Automorphisms and isomorphisms of heptavalent symmetric graphs of order $32p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we determine the automorphisms and isomorphisms of connected heptavalent symmetric graphs of order $32p$ for each prime p. As a result, we get the complete classification of such graphs, and there are two sporadic such graphs with $p = 2$ and 3.
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1. Introduction

Let X be a finite, simple graph with vertex set $V(X)$ and edge set $E(X)$. If the full automorphism group $\mathrm{Aut}(X)$ acts transitively on $V(X)$, then X is said to be *vertex-transitive*. Recall that an *arc* in a graph X is an ordered pair of adjacent vertices. Thus, a graph X is said to be *arc-transitive* or *symmetric* if X is vertex-transitive and $\mathrm{Aut}(X)$ acts transitively on the set of all arcs in X. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [27, 30] or [2, 3], respectively.

As we all known that the vertex stabilizers of symmetric graphs are the foundation of the study of such graphs, which have been definitely known for small valences, see for example [11, 15, 25, 26]. By using this structures, classifying symmetric graphs with small valencies has been received considerable attention and a lot of results have been achieved, see[9, 13, 20, 21, 22, 31, 32, 33, 34], and reference therein. In particular, the classification of pentavalent and heptavalent symmetric graphs of order $16p$ with p a prime was given in [13, 12]. Thus, as a

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natural continuation, we classify heptavalent symmetric graphs of order $32p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and two orbits adjacent if there is an edge in X between those two orbits. In view of [23, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has at least 3 orbits on $V(X)$, N acts semiregularly on $V(X)$, and the quotient graph X_N is a connected heptavalent G/N-symmetric graph.*

For a graph X and a positive integer s, an s-arc in X is a sequence of $s+1$ vertices of which any two consecutive vertices are adjacent and any three consecutive vertices are distinct. Let $G \leq \mathrm{Aut}(X)$. Then a graph X is said to be (G, s)*-arc-transitive* if G is transitive on the set of s-arcs in X. Furthermore, if X is (G, s)-arc-transitive but not $(G, s+1)$-arc-transitive, then X is said to be (G, s)*-transitive*. The following proposition characterizes the vertex stabilizers of connected heptavalent (G, s)-transitive graphs (see [15, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G, s)-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

(1) *For $s = 1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;*

(2) *For $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\mathrm{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ or $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$;*

(3) *For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\mathrm{PSL}(3,2) \times \mathrm{S}_4$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ or $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$.*

In particular, a Sylow 3-subgroup of G_v is elementary abelian.

To extract a classification of connected heptavalent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [5], we introduce the graph $G(2p, r)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of $\mathbb{Z}_p^*$ of order r. Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, r)\}$.

Proposition 2.3. *Let X be a connected heptavalent symmetric graph of order $2p$ with p a prime. Then X is isomorphic to $K_{7,7}$ or $G(2p,7)$ with $7 \mid (p-1)$. Furthermore,* $\mathrm{Aut}(G(2p,7)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.

In view of [16, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order $4p$ for a prime p.

Proposition 2.4. *Let X be a connected heptavalent symmetric graph of order $4p$ with p a prime. Then X is isomorphic to K_8.*

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G with respect to S is defined to have vertex set $V(\mathrm{Cay}(G,S)) = G$ and edge set $E(\mathrm{Cay}(G,S)) = \{\{g, sg\} \mid g \in G, s \in S\}$. Clearly, a Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if S generates G. Furthermore, $\mathrm{Aut}(G,S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathrm{Cay}(G,S))$. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. The Cayley graph is vertex-transitive because it admits the right regular representation $R(G)$ of G as a regular group of automorphisms of $\mathrm{Cay}(G,S)$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [28]). For two subsets S and T of G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$ then S and T are said to be *equivalent*, denoted by $S \equiv T$. We may easily show that if $S \equiv T$ then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ and $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Cay}(G,T)$ is normal. The next example is about connected heptavalent symmetric Cayley graphs of order 24.

Example 2.1. Let $\mathrm{S}_4 = \langle (1,2),(1,3),(1,4)\rangle$. Then define the Cayley graph on the symmetric group S_4:

$$\mathcal{G}_{24} = \mathrm{Cay}(\mathrm{S}_4, S).$$

where $S = \{(1,2,3,4),(1,4,3,2),(1,2,4),(1,4,2),(3,4),(2,4),(1,4)(2,3)\}$. By Magma [4], $\mathrm{Aut}(\mathcal{G}_{24}) = \mathrm{S}_4.D_{14} \cong \mathrm{PGL}(2,7)$ and $\mathcal{G}_{24}$ is a connected heptavalent 1-transitive graph.

Following this construction and the result in [14, Theorem 3.1], we have the classification of heptavalent symmetric graphs of order $8p$ with p a prime.

Proposition 2.5. *Let X be a connected heptavalent symmetric graph of order $8p$ with p a prime. Then $X \cong K_{8,8} - 8K_2$ or $\mathcal{G}_{24}$.*

Construction 2.1. Let $G = 2.\mathrm{PSL}(2,7).2i$. Then by Atlas [6], G has a representation of degree 32, its suborbits are: 1^4 and 7^4. These suborbits of length 7 can form orbital graphs of valency 7 or 14. By Magma [4], up to isomorphism, there is only one orbital graph of valency 7, denoted by $\mathcal{G}_{32}$. Furthermore, $\mathrm{Aut}(\mathcal{G}_{32}) \cong \mathbb{Z}_2.(\mathrm{PGL}(2,7) \times \mathbb{Z}_2)$. Conversely, any connected heptavalent symmetric graph of order 32 admitting $G = 2.\mathrm{PSL}(2,7).2i$ as an arc-transitive automorphism group is isomorphic to $\mathcal{G}_{32}$.

In order to construct some heptavalent symmetric graphs, we need to introduce the so called coset graph (see [25, 28]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(G,H,D)$ of G with respect to H and D is defined to have vertex set $[G:H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $\mathrm{Cos}(G,H,D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G. The action of G on $V(\mathrm{Cos}(G,H,D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G in H. Clearly, $\mathrm{Cos}(G,H,D) \cong \mathrm{Cos}(G,H^\alpha,D^\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$. For more details regarding coset graphs, see for example [8, 23, 28].

