Groups in which every element centralizer is a *TI*-subgroup

Xianhe Zhao*
School of Mathematics and Statistics
Henan Normal University
Xinxiang Henan 453007
P.R. China
zhaoxianhe989@163.com

Yuxin Zhao
School of Mathematics and Statistics
Henan Normal University
Xinxiang Henan 453007
P.R. China
zhaoyuxin501@163.com

Ruifang Chen
School of Mathematics and Statistics
Henan Normal University
Xinxiang Henan 453007
P.R. China
fang119128@126.com

Abstract. Let G be a finite group. Recall that a subgroup H is called a TI-subgroup of G if $H \cap H^g = 1$ or H for every element g of G. We call a group G a CTI-group if its every element centralizer is a TI-group. Clearly S_3, A_5, D_7 and Q_8 are all CTI-groups. In this paper, we investigate the structure of a CTI-group G and prove that a CTI-group G is a nilpotent group or a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8, or $G \cong PSL(2, 2^n)$ with $n > 1$.

Keywords: finite groups, CTI-groups, TI-subgroups, solvable groups, centralizers.
MSC 2020: 20D25, 20E99, 20F18.

1. Introduction

All groups considered in this paper are finite. The properties of some element centralizers have a profound influence on the structure of a group G, and there are many results in this regard, see [1, 2, 9, 10, 16, 3, 17, 18, 12, 13, 4]. Among these results, a classic result was derived from Brauer-Fowler ([4]) in 1955, which showed that if G was a group of even order with center of odd order, then there existed an element $x \in G \setminus Z(G)$ such that $|G| < |C_G(x)|^3$. As for the number of element centralizers, Belcastro and Sherman in [3] asserted that there was

*. Corresponding author

no a finite group with 2 or 3 element centralizers, furthermore, there are also many results relevant to this topic, see [1, 2, 9, 10, 16]. Except for these, [12, 13] concerned the group G whose element centralizers were nilpotent, even abelian, [17, 18] characterized the group G all of whose centralizers were maximal or second maximal, and so on.

On the other hand, in 1979, Walls in [15] introduced the definition of the TI-subgroup(a subgroup H of G is called a TI-subgroup if $H \cap H^g = 1$ or H for every element g of G) and classified groups all of whose subgroups were TI-subgroups. Based on this, some authors investigated the finite groups in which some subgroups were assumed to be TI-subgroups, such as abelian subgroups [6], non-abelian subgroups [11], etc.

Combining the above two aspects of studies, in this paper, we investigate the finite groups in which every element centralizer is a TI-group. For convenience, we call a group G a CTI-group if and only if its every element centralizer is a TI-subgroup of G. In fact, there exists many CTI-groups in finite groups, some examples are as follows:

Example 1. In S_3, for every non-central element $x \in S_3$, it is clear that $|C_{S_3}(x)| = 2$ or 3, so $C_{S_3}(x)$ is a TI-group, and thus S_3 is a CTI-group.

Example 2. In A_5, for every non-central element $x \in A_5$, it is easy to find that $|C_{A_5}(x)| = 3$ or 5 or 4, thus:

$$C_{A_5}(x) \cap C_{A_5}(x)^g = 1 \text{ or } C_{A_5}(x), \text{ for any } g \in A_5.$$

So, A_5 is a CTI-group.

Example 3. The dihedral group D_7 is a CTI-group since $|C_G(x)| = 2$ or 7 for every non-central element $x \in D_7$.

Example 4. The quaternion group Q_8 are CTI-groups since every subgroup of Q_8 is normal.

A natural question is:

What about the structure of a *CTI*-group G?

In this paper, we prove the following result:

Theorem 3.1. *Let G be a CTI-group, then one of the following statements holds:*

(1) *G is nilpotent;*

(2) *G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8;*

(3) *$G \cong PSL(2, 2^n)$ with $n > 1$.*

2. Preliminaries

Here, we present some useful results needed during the proof of Theorem 3.1.

