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Recurrent Hopf hypersurfaces in complex 2-plane Grassmannians

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Abstract. In this paper, it is proved that if the shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent, it is Reeb parallel. Another recurrent hypersurfaces are also classified.

Keywords: recurrent, Hopf hypersurface, complex two-plane Grassmannian, shape operator.

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1. Introduction

Let A be the shape operator of a hypersurface (M, g) isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The hypersurface M is called recurrent if

$$\nabla_X A = \omega(X)A \tag{1}$$

for a certain one-form ω and any vector field $X \in \mathfrak{X}(M)$, where ∇ and $\mathfrak{X}(M)$ denote the Levi-Civita connection of g and the set of all tangent vector fields on M. The recurrence condition for a tensor field T of type (r, s) was first introduced in [8, 21] in which a geometric interpretation of it in terms of the holonomy group was provided. Hamada in [4] considered (1) on a real hypersurface in a complex projective space $\mathbb{C}P^n$ and proved that recurrent hypersurfaces do not exist. The other type of geometric meaning of recurrent shape operator A of a real hypersurface was indicated by Suh in [14], namely (1) implies that the eigenspaces of the shape operator A are invariant with respect to any parallel translation along any curve γ in M.

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined as the set of all two-dimensional linear subspaces in the complex Euclidean space $\mathbb{C}^{m+2}$ which is identified with the homogeneous space $SU(m+2)/S(U(2)\times U(m))$. $G_2(\mathbb{C}^{m+2})$ is known as a compact irreducible Hermitian symmetric space of rank two equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ with a canonical basis $\{J_1, J_2, J_3\}$ which does not contain J (see [1]). In this paper m is assumed to be an integer greater than two.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with N and A a unit normal vector field and the shape operator respectively, and g and ∇ be the induced metric and the corresponding Levi-Civita connection on M, respectively. $\xi := -JN$ is called the Reeb vector field and the almost contact metric 3-structure vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$ for $\nu \in \{1,2,3\}$. It is denoted by $\mathfrak{D}^\perp$ the distribution defined by $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ and $\mathfrak{D}$ its orthogonal complement distribution satisfying $T_pM = \mathfrak{D}_p \oplus \mathfrak{D}_p^\perp$ at every point $p \in M$. A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be Hopf if ξ is an eigenvector field of the shape operator at each point, i.e., $A\xi = \alpha\xi$ and $\alpha = g(A\xi, \xi)$ is called the Hopf principal curvature. Classification result for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ was obtained by Berndt and Suh [2].

Theorem 1.1 ([2])**.** *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then $\mathfrak{D}^\perp$ is invariant under the shape operator if and only if*

(A) M is an open part of a tube around a totally geodesic

$$G_2(\mathbb{C}^{m+1}) \textit{ in } G_2(\mathbb{C}^{m+2}),$$

or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^m$ in $G_2(\mathbb{C}^{m+2})$.

Applying such a theorem, Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ under some other conditions were extensively studied. Here we recall some results related to (1). A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is called parallel if

$$\nabla_X A = 0, \tag{2}$$

for any vector field X. We may regard the parallel condition as a special case of the recurrent condition. Suh in [15] proved that there exist no real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator. Generalizing this result, some other types of parallelism were introduced. A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is called Reeb (resp. $\mathfrak{D}^\perp$ or $\mathfrak{D}$) parallel if (2) holds only for X belonging to the Reeb distribution (resp. $\mathfrak{D}^\perp$ or $\mathfrak{D}$). Under such weaker conditions, Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ were considered in [5, 7, 9] and main theorem in [15] was extended. In view of the weakness of (1) than (2), it is very natural to consider recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Suh in [16] first considered this problem and proved that there exist no recurrent real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$ (resp. $\mathfrak{D}^\perp$)-invariant shape operator. Later, another nonexistence theorem for recurrent real hypersurfaces was obtained in [7], namely there do not exist any Hopf recurrent hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Since the distributions generated by the characteristic vector field ξ and $\mathfrak{D}^\perp$ are most important distributions on real hypersurface in $G_2(\mathbb{C}^{m+2})$, in this paper, we consider some new conditions which are much weaker than (1). A real hypersurface in $G_2(\mathbb{C}^{m+2})$ is called Reeb (or $\mathfrak{D}^{\perp}$) recurrent if (1) is valid for X belonging to the Reeb distribution (or $\mathfrak{D}^{\perp}$). The relationships between parallelism and recurrence of the shape operator A of real hypersurfaces are given as follows.

