More on the weakly S-2-prime ideals of commutative rings

Sanem Yavuz
Department of Mathematics
Yildiz Technical University
Istanbul
Türkiye
ssanemy@gmail.com

Bayram Ali Ersoy
Department of Mathematics
Yildiz Technical University
Istanbul
Türkiye
ersoya@yildiz.edu.tr

Ünsal Tekir
Department of Mathematics
Marmara University
Istanbul
Türkiye
utekir@marmara.edu.tr

Ece Yetkin Çelikel*
Department of Software Engineering
Faculty of Engineering
Hasan Kalyoncu University
Gaziantep
Türkiye
yetkinece@gmail.com
ece.celikel@hku.edu.tr

Abstract. Prime ideals and their generalizations are fundamental in various research areas, especially in commutative algebra. The study of weakly prime ideals is marked the beginning of this generalization. Subsequent research has further expanded these concepts, with recent attention on weakly 2-prime and S-2-prime ideals. This study aims for new characterizations of weakly S-2-prime ideals, a generalization that includes both weakly 2-prime and S-2-prime ideals. To achieve this goal, we construct an ideal disjoint with a multiplicatively closed subset of commutative rings. We explore several characterizations concerning weakly S-2-prime ideals and investigate this class of ideals in polynomial and formal power series rings. Besides, we examine several new results regarding the trivial extension and amalgamated algebra along an ideal with respect to a ring homomorphism concerning weakly S-2-prime ideals.

∗. Corresponding author

Keywords: 2-prime ideal, S-prime ideal, weakly 2-prime ideal, S-2-prime ideal, weakly S-2-prime ideal.

MSC 2020: 13A15, 13C05, 13A99.

1. Introduction

In this paper, we suppose that all rings are commutative with a non-zero identity. For any proper ideal Q of a ring R, the radical of Q is defined by intersection of all prime ideals containing Q, denoted by $\sqrt{Q}$ which is equivalent to the set $\{\alpha \in R : \alpha^n \in Q$ for some $n \in \mathbb{N}\}$. In particular, $Nil(R)$ is the set of all nilpotent elements of R called the nilradical of R and described by $Nil(R) := \sqrt{0_R} = \{\alpha \in R : \alpha^n = 0$ for some positive integer $n\}$. Furthermore, a ring R is called a reduced ring if it has no non-zero nilpotent elements (that is $Nil(R) = 0$).

The notion of prime ideals and its generalizations play a central role in commutative algebra, and so this concept has been generalized and investigated in many aspects. In 2003, Anderson and Smith [4] introduced the notion of weakly prime ideals. A proper ideal Q of a ring R is called weakly prime if $0 \neq \alpha\beta \in Q$ for some α, $\beta \in R$ implies that $\alpha \in Q$ or $\beta \in Q$. It is well known that prime ideals are weakly prime ideals but the other statement is not generally true, see [4]. On the other hand, S-prime ideals, which are extentions of prime ideals, were introduced by Hamed and Malek [10]. Remember that a subset S of R is called a multiplicatively closed subset (in briefly m.c.s) if S is closed under multiplication and $1 \in S$. Let S be an m.c.s of R and Q be an ideal with $Q \cap S = \emptyset$. In their work, Hamed and Malek defined an S-prime ideal of R as follows: if there exists an $s \in S$ such that for all α, $\beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha \in Q$ or $s\beta \in Q$. Later, Almahdi et al. [1] defined weakly S-prime ideals, which further generalized S-prime ideals. An ideal Q disjoint with S is considered a weakly S-prime ideal if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha \in Q$ or $s\beta \in Q$.

Furthermore, Beddani and Messirdi [6] introduced and studied 2-prime ideals, which offer another generalization of prime ideals and this concept has also been investigated by Nikandish et al. [15]. A proper ideal Q of R is called a 2-prime ideal if $\alpha, \beta \in R$ such that $\alpha\beta \in Q$, then either $\alpha^2 \in Q$ or $\beta^2 \in Q$. Additionally, Koç [13] described weakly 2-prime ideals as a generalization of 2-prime ideals and explored this notion in the context of compactly packedness and coprimely packedness in trivial extensions. Moreover, Issoual et al. [12] further examined properties of this class of ideals. In their framework, a proper ideal Q is called a weakly 2-prime ideal of R if $0 \neq \alpha\beta \in Q$ for some $\alpha, \beta \in R$, then either $\alpha^2 \in Q$ or $\beta^2 \in Q$.

As a recent research [16], the concept of S-2-prime ideals, which generalize both S-prime and 2-prime ideals, is introduced. An ideal Q of R with $Q \cap S = \emptyset$ is called an S-2-prime ideal of R if there exists an $s \in S$ such that for all $\alpha, \beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. As a more recent development, in

[17], the notion of weakly S-2-prime ideals, which generalize both S-2-prime and weakly 2-prime ideals, is defined. An ideal Q of R with $Q \cap S = \emptyset$ is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$.

