

On τ -supplemented Krasner hypermodules

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Abstract. In this study we define the radical of the Krasner hypermodules in the subcategory ${}_R\mathcal{H}mod$, then we use short exact sequences in homological algebra for Krasner hypermodules. Besides, by studying the concept of τ -supplements in module theory we will generalize it to the Krasner R -hypermodules by using short exact sequences and a subcategory of ${}_R\mathcal{H}mod$.

Keywords: subcategory of hypermodules, preradical, supplemented hypermodules, τ -supplemented hypermodules.

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1. Introduction

In this text, we assume that all hypermodules are left Krasner R -hypermodules, all hyperrings are Krasner hyperrings, and all homomorphisms are strong R -homomorphisms. In order to formalize the theory of hypercomposition algebra, a new branch of abstract algebra began its development in 1934, when F. Marty introduced the concept of hypergroups as a proper generalization of the concept of groups. The combination of two elements is extended by replacing the operation defined in groups with a multivalued operation (or hyperoperation), with the result being a subset of the fundamental set. As a result, new algebraic hypercompositional structures and the properties of classical structures are defined and the properties of classical structures are defined while the properties of classical structures are preserved or not for similar hyperstructures. This also applies to modules and expanded into hypermodules by Krasner [15] today

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known as Krasner hypermodules. Their additive part is canonical hypergroup. Fundamental aspects of hypermodule theory are covered in [3], [4], [5], [7], [8], [9], [18], [22] and [23]. Recently, the concept of smallness in module theory has been carried over and investigated in [13] on the class of hypermodules in the context of similarities and differences. The concept of small subhypermodules was defined in [6]. An R -subhypermodule K of H is said to be *small* in H if $K + T = H$ for each subhypermodule T of H , implies $T = H$, and it is denoted by $K \ll H$. An R -hypermodule H is said to be *hollow* if each subhypermodule is small in H [13]. An R -hypermodule H is said to be *local* if H has a proper subhypermodule that contains all proper subhypermodules of H [12]. A hypermodule H is said to be *supplemented* if for each subhypermodule K of H , there is a subhypermodule T of H provided that $H = K + T$ and $K \cap T \ll T$ [13]. Here T is said to be the *supplement* of K in H .

According to this study the class of τ -supplemented hypermodules has been classified in a category theory. In this way, the functor τ was obtained and the class of τ -supplemented hypermodules was defined. In section 2, the algebraic and categorical properties of Krasner hypermodules will be included and the concept of supplement in subhypermodules will be emphasized. In section 3, (pre)radicals will be presented in the $R_S hmod$, which is a special hypermodule subcategory. In section 4, τ -supplemented hypermodules are examined.

2. Preliminaries

This section briefly reviews the basic concepts and results on Krasner hypermodules, which are used throughout this article for clarity. We start with relevant basic definitions of the topic in hypercomposition algebra, which are presented in several books [10], [11] and overview articles [2], [19], [20] and [21]. We also refer the reader to [1] for generalizing the concept of τ -supplement in module theory and the studies related to the category of hypermodules.

Hypermodules. Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H . The function $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* on H . The image of the element $(a, b) \in H \times H$ under this operation is not a single element, but a non-empty subset of the set H . Thanks to this idea, the theory of hyperstructures was introduced by Marty in [17] as a natural and interesting generalization of the theory of algebraic structures. Following [17], Marty defines hypergroups using the hyperoperation on a set. Let H be a non-empty set and a function $+$: $H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation on H . Then, $(H, +)$ is called a *hypergroupoid*. Moreover, for any non-empty subsets X and Y of H , define

$$X + Y = \bigcup \{x + y \mid x \in X \text{ and } y \in Y\}.$$

We simply write $a + X$ and $X + a$ instead of $\{a\} + X$ and $X + \{a\}$, respectively, for any $a \in H$ and any non-empty subset X of H . A hypergroupoid $(H, +)$ is called a

- (1) *Semihypergroup* if for every $a, b, c \in H$, we have $a + (b + c) = (a + b) + c$;
- (2) *Quasihypergroup* if for every $x \in H$, $x + H = H = H + x$.

If the hypergroupoid $(H, +)$ is a semihypergroup and quasihypergroup, then it is called a *hypergroup*. A non-empty subset S of a hypergroup $(H, +)$ is called a *subhypergroup* of H if for every $a \in S$, $a + S = S = S + a$, written as $S \leq H$.

A hypergroup $(H, +)$ is called a *canonical hypergroup* if

- (1) for every $a, b \in H$, $a + b = b + a$, that is, it is commutative;
- (2) there exists a unique $0 \in H$ such that for each $a \in H$ there exists a unique element a' in H , denoted by $-a$, such that $0 \in a + (-a)$;
- (3) for every $a, b, c \in H$, if $c \in a + b$, then $a \in c + (-b) := c - b$.

If $(H, +)$ is a canonical hypergroup, then $a + 0 = a$ for all $a \in H$.

