The ranks of classes and nX-complementary generations of the Tits group ${}^2\mathrm{F}_4(2)'$

Ayoub B. M. Basheer
School of Mathematical and Computer Sciences
University of Limpopo (Turfloop)
P Bag X1106, Sovenga 0727
South Africa
and
Mathematics Program
Faculty of Education and Arts
Sohar University
P. O. Box 311, Sohar
Oman
ayoub.basheer@ul.ac.za

Malebogo J. Motalane
School of Mathematical and Computer Sciences
University of Limpopo (Turfloop)
P Bag X1106, Sovenga 0727
South Africa
john.motalane@ul.ac.za

Mahlare G. Sehoana*
School of Mathematical and Computer Sciences
University of Limpopo (Turfloop)
P Bag X1106, Sovenga 0727
South Africa
mahlare.sehoana@ul.ac.za

Thekiso T. Seretlo
School of Mathematical and Statistical Sciences
PAA Focus Area
North-West University (Mahikeng)
P. Bag X2046, Mmabatho 2790
South Africa
thekiso.seretlo@nwu.ac.za

Abstract. Let G be a finite non-abelian simple group. The *rank* of non-trivial conjugacy class X of G, denoted by $\mathrm{rank}(G{:}X)$, is defined to be the minimal number of elements of X generating G. Also, a group G is said to be nX-complementary generated if given an arbitrary non-identity element $x \in G$ then there exists an element $y \in nX$ such that $G = \langle x, y\rangle$. In this paper we establish the ranks of all the conjugacy classes of

*. Corresponding author

the Tits group ${}^2\mathrm{F}_4(2)'$ and also classify all the non-trivial conjugacy classes of ${}^2\mathrm{F}_4(2)'$ whether they are complementary generators of ${}^2\mathrm{F}_4(2)'$ or not.

Keywords: conjugacy classes, nX-complementary generation, rank, structure constant, Tits group.

MSC 2020: 20C15, 20D06.

1. Introduction

A finite group G can be generated in many different ways. For example the probabilistic generation, $\frac{3}{2}$-generation, (p, q, r)-generations, ranks of non-trivial classes of G, nX-complementary generation and many other methods. Generation of finite groups by suitable subsets has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [42] for details). Also Di Martino et al. [32] established a useful connection between generation of groups by conjugate elements and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [40], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions.

A finite group G is said to be (l, m, n)-*generated* if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of lX in G containing elements of order l. The same applies to $[y]$ and $[z]$. Since a group G can be generated by a number of elements in a given conjugacy class X, there is a considerable interest of finding the minimal number of elements of X generating G. This minimal number is given as the rank of X in G and is denoted by $\mathrm{rank}(G{:}X)$.

Moori in the articles [33, 34] and [36], computed the ranks of involution classes of the Fischer sporadic simple group Fi_{22}. Furthermore, Moori and Basheer in [8] computed the rank of the class $3A$ of A_n, $n \geq 5$. They proved by mathematical induction that $\mathrm{rank}(A_n{:}3A)$ is $\frac{n-1}{2}$ if n is odd and is $\frac{n}{2}$ if n is even. Ali and Ibrahim in [1, 2, 3] determined the ranks of Conway group Co_1, the Higman-Sims group HS, McLaughlin group McL, Conway's sporadic simple groups Co_2 and Co_3. In 2008, Ali and Moori [4] established the ranks of conjugacy classes of the Janko groups J_1, J_2, J_3 and J_4. Recently, Motalane [38] computed the ranks of the classes of the Mathieu group M_{23} and the Alternating group A_{11}.

For a non-trivial conjugacy class nX of a finite non-abelian group G, we say that G is nX-**complementary generated** if for any $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. We say y is a complementary. The motivation of studying this kind of generation comes from a conjecture by Brenner-Guralnick-Wiegold [20] that every finite simple group can be generated by an arbitrary non-trivial element together with another suitable element.

In a series of papers [5, 24, 26, 27, 28, 30, 35] and [37], the nX-complementary generations of the sporadic simple groups Th, Co_1, J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22} have been investigated.

The aim of this paper is two fold. Firstly, we intend to establish the ranks of all nontrivial conjugacy classes of an exceptional group of Lie type, namely, the Tits group ${}^2\mathrm{F}_4(2)'$. Secondly, we establish all the nX-complementary generations of ${}^2\mathrm{F}_4(2)'$, where nX is a non-trivial conjugacy class of elements of order n as in the Atlas [23]. We follow the methods used in the papers [6, 7, 9, 10, 11, 12, 13,14] and [15]. Note that, in general, if G is a $(2, 2, n)$-generated group then G is a dihedral group and therefore G is not simple. Also by [22], if G is a non-abelian (l, m, n)-generated group then either $G \cong A_5$ or $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$. Thus, for our purpose of establishing the nX-complementary generations of $G = {}^2\mathrm{F}_4(2)'$, the only cases we need to consider are when $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$.

The main results in this paper can be summarized in Theorems 1.1 and 1.2.

Theorem 1.1. *Let nX be a non-trivial conjugacy class of the exceptional Tits group ${}^2\mathrm{F}_4(2)'$. Then*

1. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) = \mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) = 3$,
2. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}nX) = 2$ *for all* $nX \notin \{2A, 2B\}$.

Theorem 1.2. *The group ${}^2\mathrm{F}_4(2)'$ is nX-complementary generated if and only if $n \geq 3$.*

The group ${}^2\mathrm{F}_4(2)'$ as outlined in the Atlas [23] is a simple group of order $17971200 = 2^{11} \times 3^3 \times 5^2 \times 13$. It has exactly 22 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Table 1 depicts representatives of the maximal subgroups of ${}^2\mathrm{F}_4(2)'$ and their orders.

