

New characteristics and applications of the EP, normal and Hermitian matrices

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Abstract. In this paper, we present the EP-matrices, normal, and new features of the Hermitian matrix. We are going to push it to a higher order form. This paper describes EP-matrices, normal matrices and Hermitian matrix equivalent forms by using Core inverse. We also give several special equivalent facts by use decompositions. We also give other special equivalent facts. A new equivalent condition for the reverse order law is also obtained.

Keywords: EP matrix, Hermitian matrix, generalized inverse, decompositions, normal matrix, core inverse.

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1. Introduction

In this paper, $\mathbb{C}^{m \times n}$ and $\mathbb{R}^{m \times n}$ stand for the sets of all $m \times n$ matrices over the fields of complex numbers and real numbers, respectively. A^* stands for the conjugate transpose of $A \in \mathbb{C}^{m \times n}$ and $r(A)$ stand for the rank of $A \in \mathbb{C}^{m \times n}$. $\det(A)$ stands for the determinant of $A \in \mathbb{C}^{m \times m}$, and I_m denotes the identity matrix of order m .

The Core inverse denoted by $A^\oplus \in \mathbb{C}^{m \times m}$ of A is the unique matrix satisfying the following matrix equations, $r(A^2) = r(A)$

- (1) $AA^\oplus A = A$,
- (2) $A(A^\oplus)^2 = A^\oplus$,
- (3) $(AA^\oplus)^* = AA^\oplus$ ([1]).

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The Moore-penrose inverse denoted by $A^\dagger$ of A is the unique matrix satisfying the following matrix equations

- (1) $AA^\dagger A = A$,
- (2) $A^\dagger AA^\dagger = A^\dagger$,
- (3) $(AA^\dagger)^* = AA^\dagger$,
- (4) $(A^\dagger A)^* = A^\dagger A$ ([15]).

Recall that a matrix $A \in \mathbb{C}^{m \times m}$ is said to be EP-matrix if and only if it satisfies the following equality

$$(1) \quad AA^\dagger = A^\dagger A.$$

A matrix $A \in \mathbb{C}^{m \times m}$ is said to be Hermitian matrix if and only if it satisfies the following equality

$$(2) \quad A^* = A.$$

A matrix $A \in \mathbb{C}^{m \times m}$ is said to be normal matrix if and only if it satisfies the following equality

$$(3) \quad AA^* = A^* A.$$

In addition to the definitions in (1), (2) and (3), a Hermitian matrix can be characterized by some other matrix equalities and facts. It can be expressed by the following facts

- (4) $f(A, A^\dagger) = 0 \Leftrightarrow AA^\dagger = A^\dagger A$,
- (5) $f(A, A^*) = 0 \Leftrightarrow A = A^*$,
- (6) $f(A, A^*) = 0 \Leftrightarrow AA^* = A^* A$,

where $f(\cdot)$ is the certain ordinary algebraic operation of A and A^* , A and $A^\dagger$. Due to the arbitrariness of matrix expressions, there does not exist a general and significant method to construct the matrix equations in (4) except some special cases. We already have the following facts from [6][22]:

- (7) $A^2 = AA^* \Leftrightarrow A = A^*$,
- (8) $A^2 = A^* A \Leftrightarrow A = A^*$,
- (9) $A^3 = AA^* A \Leftrightarrow A = A^*$.

Lemma 1.1 ([13]). *Let $A \in \mathbb{C}^{m \times n}$. The singular value decomposition of A is*

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V,$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are two unitary matrices, namely, $UU^* = U^*U = I_m$, $VV^* = V^*V = I_n$, and $\Sigma \in \mathbb{R}^{s \times s}$ is a positive diagonal matrix composed of the singular values of A with $s = r(A)$. In particular, if $A \in \mathbb{C}^{m \times m}$, then A admits the following decomposition:

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} (VU) U^* = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*,$$

where VU can be decomposed as $VU = \begin{bmatrix} K & L \\ K_1 & L_1 \end{bmatrix}$ with $(VU)(VU)^* = I_m$, $K \in \mathbb{C}^{s \times s}$, $L \in \mathbb{C}^{s \times (m-s)}$, and

$$KK^* + LL^* = I_s, \quad KK^* \leq I_s, \quad LL^* \leq I_s,$$

where $KK^* \leq I_s$, $LL^* \leq I_s$ mean that $I_s - KK^*$ and $I_s - LL^*$ are positive semi-definite.

So, A is Hermitian if and only if $L = 0$ and $\Sigma K = K^* \Sigma$; A is normal if and only if $L = 0$ and $\Sigma K = K \Sigma$; and finally, A is EP if and only if $L = 0$.

In [2, 6, 7, 10, 13, 15, 21], the properties of the EP-matrix, normal matrix, Hermitian matrix and their equivalence relation are described. Based on this background, we discuss their other equivalent forms. In [9] [18] and [19], the equivalent analysis of different inversion laws of matrix is described. This paper also gives an equivalent form of the reverse order law.

This paper is organized as follows

The second part gives some equivalent forms of the EP-matrix and corollary:

- (I) A is EP.
- (II) $AA^\dagger A^\oplus = A^\dagger A^\oplus A$.
- (III) $AA^\oplus A^* = A^* AA^\oplus$.
- (IV) $AA^\oplus A^\dagger = A^\oplus A^\dagger A$.
- (V) $A^\dagger AA^\oplus = A^\oplus A^\dagger A$ and the third part gives some equivalent forms of the normal matrix and corollary.

The fourth part describes the relevant properties of the Hermitian matrix

- (I) A is Hermitian,
- (II) $AA^\oplus = A^* A^\oplus$,
- (III) $AA^\oplus = A^\dagger A^*$,
- (IV) $A^\dagger A = A^\oplus A^*$,
- (V) $AA^\oplus = A^* A^\dagger$.

