The residuated lattice-orderability of idempotent monoids

Wei Chen
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, Fujian 363000
P.R. China
chenwei6808467@126.com

Abstract. In this paper, we study a class of residuated lattices, namely, conical idempotent residuated lattices. We give necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical idempotent residuated lattice.

Keywords: conical idempotent residuated lattice, semilattice, idempotent semigroup.
MSC 2020: 06F05, 20M10.

1. Introduction

Let $(\mathfrak{P}, \leq)$ be a poset. A (binary) operation $\circ$ is called *residuated* if there exist (binary) operations $\rhd$ and $\lhd$ on $\mathfrak{P}$ such that

$$(\forall x, y \in \mathfrak{P})\; x \circ y \leq z \Longleftrightarrow y \leq x \rhd z \Longleftrightarrow x \leq z \lhd y.$$

In this case, the operation pair $(\rhd, \lhd)$ is called a pair of *residuals* of the operation $\circ$. It is well known that an operation $\circ$ on the poset $(\mathfrak{P}, \leq)$ is residuated if and only if $\circ$ is order preserving in each argument and such that, for all $a, b \in \mathfrak{P}$, both $\{p \in \mathfrak{P} \mid a \circ p \leq b\}$ and $\{q \in \mathfrak{P} \mid q \circ a \leq b\}$ contain a greatest element(denote by $a \rhd b$ and $b \lhd a$, respectively). A *residuated lattice* is defined as an algebra $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ satisfying the following conditions:

($RL1$) $(\mathfrak{L}, \wedge, \vee)$ is a lattice;
($RL2$) $(\mathfrak{L}, \circ)$ is a monoid with identity e; and
($RL3$) the operation pair $(\rhd, \lhd)$ is a pair of residuals of the operation $\circ$.

In this case, we call the lattice $(\mathfrak{L}, \wedge, \vee)$ and the semigroup $(\mathfrak{L}, \circ)$ the *lattice reduct* and the *semigroup reduct* of the residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$, respectively. Sometime, residuated lattices are also called *residuated lattice-ordered monoids*. A residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ is called *idempotent* if the semigroup reduct of $\mathfrak{L}$ is an idempotent semigroup. Moreover, an idempotent residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ is called *commutative* if the semigroup reduct of $\mathfrak{L}$ is commutative and $\mathfrak{L}$ is called a *commutative idempotent residuated chain* if its lattice reduct is a chain (see [18] and [19]). As in [1] and [8], a residuated lattice $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ is said to be *conical*, if for each $a \in \mathfrak{L}$, $a \leq e$ or $a \geq e$, where $\leq$ is the order on the lattice reduct of $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$.

Idempotent residuated lattices play a crucial role in residuated lattice theory. On the one hand they include several important algebraic counterparts of substructural logics, e.g., Brouwerian algebras, i.e. algebras of the positive intuitionistic logic, Heyting algebras, i.e. algebras of the propositional intuitionistic logic and positive Sugihara monoids (see [16]) and on the other hand, the knowledge of idempotent residuated lattices can increase our comprehension of residuated lattices (see [11, 13, 14]). Different kinds of idempotent residuated lattices have been introduced and studied in the literature, first to handle idempotent residuated chains, and then for conical idempotent residuated lattices and for finite idempotent residuated lattices (see[1-6,8-10,12,15-19]). In an earlier paper [1] and [3], we investigated conical idempotent residuated lattices from semigroup perspectives. We established a structure theorem and decomposition theorem for conical idempotent residuated lattices. Recently, Gil-Férez, Jipsen and Metcalfe in [12] have used semigroup reducts to give a complete structural description of finite description of finite idempotent residuated chains. More recently, Fussner and Galatos in [6] have shown that non-isomorphic idempotent residuated chains may have the same semigroup reduct. This paper is a continuation of [1] and [3]. Conical idempotent residuated lattices can be considered as a class of lattice-ordered idempotent monoids. The following question naturally arises : How can we characterize the class of monoids that are semigroup reducts of conical idempotent residuated lattices? The purpose of this paper is to solve this question.

We proceed as follows: In Section 2, we recall some definitions and basic facts needed in later proofs. In Section 3, we provide the necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical idempotent residuated lattice which generalize [2, Theorem 5.2].

2. Preliminaries

In this section, we shall first recall some basic definitions and facts on semigroups. For further information on semigroups, we refer to any standard text book, for example, the book by Howie [7]. After this, we recall the concepts of residuated lattices.

Let $\mathfrak{S}$ be a semigroup and $\mathfrak{S}^1$ the semigroup obtained from $\mathfrak{S}$ by adding an identity if $\mathfrak{S}$ has no identity, otherwise we put $\mathfrak{S}^1 = \mathfrak{S}$. In the theory of semigroups, the Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ are of fundamental importance. They are defined in the following way:

$$\mathcal{L} = \{(a, b) \in \mathfrak{S} \times \mathfrak{S} | \mathfrak{S}^1 a = \mathfrak{S}^1 b\},$$
$$\mathcal{R} = \{(a, b) \in \mathfrak{S} \times \mathfrak{S} | a\mathfrak{S}^1 = b\mathfrak{S}^1\},$$

$$\mathcal{J} = \{(a,b) \in \mathfrak{S} \times \mathfrak{S} | \mathfrak{S}^1 a \mathfrak{S}^1 = \mathfrak{S}^1 b \mathfrak{S}^1\},$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R},$$
$$\mathcal{D} = \mathcal{L} \vee \mathcal{R}.$$

Evidently, $\mathcal{L}$ is a right congruence while $\mathcal{R}$ is a left congruence. Moreover, we have

Lemma 2.1 ([7]). *The relations $\mathcal{L}$ and $\mathcal{R}$ commute and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

If $\mathcal{K}$ is one of Green's relations, we shall denote by K_a the $\mathcal{K}$-class of $\mathfrak{S}$ containing $a \in \mathfrak{S}$.

An element x of $\mathfrak{S}$ is called an *idempotent* if $x^2 = x$. $\mathfrak{S}$ is called a *idempotent semigroup* if every element of $\mathfrak{S}$ is an idempotent. idempotent semigroups are also called bands. Moreover, an idempotent semigroup $\mathfrak{B}$ is called a *semilattice* if it is commutative; $\mathfrak{B}$ is called a *rectangular band* if $\mathfrak{B}$ satisfies the identity: $abc \approx ac$.

Now let $\mathfrak{B}$ be an idempotent semigroup. By [7, Proposition 2.1.4 and Theorem 4.1.3], on $\mathfrak{B}$, $\mathcal{D} = \mathcal{J}$ and is a congruence such that $\mathfrak{B}/\mathcal{D}$ is a semilattice, which is usually called *the structure semilattice* of $\mathfrak{B}$. Denote the semilattice $\mathfrak{B}/\mathcal{D}$ by Y. Consider the natural homomorphism $\mathcal{D}^\natural : \mathfrak{B} \to Y$ induced by $\mathcal{D}$. For $\alpha \in Y$, we use D_α to stand for $\alpha {\mathcal{D}^\natural}^{-1}$. Obviously, each D_α is a $\mathcal{D}-$class of $\mathfrak{B}$ and further a rectangular band. It is easy to see that $D_\alpha D_\beta := \{ab \mid a \in D_\alpha, b \in D_\beta\} \subseteq D_{\alpha\beta}$. Thus, we have

Lemma 2.2 ([7]). *Every idempotent semigroup is a semilattice of rectangular bands.*

Let $(\mathfrak{L}, \circ)$ be an idempotent semigroup with an identity. By the arguments before Lemma 2.2, $\mathcal{D}$ is a semilattice congruence on $\mathfrak{L}$. In other words, the quotient semigroup $(\mathfrak{L}/\mathcal{D}, \cdot)$, for short, $\mathfrak{L}/\mathcal{D}$, is a semilattice. For simplicity, we also write the element $a\mathcal{D}^\natural$ of the semigroup $\mathfrak{L}/\mathcal{D}$ as D_a, for $a \in \mathfrak{L}$. On $\mathfrak{L}/\mathcal{D}$, define: for $a, b \in \mathfrak{L}$,

$$D_a \leq^* D_b \text{ if and only if } D_a \cdot D_b = D_a.$$

By [7, Proposition 1.3.2, p. 14], $\leq^*$ is an order on the semilattice $\mathfrak{L}/\mathcal{D}$.

Now, we shall list some basic concepts of residuated lattices used in the sequel. For further information on residuated lattices, we refer to [1] and [10].

Let $(\mathfrak{L}, \wedge, \vee, \circ, \rhd, \lhd, e)$ be a residuated lattice and assume the lattice reduct of $\mathfrak{L}$ is the lattice $(\mathfrak{L}, \leq)$. For convenience, we simply write $a \circ b$ as ab for $a, b \in \mathfrak{L}$. It's well-known that the sets $\{c \in \mathfrak{L} \mid ac \leq b\}$ and $\{c \in \mathfrak{L} \mid ca \leq b\}$ have both a greatest element, in notation, $a \rhd b$ and $b \lhd a$, respectively.

In a residuated lattice term, we assume that multiplication has priority over the division operations, which, in turn, have priority over the lattice operations. So, for example, we write $x \lhd yz \wedge u \rhd v$ for $[x \lhd (yz)] \wedge (u \rhd v)$.

Now, let $(\mathfrak{P}, \leq)$ be a poset and assume $x, y \in \mathfrak{P}$ with $y < x$. We say x *covers* y, in notation, $y \prec x$, if for any $z \in \mathfrak{P}$, $y \leq z \leq x$ implies either $x = z$ or $y = z$. We define $[x, y] = \{u \in \mathfrak{P} \mid x \leq u \leq y\}$ for $x, y \in \mathfrak{P}$ with $x \leq y$. For $a, b \in \mathfrak{P}$, $a \parallel b$ means that a and b are not comparable under $\leq$. Write $P_a^{nc} = \{x \in \mathfrak{P} \mid x \parallel a\}$. b is called a *lower square point* of a if $b < a$ and for any $d \in \mathfrak{P}$ with $b \leq d < a$, there exists $d' \in \mathfrak{P}$ such that $d \parallel d'$. Similarly, we can define *upper square points*. We denote by P_a^l the set of lower square points of a and by P_a^u the set of upper square points of a. Put $P_a = P_a^{nc} \cup P_a^l \cup P_a^u \cup \{a\}$. In what follows, we denote by $P_a^{\top}$ the least upper bound of P_a and by $P_a^{\perp}$ the greatest lower bound of P_a if they exist. (P_a) is obtained by adjoining $P_a^{\top}$ and/or $P_a^{\perp}$, whenever they exist, to P_a. We call the above (P_α) the *square point set* of $\mathfrak{P}$.

Definition 2.1 ([1])**.** *Let $\{\mathfrak{L}^+, \mathfrak{L}^-, \{e\}\}$ be a family of pairwise disjoint subsets of $\mathfrak{L}$ such that $\mathfrak{L} = \mathfrak{L}^+ \cup \mathfrak{L}^- \cup \{e\}$. A partition $\{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ on $\mathfrak{L}$ is called a* ***conical semilattice partition*** *on $\mathfrak{L}$ provided*

(1) $(\mathfrak{Y}, \leq^*)$ *is a semilattice with greatest element* 1*; and*

(2) *The partition satisfies the following conditions:*

(CSP1) $\mathfrak{L}_1 = \{e\}$, *where we let* $e = a_1 = b_1$.

(CSP2) *For each* $\alpha \in \mathfrak{Y} \setminus \{1\}$, $|\mathfrak{L}_\alpha \cap \mathfrak{L}^+| \leq 1$ *and* $|\mathfrak{L}_\alpha \cap \mathfrak{L}^-| \leq 1$. *We shall denote the element in* $\mathfrak{L}_\alpha \cap \mathfrak{L}^+$ *and* $\mathfrak{L}_\alpha \cap \mathfrak{L}^-$ *by* a_α *and* b_α*, respectively.*

(CSP3) *For every* $\alpha, \beta \in \mathfrak{Y}$ *with* $\alpha \parallel^* \beta$, *we have* $|\mathfrak{L}_\alpha| = |\mathfrak{L}_\beta| = |\mathfrak{L}_{\alpha\wedge\beta}| = 1$ *and*

$$\mathfrak{L}_\alpha \cap \mathfrak{L}^+ \neq \emptyset \Leftrightarrow \mathfrak{L}_\beta \cap \mathfrak{L}^+ \neq \emptyset \Leftrightarrow \mathfrak{L}_{\alpha\wedge\beta} \cap \mathfrak{L}^+ \neq \emptyset.$$

(CSP4) $\alpha \vee \beta$ *exists and* $\mathfrak{L}_{\alpha\vee\beta} = \mathfrak{L}_{\alpha\vee\beta} \cap \mathfrak{L}^-$ *for every* $\alpha, \beta \in \mathfrak{Y}$ *such that* $\alpha \parallel^* \beta$ *and* $\mathfrak{L}_\alpha \cap \mathfrak{L}^- \neq \emptyset$.