Table 1: Maximal subgroups of ${}^2\mathrm{F}_4(2)'$

Maximal Subgroup	Order
$L_3(3):2 = M_1$	$11232 = 2^5 \times 3^3 \times 13$
$L_3(3):2 = M_2$	$11232 = 2^5 \times 3^3 \times 13$
$2^{\cdot}(2^8){:}5{:}4 = M_3$	$10240 = 2^{11} \times 5$
$L_2(25) = M_4$	$7800 = 2^3 \times 3 \times 5^2 \times 13$
$2^{2\cdot}(2^8){:}S_3 = M_5$	$6144 = 2^{11} \times 3$
$A_6^{\cdot}2^2 = M_6$	$1440 = 2^5 \times 3^2 \times 5$
$A_6^{\cdot}2^2 = M_7$	$1440 = 2^5 \times 3^2 \times 5$
$5^2{:}4A_4 = M_8$	$1200 = 2^4 \times 3 \times 5^2$

Using Equation (4) on GAP [31], we calculated the values of $h(g, M_i)$, where g is a representative of a non-trivial conjugacy class of ${}^2\mathrm{F}_4(2)'$ and over all the maximal subgroups M_i of ${}^2\mathrm{F}_4(2)'$. We list these values in Table 2.

Table 2: The values $h(g, M_i)$, $1 \leq i \leq 8$, for non-identity classes and maximal subgroups of ${}^2\mathrm{F}_4(2)'$

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8
$2A$	0	0	91	0	45	256	256	256
$2B$	48	48	27	64	61	80	80	32
$3A$	7	7	0	9	9	6	6	18
$4A$	4	4	15	16	13	12	12	8
$4B$	0	0	3	0	5	0	0	0
$4C$	4	4	7	0	5	12	12	8
$5A$	0	0	5	4	0	5	5	1
$6A$	3	3	0	1	1	2	2	2
$8A$	0	0	3	0	1	0	0	0
$8B$	0	0	3	0	1	0	0	0
$8C$	2	2	1	0	1	2	2	0
$8D$	2	2	1	0	1	2	2	0
$10A$	0	0	1	0	0	1	1	1
$12A$	1	1	0	1	1	0	0	2
$12B$	1	1	0	1	1	0	0	2
$13A$	1	1	0	3	0	0	0	0
$13B$	1	1	0	3	0	0	0	0
$16A$	0	0	1	0	1	0	0	0
$16B$	0	0	1	0	1	0	0	0
$16C$	0	0	1	0	1	0	0	0
$16D$	0	0	1	0	1	0	0	0

2. Preliminaries

Let G be a finite group and $C_1, C_2, \ldots, C_k$ (not necessarily distinct) for $k \geq 3$ be conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \ldots \times C_{k-1}$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G using Equation (1),

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{1}$$

Also, for a fixed $g_k \in C_k$, we denote by $\Delta^*_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ satisfying

$$g_1 g_2 \ldots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \ldots, g_{k-1} \rangle. \tag{2}$$

Definition 2.1. *If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, then the group G is said to be $(C_1, C_2, \ldots, C_k)$-**generated**.*

Furthermore, if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \ldots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \ldots, h_{k-1})$ which are in $C_1 \times C_2 \times \ldots \times C_{k-1}$ such that

$$h_1 h_2 \ldots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \ldots, h_{k-1}\rangle \leq H. \tag{3}$$

The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \cap C_i$.

Lastly, for conjugacy classes $c_1, c_2, \ldots, c_k$ of a proper subgroup H of G and a fixed $g_k \in c_k$, let $\Sigma^*_H(c_1, c_2, \ldots, c_k)$ represents the number of tuples $(h_1, h_2, \ldots, h_{k-1}) \in c_1 \times c_2 \times \ldots \times c_{k-1}$ such that $h_1 h_2 \ldots h_{k-1} = g_k$ and $\langle h_1, h_2, \ldots, h_{k-1}\rangle = H$.

When it is clear from the context which conjugacy classes of H are considered, we will use the notation $\Sigma(H)$ and $\Sigma^*(H)$ to denote $\Sigma_H(c_1, c_2, \ldots, c_k)$ and $\Sigma^*_H(c_1, c_2, \ldots, c_k)$, respectively.

Theorem 2.1. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular,*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|}, \tag{4}$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof of Theorem 2.1. See [27] and [29, Theorem 2.1].

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta^*_G(C_1, C_2, \ldots, C_k)$, namely $\Delta^*_G(C_1, C_2, \ldots, C_k) \geq \Theta_G(C_1, C_2, \ldots, C_k)$, where

$$\Theta_G(C_1, C_2, \ldots, C_k) = \Delta_G(C_1, C_2, \ldots, C_k) - \sum h(g_k, H)\Sigma^*_H(C_1, C_2, \ldots, C_k),$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \ldots, C_k$.

In the following, Theorem 2.2 and Lemma 2.2 are in many cases useful in determining whether G is nX-complementary generated, while Lemma 2.1 is in some cases useful in establishing non-generation for finite groups.

Theorem 2.2 ([8, Lemma 2.5])**.** *Let G be a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.*

Lemma 2.1 (e.g., see Ali and Moori [4] or Conder et al. [21])**.** *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$ and $g_k \in C_k$ then $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$ and therefore, G is not $(C_1, C_2, \ldots, C_k)$-generated.*

Lemma 2.2 ([25])**.** *If G is sY-complementary generated and $(rX)^n = sY$ then G is rX -complementary generated.*

Proof of Theorem 2.2. Let rX and sY be non-trivial conjugacy classes of G such that $(rX)^n = sY$ for some integer n. If G is not rX-complementary generated then there exits an element x of prime order such that $\langle x, y\rangle < G$ for all $y \in rX$. Since $x, y^n \in \langle x, y\rangle$, it follows that $\langle x, y^n\rangle \leq \langle x, y\rangle < G$ for all $y^n \in sY$. Thus the results follows by method of contrapositive.