We aim to $A^5 = (AA^*)^2 A$ and $A^7 = (AA^*)^3 A \Leftrightarrow A = A^*$ and higher ones. A new equivalent condition for the reverse order law is also obtained $(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (B^* A^* A B B^* A^*)^\dagger = (A^*)^\dagger (B B^*)^\dagger (A^* A)^\dagger (B^*)^\dagger$.

2. Related results of the EP-matrix

In this section we give the equivalent form of the EP-matrix of the form $RS = XY$ and $XY = YX$, where $R, S, X, Y \in A^\dagger, A, A^\oplus, A^*$, and some conclusions are obtained.

Lemma 2.1 ([21]). *Let $A \in \mathbb{C}^{m \times m}$. Then, the following results hold:*

$$(I) \quad A^2 A^\dagger = AA^\dagger A \Leftrightarrow AA^\dagger = A^\dagger A;$$

$$(II) \quad A^\dagger A^2 = AA^\dagger A \Leftrightarrow AA^\dagger = A^\dagger A.$$

We present necessary and sufficient conditions for a matrix to be EP by referring to the commutativity property $XY = YX$, where X and Y are transforms of A from the set $A^\dagger, A, A^\oplus, A^*$.

Theorem 2.1. *Let $A \in \mathbb{C}^{m \times m}$. The index is one. Then, the following conditions are equivalent:*

- (I) A is EP;
- (II) $AA^\dagger A^\oplus = A^\dagger A^\oplus A$, $r(A) = r(A^2)$;
- (III) $AA^\oplus A^* = A^* AA^\oplus$, $r(A) = r(A^2)$;
- (IV) $AA^\oplus A^\dagger = A^\oplus A^\dagger A$, $r(A) = r(A^2)$;
- (V) $A^\dagger AA^\oplus = A^\oplus A^\dagger A$, $r(A) = r(A^2)$;
- (VI) $A^\dagger A^\dagger A^\oplus = A^\oplus A^\dagger A^\dagger$, $r(A) = r(A^2)$;
- (VII) $A^\dagger A^\oplus A^\dagger = A^\oplus A^\dagger A^\dagger$, $r(A) = r(A^2)$;
- (VIII) $A^\dagger A^\oplus A^\oplus = A^\oplus A^\oplus A^\dagger$, $r(A) = r(A^2)$;
- (IX) $AA^\oplus A^\dagger = A^\dagger AA^\oplus$, $r(A) = r(A^2)$;
- (X) $A^\dagger A^\oplus = A^\oplus A^\dagger$, $r(A) = r(A^2)$;
- (XI) $A^\dagger A^\oplus A^\oplus = A^\oplus A^\dagger A^\oplus$, $r(A) = r(A^2)$.

Proof. By the Hartwig-Spindelböck decomposition of A , one has

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*.$$

By calculating, we can get the Core inverse of A , that is

$$A^\oplus = U \begin{bmatrix} K^{-1} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

On the other hand, the Moore-Penrose inverse of A is

$$A^\dagger = U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^*.$$

It is easy to figure out that $AA^\dagger A^\oplus$ and $A^\dagger A^\oplus A$ are

$$\begin{aligned} AA^\dagger A^\oplus &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^* U \begin{bmatrix} K^{-1} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= U \begin{bmatrix} \Sigma K K^* \Sigma^{-1} K^{-1} \Sigma^{-1} + \Sigma L L^* \Sigma^{-1} K^{-1} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \end{aligned}$$

$$\begin{aligned} A^\dagger A^\oplus A &= U \begin{bmatrix} K^* \Sigma^{-1} & 0 \\ L^* \Sigma^{-1} & 0 \end{bmatrix} U^* U \begin{bmatrix} K^{-1} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} \\ &= U \begin{bmatrix} K^* \Sigma^{-1} & K^* \Sigma^{-1} K^{-1} L \\ L^* \Sigma^{-1} & L^* \Sigma^{-1} K^{-1} L \end{bmatrix} U^*. \end{aligned}$$

By $AA^\dagger A^\oplus = A^\dagger A^\oplus A$, one has

$$\begin{cases} L^* \Sigma^{-1} = 0, \\ K^* \Sigma^{-1} K^{-1} L = 0, \\ L^* \Sigma^{-1} K^{-1} L = 0, \\ K^* \Sigma^{-1} = \Sigma K K^* \Sigma^{-1} K^{-1} \Sigma^{-1} + \Sigma L L^* \Sigma^{-1} K^{-1} \Sigma^{-1}. \end{cases}$$

The condition $AA^\dagger A^\oplus = A^\dagger A^\oplus A$ is clearly equivalent to $L = 0$, $K^* \Sigma^{-1} = \Sigma K K^* \Sigma^{-1} K^{-1} \Sigma^{-1}$, multiply the equation to the right by $\Sigma K \Sigma$, can get $K^* K \Sigma = \Sigma K K^*$, because of $K K^* = K^* K = I_r$, so the equation works.

It is easy to figure out $AA^\oplus A^*$ and $A^* AA^\oplus$ are

$$\begin{aligned} AA^\oplus A^* &= U \begin{bmatrix} \Sigma K K^{-1} \Sigma^{-1} K^* \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*, \\ A^* AA^\oplus &= U \begin{bmatrix} K^* \Sigma \Sigma K K^{-1} \Sigma^{-1} & 0 \\ L^* \Sigma \Sigma K K^{-1} \Sigma^{-1} & 0 \end{bmatrix} U^*, \end{aligned}$$

$AA^\oplus A^* = A^* AA^\oplus \Rightarrow K^* \Sigma = K^* \Sigma$, $L = 0$. So, A is EP.