Definition 2.2 ([1])**.** *Let $\{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ be a conical semilattice partition. A subset $\mathfrak{X}$ of $\mathfrak{Y}$ is called a* ***band subset*** *of the partition $\{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ if $|\mathfrak{L}_\alpha| = 2$ for every $\alpha \in \mathfrak{X}$.*

Definition 2.3 ([1])**.** *Let $\mathfrak{L}$ be a nonempty set. The conical semilattice partition $\pi = \{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\}$ on $\mathfrak{L}$ with band subset $\mathfrak{X}$ is called a* **CLOB-system** *provided the mapping*

$$\Phi : \mathfrak{C} \to \mathfrak{Y}; \quad (\alpha, \beta) \mapsto \Phi(\alpha, \beta),$$

where $\mathfrak{C} = \{(\alpha, \beta) \in \mathfrak{Y} \times \mathfrak{Y} \mid \alpha \parallel^* \beta, \mathfrak{L}_\alpha \cap \mathfrak{L}^+ \neq \emptyset\}$, satisfies the following conditions:

$(B1)$ *If $(\alpha, \beta) \in \mathrm{Dom}\Phi$, then $\alpha <^* \Phi(\alpha, \beta)$, $\beta <^* \Phi(\alpha, \beta)$ and $\mathfrak{L}_{\Phi(\alpha,\beta)} \cap (\mathfrak{L}^+ \cup \{e\}) \neq \emptyset$.*

$(B2)$ *If $(\alpha, \beta) \in \mathrm{Dom}\Phi$ and $\gamma \in \mathfrak{Y}$ such that $\alpha, \beta <^* \gamma <^* \Phi(\alpha, \beta)$, then $\mathfrak{L}_\gamma = \mathfrak{L}_\gamma \cap \mathfrak{L}^-$.*

In what follows, we denote this system in Definition 2.3 by $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi)$.

Lemma 2.3 ([1]). *Let $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi)$ be a CLOB-system. Define an order $\leq$ on $\mathfrak{L}$ as follows: for $a\in\mathfrak{L}_\alpha, b\in\mathfrak{L}_\beta$,*

$$a\leq b \quad \text{if and only if} \quad \begin{array}{l}\text{either condition } (O1)\text{: } \alpha\leq^*\beta \text{ and } a=b_\alpha \text{ or}\\ \text{condition } (O2)\text{: } \beta\leq^*\alpha \text{ and } b=a_\beta.\end{array}$$

Define a multiplication $\circ$ on $\mathfrak{L}$ in the following ways: for $a_\alpha,b_\alpha,a_\beta,b_\beta\in\mathfrak{L}$,

$$a_\alpha\circ a_\beta=a_{\alpha\wedge\beta};$$
$$b_\alpha\circ b_\beta=b_{\alpha\wedge\beta};$$
$$a_\alpha\circ b_\beta=\begin{cases}a_\alpha, & \text{if } \alpha<^*\beta \text{ or } \alpha=\beta\in\mathfrak{X},\\ b_\beta, & \text{if } \beta<^*\alpha \text{ or } \alpha=\beta\notin\mathfrak{X};\end{cases}$$
$$b_\beta\circ a_\alpha=\begin{cases}a_\alpha, & \text{if } \alpha<^*\beta \text{ or } \alpha=\beta\notin\mathfrak{X},\\ b_\beta, & \text{if } \beta<^*\alpha \text{ or } \alpha=\beta\in\mathfrak{X}.\end{cases}$$

Then, $(\mathfrak{L},\circ,\leq)$ is an ordered idempotent semigroup (see [1]).

Definition 2.4 ([1]). *A CLOB-system $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi)$ is called a **CRLOB-system** provided the mappings*

$$\psi:\{\alpha\in\mathfrak{Y}\setminus\{1\}:\mathfrak{L}_\alpha\cap\mathfrak{L}^+\neq\emptyset\}\to\mathfrak{Y};\quad \alpha\mapsto\psi(\alpha),$$
$$\varphi:\{\alpha\in\mathfrak{Y}\setminus\{1\}:\mathfrak{L}_\alpha\cap\mathfrak{L}^-\neq\emptyset\}\to\mathfrak{Y};\quad \alpha\mapsto\varphi(\alpha)$$

and

$$\Psi:\mathfrak{R}\to\mathfrak{Y};(\alpha,\beta)\mapsto\Psi(\alpha,\beta),$$

where $\mathfrak{R}=\{(\alpha,\beta)\in\mathfrak{Y}\times\mathfrak{Y}\mid\alpha\parallel^*\beta,\mathfrak{L}_\alpha\cap\mathfrak{L}^-\neq\emptyset$ or $\beta<^*\alpha,\mathfrak{L}_\beta\cap\mathfrak{L}^-\neq\emptyset$ and $\alpha\in P_\beta\}$, satisfy the following conditions:

$(C1)$ *If $\alpha\in\mathrm{Dom}\psi$, then $\psi(\alpha)<^*\alpha$ and $\mathfrak{L}_{\psi(\alpha)}\cap\mathfrak{L}^-\neq\emptyset$.*
$(C2)$ *If $\alpha\in\mathrm{Dom}\varphi$, then $\alpha<^*\varphi(\alpha)$ and $\mathfrak{L}_{\varphi(\alpha)}\cap(\mathfrak{L}^+\cup\{e\})\neq\emptyset$.*
$(C3)$ *If $\alpha\in\mathrm{Dom}\psi$ and $\beta\in\mathfrak{Y}$ such that $\psi(\alpha)<^*\beta<^*\alpha$, then $\mathfrak{L}_\beta=\mathfrak{L}_\beta\cap\mathfrak{L}^+$.*
$(C4)$ *If $\alpha\in\mathrm{Dom}\varphi$ and $\beta\in\mathfrak{Y}$ such that $\alpha<^*\beta<^*\varphi(\alpha)$, then $\mathfrak{L}_\beta=\mathfrak{L}_\beta\cap\mathfrak{L}^-$.*
$(C5)$ *If $(\alpha,\beta)\in\mathrm{Dom}\Psi$, then $\alpha\wedge\Psi(\alpha,\beta)\leq^*\beta$.*
$(C6)$ *If $(\alpha,\beta)\in\mathrm{Dom}\Psi$ and $\gamma\in\mathfrak{Y}$ such that $\mathfrak{L}_\gamma\cap\mathfrak{L}^-\neq\emptyset$ and $\alpha\wedge\gamma\leq^*\alpha\wedge\beta$, then $\gamma\leq^*\Psi(\alpha,\beta)$.*

We shall denote this system by $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$ and the related ordered idempotent semigroup $(\mathfrak{L},\circ,\leq)$ by $CRLOB(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$.

Theorem 2.1 ([1]). *Let $(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$ be a CRLOB-system. Then $CRLOB(\mathfrak{L};\pi,\mathfrak{Y};\mathfrak{X},\Phi;\psi,\varphi,\Psi)$ is a conical idempotent residuated lattice. Conversely, any such residuated lattice can be constructed in this manner.*

3. Residuated lattice-orderability of an idempotent semigroup with an identity

In this section we shall give necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical residuated lattice, which generalize [2, Theorem 5.2].

Let $(\mathfrak{Y}, \leq^*)$ be a semilttice. The subset $\mathfrak{U}$ of $\mathfrak{Y}$ is called an *R-sublattice* of $\mathfrak{Y}$ if $\mathfrak{U}$ is a sublattice of $\mathfrak{Y}$ and satisfies the condition (RS): If $\alpha, \beta \in \mathfrak{U}$, $\alpha \nleq^* \beta$, then there exists δ in $\mathfrak{U}$ such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in \mathfrak{U}$ such that $\delta' \wedge \alpha \leq^* \alpha \wedge \beta$, $\delta' \leq^* \delta$. The condition (RS) indeed is to ensure that the set $\{c \in \mathfrak{U} \mid a \wedge c \leq^* b\}$ has a greatest element, for all $a, b \in \mathfrak{U}$.

Let (P_α) be the square point set of $\mathfrak{Y}$. Let $(P_\alpha)^+ = \{\beta \in (P_\alpha) \mid (\exists \gamma, \gamma' \in (P_\alpha))\gamma \vee \gamma'$ does not exists and $\beta \leq^* \gamma\}$ and let $(P_\alpha)^- = (P_\alpha) \setminus (P_\alpha)^+$. (P_α) is called a *pre-sublattice* of $\mathfrak{Y}$ if (P_α) satisfies the following conditions:

$(P1)$ (P_α) is not a sublattice of $\mathfrak{Y}$ and $((P_\alpha)^+ \cup \{\beta\}, \leq^*)$ is a lattice for some $\beta \in \mathfrak{Y}$;

$(P2)$ If $\beta \in (P_\alpha)^+$ and $\gamma \in (P_\alpha)^-$, then $\beta <^* \gamma$;

$(P3)$ If $(P_\alpha)^- \neq \emptyset$, then $(P_\alpha)^-$ is an R-sublattice of $\mathfrak{Y}$.

Lemma 3.1. *Let $(\mathfrak{Y}, \leq^*)$ be a semilattice.*

(1) *If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $\beta, \gamma \in (P_\alpha)$, then $\beta \wedge \gamma \in (P_\alpha)$ and $\beta \vee \gamma \in (P_\alpha)$ whenever $\beta \vee \gamma$ exists.*

(2) *If $\alpha, \alpha', \gamma \in \mathfrak{Y}$ such that $\alpha \parallel^* \alpha'$ and $\gamma \in P_\alpha$, then $P_\alpha = P_\gamma$.*

(3) *If $\alpha, \beta, \alpha', \beta' \in \mathfrak{Y}$ such that $\alpha \parallel \alpha', \beta \parallel \beta'$ and $P_\alpha \cap P_\beta \neq \emptyset$, then $P_\alpha = P_\beta$ and $(P_\alpha) = (P_\beta)$.*

(4) *If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^+$, then $\beta \wedge \gamma \in (P_\alpha)^+$ and $\beta \vee \gamma \in (P_\alpha)^+$ whenever $\beta \vee \gamma$ exists.*

(5) *If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^-$, then $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)^-$.*

Proof. (1) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $\beta, \gamma \in (P_\alpha)$. We consider the following cases:

- $\beta \leq^* \gamma$ or $\gamma \leq^* \beta$. Then, $\beta \wedge \gamma \in \{\beta, \gamma\} \subseteq (P_\alpha)$.
- $\beta \parallel^* \gamma$. Then, $\beta \wedge \gamma <^* \beta$. If $\beta \wedge \gamma \parallel^* \alpha$, then $\beta \wedge \gamma \in (P_\alpha)$. If $\alpha \leq^* \beta \wedge \gamma$, then $\alpha \leq^* \beta \wedge \gamma <^* \beta$. Since $\beta \in (P_\alpha)$, $\beta \wedge \gamma \in (P_\alpha)$. If $\beta \wedge \gamma <^* \alpha$, then we can claim that $\beta \wedge \gamma \in (P_\alpha)$. Otherwise if $\beta \wedge \gamma \notin (P_\alpha)$, then for all $\xi \in (P_\alpha)$, $\beta \wedge \gamma <^* \xi$. If $P_a^\perp$ exists, then $\beta \wedge \gamma <^* P_a^\perp$. Since $P_a^\perp <^* \beta, \gamma$, $P_a^\perp \leq^* \beta \wedge \gamma$. It's a contradiction. If $P_a^\perp$ doesn't exist, then there exists $\delta \in \mathfrak{Y}$ such that $\beta \wedge \gamma <^* \delta <^* \alpha$ and $\delta <^* \eta$, for all $\eta \in (P_\alpha)$. Since $\beta, \gamma \in (P_\alpha)$, $\delta <^* \beta, \gamma$. It follows that $\delta \leq^* \beta \wedge \gamma$, which contrary to $\beta \wedge \gamma <^* \delta$. Thus, $\beta \wedge \gamma \in (P_\alpha)$.

Similarly, if $\beta \vee \gamma$ exists, then $\beta \wedge \gamma \in (P_\alpha)$.

(2) Let $\alpha, \alpha', \gamma \in \mathfrak{Y}$ such that $\alpha \parallel^* \alpha'$ and $\gamma \in P_\alpha$. To prove that $P_\alpha = P_\gamma$, we consider the following cases:

- $\gamma \parallel^* \alpha$. Firstly, we will prove that $P_\gamma \subseteq P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta \parallel^* \gamma$. If $\delta \parallel^* \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then since $\gamma \parallel^* \alpha$, for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, $\zeta \parallel^* \gamma$. It follows that $\delta \in P_\alpha$. Similarly, if $\alpha <^* \delta$, then $\delta \in P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta <^* \gamma$. Then, since $\gamma \parallel^* \alpha$, either $\delta \parallel^* \alpha$, or $\delta <^* \alpha$. If $\delta \parallel^* \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, either $\zeta \parallel^* \gamma$ or $\zeta <^* \gamma$. If $\zeta <^* \gamma$, then since $\delta \leq^* \zeta$ and $\delta \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. Similarly, if $\delta \in P_\gamma$ such that $\gamma \leq^* \delta$, then $\delta \in P_\alpha$. It follows that $P_\gamma \subseteq P_\alpha$. Similarly, $P_\alpha \subseteq P_\gamma$. Consequently, $P_\alpha = P_\gamma$.

