Inequalities of DVT-type – the two-dimensional case continued

Barbora Batíková
Department of Mathematics
CULS
Kamýcká 129
165 21 Praha 6 - Suchdol
Czech Republic
batikova@tf.czu.cz

Tomáš J. Kepka
Faculty of Education
Charles University
M. Rettigové 4
116 39 Praha 1
Czech Republic
tomas.kepka@pedf.cuni.cz

Petr C. Němec
Department of Mathematics
CULS
Kamýcká 129
165 21 Praha 6 - Suchdol
Czech Republic
nemec@tf.czu.cz

Abstract. In this note, particular two-dimensional inequalities dealing with two n-tuples of integer numbers under relatively general assumptions are investigated. Moreover, systems of integers for which the equality holds are completely described.

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1. Introduction

In [4], A. Drápal and V. Valent proved that in a finite quasigroup Q of order n the number of associative triples $a(Q) \geq 2n - i(Q) + (\delta_1 + \delta_2)$, where $i(Q)$ is the number of idempotents in Q, i.e., $i(Q) = |\{\, x \in Q \,|\, xx = x \,\}|$, $\delta_1 = |\{\, z \in Q \,|\, zx \neq x$ for all $x \in Q \,\}|$ and $\delta_2 = |\{\, z \in Q \,|\, xz \neq x$ for all $x \in Q \,\}|$. This important result is an easy consequence of the inequality

$$\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) - \sum_{i=1}^{k}(a_i + b_i) \geq 3n - 2k + (r + s),$$

where $n \geq k \geq 0$, $a_1, \dots, a_n, b_1 \dots, b_n$ are non-negative integers such that $\sum a_i = n = \sum b_i$, $a_i \geq 1$ and $b_i \geq 1$ for $1 \leq i \leq k$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$. It should be noted that quasigroups with small $a(Q)$ may have applications in cryptography [5]. The lengthy and complicated proof of this DVT-inequality (inequality of Drápal-Valent type) in [4] is based on highly semantically involved insight.

In [6], a very short elementary arithmetical proof of a more general inequality of this type was found under assumption that $\sum_{i=1}^{n} a_i \geq n$, $\sum_{i=1}^{n} b_i \geq n$. This inequality is two-dimensional in the sense that it works with two n-tuples of integers. The approach in [6] opens a road to investigation of similar DVT-inequalities which could be useful in further investigations of estimates in non-associative algebra and they are also of independent interest. Hence they deserve a thorough examination; however, the research is only at its beginning. In [1] and [2], the one-dimensional case working with one n-tuple of real numbers was investigated. In [3], the investigation of the two-dimensional case working with two n-tuples of integer numbers was begun. The main aim of this note is to show that

$$2\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) \geq 3\sum_{i=1}^{n}(a_i + b_i) + 2r + 2s,$$

where $a_1, \dots, a_n, b_1 \dots, b_n$ are integers such that $\sum_{i=1}^{n} |a_i| \geq n$, $\sum_{i=1}^{n} |b_i| \geq n$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$. Moreover, the case when the equality holds is completely described.

2. The inequalities

Throughout this section, let $n \geq 1$ and $a_1, \dots, a_n, b_1, \dots, b_n$ be integers. Put $\alpha = (a_1, \dots, a_n)$, $\beta = (b_1, \dots, b_n)$, $I = \{1, \dots, n\}$, $A = \{\, i \in I \mid a_i \geq 0, b_i \geq 0, a_i + b_i \geq 3\}$, $B_1 = \{\, i \in I \mid (a_i, b_i) = (2, 0)\}$, $B_2 = \{\, i \in I \mid (a_i, b_i) = (0, 2)\}$, $B_3 = \{\, i \in I \mid (a_i, b_i) = (1, 1)\}$, $B = B_1 \cup B_2 \cup B_3$, $C_1 = \{\, i \in I \mid (a_i, b_i) = (2, -1)\}$, $C_2 = \{\, i \in I \mid (a_i, b_i) = (-1, 2)\}$, $C = C_1 \cup C_2$, $D_1 = \{\, i \in I \mid (a_i, b_i) = (0, 1)\}$, $D_2 = \{\, i \in I \mid (a_i, b_i) = (1, 0)\}$, $D = D_1 \cup D_2$ and $E = \{\, i \in I \mid (a_i, b_i) = (0, 0)\}$. For $X = A, B_1, \dots, E$, denote $\mathrm{x} = |X|$. Further, for integers x, y put $z(0) = 1$, $z(x) = 0$ otherwise and $t(x, y) = 2x^2 + 2y^2 + 2xy - 3x - 3y - 2z(x) - 2z(y)$. Finally, put $z(\alpha) = |\{\, i \in I \mid a_i = 0\}| = \sum_{i=1}^{n} z(a_i)$ and $t(\alpha, \beta) = \sum_{i=1}^{n} t(a_i, b_i)$.

