Exchange pre-Hilbert algebras and their connections with other algebras of logic

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Abstract. In the paper, as a generalization of well-known Hilbert algebras, exchange pre-Hilbert algebras are introduced. Their properties and characterizations are investigated. Some important results and examples are given. Moreover, connections between exchange pre-Hilbert algebras, generalized exchange algebras and BE algebras are presented. Finally, implicative and positive implicative algebras are considered. It is shown that implicative (resp. positive implicative) exchange pre-Hilbert algebras are equivalent to implicative BE algebras with (*) (resp. generalized Hilbert algebras).

Keywords: Hilbert algebra, exchange pre-Hilbert algebra, BCK, BE algebra, implicativity, positive implicativity.

MSC 2020: 03G25, 06A06, 06F35.

1. Introduction

L. Henkin [5] introduced the notion of "implicative model", as a model of positive implicative propositional calculus. In 1960, A. Monteiro [11] has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models. In 1966, K. Iséki [7] introduced the notion of a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [10]. In [9], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim introduced BE algebras. A. Rezaei et al. [12] investigated connections between Hilbert algebras and BE algebras. In 2008, A. Walendziak [13] defined commutative BE algebras and proved that they are BCK algebras. Later on, in 2010, D. Buşneag and S. Rudeanu [3] introduced the notion of a pre-BCK algebra. A BCK algebra is just a pre-BCK algebra satisfying also the antisymmetry property. In 2016, A. Iorgulescu [6] introduced new generalizations of BCK and Hilbert algebras (RML, aBE, pi-BE, pimpl-RML algebras and many others). Recently, as a generalization of Hilbert algebras, R. Bandaru et al. [1] introduced GE algebras (generalized exchange algebras) and A. Walendziak [16] introduced pre-Hilbert algebras (the definition of a pre-Hilbert algebra is inspired by Henkin's Positive Implicative Logic [5]). All of the algebras mentioned above are contained in the class of RML algebras (an RML algebra is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying the identities: $x \to x = 1 = x \to 1$ and $1 \to x = x$).

In the paper, we introduce and study exchange pre-Hilbert algebras. We give their characterizations and examples. We investigate connections between GE algebras, BE algebras and exchange pre-Hilbert algebras. Moreover, these algebras with the antisymmetry property are considered. Finally, we define and characterize implicative exchange pre-Hilbert algebras. We also define positive implicative exchange pre-Hilbert algebras and prove that they are equivalent to generalized Hilbert algebras.

The motivation of this study consists of algebraic and logical arguments. Namely, exchange pre-Hilbert algebras belong to a wide class of algebras of logic. Furthermore, the results of the paper may have applications for future studies of the relationships between some generalizations of Hilbert algebras. An additional motivation is the fact that the present paper is a continuation of previous papers: [15] on GE algebras and [16] on pre-Hilbert algebras.

2. Preliminaries. GE algebras and pre-Hilbert algebras

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ on A by: for all $x, y \in A$, $x \leq y \Longleftrightarrow x \to y = 1$.

We consider the following list of properties ([6]) that can be satisfied by $\mathcal{A}$:

(An) (Antisymmetry) $(x \leq y$ and $y \leq x) \Longrightarrow x = y$,

(B) $y \to z \leq (x \to y) \to (x \to z)$,

(C) $x \to (y \to z) \leq y \to (x \to z)$,

(D) $y \leq (y \to x) \to x$,

(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$,

(GE) (Generalized exchange) $x \to (y \to z) = x \to (y \to (x \to z))$,

(K) $x \leq y \to x$,

(L) (Last element) $x \leq 1$,

(M) $1 \to x = x$,

(Re) (Reflexivity) $x \leq x$,

(Tr) (Transitivity) $(x \leq y$ and $y \leq z) \Longrightarrow x \leq z$,

(*) $y \leq z \Longrightarrow x \to y \leq x \to z$,

(**) $y \leq z \Longrightarrow z \to x \leq y \to x$,

(pi) $x \to (x \to y) = x \to y$,

(p-1) $x \to (y \to z) \le (x \to y) \to (x \to z)$,

(p-2) $(x \to y) \to (x \to z) \le x \to (y \to z)$,

(pimpl) (Positive implicativity) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Remark 2.1. The properties in the list are the most important properties satisfied by a Hilbert algebra (the properties (An) – (**) are satisfied by a BCK algebra).