- $\gamma \leq^* \alpha$. Firstly, we will prove that $P_\gamma \subseteq P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta \parallel^* \gamma$. Then, either $\delta \parallel^* \alpha$ or $\delta <^* \alpha$. If $\delta \parallel^* \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, either $\zeta \parallel^* \gamma$ or $\gamma <^* \zeta$. If $\gamma <^* \zeta$, then $\gamma <^* \zeta <^* \alpha$. Since $\gamma \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. Let $\delta \in P_\gamma$ such that $\delta <^* \gamma$. Then, $\delta <^* \gamma \leq^* \alpha$. Suppose that $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$. Then, either $\zeta \parallel^* \gamma$ or $\zeta \nparallel^* \gamma$. If $\zeta <^* \gamma$, then since $\delta \in P_\gamma$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. If $\gamma \leq^* \zeta$, then since $\gamma \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. Let $\delta \in P_\gamma$ such that $\gamma \leq^* \delta$. If $\delta \parallel \alpha$ or $\delta = \alpha$, then $\delta \in P_\alpha$. If $\delta <^* \alpha$, then for any $\zeta \in \mathfrak{Y}$ such that $\delta \leq^* \zeta <^* \alpha$, $\gamma \leq^* \zeta <^* \alpha$. Since $\gamma \in P_\alpha$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$. It follows that $\delta \in P_\alpha$. If $\alpha <^* \delta$, then for any $\zeta \in \mathfrak{Y}$ such that $\alpha <^* \zeta \leq^* \delta$, $\gamma <^* \zeta \leq^* \delta$. Since $\delta \in P_\gamma$, there exists $\zeta' \in \mathfrak{Y}$ such that $\zeta \parallel^* \zeta'$, which implies that $\delta \in P_\alpha$. It follows that $P_\gamma \subseteq P_\alpha$. Similarly, $P_\alpha \subseteq P_\gamma$. Consequently, $P_\alpha = P_\gamma$.

- $\alpha <^* \gamma$. By similar arguments as in the prior case, we have $P_\alpha = P_\gamma$.

(3) Let $\alpha, \beta, \alpha', \beta' \in \mathfrak{Y}$ such that $\alpha \parallel \alpha', \beta \parallel \beta'$ and $P_\alpha \cap P_\beta \neq \emptyset$. Let $\gamma \in P_\alpha \cap P_\beta$. Then, by (2), $P_\alpha = P_\gamma = P_\beta$ and so $(P_\alpha) = (P_\beta)$.

(4) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^+$. Since $\beta \wedge \gamma \leq^* \beta$, $\beta \wedge \gamma \in (P_\alpha)^+$ by (1) and $(P2)$. If $\beta \vee \gamma$ exists, then by (1), $\beta \vee \gamma \in (P_\alpha)$. Since $\beta \in (P_\alpha)^+$, there exist $\delta, \delta' \in (P_\alpha)^+$ such that $\delta \vee \delta'$ doesn't exist and $\beta \leq^* \delta$. Suppose that $\beta \vee \gamma \in (P_\alpha)^-$. Then, by $(P2)$, $\delta, \delta' <^* \beta \vee \gamma$ and so there exists $\zeta \in (P_\alpha)$ such that $\delta, \delta' <^* \zeta <^* \beta \vee \gamma$. If $\zeta \in (P_\alpha)^-$, then by $(P2)$, $\beta, \gamma <^* \zeta$, and so $\beta \vee \gamma \leq^* \zeta$, which contrary to $\zeta <^* \beta \vee \gamma$. If $\zeta \in (P_\alpha)^+$, then there exist $\omega, \omega' \in (P_\alpha)^+$ such that $\omega \vee \omega'$ doesn't exist and $\zeta \leq^* \omega$. Since $\delta \vee \delta'$ doesn't exist, there exists $\xi \in (P_\alpha)$ such that $\delta, \delta' <^* \xi <^* \zeta$. Hence, by $(P2)$, $\xi \in (P_\alpha)^+$. Because $((P_\alpha)^+ \cup \{\eta\}, \leq^*)$ is a lattice for some $\eta \in \mathfrak{Y}$, $\delta \vee_{((P_\alpha)^+ \cup \{\eta\})} \delta' = \omega \vee_{((P_\alpha)^+ \cup \{\eta\})} \omega' = \eta$, which implies that $\eta <^* \xi <^* \zeta \leq^* \omega <^* \eta$. It's a contradiction. Thus, $\beta \vee \gamma \in (P_\alpha)^+$.

(5) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that (P_α) is a pre-sublattice of $\mathfrak{Y}$ and $\beta, \gamma \in (P_\alpha)^-$. Then, by (1), $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)$. Since $\beta <^* \beta \vee \gamma$, $\beta \vee \gamma \in (P_\alpha)^-$ by $(P2)$. Assume that $\beta \wedge \gamma \in (P_\alpha)^+$. Then, there exist $\delta, \delta' \in (P_\alpha)^+$ such that $\delta \vee \delta'$ doesn't exist and $\beta \wedge \gamma \leq^* \delta$. Hence, by $(P2)$, $\delta, \delta' <^* \beta, \gamma$. Thus, $\delta, \delta' \leq^* \beta \wedge \gamma$.

It follows that $\delta' \leq^* \beta \wedge \gamma = \delta$. Thus, $\delta \vee \delta' = \delta$, which contrary to $\delta \vee \delta'$ doesn't exist. Consequently, $\beta \wedge \gamma \in (P_\alpha)^-$. □

Lemma 3.2. *Let $\mathfrak{L} = CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}); \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$ be a conical idempotent residuated lattice.*

(1) If $\alpha \in \mathrm{Dom}\psi$, then for each $\beta \in \mathfrak{Y}$, $\psi(\alpha) \leq^* \beta$ or $\beta \leq^* \psi(\alpha)$. Moreover, in this case, $\psi(\alpha) \prec^* \varphi(\psi(\alpha))$.

(2) If $\alpha \in \mathrm{Dom}\varphi$, then for each $\beta \in \mathfrak{Y}$, $\varphi(\alpha) \leq^* \beta$ or $\beta \leq^* \varphi(\alpha)$. Moreover, if $\varphi(\alpha) \neq 1$, then $\psi(\varphi(\alpha)) \prec^* \varphi(\alpha)$.

(3) If $(\alpha, \beta) \in \mathrm{Dom}\Phi$ and $\alpha \vee \beta$ doesn't exist, then for each $\gamma \in \mathfrak{Y}$, $\Phi(\alpha, \beta) \leq^* \gamma$ or $\gamma \leq^* \Phi(\alpha, \beta)$.

(4) If $\alpha, \eta, \zeta \in \mathfrak{Y}$ such that $\zeta \in (P_\alpha)^+, \eta \in (P_\alpha)^-$, then $\zeta <^* \eta$ and $D_\zeta = \{a_\zeta\}$.

(5) If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $\beta, \gamma \in (P_\alpha)^+$, then $\beta \wedge \gamma \in (P_\alpha)^+$ and $\beta \vee \gamma \in (P_\alpha)^+$ whenever $\beta \vee \gamma$ exists.

(6) If $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $(P_\alpha)^+ \neq \emptyset$ and $\beta, \gamma \in (P_\alpha)^-$, then $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)^-$. Moreover, if $\beta \neq 1$, then $D_\beta = \{b_\beta\}$.

Proof. (1) Suppose that there exists $\beta \in \mathfrak{Y}$ such that $\psi(\alpha) \parallel^* \beta$. Since $D_{\psi(\alpha)}$ contains $b_{\psi(\alpha)}$ and by $(CSP3)$ of Definition 2.1, $D_\beta = \{b_\beta\}$, which, together with $a_\alpha \in D_\alpha$ by noting that $\alpha \in \mathrm{Dom}\psi$, derives $\alpha \nparallel^* \beta$. By $(C1)$ of Definition 2.4, $\psi(\alpha) <^* \alpha$, so $\beta <^* \alpha$, hence by $(CSP4)$ of Definition 2.1, $\psi(\alpha) \vee \beta$ exists and $D_{\psi(\alpha)\vee\beta} = \{b_{\psi(\alpha)\vee\beta}\}$. Thus $\psi(\alpha) <^* \psi(\alpha) \vee \beta <^* \alpha$. Therefore, by $(C3)$ of Definition 2.4, $D_{\psi(\alpha)\vee\beta} = \{a_{\psi(\alpha)\vee\beta}\}$, contrary to $D_{\psi(\alpha)\vee\beta} = \{b_{\psi(\alpha)\vee\beta}\}$. We conclude that for each $\beta \in \mathfrak{Y}$, $\varphi(\alpha) \leq^* \beta$ or $\beta \leq^* \varphi(\alpha)$. Suppose that there exists $\beta \in \mathfrak{Y}$ such that $\psi(\alpha) <^* \beta <^* \varphi(\psi(\alpha))$. Then, by $(C4)$ of Definition 2.4, $D_\beta = \{b_\beta\}$. Since $\alpha \in \mathrm{Dom}\psi$, $a_\alpha \in D_\alpha$ and by $(CSP3)$ of Definition 2.1, $\alpha \nparallel^* \beta$. Assume $\alpha <^* \beta$. Then $\psi(\alpha) <^* \alpha <^* \beta <^* \varphi(\psi(\alpha))$ and so by $(C4)$ of Definition 2.4, $D_\alpha = \{b_\alpha\}$, contrary to $\alpha \in \mathrm{Dom}\psi$. This implies $\beta <^* \alpha$ and so by $(C3)$ of Definition 2.4, $D_\beta = \{a_\beta\}$, contrary to $D_\beta = \{b_\beta\}$. Thus, $\psi(\alpha) \prec^* \varphi(\psi(\alpha))$.

(2) It is similar to (1).

(3) Suppose that $(\alpha, \beta) \in \mathrm{Dom}\Phi$ and $\alpha \vee \beta$ doesn't exist. Then, by $(B1)$ of Definition 2.3, $\alpha, \beta <^* \Phi(\alpha, \beta)$ and $D_{\Phi(\alpha,\beta)}$ contains $a_{\Phi(\alpha,\beta)}$. Since $\alpha \vee \beta$ doesn't exist, there exists $\gamma \in \mathfrak{Y}$ such that $\alpha, \beta <^* \gamma <^* \Phi(\alpha, \beta)$. Hence, by $(B2)$ of Definition 2.3, $D_\gamma = \{b_\gamma\}$, so by (2), for each $\delta \in \mathfrak{Y}$, either $\varphi(\gamma) \leq^* \delta$ or $\delta \leq^* \varphi(\gamma)$, thereby $\varphi(\gamma) <^* \Phi(\alpha, \beta)$ or $\Phi(\alpha, \beta) \leq^* \varphi(\gamma)$. If $\varphi(\gamma) <^* \Phi(\alpha, \beta)$, then $\alpha, \beta <^* \gamma <^* \varphi(\gamma) <^* \Phi(\alpha, \beta)$ and so by $(B2)$ of Definition 2.3, $D_{\varphi(\gamma)} = \{b_{\varphi(\gamma)}\}$, contrary to $D_{\varphi(\gamma)}$ contains $a_{\varphi(\gamma)}$. If $\Phi(\alpha, \beta) <^* \varphi(\gamma)$, then $\gamma <^* \Phi(\alpha, \beta) <^* \varphi(\gamma)$ and so by $(C4)$ of Definition 2.4, $D_{\Phi(\alpha,\beta)} = \{b_{\Phi(\alpha,\beta)}\}$, contrary to $D_{\Phi(\alpha,\beta)}$ contains $a_{\Phi(\alpha,\beta)}$. Hence, $\Phi(\alpha, \beta) = \varphi(\gamma)$ and so by (2), for each $\delta \in \mathfrak{Y}$, either $\Phi(\alpha, \beta) \leq^* \delta$ or $\delta \leq^* \Phi(\alpha, \beta)$.