Lemma 2.3 (Ali and Moori [4] or Conder et al. [21])**.** *Let G be a finite simple group such that G is (lX, mY, nZ)-generated. Then G is $(\underbrace{lX, lX, \cdots, lX}_{m-times}, (nZ)^m)$-generated.*

The following result is due to Scott ([21] and [39]).

Theorem 2.3 (Scott's Theorem)**.** *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G such that $g_1 g_2 \ldots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with* $\dim \mathbb{V} = n \geq 2$. *Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i\rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i\rangle}$ being the trivial character of the cyclic group $\langle g_i\rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([25]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^G_{\langle g_i\rangle}, \mathbf{1}_{\langle g_i\rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i\rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{5}$$

The following proposition gives a criterion for G to be nX-complementary generated or not.

Proposition 2.1 ([25])**.** *A finite non-abelian group G is nX-complementary generated if and only if for each conjugacy class pY of G, where p is prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that G is $(pY, nX, t_{pY}Z)$-generated. Moreover, if G is a finite simple group then G is not $2X$-complementary generated for any conjugacy class of involutions.*

3. The ranks of non-trivial conjugacy classes of ${}^2\mathrm{F}_4(2)'$

We start our investigation on the ranks of the non-trivial classes of ${}^2\mathrm{F}_4(2)'$ by looking at the two classes of involutions, namely $2A$ and $2B$. It is well-known that two involutions generate a dihedral group. Thus the lower bound for the rank of a class of involutions in a finite simple group G is 3.

Lemma 3.1. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 3A, 13A)$*-generated.*

Proof. Direct computations on GAP show that $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A) = 13$. We deduce from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ will be involved in the calculations of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A)$. Hence, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 3A, 13A) = 13$ and generation occurs immediately. □

Proposition 3.1. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) = 3$.

Proof. Since by Lemma 3.1, ${}^2\mathrm{F}_4(2)'$ is $(2A, 3A, 13A)$-generated group, it follows by applications of Lemma 2.3 that ${}^2\mathrm{F}_4(2)'$ is $(2A, 2A, 2A, (13A)^3)$-generated, i.e., $(2A, 2A, 2A, 13A)$-generated group. Thus, $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) \leq 3$. Since $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) \notin \{1, 2\}$, it follows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2A) = 3$. □

Lemma 3.2. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2B, 3A, 16A)$*-generated.*

Proof. Direct calculations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) = 112$. We deduce from Table 2 that only one maximal subgroup of ${}^2\mathrm{F}_4(2)'$, namely M_5, has a nonempty intersection with the triple $(2B, 3A, 16A)$. We further calculated $\Sigma(M_5) = 0 + 0 + 0 + 32 = 32$ and $h(g, M_5) = 1$ for $g \in 16A$. Thus,

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) \quad \geq \quad \Delta_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) - 1 \cdot \Sigma(M_5) = 112 - 32 = 80.$$

We find $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 3A, 16A) > 0$ showing that ${}^2\mathrm{F}_4(2)'$ is $(2B, 3A, 16A)$-generated. □

Proposition 3.2. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) = 3$.

Proof. By Lemma 3.2, ${}^2\mathrm{F}_4(2)'$ is $(2B, 3A, 16A)$-generated. We see from the Atlas [41] that $(16A)^3 = 16A$. Now, application of Lemma 2.3 reveals that ${}^2\mathrm{F}_4(2)'$ is $(2B, 2B, 2B, (16A)^3)$-generated, i.e., $(2B, 2B, 2B, 16A)$- generated. Thus, $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) \leq 3$. Since $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) \notin \{1, 2\}$, it follows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}2B) = 3$. □

Proposition 3.3. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}3A) = 2$.

Proof. Since by Lemma 3.1, ${}^2\mathrm{F}_4(2)'$ is $(2A, 3A, 13A)$-generated group, it follows by applications of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $(3A, 3A, (13A)^2)$-generated. Since $(13A)^2 = 13B$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(3A, 3A, 13B)$-generated. This shows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}3A) = 2$. □

Remark 3.1. The result of Proposition 3.3 can be obtained directly from Proposition 12 of [16].

Lemma 3.3. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 4A, 12A)$*-generated.*

Proof. We calculated, with GAP, the structure constant $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = 12$. We deduce from Table 2 that only the maximal subgroups M_5 and M_8 of ${}^2\mathrm{F}_4(2)'$ have nonempty intersection with the triple $(2A, 4A, 12A)$. Further calculations with GAP reveal that $\Sigma(M_5) = 0 = \Sigma(M_8)$. We see from Table 2 that $h(g, M_5) = 1$ and $h(g, M_8) = 2$ for $g \in 12A$. Therefore,

$$\begin{aligned}\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) &= \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) - 1 \cdot \Sigma^*(M_5) - 2 \cdot \Sigma^*(M_8) \\ &= 12 - 0 - 0 = 12\end{aligned}$$

showing that ${}^2\mathrm{F}_4(2)'$ is $(2A, 4A, 12A)$-generated. □

Proposition 3.4. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}4A) = 2$.

Proof. Applying Lemma 2.2 to the result in Lemma 3.3 we see that ${}^2\mathrm{F}_4(2)'$ is $(4A, 4A, (12A)^2)$-generated. Since $(12A)^2 = 12A$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(4A, 4A, 12A)$-generated and the result follows. □

Lemma 3.4. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 4C, 13B)$*-generated.*

Proof. Calculations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13B) = 26$. We see from Table 2 that all the maximal subgroups of ${}^2\mathrm{F}_4(2)'$ have an empty intersection with the triple $(2A, 4C, 13B)$. Thus, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13B) = \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13B)$ $= 26$ showing that ${}^2\mathrm{F}_4(2)'$ is $(2A, 4C, 13B)$-generated. □

Proposition 3.5. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}4C) = 2$.