From Proposition 2.1 and Theorem 2.7 of [6] we have

Lemma 2.1. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Then, the following hold:*

(i) *(M) + (B) imply (Re), (*), (**);*

(ii) *(M) + (*) imply (Tr);*

(iii) *(M) + (L) + (**) imply (K);*

(iv) *(Re) + (L) + (Ex) imply (D), (K);*

(v) *(An) + (C) imply (Ex).*

Following Iorgulescu [6], we say that $(A, \to, 1)$ is an *RML algebra* if it verifies the axioms (Re), (M), (L). We recall now the following definitions.

Definitions 2.1 ([6])**.** *Let $\mathcal{A} = (A, \to, 1)$ be an RML algebra. The algebra $\mathcal{A}$ is said to be:*

1. *a BE algebra if it verifies (Ex);*
2. *an aRML algebra if it verifies (An);*
3. *an aBE algebra if it verifies (Ex) and (An), that is, it is a BE algebra with (An);*
4. *a pre-BCK algebra if it verifies (B) and (Ex), that is, it is a BE algebra with (B);*
5. *a BCK algebra if it is a pre-BCK algebra verifying (An).*

Denote by **RML**, **BE**, **aRML**, **aBE**, **preBCK** and **BCK** the classes of RML, BE, aRML, aBE, pre-BCK and BCK algebras, respectively.

By definition, **BE** = **RML** + (Ex), **aRML** = **RML** + (An), **aBE** = **BE** + (An), **preBCK** = **BE** + (B), **BCK** = **preBCK** + (An).

It is known that $\le$ is an order relation in BCK algebras. By definition, in RML and BE algebras, $\le$ is a reflexive relation; in aRML and aBE algebras, $\le$ is reflexive and antisymmetric. Since (M) + (B) imply (Tr), see Lemma 2.1 (i) and (ii), in pre-BCK algebras, $\leq$ is reflexive and transitive (i.e., it is a pre-order relation).

Lemma 2.2. *Let* $(A, \rightarrow, 1)$ *be an algebra of type* $(2,0)$*. Then, the following hold:*

(i) *(Re) + (pi) imply (L);*

(ii) *(Ex) + (pi) imply (GE);*

(iii) *(Re) + (GE) + (L) imply (K);*

(iv) *(M) + (GE) imply (pi);*

(v) *(Re) + (pimpl) imply (B), (L);*

(vi) *(p-1) + (p-2) +(An) imply (pimpl).*

Proof. (i) – (iii) follow from Propositions 2.7 and 3.1 (ii) of [15].
(iv) by the proof of Proposition 2.4 of [15].
(v) follows from Propositions 6.4 and 6.9 of [6].
(vi) is trivial. □

Proposition 2.1. *Let* $\mathcal{A} = (A, \rightarrow, 1)$ *be an algebra of type* $(2,0)$*. We have*

(i) *(Re) + (M) + (GE) + (An) imply (Ex);*

(ii) *(M) + (L) + (p-1) imply (*) and (Tr).*

Proof. (i) Let $\mathcal{A}$ satisfy (Re), (M), (GE) and (An). Using Lemma 2.2 (iv), (i) and (iii), we conclude that $\mathcal{A}$ also satisfies (K). Let $x, y, z \in A$. Applying (GE) and (K), we get $[x \rightarrow (y \rightarrow z)] \rightarrow [y \rightarrow (x \rightarrow z)] = [x \rightarrow (y \rightarrow z)] \rightarrow [y \rightarrow (x \rightarrow (y \rightarrow z))] = 1$, that is, (C) holds in $\mathcal{A}$. By Lemma 2.1 (v), $\mathcal{A}$ satisfies (Ex).

(ii) Let $x, y, z \in A$ and suppose that $y \leq z$. By (L) and (p-1), $1 = x \rightarrow 1 = x \rightarrow (y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$. From (M) it follows that $x \rightarrow y \leq x \rightarrow z$. Therefore, (*) holds in $\mathcal{A}$. Using Lemma 2.1 (ii) we see that (Tr) also holds. □

Remark 2.2. Theorem 6.16 (b) of [6] gives

(I) (Re) + (M) + (pimpl) + (An) $\Longrightarrow$ (Ex).

Observe that Proposition 2.1 (i) is a generalization of (I). The property (GE) is just the property (16) from the proof of Theorem 6.16 (a) of [6]. Hence, we obtain: (Re) + (M) + (pimpl) imply (GE). Therefore, Proposition 2.1 (i) implies (I).