In light of the ongoing exploration into generalizations of prime ideals, we endeavor to broaden the scope by incorporating S-2-prime ideals and weakly 2-prime ideals. Motivated by the aforementioned previous studies, we introduce and investigate new characterizations and properties of the concept of weakly S-2-prime ideals in commutative rings in Section 2 (Theorem 2.1, Propositions 2.1, 2.2). Furthermore, we explore weakly S-2-prime ideals in commutative rings whose characteristic is 2 (Theorem 2.2, Corollary 2.1), drawing insights from [3]. Moreover, we examine the behavior of these ideals in polynomial and formal power series rings (Theorems 2.3, 2.4), utilizing references ([2], [8], [9]). Additionally, we delve into new findings concerning weakly S-2-prime ideals in trivial ring extensions, idealizations, and amalgamated algebras along an ideal with regard to a ring homomorphism taking advantage of ([5], [7], [11], [14]) in the subsequent section (Theorems 3.1, 3.2).

As a result, we observe that many of the results established for S-2-prime ideals and weakly 2-prime ideals are analogously obtained by weakly S-2-prime ideals, which encompass a broader scope. Furthermore, in the light of the trivial ring extensions studied in S-2-prime ideals and amalgamated algebra along an ideal with respect to a ring homomorphism studied in weakly 2-prime ideals, we elaborated on the properties of weakly S-2-prime ideals on these algebraic structures.

2. Characterizations and properties of weakly S-2-prime ideals

In this section, we investigate weakly S-2-prime ideals and present their new properties. Unless otherwise stated, R denotes a commutative ring with identity. We recall the following definitions:

Definition 2.1. *Let S be an m.c.s of a ring R and Q be an ideal of R with $Q \cap S = \emptyset$.*

(i) *[16], Q is called an S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $\alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. In this case, we say that Q is associated to s.*

(ii) *[17], Q is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all α, $\beta \in R$ with $0 \neq \alpha\beta \in Q$, we have $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. In this case, we say that Q is associated to s.*

Clearly, an S-2-prime ideal of R is a weakly S-2-prime ideal of R. However, the converse implication is not true in general see [17, Example 2.4]. Now, we will present a new characterization of weakly S-2-prime ideals.

Theorem 2.1. *Let S be an m.c.s of R and Q be an ideal of R disjoint with S. The following statements are equivalent:*

(i) Q is a weakly S-2-prime ideal of R associated to $s \in S$.

(ii) For every $r \in R$, if $r^2 \notin (Q : s)$, then $(Q : r) \subseteq (0 : r) \cup \{r \in R : sr^2 \in Q\}$.

Proof. $(i) \implies (ii)$: Let Q be a weakly S-2-prime ideal of R associated to $s \in S$. Suppose that $r \in R$ with $r^2 \notin (Q : s)$ and $\beta \in (Q : r)$. Then, we have $\beta r \in Q$. If $0 \neq \beta r \in Q$, then $s\beta^2 \in Q$ since Q is a weakly S-2-prime ideal and $r^2 \notin (Q : s)$. Then, we have $\beta \in \{r \in R : sr^2 \in Q\}$. Now if $\beta r = 0$, then $\beta \in (0 : r)$, so we have $(Q : r) \subseteq (0 : r) \cup \{r \in R : sr^2 \in Q\}$.

$(ii) \Longrightarrow (i)$: Let $0 \neq \alpha\beta \in Q$ and $s\alpha^2 \notin Q$ for all $\alpha, \beta \in R$. Then, we have $\beta \in (Q : \alpha)$ and $\beta \notin (0 : \alpha)$. From the assumption, we have $\beta \in \{r \in R : sr^2 \in Q\}$. Therefore, $s\beta^2 \in Q$ and Q is a weakly S-2-prime ideal of R associated to s. □

In the following result, we will present another characterization of a weakly S-2-prime ideal. First of all, we need the next definitions.

Definition 2.2. *Suppose that S is an m.c.s of R and Q is a weakly S-2-prime ideal of R associated to $s \in S$.*

(i) Let $\alpha, \beta \in R$. We call (α, β) an S-double-zero of Q if $\alpha\beta = 0$, $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$.

(ii) Let $AB \subseteq Q$ for some ideals A, B of R. If (α, β) is not an S-double-zero of Q for every $\alpha \in A$ and $\beta \in B$, then we call Q a free S-double-zero with regard to AB.

Note that if Q is a weakly S-2-prime ideal of R without S-double-zeros, then Q is an S-2-prime ideal of R. So, if Q is a weakly S-2-prime ideal which is not an S-2-prime ideal, then there exists an S-double-zero of Q.

Let Q be a proper ideal of R. Recall from [3] that the ideal generated by n^{th} powers of elements of Q is denoted by $Q_{[n]} = \langle \{q^n : q \in Q\} \rangle$. It is easy to see that $Q_{[n]} \subseteq Q^n \subseteq Q$ and also the equality provides if $n = 1$. Moreover, if $n!.1_R$ is a unit of R, then $Q_{[n]} = Q^n$ see [3, Theorem 5].

