On biharmonic surfaces in pseudo-Riemannian 4-dimensional space forms

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Abstract. In this paper, biharmonic pseudo-Riemannian surfaces with diagonalizable shape operetor in pseudo-Riemannian space form $N_s^4(c)$ are studied. We prove that the surfaces with light-like mean curvature vector field are pseudo-umbilical. For non light-like mean curvature vector field, we show that the pseudo-umbilical surfaces is minimal or $H^2 = |c|$. Also, we give some sufficient conditions for such surfaces with parallel mean curvature vector field to be minimal.

Keywords: Pseudo-Riemannian space forms, biharmonic surfaces, minimal surfaces, parallel mean curvature vector field, pseudo-umbilical surfaces.

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1. Introduction

Let $\phi : M_r^n \to N_s^{n+p}$ be the inclusion of a pseudo-Riemannian submanifold M_r^n with index r into a pseudo-Riemannian manifold N_s^{n+p} with index s. We say that M_r^n is a biharmonic submanifold, if its bitension field $\tau_2(\phi)$ vanishes identically, i.e. (see [3, 10, 15])

$$\tau_2(\phi) := -\Delta^\phi \tau(\phi) - \operatorname{tr} R^N(\mathrm{d}\phi, \tau(\phi))\mathrm{d}\phi = 0, \tag{1}$$

where Δ^ϕ is the rough Laplacian defined on sections of $\phi^{-1}(TN)$ and $\tau(\phi) = \operatorname{tr}\nabla^\phi \mathrm{d}\phi$ is the tension field of ϕ that vanishes for ϕ being a harmonic map. R^N, ∇^ϕ and ∇ are the curvature tensor of N_s^{n+p}, the induced connection by ϕ on the bundle $\phi^* TN_s^{n+p}$ and the connection of M_r^n, respectively.

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Notice that $\tau(\phi) = n\overrightarrow{H}$ with $\overrightarrow{H}$ the mean curvature vector field of M_r^n, then it is clear from (1) that a minimal submanifold must be biharmonic, and we call a nonminimal biharmonic submanifold a proper biharmonic submanifold.

During the last decade, a special attention has been paid to the study of biharmonic submanifolds and important progress has been made in the study of this subject. B. Y. Chen and S. Ishikawa in [5, 6] gave full classification of proper biharmonic curves in $\mathbb{E}_q^3$ and proved that there exists no proper biharmonic surface in $\mathbb{E}_q^3$. In [15], T. Sasahara classified proper biharmonic curves and surfaces in nonflat Lorentz 3-space forms.

Now, let us turn to the problem in 4-dimensional pseudo-Riemannian space forms. F. Defever et al. proved in [8] that every biharmonic hypersurface M_r^3 ($r = 0, 1, 2, 3$) of $\mathbb{E}_s^4$ with diagonalizable shape operator is minimal, and the same conclusion holds for Lorentz hypersurfaces in $\mathbb{E}_1^4$ (see [1]).

It seems then natural, as the next step, to study biharmonic surfaces M_r^2 in pseudo-Riemannian space froms $N_s^4(c)$. The structure of the surfaces often appears considerably different from that of the hypersurfaces, the mean curvature vector field $\vec{H}$ of surfaces may be light-like, except space-like and time-like ones. In general, each of them will imply different properties of surfaces. So far, there have been no many developments in this direction.

Surfaces with light-like mean curvature vector field were concerned by many geometers due to the study of trapped surfaces in 4-dimensional Lorentz manifolds in [14], which are related to the presence of a black hole. In [5, 6], B. Y. Chen and S. Ishikawa firstly studied biharmonic surfaces with light-like mean curvature vector field (i.e., marginally trapped biharmonic surfaces) in $\mathbb{E}_s^4$ and gave some examples of such surfaces in $\mathbb{E}_s^4$. Our first goal in this paper continues to study this subject in nonflat cases and obtain

Theorem 1.1. *Let M_r^2 be a biharmonic surface with light-like mean curvature vector in a pseudo-Riemannian space form $N_s^4(c)$. Assume that M_r^2 has diagonalizable shape operator, then it is pseudo-umbilical.*

In [13], C.-Z. Ouyang investigated the minimality of space-like biharmonic surfaces M^2 (i.e., $r = 0$) with parallel mean curvature vector field in pseudo-Riemannian space forms. After that, J.-C. Liu, L. Du and J. Zhang (see [11]) studied the problem for space-like pseudo-umbilical surfaces. Our second goal in this paper is to investigate the minimality of such surfaces M_r^2 with general index r and obtain