Proof. Since by Lemma 3.4, ${}^2\mathrm{F}_4(2)'$ is $(2A, 4C, 13B)$-generated group, it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $(4C, 4C, (13B)^2)$-generated. Since $(13B)^2 = 13A$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(4C, 4C, 13A)$-generated. We thus conclude that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}4C) = 2$. □

Lemma 3.5. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $(2A, 5A, 13B)$*-generated.*

Proof. See Proposition 9 of [16]. □

Proposition 3.6. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}5A) = 2$.

Proof. Since by Lemma 3.5, ${}^2\mathrm{F}_4(2)'$ is $(2A, 5A, 13B)$-generated group, it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $(5A, 5A, (13B)^2)$-generated. Since $(13B)^2 = 13A$ it follows that ${}^2\mathrm{F}_4(2)'$ is $(5A, 5A, 13A)$-generated. This shows that $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}5A) = 2$. □

Remark 3.2. Alternatively, the result of Proposition 3.6 follows immediately from either Proposition 16 or 17 of [16].

Proposition 3.7. *Let $T = \{4B, 8A, 8B, 10A\}$. For all conjugacy classes $pX \in T$ we have* $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}pX) = 2$.

Proof. We achieve the result by showing that ${}^2\mathrm{F}_4(2)'$ is $(pX, pX, 13A)$-generated for all conjugacy classes $pX \in T = \{4B, 8A, 8B, 10A\}$. We observe from Table 2 that for all $pX \in T$, all the maximal subgroups of ${}^2\mathrm{F}_4(2)'$ have an empty intersection with the triple $(pX, pX, 13A)$. Thus, we will have $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, pX, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(pX, pX, 13A)$. Now, direct computations with GAP yield

- $\Delta_{{}^2\mathrm{F}_4(2)'}(4B, 4B, 13A) = 1222$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(8A, 8A, 13A) = 13624$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(8B, 8B, 13A) = 13624$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(10A, 10A, 13A) = 17000$.

Since all of the structure constants calculated above are greater than zero, it follows that $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, pX, 13A) > 0$ for all $pX \in T$. Therefore, ${}^2\mathrm{F}_4(2)'$ is $(pX, pX, 13A)$-generated for all conjugacy classes $pX \in T$ and thus

$$\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}pX) = 2. \qquad \square$$

Lemma 3.6. *The group ${}^2\mathrm{F}_4(2)'$ is $(2A, 13X, 13Y)$-generated for $X \in \{A, B\}$.*

Proof. The proof is given in Proposition 10 of [16]. □

Proposition 3.8. $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}13X) = 2$ *for* $X \in \{A, B\}$.

Proof. The proof follows similarly from the proofs of Propositions 3.3 and 3.6. Alternatively, the result follows immediately from Proposition 19 of [16]. □

Proposition 3.9. *Let $S = \{6A, 12A, 12B\}$. For all conjugacy classes $qX \in S$ we have* $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}qX) = 2$.

Proof. We achieve the result by showing that ${}^2\mathrm{F}_4(2)'$ is $(qX, qX, 16A)$-generated for all conjugacy classes $qX \in S = \{6A, 12A, 12B\}$. Table 2 shows that only the maximal subgroup M_5 has a nonempty intersection with the triple $(qX, qX, 16A)$. Direct computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(qX, qX, 16A) = 124416$ and $\Sigma(M_5) = 0$ for all $qX \in S$. With $h(g, M_5) = 1$ for $g \in 16A$ we find that

$$\begin{aligned} \Delta^*_{{}^2\mathrm{F}_4(2)'}(qX, qX, 16A) &= \Delta_{{}^2\mathrm{F}_4(2)'}(qX, qX, 16A) - 1 \cdot \Sigma(M_5) \\ &= 124416 - 0 = 124416. \end{aligned}$$

Thus, the group ${}^2\mathrm{F}_4(2)'$ is $(qX, qX, 16A)$-generated, hence $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}qX) = 2$ for all $qX \in S$. □

Proposition 3.10. *Let* $R = \{8C, 8D, 16A, 16B, 16C, 16D\}$*. For all conjugacy classes* $rX \in R$ *we have* $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}rX) = 2$.

Proof. We obtain the result by showing that ${}^2\mathrm{F}_4(2)'$ is $(rX, rX, 16A)$-generated for all conjugacy classes $rX \in R = \{8C, 8D, 16A, 16B, 16C, 16D\}$. Table 2 shows that only the maximal subgroups M_3 and M_5 have a nonempty intersection with the triple $(rX, rX, 16A)$. Direct computations with GAP yield $\Delta_{2\,\mathrm{F4(2)'}}(rX, rX, 16A) = 69888$ and $\Sigma(M_3) = \Sigma(M_5) = 0$ for all $rX \in R$. We see from Table 2 that $h(g, M_3) = h(g, M_5) = 1$ for $g \in 16A$. Therefore,

$$\begin{aligned}\Delta^*_{{}^2\mathrm{F}_4(2)'}(rX, rX, 16A) &= \Delta_{{}^2\mathrm{F}_4(2)'}(rX, rX, 16A) - 1 \cdot \Sigma(M_3) - 1 \cdot \Sigma(M_5) \\ &= 69888 - 0 - 0 = 69888.\end{aligned}$$

Thus, the group ${}^2\mathrm{F}_4(2)'$ is $(rX, rX, 16A)$-generated, hence $\mathrm{rank}({}^2\mathrm{F}_4(2)'{:}rX) = 2$ all $rX \in R$. □

The proof of Theorem 1.1 follows directly from Propositions 3.1 to 3.10.

4. The nX-complementary generations of ${}^2\mathrm{F}_4(2)'$

In this section we apply the results discussed in Section 2 to the group $G = {}^2\mathrm{F}_4(2)'$. We determine the non-trivial conjugacy classes nX such that ${}^2\mathrm{F}_4(2)'$ is nX-complementary generated.