Proposition 2.1. *Let S be an m.c.s of R, Q be a weakly S-2-prime ideal of R associated to $s \in S$ and P be a proper ideal of R with $\alpha P \subseteq Q$ for some $\alpha \in R$. If (α, p) is not an S-double-zero of Q for all $p \in P$ and $s\alpha^2 \notin Q$, then $sP_{[2]} \subseteq Q$. Furthermore, if 2.1_R is a unit, then $sP^2 \subseteq Q$.*

Proof. Assume that Q is a weakly S-2-prime ideal of R associated to $s \in S$ and $sP_{[2]} \nsubseteq Q$. Then, there exists $p \in P$ such that $sp^2 \notin Q$. Also, we have $\alpha p \in Q$ since $\alpha P \subseteq Q$. If $\alpha p \neq 0$, it contradicts with our assumption that $s\alpha^2 \notin Q$ and $sp^2 \notin Q$. Thus, $\alpha p = 0$. Since (α, p) is not an S-double-zero of Q and $s\alpha^2 \notin Q$, we conclude that $sp^2 \in Q$, which is a contradiction. Therefore, $sP_{[2]} \subseteq Q$. The "furthermore" part is clear because of $P_{[2]} = P^2$. □

Proposition 2.2. *Let S be an m.c.s of R, Q be a weakly S-2-prime ideal of R associated to $s \in S$ and $0 \neq AB \subseteq Q$ for some ideals A and B of R. If Q*

is a free S-double-zero with regard to AB, then either $sA_{[2]} \subseteq Q$ or $sB_{[2]} \subseteq Q$. Furthermore, if 2.1_R is a unit, then either $sA^2 \subseteq Q$ or $sB^2 \subseteq Q$.

Proof. Suppose that Q is a free S-double-zero with regard to AB and $0 \neq AB \subseteq Q$. If $sA_{[2]} \nsubseteq Q$, then there exits $\alpha \in A$ such that $s\alpha^2 \notin Q$. Since Q is a free S-double-zero with regard to AB, we conclude that (α, β) is not an S-double-zero of Q for all $\beta \in B$. From Proposition 2.1, we have $sB_{[2]} \subseteq Q$. The rest of the proof is clear as $A_{[2]} = A^2$ and $B_{[2]} = B^2$. □

Let Q be an ideal of a ring R. Then, we define the set of all elements of R whose square is in Q, that is $\sqrt[2]{Q} = \{\alpha \in R : \alpha^2 \in Q\}$. Also, it is easy to see that $Q \subseteq \sqrt[2]{Q} \subseteq \sqrt{Q}$. Note that $\sqrt[2]{Q}$ may not be an ideal of R. See the next example.

Example 2.1. Suppose that F be a field whose characteristic is not 2 and $R = F[X, Y, Z]$, where X, Y, Z are indeterminates over F. Take the ideal $Q = (X^2, Y^2, Z^2)$ of R. We know that $\sqrt{Q} = (X, Y, Z)$ and also $X, Y, Z \in \sqrt[2]{Q}$. However, $(X + Y + Z)^2 = X^2 + Y^2 + Z^2 + 2XY + 2YZ + 2XZ \notin Q$. Therefore, $\sqrt[2]{Q}$ is not an ideal of R.

We might inquire under what conditions $\sqrt[2]{Q}$ becomes an ideal of R. We provide an answer to this question with the next result.

Proposition 2.3. *Let R be a ring and Q be a proper ideal of R.*

- (*i*) *$\sqrt[2]{Q}$ is an ideal of R if and only if $2(\sqrt[2]{Q})^2 \subseteq Q$.*
- (*ii*) *If $char(R) = 2$, then $\sqrt[2]{Q}$ is an ideal of R.*
- (*iii*) *Let S be an m.c.s of a ring R, $s \in S$ and $2(\sqrt[2]{Q : s})^2 \subseteq (Q : s)$. Then, for any ideal K of R, $K \subseteq \sqrt[2]{Q : s}$ if and only if $sK_{[2]} \subseteq Q$.*

Proof. (i) Let $\sqrt[2]{Q}$ be an ideal of R and α, $\beta \in \sqrt[2]{Q}$. This implies that α^2, $\beta^2 \in Q$. Since $\sqrt[2]{Q}$ is an ideal of R, we have $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \in Q$. We conclude that $2\alpha\beta \in Q$ or $2(\sqrt[2]{Q})^2 \subseteq Q$.

Conversely, let $2(\sqrt[2]{Q})^2 \subseteq Q$ and α, $\beta \in \sqrt[2]{Q}$. Then, $(r\alpha)^2 = r^2\alpha^2 \in Q$ for all $r \in R$, and thus $r\alpha \in \sqrt[2]{Q}$. Also, from assumption, we have $2\alpha\beta \in 2(\sqrt[2]{Q})^2 \subseteq Q$. This implies $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \in Q$. Hence, $\alpha + \beta \in \sqrt[2]{Q}$ and $\sqrt[2]{Q}$ is an ideal of R.

(ii) Let $char(R) = 2$. We have $2(\sqrt[2]{Q})^2 = (0) \subseteq Q$. Using ($i$), proof is clear.

(iii) Let $2(\sqrt[2]{Q : s})^2 \subseteq (Q : s)$. From ($i$), $\sqrt[2]{Q : s}$ is an ideal of R. Suppose that $K \subseteq \sqrt[2]{Q : s}$ and $sk^2 \in sK_{[2]}$ such that $k \in K$. We have $k \in \sqrt[2]{Q : s}$ that is $sk^2 \in Q$. We conclude $sK_{[2]} \subseteq Q$.

Conversely, let $sK_{[2]} \subseteq Q$ and $k \in K$. We have $sk^2 \in sK_{[2]} \subseteq Q$ and so $k \in \sqrt[2]{Q : s}$. Thus, we conclude $K \subseteq \sqrt[2]{Q : s}$. □

Now, we will present a new characterization of weakly S-2-prime ideals, and also we will benefit from this characterization to examine the weakly S-2-prime ideals on polynomial and formal power series rings.