Theorem 1.2. *Let M_r^2 be pseudo-umbilical biharmonic surfaces with non light-like mean curvature vector in $N_s^4(c)$. Assume that M_r^2 has diagonalizable shape operator, then one of the following three statements holds:*

(i) *when $c = 0$, then M_r^2 is minimal*;

(ii) *when $c > 0$, then its mean curvature vector $\overrightarrow{H}$ is space-like. Furthermore, either M_r^2 is minimal or $H^2 = c$*;

(iii) *when $c < 0$, then either $\overrightarrow{H} = 0$, i.e., M_r^2 is minimal, or $\overrightarrow{H}$ is a time-like vector with $H^2 = |c|$, where H is the mean curvature of M_r^2.*

Remark 1.1. As a corollary of Theorem 1.2, let M^2 be a pseudo-umbilical biharmonic surface in Riemannian space form $N^4(c)$. When $c \leq 0$, then M^2 must be minimal; when $c > 0$, then its mean curvature $H = 0$, or $H^2 = c$, which has been proved by [9] for $c = 0$, [2, Theorem 5.1] for $c > 0$ and [4, Theorem 2.4], for $c < 0$.

Theorem 1.3. *Let M_r^2 be a biharmonic surface with non light-like mean curvature vector in $N_s^4(c)$. Assume that M_r^2 has parallel mean curvature vector field and diagonalizable shape operator, then M_r^2 is minimal if $c = 0$, or $c < 0$ and $\vec{H}$ is space-like, or $c > 0$ and trace$A_3^2 \neq 2c$, where A_3 is the shape operator with respect to the normal frame field e_3 of M_r^2.*

2. Preliminaries

Let $N_s^{n+p}(c)$ be an $(n+p)$-dimensional pseudo-Riemannian space form with index s of constant curvature c $(0 \leq s \leq n+p)$. Let $x : M_r^n \to N_s^{n+p}(c)$ be an isometric immersion of an n-dimensional manifold M_r^n of signature $(r, n-r)(r \geq 0)$ into $N_s^{n+p}(c)$. Let ∇ and $\widetilde{\nabla}$ denote by the Levi-Civita connections of M_r^n and $N_s^{n+p}(c)$, respectively. For any tangent vector fields X, Y and normal vector field ξ of M_r^n in $N_s^{n+p}(c)$, the Gauss and Weingarten formulas are given by, respectively, (cf. [7] or [12])

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X,Y),\ \widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where B, A_ξ and D are the second fundamental form, the shape operator with respect to ξ and the normal connection, respectively. It is easy to see that B and A_ξ are related by

$$\langle B(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle. \tag{2}$$

The mean curvature vector field $\overrightarrow{H}$ and the mean curvature H of M_r^n in $N_s^{n+p}(c)$ are expressed as $\overrightarrow{H} = \frac{1}{n}\text{trace}B$, and $H = \sqrt{|\langle \overrightarrow{H}, \overrightarrow{H} \rangle|}$, respectively.

We define the covariant derivative of the second fundamental form B by

$$(\widetilde{\nabla}_X B)(Y,Z) = D_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{3}$$

Then, the Codazzi equation is given by

$$(\widetilde{\nabla}_X B)(Y,Z) = (\widetilde{\nabla}_Y B)(X,Z).$$

Let $\{e_1, e_2, \ldots, e_{n+p}\}$ be a local orthonormal frame basis on $N_s^{n+p}(c)$ such that $e_1, \ldots, e_n$ are tangent to M_r^n and $e_{n+1}, \ldots, e_{n+p}$ are normal to M_r^n. Then, the connection forms (ω_B^A) are given by (cf. [6])

$$\widetilde{\nabla} e_A = \sum_{B=1}^{n+p} \omega_A^B e_B,\ \omega_B^A = -\varepsilon_A \varepsilon_B \omega_A^B,\ A, B = 1, 2, \ldots, n+p, \tag{4}$$

where and $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1,\ A = 1, 2, \ldots, n+p$. It follows from (2) that

$$\overrightarrow{H} = \frac{1}{n} \mathrm{trace} B = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i B(e_i, e_i) = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \varepsilon_\alpha (\mathrm{trace} A_\alpha) e_\alpha, \tag{5}$$

where $A_\alpha = A_{e_\alpha}$.