Definitions 2.2 ([1])**.** *Let* $\mathcal{A} = (A, \rightarrow, 1)$ *be an algebra of type* $(2,0)$*. We say that* $\mathcal{A}$ *is:*

1. *a GE algebra* (*generalized exchange algebra*) *if it verifies (Re), (M), (GE);*

2. *an antisymmetric GE algebra* (*aGE algebra,* f*or short) if it is a GE algebra verifying (An).*

Denote by **GE** and **aGE** the classes of all GE algebras and aGE algebras, respectively.

Proposition 2.2 ([15], Corollary 3.2)**.** *Any GE algebra satisfies (Re), (M), (L), (C), (D), (K), (GE), (pi).*

Remark 2.3. Since GE algebras satisfy (L), we get **GE** = **RML** + (GE). By definition, **aGE** = **GE** + (An).

GE algebras do not have to satisfy (An), (B), (Tr), (Ex), (p-1), (pimpl); see example below.

Example 2.1 ([15])**.** Consider the set $A = \{a, b, c, d, e, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	e	1
a	1	1	c	c	1	1
b	a	1	d	d	1	1
c	a	1	1	1	1	1
d	a	1	1	1	1	1
e	a	1	1	1	1	1
1	a	b	c	d	e	1

We can observe that the properties (Re), (M), (L), (GE) are satisfied. Therefore, $(A, \to, 1)$ is a GE algebra. It does not satisfy (An) for $(x, y) = (c, d)$; (Ex) for $(x, y, z) = (a, b, c)$; (Tr), (B), (p-1), (pimpl) for $(x, y, z) = (a, e, c)$.

Definitions 2.3. *Let $\mathcal{A} = (A, \to, 1)$ be an RML algebra. The algebra $\mathcal{A}$ is called:*

1. *a pi-RML algebra if it verifies (pi),*

2. *a positive implicative RML algebra (for short, a pimpl-RML algebra) if it verifies (pimpl).*

Denote by **pi-RML** and **pimpl-RML** the classes of pi-RML and pimpl-RML algebras, respectively; similarly for the subclasses of the class of all RML algebras. Note that from [6] it follows that in RML algebras, (pimpl) implies (pi). Thus, **pimpl-RML** is a subclass of **pi-RML**. For BCK algebras, (pimpl) and (pi) are equivalent (cf. Theorem 8 of [8]), that is, **pimpl-BCK** = **pi-BCK**.

Recall that an algebra $(A, \to, 1)$ is a *Hilbert algebra* ([4]) if it verifies the axioms (An), (K), (p-1).

Remark 2.4. In [4], A. Diego proved that Hilbert algebras satisfy (Re), (M), (L), (B), (Ex), (pi), (p-2), (pimpl). Moreover, he showed that the class of all Hilbert algebras is a variety. From Remark 6.7 of [6] we see that **H** = **pimpl-BCK** = **pi-BCK**, where **H** denotes the class of all Hilbert algebras.

In [16], we introduced the following notion:

A *pre-Hilbert algebra* is an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying (M), (K) and (p-1). Let us denote by **preH** the class of pre-Hilbert algebras.

The following example shows that condition (K) cannot be dropped in the definition of pre-Hilbert algebra.

Example 2.2. Let $A = \{a, b, c, d, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	1	1	1	d	1
c	1	1	1	d	1
d	b	b	c	1	1
1	a	b	c	d	1

.

We can observe that algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (Re), (M), (L), (p-1). It does not verify (K) for $x = a$, $y = d$.

Proposition 2.3. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra verifying (M), (L) and (p-1). Then*

$$(**) \iff (K) \iff (B).$$

Proof. (**) $\Longrightarrow$ (K). By Lemma 2.1 (iii).

(K) $\Longrightarrow$ (B). By Proposition 2.1 (ii), $\mathcal{A}$ satisfies (Tr). To prove (B), let $x, y, z \in A$. From (K) and (p-1) we conclude that

$$y \to z \leq x \to (y \to z) \text{ and } x \to (y \to z) \leq (x \to y) \to (x \to z).$$

Applying (Tr), we have $y \to z \leq (x \to y) \to (x \to z)$. Thus, (B) holds in $\mathcal{A}$.

(B) $\Longrightarrow$ (**). Follows from Lemma 2.1 (i). □

Proposition 2.4 ([16], Theorem 3.9)**.** *Pre-Hilbert algebras satisfy (Re), (M), (L), (B), (C), (K), (Tr), (p-1), (p-2).*

Remark 2.5. By definition and Proposition 2.4, **preH** = **RML** + (K) + (p-1). Since (An) + (K) + (p-1) imply (M) (see [4]), a Hilbert algebra is in fact a pre-Hilbert algebra verifying (An), that is, **H** = **preH** + (An).