The other equations prove the same. $\square$

Next, give another equivalent form of the EP-matrix. We present necessary and sufficient conditions for a matrix to be EP by referring to the commutativity property $RS = YX$, where R, S, X, Y are transforms of A from the set $A^\dagger, A, A^\oplus, A^*$.

Theorem 2.2. *Let $A \in \mathbb{C}^{m \times m}$. Then, the following conditions are equivalent.*

- (I) A is EP;

- (II) $AA^\oplus = A^\dagger A$, $r(A) = r(A^2)$;
- (III) $A^\dagger A^* = A^\oplus A^*$, $r(A) = r(A^2)$;
- (IV) $A^\dagger A^\dagger = A^\oplus A^\dagger$, $r(A) = r(A^2)$;
- (V) $A^\dagger A^\dagger = A^\oplus A^\oplus$, $r(A) = r(A^2)$;
- (VI) $A^\dagger A^\oplus = A^\oplus A^\oplus$, $r(A) = r(A^2)$.

Proof. By the singular value decomposition of A , one has

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*.$$

By calculating, we can get the Core inverse of A , that is

$$A^\oplus = \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand, the Moore-Penrose inverse of A is

$$A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*.$$

It is easy to figure out that $AA^\oplus$ and $A^\dagger A$ are

$$AA^\oplus = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

$$A^\dagger A = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^* U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* = \begin{bmatrix} K^*K & K^*L \\ L^*K & L^*L \end{bmatrix}.$$

The condition $AA^\oplus = A^\dagger A$ is clearly equivalent to $L = 0$, $K^* = K^{-1}$. So, A is EP. It is easy to figure out $A^\dagger A^*$ and $A^\oplus A^*$ are

$$A^\dagger A^* = U \begin{bmatrix} K^*\Sigma^{-1}K^*\Sigma & 0 \\ L^*\Sigma^{-1}K^*\Sigma & 0 \end{bmatrix} U^*, \quad A^\oplus A^* = U \begin{bmatrix} K^{-1}\Sigma^{-1}K^*\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

The condition $A^\dagger A^* = A^\oplus A^*$ is clearly equivalent to $L = 0$ and $K^*\Sigma^{-1}K^*\Sigma = K^{-1}\Sigma^{-1}K^*\Sigma \Rightarrow K^* = K^{-1}$. So, A is EP. It is easy to figure out $A^\dagger A^\dagger$ and $A^\oplus A^\dagger$ are

$$A^\dagger A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1}K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1}K^*\Sigma^{-1} & 0 \end{bmatrix} U^*, \quad A^\oplus A^\dagger = U \begin{bmatrix} K^{-1}\Sigma^{-1}K^*\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

The condition $A^\dagger A^\dagger = A^\oplus A^\dagger$ is clearly equivalent to $L = 0$ and $K^* = K^{-1}$. So, A is EP.

The other equations prove the same. □

The following inference is obtained from Lemma 2.1.

Corollary 2.1. *Let $A \in \mathbb{C}^{m \times m}$. Then,*

- (I) $A^3A^\dagger = A^2A^\dagger A$ and $A^\dagger A^3 = AA^\dagger A^2 \Leftrightarrow AA^\dagger = A^\dagger A$.
- (II) $A^4A^\dagger = A^3A^\dagger A$ and $A^\dagger A^4 = AA^\dagger A^3 \Leftrightarrow AA^\dagger = A^\dagger A$.
- (III) $A^5A^\dagger = A^4A^\dagger A$ and $A^\dagger A^5 = AA^\dagger A^4 \Leftrightarrow AA^\dagger = A^\dagger A$.

Proof.

$$\begin{aligned} AAG &= AAH \Rightarrow AG = AH \\ A^3A^\dagger &= AA^2A^\dagger = A^2A^\dagger A = AAA^\dagger A \Rightarrow A^2A^\dagger = AA^\dagger A, \\ A^\dagger A^3 &= A^\dagger A^2 A = AA^\dagger A^2 = AA^\dagger AA \Rightarrow A^\dagger A^2 = AA^\dagger A. \end{aligned}$$

From Lemma 2.1, it come to the conclusion $AA^\dagger = A^\dagger A$. On the other hand,

$$AA^\dagger = A^\dagger A \Rightarrow A^3A^\dagger = A^2A^\dagger A \text{ and } A^\dagger A^3 = AA^\dagger A^2$$

is obvious. Hence, (I) is proved.

Similarly, we have

$$\begin{aligned} A^4A^\dagger &= A^2A^2A^\dagger = A^3A^\dagger A = A^2AA^\dagger A \Rightarrow A^2A^\dagger = AA^\dagger A, \\ A^\dagger A^4 &= A^\dagger A^2 A^2 = AA^\dagger A^3 = AA^\dagger AA^2 \Rightarrow A^\dagger A^2 = AA^\dagger A. \end{aligned}$$

From Lemma 2.1, it come to the conclusion $AA^\dagger = A^\dagger A$. Then, (II) is proved. The third equivalency can be proved in the same way. This completes the proof. $\square$

Corollary 2.2. *Let $A, B \in \mathbb{C}^{m \times m}$. Then,*

- (I) $AB^\dagger B = AA^\dagger A$ and $B^\dagger AA^\dagger = B^\dagger BB^\dagger \Leftrightarrow AA^\dagger = BB^\dagger$ and $B^\dagger B = A^\dagger A$.
- (II) $BB^\dagger A = AA^\dagger A$ and $A^\dagger AB^\dagger = B^\dagger BB^\dagger \Leftrightarrow AA^\dagger = BB^\dagger$ and $B^\dagger B = A^\dagger A$.