Construction 2.2. Let $G = \mathrm{PGL}(2,7) = \langle (1,2,6)(3,4,8), (3,8,7,6,5,4)\rangle$. Then G has a Sylow 7-subgroup $H = \langle (2,5,7,6,3,4,8)\rangle$. Let $g = (1,2)(3,5)(6,8)$. Define the coset graph:

$$\mathcal{G}_{48} = \mathrm{Cos}(G,H,HgH).$$

Moreover, $\mathrm{Aut}(\mathcal{G}_{48}) \cong \mathrm{PGL}(2,7) \times \mathrm{S}_3$.

Construction 2.3. Let $G_{(2^3,2p)} = \langle a,b,x,y,z \mid a^p = b^2 = x^2 = y^2 = z^2 = 1, a^b = a^{-1}, a^x = a^y = a^z = a, b^x = b^y = b^z = b, x^y = x^z = x, y^z = y\rangle \cong D_{2p} \times \mathbb{Z}_2^3$. Take two subsets in G:

$$S = \{b, abx, a^2bxy, a^3bxz, a^4byz, a^5by, a^6bxyz\},$$
$$T = \{abx, a^rby, a^{r^2}byz, a^{r^3}bxy, a^{r^4}bz, a^{r^5}bxz, a^{r^6}bxyz\}.$$

where $r \in \mathbb{Z}_p^*$ has order 7. Then define the Cayley graphs:

$$\mathcal{G}_{112} = \mathrm{Cay}(G_{(2^3,14)}, S), \qquad \mathcal{G}_{(2^3,2p)} = \mathrm{Cay}(G_{(2^3,2p)}, T).$$

Moreover, $\mathrm{Aut}(\mathcal{G}_{112}) \cong (\mathbb{Z}_2^3 \times D_{14}) \rtimes F_{21}$ and $\mathrm{Aut}(\mathcal{G}_{(2^3,2p)}) \cong (\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$.

The next result is about heptavalent symmetric graphs of order $16p$, which is from [12, Theorem 1.1].

Proposition 2.6. *Let X be a connected heptavalent graph of order $16p$. Then X is symmetric if and only if X is isomorphic to $\mathcal{G}_{32}$, $\mathcal{G}_{48}$, $\mathcal{G}_{112}$ or $\mathcal{G}_{(2^3,2p)}$.*

Remark. The graph of $\mathcal{G}_{32}$ is missing in [12, Theorem 1.1]. In fact, in the proof of [12, lemma 4.1], the groups $\mathbb{Z}_4.\mathrm{PSL}(2,7)$ and $2.\mathrm{PSL}(2,7).2i$ are not considered, and by Atlas [6], these two groups have representations of degree 32. With the calculation of Magma [4], these two groups give the same orbital graph $\mathcal{G}_{32}$, up to isomorphism.

From [6, pp.12-14], [29, Theorem 2] and [18, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

Proposition 2.7. *Let p be a prime, and let G be a non-abelian simple group of order $|G|\bigm|(2^{29}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p)$. Then G has 3-prime factor, 4-prime factor or 5-prime factor, and is isomorphic to one of the following groups:*

Table 1: **Non-abelian simple $\{2,3,5,7,p\}-$groups**

3-prime factor					
G	Order	G	Order	G	Order
A_5	$2^2{\cdot}3{\cdot}5$	PSL(2, 8)	$2^3{\cdot}3^2{\cdot}7$	PSU(4, 2)	$2^6{\cdot}3^4{\cdot}5$
A_6	$2^3{\cdot}3^2{\cdot}5$	PSL(2, 17)	$2^4{\cdot}3^2{\cdot}17$	PSU(3, 3)	$2^5{\cdot}3^3{\cdot}7$
PSL(2, 7)	$2^3{\cdot}3{\cdot}7$	PSL(3, 3)	$2^4{\cdot}3^3{\cdot}13$		
4-prime factor					
G	Order	G	Order	G	Order
A_7	$2^3{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2, 27)	$2^2{\cdot}3^3{\cdot}7{\cdot}13$	PSp(4, 4)	$2^8{\cdot}3^2{\cdot}5^2{\cdot}17$
A_8	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2, 31)	$2^5{\cdot}3{\cdot}5{\cdot}31$	PSp(6, 2)	$2^9{\cdot}3^4{\cdot}5{\cdot}7$
A_9	$2^6{\cdot}3^4{\cdot}5{\cdot}7$	PSL(2, 49)	$2^4{\cdot}3{\cdot}5^2{\cdot}7^2$	M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$
A_{10}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7$	PSL(2, 81)	$2^4{\cdot}3^4{\cdot}5{\cdot}41$	M_{12}	$2^6{\cdot}3^3{\cdot}5{\cdot}11$
PSL(2, 11)	$2^2{\cdot}3{\cdot}5{\cdot}11$	PSL(2, 127)	$2^7{\cdot}3^2{\cdot}7{\cdot}127$	J_2	$2^7{\cdot}3^3{\cdot}5^2{\cdot}7$
PSL(2, 13)	$2^2{\cdot}3{\cdot}7{\cdot}13$	PSL(3, 4)	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	$P\Omega^+(8,2)$	$2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$
PSL(2, 16)	$2^4{\cdot}3{\cdot}5{\cdot}17$	PSU(3, 4)	$2^6{\cdot}3{\cdot}5^2{\cdot}13$	Sz(8)	$2^6{\cdot}5{\cdot}7{\cdot}13$
PSL(2, 19)	$2^2{\cdot}3^2{\cdot}5{\cdot}19$	PSU(3, 8)	$2^9{\cdot}3^4{\cdot}7{\cdot}19$	${}^2F_4(2)'$	$2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$
PSL(2, 25)	$2^3{\cdot}3{\cdot}5^2{\cdot}13$				
5-prime factor					
G	Order	G	Order	G	Order
A_{11}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	PSL(2, 449)	$2^6{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}449$	M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$
PSL(2, 29)	$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}29$	PSL(2, 2^6)	$2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$	$P\Omega^-(8,2)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$
PSL(2, 41)	$2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}41$	PSL(4, 4)	$2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	$G_2(4)$	$2^{12}{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}13$
PSL(2, 71)	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}71$	PSL(5, 2)	$2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$		

3. Graph constructions

In this section, we discuss some heptavalent graphs of order 64 and construct some heptavalent symmetric graphs of order $32p$ with p a prime.