As it is known that a vector v tangent to $N_s^{n+p}(c)$ is called *space-like* (resp. *time-like*) if $v = 0$ or $\langle v, v \rangle > 0$ (resp. $\langle v, v \rangle < 0$). A vector v is called *light-like* if $v \neq 0$ and $\langle v, v \rangle = 0$.

A submanifold M_r^n is called *minimal* if $\overrightarrow{H} = 0$. M_r^n is called *pseudo-umbilical*, if it is umbilical with respect to the direction of $\overrightarrow{H}$ (cf. [7]), i.e.,

$$\langle B(X, Y), \overrightarrow{H} \rangle = \langle \overrightarrow{H}, \overrightarrow{H} \rangle \langle X, Y \rangle. \tag{6}$$

When $\overrightarrow{H} \neq 0$, (6) becomes $A_{\overrightarrow{H}} = \langle \overrightarrow{H}, \overrightarrow{H} \rangle I$, where I stands for the identity operator. It follows from (6) that every minimal submanifold is pseudo-umbilical.

Using a similar computation as in the proof of Theorem 4.1 in [3] (also see [7]), we obtain the following.

Lemma 2.1. *An isometric immersion $\phi : M_r^n \to N_s^{n+p}(c)$ of an n-dimensional manifold M_r^n into $N_s^{n+p}(c)$ is biharmonic if and only if*

$$\begin{cases} \Delta^D \overrightarrow{H} + \sum\limits_{i=1}^{n} \varepsilon_i B(A_{\overrightarrow{H}} e_i, e_i) - n \overrightarrow{H} c = 0, \\ n \nabla \langle \overrightarrow{H}, \overrightarrow{H} \rangle + 4 \sum\limits_{i=1}^{n} \varepsilon_i A_{D_{e_i} \overrightarrow{H}}(e_i) = 0, \end{cases} \tag{7}$$

where

$$\Delta^D = -\sum_{i=1}^{n} \varepsilon_i (D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i}). \tag{8}$$

Set

$$\begin{cases} \|\omega_{n+1}^\alpha\|^2 = \sum\limits_{i=1}^{n} \varepsilon_i \big(\omega_{n+1}^\alpha (e_i)\big)^2,\ \alpha > n+1, \\ \mathrm{trace} A_{n+1} A_\alpha = \sum\limits_{i=1}^{n} \varepsilon_i \langle A_{n+1}(A_\alpha(e_i)), e_i \rangle,\ \alpha > n+1, \\ \nabla H = \sum\limits_{i=1}^{n} \varepsilon_i (e_i H) e_i. \end{cases}$$

Using Lemma 2.1, we can prove the following lemma.

Lemma 2.2. *Let M_r^n be a biharmonic submanifold in $N_s^{n+p}(c)$ with non light-like mean curvature vector. Then, we have*

$$\begin{cases} \Delta H + H\varepsilon_{n+1}\sum\limits_{\alpha=n+2}^{n+p}\varepsilon_\alpha\|\omega_\alpha^{n+1}\|^2 + H\varepsilon_{n+1}\|A_{n+1}\|^2 - nHc = 0, \\ H\varepsilon_\alpha \mathrm{trace} A_{n+1}A_\alpha = 2\omega_{n+1}^\alpha(\nabla H) + H\sum\limits_{i=1}^{n}\varepsilon_i(\nabla_{e_i}\omega_{n+1}^\alpha)(e_i) \\ \qquad\qquad + H\sum\limits_{\beta=n+1}^{n+p}\sum\limits_{i=1}^{n}\varepsilon_i\omega_{n+1}^\beta(e_i)\omega_\beta^\alpha(e_i), \forall\alpha > n+1, \\ n\varepsilon_{n+1}H\nabla H + 2A_{n+1}(\nabla H) + 2H\sum\limits_{\alpha=n+2}^{n+p}\sum\limits_{i=1}^{n}\varepsilon_i\omega_{n+1}^\alpha(e_i)A_\alpha(e_i) = 0. \end{cases} \tag{9}$$

Proof. Since $\overrightarrow{H}$ is not light-like, we can choose a local orthonormal frame field $\{e_i\}_{i=1}^{n+p}$ such that $\overrightarrow{H} = H\,e_{n+1}$.