A hyperstructure $(R, +, \cdot)$ is called a (*Krasner*) *hyperring* if

- (1) $(R, +)$ is a canonical hypergroup.
- (2) $(R, \cdot)$ is a monoid with a bilaterally absorbing element 0, i.e.,
 - (a) $a \cdot b \in R$ for all $a, b \in R$;
 - (b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in R$;
 - (c) $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$;
 - (d) There exists an *identity element* $1_R \in R$ such that $a = a \cdot 1_R = 1_R \cdot a$ for every $a \in R$.
- (3) The multiplication distributes over the addition on both sides.

A hyperring $(R, +, \cdot)$ is called *commutative* if it is commutative concerning the multiplication.

Let $(R, +, \cdot)$ be a hyperring and I be a non-empty subset of R . I is called a *left hyperideal* (respectively, *right hyperideal*) of R provided $(I, +)$ is a subhypergroup and $ra \in I$ (respectively, $ar \in I$) for all $a \in I$, and $r \in R$. I is said to be a *hyperideal* of R if it is both a right and a left hyperideal of R .

A left Krasner hypermodule over a hyperring R (or left Krasner R -hypermodule) is a canonical hypergroup $(M, +)$ together with a map $\cdot : R \times M \rightarrow M$ such that to every (r, m) , where $r \in R$ and $m \in M$, there corresponds a uniquely determined element $rm \in M$ and the following conditions are satisfied:

- (1) $r(m_1 + m_2) = rm_1 + rm_2$;
- (2) $(r + s)m = rm + sm$;
- (3) $(rs)m = r(sm)$;

$$(4) \ 1_R m = m \text{ and } r 0_M = 0_R m = 0_M,$$

for any $m, m_1, m_2 \in M$ and $r, s \in R$. Throughout this paper, for a simple explanation, when we say hypermodule, we mean the left Krasner hypermodule. A non-empty subset N of an R -hypermodule M is called a *subhypermodule* of M , denoted by $N \leq M$ if N is an R -hypermodule under the same hyperoperations of M . It is clear that M and $\{0_M\}$ are *trivial subhypermodules* of M . It is known that a non-empty subset N of an R -hypermodule M is a subhypermodule of M if and only if $a - b \subseteq N$ and $ra \in N$ for all $a, b \in M$ and $r \in R$.

Let R be a hyperring. It follows that R is an R -hypermodule. Then, a non-empty subset I of R is a left hyperideal of R if and only if it is a subhypermodule of the hypermodule ${}_R R$.

Let M be a hypermodule over a hyperring R and K be a subhypermodule of M . Consider the set $\frac{M}{K} = \{a + K \mid a \in M\}$. Then, $\frac{M}{K}$ is a hypermodule over the hyperring R under the hyperoperation defined as $+: \frac{M}{K} \times \frac{M}{K} \longrightarrow \mathcal{P}^*(\frac{M}{K})$ and the external operation $\odot: R \times \frac{M}{K} \longrightarrow \frac{M}{K}$ defined as $(a + K) + (a' + K) = \{b + K \mid b \in a + a'\}$ and $r \odot (a + K) = ra + K$ for every $a, a', b \in M$ and $r \in R$. The hypermodule $\frac{M}{K}$ is called the *quotient hypermodule* of the hypermodule M .

Let K be an R -hypermodule and $x \in K$. Then, the subset $Rx = \{ux \mid u \in R\}$ of K is a subhypermodule of K . Let R be a hyperring and M be an R -hypermodule. For a family of subhypermodules $\{M_i\}_{i \in I}$ of M , the sum of this family is denoted by $\sum_{i \in I} M_i$ and it is the set of these elements $x \in M$ where x is an element of the set $\sum_{i \in I_0} m_i$ with finite subset $I_0 \subseteq I$ for every $i \in I_0$, $m_i \in M_i$. That is,

$$\sum_{i \in I} M_i = \{x \in M \mid x \in \sum_{i \in I_0} m_i, m_i \in M_i \text{ and } I_0 \text{ is a finite subset of } I\}.$$

It is well known that $\sum_{i \in I} M_i$ is a subhypermodule of M .

Lemma 2.1 (Modular Law [25]). *Let H be a Krasner R -hypermodule and K, T, L are subhypermodules of H with $T \leq K$. Then, $K \cap (T + L) = T + (K \cap L)$.*

Let $(K, +_1)$ and $(T, +_2)$ be two R -hypermodules. A function $\psi: K \longrightarrow T$ is said to be an R -homomorphism from K to T if satisfies the following two conditions:

1. $\psi(x +_1 y) \subseteq \psi(x) +_2 \psi(y)$;
2. $\psi(ux) = u\psi(x)$,

for each $u \in R$ and $x, y \in K$. If the inclusion (1) is an equality, then ψ is said to be a *strong* (or good) R -homomorphism.