Lemma 2.1. *$a^2 - a - 2z(a) \geq 2|a| - 2$ for every integer a.*

Proof. Obviously, $a^2 - 3a + 2 = (a-1)(a-2) \geq 0$. If $a > 0$ then $z(a) = 0$ and $a^2 - a \geq 2a - 2 = 2|a| - 2$. If $a < 0$ then $z(a) = 0$ and $a^2 \geq |a| = -a > -a - 2$, and hence $a^2 - a - 2z(a) > -2a - 2 = 2|a| - 2$. Finally, if $a = 0$ then $z(a) = 1$ and $a^2 - a - 2z(a) = -2 = 2|a| - 2$. □

Lemma 2.2. *Let $a \geq 1$ and $b \geq 0$ be integers. Then:*

(i) $t(a+1, b) > t(a, b)$.

(ii) *If* c, d *are integers such that* $c \geq a$, $d \geq b$ *and* $c + d > a + b$ *then* $t(c, d) > t(a, b)$.

(iii) *If* $i \in A$ *then* $t(a_i, b_i) \geq t(2, 1) = 5$.

(iv) $t(2, -1) = t(-1, 2) = 3$, $t(1, 0) = t(0, 1) = -3$, $t(2, 0) = t(0, 2) = t(1, 1) = 0$ *and* $t(0, 0) = -4$.

(v) *If* $I = B \cup C \cup D \cup E$ *and* $3\mathrm{c} = 3\mathrm{d} + 4\mathrm{e}$ *then* $t(\alpha, \beta) = 0$.

Proof. We have $t(a{+}1, b){-}t(a, b){=}4a{+}2b{-}1{\geq}4a{-}1{\geq}3$ and the rest is clear. □

Theorem 2.3. *Let* $\sum_{i=1}^n |a_i| \geq n$ *and* $\sum_{i=1}^n |b_i| \geq n$. *Put* $\alpha = (a_1, \dots, a_n)$, $\beta = (b_1, \dots, b_n)$. *Then*

$$2\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_ib_i) \geq 3\sum_{i=1}^{n}(a_i + b_i) + 2z(\alpha) + 2z(\beta),$$

$$2\sum_{i=1}^{n}(a_i + b_i)^2 \geq 2\sum_{i=1}^{n} a_ib_i + 3\sum_{i=1}^{n}(a_i + b_i) + 2z(\alpha) + 2z(\beta)\,.$$

The equalities hold if and only if the following conditions are satisfied:

1. $I = B \cup C \cup D$.
2. $\mathrm{d}_1 \leq \mathrm{c}_1 + \mathrm{c}_2$ *and* $k = 2\mathrm{c}_1 + 2\mathrm{c}_2 + |\mathrm{c}_1 - \mathrm{d}_1| \leq n$.
3. $\mathrm{d}_2 = \mathrm{c}_1 + \mathrm{c}_2 - \mathrm{d}_1$.
4. *If* $\mathrm{c}_1 \geq \mathrm{d}_1$ *then* $\mathrm{b}_2 = \mathrm{b}_1 + |\mathrm{c}_1 - \mathrm{d}_1|$.
5. *If* $\mathrm{c}_1 < \mathrm{d}_1$ *then* $\mathrm{b}_1 = \mathrm{b}_2 + |\mathrm{c}_1 - \mathrm{d}_1|$.
6. $2p \leq n - k$ *and* $\mathrm{b}_3 = n - k - 2p$, *where* $p = \min(\mathrm{b}_1, \mathrm{b}_2)$.

In this case, $\sum_{i=1}^n |a_i| = n = \sum_{i=1}^n |b_i|$.