Construction 3.1. Let $a=(1,2)$, $b=(3,4)$, $c=(5,6)$, $d=(7,8)$, $e=(9,10)$, $f=(11,12)$. Then $G=\langle a,b,c,d,e,f\rangle\cong\mathbb{Z}_2^6$. Define the Cayley graph as follows:

$$\mathcal{G}_{64}=\mathrm{Cay}(G,\{a,acd,bde,df,d,bcf,e\}).$$

Then by Magma [4], $\mathcal{G}_{64}$ is symmetric and $\mathrm{Aut}(\mathcal{G}_{64})\cong\mathbb{Z}_2^6\rtimes S_7$.

With the same notation as above, we have the following lemma.

Lemma 3.1. *Let X be a connected heptavalent normal Cayley graph on $\mathbb{Z}_2^6$. Then X is symmetric if and only if $X\cong\mathcal{G}_{64}$.*

Proof. By Construction 3.1, $\mathcal{G}_{64}$ is symmetric. Let $X = \mathrm{Cay}(G, S)$ be a connected heptavalent normal Cayley graph on $G = \langle a, b, c, d, e, f\rangle \cong \mathbb{Z}_2^6$. Then $\langle S\rangle = G$, $|S| = 7$. Since X is normal, we have that $\mathrm{Aut}(X)_1 = \mathrm{Aut}(G, S)$.

Suppose that X is symmetric. Then $\mathrm{Aut}(G, S)$ is transitive on S and $7 \,\big|\, |\mathrm{Aut}(G, S)|$. Let P be a Sylow 7-subgroup of $\mathrm{Aut}(G, S)$. By Proposition 2.2, $|P| = 7$. Since $\mathrm{Aut}(G) \cong \mathrm{GL}(6, 2)$ is transitive on $G\backslash\{1\}$, we may assume that $a \in S$, and since $\mathrm{Aut}(G, S)$ is transitive on S, we have that $S = a^P$. Clearly, $\mathrm{GL}(6, 2) \cong \mathrm{SL}(6, 2) \cong \mathrm{PSL}(6, 2)$. By Magma [4], a Sylow 7-subgroup of $\mathrm{PSL}(6, 2)$ is isomorphic to $\mathbb{Z}_7^2$. Thus, we only need to find every element x of order 7 in $\mathrm{Aut}(G) \cong \mathrm{PSL}(6, 2)$ such that $\langle a^{\langle x\rangle}\rangle = G$. With the calculation of Magma [4], there are 18 such elements, which form three subgroups of order 7, and the corresponding Cayley sets are: $\{a, acd, bde, df, d, bcf, e\}$, $\{a, abcf, adf, b, ae, abcdef, abf\}$, $\{a, bcdef, ab, abde, bf, cd, ad\}$. Furthermore, the three corresponding Cayley graphs are isomorphic to each other. Thus, $X \cong \mathcal{G}_{64}$. □

For a finite group G and a subset S of G, a *bi-Cayley* graph $\mathrm{BCay}(G, S)$ of G with respect to S is the bipartite graph with the vertex set $G \times \{0, 1\}$ and edge set $\{\{(g, 0), (sg, 1)\} \mid g \in G, s \in S\}$. By [7, 35], $\mathrm{BCay}(G, S)$ is connected if and only if $\langle SS^{-1}\rangle = G$ and that a bipartite graph X is a bi-Cayley graph if and only if there exists a subgroup of $\mathrm{Aut}(X)$ which acts regularly on the bipartition sets. Let $g \in G$. Then g induces an automorphism of $\mathrm{BCay}(G, S)$ as follows: $R(g) : (x, 0) \mapsto (xg, 0), (x, 1) \mapsto (xg, 1), x \in G$. Set $R(G) = \{R(g) \mid g \in G\}$. Then $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{BCay}(G, S))$ which has the bipartition sets of $\mathrm{BCay}(G, S)$ as its orbits. By [24, Lemma 2.2], for each $\alpha \in \mathrm{Aut}(G)$ and $g \in G$, we have:

$$\mathrm{BCay}(G, S) \cong \mathrm{BCay}(G, gS) \cong \mathrm{BCay}(G, Sg) \cong \mathrm{BCay}(G, S^{\alpha}).$$

The next lemma is about the existence of connected heptavalent symmetric bi-Cayley graph on the group $\mathbb{Z}_2^5$ and of order 32.

Lemma 3.2. *Any connected heptavalent bi-Cayley graph on $\mathbb{Z}_2^5$ cannot be symmetric.*

Proof. Let $G = \langle a, b, c, d, e\rangle \cong \mathbb{Z}_2^5$ and $X = \mathrm{BCay}(G, S)$ be a connected heptavalent graph. Then $|S| = 7$ and $\langle SS^{-1}\rangle = G$. Suppose to the contrary that X is symmetric.

Since all non-identity elements in G has order 2, we have that $\langle S\rangle = G$. Note that $\mathrm{Aut}(G) \cong \mathrm{GL}(5, 2)$ and all the minimal sets of generators of G have 5 elements. Thus, $\mathrm{Aut}(G)$ is transitive on the minimal sets of generators of G. Without loss of generality, we may assume that $\{1, a, b, c, d, e\} \subset S$. Set $S = \{1, a, b, c, d, x\}$. Following the above argument, $X \cong \mathrm{BCay}(G, S^{\alpha})$ with $\alpha \in \mathrm{Aut}(G)$. Each element of $G\backslash\{1, a, b, c, d, e\}$ and $\{1, a, b, c, d, e\}$ forms a connected heptavalent bi-Cayley graph. Let H be the subgroup of $\mathrm{Aut}(G)$ fixing the set $\{1, a, b, c, d, e\}$ setwise. Then $H \cong \mathrm{S}_5$. By Magma [4], H acting on $G\backslash\{1,a,b,c,d,e\}$ has four orbits and their representatives are: $\{ab\}$, $\{abc\}$, $\{abcd\}$ and $\{abcde\}$. However, by Magma [4], all the four corresponding graphs are not symmetric, a contradiction. $\square$

Construction 3.2. Let $G=2.\mathrm{PSL}(2,7).2i$. Then by Atlas [6], G has a representation of degree 96, its suborbits are: 1^{12} and 7^{12}. These suborbits of length 7 can form orbital graphs of valency 7 or 14. By Magma [4], up to isomorphism, there is only one orbital graph of valency 7, denoted by $\mathcal{G}_{96}$. Furthermore, $\mathrm{Aut}(\mathcal{G}_{96})\cong\mathbb{Z}_2.(\mathrm{PGL}(2,7)\times\mathrm{S}_3)$. Conversely, any connected heptavalent symmetric graph of order 96 admitting $G=2.\mathrm{PSL}(2,7).2i$ as an arc-transitive automorphism group is isomorphic to $\mathcal{G}_{96}$.