Categories of hypermodules. Throughout the whole text we denoted by $R\text{-hmod}$ the category whose objects are whole R -hypermodules and whose morphisms are all R -homomorphisms. The class of all R -homomorphism from

K to T is indicated by $\text{hom}_R(K, T)$. Also, $R_S - \text{hmod}$ is the category of all R -hypermodules whose morphisms are all strong R -homomorphisms. The class of all strong R -homomorphisms is denoted by $\text{hom}_R^S(K, T)$. It is clear that $R_S - \text{hmod}$ is a subcategory of $R - \text{hmod}$, this situation is indicated by the writing $R_S - \text{hmod} \preceq R - \text{hmod}$. The hyperset category denoted by $H\text{Sets}$ is a category with the following:

1. $\text{Ob}(H\text{Sets}) = \text{Ob}(\text{Sets})$;
2. $\text{Mor}(H\text{Sets})$ is the class of all multivalued functions between objects in which the composition $\theta \circ \psi$ is defined as $(\theta \circ \psi)(x) = \bigcup_{y \in \psi(x)} \theta(y)$ for each $x \in K$, and an identity morphism for an object K for all $z \in K$, $\text{id}_K(z) = \{z\}$.

Let $(K, +_1)$ and $(T, +_2)$ be R -hypermodules and ψ a multi-valued function from K to T , denoted by $\psi : K \longrightarrow P^*(T)$ providing the following statements:

1. $\psi(x +_1 y) \subseteq \psi(x) +_2 \psi(y)$;
2. $\psi(ux) = u\psi(x)$,

for each $u \in R$, and $x, y \in K$. Then, ψ is said to be a *multi-valued R -homomorphism*, shortly *R_{mv} -homomorphism*. If equality is achieved instead of coverage in the first of the above two properties, then ψ is called a *strong (or good) multi-valued R -homomorphism*, shortly a *R_{smv} -homomorphism*. We denote the class of whole R_{mv} -homomorphisms (resp., R_{smv} -homomorphisms) from K to T as $\text{Hom}_R(K, T)$ (resp., $\text{Hom}_R^S(K, T)$). $R - \text{Hmod}$ (resp., $R_S - \text{Hmod}$) specifies a category whose objects are all R -hypermodules and whose morphisms from K to T are whole R_{mv} -homomorphisms (resp., R_{smv} -homomorphisms) from K to T . Obviously, $R_S - \text{Hmod}$ is a subcategory of $R - \text{Hmod}$, that is, $R_S - \text{Hmod} \preceq R - \text{Hmod}$.

Let $\psi, \theta \in \text{Hom}_R(K, T)$ where $\leq$ is the relation on $\text{Hom}_R(K, T)$ defined as $\psi \leq \theta$ if for each $x \in K$, $\psi(x) \subseteq \theta(x)$. Here $(\text{Hom}_R(K, T), \leq)$ is a partially ordered set. Let $\psi, \theta, \eta \in \text{Hom}_R(K, T)$. Let $+$: $\text{Hom}_R(K, T) \times \text{Hom}_R(K, T) \longrightarrow \text{Hom}_R(K, T)$ be an operation on $\text{Hom}_R(K, T)$ which is defined by $(\psi, \theta) \mapsto \psi + \theta$ where $(\psi + \theta)(x) = \psi(x) +_2 \theta(x)$ for each $x \in K$ in which $+$ is the hyperoperation of the canonical hypergroup $(T, +_2)$. Here $\eta \leq \psi + \theta$ if and only if $\eta(x) \subseteq \psi(x) +_2 \theta(x)$ for each $x \in K$. A hyperoperation on $\text{Hom}_R(K, T)$ is defined as follows:

$$\psi \biguplus \theta = \{ \eta \in \text{Hom}_R(K, T) \mid \eta \leq \psi + \theta \}.$$

The hyperoperation $\biguplus$ on $\text{hom}_R(K, T)$ is defined by being restricted in the following format

$$\psi \biguplus \theta = \{ \eta \in \text{hom}_R(K, T) \mid \eta(x) \in \psi(x) + \theta(x), \text{ for each } x \in K \}.$$

In [3, Theorem 3.11], it has been proved that $(Hom_R(K, T), \uplus)$ is a commutative hypermonoid and $(hom_R(K, T), \uplus)$ is a canonical hypergroup for objects K and T in categories respectively $R - Hmod$ and $R - hmod$.