Proof. Clearly, the inequalities are equivalent to $t(\alpha, \beta) \geq 0$. Denote $I_1 = \{\, i \in I \mid a_i \geq 0, b_i \geq 0\}$, $n_1 = |I_1|$, $I_2 = \{\, i \in I \mid a_i \leq 0, b_i \leq 0\} \setminus \{(0, 0)\}$, $n_2 = |I_2|$, $I_3 = \{\, i \in I \mid a_ib_i < 0\}$ and $n_3 = |I_3|$. For $j = 1, 2, 3$ put $z_j(\alpha) = |\{\, a_i \in I_j \mid a_i = 0\}| = \sum_{i \in I_j} z(a_i)$, $z_j(\beta) = |\{\, b_i \in I_j \mid b_i = 0\}| = \sum_{i \in I_j} z(b_i)$ and $t_j = 2\sum_{i \in I_j} a_i^2 + 2\sum_{i \in I_j} b_i^2 + 2\sum_{i \in I_j} a_ib_i - 3\sum_{i \in I_j} a_i - 3\sum_{i \in I_j} b_i - 2z_j(\alpha) - 2z_j(\beta)$. Then $I = I_1 \cup I_2 \cup I_3$, $n = n_1 + n_2 + n_3$, $z_3(\alpha) = 0 = z_3(\beta)$ and $t(\alpha, \beta) = t_1 + t_2 + t_3$. The proof is divided into nine parts:

(i) First, denote $t_1(\alpha) = \sum_{i \in I_1} a_i^2 - \sum_{i \in I_1} a_i - 2z_1(\alpha)$, $t_1(\beta) = \sum_{i \in I_1} b_i^2 - \sum_{i \in I_1} b_i - 2z_1(\beta)$ and $q_1 = \sum_{i \in I_1}(a_i + b_i)^2 - 2\sum_{i \in I_1} a_i - 2\sum_{i \in I_1} b_i$. By 2.1, we get $t_1(\alpha) \geq 2\sum_{i \in I_1}|a_i| - 2n_1$, $t_1(\beta) \geq 2\sum_{i \in I_1}|b_i| - 2n_1$ and $q_1 = \sum_{i \in I_1}(a_i + b_i - 1)^2 - n_1 \geq \sum_{i \in I_1}(a_i + b_i - 1) - n_1 = \sum_{i \in I_1}|a_i| + \sum_{i \in I_1}|b_i| - 2n_1$. Then $t_1 = t_1(\alpha) + t_1(\beta) + q_1 \geq 3\sum_{i \in I_1}|a_i| + 3\sum_{i \in I_1}|b_i| - 6n_1$.

(ii) Further, we have $t_2 = 2\sum_{i\in I_2} a_i^2 + 2\sum_{i\in I_2} b_i^2 + 2\sum_{i\in I_2} a_ib_i + 3\sum_{i\in I_2}|a_i| + 3\sum_{i\in I_2}|b_i| - 2z_2(\alpha) - 2z_2(\beta) \geq 3\sum_{i\in I_2}|a_i| + 3\sum_{i\in I_2}|b_i| - 2n_2$, since $a_i = 0$ if and only if $b_i \neq 0$ for every $i \in I_2$. If $I_2 \neq \emptyset$ then $t_2 > 3\sum_{i\in I_2}|a_i| + 3\sum_{i\in I_2}|b_i| - 6n_2$.

(iii) If $i \in I_3$ and $a_i > 0$, $b_i < 0$, then $t(a_i, b_i) - 3|a_i| - 3|b_i| + 6 = 2a_i^2 + 2b_i^2 + 2a_ib_i - 6a_i + 6 = 2(b_i^2 + a_ib_i + a_i^2 - 3a_i + 3) = 2((b_i + \frac{a_i}{2})^2 + \frac{3}{4}(a_i - 2))^2 \geq 0$. Thus $t(a_i, b_i) \geq 3|a_i| + 3|b_i| - 6$ and the equality holds if and only if $(a_i, b_i) = (2, -1)$. The case $a_i < 0$, $b_i > 0$ is symmetric. Hence $t_3 \geq 3\sum_{i\in I_3}|a_i| + 3\sum_{i\in I_3}|b_i| - 6n_3$ and the equality holds if and only if $I_3 = C$.

(iv) Finally, $t(\alpha, \beta) = t_1 + t_2 + t_3 \geq 3\sum_{i=1}^n |a_i| + 3\sum_{i=1}^n |b_i| - 6n \geq 0$.