Proof. Let $M = \begin{bmatrix} 0 & A \\ B^\dagger & 0 \end{bmatrix}$, then $M^\dagger = \begin{bmatrix} 0 & B \\ A^\dagger & 0 \end{bmatrix}$. In this situation, it is easy to verify

$$M^2 = \begin{bmatrix} AB^\dagger & 0 \\ 0 & B^\dagger A \end{bmatrix}, M^2 M^\dagger = \begin{bmatrix} 0 & AB^\dagger B \\ B^\dagger AA^\dagger & 0 \end{bmatrix}, M^\dagger M^2 = \begin{bmatrix} 0 & BB^\dagger A \\ A^\dagger AB^\dagger & 0 \end{bmatrix}.$$

Moreover,

$$MM^\dagger M = \begin{bmatrix} 0 & AA^\dagger A \\ B^\dagger BB^\dagger & 0 \end{bmatrix}.$$

By Theorem 2.1, one has $M^2 M^\dagger = MM^\dagger M \Leftrightarrow MM^\dagger = M^\dagger M$. Hence,

$$AB^\dagger B = AA^\dagger A \text{ and } B^\dagger AA^\dagger = B^\dagger BB^\dagger \Leftrightarrow MM^\dagger = M^\dagger M.$$

It is easy to verify

$$M^\dagger M = \begin{bmatrix} BB^\dagger & 0 \\ 0 & A^\dagger A \end{bmatrix}, \quad MM^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ 0 & B^\dagger B \end{bmatrix}.$$

Therefore, $MM^\dagger = M^\dagger M \Leftrightarrow AA^\dagger = BB^\dagger$ and $B^\dagger B = A^\dagger A$, which implies

$$AB^\dagger B = AA^\dagger A \text{ and } B^\dagger AA^\dagger = B^\dagger BB^\dagger \Leftrightarrow AA^\dagger = BB^\dagger \text{ and } B^\dagger B = A^\dagger A.$$

On the other hand, by using $M^\dagger M^2 = MM^\dagger M \Leftrightarrow MM^\dagger = M^\dagger M$, one has

$$BB^\dagger A = AA^\dagger A \text{ and } A^\dagger AB^\dagger = B^\dagger BB^\dagger \Leftrightarrow MM^\dagger = M^\dagger M.$$

Conclusions can be drawn

$$BB^\dagger A = AA^\dagger A \text{ and } A^\dagger AB^\dagger = B^\dagger BB^\dagger \Leftrightarrow AA^\dagger = BB^\dagger \text{ and } B^\dagger B = A^\dagger A.$$

Thus, we establish the equivalent facts. $\square$

3. Related results of the normal matrix

In this section we give the equivalent form of the normal matrix of the form $RS = XY$ and $XY = YX$, where $R, S, X, Y \in A^\dagger, A, A^\oplus, A^*$, and some conclusions are obtained.

Lemma 3.1 ([23]). *Let $A \in \mathbb{C}^{m \times m}$. Then, the following results hold*

$$(I) \quad A^2 A^* = AA^* A \Leftrightarrow AA^* = A^* A;$$

$$(II) \quad A^* A^2 = AA^* A \Leftrightarrow AA^* = A^* A.$$

We present necessary and sufficient conditions for a matrix to be normal by referring to the commutativity property $XY = YX$, where X and Y are transforms of A from the set $A^\dagger, A, A^\oplus, A^*$.

Theorem 3.1. *Let $A \in \mathbb{C}^{m \times m}$. Then, the following conditions are equivalent.*

$$(I) \quad A \text{ is normal};$$

$$(II) \quad AA^* A^\oplus = A^* A^\oplus A, \quad r(A) = r(A^2);$$

$$(III) \quad AA^\oplus A^* = A^\oplus A^* A, \quad r(A) = r(A^2);$$

$$(IV) \quad A^* A^\oplus = A^\oplus A^*, \quad r(A) = r(A^2);$$

$$(V) \quad A^\dagger A^* A^\oplus = A^\oplus A^\dagger A^*, \quad r(A) = r(A^2);$$

$$(VI) \quad A^* AA^\oplus = A^\oplus A^* A, \quad r(A) = r(A^2);$$

$$(VII) \quad A^* A^\dagger A^\oplus = A^\oplus A^* A^\dagger, \quad r(A) = r(A^2);$$

$$(VIII) \quad A^* A^\oplus A^* = A^\oplus A^* A^*, \quad r(A) = r(A^2);$$

$$(IX) \quad A^* A^\oplus A^\oplus = A^\oplus A^* A^\oplus, \quad r(A) = r(A^2);$$

$$(X) \quad A^\dagger A^\oplus A^* = A^\oplus A^* A^\dagger, \quad r(A) = r(A^2).$$

Proof. Following Lemma 1.1, one has

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*.$$

Then,

$$A^* = U \begin{bmatrix} (\Sigma K)^* & 0 \\ (\Sigma L)^* & 0 \end{bmatrix} U^*.$$

Furthermore,

$$\begin{aligned} AA^* &= U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} (\Sigma K)^* & 0 \\ (\Sigma L)^* & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (\Sigma K)(\Sigma K)^* + (\Sigma L)(\Sigma L)^* & 0 \\ 0 & 0 \end{bmatrix} U^*, \\ AA^* A^\oplus &= U \begin{bmatrix} \Sigma K K^* \Sigma + \Sigma L L^* \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} K^{-1} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} \Sigma K K^* \Sigma K^{-1} \Sigma^{-1} + \Sigma L L^* \Sigma K^{-1} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

Simple computations show that

$$\begin{aligned} A^* A^\oplus A &= U \begin{bmatrix} K^* \Sigma & 0 \\ L^* \Sigma & 0 \end{bmatrix} U^* U \begin{bmatrix} K^{-1} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} K^* \Sigma & K^* \Sigma K^{-1} L \\ L^* \Sigma & L^* \Sigma K^{-1} L \end{bmatrix} U^*. \end{aligned}$$