Pre-Hilbert algebras do not have to satisfy (An), (Ex), (GE), (pi), (pimpl); see example below.

Example 2.3. Consider the set $A = \{a, b, c, d, 1\}$ and the operation $\to$ given by the following table:

$\to$	a	b	c	d	1
a	1	c	b	d	1
b	a	1	1	d	1
c	a	1	1	d	1
d	a	c	c	1	1
1	a	b	c	d	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (M), (K), (p-1). Then, $\mathcal{A}$ is a pre-Hilbert algebra. It does not verify (An) for $(x, y) = (b, c)$; (Ex) and (pimpl) for $(x, y, z) = (a, d, b)$; (pi) for $(x, y) = (a, b)$; (GE) for $(x, y, z) = (a, 1, b)$.

Remark 2.6. It is easy to see that in GE algebras, $\leq$ is a reflexive relation; in pre-Hilbert algebras, $\leq$ is a pre-order relation, in aGE algebras, $\leq$ is reflexive and antisymmetric. In Hilbert algebras, $\leq$ is an order relation.

3. Exchange pre-Hilbert algebras

We now introduce a new algebra. We say that an algebra $(A, \to, 1)$ is an *exchange pre-Hilbert algebra* if it is a pre-Hilbert algebra verifying the exchange property (Ex).

Denote by **Ex-preH** the class of exchange pre-Hilbert algebras. By definition, **Ex-preH** $=$ **preH** $+$ (Ex).

Example 3.1. Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

$\to$	a	b	c	1
a	1	c	b	1
b	1	1	1	1
c	1	1	1	1
1	a	b	c	1

The algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (Re), (M), (L), (Ex) (hence (K)), (B), (p-1). It does not verify (An) for $x = b, y = c$; (pi) for $x = a, y = b$. Hence, $\mathcal{A}$ is an exchange pre-Hilbert algebra, without (pi).

Proposition 3.1. *Any exchange pre-Hilbert algebra can be extended to an exchange pre-Hilbert algebra containing one element more.*

Proof. Let $\mathcal{A} = (A, \to, 1)$ be a pre-Hilbert algebra and let $\delta \notin A$. On the set $B = A \cup \{\delta\}$ consider the operation:

$$x \to' y = \begin{cases} x \to y, & \text{if } x, y \in A, \\ \delta, & \text{if } x \in A \text{ and } y = \delta, \\ 1, & \text{if } x = \delta \text{ and } y \in B. \end{cases}$$

Obviously, $\mathcal{B} := (B, \to', 1)$ satisfies the axioms (M), (L) and (K). Further, the axioms (p-1) and (Ex) are easily satisfied for all $x, y, z \in A$. Let at least one of x, y, z be equal to δ. First, let $x = \delta$ and $y, z \in B$. Then, $(x \to' y) \to' (x \to' z) = (\delta \to' y) \to' (\delta \to' z) = 1$ and $x \to' (y \to' z) = \delta \to' (y \to' z) = 1 = y \to' (\delta \to' z) = y \to' (x \to' z)$. Thus, (p-1) and (Ex) hold for $x = \delta$ and $y, z \in B$. Similarly, if $y = \delta$ and $x, z \in A$. Now let $z = \delta$ and $x, y \in A$. We have $x \to' (y \to' z) = x \to' (y \to' \delta) = \delta = (x \to' y) \to' (x \to' \delta) = (x \to' y) \to' (x \to' z)$ and $x \to' (y \to' z) = x \to' (y \to' \delta) = \delta = y \to' (x \to' \delta) = y \to' (x \to' z)$. Therefore, $\mathcal{B}$ satisfies (p-1) and (Ex). Hence, $\mathcal{B}$ is an exchange pre-Hilbert algebra. $\square$

Now, we give some characterizations of exchange pre-Hilbert algebras.

Theorem 3.1. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is an exchange pre-Hilbert algebra*;

(ii) *$\mathcal{A}$ satisfies (M), (L), (B), (Ex), (p-1);*

(iii) *$\mathcal{A}$ is a pre-BCK algebra satisfying (p-1);*

(iv) *$\mathcal{A}$ is a BE algebra satisfying (p-1).*

Proof. (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (iv) are obvious.