We will calculate each term in (7) individually. First, from (8) we have

$$\begin{aligned} \Delta^D\overrightarrow{H} &= -\sum_{i=1}^{n}\varepsilon_i D_{e_i}D_{e_i}(H\,e_{n+1}) + \sum_{i=1}^{n}\varepsilon_i D_{\nabla_{e_i}e_i}(H\,e_{n+1}) \\ &= -\sum_{i=1}^{n}\varepsilon_i\Big(e_ie_iHe_{n+1} + 2e_iHD_{e_i}e_{n+1} + HD_{e_i}D_{e_i}e_{n+1}\Big) \\ &= -\sum_{i=1}^{n}\varepsilon_i\Big\{\big[e_ie_iH - \varepsilon_{n+1}H\sum_{\alpha=n+2}^{n+p}\varepsilon_\alpha(\omega_{n+1}^\alpha(e_i))^2\big]e_{n+1} \\ &\quad + \sum_{\alpha=n+2}^{n+p}\big[2\omega_{n+1}^\alpha(e_i(H)e_i) + H\nabla_{e_i}(\omega_{n+1}^\alpha(e_i)) \\ &\quad + H\sum_{\beta=n+1}^{n+p}\varepsilon_i\omega_{n+1}^\beta(e_i)\omega_\beta^\alpha(e_i)\big]e_\alpha\Big\} \\ &\quad + \sum_{i=1}^{n}\varepsilon_i\Big((\nabla_{e_i}e_i)(H)e_{n+1} + H\sum_{\alpha=n+2}^{n+p}\omega_{n+1}^\alpha(\nabla_{e_i}e_i)e_\alpha\Big). \end{aligned} \tag{10}$$

Putting into (10) gives

$$\begin{aligned} \Delta^D\overrightarrow{H} =& \Big(\Delta H + \varepsilon_{n+1}H\sum_{\alpha=n+2}^{n+p}\varepsilon_\alpha\|\omega_\alpha^{n+1}\|^2\Big)e_{n+1} - \sum_{\alpha=n+2}^{n+p}\Big\{2\omega_{n+1}^\alpha(\nabla H) \\ &+ H\sum_{i=1}^{n}\big[\varepsilon_i(\nabla_{e_i}\omega_\alpha^{n+1})e_i + \sum_{\beta=n+1}^{n+p}\varepsilon_i\omega_{n+1}^\beta(e_i)\omega_\beta^\alpha(e_i)\big]\Big\}e_\alpha. \end{aligned} \tag{11}$$

Using $\overrightarrow{H} = He_{n+1}$ again, a straightforward computation yields

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon_i B(A_{\overrightarrow{H}}(e_i), e_i) =& \sum_{i=1}^{n} \varepsilon_i \Big\{ \varepsilon_{n+1} \langle A_{n+1}\big(A_{\overrightarrow{H}}(e_i)\big), e_i\rangle\, e_{n+1} \\ &+ \sum_{\alpha=n+2}^{n+p} \varepsilon_\alpha \langle A_\alpha\big(A_{\overrightarrow{H}}(e_i)\big), e_i\rangle\, e_\alpha \Big\} \\ =& \varepsilon_{n+1} H \|A_{n+1}\|^2 e_{n+1} + H \sum_{\alpha=n+2}^{n+p} \varepsilon_\alpha \mathrm{trace} A_{n+1} A_\alpha e_\alpha. \end{aligned} \tag{12}$$

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon_i A_{D_{e_i}\overrightarrow{H}}(e_i) =& \sum_{i=1}^{n} \varepsilon_i A_{D_{e_i}(H\, e_{n+1})}(e_i) \\ =& A_{n+1}(\nabla H) + H \sum_{\alpha=n+2}^{n+p} \sum_{i=1}^{n} \varepsilon_i \omega_{n+1}^{\alpha}(e_i) A_\alpha(e_i). \end{aligned} \tag{13}$$

Substituting (11), (12) and (13) into (7), we complete the proof of Lemma 2.2.