Theorem 2.2. *Let S be an m.c.s of R and Q be an ideal of R disjoint with S such that $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$. The following assertions are equivalent:*

(i) *Q is a weakly S-2-prime ideal of R associated to $s \in S$.*

(ii) *There exists an $s \in S$ such that for all $\alpha \in R - \sqrt[2]{Q:s}$, either $(Q:\alpha) \subseteq ann(\alpha)$ or $(Q:\alpha) \subseteq \sqrt[2]{Q:s}$.*

(iii) *There exists an $s \in S$ such that for all $\alpha \in R$ with $\alpha^2 \notin (Q:s)$, either $(Q:\alpha) \subseteq ann(\alpha)$ or $s(Q:\alpha)_{[2]} \subseteq Q$.*

(iv) *There exists an $s \in S$ such that $0 \neq \alpha K \subseteq Q$ for some $\alpha \in R$ and ideal K of R, either $s\alpha^2 \in Q$ or $sK_{[2]} \subseteq Q$.*

(v) *There exists an $s \in S$ such that $0 \neq JK \subseteq Q$ for some ideals J, K of R, either $sJ_{[2]} \subseteq Q$ or $sK_{[2]} \subseteq Q$.*

Proof. (i)$\Longrightarrow$(ii) Let Q be a weakly S-2-prime ideal of R associated to s and take $\alpha \in R - \sqrt[2]{Q:s}$. Then, $\alpha^2 \notin (Q:s)$. From Theorem 2.1, we have $(Q:\alpha) \subseteq (0:\alpha) \cup \sqrt[2]{Q:s}$. Since $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$, $\sqrt[2]{Q:s}$ is an ideal of R. Thus, we have either $(Q:\alpha) \subseteq ann(\alpha)$ or $(Q:\alpha) \subseteq (\sqrt[2]{Q:s})$.

(ii)$\Longleftrightarrow$(iii) Clear from Proposition 2.3 (iii).

(iii)$\Longrightarrow$(iv) Let $0 \neq \alpha K \subseteq Q$ for all $\alpha \in R$ and ideal K of R with $s\alpha^2 \notin Q$. Then, $\alpha^2 \notin (Q:s)$ and from assumption, we conclude either $K \subseteq (Q:\alpha) \subseteq ann(\alpha)$ or $sK_{[2]} \subseteq s(Q:\alpha)_{[2]} \subseteq Q$. The first case is impossible because $\alpha K \neq 0$. So, we have $sK_{[2]} \subseteq Q$.

(iv)$\Longrightarrow$(i) Suppose that $0 \neq \alpha\beta \in Q$ for all α, $\beta \in R$. Put $K = (\beta)$ in (iv).

(v)$\Longrightarrow$(ii) Let $\alpha \in R - \sqrt[2]{Q:s}$ and $\beta \in (Q:\alpha)$. If $\alpha\beta = 0$, then $\beta \in ann(\alpha)$. Suppose that $\alpha\beta \neq 0$. Put $J = (\alpha)$ and $K = (\beta)$. Then, (v) implies that there exists an $s \in S$ such that either $s\alpha^2 \in sJ_{[2]} \subseteq Q$ or $s\beta^2 \in sK_{[2]} \subseteq Q$. Since $\alpha \in R - \sqrt[2]{Q:s}$, we have $\beta \in \sqrt[2]{Q:s}$. Thus, $(Q:\alpha) \subseteq ann(\alpha) \cup \sqrt[2]{Q:s}$, so the claim is clear.

(ii)$\Longrightarrow$(v) Let $0 \neq JK \subseteq Q$ for all ideals J, K of R and $sJ_{[2]} \nsubseteq Q$. Then, there exists $\alpha \in J$ such that $s\alpha^2 \notin Q$ or $\alpha \in R - \sqrt[2]{Q:s}$. If $\alpha K \neq 0$, then by assumption, $K \subseteq (Q:\alpha) \subseteq \sqrt[2]{Q:s}$ which implies that $sK_{[2]} \subseteq Q$. So, suppose $\alpha K = 0$. Since $JK \neq 0$, there exists $\beta \in J$ such that $\beta K \neq 0$. If $\beta \in R - \sqrt[2]{Q:s}$, again by our assumption we have $sK_{[2]} \subseteq Q$. Now, we can suppose that $\beta \in \sqrt[2]{Q:s}$. Since $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$, $(\sqrt[2]{Q:s})$ is an ideal of R, $\beta + \alpha \in R - \sqrt[2]{Q:s}$. Moreover, $0 \neq (\beta+\alpha)K = \beta K \subseteq Q$. Then, our assumption yields $K \subseteq (Q:\beta+\alpha) \subseteq \sqrt[2]{Q:s}$ which implies that $sK_{[2]} \subseteq Q$, as desired. □

The following corollary is a direct consequence of Theorem 2.2 and Proposition 2.3.