Supplement subhypermodules of hypermodules. Let K and T be subhypermodules of an R -hypermodule H . Then, both subhypermodules K and T are called *independent* if $K \cap T = \{0_H\}$, so $K + T$ is denoted by $K \oplus T$ and it is called as *internal direct sum*. In addition, a subhypermodule K of H is said to be a *direct summand* of H provided that $H = K \oplus L$ for some subhypermodule L of H ([25]). A non-zero R -hypermodule H is said to be *simple* provided that the only subhypermodules of H are $\{0_H\}$ and H itself. Let R be a hyperring and H be an R -hypermodule. By $Soc(H)$ we denote the sum of all simple subhypermodules of H . An R -hypermodule H is said to be *semisimple* provided that its subhypermodules are direct summands in H ([16]). In [27], it is introduced the concept of semisimple R -hypermodule H as $H = Soc(H)$, that is, it is the sum of simple R -subhypermodules of H . It was shown that the class of semisimple hypermodules was closed under internal direct sums, factor hypermodules, and subhypermodules. It is proven in [27, Proposition 3] that $Soc(T)$ is the largest semisimple subhypermodule of a hypermodule H . The various properties of semisimple hypermodules are given in [27]. In particular, it is shown in [27, Theorem 1] that a hypermodule H is an internal direct sum of subhypermodules K and T , that is, $H = K \oplus T$ if and only if, for every element $m \in H$, there exists a unique element $k \in K$ and a unique element $t \in T$ such that m is an element of the set $k + t$. Let I be a left hyperideal and R be a hyperring. If $I \cap J \neq 0$ for every non-zero left hyperideal J of R , then I is called *essential left hyperideal* of R . Then

$$Z(M) = \{m \in M \mid Im = 0_M \text{ for some essential left hyperideal } I \text{ of } R\}$$

is called a *singular subhypermodule* of M . In [27, Corollary 2], it is proven that every simple hypermodule M is normal projective or $M = Z(M)$.

The following example is given in [27, Example 10].

Example 2.1. Let $R = \{0, 1, 2, 3\}$ with hyperoperation “+” and operation “.”:

+	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{1}	{0,1}	{3}	{2,3}
2	{2}	{3}	{0}	{1}
3	{3}	{2,3}	{1}	{0,1}

and

·	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	2	2

Then, R is an R -hypermodule. It is easy to see that the only proper subhypermodules of R are $M_0 = \{0\}$, $M_1 = \{0, 1\}$, and $M_2 = \{0, 2\}$. Therefore, M_1 and M_2 are simple subhypermodules of M , and so we can write $R = M_1 + M_2$. This means that the R -hypermodule R is semisimple.

In [13], as a generalization class of semisimplicity, the class of supplemented hypermodules was defined. Let H be a Krasner R -hypermodule and K, T be subhypermodules of H . T is said to be a *supplement* of K in H provided that T is a minimal element in the set $\{L \leq H \mid K + L = H\}$. Then, H is said to be *supplemented* provided that each subhypermodule of H has a supplement in H [13]. A subhypermodule K of a Krasner R -hypermodule H is said to be a *small subhypermodule* of H provided that $K + T \neq H$ for each proper subhypermodule T of H , denoted by $K \ll H$. So T is a supplement of K in H if and only if $K + T = H$ and $K \cap T \ll T$. In addition, K has ample supplements in H if, whenever $K + T = H$, T contains a supplement L of K in H . The Krasner R -hypermodule H is said to be *amply supplemented* provided that each subhypermodule has ample supplements in H (see [13]).

Example 2.2. Let $R = \{0, 1, 2\}$ and $A = \{0, 2\}$. Define the hyperaddition “+” and multiplication “.” by the following:

+	0	1	2
0	{0}	{1}	{2}
1	{1}	R	{1}
2	{2}	{1}	A

and

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

Then, R is a hyperring and A is the only left maximal hyperideal of R . It follows that the left R -hypermodule R is (amply) supplemented which is not semisimple.

3. Preradicals in category of ${}_R s hmod$

Our interest in this approach to the structure theory of hypermodules was noticed as a result of an observation mentioned in [1] and it was aimed to generalize to hypermodules by taking advantage of the existing category theory in module structure in [2]. In this context, we study on concept of (pre)radical for using of the concept of τ -supplement subhypermodules in category of ${}_R s hmod$.

Definition 3.1. (1) A functor τ from the category of $R_s - hmod$ to itself is called a preradical if following properties are satisfied:

- (1) $\tau(H)$ is a subhypermodule of the R -hypermodule H .
- (2) $\psi(\tau(H)) \subseteq \tau(L)$ for each strong homomorphism $\psi \in \text{Hom}_R^S(H, L)$.

Let τ be a preradical of $R_s - hmod$, where R is a hyperring. Assume that $T \leq H$ are R -hypermmodules. Applying the condition (2) of being preradical, it is clear that $\iota(\tau(T)) = \tau(T) \leq \tau(H)$, where $\iota : T \rightarrow H$ is the inclusion map.

Definition 3.2. Let τ be a preradical of $R_s - hmod$. Then, τ is called idempotent if $\tau(\tau(H)) = \tau(H)$ for every $H \in R_s - hmod$.

Example 3.1. Let R be a hyperring and M be any R -hypermmodule. By $\text{Soc}(M)$ we denote the sum of all simple subhypermmodules of M as in [27]. It follows from [27, Proposition 3] that $\text{Soc}(M)$ is the largest semisimple subhypermmodule of M . By [27, Lemma 8], we define the preradical $\tau : R_s - hmod \rightarrow R_s - hmod$ by $\tau(M) = \text{Soc}(M)$ for all $M \in R_s - hmod$. Again applying [27, Proposition 3], we deduce that τ is the idempotent preradical of $R_s - hmod$.