(iv) $\Longrightarrow$ (i). By Lemma 2.1 (iv), (Re) + (L) + (Ex) imply (K). Then, $\mathcal{A}$ satisfies (M), (K), (p-1), (Ex). Thus, $\mathcal{A}$ is an exchange pre-Hilbert algebra. $\square$

Lemma 3.1 ([15], Corollary 2.8)**.** *Any GE algebra is a pi-RML algebra.*

From Lemmas 2.2 (ii) and 3.1 we have

Proposition 3.2. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is a pi-BE algebra,*

(ii) *$\mathcal{A}$ is a GE algebra satisfying (Ex).*

Lemma 3.2. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra verifying (B), (Ex) and (pi). Then $\mathcal{A}$ satisfies (p-1).*

Proof. Let $x, y, z \in A$. By (Ex), $x \to (y \to z) = y \to (x \to z)$. Applying (B) and (pi), we get $y \to (x \to z) \leq (x \to y) \to [x \to (x \to z)] = (x \to y) \to (x \to z)$. Then, $x \to (y \to z) \leq (x \to y) \to (x \to z)$, that is, (p-1) holds. $\square$

Remark 3.1. By Proposition 3.2 and Theorem 3.1, **pi-BE** = **GE** + (Ex) and **Ex-preH** = **preBCK** + (p-1) = **BE** + (p-1). Hence **Ex-preH** + (pi) = **preBCK** + (p-1) + (pi) = **pi-preBCK**, since (B) + (Ex) + (pi) imply (p-1), see Lemma 3.2.

Proposition 3.3. *An algebra* $\mathcal{A} = (A, \to, 1)$ *of type* $(2, 0)$ *is an antisymmetric GE algebra if and only if it is a pi-aBE algebra.*

Proof. Let $\mathcal{A}$ be an aGE algebra. By Proposition 2.2, $\mathcal{A}$ satisfies (pi) and (C). Since (C) + (An) imply (Ex), see Lemma 2.1 (v), we conclude that $\mathcal{A}$ is a pi-aBE algebra. The converse follows from Proposition 3.2. □

Remark 3.2. By Proposition 3.3, **aGE** = **pi-aBE**. Hence **aGE** + (B) = **pi-aBE** + (B) = **aBE** + (B) + (pi) = **BCK** + (pi) = **pi-BCK**.

The interrelationships between the classes of algebras mentioned before are visualized in Figure 1 (see Remarks 2.3 – 2.5, 3.1 and 3.2).

