, \boxplus, 0)$. $\square$

Example 4.3. In Table 2 we describe the multioperation $\boxplus$ of the Nakano mosaic associated to the hexagon lattice H of Figure 1a.

$\boxplus$	0	a	b	a'	b'	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{a'\}$	$\{b'\}$	$\{1\}$
a	$\{a\}$	$\{0, a, b\}$	$\{a\}$	$\{1\}$	$\{1\}$	$\{a', b', 1\}$
b	$\{b\}$	$\{a\}$	$\{0, b\}$	$\{1\}$	$\{1\}$	$\{a', b', 1\}$
a'	$\{a'\}$	$\{1\}$	$\{1\}$	$\{0, a'\}$	$\{b'\}$	$\{a, b, 1\}$
b'	$\{b'\}$	$\{1\}$	$\{1\}$	$\{a'\}$	$\{0, a', b'\}$	$\{a, b, 1\}$
1	$\{1\}$	$\{b, c\}$	$\{a, c\}$	$\{a, b\}$	$\{b'\}$	$\{0, a, b, c, 1\}$

Table 2: $\mathbf{H}_\vee$: The additive Nakano polygroup associated to the hexagon ortholattice.

5. Orthomodularity and Nakano mosaics

Let `Ort` be the category formed by pairs (L, π), where L is a π-ortholattice, as objects and lattice morphisms $f : L \to L'$ such that $f \circ \pi = \pi' \circ f$ as arrows $f : (L, \pi) \to (L', \pi')$. Similarly, let $\texttt{LMsc}^d$ the category formed by pairs (A, π), where A is a π-duaizable L-mosaic as objects and mosaic morphisms $f : A \to A'$ such that $f \circ \pi = \pi' \circ f$ as arrows $f : (A, \pi) \to (A', \pi')$.

We are now in a position to synthesize the preceding developments. Building on the equivalence established in Theorem 3.1, and taking into account the role of involutive dualities, we extend our framework to ortholattices and their categorical counterpart. This leads to the following key result.

Theorem 5.1. *The categories* `Ort` *and* $\texttt{LMsc}^d$ *are equivalent.*

Proof. By restricting the source of the functor $\mathcal{E}$ defined in the proof of Theorem 3.1 to `Ort` we obtain a functor $\texttt{Ort} \to \texttt{LMsc}$, setting

$$\mathcal{E}(L, \vee, \wedge, 0, 1) := (L, \boxplus, 0),$$

where $(L, \boxplus, 0)$ denotes the Nakano mosaic associated to $(L, \vee, 0)$. Since the orthocomplementation π satisfies $x \wedge y = \pi(\pi(x) \vee \pi(y))$ and $\pi(0) = 1$, we have that $(L, \boxplus_\pi, \pi(0))$ is the Nakano mosaic associated to the bounded semilattice $(L, \wedge, 1)$. In particular, $(L, \boxplus, 0)$ is π-dualizable. Hence, the restriction of the functor $\mathcal{E}$ to `Ort` has target $\texttt{LMsc}^d$. The same arguments as in Theorem 3.1 apply here to show that it is fully faithful and essentially surjective as a functor $\texttt{Ort} \to \texttt{LMsc}^d$. $\square$

Proposition 5.1. *Let A be a π-dualizable L-mosaic. Then the π-ortholattice associated to A is π-orthomodular if and only if the implication*

$$x \in y \boxplus y \text{ and } 1 \in x \boxplus \pi(y) \implies x = y \tag{2}$$

holds, for all $x, y \in A$.

Proof. By Fact 4.3 (OM2), the π-ortholattice associated to A is π-orthomodular if and only if the conjunction of $x \leq y$ and $x \vee \pi(y) = 1$ implies $x = y$. By definition of $\vee$ in the π-ortholattice associated to A and since $1 \boxplus 1 = A$, we have that $x \vee \pi(y) = 1$ is equivalent to $1 \in x \boxplus \pi(y)$. On the other hand, we know that $x \leq y$ means that $x \in y \boxplus y$. The assertion follows. □

The following is a general result on orthomodular lattices.