By $AA^* A^\oplus = A^* A^\oplus A$, one has

$$\begin{cases} L^* \Sigma = 0 \\ K^* \Sigma K^{-1} L = 0 \\ L^* \Sigma K^{-1} L = 0 \\ K^* L = \Sigma K K^* \Sigma K^{-1} \Sigma^{-1} + \Sigma L L^* \Sigma K^{-1} \Sigma^{-1} \end{cases}.$$

Hence, it is seen that $A^* A^\oplus$ and A commute if and only if $L = 0$, $K^* \Sigma = \Sigma K K^* \Sigma K^{-1} \Sigma^{-1}$. However, $L = 0$ implies $K^* = K^{-1}$.

Multiply both sides of $K^* \Sigma = \Sigma K K^* \Sigma K^{-1} \Sigma^{-1}$ by ΣK . Can get the formula

$$K^* \Sigma^2 K = \Sigma^2.$$

Taking square roots, we arrive at $L = 0$ and $K^*\Sigma K = \Sigma$, that is $L = 0$ and $\Sigma K = K\Sigma$. So, A is normal. It is easy to figure out $AA^\oplus A^*$ and $A^\oplus A^*A$ are

$$AA^\oplus A^* = U \begin{bmatrix} K^*\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

$$A^\oplus A^*A = U \begin{bmatrix} K^{-1}\Sigma^{-1}K^*\Sigma\Sigma K & K^{-1}\Sigma^{-1}K^*\Sigma\Sigma L \\ 0 & 0 \end{bmatrix} U^*.$$

Hence, it is seen that $A^\oplus A^*$ and A commute if and only if $L = 0$, $K^*\Sigma = K^{-1}\Sigma^{-1}K^*\Sigma\Sigma K$.

Multiply both sides of this equation ΣK . Then,

$$\Sigma K = K\Sigma.$$

The other equations prove the same. $\square$

Corollary 3.1. *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then,*

$$ABB^*A^* = B^*A^*AB \Leftrightarrow (AB)(AB)^* = (AB)^*(AB).$$

Proof. The $(AB)(AB)^* = (AB)^*(AB) \Rightarrow ABB^*A^* = B^*A^*AB$ part is obvious. To show the $ABB^*A^* = B^*A^*AB \Rightarrow (AB)(AB)^* = (AB)^*(AB)$ part, multiply both sides of this equation by AB gives

$$ABABB^*A^* = ABB^*A^*AB \Rightarrow (AB)^2(AB)^* = (AB)(AB)^*(AB).$$

By Lemma 4.3, we can get

$$(AB)(AB)^* = (AB)^*(AB).$$

This proof is complete. $\square$

Corollary 3.2. *Let $A, B \in \mathbb{C}^{m \times n}$. Then*

(I) $AB^*B = AA^*A$ and $B^*BB^* = B^*A^*A^* \Leftrightarrow AA^* = BB^*$ and $B^*B = A^*A$;

(II) $AA^*A = BB^*A$ and $A^*AB^* = B^*BB^* \Leftrightarrow AA^* = BB^*$ and $B^*B = A^*A$.

Proof. Let $M = \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}$, then $M^* = \begin{bmatrix} 0 & B \\ A^* & 0 \end{bmatrix}$. In this situation, it is easy to verify

$$M^2 = \begin{bmatrix} AB^* & 0 \\ 0 & B^*A \end{bmatrix}, M^2M^* = \begin{bmatrix} 0 & AB^*B \\ B^*A^*A^* & 0 \end{bmatrix}, M^*M^2 = \begin{bmatrix} 0 & BB^*A \\ A^*AB^* & 0 \end{bmatrix}$$

and

$$MM^*M = \begin{bmatrix} 0 & AA^*A \\ B^*BB^* & 0 \end{bmatrix}.$$

By Theorem 4.3, $M^2M^* = MM^*M \Leftrightarrow MM^* = M^*M$, which implies

$$AB^*B = AA^*A \text{ and } B^*BB^* = B^*A^*A^* \Leftrightarrow MM^* = M^*M.$$

Now, it is easy to verify

$$M^*M = \begin{bmatrix} BB^* & 0 \\ 0 & A^*A \end{bmatrix}, \quad MM^* = \begin{bmatrix} AA^* & 0 \\ 0 & B^*B \end{bmatrix}.$$

Hence, $MM^* = M^*M \Leftrightarrow AA^* = BB^*$ and $B^*B = A^*A$. Thus, (I) is established.

Now, prove foemula (II), $M^*M^2 = MM^*M \Leftrightarrow MM^* = M^*M$, which implies

$$AA^*A = BB^*A \text{ and } A^*AB^* = B^*BB^* \Leftrightarrow MM^* = M^*M.$$

Conclusions can be drawn that

$$AA^*A = BB^*A \text{ and } A^*AB^* = B^*BB^* \Leftrightarrow AA^* = BB^* \text{ and } B^*B = A^*A.$$

This proof is complete. $\square$

We present necessary and sufficient conditions for a matrix to be normal by referring to the commutativity property $RS = YX$, where R, S, X, Y are transforms of A from the set $A^\dagger, A, A^\oplus, A^*$.

Theorem 3.2. *Let $A \in \mathbb{C}^{n \times n}$. Then, the following conditions are equivalent*

- (I) A is normal;
- (II) $A^*A^\dagger = A^\oplus A^*$, $r(A) = r(A^2)$;
- (III) $A^*A^\oplus = A^\dagger A^*$, $r(A) = r(A^2)$.