4. Main result

This section is devoted to classifying the connected heptavalent symmetric graphs of order $32p$ for each prime p. In what follows, we always let X be a connected heptavalent graph of order $32p$, $A=\mathrm{Aut}(X)$ and A_v the vertex stabilizer of $v\in V(X)$ in A. By Proposition 2.2, $|A_v|\,\big|\,2^{24}{\cdot}3^4{\cdot}5^2{\cdot}7$ and hence $|A|\,\big|\,2^{29}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$. Let N be a minimal normal subgroup of A. Then $N=T^k$ with T a non-abelian simple group, or $N\cong\mathbb{Z}_2^i$ with $|\mathbb{Z}_2^i|\,\big|\,32p$ or $\mathbb{Z}_p$.

Lemma 4.1. *Suppose that $p=2$. Then $X\cong\mathcal{G}_{64}$.*

Proof. Since $p=2$, we have that $|N|\,\big|\,2^{30}{\cdot}3^4{\cdot}5^2{\cdot}7$. Assume that N is nonsolvable. Then $N\cong T^k$ with T a non-abelian simple $\{2,3,5,7\}$-group. Clearly, $3\,\big|\,|N|$ and hence $N_v\neq1$. By Proposition 2.1, N has at most two orbits on $V(X)$. It follows that $2^5\,\big|\,|N|$ and $|N_v|=|N|/32$ or $|N|/64$. If $k\geq2$, then by Proposition 2.7, $N\cong\mathrm{A}_6^2$. However, by Magma [4], A_6^2 has no subgroup of order $2{\cdot}3^2{\cdot}5^2$ or $3^2{\cdot}5^2$, a contradiction. Thus, $k=1$ and $N=T$ is a non-abelian simple group. By Proposition 2.7, N is isomorphic to

$$\mathrm{PSU}(3,3),\ \mathrm{PSU}(4,2),\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10},\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2.$$

However, by Atlas [6] and Magma [4], N has no subgroup of order $|N_v|=|N|/32$ or $|N|/64$, a contradiction.

Thus, N is solvable and $N\cong\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_2^5$ or $\mathbb{Z}_2^6$. If $N\cong\mathbb{Z}_2^6$, then by Lemma 3.1, $X\cong\mathcal{G}_{64}$. If $N\cong\mathbb{Z}_2^5$, then X is a Bi-Cayley graph on N, by Lemma 3.2, this is impossible. For the remaining four cases, N has at least 4 obits on $V(X)$, by Proposition 2.1, X_N is a heptavalent symmetric graph with A/N as an arc-transitive automorphism group. Since the order of X_N is at least 8, we have that $N\ncong\mathbb{Z}_2^4$. Thus, $N\cong\mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$.

Let $N\cong\mathbb{Z}_2$. Then by Proposition 2.6, $X_N\cong\mathcal{G}_{32}$ and by Magma [4], the minimal arc-transitive subgroup of $\mathrm{Aut}(\mathcal{G}_{32})\cong\mathbb{Z}_2.(\mathrm{PGL}(2,7)\times\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_4.\mathrm{PSL}(2,7)$ or $2.\mathrm{PSL}(2,7).2i$. It forces that A/N has an arc-transitive subgroup $M/N\cong\mathbb{Z}_4.\mathrm{PSL}(2,7)$ or $2.\mathrm{PSL}(2,7).2i$. Furthermore, M/N has a normal subgroup $K/N\cong\mathrm{SL}(2,7)$. The normality of the derived subgroup K' in K

implies that $K'N/N \trianglelefteq K/N$. Since K/K' is abelian, K' is non-solvable and so is $K'N/N$. It forces that $K'N/N = K/N \cong \mathrm{SL}(2,7)$ and $K'N = K$. Note that $N \cong \mathbb{Z}_2$. Thus, $K'N = K' = K$ or $K' \times N$. For the former, we have that K is a central extension of a group of order 4 by $\mathrm{PSL}(2,7)$, this is impossible because $\mathrm{Mult}(\mathrm{PSL}(2,7)) \cong \mathbb{Z}_2$. For the later, $K = \mathrm{SL}(2,7) \times \mathbb{Z}_2$. It follows that K has a characteristic subgroup $H \cong \mathrm{SL}(2,7)$. The normality of K in M forces that $H \trianglelefteq M$. Since $|H| = 2^4{\cdot}21$, we have that H acting on $V(X)$ has at least 4 orbits and hence H is semiregular, a contradiction.

Let $N \cong \mathbb{Z}_2^2$. Then by Proposition 2.5, $X_N \cong K_{8,8} - 8K_2$ and $A/N \lesssim \mathrm{S}_8 \times \mathbb{Z}_2$. By Magma [4], $\mathrm{S}_8 \times \mathbb{Z}_2$ has minimal arc-transitive subgroups isomorphic to $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_7$, $\mathrm{PGL}(2,7)$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$. Then A/N has a subgroup $M/N \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_7$, $\mathrm{PGL}(2,7)$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$.

Assume that $M/N \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_7$. Then M/N has a normal regular subgroup $K/N \cong \mathbb{Z}_2^4$ and $K \cong \mathbb{Z}_2^2.\mathbb{Z}_2^4$. Clearly, K is regular on $V(X)$. It follows that $X \cong \mathrm{Cay}(K,S)$ and $7 \,\big|\, |\mathrm{Aut}(K,S)| \,\big|\, |\mathrm{Aut}(K)|$. By GAP [10], $K \cong \mathbb{Z}_2^6$, $\mathbb{Z}_4^3$, $\mathbb{Z}_8 \times \mathbb{Z}_2^3$, $\mathbb{Z}_4 \times \mathbb{Z}_2^4$, `SmallGroup`(64,82), `SmallGroup`(64,261) or `SmallGroup`(64,262). With the similar argument as above, we can deduce that $K \cong \mathbb{Z}_2^6$ and $X \cong \mathcal{G}_{64}$.