If $2X$ is a class of involutions of a non-abelian finite simple group G then G is not $2X$-complementary generated as if it is, then G would be $(2Y, 2X, t_{2Y}Z)$-generated. But this would mean that G is a dihedral group, which would contradict the fact that G is a simple group. Thus, in the process of investigating whether a group G is nX-complementary generated or not, we consider classes nX of G with $n \geq 3$.

Let $T = \{2A, 2B, 3A, 5A, 13A, 13B\}$ be a set of all conjugacy classes of ${}^2\mathrm{F}_4(2)'$ of elements of prime orders. This set T will be useful whenever we apply Proposition 2.1 to prove that the group ${}^2\mathrm{F}_4(2)'$ is nX-complementary generated.

Proposition 4.1. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $3A$*-complementary generated.*

Proof. We achieve the result by showing that ${}^2\mathrm{F}_4(2)'$ is $(pX, 3A, 16A)$-generated for all conjugacy classes $pX \in T$. We divide our proof into two parts.

The first part involves the conjugacy classes in $R = \{2A, 2B, 3A\} \subset T$. We observe from Table 2 that only the maximal subgroup M_5 has a nonempty intersection with the triple $(pX, 3A, 16A)$ for $pX \in R$. Computations with GAP yield the structure constants and the values of $\Sigma(M_5)$ in Table 3. We see from Table 3 that $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, 3A, 16A) > 0$ for all $pX \in R$.

The second part involves the conjugacy classes in $U = \{5A, 13A, 13B\} \subset T$. We observe from Table 2 that for $pX \in U$, all the maximal subgroups of ${}^2\mathrm{F}_4(2)'$ have an empty intersection with the triple $(pX, 3A, 16A)$. Thus, we will have $\Delta^*_G(pX, 3A, 16A) = \Delta_G(pX, 3A, 16A)$. Now, direct computations with GAP yield

Table 3: Values of $\Delta({}^2\mathrm{F}_4(2)')$, $\Sigma(M_5)$, $\Delta^*({}^2\mathrm{F}_4(2)')$ for $(pX, 3A, 16A)$, $pX \in R$

	$\Delta({}^2\mathrm{F}_4(2)')$	$\Sigma(M_5)$	$\Delta^*({}^2\mathrm{F}_4(2)')$
$(2A, 3A, 16A)$	16	0	16
$(2B, 3A, 16A)$	112	32	80
$(3A, 3A, 16A)$	1536	0	1536

- $\Delta_{{}^2\mathrm{F}_4(2)'}(5A, 3A, 16A) = 3328$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(13X, 3A, 16A) = 12800$, $X \in \{A, B\}$.

Since the structure constants computed above are greater than zero it follows that $\Delta^*_G(pX, 5Y, 21A) > 0$ for all $pX \in U$.

We conclude from the two parts that the group ${}^2\mathrm{F}_4(2)'$ is $(pX, 3A, 16A)$-generated for $pX \in T = R \cup U$. It follows, by Proposition 2.1, that ${}^2\mathrm{F}_4(2)'$ is $3A$-complementary generated. □

Proposition 4.2. *The group ${}^2\mathrm{F}_4(2)'$ is $4A$-complementary generated.*

Proof. We show that ${}^2\mathrm{F}_4(2)'$ is $(pX, 4A, tZ)$-generated for all conjugacy classes $pX \in T$ and some class tZ depending on pX. Our proof consists of six cases.
Case $(2A, 4A, 12A)$**:** Computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = 12$. That is, for a fixed $z \in 12A$ there are 12 triples (x, y, z) with $x \in 2A$ and $y \in 4A$ such that $xy = z$ and $\langle x, y\rangle \leq {}^2\mathrm{F}_4(2)'$. We can see from Table 2 that only the maximal subgroups M_5 and M_8 have a nonempty intersection with the triple $(2A, 4A, 12A)$. Further computations with GAP reveal that $\Sigma(M_5) = 0 = \Sigma(M_8)$. Therefore, none of the 12 triples generate a proper subgroup of ${}^2\mathrm{F}_4(2)'$. Hence, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4A, 12A) = 12$.
Case $(2B, 4A, 16A)$**:** We can see from Table 2 that only the maximal subgroups M_3 and M_5 have a nonempty intersection with the triple $(2B, 4A, 16A)$. Direct computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) = 56$, $\Sigma(M_3) = 16+8 = 24$ and $\Sigma(M_5) = 16$. By Table 2, $h(g, M_3) = h(g, M_5) = 1$. Thus,

$$\begin{aligned}\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) &\geq \Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) - 1\cdot\Sigma(M_3) - 1\cdot\Sigma(M_5)\\ &= 56 - 24 - 16 = 16.\end{aligned}$$

Since $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 4A, 16A) \geq 16 > 0$ it follows by Definition 2.1 that ${}^2\mathrm{F}_4(2)'$ is $(2B, 4A, 16A)$-generated.
Case $(3A, 4A, 16A)$**:** For this case we find that only one maximal subgroup, namely M_5, will be involved in the computation of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A)$. Direct computations with GAP give $\Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A) = 864$ and $\Sigma(M_5) = 32$. So, with $h(g, M_3) = 1$ we get

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A) \geq \Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4A, 16A) - 1\cdot\Sigma(M_5) = 864 - 32 = 832$$

which is clearly greater than zero. Hence, ${}^2\mathrm{F}_4(2)'$ is $(3A, 4A, 16A)$-generated.
Case $(5A, 4A, 16A)$**:** Also in this case we find that only one maximal subgroup, namely M_3, will be involved in the computation of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A)$. Computations with GAP give $\Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A) = 1856$ and $\Sigma(M_5) = 128$. So,

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A) \geq \Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4A, 16A) - 1 \cdot \Sigma(M_3) = 1856 - 128 = 1728$$

which is clearly greater than zero. Hence, ${}^2\mathrm{F}_4(2)'$ is $(5A, 4A, 16A)$-generated.
Case $(13X, 4A, 16A)$**,** $X \in \{A, B\}$**:** In this case we deal with two cases involving $13A$ and $13B$ at the same time. We see from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ has a nonempty intersection with the triple $(13X, 4A, 16A)$ for $X \in \{A, B\}$. Therefore,

$$\Delta^*_{{}^2\mathrm{F}_4(2)'}(13X, 4A, 16A) = \Delta_{{}^2\mathrm{F}_4(2)'}(13X, 4A, 16A) = 7168$$

showing generation by the triple $(13X, 4A, 16A)$ for $X \in \{A, B\}$.