(v) Now, assume that $t(\alpha, \beta) = 0$. Then $\sum_{i=1}^n |a_i| = n = \sum_{i=1}^n |b_i|$, $I_2 = \emptyset$ and $I_3 = C$. Thus $I = A \cup B \cup C \cup D \cup E$.

(vi) First, let $t(\alpha, \beta) = 0$ and $C = \emptyset$. Then $I = I_1$, $\sum_{i=1}^n a_i = n = \sum_{i=1}^n b_i$ and (see (i)) $0 = q_1 = \sum_{i=1}^n (a_i + b_i)^2 - 2\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n (a_i + b_i - 2)^2 + 2\sum_{i=1}^n a_i + 2\sum_{i=1}^n b_i - 4n = \sum_{i=1}^n (a_i + b_i - 2)^2 \geq 0$. Thus $a_i + b_i = 2$ for every $i = 1, \dots, n$, $I = B$ and $A = \emptyset = E$.

(vii) Further, let $t(\alpha, \beta) = 0$, $C \neq \emptyset$ and $E \neq \emptyset$. Take $j \in C$ and $k \in E$. If $j \in C_1$ (i.e., $(a_j, b_j) = (2, -1)$) , put $c_j = 1$, $d_j = 0$, $c_k = 1 = d_k$ and $c_i = a_i$, $d_i = b_i$ otherwise. Denote $\gamma = (c_1, \dots, c_n)$ and $\delta = (d_1, \dots, d_n)$. Then $\sum_{i=1}^n |c_i| = \sum_{i\neq j,k} |a_i| + 1 + 1 = \sum_{i\neq j,k} |a_i| + 2 + 0 = n = \sum_{i\neq j,k} |b_i| + 1 + 0 = \sum_{i\neq j,k} |b_i| + 0 + 1 = \sum_{i=1}^n |d_i|$, and hence $0 \leq t(\gamma, \delta) = \sum_{i\neq j,k} t(a_i, b_i) - 3 + 0 < \sum_{i\neq j,k} t(a_i, b_i) + 3 - 4 = t(\alpha, \beta) = 0$, a contradiction. The proof for $j \in C_2$ is similar. We have proved that if $t(\alpha, \beta) = 0$ and $C \neq \emptyset$ then $E = \emptyset$.

(viii) Now, let $t(\alpha, \beta) = 0$, $C \neq \emptyset$ and $A \neq \emptyset$. Then $E = \emptyset$ by (vii), $t(\alpha, \beta) = \sum_{i\in A} t(a_i, b_i) + 3\mathrm{c} - 3\mathrm{d}$ and $\sum_{i=1}^n |a_i| = \sum_{i\in A} a_i + 2\mathrm{b}_1 + \mathrm{b}_3 + 2\mathrm{c}_1 + \mathrm{c}_2 + \mathrm{d}_2 = n = \mathrm{a} + \mathrm{b}_1 + \mathrm{b}_2 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + \mathrm{d}_1 + \mathrm{d}_2 = \sum_{i=1}^n |b_i| = \sum_{i\in A} b_i + 2\mathrm{b}_2 + \mathrm{b}_3 + \mathrm{c}_1 + 2\mathrm{c}_1 + \mathrm{d}_1$.

In order to obtain $t(\alpha, \beta) = 0$, to each pair (a_i, b_i), $i \in C$, must correspond a pair (a_j, b_j), $j \in D$. Hence $\mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2 > \mathrm{c}_1 + \mathrm{c}_2$ and the remaining $\mathrm{d} - \mathrm{c}$ pairs (a_i, b_i), $i \in D$ (their increment to $t(\alpha, \beta)$ is $-3(\mathrm{d} - \mathrm{c})$) must compensate $\sum_{i\in A} t(a_i, b_i) \geq 5\mathrm{a}$.