(4) Suppose to the contrary that $\eta \leq^* \zeta$ or $\zeta \parallel^* \eta$. If $\eta \leq^* \zeta$, then since $\zeta \in (P_\alpha)^+$, there exist $\zeta', \delta' \in (P_\alpha)$ such that $\zeta' \vee \delta'$ doesn't exist and $\zeta \leq^* \zeta'$, hence $\eta \leq^* \zeta'$, which implies that $\eta \in (P_\alpha)^+$, contrary to $\eta \in (P_\alpha)^- = (P_\alpha) \setminus (P_\alpha)^+$. If

$\zeta \parallel^* \eta$, then since $\zeta \in (P_\alpha)^+$, there exist $\zeta', \delta' \in (P_\alpha)$ such that $\zeta' \vee \delta'$ doesn't exist and $\zeta \leq^* \zeta'$. If $\eta <^* \zeta'$, then $\eta \in (P_\alpha)^+$, contrary to $\eta \in (P_\alpha)^- = (P_\alpha) \setminus (P_\alpha)^+$. If $\eta \parallel^* \zeta'$, then since $\zeta' \vee \delta'$ doesn't exist, $\zeta' \parallel^* \delta'$, hence by $(CPS3)$ and $(CSP4)$ of Definition 2.1, $D_{\delta'} = \{a_{\delta'}\}$ and $D_{\zeta'} = \{a_{\zeta'}\}$, which imply that $D_\eta = \{a_\eta\}$. Since $\zeta' \vee \delta'$ doesn't exist, there exists $\delta \in \mathfrak{Y}$ such that $\zeta', \delta' <^* \delta <^* \Phi(\zeta', \delta')$, hence by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. By $(CSP3)$ of Definition 2.1, $\eta \nparallel^* \delta$. Since $\eta \parallel^* \zeta'$ and $\zeta' <^* \delta$, $\eta <^* \delta$. Note that $\eta \in (P_\alpha)^-$, $\eta \vee \zeta'$ and $(\eta \vee \zeta') \vee \delta'$ exist, hence we have $\zeta', \delta' <^* (\eta \vee \zeta') \vee \delta' \leq^* \delta <^* \Phi(\zeta', \delta')$. Since $\zeta' \vee \delta'$ doesn't exist, there exists $\xi \in \mathfrak{Y}$ such that $\zeta', \delta' <^* \xi <^* (\eta \vee \zeta') \vee \delta' <^* \Phi(\zeta', \delta')$. By $(B2)$ of Definition 2.3, $D_\xi = \{b_\xi\}$. By $(CSP3)$, $\xi <^* \eta$ or $\eta <^* \xi$. Assume that $\xi <^* \eta$. Then, $\zeta', \delta' <^* \xi <^* \eta <^* \delta <^* \Phi(\zeta', \delta')$. By $(B2)$, $D_\eta = \{b_\eta\}$, which contrary to $D_\eta = \{a_\eta\}$. Thus, $\eta <^* \xi$. It follows that $\eta \vee \zeta', \delta' \leq^* \xi <^* (\eta \vee \zeta') \vee \delta'$, contrary to $(\eta \vee \zeta') \vee \delta' \leq^* \xi$. Thus, $\eta \leq^* \zeta$ and $\zeta \parallel^* \eta$ are both impossible. Consequently, $\zeta <^* \eta$. Since $D_{\zeta'} = \{a_{\zeta'}\}$, by $(1, 2)$, $\psi(\zeta') \prec^* \varphi(\psi(\zeta')) \leq^* \zeta \leq^* \zeta'$ and so by $(C3)$ of Definition 2.4, $D_\zeta = \{a_\zeta\}$.

(5) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $\beta, \gamma \in (P_\alpha)^+$. Then, there exist $\zeta, \zeta', \delta, \delta' \in (P_\alpha)$ such that $\beta \leq^* \zeta$, $\gamma \leq^* \delta$, $\zeta \vee \zeta'$ and $\delta \vee \delta'$ don't exist. Hence, $\beta \wedge \gamma \leq \zeta$ which implies that $\beta \wedge \gamma \in (P_\alpha)^+$ by noting that $\beta \wedge \gamma \in (P_\alpha)$ by Lemma 3.1(1). Suppose that $\beta \vee \gamma$ exists. Then, by Lemma 3.1(1), $\beta \vee \gamma \in (P_\alpha)$. If $\beta \nparallel^* \gamma$, then $\beta \vee \gamma \in \{\beta, \gamma\} \subseteq (P_\alpha)^+$. If $\beta \parallel^* \gamma$, then by Lemma 3.1(1), $\beta \vee \gamma \in (P_\alpha)$. If $\beta \vee \gamma \leq^* \zeta$, then $\beta \vee \gamma \in (P_\alpha)^+$. If $\beta \vee \gamma \parallel^* \zeta$, then by (4), $\beta \vee \gamma \in (P_\alpha)^+$. If $\zeta <^* \beta \vee \gamma$, then $\beta \leq^* \zeta <^* \beta \vee \gamma$. Assume that $\beta \vee \gamma \notin (P_\alpha)^+$. Then, $\beta \vee \gamma \in (P_\alpha)^-$ by Lemma 3.1(1). Since $\zeta, \zeta' \in (P_\alpha)$ and $\zeta \vee \zeta'$ doesn't exist, $\zeta \parallel \zeta'$ and so $\zeta, \zeta' \in (P_\alpha)^+$. Hence, by (4), $\zeta, \zeta' <^* \beta \vee \gamma$ and by $(CSP3)$ and $(CSP4)$ of Definition 2.1, $D_\zeta = \{a_\zeta\}$, $D_{\zeta'} = \{a_{\zeta'}\}$. If $\gamma <^* \zeta'$, then for all $\eta \in \mathfrak{Y}$ such that $\zeta, \zeta' <^* \eta$, $\beta, \gamma <^* \eta$. Hence, $\beta \vee \gamma \leq^* \eta$ and so $\zeta \vee \zeta' = \beta \vee \gamma$, which contrary to $\zeta \vee \zeta'$ doesn't exist. If $\gamma \parallel^* \zeta'$, then by $(CSP2)$, $D_\gamma = \{a_\gamma\}$ and $D_\beta = \{a_\beta\}$. By $(B1)$ of Definition 2.3, $\zeta, \zeta' <^* \Phi(\zeta, \zeta')$. We claim that $\alpha' \leq^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. If $\Phi(\zeta, \zeta') = 1$, then $\alpha' \leq^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. If $\Phi(\zeta, \zeta') \neq 1$, then $\Phi(\zeta, \zeta') <^* 1$. Suppose that there exists $\gamma' \in (P_\alpha)$ such that $\Phi(\zeta, \zeta') \leq^* \gamma'$. Then, by $(B1)$ of Definition 2.3, $\zeta, \zeta' <^* \Phi(\zeta, \zeta')$, hence there exists $\xi \in \mathfrak{Y}$ such that $\zeta, \zeta' <^* \xi <^* \Phi(\zeta, \zeta')$ by noting that $\zeta \vee \zeta'$ doesn't exist. By $(B2)$ of Definition 2.3, $D_\xi = \{b_\xi\}$. It follows from the proof of (3) that $\Phi(\zeta, \zeta') = \varphi(\xi)$. By (2), $\psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta') = \varphi(\xi)$. By (1-3), $\alpha \leq^* \psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta')$ or $\psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta') \leq^* \alpha$. If $\alpha \leq^* \psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta')$, then by the definition of (P_α) and (1-3), $\beta' \leq^* \psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta')$, for all $\beta' \in (P_\alpha)$, which implies that $\gamma' \notin (P_\alpha)$ by noting that $\Phi(\zeta, \zeta') \leq^* \gamma'$, a contradiction. If $\psi(\varphi(\xi)) \prec^* \Phi(\zeta, \zeta') \leq^* \alpha$, then $\beta, \gamma \notin (P_\alpha)$, a contradiction. Thus, $\alpha' <^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. Consequently, $\alpha' \leq^* \Phi(\zeta, \zeta')$, for all $\alpha' \in (P_\alpha)$. It follows that $\beta \vee \gamma \leq^* \Phi(\zeta, \zeta')$. Since $\zeta \vee \zeta'$ doesn't exist,there exists $\omega \in \mathfrak{Y}$ such that $\zeta, \zeta' <^* \omega <^* \beta \vee \gamma \leq^* \Phi(\zeta, \zeta')$. By $(B2)$ of Definition 2.3, $D_\omega = \{b_\omega\}$. Since $D_\gamma = \{a_\gamma\}$ and $\gamma \parallel^* \zeta'$, $\gamma <^* \omega$ by $(CSP3)$ of Definition 2.1. It follows that $\beta \vee \gamma \leq^* \omega$. It's a contradiction. Consequently, $\beta \vee \gamma \in (P_\alpha)^+$.

(6) Let $\alpha, \beta, \gamma \in \mathfrak{Y}$ such that $(P_\alpha)^+ \neq \emptyset$ and $\beta, \gamma \in (P_\alpha)^-$. Then, there exist $\zeta, \zeta' \in (P_\alpha)^+$ such that $\zeta \vee \zeta'$ doesn't exist and $\beta \wedge \gamma, \beta \vee \gamma \in (P_\alpha)$ by Lemma 3.1(1). Since $\beta \in (P_\alpha)^-$ and $\beta <^* \beta \vee \gamma$, $\beta \vee \gamma \in (P_\alpha)^-$ by (4). If $\beta \nparallel^* \gamma$, then $\beta \wedge \gamma \in \{\beta, \gamma\} \subseteq (P_\alpha)^-$. Suppose that $\beta \parallel^* \gamma$. Then, by the proof of (5), $\zeta, \zeta' <^* \beta \wedge \gamma <^* \beta, \gamma <^* \Phi(\zeta, \zeta')$, and so by $(B2)$ of Definition 2.3, $D_\beta = \{b_\beta\}, D_\gamma = \{b_\gamma\}$ and $D_{\beta\wedge\gamma} = \{b_{\beta\wedge\gamma}\}$. Thus, by (4), $\beta \wedge \gamma \in (P_\alpha)^-$. If $\beta \neq 1$, then by the proof of (5), $\zeta, \zeta' <^* \beta <^* \Phi(\zeta, \zeta')$, and so by $(B2)$ of Definition 2.3, $D_\beta = \{b_\beta\}$. □

Theorem 3.1. *Let $\mathfrak{L}$ be an idempotent semigroup with identity e and $\mathfrak{Y}$ be the structure semilattice with greatest element 1 of $\mathfrak{L}$. Then, $\mathfrak{L}$ is the semigroup reduct of some conical idempotent residuated lattice if and only if $\mathfrak{L}$ satisfies the following conditions:*

$(SR1)$ *For each $\alpha \in \mathfrak{Y}$, either (P_α) is a sublattice of $\mathfrak{Y}$ or (P_α) a pre-sublattice of $\mathfrak{Y}$ and $|D_\alpha| = 1$.*

$(SR2)$ *Each $\mathcal{D}-$class of $\mathfrak{L}$ contains at most two elements.*

$(SR3)$ *If $a \in D_\alpha$ and $b \in D_\beta$ such that $\alpha <^* \beta$, then $ab = ba = a$.*

$(SR4)$ *If $\alpha, \beta \in \mathfrak{Y}$ such that $\alpha \parallel^* \beta$, then $|D_\alpha| = |D_\beta| = |D_{\alpha\wedge\beta}| = 1$.*

$(SR5)$ *If (P_α) is a pre-sublattice of $\mathfrak{Y}$, then α satisfies the following conditions:*

$(D1)$ *There exists $\alpha^+ \in \mathfrak{Y}$ satisfying*

(a) *For each $\beta \in \mathfrak{Y}$, either $\alpha^+ \leq^* \beta$ or $\beta \leq^* \alpha^+$;*

(b) *If $\alpha \neq 1$, then $\alpha <^* \alpha^+$;*

(c) *If $\alpha^+ \neq 1$, then there exists $\gamma \in \mathfrak{Y}$ such that $\gamma \prec^* \alpha^+$ and for each $\beta \in \mathfrak{Y}$, either $\gamma \leq^* \beta$ or $\beta \leq^* \gamma$;*

(d) *If $\beta \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\alpha' <^* \beta <^* \alpha^+$, then $|D_\beta| = 1$ and (P_β) is a R-sublattice;*

$(D2)$ *There exists $\alpha^* \in \mathfrak{Y}$ satisfying*

(a) *For each $\beta \in \mathfrak{Y}$, either $\alpha^* \leq^* \beta$ or $\beta \leq^* \alpha^*$;*

(b) *$\alpha^* <^* \alpha$;*

(c) *There exists $\gamma \in \mathfrak{Y}$ such that $\alpha^* \prec^* \gamma$ and for each $\beta \in \mathfrak{Y}$, either $\gamma \leq^* \beta$ or $\beta \leq^* \gamma$;*

(d) *If $\beta \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\alpha^* <^* \beta <^* \alpha'$, then $|D_\beta| = 1$ and (P_β) is a sublattice.*

$(SR6)$ *If $\alpha \in \mathfrak{Y}$ such that $|D_\alpha| = 2$, then $|D_{P_\alpha^\top}| = 1$ whenever $P_\alpha^\top$ exists and $P_\alpha^\top \neq \alpha$, $|D_{P_\alpha^\perp}| = 1$ whenever $P_\alpha^\perp$ exists and $P_\alpha^\perp \neq \alpha$, $\overline{(P_\alpha)} = \{\beta \in (P_\alpha) | \alpha \leq^* \beta\}$ is a R-sublattice of $\mathfrak{Y}$ and α satisfies $(D1)$ and $(D2)$.*

$(SR7)$ *If $\alpha \in \mathfrak{Y} \setminus \{1\}$ such that $|D_\alpha| = 1$ and (P_α) is a R-sublattice of $\mathfrak{Y}$, then α satisfies $(D1)$ or $(D2)$.*

$(SR8)$ *If $\alpha \in \mathfrak{Y} \setminus \{1\}$ such that $|D_\alpha| = 1$ and (P_α) isn't a R-sublattice of $\mathfrak{Y}$, then α satisfies $(D2)$.*

Proof. Let $\mathfrak{L}$ be a conical idempotent residuated lattice. By Theorem 2.1, $\mathfrak{L} = CRLOB(\mathfrak{L}; \pi, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$. By Lemma 4.7 of [1], $\pi = \{\mathfrak{L}_\alpha \mid \alpha \in \mathfrak{Y}\} =$

$\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$. By $(CSP1)$ and $(CSP2)$ of Definition 2.1, $\mathfrak{L}$ satisfies $(SR2)$. By Lemma 2.3 and Theorem 2.1, condition $(SR3)$ holds. By $(CSP3)$ of Definition 2.1, $\mathfrak{L}$ satisfies $(SR4)$.