Corollary 2.1. *Suppose that R is a ring whose characteristic is 2, S is an m.c.s of R and Q is an ideal of R with $Q \cap S = \emptyset$. Then, Q is a weakly S-2-prime ideal of R associated to $s \in S$ if and only if there exists an $s \in S$ such that for all ideals J, K of R with $0 \neq JK \subseteq Q$, either $sJ_{[2]} \subseteq Q$ or $sK_{[2]} \subseteq Q$.*

Let R be a ring and $R[X]$ be a polynomial ring, where X is an indeterminate over R. For any $g(x) = \sum_{j=0}^{n} \alpha_j X^j$, the content ideal of g is denoted by $c(g) = (\alpha_0, \alpha_1, ..., \alpha_n)$ [9]. If Q is an ideal of R, then $Q[X] = \{g \in R[X] : c(g) \subseteq Q\}$ is an ideal of $R[X]$. Moreover, a subset $S[X]$ of $R[X]$ is called an m.c.s of $R[X]$ if $S[X]$ closed under multiplication and $1 \in S[X]$. It is clear to verify that if S is an m.c.s of R, then $S[X]$ is an m.c.s of $R[X]$.

Theorem 2.3. *Let R be a ring with 2.1_R a unit of R and S be an m.c.s of R with $s \in S$. Suppose that Q and $(Q : s)$ are radical ideals of R. Then, Q is a weakly S-2-prime ideal of R associated to s if and only if $Q[X]$ is a weakly $S[X]$-2-prime ideal of $R[X]$ associated to s.*

Proof. Let Q be a weakly S-2-prime ideal of R associated to s. It is clear that $Q[X] \cap S[X] = \emptyset$. Since $\sqrt[2]{Q : s} \subseteq \sqrt{Q : s} = (Q : s)$, this gives $2(\sqrt[2]{Q : s})^2 \subseteq (Q : s)$. Let $0 \neq gh \in Q[X]$ for all $g, h \in R[X]$. This implies $c(gh) \subseteq Q$. Suppose $deg(g) = k$. From Dedekind-Mertens Theorem [9, Theorem 28.1], we have $c(g)c(h)^{k+1} = c(gh)c(h)^k \subseteq Q$. Since Q is a radical ideal, we have $0 \neq c(g)c(h) \subseteq Q$. From Theorem 2.2 (v), $sc(g)_{[2]} \subseteq Q$ or $sc(h)_{[2]} \subseteq Q$. As 2.1_R is a unit of R, we have $sc(g^2) \subseteq sc(g)^2 = sc(g)_{[2]} \subseteq Q$ or $sc(h^2) \subseteq sc(h)^2 = sc(h)_{[2]} \subseteq Q$. Since $sc(g^2) \subseteq Q$, we have $g^2 \in (Q : s)[X]$. That is, $sg^2 \in Q[X]$. Similarly, we can achieve $sh^2 \in Q[X]$. Hence, $Q[X]$ is a weakly $S[X]$-2-prime ideal of $R[X]$ associated to s. The converse part is straightforward by taking constant polynomials. □

From [2], a ring R is called Gaussian ring if $c(gh) = c(g)c(h)$ for every g, $h \in R[X]$. Then, we can remove the condition "Q is a radical ideal of R" in the Theorem 2.3 provided that R is a Gaussian ring.

Let R be a ring and $R[[X]]$ be a ring of formal power series, where X is an indeterminate over R. For any $g = \sum_{j=0}^{\infty} \alpha_j X^j \in R[[X]]$, the content ideal of g is denoted by $c(g) = \langle \{\alpha_j : j \in \mathbb{N} \cup \{0\}\} \rangle$. If Q is an ideal of R, then $Q[[X]] = \{g \in R[[X]] : c(g) \subseteq Q\}$ is an ideal of $R[[X]]$. Note that if S is an m.c.s of R, then $S[[X]]$ is an m.c.s of $R[[X]]$.

In [8], the authors established a version of the Dedekind-Mertens Theorem for Noetherian formal power series rings. We will now examine the weakly S-2-prime ideals in Noetherian formal power series rings.

Theorem 2.4. *Let R be a Noetherian ring, S be an m.c.s of R with $s \in S$ and 2.1_R be a unit of R. Suppose that Q and $(Q : s)$ are radical ideals of R. Then, Q is a weakly S-2-prime ideal of R associated to s if and only if $Q[[X]]$ is a weakly $S[[X]]$-2-prime ideal of $R[[X]]$ associated to s.*

Proof. $\Longleftarrow$: Proof is straightforward by taking constant power series.

$\Longrightarrow$: Let Q be a weakly S-2-prime ideal of R associated to s. It is clear that $Q[[X]] \cap S[[X]] = \emptyset$. Let $0 \neq gh \in Q[[X]]$ for all g, $h \in R[[X]]$. This implies $c(gh) \subseteq Q$. Let $\mu(c(g))$ denotes the minimal number of the generators of $c(g)$. As R is a Noetherian ring, we can choose k as maximum of the numbers

$\mu(c(g)_m)$, taken over all maximal ideals m of R. From [8, Theorem 2.6], we have $c(g)c(h)^k = c(gh)c(h)^{k-1} \subseteq Q$. Since Q is a radical ideal, we have $0 \neq c(g)c(h) \subseteq Q$. Since 2.1_R is a unit of R and $2(\sqrt[2]{Q:s})^2 \subseteq (Q:s)$, by the similar argument in the Theorem 2.3, we conclude that $sc(g^2) \subseteq Q$ or $sc(h^2) \subseteq Q$. Since $sc(g^2) \subseteq Q$, we have $g^2 \in (Q:s)[[X]]$. That is, $sg^2 \in Q[[X]]$. Similarly, we can achieve $sh^2 \in Q[[X]]$. Hence, $Q[[X]]$ is a weakly $S[[X]]$-2-prime ideal of $R[[X]]$ associated to s. □

In the previous theorem, the condition that Q and $(Q:s)$ are radical ideals of R does not ensure that these ideals are weakly S-2-prime ideals of R.