3. Proof of Main Theorems

Proof of Theorem 1.1. Since A_3 and A_4 are diagonalizable, we choose a local orthonormal frame field $\{e_1, \ldots, e_4\}$ such that $e_1,\ e_2$ are tangent to M_r^2 and

$$A_3 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},\ A_4 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}. \tag{14}$$

Since $\overrightarrow{H}$ is light-like, we can set

$$\overrightarrow{H} = f(e_3 + e_4) \tag{15}$$

with e_3 being space-like, e_4 being time-like and f being a non-zero function on M_r^2 (in fact, we can assume e_3 be time-like and e_4 be space-like, the proof idea is the same). By a direct computation, we have

$$\Delta^D(fe_3) = (\Delta f)e_3 - 2\omega_3^4(\nabla f)e_4 - f(\mathrm{trace}\nabla\omega_3^4)e_4 - f\|\omega_3^4\|^2 e_3.$$

$$\Delta^D(fe_4) = (\Delta f)e_4 - 2\omega_3^4(\nabla f)e_3 - f(\mathrm{trace}\nabla\omega_3^4)e_3 - f\|\omega_3^4\|^2 e_4.$$

Then, we have

$$\begin{aligned}\Delta^D\overrightarrow{H} =& \Delta^D(fe_3) + \Delta^D(fe_4) \\ =& \big(\Delta f - 2\omega_3^4(\nabla f) - f\mathrm{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2\big)\, e_3 \\ &+ \big(\Delta f - 2\omega_3^4(\nabla f) - f\mathrm{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2\big)\, e_4, \end{aligned} \tag{16}$$

where $\mathrm{trace}\nabla\omega_3^4 = \sum_{i=1}^2 \varepsilon_i(\nabla_{e_i}\omega_3^4)(e_i)$. We also have

$$\begin{aligned}\sum_{i=1}^{2}\varepsilon_i B(A_{\overrightarrow{H}}(e_i), e_i) =& f(\mathrm{trace}A_3^2 + \mathrm{trace}A_3A_4)e_3 \\ &- f(\mathrm{trace}A_4^2 + \mathrm{trace}A_3A_4)e_4.\end{aligned} \tag{17}$$

$$\sum_{i=1}^{2}\varepsilon_i A_{D_{e_i}\overrightarrow{H}}(e_i) = (A_3 + A_4)\Big(\nabla f + f\sum_{i=1}^{2}\varepsilon_i\omega_3^4(e_i)e_i\Big). \tag{18}$$

Substituting (15), (16), (17) and (18) into (7), we get

$$\begin{cases}\Delta f - 2\omega_3^4(\nabla f) - f\mathrm{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2 - 2fc \\ \qquad\qquad +f(\mathrm{trace}A_3^2 + \mathrm{trace}A_3A_4) = 0, \\ \Delta f - 2\omega_3^4(\nabla f) - f\mathrm{trace}\nabla\omega_3^4 - f\|\omega_3^4\|^2 - 2fc \\ \qquad\qquad -f(\mathrm{trace}A_4^2 + \mathrm{trace}A_3A_4) = 0, \\ (A_3 + A_4)\Big(\nabla f + f\sum_{i=1}^2 \varepsilon_i\omega_3^4(e_i)e_i\Big) = 0.\end{cases} \tag{19}$$

Using the first and second equations in (19), we have

$$\mathrm{trace}A_3^2 + \mathrm{trace}A_4^2 + 2\mathrm{trace}A_3A_4 = 0, \tag{20}$$

which together with (14), we get

$$\lambda_1 = -\mu_1,\ \lambda_2 = -\mu_2.$$

Note that

$$A_{\vec{H}} = A_{f(e_3+e_4)} = \begin{pmatrix} f(\lambda_1 + \mu_2) & 0 \\ 0 & f(\lambda_2 + \mu_2)\end{pmatrix} = \mathbf{0}. \tag{21}$$

Then, $\langle B(e_i, e_j), \overrightarrow{H}\rangle = \langle A_{\overrightarrow{H}}(e_i), e_j\rangle = 0,\ i, j = 1, 2$. Also, $\langle\overrightarrow{H}, \overrightarrow{H}\rangle\langle e_i, e_j\rangle = 0$ because of $\overrightarrow{H}$ being light-like. So, $\langle B(e_i, e_j), \overrightarrow{H}\rangle = \langle\overrightarrow{H}, \overrightarrow{H}\rangle\langle e_i, e_j\rangle$. We have from (6) that M_r^2 is pseudo-umbilical and complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Choose a local orthonormal frame field $\{e_i\}_{i=1}^4$ such that $\overrightarrow{H} = He_3$ with $\langle e_3, e_3\rangle = \varepsilon_3 = \pm 1$. Then, (9) is simplified as

$$\begin{cases}\Delta H + H\varepsilon_3\varepsilon_4\|\omega_4^3\|^2 + H\varepsilon_3\|A_3\|^2 - 2Hc = 0, \\ \varepsilon_4 H\mathrm{trace}A_3A_4 = 2\omega_3^4(\nabla H) + H\sum\limits_{i=1}^{2}\varepsilon_i(\nabla_{e_i}\omega_3^4)(e_i), \\ \varepsilon_3\nabla H + A_3(\nabla H) + H\sum\limits_{i=1}^{2}\varepsilon_i\omega_3^4(e_i)A_4(e_i) = 0.\end{cases} \tag{22}$$