Now, we prove that $\mathfrak{L}$ satisfies $(SR1)$. Let $\alpha \in \mathfrak{Y}$ and suppose that (P_α) isn't a sublattice of $\mathfrak{L}$. Then, there exist $\beta, \gamma \in (P_\alpha)$ such that $\beta \vee \gamma$ doesn't exist, hence $(P_\alpha)^+ \neq \emptyset$, so by Lemma 3.2(4,5), for all $\beta', \gamma' \in (P_\alpha)^+$, $D_{\beta'} = \{a_{\beta'}\}$, $\beta' \wedge \gamma' \in (P_\alpha)^+$ and $\beta' \vee \gamma' \in (P_\alpha)^+$ whenever $\beta' \vee \gamma'$ exists. To show that $((P_\alpha)^+ \cup \{\Phi(\beta,\gamma)\}, \leq^*)$ is a lattice, we consider the following cases:

- $\Phi(\beta,\gamma) \neq 1$. By the proof of Lemma 3.2(5), $\alpha' <^* \Phi(\beta,\gamma)$, for all $\alpha' \in (P_\alpha)$. Suppose that $\beta', \gamma' \in (P_\alpha)^+$ such that $\beta' \vee \gamma'$ doesn't exist. Then, by Lemma 3.2(3), either $\Phi(\beta',\gamma') \leq^* \Phi(\beta,\gamma)$ or $\Phi(\beta,\gamma) \leq^* \Phi(\beta',\gamma')$. Assume that $\Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Then, $\Phi(\beta',\gamma') \neq 1$, so by the proof of Lemma 3.2(5), we have $\alpha' <^* \Phi(\beta',\gamma')$, for all $\alpha' \in (P_\alpha)$, hence $\beta, \gamma <^* \Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Thus, by $(B2)$ of Definition 2.3, $D_{\Phi(\beta',\gamma')} = \{b_{\Phi(\beta',\gamma')}\}$, contrary to $D_{\Phi(\beta',\gamma')}$ contains $a_{\Phi(\beta',\gamma')}$ by $(B1)$ of Definition 2.3. If $\Phi(\beta,\gamma) <^* \Phi(\beta',\gamma')$, then $\beta', \gamma' <^* \Phi(\beta,\gamma) <^* \Phi(\beta',\gamma')$, hence by $(B2)$ of Definition 2.3, $D_{\Phi(\beta,\gamma)} = \{b_{\Phi(\beta,\gamma)}\}$, contrary to $D_{\Phi(\beta,\gamma)}$ contains $a_{\Phi(\beta,\gamma)}$ by $(B1)$ of Definition 2.3, which implies $\Phi(\beta,\gamma) <^* \Phi(\beta',\gamma')$ is impossible. Thus, $\Phi(\beta,\gamma) = \Phi(\beta',\gamma')$. If there exists $\delta \in (P_\alpha)$ such that $\beta, \gamma <^* \delta$, then $\beta, \gamma <^* \delta <^* \Phi(\beta,\gamma)$, hence by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. Thus, by Lemma 3.1(4), $\delta \notin (P_\alpha)^+$, which means that $((P_\alpha)^+ \cup \{\Phi(\beta,\gamma)\}, \leq^*)$ is a lattice. This proves that $(P1)$ holds.

- $\Phi(\beta,\gamma) = 1$. Then, $\alpha' \leq^* \Phi(\beta,\gamma)$, for all $\alpha' \in (P_\alpha)$ and there exists $\delta \in \mathfrak{Y}$ such that $\beta, \gamma <^* \delta <^* \Phi(\beta,\gamma)$ by noting that $\beta \vee \gamma$ doesn't exist. By $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. It follows from the proof of Lemma 3.2(3) that $\Phi(\beta,\gamma) = \varphi(\delta)$. Suppose that $\beta', \gamma' \in (P_\alpha)^+$ and $\beta' \vee \gamma'$ doesn't exist. Then, $\Phi(\beta',\gamma') \leq^* \Phi(\beta,\gamma) = 1$. Assume $\Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Then, $\Phi(\beta',\gamma') \neq 1$, so by the proof of the prior case, we have $\alpha' <^* \Phi(\beta',\gamma')$, for all $\alpha' \in (P_\alpha)$, hence $\beta, \gamma <^* \Phi(\beta',\gamma') <^* \Phi(\beta,\gamma)$. Thus, by $(B2)$ of Definition 2.3, $D_{\Phi(\beta',\gamma')} = \{b_{\Phi(\beta',\gamma')}\}$, contrary to $D_{\Phi(\beta',\gamma')}$ contains $a_{\Phi(\beta',\gamma')}$ by $(B1)$ of Definition 2.3. We conclude that $\Phi(\beta',\gamma') = \Phi(\beta,\gamma) = 1$. Furthermore, by Lemma 3.2(4), we have $\delta \notin (P_\alpha)^+$, which means that $((P_\alpha)^+ \cup \{\Phi(\beta,\gamma)\}, \leq^*)$ is a lattice. This proves that $(P1)$ holds.

We conclude that $(P1)$ holds. By Lemma 3.2(4), (P_α) satisfies $(P2)$. We shall show that (P_α) satisfies $(P3)$. By Lemma 3.2(6), $(P_\alpha)^-$ is a sublattice of $\mathfrak{Y}$. Let $\delta, \delta' \in (P_\alpha)^-$ be such that $\delta \parallel^* \delta'$ or $\delta' <^* \delta$. If $\delta = 1$, then $\delta' <^* \delta = 1$ and for all $\zeta \in (P_\alpha)^-$ such that $\zeta \wedge 1 \leq^* 1 \wedge \delta'$, we have $\zeta \leq^* \delta'$. If $\delta \neq 1$, then by Lemma 3.2(6), $D_{\delta'} = \{b_{\delta'}\}$ and $D_\delta = \{b_\delta\}$. So, $(\delta, \delta') \in \mathrm{Dom}\Psi$, hence by $(C5)$ of Definition 2.4, $\delta \wedge \Psi(\delta,\delta') \leq^* \delta'$, which implies that $\delta \wedge \Psi(\delta,\delta') \leq^* \delta \wedge \delta'$. Since $\delta \wedge \delta' \leq^* \delta \wedge \delta'$, by $(C6)$ of Definition 2.4, $\delta' \leq^* \Psi(\delta,\delta')$, hence $\delta \wedge \delta' \leq^* \delta \wedge \Psi(\delta,\delta')$, thus $\delta \wedge \Psi(\delta,\delta') = \delta \wedge \delta'$. Let $\gamma' \in (P_\alpha)^-$ such that $\delta \wedge \gamma' \leq^* \delta \wedge \delta'$. By

Lemma 3.2(6), $D_{\gamma'} = \{b_{\gamma'}\}$, hence by $(C6)$ of Definition 2.4, $\gamma' \leq^* \Psi(\delta, \delta')$. We shall show $\Psi(\delta, \delta') \in (P_\alpha)$. For this, we consider the following cases:

- $\Psi(\delta, \delta') \parallel^* \alpha$. Then $\Psi(\delta, \delta') \in (P_\alpha)$.
- $\Psi(\delta, \delta') \leq^* \alpha$. Then $\delta' \leq^* \Psi(\delta, \delta') \leq^* \alpha$, and so $\Psi(\delta, \delta') \in (P_\alpha)$.
- $\alpha <^* \Psi(\delta, \delta')$ and $\Psi(\delta, \delta') = \delta'$. Then, $\Psi(\delta, \delta') \in (P_\alpha)$.
- $\alpha <^* \Psi(\delta, \delta')$ and $\Psi(\delta, \delta') \neq \delta'$. Then, by $(C6)$ of Definition 2.4, $\delta' <^* \Psi(\delta, \delta')$. If $\Psi(\delta, \delta') \not\parallel^* \delta$, then $\Psi(\delta, \delta') \wedge \delta \in \{\Psi(\delta, \delta'), \delta\}$, contrary to $\Psi(\delta, \delta') \wedge \delta = \delta' \wedge \delta \leq^* \delta'$, thus $\Psi(\delta, \delta') \parallel^* \delta$, which implies that $\Psi(\delta, \delta') \in P_\delta$ and $\delta \in P_\alpha$. By the similar argument as in the proof of Lemma 3.1(2), $P_\delta \subseteq P_\alpha$ and so $\Psi(\delta, \delta') \in P_\alpha \subseteq (P_\alpha)$.

We conclude that $\Psi(\delta, \delta') \in (P_\alpha)$. Since $\delta' \leq^* \Psi(\delta, \delta')$ and $\delta' \in (P_\alpha)^-$, by $(P2)$, $\Psi(\delta, \delta') \in (P_\alpha)^-$, which implies that $(P_\alpha)^-$ satisfies (RS). Thus, $(P_\alpha)^-$ is a R-sublattice of $\mathfrak{Y}$. This proves that $(P3)$ holds. Thereby, (P_α) is a pre-sublattice of $\mathfrak{Y}$. By Lemma 3.2(4,6) and $(CSP1)$ of Definition 2.1, $|D_\alpha| = 1$. Consequently, $\mathfrak{L}$ satisfies $(SR1)$.