Fact 5.1 ([8], Theorem 9)**.** *Let L be a π-orthomodular lattice and $x, y \in L$. Then the interstection of all sublattices L' of L such that $x, y \in L'$ and $\pi(L') = L'$ is a modular lattice.*

From the above result and Fact 4.4 we deduce:

Proposition 5.2. *Let A be a π-dualizable L-mosaic satisfying the implication* (2) *for all $x, y \in A$. Then for all $x, y \in A$ the smallest L-submosaic A' of A containing x, y and closed under π is a polygroup.*

Proof. Fix $x, y \in A$ and let A' be as in the statement. Since A' is an L-mosaic, we may consider its associated ortholattice structure. Call this lattice L'. Let further L be the ortholattice structure associated to the L-mosaic A. By Theorem 5.1, the lattice L' is the smallest of all sublattices of L such that $x, y \in L'$ and $\pi(L') = L'$, hence, by Fact 5.1, L' is a modular lattice. It now follows by Fact 4.4 that A' is a polygroup. □

6. Further research: the connection with the quantum world

Since the 1990s, *dagger compact categories* have played a central role in the development of topological quantum field theories, as introduced by John Baez and James Dolan [2]. Later, in the early 2000s, Samson Abramsky and Bob Coecke identified these categories as fundamental structures in their framework of categorical quantum mechanics [1].

Building on these foundational ideas, recent work has explored alternative algebraic approaches to quantum theories. In this context, we highlight the contribution of Jenčová and Jenča [7], which served as an initial inspiration for our study. Their research underscores the potential role of hypercompositional structures in the analysis of quantum phenomena. Specifically, their approach employs the concept of *effect algebras*—algebraic structures with a partial operation (see [6] for further details).

Both partial operations and multioperations can be naturally modeled by monoids in the category `Rel` (see [10] for further details). At this point, it should be noted that the category `Rel` possess a dagger compact structure. While partial algebraic structures, such as effect algebras, are already well-integrated into quantum theories, the study of algebraic structures with multioperations remains less developed. This work aims to encourage further research into this promising direction, which we also plan to explore in future investigations.

Acknowledgments

The authors are grateful towards the organizers and participants of both the *"4th Symposium On Hypercompositional Algebra-New Developments And Applications"* and the *"20th International Conference on Quantum Physics and Logic"*, where the research contained in this paper has been fruitfully discussed. In addition, the authors would like to thank the anonymous referee for many suggestions which greatly improved the final version of the manuscript.

References

[1] S. Abramsky, B. Coecke, *A categorical semantics of quantum protocols*, Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004, 415–425.

[2] J. C. Baez, J. Dolan, *Higher-dimensional Algebra and Topological Quantum Field Theory*, J. Math. Phys., 36 (1995), 6073–6105.

[3] M. Baker, N. Bowler, *Matroids over partial hyperstructures*, Adv. Math., 343 (2019), 821-863.

[4] S. D. Comer, *Combinatorial aspects of relations*, Algebr. Univ., 18 (1984), 77-94.

[5] P. Corsini, V. Leorenau, *applications of hyperstructure theory*, Springer, New York, USA, 2003.

[6] D. J. Foulis, M. K. Bennett, *Effect algebras and unsharp quantum logics*, Found. Phys., 24 (1994), 1331-1352.

[7] A. Jenčová, G. Jenča, *On monoids in the category of sets and relations*, Internat. J. Theoret. Phys., 56 (2017), 3757-3769.

[8] G. Kalmbach, *Orthomodular lattices*. L.M.S. monographs. Academic Press, New York, USA, 1983.

[9] K. Kuhlmann, A. Linzi, H. Stojałowska, *Orderings and valuations in hyperfields*, J. Algebra, 611 (2022), 399-421.

[10] A. Linzi, *Polygroup objects in regular categories*, AIMS Math., 9 (2024), 11247-11277.

[11] S. Nakamura, M. L. Reyes, *Categories of hypermagmas, hypergroups, and related hyperstructures*, J. Algebra, 676 (2025), 408-474.

[12] T. Nakano, *Rings and partly ordered systems*, Math. Z., 99 (1967), 355-376.

[13] J. Von Neumann, *Mathematical foundations of quantum mechanics*, Vol. 183, Princeton University Press, Princeton, USA, 1932.

Accepted: September 5, 2025