Lemma 2.1 ([5, Chapter 14, Theorem 1.5])**.** *If G is a solvable CN-group, then one of the following holds:*

(1) *G is nilpotent.*

(2) *G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and a generalized quaternion group.*

(3) *G is a 3-step group*[1]*.*

Lemma 2.2 ([14, Theorem 2])**.** *Let G be a group of even order and its Sylow 2-subgroups are TI-subgroups. Then one of the following three statements is true:*

(1) *A Sylow 2-subgroup of G is a normal subgroup;*

(2) *A Sylow 2-subgroup of G is either a cyclic group or a generalized (or ordinary) quaternion group;*

(3) *G contains normal subgroups G_1 and G_2 such that*

$$G \supseteq G_1 \supset G_2 \supseteq \{1\}$$

where both G/G_1 and G_2 are of odd order and G_1/G_2 is isomorphic to one of the groups $PSL(2, 2^n)$, $U_3(2^n)$ or $Sz(q)$.

The main results of [13] indicate that the class of ZT-groups[2] consists of simple groups $PSL(2, 2^n)$ and simple groups $Sz(q)$, combining [12, Part I, Theorem 4] and [12, Part III, Theorem 5], we have:

Lemma 2.3. *If G is a non-solvable CN-group*[3]*, then the maximal solvable normal subgroup N of G is a 2-group and G/N is one of the following types:*

$$PSL(2, q) \text{ with } q = 2^n \pm 1 \text{ or } 2^n, Sz(q), PSL(3, 4) \text{ or } M_9.$$

where M_9 is the projective group of one variable over near-field of 9 elements.

Lemma 2.4. *Let $G = Sz(q)$ and $S \in Syl_2(G)$. For elements x, y of S with order 4, we have:*

---

1. We shall call G a 3-step group (with respect to the prime p) provided: (i) $O_{pp'}(G)$ is a Frobenius group with kernel $O_p(G)$ and cyclic complement of odd order. (ii) $G = O_{pp'p}(G)$ and $O_{pp'}(G) \subset G$. (iii) $G/O_p(G)$ is a Frobenius group with kernel $O_{pp'(G)}/O_p(G)$(see [5]).
2. A permutation group is called a ZT-group if it is a doubly transitive group of odd degree containing no regular normal subgroup, and if no non-identity element leaves more than three letters invariant(see [12]).
3. A group G is a CN-group if the centralizer of every non-identity element is nilpotent(see [12]).

(1) *If* $C_S(x) \neq C_S(y)$, *then* $C_S(x) \cap C_S(y) = Z(S)$.

(2) $C_G(x)$ *is not a TI-subgroup of* G.

Proof. (1) Clearly $|S|=q^2$, in view of [8, Ch. 11, Lemma 5.9 and Lemma 11.7] and [16, Theorem 1.2], we have $|Z(S)| = q$ and $|C_S(x)| = 2q$ for $x \in S - Z(S)$. Let $|\mathrm{Cent}(S)|$ be the number of centralizers of S, clearly $|\mathrm{Cent}(S)|=q$ by [16, Theorem 1.2]. Assume that $C_S(x_i)(0 \leq i \leq q-1)$ are all distinct centralizers, where $C_S(x_0) = S$, we have $S = C_S(x_1) \cup C_S(x_2) \cup \ldots \cup C_S(x_{q-1})$, so

$$\begin{aligned} |S| &\leq |C_S(x_1)| + |C_S(x_2)| + \ldots + |C_S(x_{q-1})| - (q-2) \cdot |Z(S)| \\ &= (q-1) \cdot |C_S(x_1)| - (q-2) \cdot |Z(S)| = q^2 = |S|, \end{aligned}$$

which means

$$C_S(x_i) \cap C_S(x_j) = Z(S), \text{ for } i \neq j.$$

(2) Otherwise, there exists an element u of G with order 4 such that $C_G(u)$ is a TI-subgroup of G. Clearly $u \notin Z(S)$, so there is an $s (\in S)$ such that $u^s \neq u$. By (1), we have $C_S(u) \cap C_S(u^s) = Z(S)$, it follows that $C_G(u) = C_G(u^s)$ since $C_G(u)$ is a TI-subgroup of G, and hence $C_S(u) = C_S(u^s)$, a contradiction. □

3. Main results

In this section, we give some properties about CTI-groups.