Example 3.2. Let R be a hyperring and M be any R -hypermmodule. By [27, Theorem 7], we consider the singular subhypermmodule $Z(M)$ of M , that is

$$Z(M) = \{m \in M \mid Im = 0_M \text{ for some essential left hyperideal } I \text{ of } R\}.$$

Here, I is an essential left hyperideal of R if $I \cap J \neq 0$ for every non-zero left hyperideal J of R . For any R -hypermmodules $M, N \in R_s - hmod$, let $f : M \rightarrow N$ be any strong homomorphism and $m \in Z(M)$. Therefore, there exists an essential left hyperideal I of R such that $Im = 0_M$. It follows that $If(m) = f(Im) = 0_N$ and so $f(m) \in Z(N)$. It means that $Z(\cdot)$ defines the preradical in $R_s - hmod$. Since $Z(Z(M)) = Z(M)$ for any hypermodule $M \in R_s - hmod$, the preradical $Z(\cdot)$ is idempotent in $R_s - hmod$.

Example 3.3. Given the hyperring $R = \{0, 1, 2\}$ and the left hyperideal $A = \{0, 2\}$ in Example 2.2, define the preradical $\tau : R_s - hmod \rightarrow R_s - hmod$ by $\tau(N) = AN$ for all $N \in R_s - hmod$, where $AN = \{m \in N \mid m \in \sum_{i=1}^n r_i m_i, m_i \in N, r_i \in A, 1 \leq i \leq n \text{ and } n \in \mathbb{Z}^+\}$. It follows that τ is preradical of $R_s - hmod$. Since $AA = \{0\}$, we deduce that τ is not idempotent preradical.

Definition 3.3. Let τ be a preradical of $R_s - hmod$. Then, τ is called radical if $\tau(\frac{H}{\tau(H)}) = \{0\}$ for every $H \in R_s - hmod$.

Let M be an R -hypermmodule. By $\text{Rad}(M)$ we denote the intersection of all maximal subhypermmodules of M . Now we shall show that $\text{Rad}(\cdot)$ induces a radical of $R_s - hmod$. Firstly we need the following lemma.

Lemma 3.1. Let M be a hypermodule and N be a subhypermmodule of M . Assume that $\{M_i\}_{i \in I}$ is a family of subhypermmodules of M containing N . Then, $\bigcap_{i \in I} \frac{M_i}{N} = \frac{\bigcap_{i \in I} M_i}{N}$.

Proof. The proof is straightforward. $\square$

Example 3.4. Let R be any hyperring. Define $\tau : R_s - hmod \longrightarrow R_s - hmod$ by $\tau(M) = Rad(M)$ for all $M \in R_s - hmod$. It follows from [23] that τ is the preradical of $R_s - hmod$.

Next we show that τ is the radical of $R_s - hmod$. Let M be any hypermodule over the hyperring R and $\{M_i\}_{i \in I}$ be the set of all maximal subhypermodules of M . Since $Rad(M)$ is contained in all maximal hypermodules of M , we get that

$$Rad\left(\frac{M}{Rad(M)}\right) = \bigcap_{i \in I} \frac{M_i}{Rad(M)} = \frac{\bigcap_{i \in I} M_i}{Rad(M)} = 0$$

by Lemma 3.1. Hence, τ is radical.

Now we give the following lemma and we will use it throughout the paper.

Lemma 3.2. *For a hyperring R , let τ be a preradical of $R_s - hmod$. If $T \leq H$ are R -hypermodules, then $\frac{\tau(H)+T}{T} \leq \tau\left(\frac{H}{T}\right)$. Moreover, if τ is radical and $T \leq \tau(H)$, $\tau\left(\frac{H}{T}\right) = \frac{\tau(H)}{T}$.*

Proof. Consider the canonical strong projection $\Psi : H \longrightarrow \frac{H}{T}$. Since τ is a preradical of $R_s - hmod$, we can write $\Psi(\tau(H)) = \frac{\tau(H)+T}{T} \subseteq \tau\left(\frac{H}{T}\right)$ as required.

Let τ be a radical of $R_s - hmod$ and $T \leq \tau(H)$. Therefore, $\frac{\tau(H)+T}{T} = \frac{\tau(H)}{T} \subseteq \tau\left(\frac{H}{T}\right)$. Note that

$$\frac{\frac{H}{T}}{\frac{\tau(H)}{T}} \cong \frac{H}{\tau(H)}.$$

Consider the canonical strong projection $\Phi : \frac{H}{T} \longrightarrow \frac{\frac{H}{T}}{\frac{\tau(H)}{T}}$. It follows that

$$\Phi\left(\tau\left(\frac{H}{T}\right)\right) = \frac{\tau\left(\frac{H}{T}\right) + \frac{\tau(H)}{T}}{\frac{\tau(H)}{T}} \subseteq \tau\left(\frac{\frac{H}{T}}{\frac{\tau(H)}{T}}\right).$$

Since τ is a radical of $R_s - hmod$, $\tau\left(\frac{H}{\tau(H)}\right) = \{0\}$ and so $\tau\left(\frac{\frac{H}{T}}{\frac{\tau(H)}{T}}\right) = \left\{\frac{\tau(H)}{T}\right\}$.