Example 2.2. Let $R = \mathbb{Z}_{12}$ be a ring, $S = \{\overline{1}, \overline{5}\}$ an m.c.s of R, $Q = (\overline{6})$ an ideal of R disjoint with S. We can achieve $\sqrt{Q} = (\overline{6})$ and $(Q:s) = \{\overline{0}, \overline{6}\}$. Q and $(Q:s)$ are radical ideals of R since $Q = \sqrt{Q} = \{\overline{0}, \overline{6}\}$ and $(Q:s) = \sqrt{(Q:s)} = \{\overline{0}, \overline{6}\}$. However, Q and $(Q:s)$ are not weakly S-2-prime ideals of R since $0 \neq \overline{2}.\overline{3} \in Q$ (and $(Q:s)$) but $s.\overline{2}^2 \notin Q$ (and $\notin (Q:s)$) and $s.\overline{3}^2 \notin Q$ (and $\notin (Q:s)$) for all $s \in S$.

3. Idealization and amalgamation properties on the weakly S-2-prime ideals

In this part, we examine the class of weakly S-2-prime ideals of characteristics over $R(+)M$ constructions. Let M be an R-module. The trivial extension or idealization $R(+)M = \{(r,m) : r \in R,\ m \in M\}$ is a commutative ring with componentwise addition and multiplication described by $(\alpha, m)(\beta, m') = (\alpha\beta, \alpha m' + \beta m)$ for all (α, m), $(\beta, m') \in R(+)M$ (see [5, 14]).

Theorem 3.1. *Let S be an m.c.s of a ring R, Q an ideal of R with $Q \cap S = \emptyset$ and M an R-module. Then, the following statements are equivalent:*

(i) $Q(+)M$ is a weakly $(S(+)0)$-2-prime ideal (and weakly $(S(+)M)$-2-prime ideal) of $R(+)M$.

(ii) Q is a weakly S-2-prime ideal of R and for every S-double-zero (α, β) of Q, we have $\alpha M = 0 = \beta M$.

Proof. The proof is clear from [17, Theorem 2.14]. □

Example 3.1. Suppose that R is a reduced ring, M an $R-$module and S an m.c.s of R. The unique ideal of $R(+)M$ which has the form $Q(+)M$ which is weakly $(S(+)0)$-2-prime ideal (resp. weakly $(S(+)M)$-2-prime ideal) and not $(S(+)0)$-2-prime ideal (resp. not $(S(+)M)$-2-prime ideal) is $0(+)M$. Indeed, if $Q(+)M$ is weakly $(S(+)0)$-2-prime ideal (resp. weakly $(S(+)M)$-2-prime ideal) and not $(S(+)0)$-2-prime ideal (resp. not $(S(+)M)$-2-prime ideal), then from [16, Theorem 6] and [17, Theorem 2.14], we have Q is a weakly S-2-prime ideal and not S-2-prime ideal of R. From [17, Theorem 2.6], we know that if R is a reduced ring and Q is a weakly S-2-prime ideal, then either Q is an S-2-prime ideal or $Q = 0$. Then, we have $Q = 0$.

Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Describe the subring of $A \times B$ as:

$$A \bowtie^f J = \{(\alpha, f(\alpha) + j) : \alpha \in A,\ j \in J\}$$

called the amalgamation of A with B along the ideal J with regard to f. This structure is presented and examined by [7].

We will examine this amalgamation algebra property for weakly S-2-prime ideals.

Theorem 3.2. *Suppose that $f : A \to B$ is a ring homomorphism, J is an ideal of B, Q is an ideal of A and S is an m.c.s of A.*

(1) *If $Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$, then Q is a weakly S-2-prime ideal of A.*

(2) *If Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A, then the following assertions are equivalent:*

  (i) *$Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$,*

  (ii) *For each S-double-zero $(\alpha, \beta) \in A \times A$ of Q, we have $f(\alpha)J = 0 = f(\beta)J$ and $J^2 = 0$.*

We need the following lemmas to verify the theorem above.

Lemma 3.1. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and Q be an ideal of A. Then,*

$$(Q \bowtie^f J)^2 = Q^2 \bowtie^f (f(Q)J + J^2).$$

Proof. See [11, Lemma 3.4]. □

Lemma 3.2. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B, Q be an ideal of A and S be an m.c.s of A. The following assertions are equivalent:*

(i) *Q is an S-2-prime ideal of A.*

(ii) *$Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

Proof. Let Q be an S-2-prime ideal of A and $(\alpha, f(\alpha) + j_1)(\beta, f(\beta) + j_2) \in Q \bowtie^f J$ for all $(\alpha, f(\alpha)+j_1), (\beta, f(\beta)+j_2) \in A \bowtie^f J$. Then, $(\alpha\beta, f(\alpha\beta)+f(\alpha)j_2+f(\beta)j_1 + j_1j_2) \in Q \bowtie^f J$. We conclude that $\alpha\beta \in Q$. From assumption, there exists an $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Therefore, $(s, f(s))(\alpha, f(\alpha) + j_1)^2 = (s\alpha^2, f(s\alpha^2) + 2f(s\alpha)j_1 + f(s)j_1^2) \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta) + j_2)^2 = (s\beta^2, f(s\beta^2) + 2f(s\beta)j_2 + f(s)j_2^2) \in Q \bowtie^f J$ for an $(s, f(s)) \in (S \bowtie^f 0)$. Hence, $Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.
The converse part is similarly verified. □

Proof of Theorem 3.2. (1) Suppose that $Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$. Let $0 \neq \alpha\beta \in Q$ for all $\alpha,\ \beta \in A$. We have $0 \neq (\alpha, f(\alpha))(\beta, f(\beta)) \in Q \bowtie^f J$. From assumption, there exists an $(s, f(s)) \in (S \bowtie^f 0)$ such that $(s, f(s))(\alpha, f(\alpha))^2 \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta))^2 \in Q \bowtie^f J$. This implies that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Hence, Q is a weakly S-2-prime ideal of A.