Theorem 1.2. *The shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb (or $\mathfrak{D}^{\perp}$) recurrent if and only if it is Reeb (or $\mathfrak{D}^{\perp}$) parallel.*

Not all operators on a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent if and only if it is Reeb parallel (for example, see the Ricci operators in [18, 19]). The recurrence condition (1) for some other operators on real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ can be seen in [6, 12, 20]. By using results in [5, 9] and Theorem 1.2, Reeb (or $\mathfrak{D}^{\perp}$) recurrent real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are classified in this paper.

2. Preliminaries

2.1 Real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section, first we recall some fundamental formulas shown in [1, 2, 3, 17]. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with real codimension one and N be a unit normal vector field. On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler structure J of $G_2(\mathbb{C}^{m+2})$. Let $\{J_1, J_2, J_3\}$ be a canonical local basis of quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$. In this paper we put

$$JX = \phi X + \eta(X)N,\ J_\nu X = \phi_\nu X + \eta_\nu(X)N, \tag{3}$$

for any vector field X, $\nu \in \{1, 2, 3\}$. From the first term of (3), it follows that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(X) = g(X, \xi), \tag{4}$$

where the Reeb vector field ξ is determined by $\xi := -JN$. From the condition

$$J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$$

we have an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ as the following

$$\begin{aligned}
&\phi_\nu^2 = -\mathrm{id} + \eta_\nu \otimes \xi_\nu,\ \eta_\nu(\xi_\nu) = 1,\ \phi_\nu \xi_\nu = 0,\\
&\phi_\nu \xi_{\nu+1} = \xi_{\nu+2},\ \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2},\\
&\phi_\nu \phi_{\nu+1} = \phi_{\nu+2} + \eta_{\nu+1} \otimes \xi_\nu,\\
&\phi_{\nu+1}\phi_\nu = -\phi_{\nu+2} + \eta_\nu \otimes \xi_{\nu+1},
\end{aligned} \tag{5}$$

where the index is taken modulo three. According to condition $J_\nu J = J J_\nu$, the relationships between two almost contact metric structures are given by

$$\begin{aligned}
&\phi\phi_\nu = \phi_\nu\phi + \eta_\nu \otimes \xi - \eta \otimes \xi_\nu,\\
&\phi\xi_\nu = \phi_\nu \xi,\ \eta_\nu(\phi X) = \eta(\phi_\nu X),
\end{aligned} \tag{6}$$

for any vector field X. Because J is parallel with respect to the Riemannian connection of $G_2(\mathbb{C}^{m+2})$, we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi,\ \nabla_X\xi = \phi AX, \tag{7}$$

for any vector fields X and Y, where we have applied the Gauss and Weingarten formulas. Similarly, since J_ν is a quaternionic Kähler structure of $G_2(\mathbb{C}^{m+2})$, we have

$$\begin{aligned} \nabla_X\xi_\nu &= q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,\\ \nabla_X\phi_\nu &= q_{\nu+2}(X)\phi_{\nu+1} - q_{\nu+1}(X)\phi_{\nu+2} + \eta_\nu\otimes AX - g(AX,\cdot)\xi_\nu, \end{aligned} \tag{8}$$

for any vector field X. The Codazzi equation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X =& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\\ &+ \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu)\\ &+ \sum_{\nu=1}^{3}(\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X)\\ &+ \sum_{\nu=1}^{3}(\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X))\xi_\nu, \end{aligned} \tag{9}$$

for any vector fields X, Y.

2.2 Some key lemmas

We need the following some important results.

Lemma 2.1 ([3]). *If M is a connected and oriented Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, then we have*

$$\mathrm{grad}\alpha = \xi(\alpha)\xi + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi, \tag{10}$$

$$\begin{aligned} 2A\phi AX =& \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\sum_{\nu=1}^{3}(\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu\\ &+ \eta_\nu(\xi)\phi_\nu X - 2\eta(X)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi), \end{aligned} \tag{11}$$

for any vector field X, where grad *denotes the gradient operator.*

Proposition 2.1 ([2]). *Let M be a connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $A\mathfrak{D}\subset\mathfrak{D}$ and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1\in\mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \frac{\pi}{4\sqrt{2}}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0,$$

with some $r \in (0, \frac{\pi}{\sqrt{8}})$. *The corresponding multiplicities are*

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = m(\mu) = 2m - 2$$

and the corresponding eigenspaces are

$$\begin{aligned}
&T_\alpha = \mathbb{R}\xi = \mathbb{R}\xi_1 = \mathbb{R}JN = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},\\
&T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2, \xi_3\},\\
&T_\lambda = \{X : X \perp \mathbb{H}\xi, JX = J_1X\},\ T_\mu = \{X : X \perp \mathbb{H}\xi, JX = -J_1X\},
\end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ *and* $\mathbb{H}\xi$ *denote the real, complex and quaternionic span of the Reeb vector field* ξ, *respectively, and* $\mathbb{C}^\perp\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$.