Proof. From Lemma 1.1, it is easy to figure out that $A^*A^\dagger$ and $A^\oplus A^*$ are

$$A^*A^\dagger = U \begin{bmatrix} K^*\Sigma K^*\Sigma^{-1} & 0 \\ L^*\Sigma K^*\Sigma^{-1} & 0 \end{bmatrix} U^*, \quad A^\oplus A^* = U \begin{bmatrix} K^{-1}\Sigma^{-1}K^*\Sigma & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

The condition $A^*A^\dagger = A^\oplus A^*$ is clearly equivalent to $L = 0$. Moreover, by $K^* = K^{-1}$, we get $\Sigma K^{-1}\Sigma^{-1} = \Sigma^{-1}K^{-1}\Sigma$, which implies $\Sigma K = K\Sigma$. So, A is normal.

On the other hand, it is easy to figure out that $A^*A^\oplus$ and $A^\dagger A^*$ are

$$A^*A^\oplus = U \begin{bmatrix} K^*\Sigma K^{-1}\Sigma^{-1} & 0 \\ L^*\Sigma K^{-1}\Sigma^{-1} & 0 \end{bmatrix} U^*, \quad A^\dagger A^* = U \begin{bmatrix} K^*\Sigma^{-1}K^*\Sigma & 0 \\ L^*\Sigma^{-1}K^*\Sigma & 0 \end{bmatrix} U^*.$$

The condition $A^*A^\oplus = A^\dagger A^*$ is clearly equivalent to $L = 0$. Moreover, by $K^* = K^{-1}$, we get $\Sigma K^{-1}\Sigma^{-1} = \Sigma^{-1}K^{-1}\Sigma$, which implies $\Sigma K = K\Sigma$. So, A is normal. This proof is complete. $\square$

Corollary 3.3. *Let $A \in \mathbb{C}^{m \times m}$. Then*

- (I) $A^3 A^* = A^2 A^* A \Leftrightarrow AA^* = A^* A$;
- (II) $A^4 A^* = A^3 A^* A, A^3 A^* = A^2 A^* A \Leftrightarrow AA^* = A^* A$;
- (III) $A^5 A^* = A^4 A^* A, A^4 A^* = A^3 A^* A \Leftrightarrow AA^* = A^* A$;
- (IV) $A^6 A^* = A^5 A^* A, A^5 A^* = A^4 A^* A \Leftrightarrow AA^* = A^* A$.

Proof. The $AA^* = A^* A \Rightarrow A^3 A^* = A^2 A^* A$ is obvious. Now, we show the $A^3 A^* = A^2 A^* A \Rightarrow AA^* = A^* A$ part. in [15]. It is clearly that

$$\begin{aligned} AAG &= AAH \Rightarrow AG = AH \\ A^3 A^* &= A^2 A^* A \Rightarrow AA^2 A^* = AAA^* A \Rightarrow A^2 A^* = AA^* A. \end{aligned}$$

By Lemma 4.3, one has

$$A^2 A^* = AA^* A \Rightarrow A^* A = AA^*.$$

It is easy to see that

$$A^4 A^* = A^3 A^* A \Rightarrow A^3 A^* = A^2 A^* A \Rightarrow A^2 A^* = AA^* A \Rightarrow A^* A = AA^*.$$

Hence, $A^4 A^* = A^3 A^* A \Leftrightarrow AA^* = A^* A$.

The same can be said for the other equations. □

4. Related results of the Hermitian matrix

In this section we give the equivalent form of the Hermitian of the form $RS = XY$, where $R, S, X, Y \in A^\dagger, A, A^\oplus, A^*$, and some conclusions are obtained, and we also get a new equivalent form of the reverse order law.

Lemma 4.1. *Let $A \in \mathbb{C}^{m \times n}$ and $B, C \in \mathbb{C}^{n \times p}$. Then, the following results hold*

- (I) $A^* AB = A^* AC \Leftrightarrow AB = AC$. In particular, $A^* A = 0 \Leftrightarrow AA^* A = 0 \Leftrightarrow A = 0$ ([14]).
- (II) *The principal k th root of positive semi-definite matrix exists and is unique* ([16]).

Lemma 4.2. *Let $A \in \mathbb{C}^{m \times m}$. Then, the following results hold*

- (I) $AA^* A = A^* AA^* \Leftrightarrow A = A^*$ ([3]).
- (II) $A^3 = AA^* A \Leftrightarrow A = A^*$ ([3]).

Lemma 4.3 ([3]). *Let $A \in \mathbb{C}^{m \times m}$. Then*

- (I) $(AA^* A)^2 = (AA^*)^3 \Leftrightarrow A = A^*$,

$$(II) (AA^*A)^2 = (A^*A)^3 \Leftrightarrow A = A^*,$$

$$(III) (AA^*A)^3 = (AA^*)^2 A (A^*A)^2 \Leftrightarrow A = A^*,$$

$$(IV) A^3 = A^*AA^* \text{ and } A^5 = (A^*A)^2 A^* \Leftrightarrow A = A^*,$$

$$(V) A^5 = (AA^*)^2 A \text{ and } A^7 = (AA^*)^3 A \Leftrightarrow A = A^*.$$

Next, the above results are pushed to higher order form in order to better judge the equivalence conditions of Hermitian.