(2) Suppose that Q is a weakly S-2-prime ideal and which is not an S-2-prime ideal of A. Let $Q \bowtie^f J$ be a weakly $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$ and $(\alpha, \beta) \in A \times A$ be an S-double-zero of Q. Suppose $f(\alpha) \notin\ ann(J)$. So, there exists $j \in J$ such that $f(\alpha)j \neq 0$. As a result, $(0, 0) \neq (\alpha, f(\alpha))(\beta, f(\beta) + j) = (0, f(\alpha\beta)+f(\alpha)j) \in Q \bowtie^f J$. From assumption, there exists an $(s, f(s)) \in (S \bowtie^f 0)$ such that $(s, f(s))(\alpha, f(\alpha))^2 \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta) + j)^2 \in Q \bowtie^f J$. This implies that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. It is a contradiction, so $f(\alpha)J = 0$. Similarly, we conclude $f(\beta)J = 0$. Moreover, from Lemma 3.2, $Q \bowtie^f J$ is not an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$. From [17, Theorem 2.6], we know that $(Q \bowtie^f J)^2 = 0$. We have from Lemma 3.1, $(Q \bowtie^f J)^2 = Q^2 \bowtie^f (f(Q)J + J^2) = 0$. We have $J^2 = 0$, because of $Q^2 = 0$.

Conversely, assume that $(\alpha, f(\alpha) + i)$, $(\beta, f(\beta) + j) \in A \bowtie^f J$ such that $(0, 0) \neq (\alpha, f(\alpha) + i)(\beta, f(\beta) + j) \in (Q \bowtie^f J)$.

Case 1. $\alpha\beta \neq 0$: Since Q is a weakly S-2-prime ideal of A, there exists an $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Hence, $(s, f(s))(\alpha, f(\alpha) + i)^2 \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta) + j)^2 \in Q \bowtie^f J$ for an $(s, f(s)) \in (S \bowtie^f 0)$, as desired.

Case 2. $\alpha\beta = 0$: We know that Q is a weakly S-2-prime ideal and which is not an S-2-prime ideal of A, so we have S-double-zero of Q. Without loss of the generality, we can assume $s\alpha^2 \notin Q$ and $s\beta^2 \notin Q$. Thus, (α, β) is an S-double-zero of Q and from assumption, we have $f(\alpha)J = 0 = f(\beta)J$. Then, $(\alpha, f(\alpha) + i)(\beta, f(\beta) + j) = (\alpha\beta, f(\alpha\beta) + f(\alpha)j + f(\beta)i + ij) = (0, ij) = (0, 0)$ because of $J^2 = 0$. It is a contradiction.

In view of Theorem 3.2 (2) and Lemma 3.2, we conclude the following corollary.

Corollary 3.1. *Suppose that $f : A \to B$ is a ring homomorphism, J is an ideal of B with $J^2 = 0$, Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A, where S is an m.c.s of A, for each $(\alpha, \beta) \in A \times A$ as an S-double-zero of Q, $(f(\alpha), f(\beta)) \in ann(J) \times ann(J)$. Then, $Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal which is not an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

Corollary 3.2. *Suppose that (A, M) is a local ring with a maximal ideal M and S is an m.c.s of A. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B with $f(M)J = 0$. Then, the following assertions are equivalent:*

(*i*) *Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A and $J^2 = 0$.*

(*ii*) *$Q \bowtie^f J$ is a weakly $(S \bowtie^f 0)$-2-prime ideal which is not an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.*

Proof. (i)$\Longrightarrow$(ii) Let Q be a weakly S-2-prime ideal which is not an S-2-prime ideal of A and take $(\alpha,\beta)\in A\times A$ as an S-double-zero of Q. We claim $\alpha,\beta\in M$. Let $\alpha\notin M$. Thus, α is invertible and so $\beta=0$, which is a contradiction. Hence, $(\alpha,\beta)\in M\times M$ and from hypothesis, we have $f(\alpha)J=0=f(\beta)J$. The result is clear from Theorem 3.2 (2) and Lemma 3.2.