Assume that $M/N \cong \mathrm{PGL}(2,7)$ or $M/N \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$. By Atlas [6], the Schur multiplier $\mathrm{Mult}(\mathrm{PSL}(2,7)) = \mathbb{Z}_2$ and $\mathrm{Aut}(N) \cong \mathrm{GL}(2,2)$ is solvable. Thus, $\mathrm{PSL}(2,7)$ commutes with N. It is easy to see that M has a normal subgroup $K \cong \mathbb{Z}_2$. It follows that X_K is a connected heptavalent symmetric graph of order 32, and by Proposition 2.6, $X_N \cong \mathcal{G}_{32}$. With the similar argument as above, we can also deduce that this is also impossible.

Let $N \cong \mathbb{Z}_2^3$. Then $X_N \cong K_8$ and $A/N \lesssim \mathrm{S}_8$. By Magma [4], $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$ and $\mathrm{PSL}(2,7)$ are minimal arc-transitive subgroups of S_8. Thus, we may assume that A/N has a subgroup $M/N \cong \mathrm{PSL}(2,7)$ or $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$. If $M/N \cong \mathrm{PSL}(2,7)$, then since $\mathrm{Aut}(N) \cong \mathrm{GL}(3,2) \cong \mathrm{SL}(3,2) \cong \mathrm{PSL}(2,7)$ and $\mathrm{Mult}(\mathrm{PSL}(2,7)) \cong \mathbb{Z}_2$, we have that $M \cong \mathrm{ASL}(3,2)$, $\mathbb{Z}_2^2 \times \mathrm{SL}(2,7)$ or $\mathbb{Z}_2^3 \times \mathrm{PSL}(2,7)$. However, by Magma [4], there is no connected heptavalent symmetric graph on these three groups, a contradiction. If $M/N \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$, then M has normal regular subgroup $K \cong \mathbb{Z}_2^3.\mathbb{Z}_2^3$. It follows that X is a normal Cayley graph on K, that is, $X \cong \mathrm{Cay}(K,S)$. The normality of X implies that $7 \,\big|\, |\mathrm{Aut}(K)|$. By GAP [10], there are 7 such groups: $\mathbb{Z}_2^6$, $\mathbb{Z}_4^3$, $\mathbb{Z}_8 \times \mathbb{Z}_2^3$, $\mathbb{Z}_4 \times \mathbb{Z}_2^4$, `SmallGroup`(64,82), `SmallGroup`(64,261) and `SmallGroup`(64,262). Since X is normal, all elements in S have order 2 and hence K must be generated by elements of order 2. Thus, $K \ncong \mathbb{Z}_4^3$, $\mathbb{Z}_8 \times \mathbb{Z}_2^3$ or $\mathbb{Z}_4 \times \mathbb{Z}_2^4$ and by GAP [10], $K \ncong$`SmallGroup`(64,82) or `SmallGroup`(64,262). Assume that K is isomorphic to `SmallGroup`(64,261). Then by GAP [10], $|\mathrm{Aut}(K)| = 2^{15}{\cdot}3{\cdot}7$. Let p be a Sylow 7-subgroup of $\mathrm{Aut}(K,S) \leq \mathrm{Aut}(K)$. Take $s \in S$. The normality of X implies that the order $o(s) = 2$ and $S = s^P$. With the calculation of GAP [10], the orbits of P acting on the all the elements of K are: 1^5 and 7^6. It follows that S has six choices. On the other hand, the connectivity of X forces that $\langle S\rangle = K$. However, all these six choices cannot generate the group K, a contradiction. Thus, $K \cong \mathbb{Z}_2^6$ and $X \cong \mathcal{G}_{64}$. $\square$

Lemma 4.2. *Suppose that $p = 3$. Then $X \cong \mathcal{G}_{96}$.*

Proof. Since $p = 3$, we have that $|V(X)| = 64{\cdot}3$ and $|N| \,\big|\, 2^{29}{\cdot}3^5{\cdot}5^2{\cdot}7$. Assume that N is non-solvable, then $N = T^k$ with T a non-abelian simple $\{2,3,5,7\}$-group. Since $|N|$ has at least 3 prime factors, we have that $N_v \neq 1$. By Proposition 2.1, N has at most two orbits on $V(X)$. It forces that $2^4{\cdot}3 \,\big|\, |N|$ and $|N_v| = |N|/(16{\cdot}3)$ or $|N|/(32{\cdot}3)$. If $k \geq 2$, then $|T|^2 \,\big|\, |N|$ and $7 \nmid |T|$. By Proposition 2.7, $N \cong \mathrm{A}_5^2$ or A_6^2. Thus, $|N_v| = 3{\cdot}5^2$ for $N \cong \mathrm{A}_5^2$ and $|N_v| = 2^2{\cdot}3^2{\cdot}5^2$ or $2{\cdot}3^2{\cdot}5^2$ for $N \cong \mathrm{A}_6^2$. However, by Magma [4], A_5^2 and A_6^2 has no subgroups of such orders, a contradiction. It follows that $k = 1$ and $N = T$ is a non-abelian simple group. Checking the orders of the simple groups in Proposition 2.7, N is isomorphic to

$$\mathrm{PSU}(3,3),\ \mathrm{PSU}(4,2),\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10},\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2).$$

However, by Atlas [6] and Magma [4], all the groups listed above have no subgroups of order $|N_v| = |N|/(16{\cdot}3)$ or $|N|/(32{\cdot}3)$, a contradiction.

Thus, N is solvable, and $N \cong \mathbb{Z}_3$, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^5$. Note that there is no heptavalent regular graph of order 3 or 6. Thus, $N \ncong \mathbb{Z}_2^5$ or $\mathbb{Z}_2^4$. By Proposition 2.4, there is no connected heptavalent symmetric graph of order 12, so we have $N \ncong \mathbb{Z}_2^3$. Thus, we have that $N \cong \mathbb{Z}_3$, $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$.