Proposition 2.2 ([2]). *Let* M *be a connected Hopf hypersurface in* $G_2(\mathbb{C}^{m+2})$ *with* $A\mathfrak{D} \subset \mathfrak{D}$ *and* $\xi \in \mathfrak{D}$. *Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$, *and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r),$$

with some $r \in (0, \frac{\pi}{4})$. *The corresponding multiplicities are*

$$m(\alpha) = 1,\ m(\beta) = m(\gamma) = 3,\ m(\lambda) = m(\mu) = 4n - 4$$

and the corresponding eigenspaces are

$$\begin{aligned}
&T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\},\ T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\},\\
&T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\phi_1\xi, \phi_2\xi, \phi_3\xi\},\ T_\lambda,\ T_\mu,
\end{aligned}$$

where $T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp$, $\mathfrak{J}T_\lambda = T_\lambda$, $\mathfrak{J}T_\mu = T_\mu$, $JT_\lambda = T_\mu$.

3. Reeb recurrent hypersurfaces

Suppose that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ whose shape operator A is Reeb recurrent. From (1), we have

$$(\nabla_\xi A)X = \omega(\xi)AX \tag{12}$$

for a certain one-form ω and any vector field $X \in \mathfrak{X}(M)$. Using this and setting $Y = \xi$ in the Codazzi equation (9) we get

$$(\nabla_X A)\xi = \omega(\xi)AX - \phi X + \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu),$$

for any $X \in \mathfrak{X}(M)$. As M is Hopf, by using $A\xi = \alpha\xi$ and (7) we have

$$(\nabla_X A)\xi = X(\alpha)\xi + \alpha\phi AX - A\phi AX.$$

The subtraction of the above equality from the previous one gives

$$\begin{aligned}\omega(\xi)AX =& X(\alpha)\xi + \alpha\phi AX - A\phi AX + \phi X \\ & - \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu),\end{aligned} \tag{13}$$

for any vector field X. It was proved by Lee and Loo in [13] that $\xi(\alpha) = 0$ on a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. So, using again $A\xi = \alpha\xi$ and setting $X = \xi$ in (13) we have $\alpha\omega(\xi) = 0$. In view of the above equality, let us consider a subset of M defined as $\Omega = \{p \in M : \omega(\xi) \neq 0 \text{ at } p\}$. In what follows in this section, we work on Ω. It follows directly that $\alpha = 0$ on Ω. Putting this into (10) we obtain

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi = 0. \tag{14}$$

Simplifying (13) by using $\alpha = 0$ implies

$$A\phi AX = -\omega(\xi)AX + \phi X - \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu).$$

Similarly, simplifying (11) by using $\alpha = 0$ and (14) we obtain

$$A\phi AX = \phi X + \sum_{\nu=1}^{3}(\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X).$$

The substraction of the above equality from the previous one gives

$$AX = \frac{2}{\omega(\xi)}\sum_{\nu=1}^{3}(\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi\xi_\nu), \tag{15}$$

for any vector field X.

Lemma 3.1. *On a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\alpha = 0$, either $\xi \in \mathfrak{D}^\perp$ or $\xi \in \mathfrak{D}$.*

Proof. Without loss of generality, we assume $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ with X_0 a unit vector field orthogonal to $\mathfrak{D}^\perp$ satisfying $\eta(X_0)\eta(\xi_1) \neq 0$. Using this in (14) gives $\phi_1\xi = 0$ due to $\eta(\xi_1) \neq 0$. It follows directly that

$$0 = g(\phi_1\xi, \phi_1\xi) = \|\xi_1\|^2 - \eta^2(\xi_1) = 1 - \eta^2(\xi_1).$$

But, on the other hand we also have

$$1 = \|\xi\|^2 = g(\eta(X_0)X_0 + \eta(\xi_1)\xi_1, \eta(X_0)X_0 + \eta(\xi_1)\xi_1) = \eta^2(X_0) + \eta^2(\xi_1).$$

By the above two equalities, obviously, $\eta(X_0) = 0$, this contradicts our assumption. Then we obtain the desired result. □