We see from these cases that ${}^2\mathrm{F}_4(2)'$ is $(pX, 4A, tZ)$-generated for all $pX \in T$. Thus, by Proposition 2.1, the group ${}^2\mathrm{F}_4(2)'$ is $4A$-complementary generated. □

Proposition 4.3. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $4B$*-complementary generated.*

Proof. Here, we show that ${}^2\mathrm{F}_4(2)'$ is $(pX, 4B, 13A)$-generated for all $pX \in T$. From GAP, we get the following structure constants:

- $\Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4B, 13A) = 13$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4B, 13A) = 65$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4B, 13A) = 1352$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4B, 13A) = 2392$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(13A, 4B, 13A) = 10192$,
- $\Delta_{{}^2\mathrm{F}_4(2)'}(13B, 4B, 13A) = 10192$.

From the calculations above we see that $\Delta_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A) > 0$ for all $pX \in T$. From Table 2, we can see that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ contains elements from $4B$ and $13A$. Therefore, maximal subgroups of ${}^2\mathrm{F}_4(2)'$ make no contribution in the calculations of $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A)$. Hence, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(pX, 4B, 13A) > 0$. Therefore, by Definition 2.1, the group ${}^2\mathrm{F}_4(2)'$ is $(pX, 4B, 13A)$-generated for all the conjugacy classes $pX \in T$. It follows by Proposition 2.1 that ${}^2\mathrm{F}_4(2)'$ is $4B$-complementary generated. □

Proposition 4.4. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $4C$*-complementary generated.*

Proof. We will treat cases involving classes in T separately.
Case $(2A, 4C, 13A)$**:** We observe from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ contains elements from $2A$ and $13A$ together. Thus, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13A) = 26$ and generation by $(2A, 4C, 13A)$ follows since $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2A, 4C, 13A) > 0$.
Case $(2B, 4C, 16A)$**:** In this case the maximal subgroups M_3 and M_5 have the required fusions. Computations with GAP yield $\Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4C, 16A) = 168$, $\Sigma(M_3) = 16 + 8 = 24$ and $\Sigma(M_5) = 16$. Therefore, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(2B, 4C, 16A) \geq \Delta_{{}^2\mathrm{F}_4(2)'}(2B, 4C, 16A) - 1 \cdot \Sigma(M_3) - 1 \cdot \Sigma(M_5) = 168 - 24 - 16 = 128$ and thus, ${}^2\mathrm{F}_4(2)'$ is $(2B, 4C, 16A)$-generated.
Case $(3A, 4C, 16A)$**:** We observe from Table 2 that only the maximal subgroup M_5 has a nonempty intersection with the triple $(3A, 4C, 16A)$. Computations with GAP reveal that $\Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4C, 16A) = 2592$ and $\Sigma^*(M_5) = \Sigma(M_5) = 32$. Therefore, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(3A, 4C, 16A) = \Delta_{{}^2\mathrm{F}_4(2)'}(3A, 4C, 16A) - 1 \cdot \Sigma^*(M_5) = 2592 - 32 = 2560$ showing that ${}^2\mathrm{F}_4(2)'$ is $(3A, 4C, 16A)$-generated.
Case $(5A, 4C, 13A)$**:** We observe from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ has a nonempty intersection with the triple $(5A, 4C, 13A)$. Therefore, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(5A, 4C, 13A) = \Delta_{{}^2\mathrm{F}_4(2)'}(5A, 4C, 13A) = 4784$ showing that ${}^2\mathrm{F}_4(2)'$ is $(5A, 4C, 13A)$-generated.
Case $(13A, 4C, 16A)$**:** Again, from Table 2, we see that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ contains elements from $13A$ and $16A$ together. So, $\Delta^*_{{}^2\mathrm{F}_4(2)'}(13A, 4C, 16A) = \Delta_{{}^2\mathrm{F}_4(2)'}(13A, 4C, 13A) = 21504$, hence ${}^2\mathrm{F}_4(2)'$ is $(13A, 4C, 16A)$-generated.
Case $(13B, 4C, 16A)$**:** The proof in this case is similar to the proof of $(13A, 4C, 16A)$.

Thus, ${}^2\mathrm{F}_4(2)'$ is $(pX, 4C, tZ)$-generated, where $tZ = 13A$ when $pX \in \{2A, 5A\}$ and $tZ = 16A$ when $pX \in \{2B, 3A, 13A, 13B\}$, for all the conjugacy classes $pX \in T$. It follows by Proposition 2.1 that ${}^2\mathrm{F}_4(2)'$ is $4C$-complementary generated. $\square$

Proposition 4.5. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $5A$*-complementary generated.*