Thus, $\frac{\tau\left(\frac{H}{T}\right) + \frac{\tau(H)}{T}}{\frac{\tau(H)}{T}} \subseteq \frac{\tau(H)}{T}$. It means that $\tau\left(\frac{H}{T}\right) = \frac{\tau(H)}{T}$. $\square$

4. τ -supplemented hypermodules

Recall that an R -module M is τ -supplemented if every submodule U of M has a τ -supplement V in M , that is, $M = U + V$ and $U \cap V \subseteq \tau(V)$, where τ is a preradical of $R - Mod$. Now we transfer this definition to hypermodules as follows:

Definition 4.1. Let R be a hyperring and τ be a preradical of $R_s - hmod$. An R -hypermodule M is called τ -supplemented if every subhypermodule U has a τ -supplement V in M , that is, $M = U + V$ and $U \cap V \subseteq \tau(V)$. M is called amply τ -supplemented if for every subhypermodules K, T of H with $K + T = M$, there exists a τ -supplement V of K with $V \leq T$.

It is clear that every semisimple hypermodule is (amply) τ -supplemented. Observe from Example 2.2 that the R -hypermodule R is a *Soc*-supplemented and *Rad*-supplemented hypermodule which is not semisimple.

Lemma 4.1. Let M be an R -hypermodule and τ be a preradical of $R_s - hmod$. Assume that $M = \tau(M)$. Then, M is τ -supplemented.

Proof. Let $U \leq M$. Therefore, $M = U + M$ and $U \cap M = U \subseteq M = \tau(M)$. It means that M is a τ -supplement of U in M . Hence, M is τ -supplemented. $\square$

Theorem 4.1. Let τ be a radical of $R_s - hmod$ and T be an R -hypermodule. Then, a subhypermodule K of T is a τ -supplement in T if and only if, for every strong homomorphism $g : K \rightarrow L$ of hypermodules with $\tau(L) = 0$, there exists a strong homomorphism $h : T \rightarrow L$ such that $g = h\iota$, where $\iota : K \rightarrow T$ is the inclusion map.

Proof. ($\Rightarrow$) Let K be a τ -supplement of some hypermodule U of T . Therefore, we can write $\frac{T}{\tau(K)} = \frac{K}{\tau(K)} \oplus \frac{U+\tau(K)}{\tau(K)}$. Assume that $g : K \rightarrow L$ is a strong homomorphism of hypermodules, where $\tau(L) = 0$. It follows that there exists a strong homomorphism $h : T \rightarrow \frac{K}{\tau(K)}$ with $h(x) = x + \tau(K)$ for all $x \in T$. Then, $g(\tau(K)) \subseteq \tau(g(K)) \subseteq \tau(L) = 0$ and thus the strong homomorphism $g : K \rightarrow L$ factors through the canonical strong epimorphism $\pi : K \rightarrow \frac{K}{\tau(K)}$ with $g = g'\pi$, where $g' : \frac{K}{\tau(K)} \rightarrow L$. Put $\alpha = g'h$. Hence, $\alpha\iota = g$.

($\Leftarrow$) Since τ is a radical of $R_s - hmod$, we have $\tau(\frac{K}{\tau(K)}) = 0$ and so, by assumption, there exists a strong homomorphism $h : T \rightarrow \frac{K}{\tau(K)}$ with $h\iota = \pi$, where $\pi : K \rightarrow \frac{K}{\tau(K)}$ is the canonical strong homomorphism. It follows that $h(k) = k + \tau(K)$ for all $k \in K$. Thus, h induces a strong homomorphism $f : \frac{T}{\tau(K)} \rightarrow \frac{K}{\tau(K)}$ such that $f(k + \tau(K)) = h(k)$. So, $\frac{K}{\tau(K)}$ is a direct summand of $\frac{T}{\tau(K)}$ and then we can write $\frac{T}{\tau(K)} = \frac{K}{\tau(K)} \oplus \frac{U}{\tau(K)}$ for some subhypermodule $\frac{U}{\tau(K)}$ of $\frac{T}{\tau(K)}$. Hence, $T = K + U$ and $K \cap U \subseteq \tau(K)$. It means that K is a τ -supplement in T . $\square$

Corollary 4.1. Let τ be a radical for $R_s hmod$ and $K \leq T$, where $T \in R_s hmod$.

1. If K is a τ -supplement in T and $\tau(K) = \{0\}$, then K is a direct summand of T .
2. If $\tau(T) = \{0_H\}$, then each τ -supplement subhypermodule of T is a direct summand.

3. If K is a τ -supplement in T and $W \leq K$, then $\frac{K}{W}$ is a τ -supplement in $\frac{T}{W}$.

Proof. (1) Let K be a τ -supplement of L in T . Therefore, $T = L + K$ and $L \cap K \subseteq \tau(K) = 0$ and so $T = L \oplus K$, as required.