Since $\overrightarrow{H}$ is non light-like, then either $\overrightarrow{H}=0$, or $\langle\overrightarrow{H},\overrightarrow{H}\rangle\neq 0$.

Suppose that $\langle\overrightarrow{H},\overrightarrow{H}\rangle\neq 0$. Since M^2_r is pseudo-umbilical, it follows from (2) and (6) that $A_{\overrightarrow{H}}=\langle\overrightarrow{H},\overrightarrow{H}\rangle I$. On one hand, we obtain

$$\sum_{i=1}^{2}\varepsilon_i(A_{D_{e_i}\overrightarrow{H}})(e_i)=\sum_{i=1}^{2}\varepsilon_i\left(\nabla_{e_i}A_{\overrightarrow{H}}\right)(e_i)-\nabla\langle\overrightarrow{H},\overrightarrow{H}\rangle. \tag{23}$$

On the other hand, it follows that

$$\sum_{i=1}^{2}\varepsilon_i\left(\nabla_{e_i}A_{\overrightarrow{H}}\right)(e_i)=\nabla\langle\overrightarrow{H},\overrightarrow{H}\rangle. \tag{24}$$

Putting (23) and (24) into the second equation of (7), we obtain $\nabla\langle\overrightarrow{H},\overrightarrow{H}\rangle=0$, i.e., H is a non zero constant. Then, (22) becomes

$$\begin{cases}\varepsilon_4\varepsilon_3\|\omega_4^3\|^2+\varepsilon_3\|A_3\|^2-2c=0,\\ \varepsilon_4\mathrm{trace}A_3A_4=\sum\limits_{i=1}^{2}\varepsilon_i(\nabla_{e_i}\omega_3^4)(e_i),\\ \sum\limits_{i=1}^{2}\varepsilon_i\omega_3^4(e_i)A_4(e_i)=0.\end{cases} \tag{25}$$

According to (4), we put

$$\omega_3^4=-\varepsilon_3\varepsilon_4\omega_4^3=\tilde{f}_1\omega_1+\tilde{f}_2\omega_2, \tag{26}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are some functions. Note that $\overrightarrow{H}=He_3$, it follows from (5) that $\mathrm{trace}\,A_4=0$. Then, we can express the matrix representation of A_4 as

$$A_4=\begin{pmatrix}\tilde{g}_{11} & 0\\ 0 & -\tilde{g}_{11}\end{pmatrix}. \tag{27}$$

We claim that $\|\omega_4^3\|^2=0$.

Assume on the contrary that $\|\omega_4^3\|^2\neq 0$, i.e., $\tilde{f}_1^2+\varepsilon_1\varepsilon_2\tilde{f}_2^2\neq 0$. Making use of the third equation of (25) and (26), we have

$$\varepsilon_1\varepsilon_2\tilde{f}_1A_4(e_1)+\tilde{f}_2A_4(e_2)=0, \tag{28}$$

which together with (27), we obtain $\tilde{f}_1\tilde{g}_{11}=0$, and $\varepsilon_1\varepsilon_2\tilde{f}_2\tilde{g}_{11}=0$. So, we have

$$\big(\tilde{f}_1^2+\varepsilon_1\varepsilon_2\tilde{f}_2^2\big)\tilde{g}_{11}=0,$$

which shows that $\tilde{g}_{11}=0$. Thus, $A_4=0$.