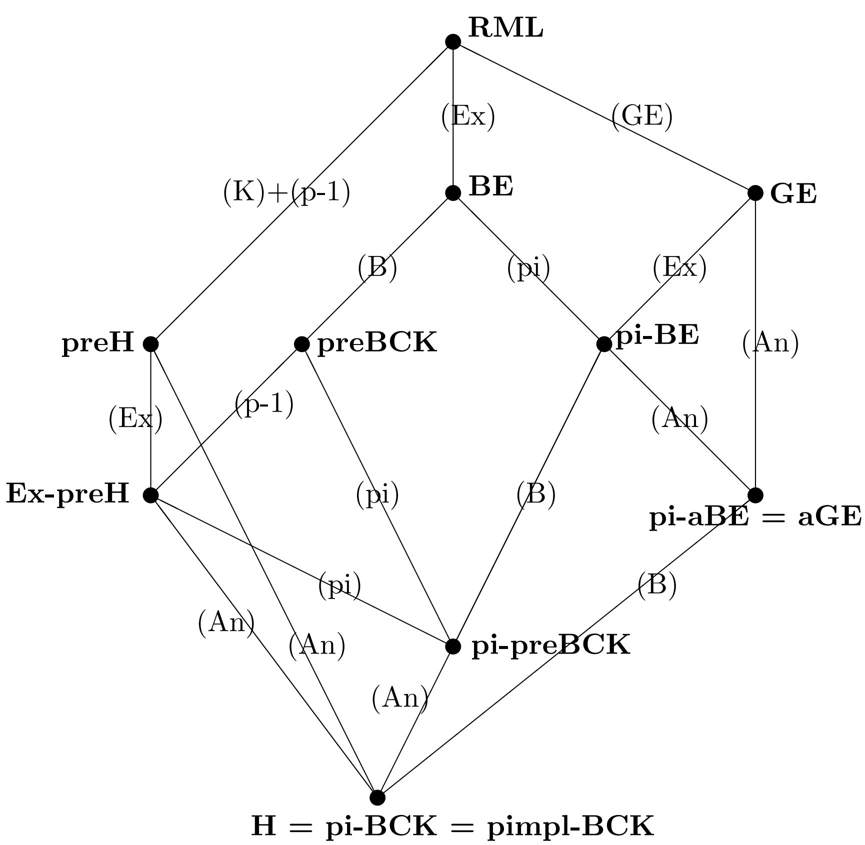

Figure 1: The hierarchy between **RML** and **H**

Consider now the property of positive implicativity for exchange pre-Hilbert algebras. Note that positive implicative GE algebras and pre-Hilbert algebras were studied in [15] and [16], respectively.

Remark 3.3. By Remark 4.9 of [15], **pimpl-RML** = **pimpl-GE** and by Remark 4.9 of [16], **pimpl-preH** = **pimpl-RML**. Therefore, **pimpl-preH** = **pimpl-GE** = **pimpl-RML**.

Theorem 3.2. *Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. The following statements are equivalent:*

(i) *$\mathcal{A}$ is a positive implicative exchange pre-Hilbert algebra;*

(ii) *$\mathcal{A}$ satisfies (Re), (M), (Ex) and (pimpl);*

(iii) *$\mathcal{A}$ is a pimpl-BE algebra;*

(iv) *$\mathcal{A}$ is a pimpl-pre-BCK algebra.*

Proof. (i) $\Longrightarrow$ (ii). By definitions.

(ii) $\Longrightarrow$ (iii). If $\mathcal{A}$ satisfies (Re), (M), (Ex) and (pimpl), then $\mathcal{A}$ also satisfies (L), since (Re) + (pimpl) imply (L) by Lemma 2.2 (v). Therefore, $\mathcal{A}$ is a pimpl-BE algebra.

(iii) $\Longrightarrow$ (iv). By Lemma 2.2 (v), (Re) + (pimpl) imply (B). Hence $\mathcal{A}$ is a pimpl-pre-BCK algebra.

(iv) $\Longrightarrow$ (i). Obvious. □

Remark 3.4. From Theorem 3.2 it follows that **pimpl-Ex-preH** = **pimpl-BE** = **pimpl-preBCK**.

In 2012, R. A. Borzooei and J. Shohani [2] introduced the notion of a generalized Hilbert algebra. Following [2], a *generalized Hilbert algebra* is an algebra $(A, \to, 1)$ satisfying (Re), (M), (Ex), (pimpl). From Theorem 3.2 we conclude that generalized Hilbert algebras are just pimpl-pre-BCK algebras. Moreover, we have

Corollary 3.1. *An algebra is a positive implicative exchange pre-Hilbert algebra if and only if it is a generalized Hilbert algebra.*

4. The implicative property (im)

The implicative BCK algebras were introduced and investigated by K. Iséki and S. Tanaka [8]. It is well-known that any bounded implicative BCK algebra is a Boolean algebra. Note that the implicative property for some generalizations of BCK algebras were studied in [14].

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. We consider the following property:

(im) (Implicativity) $(x \to y) \to x = x$.

Lemma 4.1 ([14], Proposition 3.5). *Let* $(A, \to, 1)$ *be an algebra of type* $(2, 0)$. *Then:*

(i) *(Re) + (im) imply (M),*

(ii) *(M) + (im) imply (L),*

(iii) *(im) implies (pi).*

Similarly as in the case of BCK algebras, we say that an RML algebra (in particular, an exchange pre-Hilbert algebra) $(A, \to, 1)$ is *implicative* if it satisfies (im).

Denote by **im-RML** the class of implicative RML algebras; similarly for subclasses of the class of all RML algebras.

Remarks 4.1. (1) By definitions, **im-RML** = **RML** + (im), **im-GE** = **im-RML** + (GE).

(2) By Lemma 2.2 (ii), (Ex) + (pi) imply (GE). Hence (Ex) + (im) imply (GE), because (im) implies (pi). Consequently, **im-BE** = **im-RML** + (Ex) = **im-RML** + (GE) + (Ex) = **im-GE** + (Ex).

Now we give several characterizations of implicative exchange pre-Hilbert algebras.

Theorem 4.1. *Let* $\mathcal{A} = (A, \to, 1)$ *be an algebra of type* $(2, 0)$. *The following are equivalent:*

(i) $\mathcal{A}$ *is an implicative exchange pre-Hilbert algebra;*

(ii) $\mathcal{A}$ *satisfies (Re), (B), (Ex), (im);*

(iii) $\mathcal{A}$ *is an implicative pre-BCK algebra;*

(iv) $\mathcal{A}$ *is an implicative BE algebra satisfying* (*).