Let $N \cong \mathbb{Z}_3$. Then X_N is a heptavalent symmetric graph of order 32. By Proposition 2.6, $X_N \cong \mathcal{G}_{32}$. Since $\mathbb{Z}_4.\mathrm{PSL}(2,7)$ and $2.\mathrm{PSL}(2,7).2i$ are the minimal arc-transitive subgroup of $\mathrm{Aut}(\mathcal{G}_{32})$, we have that A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_4.\mathrm{PSL}(2,7)$ or $2.\mathrm{PSL}(2,7).2i$, and M/N has a normal subgroup $K/N \cong \mathrm{SL}(2,7)$. By "N/C-Theorem" (see [17, Chapter I, Theorem 4.5]), $K/C_K(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_2$. It is easy to see that $C_K(N) = K$ and hence $K \cong \mathrm{SL}(2,7) \times \mathbb{Z}_3$. It forces that K has a characteristic subgroup $H \cong \mathrm{SL}(2,7)$, which is normal in M. Since H has at least 3 orbits on $V(X)$, by Proposition 2.1, H is semiregular and $|H| \,\big|\, 32p$, a contradiction.

Let $N \cong \mathbb{Z}_2$. Then X_N is a heptavalent symmetric graph of order 48. By Proposition 2.6, $X_N \cong \mathcal{G}_{48}$ and $A/N \lesssim \mathrm{PGL}(2,7) \times \mathrm{S}_3$. By Magma [4], $\mathrm{PGL}(2,7) \times \mathrm{S}_3$ has minimal arc-transitive subgroups $\mathrm{PGL}(2,7)$ and $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$. It follows that A/N has an arc-transitive subgroup $M/N \cong \mathrm{PGL}(2,7)$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_2$. In both cases, M/N has a normal subgroup $K/N \cong \mathrm{PSL}(2,7)$. Note that $\mathrm{Mult}(\mathrm{PSL}(2,7)) \cong \mathbb{Z}_2$ by Atlas [6]. Thus, $K \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$ or $\mathrm{SL}(2,7)$. For $K \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$, we have that K has a characteristic subgroup $H \cong \mathrm{PSL}(2,7)$, which is normal in M. However, H acting on $V(X)$ has 4 orbits and hence is semiregular by Proposition 2.1. This is impossible because $H_v \cong \mathbb{Z}_7$. For $K \cong \mathrm{SL}(2,7)$, if $M/N \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$ then $M \cong \mathrm{SL}(2,7) \times \mathbb{Z}_2$ or $\mathbb{Z}_4.\mathrm{PSL}(2,7)$. By Magma [4], there is no such graph admitting $\mathrm{SL}(2,7) \times \mathbb{Z}_2$ as an arc-transitive subgroup, and there is only one heptavalent symmetric orbital graph on $\mathbb{Z}_4.\mathrm{PSL}(2,7)$, which is isomorphic to $\mathcal{G}_{96}$. If $M/N \cong \mathrm{PGL}(2,7)$, then by Atlas [6], $M \cong 2.\mathrm{PGL}(2,7)$ or $2.\mathrm{PSL}(2,7).2i$. By Magma [4], the only possibility is $M \cong 2.\mathrm{PSL}(2,7).2i$, and by Construction 3.2, $X \cong \mathcal{G}_{96}$.

Let $N \cong \mathbb{Z}_2^2$. Then X_N is a heptavalent symmetric graph of order 24. By Proposition 2.5, $X_N \cong \mathcal{G}_{24}$ and $A/N \lesssim \mathrm{PGL}(2,7)$. Clearly, A/N has an arc-transitive subgroup $M/N \cong \mathrm{PSL}(2,7)$. Since $\mathrm{Mult}(\mathrm{PSL}(2,7)) \cong \mathbb{Z}_2$, we have that $M \cong \mathrm{SL}(2,7) \times \mathbb{Z}_2$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_2^2$. Thus, M has a normal subgroup $K \cong \mathbb{Z}_2$. With a similar argument as above, we can easily deduce that this is impossible. $\square$

Lemma 4.3. *Suppose that $p \geq 5$. Then there is no new graph.*

Proof. Since $p \geq 5$, we have that $|N| \mid 2^{29}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$. we separate the proof into two cases: A has a solvable minimal normal subgroup; A has no solvable normal subgroup.

Case 1: A has a solvable minimal normal subgroup.

Since A has a solvable minimal normal subgroup, we may assume that N is solvable, and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_2^5$ or $\mathbb{Z}_p$. Note that there is no heptavalent regular graph of odd order. Thus $N \ncong \mathbb{Z}_2^5$. By Propositions 2.4 and 2.5, there is no heptavalent symmetric graph of order $4p$ or $8p$ with $p \geq 5$. Thus, $N \ncong \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$ and hence $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^4$ or $\mathbb{Z}_p$.

Let $N \cong \mathbb{Z}_p$. Then X_N has order 32, and by Proposition 2.6, $X_N \cong \mathcal{G}_{32}$ and $A/N \leq \mathbb{Z}_2.(\mathrm{PGL}(2,7) \times \mathbb{Z}_2)$. By Magma [4], $\mathbb{Z}_2.(\mathrm{PGL}(2,7) \times \mathbb{Z}_2)$ has two minimal arc-transitive subgroups: $\mathbb{Z}_4.\mathrm{PSL}(2,7)$ and $2.\mathrm{PSL}(2,7).2i$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_4.\mathrm{PSL}(2,7)$ or $2.\mathrm{PSL}(2,7).2i$ and M/N has a normal subgroup $K/N \cong \mathrm{SL}(2,7)$. By "N/C-Theorem" (see [17, Chapter I, Theorem 4.5]), $K/C_K(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$. It is easy to see that $C_K(N) = K$ and hence $K \cong \mathrm{SL}(2,7) \times \mathbb{Z}_p$. It forces that K has a characteristic subgroup $H \cong \mathrm{SL}(2,7)$, which is normal in M. Since H has at least p orbits on $V(X)$, by Proposition 2.1 H is semiregular and $|H| \mid 32p$, a contradiction.

Let $N \cong \mathbb{Z}_2^4$. Then X_N has order $2p$, and by Proposition 2.3, $X_N \cong K_{7,7}$ or $G(2p,7)$ and $A/N \lesssim (\mathrm{S}_7 \times \mathrm{S}_7) \rtimes \mathbb{Z}_2$ or $(\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.