(2) It follows from (1).

(3) Let $V \leq T$ satisfying conditions $K + V = T$ and $K \cap V \leq \tau(K)$. Then, we can write $\frac{K}{W} + \frac{V+W}{W} = \frac{T}{W}$ and $\frac{K}{W} \cap \frac{(V+W)}{W} = \frac{(K \cap V) + W}{W} \leq \frac{\tau(K) + W}{W} \leq \tau(\frac{K}{W})$. $\square$

Recall from [12] that a subhypermodule K of H is called *essential* in H if $K \cap T = \{0_H\}$ implies $T = \{0_H\}$ for each nonzero subhypermodule T of H , denoted by $K \trianglelefteq H$.

Theorem 4.2. Let $H \in_{R_S} \text{hmod}$ with a τ -supplemented hypermodule. Then, we have:

1. Each subhypermodule $K \leq H$ with $K \cap \tau(H) = \{0_H\}$ is a direct summand. Especially, if $\tau(H) = \{0_H\}$, then H is semisimple.
2. Each factor hypermodule and each direct summand of H is τ -supplemented.
3. $\frac{H}{\tau(H)}$ is a semisimple hypermodule.
4. $H = V \oplus T$ where T is semisimple and $\tau(V) \trianglelefteq V$.

Proof. (1) Since $\tau(K) \leq K \cap \tau(H)$, it follows from [27, Theorem 9] and Corollary 4.1-(1).

(2) and (3) Clear by Corollary 4.1.

(4) Let $T \cap \tau(H) = \{0_H\}$ and $T \oplus \tau(H) \trianglelefteq H$. It follows that $\tau(T) = \{0_H\}$. Since H is τ -supplemented module, there is a subhypermodule V of H such that $T + V = H$ and $T \cap V \leq \tau(V)$. Then, $T \cap V = T \cap (T \cap V) \leq T \cap \tau(V) \leq T \cap \tau(H) = \{0_H\}$, $H = T \oplus V$ and $\tau(H) = \tau(T) \oplus \tau(V) = \tau(V)$. Therefore, $T \oplus \tau(V) \trianglelefteq T \oplus V$. It follows from $\tau(V) \trianglelefteq V$. By (1), T is semisimple. $\square$

Theorem 4.3. Let $H \in_{R_S} \text{hmod}$. Then, the following statements hold.

1. Let $K, V \leq H$, where K is τ -supplemented. If $K + V$ has a τ -supplement in H , then V has a τ -supplement in H .
2. If K and T are τ -supplemented, then $K + T$ is τ -supplemented.
3. Any finite sum of τ -supplemented hypermodules is τ -supplemented.

Proof. (1) By hypothesis, there is $T \leq H$ provided that $(K + V) + T = H$ and $(K + V) \cap T \leq \tau(T)$. Since $(V + T) \cap K$ has a τ -supplement in K , we have $(V + T) \cap K + W = K$ and $(V + T) \cap W \leq \tau(W)$ for $W \leq K$. As $V + T + W = H$, W is a τ -supplement of $V + T$ in H . To show that $T + W$ is a τ -supplement of V in H , we must prove that $V \cap (T + W) \leq \tau(T + W)$. Note that $W + V \leq K + V$. So $T \cap (W + V) \leq T \cap (K + V) \leq \tau(T)$.

- (2) Let $V \leq K + T$. By hypothesis, $K + T + V$ has a trivial τ -supplement in $K + T$ and by using (1), $T + V$ has a τ -supplement in $K + T$. It follows from again (1) that V has a τ -supplement in $K + T$. Hence, $K + T$ is τ -supplemented.
- (3) It is obtained directly by (1). $\square$

Theorem 4.4. *Let $H \in_{R_S} hmod$. If H is an amply τ -supplemented hypermodule, then following statements hold.*

1. *Direct summands of H are amply τ -supplemented.*
2. *Factor hypermodules of H are amply τ -supplemented.*

Proof. (1) Let $H = K \oplus T$ and let V, W be subhypermodules of T with $T = V + W$. Since $H = K + V + W$, there is a subhypermodule Z of W such that $Z + K + V = H$ and $Z \cap (K + V) \leq \tau(Z)$. It follows that $Z \cap V \leq Z \cap (V + K) \leq \tau(Z)$ and $T = T \cap H = T \cap (Z + (K \oplus V)) = T \cap (K \oplus V) + Z = V + Z$. Therefore, Z is a τ -supplement of V in T and $Z \leq W$.