Theorem 4.1. *Let $A \in \mathbb{C}^{m \times m}$. Then,*

$$(I) A^7 = (A^*A)^3 A^* \text{ and } A^9 = (A^*A)^4 A^* \Leftrightarrow A = A^*;$$

$$(II) A^9 = (AA^*)^4 A \text{ and } A^{11} = (AA^*)^5 A \Leftrightarrow A = A^*;$$

$$(III) A^{11} = (A^*A)^5 A^* \text{ and } A^{13} = (A^*A)^6 A^* \Leftrightarrow A = A^*;$$

$$(IV) A^{13} = (AA^*)^6 A \text{ and } A^{15} = (AA^*)^7 A \Leftrightarrow A = A^*;$$

$\vdots$

$$(V) A^{2k-3} = (A^*A)^k A^* \text{ and } A^{2k-1} = (A^*A)^k A^* \Leftrightarrow A = A^*;$$

$$(VI) A^{2k-1} = (AA^*)^k A \text{ and } A^{2k+1} = (AA^*)^k A \Leftrightarrow A = A^*.$$

Proof. We are supported by $A = A^*$, it's easy to get the left-hand side. Now, let us derive from the left-hand formula. Since $A^7 = (A^*A)^3 A^*$, one has $A^9 = A^7 A^2 = (A^*A)^3 A^* A A = (A^*A)^4 A^*$, which is equivalent to $A^2 = AA^*$ by applying Lemma(4.1)(I) four times. By the formula (7), one has $A = A^*$. By $A^9 = (AA^*)^4 A$, one can get $A^{11} = A^9 A^2 = (AA^*)^4 A A A$. Hence, $(AA^*)^5 A = (AA^*)^4 A A A$, which is equivalent to $A^3 = AA^* A$ by applying Lemma(4.1) (I) four times. By the Lemma (4.2), one has $A = A^*$.

Notice that $A^{11} = (A^*A)^5 A^*$. Then, $A^{13} = A^{11} A^2 = (A^*A)^5 A^* A^2$. Hence, which is equivalent to $A^2 = AA^*$ by applying Lemma(4.1)(I) five times. By the formula (7), one has $A = A^*$.

And so on, by the same proofs, we can get

$$A^{2k-3} = (A^*A)^k A^* \text{ and } A^{2k-1} = (A^*A)^k A^* \Leftrightarrow A = A^*,$$

$$A^{2k-1} = (AA^*)^k A \text{ and } A^{2k+1} = (AA^*)^k A \Leftrightarrow A = A^*.$$

This completes the proof. $\square$

Next, an equivalent form of the matrix is given. We present necessary and sufficient conditions for a matrix to be Hermitian by referring to the commutativity property $RS = YX$, where R, S, X, Y are transforms of A from the set $A^\dagger, A, A^\oplus, A^*$.

Theorem 4.2. *Let $A \in \mathbb{C}^{n \times n}$. Then, the following conditions are equivalent:*

- (I) A is Hermitian;
- (II) $AA^\oplus = A^*A^\oplus$, $r(A) = r(A^2)$;
- (III) $AA^\oplus = A^\dagger A^*$, $r(A) = r(A^2)$;
- (IV) $A^\dagger A = A^\oplus A^*$, $r(A) = r(A^2)$;
- (V) $AA^\oplus = A^*A^\dagger$, $r(A) = r(A^2)$.

Proof. By the singular value decomposition of A , one has

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*.$$

By calculating, we can get the Core inverse

$$A^\oplus = U \begin{bmatrix} K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

On the other hand, the Moore-Penrose inverse of A is

$$A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*.$$

It is easy to figure out that $AA^\oplus$ and $A^*A^\oplus$ are

$$AA^\oplus = U \begin{bmatrix} \Sigma K K^{-1}\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad A^*A^\oplus = U \begin{bmatrix} K^*\Sigma K^{-1}\Sigma^{-1} & 0 \\ L^*\Sigma K^{-1}\Sigma^{-1} & 0 \end{bmatrix} U^*.$$

The condition $AA^\oplus = A^*A^\oplus$ is clearly equivalent to $L = 0$, $I = K^*\Sigma K^{-1}\Sigma^{-1}$, which implies $\Sigma K = K^*\Sigma$, and hence A is Hermitian.

On the other hand, it is easy to figure out that $AA^\oplus$ and $A^\dagger A^*$ are

$$AA^\oplus = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad A^\dagger A^* = U \begin{bmatrix} K^*\Sigma^{-1}K^*\Sigma & 0 \\ L^*\Sigma^{-1}K^*\Sigma & 0 \end{bmatrix} U^*.$$

The condition $AA^\oplus = A^\dagger A^*$ is clearly equivalent to $L = 0$, $I = K^*\Sigma^{-1}K^\dagger\Sigma$ which implies $\Sigma K = K^*\Sigma$. So, A is Hermitian.

The other equations can be proved the same. □

By Lemma 4.2, have the following inference.

Corollary 4.1. *Let $A \in \mathbb{C}^{m \times m}$ and $B \in \mathbb{C}^{m \times m}$. Then,*

$$BAB B A B B = B A (B B A)^* B B \Leftrightarrow A^* B^* B^* = B B A.$$

Proof. If $A^*B^*B^* = BBA$, we can get $BABBBABB = BA(BBA)^*BB$. If $BABBBABB = BA(BBA)^*BB$, multiply left by B and right by A on both sides of this equation, we can get

$$BBABBBABBA = BBA(BBA)^*BBA \Rightarrow (BBA)^3 = BBA(BBA)^*BBA.$$

According to Lemma 4.2, one has $A^3 = AA^*A \Leftrightarrow A = A^*$, by which we can get $BBA = (BBA)^* = A^*B^*B^*$. This completes the proof. $\square$

Theorem 4.3. Let $A, B \in \mathbb{C}^{m \times n}$. Then,

$$(I) A^*AA^* = B^*BB^* \Leftrightarrow A = B,$$

$$(II) A^*BA^* = B^*BB^* \text{ and } BB^*B = BA^*B \Leftrightarrow A = B.$$

Proof. Let

$$X = \begin{bmatrix} 0 & B \\ A^* & 0 \end{bmatrix} \text{ and } X^* = \begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix}.$$