Proposition 3.1. *If* G *is a* CTI*-group and* x *a non-identity element of* G, *then* $N_G(\langle y \rangle) \leq N_G(C_G(y)) = N_G(C_G(x))$ *for every* $1 \neq y \in C_G(x)$.

Proof. (1) $N_G(\langle y \rangle) \leq N_G(C_G(y))$ for any $1 \neq y \in C_G(x)$.

For every $1 \neq y \in C_G(x)$ and $g \in N_G(\langle y \rangle)$, we have $y^g \in C_G(y)^g \cap C_G(y)$. Notice that $C_G(y)$ is a TI-subgroup of G, we have $C_G(y) = C_G(y)^g$, consequently $g \in N_G(C_G(y))$, and therefore $N_G(\langle y \rangle) \leq N_G(C_G(y))$ by the arbitrariness of g.

(2) $N_G(C_G(y)) = N_G(C_G(x))$ for every $1 \neq y \in C_G(x)$.

On the one hand, $x^g \in C_G(y)^g = C_G(y)$ for every $1 \neq y \in C_G(x)$ and $g \in N_G(C_G(y))$, and hence $y \in C_G(x) \cap C_G(x)^g$, therefore $C_G(x) = C_G(x)^g$ since $C_G(x)$ is a TI-subgroup of G, so $N_G(C_G(y)) \leq N_G(C_G(x))$. On the other hand, by using a similar argument to the above, we have $N_G(C_G(x)) \leq N_G(C_G(y))$, consequently $N_G(C_G(y)) = N_G(C_G(x))$. □

Proposition 3.2. *If* G *is a* CTI*-group, then* G *is a* CN*-group. Especially, if* $Z(G) \neq 1$, *then* G *is nilpotent.*

Proof. For every $1 \neq x \in G$ and $1 \neq y \in C_G(x)$, it is clear that $\langle y \rangle \trianglelefteq C_G(y)$ and $C_G(y) \trianglelefteq N_G(C_G(y))$. Also, $N_G(C_G(y)) = N_G(C_G(x))$ by Proposition 3.1, so $\langle y \rangle$ is a subnormal subgroup of $C_G(x)$, therefore G is a CN-group. Especially, clearly G is nilpotent if $Z(G) \neq 1$. □

Proposition 3.3. *If G is a CTI-group, p a prime factor of $|G|$ and x a p-element of G, then $P \leq N_G(C_G(x))$ with $x \in P \in Syl_p(G)$. Especially, if $x \in Z(P)$, then $N_G(P) = N_G(C_G(x))$.*

Proof. For every p-element $x \in P \in Syl_p(G)$, it is clear that $Z(P) \leq C_G(x)$. Take $1 \neq y \in Z(P)$, so $P \leq N_G(\langle y \rangle) \leq N_G(C_G(x))$ by Proposition 3.1.

Especially, if $x \in Z(P)$, then $P \leq C_G(x)$. In fact, $P = P^g \leq C_G(x)^g$ for every $g \in N_G(P)$, it shows that $P \leq C_G(x) \cap C_G(x)^g$. Notice that $C_G(x)$ is a TI-subgroup of G, we have $C_G(x) = C_G(x)^g$, it follows that $g \in N_G(C_G(x))$, so $N_G(P) \leq N_G(C_G(x))$ by the arbitrariness of g. On the other hand, obviously $N_G(C_G(x)) \leq N_G(P)$ since $C_G(x)$ is nilpotent by Proposition 3.2. Therefore, $N_G(P) = N_G(C_G(x))$. $\square$

Proposition 3.4. *If G is a CTI-group, p a prime factor of $|G|$ and x a p-element of G, then $x^G \cap C_G(x) \subseteq x^G \cap P$, where $x \in P \in Syl_p(G)$. Especially, $|x^G \cap C_G(x)|\,\big|\,|x^G|$.*