Since $\overrightarrow{H}=He_3$ and M^2_r is pseudo-umbilical, $A_3=\varepsilon_3HI$. It follows from (2) that

$$\begin{aligned}\varepsilon_1B(e_1,e_1)&=\varepsilon_1\varepsilon_3\langle A_3(e_1),e_1\rangle\,e_3+\varepsilon_1\varepsilon_4\langle A_4(e_1),e_1\rangle\,e_4=\varepsilon_3He_3,\\ \varepsilon_1B(e_1,e_2)&=\varepsilon_1\varepsilon_3\langle A_3(e_1),e_2\rangle\,e_3+\varepsilon_1\varepsilon_4\langle A_4(e_1),e_2\rangle\,e_4=0,\\ \varepsilon_2B(e_2,e_2)&=\varepsilon_2\varepsilon_3\langle A_3(e_2),e_2\rangle\,e_3+\varepsilon_2\varepsilon_4\langle A_4(e_2),e_2\rangle\,e_4=\varepsilon_3He_3,\end{aligned}$$

which imply that

$$B(e_1, e_1) = \varepsilon_1\varepsilon_3 H e_3, \quad B(e_1, e_2) = 0, \quad B(e_2, e_2) = \varepsilon_2\varepsilon_3 H e_3.$$

Using these three equations and (3), we compute and get

$$(\widetilde{\nabla}_{e_2} B)(e_1, e_1) = \varepsilon_1\varepsilon_3 H \omega_3^4(e_2) e_4.$$
$$(\widetilde{\nabla}_{e_1} B)(e_2, e_1) = -\varepsilon_3 H\big(\varepsilon_1\omega_2^1(e_1) + \varepsilon_2\omega_1^2(e_1)\big)e_3.$$

Then, the Codazzi equation $(\widetilde{\nabla}_{e_2} B)(e_1, e_1) = (\widetilde{\nabla}_{e_1} B)(e_2, e_1)$ and (26) deduce to $H\tilde{f}_2 = 0$. Similarly, the Codazzi equation $(\widetilde{\nabla}_{e_1} B)(e_2, e_2) = (\widetilde{\nabla}_{e_2} B)(e_1, e_2)$ gives $H\tilde{f}_1 = 0$. These two equations imply that $H = 0$, which is a contradiction.

Since $\|\omega_4^3\|^2 = 0$ and $A_3 = \varepsilon_3 H I$, then the first equation of (25) becomes

$$\varepsilon_3 H^2 - c = 0. \tag{29}$$

When $c = 0$, (29) implies $H = 0$, a contradiction. So, $\overrightarrow{H} = 0$ and M_1^2 is minimal.

When $c > 0$, it follows from (29) that $\varepsilon_3 = 1$. Then, the mean curvature vector $\overrightarrow{H}$ $(= He_3)$ is space-like, and $H^2 = c$.

When $c < 0$, we know from (29) that $\overrightarrow{H}$ is a time-like vector with $H^2 = |c|$.

We complete the proof of Theorem 1.2.

Proof of Theorem 1.3. Choose a orthonormal frame field $\{e_1, \ldots, e_4\}$ such that $\overrightarrow{H} = He_3$. Since A_3 is diagonalizable, we denote by

$$A_3(e_1) = \lambda e_1,\ A_3(e_2) = \mu e_2. \tag{30}$$

Then, $\lambda + \mu = 2H$ and $\mathrm{tr}A_4 = 0$. Since $D_{e_i}\vec{H} = 0$, we easily prove that $e_i(H)e_3 + H\omega_3^4(e_i)e_4 = 0$ for $i = 1, 2$, which means that H is a constant and $\omega_3^4 = 0$. Thus, it follows from (30) that (22) is simplified to

$$H(\varepsilon_3(\lambda^2 + \mu^2) - 2c) = 0. \tag{31}$$

When $c = 0$, we have from (31) that $H(\lambda^2 + \mu^2) = 0$, which implies that $H = 0$ or $\lambda^2 + \mu^2 = 0$. This together with $\lambda + \mu = 2H$ gives that $H = 0$, that is, M_r^2 is minimal.

When $c < 0$, then $(\lambda^2 + \mu^2) - 2c \neq 0$ since $\varepsilon_3 = 1$. Thus, it follows from (31) that $H = 0$.

We claim that $\vec{H}$ is space-like when $c > 0$.

Suppose on contrary that $\vec{H}$ is time-like, then $\varepsilon_3 = -1$ and $H \neq 0$. Then, using (31) leads to $(\lambda^2 + \mu^2) + 2c = 0$, which contradicts the assumption that $(\lambda^2 + \mu^2) + 2c \neq 0$. So, $\vec{H}$ is space-like.

Since $\vec{H}$ is space-like, then $\varepsilon_3 = 1$. This which together with (31) gives

$$H(\lambda^2 + \mu^2 - 2c) = 0, \tag{32}$$

which implies that $H = 0$, and we complete the proof of Theorem 1.3.

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