In this situation, it is easy to verify

$$XX^*X = \begin{bmatrix} 0 & BB^*B \\ A^*AA^* & 0 \end{bmatrix}, X^*XX^* = \begin{bmatrix} 0 & AA^*A \\ B^*BB^* & 0 \end{bmatrix},$$

$$X^3 = \begin{bmatrix} 0 & BA^*B \\ A^*BA^* & 0 \end{bmatrix}.$$

From Lemma 4.2, we can get

$$X^3 = XX^*X = X^*XX^* \Rightarrow X = X^* \Rightarrow A = B.$$

So, we get the following:

$$A^*AA^* = B^*BB^* \Rightarrow A = B,$$

$$A^*BA^* = B^*BB^* \text{ and } BB^*B = BA^*B \Rightarrow A = B.$$

This completes the proof. $\square$

Theorem 4.4. Let $A, B \in \mathbb{C}^{n \times n}$ be two Hermitian matrices. Then, the following seven statements are equivalent:

$$(I) BBA = ABB;$$

$$(II) (BBA)(ABB)(BBA) = (ABB)(BBA)(ABB);$$

$$(III) (BBA)^3 = (BBA)(ABB)(BBA);$$

$$(IV) ((BBA)(ABB)(BBA))^2 = ((BBA)(ABB))^3;$$

$$(V) ((BBA)(ABB)(BBA))^3 = ((BBA)(ABB))^2(BBA)((ABB)(BBA))^2;$$

- (VI) $(BBA)^3 = (BBA)(ABB)(BBA)$ and
 $(BBA)^5 = (BBA)[(ABB)(BBA)^2](BBA);$
- (VII) $(BBA)^5 = (BBA)[(ABB)(BBA)^2](BBA)$ and
 $(BBA)^7 = (BBA)[(ABB)(BBA)^4](BBA).$

Proof. If $BBA = ABB$, the other equations are obviously true. Now, let us verify $BBA = ABB$ with something else. Since $(BBA)(ABB)(BBA) = (ABB)(BBA)(ABB)$, one has $(BBA)(BBA)^*(BBA) = (BBA)^*(BBA)(BBA)^*$. It follows from replacing A with BBA in Lemma 4.2. We have that

$$\begin{aligned} (BBA)(BBA)^*(BBA) &= (BBA)^*(BBA)(BBA)^* \Rightarrow (BBA)^* \\ &= A^*B^*B^* = ABB = BBA \end{aligned}$$

and we can do the same thing with the rest. This completes the proof. $\square$

A typical matrix equality for the case of the product of two matrices of appropriate sizes is

$$(AB)^\dagger = B^\dagger A^\dagger,$$

which is usually called the reverse-order law for the Moore-Penrose inverse of a matrix product. The reverse-order law does not necessarily hold. So, have a fact

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (ABB^*A^*AB)^\dagger = B^\dagger(A^*A)^\dagger(BB^*)^\dagger A^\dagger$$

(see [3]). In [9, 12, 13, 18, 22], some equivalent conditions for the reverse-order law are also given. The result of theorem 4.3 applies here to the inverse order law. In the following, we will give a new equivalent condition for the reverse-order law.

Theorem 4.5. *Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$. Then,*

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger(BB^*)^\dagger(A^*A)^\dagger(B^*)^\dagger.$$

Proof. The implying

$$(AB)^\dagger = B^\dagger A^\dagger \Rightarrow (B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger(BB^*)^\dagger(A^*A)^\dagger(B^*)^\dagger$$

is obvious. Now, we will prove the reverse part. Utilization Theorem 4.3 (1), $(AB)^\dagger$ instead of A , $B^\dagger A^\dagger$ instead of B , one has

$$((AB)^\dagger)^*(AB)^\dagger((AB)^\dagger)^* = ((B^\dagger A^\dagger)^*(B^\dagger A^\dagger)((B^\dagger A^\dagger)^*)^* \Rightarrow (AB)^\dagger = B^\dagger A^\dagger.$$

Hence, $(B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger(BB^*)^\dagger(A^*A)^\dagger(B^*)^\dagger \Rightarrow (AB)^\dagger = B^\dagger A^\dagger.$

This completes the proof. $\square$

5. Conclusions

Obviously, the results in the above theorem and corollaries offer lots of equivalent facts about EP-matrices, normal matrices, and Hermitian matrices. This fact can be described in the implication form $f(A, A^*) = 0 \Leftrightarrow AA^* = A^*A$ and $f(A, A^\dagger) = 0 \Leftrightarrow AA^\dagger = A^\dagger A$. In this note, applied the core inverse to provide new characterizations. We show servers special cases of the equivalent facts:

- (I) $AA^\oplus = A^*A^\oplus \Leftrightarrow A = A^*$.
- (II) $AA^\oplus = A^\dagger A^* \Leftrightarrow A = A^*$ and (I) A is EP.
- (II) $AA^\dagger A^\oplus = A^\dagger A^\oplus A$.
- (III) $AA^\oplus A^* = A^*AA^\oplus$.
- (IV) $AA^\oplus A^\dagger = A^\oplus A^\dagger A$.
- (V) $A^\dagger AA^\oplus = A^\oplus A^\dagger A$.

without assuming the invertibility of A through the skillful use of decompositions of matrices. Some equivalent forms related to $A^\oplus$, $A^\dagger$, A^* can also be obtained. A new equivalent condition for the reverse order law is also obtained: $(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (B^*A^*ABB^*A^*)^\dagger = (A^*)^\dagger(BB^*)^\dagger(A^*A)^\dagger(B^*)^\dagger$. We can also use the new generalized inverse to study the equivalent form of EP, normal and Hermitian matrices.

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