Proof. By Propositions 8, 13 and 16 of [16], the group ${}^2\mathrm{F}_4(2)'$ is $(pX, 5A, 13A)$-generated for all $pX \in \{2A, 2B, 3A, 5A\} \subset T$. For the remaining classes $13A$ and $13B$ in T we have $\Delta_{{}^2\mathrm{F}_4(2)'}(13X, 5A, 16A) = 27648$. We can see from Table 2 that no maximal subgroup of ${}^2\mathrm{F}_4(2)'$ has a nonempty intersection with the triple $(13X, 5A, 16A)$ for $X \in \{A, B\}$. Hence, generation by the triple $(13X, 5A, 16A)$ occurs because $\Delta^*_{{}^2\mathrm{F}_4(2)'}(13X, 5A, 16A) = \Delta_{{}^2\mathrm{F}_4(2)'}(13X, 5A, 16A) = 27648$. Thus, ${}^2\mathrm{F}_4(2)'$ is $(pX, 5A, tZ)$-generated, where $tZ = 13A$ when $pX \in \{2A, 2B, 3A, 5A\}$ and $tZ = 16A$ when $pX \in \{13A, 13B\}$, for all the conjugacy classes $pX \in T$. We conclude, by Proposition 2.1, that ${}^2\mathrm{F}_4(2)'$ is $5A$-complementary generated. $\square$

Proposition 4.6. *The group* ${}^2\mathrm{F}_4(2)'$ *is* $6A$*-complementary generated.*

Proof. By Proposition 4.1, the group ${}^2\mathrm{F}_4(2)'$ is $3A$-complementary generated. Since $(6A)^2 = 3A$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $6A$-complementary generated. □

Proposition 4.7. *The group ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{A, B\}$.*

Proof. By Proposition 4.3, the group ${}^2\mathrm{F}_4(2)'$ is $4B$-complementary generated. Since $(8A)^2 = (8B)^2 = 4B$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{A, B\}$. □

Proposition 4.8. *The group ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{C, D\}$.*

Proof. By Proposition 4.4, the group ${}^2\mathrm{F}_4(2)'$ is $4C$-complementary generated. Since $(8C)^2 = (8D)^2 = 4C$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{C, D\}$. □

Proposition 4.9. *The group ${}^2\mathrm{F}_4(2)'$ is $10A$-complementary generated.*

Proof. By Proposition 4.5, the group ${}^2\mathrm{F}_4(2)'$ is $5A$-complementary generated. Since $(10A)^2 = 5A$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $10A$-complementary generated. □

Proposition 4.10. *The group ${}^2\mathrm{F}_4(2)'$ is $12X$-complementary generated for $X \in \{A, B\}$.*

Proof. By Proposition 4.2, the group ${}^2\mathrm{F}_4(2)'$ is $4A$-complementary generated. Since $(12A)^3 = (12B)^3 = 4A$ it follows by application of Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $12X$-complementary generated for $X \in \{A, B\}$. □

Proposition 4.11. *The group ${}^2\mathrm{F}_4(2)'$ is $13Y$-complementary generated for $Y \in \{A, B\}$.*

Proof. By Propositions 9, 14, 17 and 18 of [16] and for $X, Y \in \{A, B\}$, the group G is $(2X, 13Y, 13B)$-, $(3A, 13Y, 13B)$-, $(5A, 13Y, 13B)$- and $(13X, 13Y, 13B)$-generated, respectively. Thus ${}^2\mathrm{F}_4(2)'$ is $(pX, 13Y, 13B)$-generated for all $pX \in T$, hence, it is $13Y$-complementary generated for $Y \in \{A, B\}$. □

Proposition 4.12. *The group ${}^2\mathrm{F}_4(2)'$ is $16X$-complementary generated for $X \in \{A, B, C, D\}$.*

Proof. Since, by Proposition 4.7, the group ${}^2\mathrm{F}_4(2)'$ is $8X$-complementary generated for $X \in \{A, B\}$ and since $(16A)^2 = (16C)^2 = 8A$ and $(16B)^2 = (16D)^2 = 8B$, it follows by Lemma 2.2 that ${}^2\mathrm{F}_4(2)'$ is $16X$-complementary generated for $X \in \{A, B, C, D\}$. □

The results established in the Propositions 4.1 to 4.12 show that the Tits group ${}^2\mathrm{F}_4(2)'$ is nX-complementary generated if and only if $n \geq 3$. Hence Theorem 1.2 is proved.

5. Conclusion

The ranks of the nontrivial conjugacy classes nX of the Tits group ${}^2\mathrm{F}_4(2)'$ are all equal to 2 except when nX is an involutory class. Furthermore, all these conjugacy classes of rank 2 are the nX-complementary generators of ${}^2\mathrm{F}_4(2)'$. However, $rank(G : nX) = 2$ does not necessarily imply nX-complementary generation. Since the Tits group is classified under the Twisted Chevalley groups, it would be interesting to find if all the conjugacy classes nX of rank 2 in the other Twisted Chevalley groups are the nX-complementary generators.

References

[1] F. Ali, M. A. F. Ibrahim, *On the ranks of HS and McL*, Util. Math., 70 (2006), 187-195.

[2] F. Ali, M. A. F. Ibrahim, *On the ranks of Conway group Co_1*, Proc. Japan Acad., 81A (2005), 95-98.

[3] F. Ali, M. A. F. Ibrahim, *On the ranks of Conway groups Co_2 and Co_3*, J. Algebra Appl., 45 (2005), 557-565.

[4] F. Ali, J. Moori, *On the ranks of Janko groups J_1, J_2, J_3 and J_4*, Quaest. Math., 31 (2008), 37-44.

[5] A. R. Ashrafi, *(p, q, r)-Generations and nX-Complementary Generations of the Thompson Group Th*, SUT J. Math., 39 (2003), 41–54

[6] A. B. M. Basheer, *The ranks of the classes of A_{10}*, Bull. Iranian Math. Soc., 43 (2017), 2125-2135.

[7] A. B. M. Basheer, J. Moori, *A survey on some methods of generating finite simple groups,* London Math. Soc. Lecture Note Ser., published as a chapter of the book of Groups St Andrews 2017 by Cambridge University Press, 455 (2019), 106-118.

[8] A. B. M. Basheer, J. Moori, *On the ranks of finite simple groups*, Khayyam J. Math., 2 (2016), 18-24.