Now, suppose that $\mathrm{d}_1 \geq \mathrm{c}_1$ and $\mathrm{d}_2 \geq \mathrm{c}_2$. Then for every $i \in C_1$ we can choose $j_i \in D_1$ and for every $i \in C_2$ we can choose $j_i \in D_2$. Put $K = I \setminus \{i, j_i \,|\, i \in C\}$. Then $K = \{\, i \in K \,|\, a_i \geq 0,\, b_i \geq 0 \,\}$, $|K| = n - 2\mathrm{c}$, $\sum_{i\in K} |a_i| = \sum_{i\in A} a_i + 2\mathrm{b}_1 + \mathrm{b}_3 + d_2 - \mathrm{c}_2 = |K| = \sum_{i\in A} a_i + 2\mathrm{b}_1 + \mathrm{b}_3 + 2\mathrm{c}_1 + \mathrm{c}_2 + \mathrm{d}_2 - 2\mathrm{c}_1 - 2\mathrm{c}_2 = n - 2\mathrm{c} = |K| = \sum_{i\in A} b_i + 2\mathrm{b}_2 + \mathrm{b}_3 + \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 - 2\mathrm{c}_1 - 2\mathrm{c}_2 = \sum_{i\in A} b_i + 2\mathrm{b}_2 + \mathrm{b}_3 + \mathrm{d}_1 - \mathrm{c}_1 = \sum_{i\in K} |b_i|$ and $0 = t(\alpha, \beta) = \sum_{i\in K} t(a_i, b_i) + \sum_{i\in C_1}(t(a_i, b_i) + t(a_{j_i}, b_{j_i})) + \sum_{i\in C_2}(t(a_i, b_i) + t(a_{j_i}, b_{j_i})) = \sum_{i\in K} t(a_i, b_i) + \mathrm{c}_1(t(2, -1) + t(0, 1)) + \mathrm{c}_2(t(-1, 2) + t(1, 0)) = \sum_{i\in K} t(a_i, b_i)$. By (vi) (for K instead of I) we obtain $K = \{\, i \in K \,|\, a_i + b_i = 2 \,\}$, a contradiction with $\emptyset \neq A \subseteq K$.

Further, suppose that $\mathrm{d}_1 > \mathrm{c}_1$ and $\mathrm{d}_2 < \mathrm{c}_2$. Again, for every $i \in C_1$ we can choose $j_i \in D_1$, and for every $i \in D_2$ we can choose $j_i \in C_2$. Taking into account that the increment of pairs $(2, -1), (0, 1)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 2 and $t(2, -1) + t(0, 1) = 3 - 3 = 0$, the increment of pairs $(1, 0), (-1, 2)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 2 and $t(1, 0) + t(-1, 2) = -3 + 3 = 0$, the increment of pairs $(2, 0), (0, 2)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 2 and $t(2, 0) = 0 = t(0, 2)$, and the increment of pair $(1,1)$ to $\sum_{i=1}^n |a_i|$, $\sum_{i=1}^n |b_i|$ and n is 1 and $t(1,1) = 0$, we may assume without loss of generality that $\mathrm{b}_3 = 0$, $\min(\mathrm{b}_1, \mathrm{b}_2) = 0$, $\mathrm{c}_1 = 0$ and $\mathrm{d}_2 = 0$. Of course, $\mathrm{d}_1 > \mathrm{c}_2$ and $C = C_2$, $D = D_1$. Now, for each $i \in C_2$ we can choose $j_i \in D_1$. Put $L = \{\, j_i \mid i \in C_2 \,\} \subseteq D_1$ and $K = I \setminus (C_2 \cup L)$. For every $i \in C_2$ put $c_i = 0$, $d_i = 2$, $c_{j_i} = 1$ and $d_{j_i} = 1$. Further, put $c_i = a_i$, $d_i = b_i$ for every $i \in K$ and $\gamma = (c_1, \dots, c_n)$, $\delta = (d_1, \dots, d_n)$. Then $(c_i, d_i) = (0,2)$ for every $i \in C_2$, $(c_i, d_i) = (1,1)$ for every $i \in L$ and $c_i \geq 0$, $d_i \geq 0$ for every $i \in I$, $\sum_{i=1}^n |c_i| = \sum_{i \in K} |a_i| + \mathrm{c}_2 = \sum_{i=1}^n |a_i| = n = \sum_{i=1}^n |b_i| = \sum_{i \in K} |b_i| + \mathrm{c}_2 + 2\mathrm{c}_2 = \sum_{i=1}^n |d_i|$ and $t(\gamma, \delta) = \sum_{i \in K} t(a_i, b_i) + \sum_{i \in C_2} t(c_i, d_i) + \sum_{i \in L} t(c_i, d_i)) = \sum_{i \in K} t(a_i, b_i) = \sum_{i \in K} t(a_i, b_i) + 3\mathrm{c}_2 - 3\mathrm{c}_2 = \sum_{i \in K} t(a_i, b_i) + \sum_{i \in C_2} t(a_i, b_i) + \sum_{i \in L} t(a_i, b_i) = \sum_{i=1}^n t(a_i, b_i) = 0$. By (vi) for γ, δ, we obtain $K = \{\, i \in K \mid a_i + b_i = 2 \,\}$, a contradiction with $\emptyset \neq A \subseteq K$. The proof for $\mathrm{d}_1 < \mathrm{c}_1$, $\mathrm{d}_2 > \mathrm{c}_2$ is similar. We have proved that if $t(\alpha, \beta) = 0$ and $C \neq \emptyset$ then $A = \emptyset$.