Assume that $X_N \cong K_{7,7}$. Then $p = 7$. By Magma [4], $(\mathrm{S}_7 \times \mathrm{S}_7) \rtimes \mathbb{Z}_2$ has minimal arc-transitive subgroups $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ and $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. By "N/C-Theorem", $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{GL}(4,2)$. Since $|\mathrm{GL}(4,2)| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$, we have that $7 \mid |C_M(N)|$. Let H be a Sylow 7-subgroup of $C_M(N)$. Then $H = \mathbb{Z}_7^2$ or $\mathbb{Z}_7$ is normal in M. Considering the quotient graph X_H. Note that $p = 7$. Thus, X_H has order 32. By Propositions 2.1 and 2.6, H is semiregular, $H \cong \mathbb{Z}_7$ and $X_H \cong \mathcal{G}_{32}$. In particular, M/H is an arc-transitive subgroup of X_H. Clearly, M/H is solvable. However, by Magma [4], $\mathrm{Aut}(\mathcal{G}_{32}) \cong \mathbb{Z}_2.(\mathrm{PGL}(2,7) \times \mathbb{Z}_2)$ has no solvable arc-transitive subgroup, a contradiction.

Assume that $X_N \cong G(2p,7)$. Then $A/N \lesssim (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$ with $7 \mid (p-1)$. Since $G(2p,7)$ is 1-regular, we have $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. By "N/C-Theorem",

$A/C_A(N) \lesssim \mathrm{Aut}(N) \cong \mathrm{GL}(4,2)$. Since $7 \,\big|\, (p-1)$ and $p \nmid |\mathrm{GL}(4,2)|$, we have $p \,\big|\, |C_A(N)|$. Let P be a Sylow p-subgroup of $C_A(N)$. Then $P \cong \mathbb{Z}_p$ and P is characteristic in $C_A(N)$. The normality of $C_A(N)$ implies that $P \trianglelefteq A$. Thus, X_P has order 32 and $X_P \cong \mathcal{G}_{32}$. By Proposition 2.1, $A/P \cong (\mathbb{Z}_2^4 \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$ is a solvable arc-transitive subgroup of X_P. However, by Magma [4], $\mathrm{Aut}(\mathcal{G}_{32}) \cong \mathbb{Z}_2.(\mathrm{PGL}(2,7) \times \mathbb{Z}_2)$ has no solvable arc-transitive subgroup, a contradiction.

Let $N \cong \mathbb{Z}_2$. Then since $p \geq 5$ and by Proposition 2.6, $X_N \cong \mathcal{G}_{112}$ or $\mathcal{G}_{(2^3,2p)}$ and $A/N \lesssim (\mathbb{Z}_2^3 \times D_{14}) \rtimes F_{21}$ or $(\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$. In both cases, we can easily deduce that $(\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$ is the unique minimal arc-transitive subgroup. Thus, we may assume that A/N has an arc-transitive subgroup $M/N \cong (\mathbb{Z}_2^3 \times D_{2p}) \rtimes \mathbb{Z}_7$ with $p = 7$ or $7 \,\big|\, (p-1)$. Since $N \cong \mathbb{Z}_2$, we have that N lies in the center of M. It follows that a Sylow p-subgroup, say P, commutes with N. Thus, P is normal in M and by Proposition 2.1, X_P is heptavalent graph of order 32 with M/P as an arc-transitive subgroup. By Proposition 2.6, $X_P \cong \mathcal{G}_{32}$. Clearly, M/P is solvable. This is impossible because $\mathrm{Aut}(\mathcal{G}_{32})$ has no solvable arc-transitive subgroup by Magma [4].

Case 2: Suppose that A has no solvable normal subgroup.

Since every normal subgroup of A is non-solvable, N is non-solvable. Thus, $N = T^k$ with T a non-abelian simple group and $|T| \,\big|\, |N| \,\big|\, 2^{29}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}p$. It follows that T is isomorphic to one of the groups listed in Proposition 2.7. Note that T has at least 3 prime factors. By checking the order of T, we have that $T_v \neq 1$ and hence $N_v \neq 1$. By Proposition 2.1, N has at most two orbits on $V(X)$, that is, $|N_v| = |N|/16p$ or $|N|/32p$. The normality of N in A implies that $N_v \trianglelefteq A_v$.

Assume that $k \geq 2$. Note that $p \geq 5$. Thus, $N \cong \mathrm{A}_5^2$, A_5^3 or A_6^2 for $p = 5$; $N \cong \mathrm{PSL}(2,7)^2$, $\mathrm{PSL}(2,8)^2$, A_7^2, A_8^2 or $\mathrm{PSL}(3,4)^2$ for $p = 7$. Let $N \cong \mathrm{A}_5^2$, A_6^2, A_5^3, $\mathrm{PSL}(2,8)$, A_7^2 or $\mathrm{PSL}(3,4)^2$. Then $|N_v| = |N|/16p$ or $|N|/32p$. By Magma [4], N has no subgroups of such orders, a contradiction. Let $N \cong \mathrm{PSL}(2,7)^2$. Then by Atlas [6], $N_v \cong \mathrm{A}_4 \times F_{21}$, $\mathrm{S}_3 \times F_{21}$. By Proposition 2.2, A_v has no such normal subgroups, a contradiction. Let $N \cong \mathrm{A}_8^2$. Then $|N_v| = 2^8{\cdot}3^4{\cdot}5^2{\cdot}7$ or $2^7{\cdot}3^4{\cdot}5^2{\cdot}7$. By Magma [4], $N_v \cong \mathrm{A}_8 \times ((A_5 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2)$ or $\mathrm{A}_7 \times \mathrm{S}_6$. However, A_v has no such normal subgroups, a contradiction.

Thus, $k = 1$ and N is a non-abelian simple group. Since N has at most two orbits, we have $16p \,\big|\, |N|$. It forces that N is isomorphic to the groups listed in Proposition 2.7 except for

A_5, A_6, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, A_7, $\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,13)$, $\mathrm{PSL}(2,19)$,

$\mathrm{PSL}(2,25)$, $\mathrm{PSL}(2,27)$, $\mathrm{PSL}(2,29)$, $\mathrm{PSL}(2,41)$ and $\mathrm{PSL}(2,71)$.

Note that $p \geq 5$. Since $|N_v| = |N|/16p$ or $|N|/32p$, we have that a Sylow 3-subgroup of N_v is also a Sylow 3-subgroup of N. By Proposition 2.2, a Sylow 3-subgroup of A_v is elementary abelian, and so is that of N_v. With the calculation of Magma [4], $N \ncong \mathrm{PSL}(2,17)$, $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(4,2)$, $\mathrm{PSU}(3,3)$, A_9, A_{10}, $\mathrm{PSU}(3,8)$, $\mathrm{PSp}(6,2)$, M_{12}, J_2, $\mathrm{P}\Omega^+(8,2)$, ${}^2F_4(2)'$, A_{11}, $\mathrm{PSL}(4,4)$, $\mathrm{P}\Omega^-(8,2)$ or $G_2(4)$.