(2) Let $V \leq H$ and $\frac{H}{V} = \frac{K}{V} + \frac{T}{V}$ with $V \leq K \leq H$, $V \leq T \leq H$. It follows from $H = K + T$ that there is a subhypermodule $W \leq T \leq H$. It follows from $H = K + T$ that there is a subhypermodule $W \leq T$ provided that $K + W = H$ and $K \cap W \leq \tau(W)$. We must prove that $\frac{W+V}{V}$ is a τ -supplement of $\frac{K}{V}$ in $\frac{H}{V}$. It is clear that $\frac{W+V}{V} + \frac{K}{V} = \frac{H}{V}$ and so $\frac{K}{V} \cap \frac{W+V}{V} = \frac{(K \cap W) + V}{V} \leq \frac{\tau(W) + V}{V} \leq \tau(\frac{W+V}{V})$. $\square$

The following corollary follows from directly using Theorem 4.4.

Corollary 4.2. *Let $H \in_{R_S} hmod$. If H is amply τ -supplemented, then following statements hold.*

1. *If K is a τ -supplement in H and $\tau(K) = \{0_K\}$, then K is amply τ -supplemented.*
2. *If $\tau(H) = \{0\}$, then each τ -supplement subhypermodule of H is amply τ -supplemented.*

Recall from [22] that a hypermodule H is called *normal π -projective* if for each (K, T) of subhypermodules of H providing $K + T = H$, there is a strong homomorphism $\gamma : H \rightarrow H$ with $Im(\gamma) \leq K$ and $Im(1 - \gamma) \leq T$, where 1 denotes the identity strong homomorphism of H .

Proposition 4.1. *Let $H \in_{R_S} hmod$. Then, following statements hold.*

1. *If each subhypermodule of H is τ -supplemented, then H is an amply τ -supplemented hypermodule.*
2. *If H is normal π -projective and τ -supplemented, then H is amply τ -supplemented.*

Proof. (1) Let $K, T \leq H$ with $H = K + T$. As K is τ -supplemented, there is $W \leq K$ with $(K \cap T) + W = K$ and $K \cap T \cap W \leq \tau(W)$. It follows that $T \cap W = K \cap T \cap W \leq \tau(W)$ and $H = W + T$.

(2) Let $H = K + T$. By hypothesis there exists a strong endomorphism γ of H with $Im(\gamma) \leq K$ and $Im(1 - \gamma) \leq T$. As H is τ -supplemented, there is a subhypermodule $V \leq H$ with $V + K = H$ and $V \cap K \leq \tau(V)$. Therefore, $H = Im(\gamma) + Im(1 - \gamma) \leq K + (1 - \gamma)(K + V) \leq K + (1 - \gamma)(V)$. So $H = K + (1 - \gamma)(V)$. It can be shown that $(1 - \gamma)(V) \leq T$ and $K \cap (1 - \gamma)(V) = (1 - \gamma)(K \cap V)$. Since $K \cap V \leq \tau(V)$, we have $K \cap (1 - \gamma)(V) \leq \tau((1 - \gamma)(V))$. Thus, $(1 - \gamma)(V)$ is a τ -supplement of K in H and $(1 - \gamma)(V) \leq T$. Hence, H is amply τ -supplemented. $\square$

Finally, we give examples of a $\tau = Rad$ -supplemented Krasner hypermodule and an amply $\tau = Rad$ -supplemented Krasner hypermodule in ${}_S hmod$ without completing the article in the following.

Example 4.1. 1. Consider $\mathbb{Q}$ is quotient field of $\mathbb{Z}$. Then, we take H as a Krasner $\mathbb{Z}$ -hypermodule $\mathbb{Q}$. It can be seen clearly from [13] and [26] that the Krasner $\mathbb{Z}$ -hypermodule H is Rad -supplemented but not supplemented.

2. Consider the $\mathbb{Z}$ -module $\mathbb{Z}_{p^\infty} = \{ \frac{m}{p^n} \mid m \in \mathbb{Z}, n \geq 0, 0 \leq \frac{m}{p^n} < 1, p \nmid m \}$ and a submodule $H = \langle \frac{1}{p} + \mathbb{Z} \rangle$. By [12, Example 2.5], it can be constructed a $\mathbb{Z}$ -hypermodule $(\mathbb{Z}_{p^\infty}, \oplus, \odot)$. Since $\tau(\mathbb{Z}_{p^\infty}) = Rad(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$ and each subhypermodule of $\mathbb{Z}_{p^\infty}$ is the form $\langle \frac{1}{p^m} + \mathbb{Z} \rangle$ where $m \geq 1$, $\mathbb{Z}_{p^\infty}$ is amply τ -supplemented, but not local.

5. Conclusions

In essence of this study, the properties of the concept of τ -supplemented module, one of the most basic topics of module theory, in the structure of subcategory hypermodules were investigated. We first characterized the τ -supplemented hypermodules by generalizing the supplemented hypermodules with the help of preradical, which we defined in the subcategory ${}_R hmod$. The basic properties of τ -supplemented hypermodules have been provided. The connection between the notion of τ -supplemented hypermodules and the notion of supplemented hypermodules can be obtained with the help of the basic algebraic properties provided by the concept of τ -supplement subhypermodule. In particular, we have shown that the class of amply τ -supplemented is closed under direct summands, factor hypermodules and finite sums. Also, we prove that every subhypermodule τ -supplemented hypermodule is τ -supplemented.

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