On Ω, as $\alpha = 0$, Lemma 12 is applicable. First, we consider the case $\xi \in \mathfrak{D}$. It has been proved by Lee and Suh in [11] that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$ is locally congruent to an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^m$ in $G_2(\mathbb{C}^{m+2})$. In fact, if $\xi \in \mathfrak{D}$, $\mathfrak{D}$ is invariant under the shape operator A. According to Proposition 2.2 we have $\alpha = -2\tan(2r)$ satisfying $r \in (0, \pi/4)$. Obviously, α is never zero. Therefore, on Ω, by Lemma 3.1, it is necessarily that $\xi \in \mathfrak{D}^\perp$. In this case, without loss of generality we may assume $\xi = \xi_1$. From (15), with the aid of (5), we have $A\xi = 0$, $A\xi_2 = 0$ and $A\xi_3 = 0$.

This means that $\mathfrak{D}$ is invariant under the shape operator and hence now on Ω, M is of type (A) in Theorem 1.1. It has been proved by Lee, Choi and Woo in [9, Remark 4.5] that the shape operator A of real hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$ is necessarily Reeb parallel. This implies $\omega(\xi) = 0$ on Ω, and it contradicts the definition of Ω.

Theorem 3.1. *The shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent if and only if it is Reeb parallel.*

Reeb parallel Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ were classified in [9]. Applying these results and Theorem 12, the following two theorems are valid.

Corollary 3.1. *The shape operator of a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent with $\alpha \neq 0$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \pi/2\sqrt{2})$ but $r \neq \pi/4\sqrt{2}$.*

If $\alpha = 0$, the situation is complex and some additional assumption is needed.

Corollary 3.2. *The shape operator of a Hopf hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent with $\alpha = 0$ and $\|A\|^2 \leq 4m$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r = \pi/4\sqrt{2}$.*

4. $\mathfrak{D}^\perp$-recurrent hypersurfaces

Suppose that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ whose shape operator A is $\mathfrak{D}^\perp$-recurrent. From (1), we have

$$(\nabla_{\xi_\kappa} A)X = \omega(\xi_\kappa)AX, \tag{16}$$

for a certain one-form ω, any vector field X and any $\kappa \in \{1, 2, 3\}$. Setting $Y = \xi_\kappa$ in the Codazzi equation (9) we get

$$\begin{aligned}(\nabla_X A)\xi_\kappa =& \omega(\xi_\kappa)AX + \eta(X)\phi\xi_\kappa - \eta(\xi_\kappa)\phi X - 2\eta_\kappa(\phi X)\xi \\ &+ \sum_{\nu=1}^{3}(\eta_\nu(X)\phi_\nu\xi_\kappa - \eta_\nu(\xi_\kappa)\phi_\nu X - 2\eta_\kappa(\phi_\nu X)\xi_\nu)\end{aligned} \tag{17}$$

$$+\sum_{\nu=1}^{3}(\eta_\nu(\phi X)\phi_\nu\phi\xi_\kappa-\eta_\nu(\phi\xi_\kappa)\phi_\nu\phi X)$$
$$+\sum_{\nu=1}^{3}(\eta(X)\eta_\nu(\phi\xi_\kappa)-\eta(\xi_\kappa)\eta_\nu(\phi X))\xi_\nu,$$

for any vector field X and any $\kappa\in\{1,2,3\}$. Taking the inner product of (4) with ξ and using

$$\eta((\nabla_X A)\xi_\kappa)=g((\nabla_X A)\xi,\xi_\kappa)=X(\alpha)\eta(\xi_\kappa)+\alpha\eta_\kappa(\phi AX)-\eta_\kappa(A\phi AX)$$

we obtain

$$\begin{aligned}
&X(\alpha)\eta(\xi_\kappa)+\alpha\eta_\kappa(\phi AX)-\eta_\kappa(A\phi AX)\\
&=\omega(\xi_\kappa)\eta(AX)-2\eta_\kappa(\phi X)\\
&+\sum_{\nu=1}^{3}(\eta_\nu(X)\eta(\phi_\nu\xi_\kappa)-\eta_\nu(\xi_\kappa)\eta(\phi_\nu X)-2\eta_\kappa(\phi_\nu X)\eta(\xi_\nu))\\
&+\sum_{\nu=1}^{3}(\eta_\nu(\phi X)\eta(\phi_\nu\phi\xi_\kappa)-\eta_\nu(\phi\xi_\kappa)\eta(\phi_\nu\phi X))\\
&+\sum_{\nu=1}^{3}(\eta(X)\eta_\nu(\phi\xi_\kappa)\eta(\xi_\nu)-\eta(\xi_\kappa)\eta_\nu(\phi X)\eta(\xi_\nu)),
\end{aligned}$$