[9] A. B. M. Basheer, T. T. Seretlo, *(p,q,r)-generations of the Mathieu group M_{22}*, Southeast Asian Bull. Math, 45 (2021), 11-28.

[10] A. B. M. Basheer, T. T. Seretlo, *The (p, q, r)-generations of the alternating group A_{10}*, Quaest. Math., 43 (2020), 395-408.

[11] A. B. M. Basheer, T. T. Seretlo, *On two generation methods for the simple linear group $PSL(3, 5)$* Khayyam J. Math., 5 (2019), 125-139.

[12] A. B. M. Basheer, M. J. Motalane, T. T. Seretlo, *The (p, q, r)-generations of the Mathieu group M_{23}*, Italian J. Pur and Appl. Math., 52 (2024), 110–137.

[13] A. B. M. Basheer, J. Motalane, T. T. Seretlo, *The (p, q, r)-generations of the alternating group A_{11},* Khayyam J. Math., 7 (2021), 165-186.

[14] A. B. M. Basheer, M. J. Motalane, T. T. Seretlo, *The (p, q, r)-generations of the sympletic group $Sp(6, 2)$*, Alg. Struc. Appl., 8 (2021), 31-49.

[15] A. B. M. Basheer, M. J. Motalane, M. G. Sehoana, T. T. Seretlo, *The nX-complementary generations of the classes of the Chevalley group $G_2(3)$*, Math. Interdisc. Res., 9 (2024), 443-460.

[16] A. B. M. Basheer, M. J. Motalane, M. G. Sehoana, T. T. Seretlo, *On (p, q, r)-generations of the Tits group*, Contrib. Math., 11 (2025), 26-35.

[17] A. B. M. Basheer, M. J. Motalane, M. G. Sehoana, T. T. Seretlo, *The nX-comple- mentary generations the Chevalley group $G_2(4)$*, Electron. J. Math., 10 (2025), 11-22.

[18] A. B. M. Basheer, M. J. Motalane, M. G. Sehoana, T. T. Seretlo, *The ranks of the conjugacy classes of the Chevalley group $G_2(3)$*. Palest. J. Math., 14 (2025), 115-126.

[19] A. B. M. Basheer, M. J. Motalane, M. G. Sehoana, T. T. Seretlo, *The ranks of the conjugacy classes of the Chevalley group $G_2(4)$*, Alg. Struc. Appl., 12 (2025), 243-261.

[20] J. L. Brenner, R. M. Guralnick, J. Wiegold, *Two generator groups III*, Amer. Math. Soc., 33 (1984), 82-89.

[21] M. D. E. Conder, R. A. Wilson and A. J. Woldar, *The symmetric genus of sporadic groups*, Proc. Amer. Math. Soc., 116 (1992), 653-663.

[22] M. D. E. Conder, *Some results on quotients of triangle groups*, Bull. Aust. Math. Soc., 30 (1984), 167-174.

[23] J. H. Conway et al., *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[24] M. R. Darafsheh, A. R. Ashrafi, G. A. Moghani, *nX-Complementary generations of the sporadic group Co_1*, ACTA Math. Vietnam, 29 (2004), 57-75

[25] S. Ganief, *2-Generations of the sporadic simple groups*, PhD Thesis, University of Natal, South Africa, 1997.

[26] S. Ganief, J. Moori, *2-generations of the fourth Janko group J_4*, J. Algebra, 212 (1999), 305-322.

[27] S. Ganief, J. Moori, *2-generations of the smallest Fischer group Fi_{22}*, Nova J. Math. Game Theory Algebra, 6 (1997), 127-145.

[28] S. Ganief, J. Moori, *Generating pairs for the Conway groups Co_2 and Co_3*, J. Group Theory, 1 (1998), 237-256.

[29] S. Ganief, J. Moori, (p, q, r)*-generations of the smallest Conway group* Co_3, J. Algebra, 188 (1997), 516-530.

[30] S. Ganief, J. Moori, *(p,q,r)-generations and nX-complementary generations of the sporadic groups HS and McL.*, J. Algebra, 188 (1997), 531-546.

[31] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.10.2;* 2019. (http://www.gap-system.org)

[32] L. Di Martino, M. Pellegrini, A. Zalesski, *On generators and representations of the Sporadic simple groups*, Comm. Algebra, 42 (2014), 880-908.

[33] J. Moori, *On the ranks of the Fischer group* Fi_{22}, Math. Japonica, 43 (1996), 365-367.

[34] J. Moori, *Subgroups of 3-transposition groups generated by four 3-transpositions*, Quaest. Math., 17 (1994), 483-94.

[35] J. Moori, (p, q, r)*-generations for the Janko groups* J_1 *and* J_2, Nova J. Algebra and Geometry, 2 (1993), 277-285.

[36] J. Moori, *Generating sets for* F_{22} *and its automorphism group*. J. Algebra, 159 (1993), 488-499.

[37] J. Moori, $(2, 3, p)$*-generations for the Fischer group* Fi_{22}, Comm. Algebra, 2 (1994), 4597-4610.

[38] M. J. Motalane, (p, q, r)*-Generations and Conjugacy Class Ranks of certain Simple Groups of the form* $Sp(6, 2)$*,* M_{23} *and* A_{11}, PhD Thesis, University of Limpopo, 2021.

[39] L. L. Scott, *Matrices and cohomology*, Ann. of Math., 105 (1977), 473-492.

[40] J. Ward, *Generation of simple groups by conjugate involutions*, PhD Thesis, University of London, 2009.

[41] R. A. Wilson et al., *Atlas of finite group representations*, (http://brauer.maths.qmul.ac.uk/Atlas/v3/)

[42] A. J. Woldar, *Representing* M_{11}, M_{12}, M_{22} *and* M_{23} *on surfaces of least genus*, Comm. Algebra, 18 (1990), 15-86.

Accepted: May 12, 2025