Proof. Obviously, $x^G \cap C_G(x) = x^G \cap P_1 \subseteq x^G \cap P$ by Proposition 3.2 and Proposition 3.3, where $P_1 \in Syl_p(C_G(x))$. Especially, $x^G \subseteq C_G(x)^G$, while $C_G(x)$ is a TI-subgroup, so $|x^G \cap C_G(x)|\,\big|\,|x^G|$. $\square$

Proposition 3.5. *If G is a CTI-group, then any two Sylow subgroups of G have a trivial intersection.*

Proof. Let p be a prime factor of $|G|$ and $P \in Syl_p(G)$. Take $1 \neq x \in Z(P)$, clearly $P \leq C_G(x)$. If there exists a $g \in G$ such that $1 \neq u \in P \cap P^g$, then $u \in P \cap P^g \leq C_G(x) \cap C_G(x)^g$, and hence $C_G(x) = C_G(x)^g$ since $C_G(x)$ is a TI-subgroup of G, therefore $P = P^g$ since $C_G(x)$ is nilpotent by Proposition 3.2. $\square$

Now, we would prove Theorem 3.1.

Theorem 3.1. *Let G be a CTI-group, then one of the following statements holds:*

(1) *G is nilpotent;*

(2) *G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8;*

(3) *$G \cong PSL(2, 2^n)$ with $n > 1$.*

Proof. The proof is divided into two aspects according to the solvablity of G.

(1) If G is solvable, then G is nilpotent or G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and Q_8.

By Proposition 3.2, we have G is a CN-group. By Lemma 2.1, we have:

(a) G is nilpotent; or

(b) G is a Frobenius group whose complement is either cyclic or the direct product of a cyclic group of odd order and a generalized quaternion group; or

(c) G is a 3-step group.

For (b), if $G(= K \rtimes H)$ is a Frobenius group and H is the direct product of an odd order cyclic group and a generalized quaternion group Q_{4n}, we assert $n = 2$. Otherwise, let $Q_{4n} = \langle a, b\rangle$, where $a^{2n} = 1$, $b^2 = a^n$, and $b^{-1}ab = a^{-1}$. Notice that $C_{Q_{4n}}(b) = \langle b\rangle \ntrianglelefteq Q_{4n}$. However, $b^2 \in C_{Q_{4n}}(b) \cap C_{Q_{4n}}(b)^g \leq C_G(b) \cap C_G(b)^g$ for any $g \in Q_{4n}$, so $C_{Q_{4n}}(b) = C_{Q_{4n}}(b)^g$ since $C_G(b)$ is a TI-subgroup of G, and hence $C_{Q_{4n}}(b) \trianglelefteq Q_{4n}$, a contradiction. Therefore, $n = 2$, and so $Q_{4n} = Q_8$.

For (c), if G is a 3-step group with respect to the prime p, we have $O_{pp'}(G) = O_p(G) \rtimes H$ is a Frobeius group. For every $P_1, P_2 \in Syl_p(G)$, clearly $O_p(G) \leq P_1 \cap P_2$, and hence P_1 is a normal subgroup of G since G is a CTI-group and Proposition 3.2, in contradiction with the definition of 3-step group.

(2) If G is non-solvable, then $G \cong PSL(2, 2^n)$ with $n > 1$.

By Proposition 3.2 and Lemma 2.3, we have the maximal solvable normal subgroup N of G is a 2-group and G/N is one of the following types:

$$(*) \qquad PSL(2, q) \text{ with } q = 2^n \pm 1 \text{ or } 2^n, Sz(q), PSL(3, 4) \text{ or } M_9.$$