for any vector field X and any $\kappa\in\{1,2,3\}$. Setting $X=\xi$ in the above equality and using $A\xi=\alpha\xi$, we obtain

$$\alpha\omega(\xi_\kappa)+4\sum_{\nu=1}^{3}\eta(\xi_\nu)\eta_\nu(\phi\xi_\kappa)=0,$$

for any $\kappa\in\{1,2,3\}$, where we applied again the fact that the Hopf condition implies $\xi(\alpha)=0$ (see [13]). Applying (5), it follows that $\alpha\omega(\xi_\kappa)=0$, for any $\kappa\in\{1,2,3\}$. This leads us to consider a subset of M defined as follows:

$$\Omega=\{p\in M:\omega(\xi_\mu)\neq 0 \text{ at } p \text{ for some } \mu\in\{1,2,3\}\}.$$

On Ω, we get $\alpha=0$ and hence from Lemma 3.1 we first consider the case $\xi\in\mathfrak{D}$. In this case, from [11] M is of type (B) in Theorem 1.1. But, from Proposition 2.2, α is never zero and this contradicts with $\alpha=0$ on Ω. Hence we must have $\xi\in\mathfrak{D}^\perp$ due to Lemma 3.1. Without loss of generality we may assume $\xi=\xi_1$.

Setting $X=\xi_1$ in (4) and using the $\mathfrak{D}^\perp$ recurrent assumption, we get

$$\omega(\xi_1)A\xi_\kappa=\phi_\kappa\xi_1+\phi_1\xi_\kappa-4\eta_2(\xi_\kappa)\xi_3+4\eta_3(\xi_\kappa)\xi_2,$$

for any $\kappa\in\{1,2,3\}$, where we used (5). We see that $\omega(\xi_1)\neq 0$ at every point in Ω. Otherwise, if there is a point in Ω at which $\omega(\xi_1)=0$. We have

$$\phi_\kappa\xi_1+\phi_1\xi_\kappa-4\eta_2(\xi_\kappa)\xi_3+4\eta_3(\xi_\kappa)\xi_2=0.$$

Setting $\kappa = 2$ or $\kappa = 3$ in this equality we obtain $\xi_3 = 0$ and $\xi_2 = 0$, respectively, and both these are impossible. Now, taking into account $\omega(\xi_1) \neq 0$, we obtain

$$A\xi_\kappa = \frac{1}{\omega(\xi_1)}(\phi_\kappa \xi_1 + \phi_1 \xi_\kappa - 4\eta_2(\xi_\kappa)\xi_3 + 4\eta_3(\xi_\kappa)\xi_2).$$

This means $A\xi_\kappa \in \mathfrak{D}^\perp$, for any $\kappa \in \{1, 2, 3\}$, or equivalently, $\mathfrak{D}$ is invariant under the shape operator. Consequently, from Theorem 1.1 now M is of type (A) and Proposition 2.1 is valid. Since $\alpha = 0$ on Ω, we have $r = \pi/4\sqrt{2}$ and in this case $\beta = \sqrt{2}$. According to this proposition we get

$$(\nabla_{\xi_2} A)\xi = -A\phi A\xi_2 = -\sqrt{2}A\phi\xi_2 = \sqrt{2}A\xi_3 = 2\xi_3.$$

However, as the shape operator is $\mathfrak{D}^\perp$ recurrent, from (16) we have

$$(\nabla_{\xi_2} A)\xi = \omega(\xi_2)A\xi = 0.$$

This equality contradicts the previous one. Finally, we see that Ω is empty and hence $\omega(\xi_\kappa) = 0$, for any $\kappa \in \{1, 2, 3\}$.

Theorem 4.1. *The shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}^\perp$-recurrent if and only if it is $\mathfrak{D}^\perp$-parallel.*

The following corollary follows immediately from Theorem 4.1 and main theorem in [5].

Corollary 4.1. *There exist no Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}^\perp$-recurrent shape operator.*

Remark 4.1. *This corollary covers main results in [7, 16].*

5. Conclusion

The study of recurrent condition of some tensor fields on a Riemannian manifold has been an interesting topic for the last sixty years. In this paper the author classified recurrent condition of the shape operator of a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ along two special distributions of the tangent bundle. This generalized some recent results in this field (for example see Remark 4.1). Besides this, main results in paper motivate the study of other operators in Hopf hypersurface in complex two-plane Grassmannians.

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