Next, we prove that $\mathfrak{L}$ satisfies $(SR5)$. Let (P_α) be a pre-sublattice of $\mathfrak{Y}$. Then, there exist $\beta, \gamma \in (P_\alpha)$ such that $\beta \parallel^* \gamma$ and $\beta \vee \gamma$ doesn't exist, hence by $(CSP3)$ and $(CSP4)$ of Definition 2.1, $D_\beta = \{a_\beta\}, D_\gamma = \{a_\gamma\}$. If $\alpha \neq 1$, then by Lemma 3.2(4,6), $D_\alpha = \{a_\alpha\}$ or $D_\alpha = \{b_\alpha\}$. Let $\alpha^+ = \Phi(\beta, \gamma)$. If $\alpha \neq 1$, then by the proof of $|D_\alpha| = 1$ in the case that $\mathcal{L}$ satisfies $(SR1)$, $\alpha <^* \Phi(\beta, \gamma)$ and by Lemma 3.2(3), for each $\beta' \in \mathfrak{Y}$, either $\beta' \leq^* \Phi(\beta, \gamma)$ or $\Phi(\beta, \gamma) \leq^* \beta'$. This proves that $\Phi(\beta, \gamma)$ satisfies $(D1)(a)$ and (b). Since $\beta \vee \gamma$ doesn't exist, there exists $\delta \in \mathfrak{Y}$ such that $\beta, \gamma <^* \delta <^* \Phi(\beta, \gamma)$ and by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$. If $\Phi(\beta, \gamma) \neq 1$, then by the proof of Lemma 3.2(3), $\Phi(\beta, \gamma) = \varphi(\delta)$ and by Lemma 3.2(2), $\psi(\Phi(\beta, \gamma)) = \psi(\varphi(\delta)) \prec^* \varphi(\delta) = \Phi(\beta, \gamma)$ and for each $\gamma' \in \mathfrak{Y}$, either $\psi(\Phi(\beta, \gamma)) \leq^* \gamma'$ or $\gamma' \leq^* \psi(\Phi(\beta, \gamma))$. This shows that $\Phi(\beta, \gamma)$ satisfies $(D1)(c)$. If $\beta' \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\alpha' <^* \beta' <^* \Phi(\beta, \gamma)$, then $\beta' \notin (P_\alpha)$. Furthermore, we can claim that there exists $\delta' \in \mathfrak{Y}$ such that $\alpha <^* \delta' <^* \beta'$ and for each $\gamma' \in \mathfrak{Y}$, either $\delta' \leq^* \gamma'$ or $\gamma' \leq^* \delta'$. Otherwise, for any $\delta' \in \mathfrak{Y}$ such that $\alpha <^* \delta' <^* \beta'$, there exists $\gamma' \in \mathfrak{Y}$ such that $\delta' \parallel^* \gamma'$. Assume $\zeta \in \mathfrak{Y}$ such that $\alpha' \leq^* \zeta$, for all $\alpha' \in (P_\alpha)$. If $\beta' \parallel^* \zeta$, then $\beta' \in (P_\alpha)$, contrary to $\beta' \notin (P_\alpha)$. If $\zeta <^* \beta'$, then there exists $\gamma' \in \mathfrak{Y}$ such that $\zeta \parallel^* \gamma'$, so $\gamma' <^* \beta'$. Since $\alpha <^* \beta'$, $\alpha \parallel^* \gamma'$ or $\alpha <^* \gamma' <^* \beta'$, hence $\gamma' \in (P_\alpha)$, which implies that $\gamma' \leq^* \zeta$, contrary to $\zeta \parallel^* \gamma'$. Consequently, $\beta' \leq^* \zeta$, which derives that $\beta' = P_\alpha^\top \in (P_\alpha)$, contrary to $\beta' \notin (P_\alpha)$. We conclude that there exists $\delta' \in \mathfrak{Y}$ such that $\alpha <^* \delta' <^* \beta'$ and for each $\gamma' \in \mathfrak{Y}$, either $\delta' \leq^* \gamma'$ or $\gamma' \leq^* \delta'$, which implies that $\alpha' \leq^* \delta'$, for all $\alpha' \in (P_\alpha)$, and $\delta' \leq^* \xi$, for all $\xi \in (P_{\beta'})$. If $\Phi(\beta, \gamma) \neq 1$, then by the proof of $(P1)$ and Lemma 3.2, $\beta, \gamma <^* \delta' \leq^* \xi \leq^* \psi(\Phi(\beta, \gamma)) \prec^* \Phi(\beta, \gamma)$, for all $\xi \in (P_{\beta'})$, so by $(B2)$ of Definition 2.3, $D_\xi = \{b_\xi\}$, for all $\xi \in (P_{\beta'})$. If $\Phi(\beta, \gamma) = 1$, then $\beta, \gamma <^* \delta' \leq^* \xi \leq^* \Phi(\beta, \gamma)$, for all $\xi \in (P_{\beta'})$, hence by $(B2)$ of Definition 2.3 and $(CSP1)$ of Definition 2.1, $D_\xi = \{b_\xi\}$, for all $\xi \in (P_{\beta'})$. We conclude that $D_\xi = \{b_\xi\}$, for all $\xi \in (P_{\beta'}) \setminus \{1\}$. By similar arguments as in the proof of $(P3)$, $(P_{\beta'})$ is a R-sublattice of $\mathfrak{Y}$, which implies that $(D1)(d)$ holds. We have proved α satisfies $(D1)$. Let $\alpha^* = \psi(\beta)$. Then, by Lemma 3.2(1), $\psi(\beta)$ satisfies $(D2)(a)$ and $\psi(\beta) \prec^* \varphi(\psi(\beta)) \leq^* \alpha'$, for all $\alpha' \in (P_\alpha)$, which implies that $\psi(\beta)$ satisfies $(D2)$ $(b-c)$. If $\gamma' \in \mathfrak{Y}$ such that, for all $\alpha' \in (P_\alpha)$, $\psi(\beta) <^* \gamma' <^* \alpha'$, then $\gamma' \notin (P_\alpha)$ and $\psi(\beta) <^* \gamma' <^* \beta$, so by $(C3)$ of Definition 2.4, $D_{\gamma'} = \{a_{\gamma'}\}$ and $|D_{\gamma'}| = 1$ and by Lemma 3.2(1), $\psi(\beta) \prec^* \varphi(\psi(\beta)) \leq^* \xi$, for all $\xi \in (P_{\gamma'})$. By similar arguments as in the proof of $(D1)(d)$, there exists $\zeta \in \mathfrak{Y}$ with $\gamma' <^* \zeta <^* \alpha$ such that for each $\zeta' \in \mathfrak{Y}$, either $\zeta' \leq^* \zeta$ or $\zeta \leq^* \zeta'$, which implies that $\zeta' \leq^* \alpha'$, for all $\alpha' \in (P_\alpha)$, and $\xi \leq^* \zeta$, for all $\xi \in (P_{\gamma'})$. Suppose that $(P_{\gamma'})$ isn't a sublattice of $\mathfrak{Y}$. Then, there exist $\xi, \xi' \in P_{\gamma'}$ such that $\xi \parallel^* \xi'$ and $\xi \vee \xi'$ doesn't exist. By $(CSP3)$ and $(CSP4)$ of Definition 2.1, $D_\xi = \{a_\xi\}$ and $D_{\xi'} = \{a_{\xi'}\}$. Since $\beta \parallel^* \gamma$, by Lemma 3.2(3), either $\Phi(\xi, \xi') <^* \beta$ or $\beta <^* \Phi(\xi, \xi')$. If $\Phi(\xi, \xi') <^* \beta$, then there exists $\delta \in \mathfrak{Y}$ such that $\psi(\beta) <^* \xi, \xi' <^* \delta <^* \Phi(\xi, \xi') <^* \beta$, hence by $(C3)$ of Definition 2.4 $D_\delta = \{a_\delta\}$, but by $(B2)$ of Definition 2.3, $D_\delta = \{b_\delta\}$, a contradiction. If $\beta <^* \Phi(\xi, \xi')$, then $\xi, \xi' <^* \beta <^* \Phi(\xi, \xi')$, hence by $(B2)$ of Definition 2.3, $D_\beta = \{b_\beta\}$, contrary to $D_\beta = \{a_\beta\}$. Consequently, $(P_{\gamma'})$ is a sublattice of $\mathfrak{Y}$. Thus, α satisfies $(D2)$. This proves that $\mathfrak{L}$ satisfies $(SR5)$.

We shall prove that $\mathfrak{L}$ satisfies $(SR6)$. Let $\alpha \in \mathfrak{Y}$ such that D_α contains two elements. Then, by $(CSP2)$ of Definition 2.1, $D_\alpha = \{a_\alpha, b_\alpha\}$. By Lemma 3.2(1,2), for all $\gamma \in (P_\alpha) \setminus \overline{(P_\alpha)}$, $\psi(\alpha) \prec^* \varphi(\psi(\alpha)) \leq^* \gamma <^* \alpha$, hence by $(C3)$ of Definition 2.4, $D_\gamma = \{a_\gamma\}$, which implies that $|D_{P_\alpha^\perp}| = 1$ if $P_\alpha^\perp$ exists and $P_\alpha^\perp \neq \alpha$. If $\varphi(\alpha) \neq 1$, then by Lemma 3.2(1,2), for all $\beta \in \overline{(P_\alpha)} \setminus \{\alpha\}$, $\alpha <^* \beta \leq^* \psi(\varphi(\alpha)) \prec^* \varphi(\alpha)$, hence by $(C4)$ of Definition 2.4, $D_\beta = \{b_\beta\}$, which implies that $|D_{P_\alpha^\top}| = 1$ whenever $P_\alpha^\top$ exists and $P_\alpha^\top \neq \alpha$. If $\varphi(\alpha) = 1$, then by Lemma 3.2(2), for all $\beta \in \overline{(P_\alpha)} \setminus \{\alpha, 1\}$, $\alpha <^* \beta <^* \varphi(\alpha) = 1$, hence by $(C4)$ of Definition 2.4, $D_\beta = \{b_\beta\}$, which implies that $|D_{P_\alpha^\top}| = 1$ whenever $P_\alpha^\top$ exists and $P_\alpha^\top \neq \alpha$. By similar arguments as in the proof that $(P_\alpha)^-$ is a R-sublattice of $\mathfrak{Y}$, $\overline{(P_\alpha)}$ is a R-sublattice of $\mathfrak{Y}$. Let $\alpha^+ = \varphi(\alpha)$ and $\alpha^* = \psi(\alpha)$. By similar arguments as in the proof of $(SR5)$, α satisfies $(D1)$ and $(D2)$. This proves that $\mathfrak{L}$ satisfies $(SR6)$. Similarly, $\mathfrak{L}$ satisfies $(SR7)$.

Next we shall show that $\mathfrak{L}$ satisfies $(SR8)$. Let $\alpha \in \mathfrak{Y} \setminus \{1\}$ be such that $|D_\alpha| = 1$ and (P_α) isn't a R-sublattice of $\mathfrak{Y}$. If (P_α) is a pre-sublattice of $\mathfrak{Y}$, then by $(SR5)$, α satisfies $(D2)$. Let (P_α) be a sublattice of $\mathfrak{Y}$. If $D_\alpha = \{a_\alpha\}$, then we let $\alpha^* = \psi(\alpha)$, so by similar arguments to those in the case that (P_α) is a pre-sublattice of $\mathfrak{Y}$, α satisfies $(D2)$. If $D_\alpha = \{b_\alpha\}$, then we consider the following cases:

- For all $\beta \in (P_\alpha)$, $D_\beta = \{b_\beta\}$. Then, by similar arguments to the proof of $(D1)(d)$ in the case that (P_α) is a pre-sublattice of $\mathfrak{Y}$, (P_α) is a R-sublattice of $\mathfrak{Y}$, contrary to (P_α) isn't a R-sublattice of $\mathfrak{Y}$.

- There exists $\beta \in (P_\alpha)$ such that D_β contains a_β. Since $D_\alpha = \{b_\alpha\}$, by $(CSP3)$ of Definition 2.1, $\alpha \not\parallel^* \beta$. If $\alpha <^* \beta$, then by Lemma 3.2 and $\beta \in (P_\alpha)$, $\psi(\beta) \prec^* \varphi(\psi(\beta)) \leq^* \alpha <^* \beta$, thus by $(C3)$ of Definition 2.4, $D_\alpha = \{a_\alpha\}$, contrary to $D_\alpha = \{b_\alpha\}$. If $\beta <^* \alpha$, then we let $\alpha^* = \psi(\beta)$, so by similar arguments to those in the case that (P_α) is a pre-sublattice of $\mathfrak{Y}$, α satisfies $(D2)$.

We conclude that $\mathfrak{L}$ satisfies $(SR8)$.

Conversely, suppose that $\mathfrak{L} = \bigcup_{\alpha \in \mathfrak{Y}} D_\alpha$ is an idempotent semigroup with an identity e and satisfies conditions (SR1-8), where $\mathfrak{Y}$ is the structure semilattice with greatest element 1 of $\mathfrak{L}$. By a routine calculation, $D_1 = \{e\}$. We let $e = a_1 = b_1$. Put $\mathscr{D} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$, $Y_1 = \{\alpha \in \mathfrak{Y} \mid |D_\alpha| = 2\}$ and $P = \{\alpha \in \mathfrak{Y} \mid |D_\alpha| = 1, (P_\alpha)$ is pre-sublattice of $\mathfrak{Y}$ and there exists $\alpha' \in \mathfrak{Y}$ such that $\alpha \parallel \alpha'\}$. We define a binary relation $\sim$ on P as follows: for $\alpha, \beta \in P$, $\alpha \sim \beta$ if and only if $P_\alpha = P_\beta$. By lemma 3.1(2,3), $\sim$ is an equivalence relation on P. It follows that $P_\alpha = \{\beta \in P \mid \beta \sim \alpha\}$ for every $\alpha \in P$. Let $Y_2 \subseteq P$ be such that for every $\alpha \in P$, $|Y_2 \cap P_\alpha| = 1$. For any $\alpha \in Y_1$, we denote one element of D_α by a_α and the other element of D_α by b_α. If $\beta \in \overline{(P_\alpha)} \setminus \{\alpha, 1\}$, then we let $D_\beta = \{b_\beta\}$. If $\beta' \in (P_\alpha) \setminus \overline{(P_\alpha)}$, then we let $D_{\beta'} = \{a_{\beta'}\}$. For any $\alpha \in Y_2$, let $D_\gamma = \{a_\gamma\}$, for all $\gamma \in (P_\alpha)^+$, and let $D_{\gamma'} = \{b_{\gamma'}\}$, for all $\gamma' \in (P_\alpha)^- \setminus \{1\}$. Let $\alpha \in Y_1 \cup Y_2$. We distinguish two cases.

- For each $\beta \in \mathfrak{Y}$ such that $\alpha' <^* \beta$ for all $\alpha' \in (P_\alpha)$, $|D_\beta| = 1$ and (P_β) is a R-sublattice of $\mathfrak{Y}$. By conditions $(SR5-6)$, we can choose $\alpha^* \in \mathfrak{Y}$ and 1 is interpreted as α^+. Hence, we obtain the closed interval $H_\alpha = [\alpha^*, 1]$ and by $(SR5-6)$, there exists element γ in $\mathfrak{Y}$ such that $\alpha^* \prec^* \gamma$. For any $\beta \in \mathfrak{Y}$ such that $\alpha' <^* \beta <^* 1$, for all $\alpha' \in (P_\alpha)$, let $D_\beta = \{b_\beta\}$. For any $\gamma' \in \mathfrak{Y}$ such that $\alpha^* <^* \gamma' <^* \alpha'$, for all $\alpha' \in (P_\alpha)$, let $D_{\gamma'} = \{a_{\gamma'}\}$. If $|D_{\alpha^*}| = 1$, then we define $D_{\alpha^*} = \{b_{\alpha^*}\}$.
- There exists $\beta \in \mathfrak{Y}$ such that $\alpha' <^* \beta$, for all $\alpha' \in (P_\alpha)$ and $|D_\beta| = 2$ or $\alpha' <^* \beta$, for all $\alpha' \in (P_\alpha)$ and (P_β) isn't a R-sublattice of $\mathfrak{Y}$. By conditions $(SR5-6)$, we can choose α^* and α^+ in $\mathfrak{Y} \setminus \{1\}$. Hence, we obtain the closed interval $H_\alpha = [\alpha^*, \alpha^+]$ and by $(SR5-6)$, there exist elements β, γ in $\mathfrak{Y}$ such that $\beta \prec^* \alpha^+$ and $\alpha^* \prec^* \gamma$. For any $\beta' \in \mathfrak{Y}$ such that $\alpha' <^* \beta' <^* \alpha^+$, for all $\alpha' \in (P_\alpha)$, let $D_{\beta'} = \{b_{\beta'}\}$. If $|D_{\alpha^+}| = 1$, then we define $D_{\alpha^+} = \{a_{\alpha^+}\}$. For any $\gamma' \in \mathfrak{Y}$ such that $\alpha^* <^* \gamma' <^* \alpha'$, for all $\alpha' \in (P_\alpha)$, let $D_{\gamma'} = \{a_{\gamma'}\}$. If $|D_{\alpha^*}| = 1$, then we define $D_{\alpha^*} = \{b_{\alpha^*}\}$.