Proof. (i) $\Longrightarrow$ (ii). Obvious.

(ii) $\Longrightarrow$ (iii). By Lemma 4.1, $\mathcal{A}$ also satisfies (M) and (L). Thus, $\mathcal{A}$ is an implicative pre-BCK algebra.

(iii) $\Longrightarrow$ (iv). Clearly, $\mathcal{A}$ is an implicative BE algebra. Moreover, $\mathcal{A}$ satisfies (*) by Lemma 2.1 (i).

(iv) $\Longrightarrow$ (i). From Lemma 2.1 (iv) we see that $\mathcal{A}$ satisfies (D) and (K). Observe that $\mathcal{A}$ also satisfies (B). Let $x, y, z \in A$. By (D), $y \leq (y \to z) \to z$. Hence, applying (*) and (Ex), we have

$$x \to y \leq x \to [(y \to z) \to z] = (y \to z) \to (x \to z),$$

that is, $x \to y \le (y \to z) \to (x \to z)$. From (Ex) we conclude that $y \to z \le (x \to y) \to (x \to z)$, that is, (B) holds in $\mathcal{A}$. By Lemma 4.1 (iii), (im) implies (pi). Thus, $\mathcal{A}$ satisfies (B), (Ex), (pi). Hence, using Lemma 3.2, we deduce that (p-1) holds. Consequently, $\mathcal{A}$ is an implicative exchange pre-Hilbert algebra. $\square$

Example 4.1 ([15]). Let $A = \{a, b, c, d, e, 1\}$ and $\to$ be defined as follows:

$\to$	a	b	c	d	e	1
a	1	1	e	d	e	1
b	1	1	d	d	d	1
c	1	1	1	1	1	1
d	a	b	b	1	1	1
e	a	a	a	1	1	1
1	a	b	c	d	e	1

.

It is easy to see that the properties (Re), (B), (Ex), (im) are satisfied; (An) is not satisfied for $(x, y) = (a, b)$, (pimpl) is not satisfied for $(x, y, z) = (a, b, c)$, Therefore, $(A, \to, 1)$ is an implicative exchange pre-Hilbert algebra that is not positive implicative.

Lemma 4.2 ([17], Lemma 10). *If* $(A, \to, 1)$ *is an implicative aBE algebra, then* $(x \to y) \to y = x$ *or* $y \to x \neq 1$*, for all* $x, y \in A$.

Proposition 4.1. Any *implicative aBE algebra satisfies* (*).

Proof. Let $\mathcal{A} = (A, \to, 1)$ be an implicative aBE algebra. By Lemma 2.1 (iv), $\mathcal{A}$ satisfies (K). Let $x, y, z \in A$ and $y \le z$. From Lemma 4.2 it follows that $(z \to y) \to y = z$. Hence, by (Ex), we get $x \to z = x \to ((z \to y) \to y) = (z \to y) \to (x \to y)$. Applying (K), we obtain $(x \to y) \to (x \to z) = (x \to y) \to ((z \to y) \to (x \to y)) = 1$. Thus, $x \to y \le x \to z$, that is, (*) holds in $\mathcal{A}$. $\square$

From Theorem 4.1 and Proposition 4.1 we have

Corollary 4.1. *The class of implicative aBE algebras coincides with the class of implicative BCK algebras.*

Remark 4.2. (1) From Theorem 4.1 we obtain **im-Ex-preH** = **im-preBCK** = **im-BE** + (B).

(2) By definition, **im-BCK** = **im-preBCK** + (An). Hence **im-Ex-preH** + (An) = **im-BCK**. From Proposition 4.17 of [15] and Corollary 4.1 we see that **im-BCK** = **im-aBE** = **im-aGE**.

(3) Since **H** = **pi-BCK** and (im) implies (pi), we obtain **H** + (im) = **pi-BCK** + (im) = **BCK** + (pi) + (im) = **im-BCK** + (pi) = **im-BCK**.

By definitions and Remarks 3.3, 3.4, 4.1 and 4.2 we can draw Figure 2.