It is obvious that the Sylow 2-subgroup of G is not a normal subgroup, a cyclic group or a generalized (or ordinary) quaternion group, so Lemma 2.2 shows that there exists a normal subgroups series

$$G \supseteq G_1 \supset G_2 \supseteq \{1\}$$

such that $G_1/G_2 \cong PSL(2, 2^n)$, $U_3(2^n)$ or $Sz(q)$, where both G/G_1 and G_2 are of odd order. Applying Lemma 2.3 once again, we have $G_2 = 1$, and hence

$$(**) \qquad G_1 \cong PSL(2, 2^n),\ U_3(2^n) \text{ or } Sz(q).$$

Moreover, notice that $N \trianglelefteq G_1$, so $N = 1$ by (**), and hence $G \cong PSL(2, 2^n)$ or $Sz(q)$ by (*) and (**). By Lemma 2.4(2), we have $G \cong PSL(2, 2^n)$ with $n > 1$. □

By Lemma 3.1(2) and [8, Chapter 11], we have:

Corollary 3.1. *Let $G = Sz(q)$, $C_G(x)$ is not a TI-subgroup of $Sz(q)$ if and only if $o(x) = 4$.*

By Theorem 3.1 and [7, Chapter 2, Theorem 8.2-Theorem 8.5], we have:

Corollary 3.2. *A group G is a simple CTI-group if and only if $G \cong PSL(2, 2^n)$ with $n > 1$.*

4. Conclusion

The classification of finite groups is one of the important topics in finite group theory. Based on the study of the TI-subgroups and the element centralizers, in this paper, we introduced the definition of the CTI-group, subsequently, studied the properties of CTI-groups, and then finished the classification of the CTI-groups.

Acknowledgment

The authors would like to thank the anonymous referees and the editor for their valuable suggestions and comments that helped improve the paper significantly.

The research of the work was supported by the National Natural Science Foundation of China (12471018).

References

[1] A.R. Ashrafi, *On finite groups with a given number of centralizers*, Algebra Colloq., 7 (2000), 139-146.

[2] S.J. Baishya, *On groups with same number of centralizers*, Comm. Algebra, (2023), 1-9.

[3] S.M. Belcastro, G.J. Sherman, *Counting centralizers in finite groups*, Math. Mag., 5 (1994), 111-114.

[4] R. Brauer, K.A. Fowler, *On groups of even order*, Ann. of Math., 62 (1955), 565-583.

[5] D. Gorenstein, *Finite groups*, Chelsea Publish Company, New-York, 1980.

[6] X.Y. Guo, S.R. Li, P. Flavell, *Finite groups whose abelian subgroups are TI-subgroups*, J. Algebra, 307 (2007), 565-569.

[7] B. Huppert, *Endliche gruppen I*, Springer-Verlag, New York, 1967.

[8] N. Huppert, N. Blackburn, *Finite groups III*, Springer Science, Business Media, 2012.

[9] K. Khoramshahi, M. Zarrin, *Groups with the same number of centralizers*, J. Algebra Appl., 20 (2021), 2150012.

[10] J.C.M. Pezzott, I. N. Nakaoka, *A note on the number of centralizers in finite AC-groups*, J. Algebra Appl., 22 (2023).

[11] J.T. Shi, C. Zhang, *Finite groups in which all nonabelian subgroups are TI-subgroups*, J. Algebra Appl., 13 (2014), 1350074.

[12] M. Suzuki, *Finite groups with nilpotent centralizers*, Trans. Amer. Math. Soc., 99 (1961), 425-470.

[13] M. Suzuki, *On a class of doubly transitive groups*, Anna. Math., 75 (1962), 105-145.

[14] M. Suzuki, *Finite groups of even order in which Sylow 2-groups are independent*, Ann. Math., 80 (1964), 58-77.

[15] G. Walls, *Trivial intersection groups*, Arch. Math., 32 (1979), 1-4.

[16] M. Zarrin, *On element-centralizers in finite groups*, Arch. Math., 93 (2009), 497-503.

[17] X.H. Zhao, R.F. Chen, X.Y. Guo, *Groups in which the centralizer of any non-central element is maximal*, J. Group Theory, 23 (2020), 871-878.

[18] X.H. Zhao, Y.X. Zhao, R.F. Chen, H.P. Qu, X.Y. Guo, *Finite groups whose centralizers of non-central elements are second maximal*, J. Algebra Appl., 2024.

Accepted: December 2, 2024