Let $\alpha, \beta \in Y_1 \cup Y_2$ be such that $\alpha <^* \beta$. If $|H_\alpha \cap H_\beta| > 2$, we choose α^+ to be γ which covers β^*. Then, we obtain a family of closed intervals $\{H_\alpha \mid \alpha \in Y_1 \cup Y_2\}$ such that for $\alpha, \beta \in Y_1 \cup Y_2$ with $\alpha \neq \beta$, $|H_\alpha \cap H_\beta| \leq 2$.

We arbitrarily choose $\alpha \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ such that $\alpha \neq 1$. We distinguish two cases.

Case A. (P_α) is a R-sublattice. By $(SR7)$, α satisfies $(D1)$ or $(D2)$. We need to consider two subcases:

(1) α satisfies $(D2)$.

- If there exists $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then there exists $\zeta \in H_\gamma$ such that $\zeta \prec^* \gamma^+$. Hence, when ζ is interpreted as α^*, we have the set $H_\alpha = \{\beta \in \mathfrak{Y} \mid [\alpha^*, \beta] \subseteq (\mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \cup \{\alpha^*, \gamma^+\}\}$. For each $\beta \in H_\alpha$ such that $\gamma^+ <^* \beta$, let $D_\beta = \{a_\beta\}$.

- If there doesn't exist $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then we can claim that $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. Otherwise, if $\alpha^* \in \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, then there exists $\gamma' \in Y_1 \cup Y_2$ such that $\alpha^* \in H_{\gamma'}$ and $\gamma' <^* \alpha$, hence there exist $\delta, \delta', \delta'' \in Y_1 \cup Y_2$ such that $\gamma' <^* \delta' <^* \delta <^* \delta'' <^* \alpha$, so $\alpha^* <^* \delta$ and $\delta \notin (P_\alpha)$, which imply that $\delta <^* \alpha'$, for all $\alpha' \in (P_\alpha)$. By $(D2)(d)$, $|D_\delta| = 1$ and (P_δ) is a sublattice of $\mathfrak{Y}$, contrary to $|D_\delta| = 2$ or $|D_\delta| = 1$ and (P_δ) is pre-sublattice of $\mathfrak{Y}$. Consequently, $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ and so $[\alpha^*, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ has the least element β, then by $(D2)(c)$, there exists $\xi \in \mathfrak{Y}$ such that $\beta \prec^* \xi$ and for each $\xi' \in \mathfrak{Y}$, either $\xi \leq^* \xi'$ or $\xi' \leq^* \xi$. Furthermore, we have for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice; whereas, if (P_ζ) isn't a R-sublattice, then by $(SR8)$, there exists $\zeta^* \in \mathfrak{Y}$ such that $(D2)(a-d)$, by similar arguments as in case: $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, $\zeta^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, which, together with $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, implies that $\zeta^* \in \{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ and $\zeta^* <^* \xi$, contrary to ξ is the least element of the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. We conclude that for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice. Thereby, when ξ is interpreted as β^+, we have the set $H_\beta = \{\beta' \in \mathfrak{Y} \mid [\beta', \beta^+] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$; when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\beta' \in H_\beta$, let
$$D_{\beta'} = \begin{cases} \{b_{\beta'}\} & \text{if } \beta' \neq \beta^+, \\ \{a_{\beta'}\} & \text{if } \beta' = \beta^+. \end{cases}$$
For each $\alpha' \in H_\alpha$, let
$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\} & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\} & \text{if } \alpha' = \alpha^*. \end{cases}$$
If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ hasn't the least element, then we choose β in the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. Hence, when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$, let
$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\} & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\} & \text{if } \alpha' = \alpha^*. \end{cases}$$

(2) α doesn't satisfy $(D1)$. Then, α satisfies $(D2)$. We can claim that for each $\beta \in \mathfrak{Y}$ such that $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_β) is a R-sublattice. Otherwise, if for some $\beta \in \mathfrak{Y}$ such that $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_β) isn't a R-sublattice, then by condition $(SR8)$, there exists $\beta^* \in \mathfrak{Y}$ satisfying $(D2)$, which, together with $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, implies that we can choose α^* in $\mathfrak{Y}$, a contradiction. Consequently, for each $\beta \in \mathfrak{Y}$ such that $[\beta, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_β) is a R-sublattice. We consider the following cases:

- If there exists $\beta \in Y_1 \cup Y_2$ such that $\alpha <^* \beta$ and $[\alpha, \beta] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\beta$. Hence, there exists $\eta \in H_\beta$ such that $\beta^* \prec^* \eta$. Thus, when η is interpreted as α^+, we have the set $H_\alpha = \{\gamma \in \mathfrak{Y} \mid [\gamma, \alpha^+] \subseteq (\mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \cup \{\alpha^+, \beta^*\}\}$. For each $\gamma \in H_\alpha$ such that $\gamma <^* \beta^*$, let $D_\gamma = \{b_\gamma\}$.

- If there doesn't exist $\beta \in Y_1 \cup Y_2$ such that $\alpha <^* \beta$ and $[\alpha, \beta] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\beta$, then by similar arguments as in (1), $\alpha^+ \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. This means that there exists $\gamma \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ satisfying $(D1)(a-d)$. If $\gamma = 1$, then when 1 is interpreted as α^+, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha', 1] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$ such that $\alpha' <^* 1$, let $D_{\alpha'} = \{b_{\alpha'}\}$. If $\gamma \neq 1$, then by $(D1)(c)$, there exists $\eta \in \mathfrak{Y}$, such that $\eta \prec^* \gamma$ and for each $\beta' \in \mathfrak{Y}$, either $\eta \leq^* \beta'$ or $\beta' \leq^* \eta$. Hence, when γ is interpreted as α^+, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha', \alpha^+] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$; when η is interpreted as β^*, we have the set $H_\beta = \{\beta' \in \mathfrak{Y} \mid [\beta^*, \beta'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$, let
$$D_{\alpha'} = \begin{cases} \{b_{\alpha'}\} & \text{if } \alpha' \neq \alpha^+, \\ \{a_{\alpha'}\} & \text{if } \alpha' = \alpha^+. \end{cases}$$
For each $\beta' \in H_\beta$, let
$$D_{\beta'} = \begin{cases} \{a_{\beta'}\} & \text{if } \beta' \neq \beta^*, \\ \{b_{\beta'}\} & \text{if } \beta' = \beta^*. \end{cases}$$

Case B. (P_α) isn't a R-sublattice. Then, by $(SR8)$, α satisfies $(D2)$. We need to consider two subcases:

- If there exists $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then there exists $\zeta \in H_\gamma$ such that $\zeta \prec^* \gamma^+$. Hence, when ζ is interpreted as α^*, we have the set $H_\alpha = \{\beta \in \mathfrak{Y} \mid [\alpha^*, \beta] \subseteq (\mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \cup \{\alpha^*, \gamma^+\}\}$. For each $\beta \in H_\alpha$ such that $\gamma^+ <^* \beta$, let $D_\beta = \{a_\beta\}$.

- If there doesn't exist $\gamma \in Y_1 \cup Y_2$ such that $\gamma <^* \alpha$ and $[\gamma, \alpha] \cap (\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta) \subseteq H_\gamma$, then we can claim that $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. Otherwise, if $\alpha^* \in \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, then there exists $\gamma' \in Y_1 \cup Y_2$ such that $\alpha^* \in H_{\gamma'}$ and $\gamma' <^* \alpha$, hence there exist $\delta, \delta', \delta'' \in Y_1 \cup Y_2$ such that $\gamma' <^* \delta' <^* \delta <^* \delta'' <^*$

α, so $\alpha^* <^* \delta$ and $\delta \notin (P_\alpha)$, which imply that $\delta <^* \alpha'$, for all $\alpha' \in (P_\alpha)$. By $(D2)(d)$, $|D_\delta| = 1$ and (P_δ) is a sublattice of $\mathfrak{Y}$, contrary to $|D_\delta| = 2$ or $|D_\delta| = 1$ and (P_δ) is a pre-sublattice of $\mathfrak{Y}$. Consequently, $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ and so $[\alpha^*, \alpha] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$. If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ has the least element β, then by $(D2)(c)$, there exists $\xi \in \mathfrak{Y}$ such that $\beta \prec^* \xi$ and for each $\xi' \in \mathfrak{Y}$, either $\xi \leq^* \xi'$ or $\xi' \leq^* \xi$. Furthermore, we have for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice; whereas, if (P_ζ) isn't a R-sublattice, then by $(SR8)$, there exists $\zeta^* \in \mathfrak{Y}$ such that $(D2)(a-d)$, by similar arguments as in case: $\alpha^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, $\zeta^* \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, which, together with $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, implies that $\zeta^* \in \{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ and $\zeta^* <^* \xi$, contrary to ξ is the least element of the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. We conclude that for any $\zeta \in \mathfrak{Y}$ such that $\zeta <^* \xi$ and $[\zeta, \xi] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$, (P_ζ) is a R-sublattice. Thereby, when ξ is interpreted as β^+, we have the set $H_\beta = \{\beta' \in \mathfrak{Y} \mid [\beta', \beta^+] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$; when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\beta' \in H_\beta$, let

$$D_{\beta'} = \begin{cases} \{b_{\beta'}\}, & \text{if } \beta' \neq \beta^+, \\ \{a_{\beta'}\}, & \text{if } \beta' = \beta^+. \end{cases}$$

For each $\alpha' \in H_\alpha$, let

$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\}, & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\}, & \text{if } \alpha' = \alpha^*. \end{cases}$$

If the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$ hasn't the least element, then we choose β in the set $\{\beta' \in \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta | \beta'$ satisfies $(D2)(a-d)$ for $\alpha\}$. Hence, when β is interpreted as α^*, we have the set $H_\alpha = \{\alpha' \in \mathfrak{Y} \mid [\alpha^*, \alpha'] \subseteq \mathfrak{Y} \setminus \bigcup_{\delta \in Y_1 \cup Y_2} H_\delta\}$. For each $\alpha' \in H_\alpha$, let

$$D_{\alpha'} = \begin{cases} \{a_{\alpha'}\}, & \text{if } \alpha' \neq \alpha^*, \\ \{b_{\alpha'}\}, & \text{if } \alpha' = \alpha^*. \end{cases}$$

We repeat the proceeding above, replacing $\bigcup_{\delta \in Y_1 \cup Y_2} H_\delta$ by corresponding subsets of $\mathfrak{Y}$. Then, we obtain a family sets $\{H_\alpha \mid \alpha \in Y_3\}$ such that, for all $\beta, \gamma \in Y_1 \cup Y_2 \cup Y_3$ with $\beta \neq \gamma$, $|H_\beta \cap H_\gamma| \leq 2$. We can claim that $\mathfrak{Y} \setminus \{1\} \subseteq \bigcup_{\alpha \in Y_1 \cup Y_2 \cup Y_3} H_\alpha$. Let $\mathscr{H}$ be a subset of power set $\mathscr{P}(\mathfrak{Y} \setminus \{1\})$ such that for $A \in \mathscr{H}$ and $\beta \in A$, there exists $\alpha \in Y_1 \cup Y_2 \cup Y_3$ such that $\beta \in H_\alpha$. Then $(\mathscr{H}, \subseteq)$ is a partial ordered set. Suppose that $\{A_i \mid i \in I\}$ is a totally ordered subset of $\mathscr{H}$. It's easy to see $\bigcup_{i \in I} A_i$ is a upper bound of $\{A_i \mid i \in I\}$ and $\bigcup_{i \in I} A_i \in \mathscr{H}$. By Zorn's Lemma, $\mathscr{H}$ has a maximal element B. Furthermore, we have $B = \mathfrak{Y} \setminus \{1\}$. Otherwise, if $B \neq \mathfrak{Y} \setminus \{1\}$, then there exists $\beta \in \mathfrak{Y} \setminus \{1\}$

such that $\beta \notin B$, so by the procedure above, there exists $\alpha \in Y_1 \cup Y_2 \cup Y_3$ such that $\beta \in H_\alpha$, thus $B \cup \{\beta\} \in \mathscr{H}$ contrary to B is a maximal element of $\mathscr{H}$. Consequently, $\mathfrak{Y} \setminus \{1\} \subseteq \bigcup_{\alpha \in Y_1 \cup Y_2 \cup Y_3} H_\alpha$. Let $\mathfrak{L}^+ = \{a_\alpha \mid \alpha \in \mathfrak{Y} \setminus \{1\}\}$ and $\mathfrak{L}^- = \{b_\beta \mid \beta \in \mathfrak{Y} \setminus \{1\}\}$. Since $\mathcal{D}$ is a congruence on $\mathfrak{L}$, $\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$ is a partition on $\mathfrak{L}$. By $(SR4)$ and the proceeding above, $\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$ is a conical semilattice partition on $\mathfrak{L}$. Let $\mathfrak{X} = \{\alpha \in \mathfrak{Y} \mid |D_\alpha| = 2$ and $(D_\alpha, \cdot)$ is a left zero semigroup$\}$ be a subset of $\mathfrak{Y}$. Then, $\mathfrak{X}$ is a band subset of $\{D_\alpha \mid \alpha \in \mathfrak{Y}\}$. Let $\mathfrak{C} = \{(\alpha, \beta) \in \mathfrak{Y} \times \mathfrak{Y} \mid \alpha \parallel^* \beta, a_\alpha \in D_\alpha\}$. We define a mapping Φ from $\mathfrak{C}$ to $\mathfrak{Y}$ by

$$\Phi(\alpha, \beta) = \begin{cases} \alpha \vee \beta & \text{if } \alpha \vee \beta \text{ exists,} \\ \gamma^+ \ (\alpha \in H_\gamma) & \text{otherwise.} \end{cases}$$

By the definition of Φ and Definition 2.3, $\pi_{\mathcal{D}} = \{D_\alpha \mid \alpha \in \mathfrak{Y}\}$ is a $CLOB-$system.