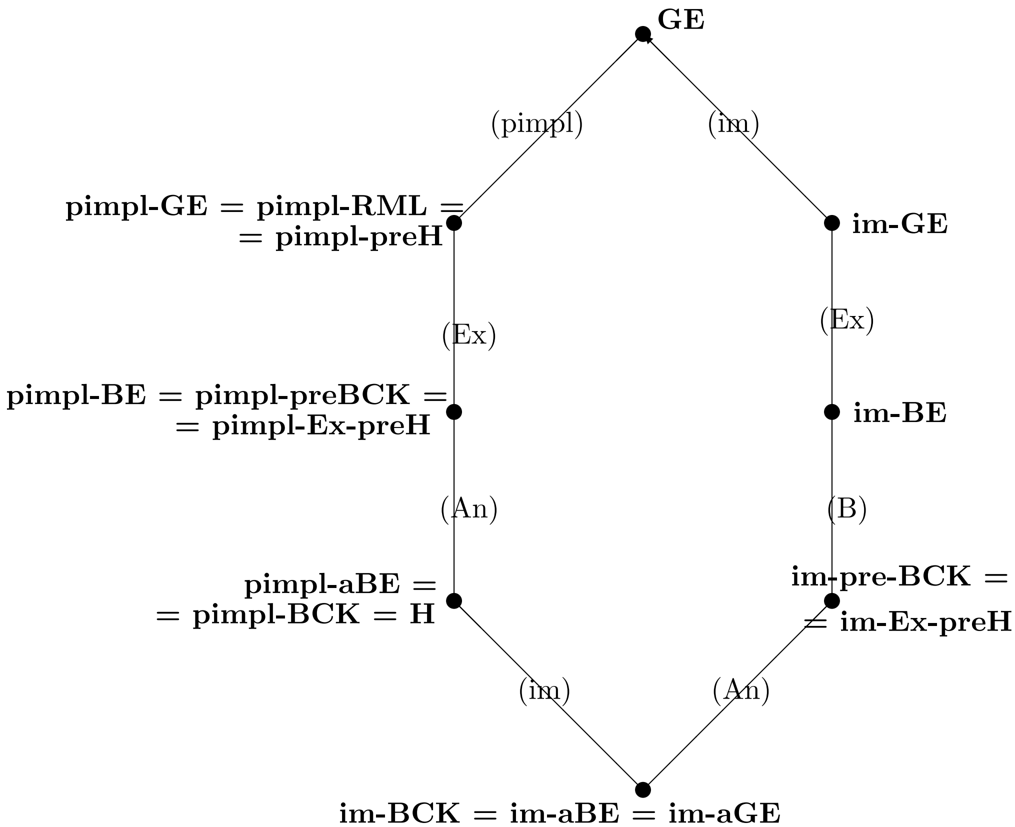

Figure 2: The hierarchy between **GE** and **im-BCK**

Example 4.2. Let $\mathbb{Z}$ be the set of integers and let for $x, y \in \mathbb{Z}$ the symbol $x \mid y$ means that x divides y. Then, the relation $\mid$ is a pre-order on $\mathbb{Z}$ which is not an order (for example, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$). We define the operation $\to$ by

$$x \to y = \begin{cases} 0, & \text{if } x \mid y, \\ y, & \text{otherwise.} \end{cases}$$

Obviously, $x \to x = 0$ and $0 \to x = x$ for each $x \in \mathbb{Z}$. Then, $\mathcal{Z} = (\mathbb{Z}, \to, 0)$ satisfies (Re) and (M). To prove (Ex), let $x, y, z \in \mathbb{Z}$. We will consider three cases:
Case 1. Let $y \mid z$. Then, $x \to (y \to z) = x \to 0 = 0 = y \to (x \to z)$, since $y \mid x \to z$.

Case 2. Let $y \nmid z$ and $x \mid z$. Then, $x \to (y \to z) = x \to z = 0 = y \to 0 = y \to (x \to z)$.

Case 3. Let $y \nmid z$ and $x \nmid z$. We have $x \to (y \to z) = x \to z = z = y \to z = y \to (x \to z)$.
Thus, $\mathcal{Z}$ satisfies (Ex). Similarly, by routine calculation we can show that $\mathcal{Z}$ also satisfies (pimpl). Consequently, $\mathcal{Z}$ is a positive implicative exchange pre-Hilbert algebra. Observe that $\mathcal{Z}$ is not implicative. Indeed, $(2 \to 1) \to 2 = 1 \to 2 = 0 \neq 2$.

Acknowledgments

The author is indebted to the referee for his/her very careful reading and suggestions.

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Accepted: October 30, 2024