(ii)$\Longrightarrow$(i) Let $Q\bowtie^f J$ be a weakly $(S\bowtie^f 0)$-2-prime ideal which is not an $(S\bowtie^f 0)$-2-prime ideal of $A\bowtie^f J$. Then, from Theorem 3.2 (1) and Lemma 3.2, we have Q is a weakly S-2-prime ideal which is not an S-2-prime ideal of A. From Theorem 3.2 (2), we have $J^2=0$. □

Corollary 3.3. *Suppose that (A,M) is a local ring with a maximal ideal M and S is an m.c.s of A. Let $f:A\to B$ be a ring homomorphism and J be an ideal of B with $f(M)J=0$ and $M^2=0$. If $J^2=0$, then every ideal of $A\bowtie^f J$ disjoint with $(S\bowtie^f 0)$ is a weakly $(S\bowtie^f 0)$-2-prime ideal.*

Proof. It is known that $A\bowtie^f J$ is a local ring with the maximal ideal $M\bowtie^f J$. From Lemma 3.1, $(M\bowtie^f J)^2=M^2\bowtie^f (f(M)J+J^2)=0$. From [13, Lemma 1], every ideal of $A\bowtie^f J$ disjoint with $(S\bowtie^f 0)$ is an $(S\bowtie^f 0)$-2-prime and weakly $(S\bowtie^f 0)$-2-prime ideal. □

Corollary 3.4. *Suppose that $f:A\to B$ is a ring homomorphism, S is an m.c.s of A, J is an ideal of B and Q is an ideal of A with $Nil(A)\not\supseteq Q$. Then, the following assertions are equivalent:*

(i) *$Q\bowtie^f J$ is an $(S\bowtie^f 0)$-2-prime ideal of $A\bowtie^f J$.*

(ii) *$Q\bowtie^f J$ is a weakly $(S\bowtie^f 0)$-2-prime ideal of $A\bowtie^f J$.*

Proof. (i)$\Longrightarrow$(ii) It is clear.

(ii)$\Longrightarrow$(i) Let $Q\bowtie^f J$ be a weakly $(S\bowtie^f 0)$-2-prime ideal of $A\bowtie^f J$. From Theorem 3.2 (1), Q is a weakly S-2-prime ideal of A. From [17, Theorem 2.6], we know that if Q is a weakly S-2-prime ideal of A, then either Q is an S-2-prime ideal of A or $Nil(A)\supseteq Q$. Therefore, Q is an S-2-prime ideal of A. From Lemma 3.2, $Q\bowtie^f J$ is an $(S\bowtie^f 0)$-2-prime ideal of $A\bowtie^f J$. □

4. Conclusion

In this study, we investigate new characterizations and properties of weakly S-2-prime ideals in commutative rings as generalizations of S-2-prime ideals with the help of S-2-prime and weakly 2-prime ideals. Besides, we examine the properties of weakly S-2-prime ideals in rings with characterization 2 and in polynomial and power series rings. Also, we delve into this ideal in idealization and amalgamated algebras along an ideal with regard to a ring homomorphism and we obtain many results. For future work, other generalizations of S-2-prime ideals can be worked on modules with the help of ideal reduction and ideal expansion functions in the light of this paper.

Acknowledgement

The authors would like to thank the editor and referees for their valuable comments, which have enhanced this research paper.

References

[1] F. A. A. Almahdi, E. M. Bouba, M. Tamekkante, *On weakly S-prime ideals of commutative rings*, An. Şiinţ. Univ. "Ovidius" Constanţa Ser. Mat., 29 (2021), 173-186.

[2] D. D. Anderson, V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra, 26 (1998), 2265-2272.

[3] D. D. Anderson, K. R. Knopp, R. L. Lewin, *Ideals generated by powers of elements*, Bull. Aust. Math. Soc., 49 (1994), 373-376.

[4] D. D. Anderson, E. Smith, *Weakly prime ideals*, Houston J. Math., 29 (2003), 831-840.

[5] D. D. Anderson, M. Winders, *Idealization of a module*, J. Commut. Algebra, 1 (2009), 3-56.

[6] C. Beddani, W. Messirdi, 2*-prime ideals and their applications*, J. Algebra Appl., 15 (2016), 1650051.

[7] M. D'Anna, C. A. Finocchiaro, C. A. Fontana, *Amalgamated algebras along an ideal*, Commutative Algebra and its Applications, (2009), 155-172.

[8] N. Epstein, J. Shapiro, *A Dedekind-Mertens theorem for power series rings*, Proc. Amer. Math. Soc., 144 (2016), 917-924.

[9] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, Inc., New York, 1972.

[10] A. Hamed, A. Malek, *S-prime ideals of a commutative ring*, Beitr. Algebra Geom., 61 (2020), 533-542.

[11] M. Issoual, N. Mahdou, *Amalgamated algebras along an ideal defined by* 2*-absorbing-like conditions*, Indian J. Math., 63 (2021), 59-77.

[12] M. Issoual, N. Mahdou, Ü. Tekir, S. Koç, *More on the weakly* 2*-prime ideals of commutative rings*, Filomat, 38 (2024), 6099-6108.

[13] S. Koç, *On weakly* 2*-prime ideals in commutative rings*, Comm. Algebra, 49 (2021), 3387-3397.

[14] M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math., New York, 1962.

[15] R. Nikandish, M. J. Nikmehr, A. Yassine, *More on the* 2*-prime ideals of commutative rings*, Bull. Korean Math. Soc., 57 (2020), 117-126.

[16] S. Yavuz, B. A. Ersoy, Ü. Tekir, E. Yetkin Çelikel, *On* S*-2-prime ideals of commutative rings*, Mathematics, 12 (2024), 1636.

[17] S. Yavuz, B. A. Ersoy, Ü. Tekir, E. Yetkin Çelikel, *Weakly* S*-2-prime ideals of commutative rings*, Moroccan Journal of Algebra and Geometry with Applications, (2024), 1-8.

Accepted: February 14, 2025