(ix) Finally, let $t(\alpha, \beta) = 0$. By (vi), (vii) and (viii), we have $A = \emptyset = E$, $I = B \cup C \cup D$, $0 = t(\alpha, \beta) = 3\mathrm{c} - 3\mathrm{d}$ and $\mathrm{c} = \mathrm{c}_1 + \mathrm{c}_2 = \mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2$. Hence $\mathrm{d}_1 \leq \mathrm{c}_1 + \mathrm{c}_2$ and $\mathrm{d}_2 = \mathrm{c}_1 + \mathrm{c}_2 - \mathrm{d}_1$. Further, $\sum_{i=1}^n |a_i| = 3\mathrm{c}_1 + 2\mathrm{c}_2 - \mathrm{d}_1 + 2\mathrm{b}_1 + \mathrm{b}_3 = n = 2\mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{b}_1 + \mathrm{b}_2 + \mathrm{b}_3 = \sum_{i=1}^n |b_i| = \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 + 2\mathrm{b}_2 + \mathrm{b}_3$, and hence $\mathrm{c}_1 + b_1 = \mathrm{d}_1 + \mathrm{b}_2$. If $\mathrm{c}_1 \geq \mathrm{d}_1$ then $\mathrm{b}_2 = b_1 + |\mathrm{c}_1 - \mathrm{d}_1|$, and if $\mathrm{c}_1 < d_1$ then $\mathrm{b}_1 = \mathrm{b}_2 + |\mathrm{c}_1 - \mathrm{d}_1|$. As $|C \cup D| = 2\mathrm{c}_1 + 2\mathrm{c}_2$, we obtain $k = 2\mathrm{c}_1 + 2\mathrm{c}_2 + |\mathrm{c}_1 - \mathrm{d}_1| \leq n$. Now, denote $p = \min(\mathrm{b}_1, \mathrm{b}_2)$. Then $2p \leq n - k$ and $\mathrm{b}_3 = n - k - 2p$. Indeed, if $\mathrm{c}_1 \geq \mathrm{d}_1$ then $n - k = n - 2\mathrm{c}_1 - 2\mathrm{c}_2 - \mathrm{c}_1 + \mathrm{d}_1 = \mathrm{b}_1 + \mathrm{b}_2 + \mathrm{b}_3 - \mathrm{c}_1 + \mathrm{d}_1 = \mathrm{b}_1 + \mathrm{b}_1 + \mathrm{b}_3 = 2p + \mathrm{b}_3$. Thus $2p \leq n - k$ and $\mathrm{b}_3 = n - k - 2p$. The proof in case $\mathrm{c}_1 < \mathrm{d}_1$ is similar.

Conversely, assume that the conditions (1) – (6) are satisfied. If $\mathrm{c}_1 \geq \mathrm{d}_1$ then $\sum_{i=1}^n |a_i| = 3\mathrm{c}_1 + 2\mathrm{c}_2 - \mathrm{d}_1 + 2\mathrm{b}_1 + \mathrm{b}_3 = 3\mathrm{c}_1 + 2\mathrm{c}_2 - \mathrm{d}_1 + 2p + n - k - 2p = 2\mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{c}_1 - \mathrm{d}_1 + n - k = n$ and $\sum_{i=1}^n |b_i| = \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 + 2\mathrm{b}_2 + \mathrm{b}_3 = \mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{d}_1 + 2p + 2(\mathrm{c}_1 - \mathrm{d}_1) + n - k - 2p = 2\mathrm{c}_1 + 2\mathrm{c}_2 + \mathrm{c}_1 - \mathrm{d}_1 + n - k = n$. The proof for $\mathrm{c}_1 < \mathrm{d}_1$ is similar. Finally, $t(\alpha, \beta) = 0$ by 2.2(v). □

Remark 2.4. If $\sum_{i=1}^n |a_i| + \sum_{i=1}^n |b_i| \geq 2n$ then the inequalities in Theorem 2.3 hold.