Now we define two mappings ψ and φ. Put Dom$\psi = \{\alpha \in \mathfrak{Y} \setminus \{1\} \mid a_\alpha \in D_\alpha\}$ and Dom$\varphi = \{\alpha \in \mathfrak{Y} \setminus \{1\} \mid b_\alpha \in D_\alpha\}$. For any $\alpha \in$ Domψ, there exists $\beta \in Y_1 \cup Y_2 \cup Y_3$ such that $\alpha \in H_\beta$ and $\alpha \neq \beta^*$. Let

$$\psi(\alpha) = \begin{cases} \beta^* & \text{if } \alpha \neq \beta^+, \\ \gamma \ (\gamma \prec^* \beta^+) & \text{if } \alpha = \beta^+. \end{cases}$$

For any $\alpha \in$ Domφ, there exists $\beta \in Y_1 \cup Y_2 \cup Y_3$ such that $\alpha \in H_\beta$ and $\alpha \neq \beta^+$. Let

$$\varphi(\alpha) = \begin{cases} \beta^+ & \text{if } \alpha \neq \beta^*, \\ \gamma \ (\beta^* \prec^* \gamma) & \text{if } \alpha = \beta^*. \end{cases}$$

It's easy to see that ψ and φ are well defined. Put $\mathfrak{R} = \{(\alpha, \beta) \in \mathfrak{Y} \times \mathfrak{Y} \mid \alpha \parallel^* \beta, b_\alpha \in D_\alpha$ or $\beta <^* \alpha, \alpha \in P_\beta, b_\beta \in D_\beta\}$. Now we define a mapping Ψ from $\mathfrak{R}$ to $\mathfrak{Y}$. If $(\alpha, \beta) \in \mathfrak{R}$, then there exists $\gamma \in Y_1 \cup Y_2 \cup Y_3$ such that $\alpha, \beta \in H_\gamma$. We consider the following cases:

- If $\gamma \in Y_1$, then by the definition of $\mathfrak{R}$ and H_γ, $\alpha, \beta \in \{\xi \in H_\gamma | \gamma \leq^* \xi\}$. If $\alpha, \beta \in \overline{(P_\gamma)}$,then since by the definition of $\overline{(P_\gamma)}$, $\overline{(P_\gamma)}$ is a R-sublattice, there exists a unique element δ in $\overline{(P_\gamma)}$ such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in \overline{(P_\gamma)}$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$. If $\alpha, \beta \in \{\xi \in H_\gamma | \gamma \leq^* \xi\} \setminus \overline{(P_\gamma)}$, then by the definition of H_γ, (P_β) is a R-sublattice, hence there exists a unique element δ in (P_β) such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$.

- If $\gamma \in Y_2$, then by the definition of $\mathfrak{R}$ and H_γ, $\alpha, \beta \in (H_\gamma)^- = \{\alpha' \in H_\gamma | (\exists \beta' \in (P_\gamma)^-) \beta' \leq^* \alpha'\}$. If $\alpha, \beta \in (P_\gamma)^-$, then since by the definition of $(P_\gamma)^-$, $(P_\gamma)^-$ is a R-sublattice, there exists a unique element δ in $(P_\gamma)^-$ such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)^-$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$,

$\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$. If $\alpha, \beta \in (H_\gamma)^- \setminus (P_\gamma)^-$, then (P_β) is a R-sublattice by the definition of H_γ. Hence, there exists a unique element δ in (P_β) such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$.

- If $\gamma \in Y_3$, then by the definition of $\mathfrak{R}$ and H_γ, (P_β) is a R-sublattice. Hence. there exists a unique element δ in (P_β) such that $\alpha \wedge \delta = \alpha \wedge \beta$ and for all $\delta' \in (P_\beta)$ such that $\delta' \wedge \alpha \leq \alpha \wedge \beta$, $\delta' \leq^* \delta$. We define $\Psi(\alpha, \beta) = \delta$.

By Definition 2.4 and a routine computation, we can prove that $(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$ is a $CRLOB$-system. By Theorem 2.1, $\mathfrak{L} = CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$ is a conical idempotent residuated lattice. Finally, we shall prove that $(\mathfrak{L}, \cdot)$ is the semigroup reduct of $CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$. Let $c \in D_\alpha, d \in D_\beta$. We consider the following cases:

- If $\alpha <^* \beta$ and $c = a_\alpha, d = a_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ a_\beta = a_{\alpha\wedge\beta} = a_\alpha = c$ and $d \circ c = a_\beta \circ a_\alpha = a_{\alpha\wedge\beta} = a_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha <^* \beta$ and $c = b_\alpha, d = b_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = b_\alpha \circ b_\beta = b_{\alpha\wedge\beta} = b_\alpha = c$ and $d \circ c = b_\beta \circ b_\alpha = b_{\alpha\wedge\beta} = b_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha <^* \beta$ and $c = a_\alpha, d = b_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ b_\beta = a_\alpha = c$ and $d \circ c = b_\beta \circ a_\alpha = a_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha <^* \beta$ and $c = b_\alpha, d = a_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = b_\alpha \circ a_\beta = b_\alpha = c$ and $d \circ c = a_\beta \circ b_\alpha = b_\alpha = c$. By $(SR3)$, $c \cdot d = d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha \parallel^* \beta$ and $c = a_\alpha, d = a_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ a_\beta = a_{\alpha\wedge\beta}$ and $d \circ c = a_\beta \circ a_\alpha = a_{\alpha\wedge\beta}$. Since $c = a_\alpha, d = a_\beta$, $c \cdot d, d \cdot c \in D_{\alpha\wedge\beta}$. Since by $(SR4)$, $|D_{\alpha\wedge\beta}| = 1$, $c \cdot d = a_{\alpha\wedge\beta}$ and $d \cdot c = a_{\alpha\wedge\beta}$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha \parallel^* \beta$ and $c = b_\alpha, d = b_\beta$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = b_\alpha \circ b_\beta = b_{\alpha\wedge\beta}$ and $d \circ c = b_\beta \circ b_\alpha = b_{\alpha\wedge\beta}$. Since $c = b_\alpha, d = b_\beta$, $c \cdot d, d \cdot c \in D_{\alpha\wedge\beta}$. Since by $(SR4)$, $|D_{\alpha\wedge\beta}| = 1$, $c \cdot d = b_{\alpha\wedge\beta}$ and $d \cdot c = b_{\alpha\wedge\beta}$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha = \beta, \alpha \in \mathfrak{X}$ and $c = a_\alpha, d = b_\alpha$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ b_\alpha = a_\alpha = c$ and $d \circ c = b_\alpha \circ a_\alpha = b_\alpha = d$. Since $c, d \in D_\alpha$, by the definition of $\mathfrak{X}$, $c \cdot d = c$ and $d \cdot c = d$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

- If $\alpha = \beta, \alpha \notin \mathfrak{X}$ and $c = a_\alpha, d = b_\alpha$, then by the definition of $\circ$ in Lemma 2.3, $c \circ d = a_\alpha \circ b_\alpha = b_\alpha = d$ and $d \circ c = b_\alpha \circ a_\alpha = a_\alpha = c$. Since $c, d \in D_\alpha$, by the definition of $\mathfrak{X}$, $c \cdot d = d$ and $d \cdot c = c$. Thus, $c \cdot d = c \circ d$ and $d \cdot c = d \circ c$.

We conclude that $(\mathfrak{L}, \cdot)$ is the semigroup reduct of $CRLOB(\mathfrak{L}; \pi_{\mathcal{D}}, \mathfrak{Y}; \mathfrak{X}, \Phi; \psi, \varphi, \Psi)$. □

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.1. *A semilattice $\mathfrak{Y}$ with greatest element* 1 *is the semigroup reduct of some conical commutative idempotent residuated lattice if and only if $\mathfrak{Y}$ satisfies conditions* $(SR1)$, $(SR5)$ *and* $(SR7-8)$.

4. Conclusions

In this paper, we have studied conical idempotent residuated lattices. Using square point sets and the structure theorem of conical idempotent residuated lattices, we have obtained necessary and sufficient conditions for an idempotent semigroup with an identity to be the semigroup reduct of some conical idempotent residuated lattice, which generalize [2, Theorem 5.2]. Note that the work presented here heavily relies on the fact that idempotent residuated lattices are conical. In the future, the structure and decomposition of non-conical idempotent residuated lattices may be investigated.

Acknowledgment

The author would like to thank the referee heartily for his/her careful reading and valuable suggestions which lead to a substantial improvement of this paper. This work is supported by grants of the NSF of China # 11171294 , China # 11571158, Fujian Province # 2024J01801; is supported by Fujian Key Laboratory of Granular Computing and Applications (Minnan Normal University).

References

[1] W. Chen, X. Zhao, X. Guo, *Conical residuated lattice-ordered idempotent monoids*, Semigroup Forum, 79 (2009), 244-278.

[2] W. Chen, X. Zhao, *The structure of idempotent residuated chains*, Czech. Math. J., 59 (2009), 453-479.

[3] W. Chen, Y. Chen, *Variety generated by conical residuated lattice-ordered idempotent monoids*, Semigroup Forum, 98 (2019), 431-455.

[4] W. Chen, *On semiconic idempotent commutative residuated lattices*, Algebra Univers. 81, 36 (2020). https://doi.org/10.1007/s00012-020-00666-6

[5] W. Fussner, N. Galatos, *Semiconic idempotent logic I: Structure and local deduction theorems*, Ann. Pure. Appl. Logic, 175 (2024), 103443. https://doi.org/10.1016/j.apal.2024.103443

[6] W. Fussner, N. Galatos, *Semiconic idempotent logic II: Beth definability and deductive interpolation.* https://doi.org/10.48550/arXiv.2208.09724.2023.

[7] J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, New series, Vol.12, Oxford Univ. Press, New York, 1995.

[8] A. Hsieh, J. G. Raftery, *Semiconic idempotent residuated structures*, Algebra Univers., 61 (2009), 413-430.

[9] N. Galatos, J. G. Raftery, *Idempotent residuated structures, some category equivalences and their applications*, Trans. Amer. Math. Soc., 367 (2015), 3189-3223.

[10] N. Galtos, P. Jipsen, T. Kowalski, H. Ono, *Residuated lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics, Elsevier, Amsterdam, 2007.

[11] N. Galatos, S. Ugolini, *Gluing residuated lattices*, Order, 40 (2023), 623-664. https://doi.org/10.1007/s11083-023-09626-w

[12] J. Gil-Férez, P. Jipsen, G. Metcalfe, *Structure theorems for idempotent residuated lattices*, Algebra Univers., 81 (2020), 433-449.

[13] S. Jenei, *Group representation for even and odd involutive commutative residuated chains*, Stud. Logica, 110 (2022), 881-922. https://doi.org/10.1007/s11225-021-09981-y.

[14] P. Jipsen, F. Montagna, *The Blok-Ferreirim theorem for normal GBL-algebras and its application*, Algebra Univers., 60 (2009), 381-404.

[15] P. Jipsen, O. Tuyt, D. Valota, *The structure of finite commutative idempotent involutive residuated lattices*, Algebra Univers., 82, 57 (2021). https://doi.org/10.1007/s00012-021-00751-4

[16] J. S. Olson, J. G. Raftery, *Positive Sugihara monoids*, Algebra Univers., 57 (2007), 75-99.

[17] J. S. Olson, *The subvariety lattice for representable idempotent commutative residuated lattices*, Algebra Univers., 67 (2012), 43-58.

[18] J. G. Raftery, *Representable idempotent commutative residuated lattices*, Trans. Ameri. Math. Soc., 359 (2007), 4405-4427.

[19] D. Stnovský, *Commutative idempotent residuated lattices*, Czech. Math. J., 57 (2007), 191-200.

Accepted: March 2, 2025