Let $N \cong \mathrm{A}_8$. Then $|N_v| = 2^2{\cdot}3^2{\cdot}7$ or $2{\cdot}3^2{\cdot}7$ for $p = 5$ or $|N_v| = 2^2{\cdot}3^2{\cdot}5$ or $2{\cdot}3^2{\cdot}5$ for $p = 7$. By Magma [4], $|N_v| = 2^2{\cdot}3^2{\cdot}5$ and $N_v \cong \mathrm{A}_5 \times \mathbb{Z}_3$. The normality of N in A implies that N_v is normal in A_v. However, by Proposition 2.2, A_v has no normal subgroup isomorphic $\mathrm{A}_5 \times \mathbb{Z}_3$, a contradiction.

Let $N \cong \mathrm{PSL}(2, 16)$ or $\mathrm{PSL}(2, 31)$. Then $|N_v| = 15$ for $N \cong \mathrm{PSL}(2, 16)$, $|N_v| = 15$ or 30 for $N \cong \mathrm{PSL}(2, 31)$. By Magma [4], $N_v \cong \mathbb{Z}_{15}$ or D_{30}. However, A_v has no such normal subgroup, a contradiction.

Let $N \cong \mathrm{PSL}(2, 49)$. Then $|N_v| = 3{\cdot}5^2{\cdot}7$. By Magma [4], $\mathrm{PSL}(2, 49)$ has no subgroup of order $3{\cdot}5^2{\cdot}7$, a contradiction.

Let $N \cong \mathrm{PSL}(2, 81)$. Then $|N_v| = 3^4{\cdot}5$. By Magma [4], $N_v \cong \mathbb{Z}_3^4 \rtimes \mathbb{Z}_5$. However, A_v has no such normal subgroup, a contradiction.

Let $N \cong \mathrm{PSL}(2, 127)$, $\mathrm{PSL}(3, 4)$, M_{11}, $\mathrm{Sz}(8)$, $\mathrm{PSL}(2, 449)$, $\mathrm{PSL}(2, 2^6)$,. Then $|N_v| = |N|/16p$ or $|N|/32p$. By Magma [4], N has no subgroups of such orders, a contradiction.

Let $N \cong \mathrm{PSU}(3, 4)$. Then $|N_v| = 2^2{\cdot}3{\cdot}5^2$ or $2{\cdot}3{\cdot}5^2$. By Atlas [6], $N_v \cong \mathrm{A}_5 \times \mathbb{Z}_5$ or $\mathbb{Z}_5^2 \rtimes \mathrm{S}_3$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{PSp}(4, 4)$. Then $|N_v| = 2^4{\cdot}3^2{\cdot}5^2$ or $2^3{\cdot}3^2{\cdot}5^2$. By Atlas [6], the only possibility is: $|N_v| = 2^4{\cdot}3^2{\cdot}5^2$ and $N_v \cong \mathrm{A}_5^2$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{PSL}(5, 2)$. Then $|N_v| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$ or $2^5{\cdot}3^2{\cdot}5{\cdot}7$. By Magma [4], the only possibility is: $|N_v| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$ and $N_v \cong \mathrm{PSL}(4, 2)$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{M}_{22}$. Then $|N_v| = 2^2{\cdot}3^2{\cdot}5{\cdot}7$ or $2^3{\cdot}3^2{\cdot}5{\cdot}7$. By Magma [4], the only possibility is: $|N_v| = 2^3{\cdot}3^2{\cdot}5{\cdot}7$ and $N_v \cong \mathrm{A}_7$. Clearly, N has two orbits on $V(X)$ and $p = 11$. By "N/C-Theorem", $A/C_A(N) \lesssim \mathrm{Aut}(N)$. If $C_A(N) \neq 1$, then $C_A(N) \cong C_A(N)N/N \leq A/N$. It forces that $|C_A(N)| \,\big|\, |A_v/N_v|$. By Proposition 2.2, $C_A(N)$ is a $\{2, 3, 5\}$-group. Thus, $C_A(N)$ acting on $V(X)$ has at least p orbits. By Proposition 2.1, $C_A(N)$ is semiregular and $|C_A(N)| \,\big|\, 32p$. It follows that $C_A(N)$ is solvable, which is contrary to our assumption. Thus, $C_A(N) = 1$ and $A \lesssim \mathrm{Aut}(N) \cong \mathrm{M}_{22} \rtimes \mathbb{Z}_2$. The intransitivity of N implies that $A \cong \mathrm{M}_{22} \rtimes \mathbb{Z}_2$. Note that $|V(X)| = 32{\cdot}11 = 352$. By Atlas [6], $\mathrm{M}_{22} \rtimes \mathbb{Z}_2$ has a representation of degree 352, its lengths of suborbits are: $1, 15, 35, 70, 105, 126$. Thus, there is no connected heptavalent orbital graph on $\mathrm{M}_{22} \rtimes \mathbb{Z}_2$, a contradiction. □

Combining Lemmas 4.1, 4.2 and 4.3, we have the complete classification of connected heptavalent symmetric graphs of order $32p$ for each prime p.

Theorem 4.1. *Let p be a prime. Then up to isomorphism, there are only two connected heptavalent symmetric graphs of order $32p$, that is, $\mathcal{G}_{64}$ with $p = 2$ and $\mathcal{G}_{96}$ with $p = 3$.*

5. Conclusion

As is known to all, arc-transitive graphs have much higher symmetries and much larger full automorphism groups, and for prime valent arc-transitive graphs, the structure of their vertex stabilizers can be definitely controlled. Thus, the characterization and classification of such graphs can be achieved, and this reveals not only the local action but also global action of the full automorphism group acting on vertices and arcs. In the paper, we classify the arc-transitive graphs of order $32p$ and valency seven for each prime p. As a natural continuation, could we find and construct infinite families of arc-transitive graphs of order $32p$ with some more larger prime valency? Furthermore, we will classify arc-transitive graphs of order $32p$ and more general prime valencies.

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