Theorem 2.5. *If $\sum_{i=1}^n a_i \geq n$ and $\sum_{i=1}^n b_i \geq n$ then the inequalities in Theorem 2.3 hold and the equalities hold if and only if $I = B$, $2\mathrm{b}_1 \leq n$, $\mathrm{b}_2 = \mathrm{b}_1$ and $\mathrm{b}_3 = n - 2\mathrm{b}_1$.*

Proof. The inequalities follow from Theorem 2.3. Now, suppose that $t(\alpha, \beta) = 0$. Then $\sum_{i=1}^n |a_i| = n = \sum_{i=1}^n |b_i|$. If $C \neq \emptyset$ then $\sum_{i=1}^n a_i < \sum_{i=1}^n |a_i| = n$ or $\sum_{i=1}^n b_i < \sum_{i=1}^n |b_i| = n$, a contradiction. Thus $C = \emptyset$ and the rest follows from Theorem 2.3 and its proof. □

Remark 2.6. (i) The situation $\sum_{i=1}^n |a_i| \geq n$, $\sum_{i=1}^n |b_i| \geq n$, $t(\alpha, \beta) = 0$ is completely described by conditions (1) – (6). In order to find all such pairs α, β for given n, choose non-negative integers $\mathrm{c}_1, \mathrm{c}_2, \mathrm{d}_1, p$ such that $\mathrm{d}_1 \leq \mathrm{c}_1 + \mathrm{c}_2$, $k = 2\mathrm{c}_1 + 2\mathrm{c}_2 + |\mathrm{c}_1 - \mathrm{d}_1| \leq n$ and $2p \leq n - k$, calculate $\mathrm{d}_2 = \mathrm{c}_1 + \mathrm{c}_2 - \mathrm{d}_1$, $\mathrm{b}_1 = p$ and $\mathrm{b}_2 = p + |\mathrm{c}_1 - \mathrm{d}_1|$ if $\mathrm{c}_1 \geq \mathrm{d}_1$, $\mathrm{b}_2 = p$ and $\mathrm{b}_1 = p + |\mathrm{c}_1 - \mathrm{d}_1|$ if $\mathrm{c}_1 < \mathrm{d}_1$,

$b_3 = n - k - 2p$ and take c_1 pairs (2,-1), c_2 pairs (-1,2), d_1 pairs (0,1), d_2 pairs (1,0), b_1 pairs (2,0), b_2 pairs (0,2) and b_3 pairs (1,1).

(ii) For instance, for $n = 17$ choose, e.g., $c_1 = 3$, $c_2 = 2$, $d_1 = 4$ and $p = 2$. Then $d_2 = 1$, $b_2 = 2$, $b_1 = 3$, $b_3 = 2$ and we obtain one type of α, β. In this way, to each choice of c_1, c_2, d_1, p satisfying $d_1 \leq c_1 + c_2$, $k = 2c_1 + 2c_2 + |c_1 - d_1| \leq n$ and $2p \leq n - k$ corresponds one type of α, β such that $\sum_{i=1}^{n} |a_i| \geq n$, $\sum_{i=1}^{n} |b_i| \geq n$ and $t(\alpha, \beta) = 0$. All $n!$ pairs α, β of this type can be obtained by permutations of I.

(iii) By Theorem 2.5, the situation $\sum_{i=1}^{n} a_i \geq n$, $\sum_{i=1}^{n} b_i \geq n$, $t(\alpha, \beta) = 0$ is completely described.

(iv) For instance, for $n = 5$ choose, e.g., $p = 2$. Then we obtain one type of α, β, namely 2 pairs (2,0), 2 pairs (0,2) and one pair (1,1). In this way, to each choice of p such that $2p \leq n$ corresponds one type of α, β such that $\sum_{i=1}^{n} a_i \geq n$, $\sum_{i=1}^{n} b_i \geq n$ and $t(\alpha, \beta) = 0$, namely p pairs (2,0), p pairs (0,2) and $n - 2p$ pairs (1,1). All $n!$ pairs of this type can be obtained by permutations of I.

3. Conclusions

In the paper, two relatively complicated inequalities concerning two n-tuples of integers are proved and the case when the equality holds is solved. Inequalities of similar type already proved useful in obtaining some estimates of the number of non-associative triples in quasigroups and hence the investigation of such inequalities can lead to further applications.

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