Non-existence of integer solutions for the Diophantine equation $p^x + p^y + n^z = w^2$, where p is an odd prime number and n is a positive integer

Suton Tadee
Department of Mathematics
Faculty of Science and Technology
Thepsatri Rajabhat University
Lopburi, 15000
Thailand
suton.t@lawasri.tru.ac.th

Apirat Siraworakun*
Department of Mathematics
Faculty of Science and Technology
Thepsatri Rajabhat University
Lopburi, 15000
Thailand
apirat.si@lawasri.tru.ac.th

Abstract. In this research, we investigate some conditions for the non-existence of integer solutions of the Diophantine equation $p^x + p^y + n^z = w^2$, where p is an odd prime number and n is a positive integer. Moreover, numerous examples to illustrate these cases are provided.

Keywords: Diophantine equation, Legendre symbol, congruence.

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1. Introduction

One of the famous Diophantine equations is the exponential Diophantine equation $a^x + b^y = w^2$, where a and b are positive integers. Many authors investigated the non-negative integer solutions of the equation, where a and b are specified as positive integers (c.f. [1], [3] and [15]). Positive integer a or b is studied as variable under certain conditions in various manuscripts. In [5], [11], [14] and [17], either a or b is a fixed number and in [4], [7], [8] [9], [10] both a and b are variables that satisfy some conditions. Moreover, the non-existence of positive integer solutions to the Diophantine equation is studied; see [16].

The Diophantine equation $a^x + b^y + c^z = w^2$, where a, b and c are positive integers, was constructed and studied. In [2], Bacani and Rabago gave all non-negative integer solutions of the equation $3^x + 5^y + 7^z = w^2$. Similarly, a, b and c are considered in other papers as variables satisfying some conditions.

*. Corresponding author

In [6], all integer solutions of the equation $p^x + (p+1)^y + (p+2)^z = w^2$ are provided, where p is a prime number and $1 \leq x, y, z \leq 2$. Recently, Pandichelvi and Sandhya [12] showed integer solutions of the equation $p_1^x + p_2^y + p_3^z = M^2$, where p_1, p_2, p_3 are prime numbers and $x, y, z \in \{1, 2\}$.

In this research, the Diophantine equation

$$p^x + p^y + n^z = w^2 \tag{1}$$

is studied, where p is an odd prime number and n is a positive integer. We found some conditions for the contradiction of the existence of solutions of (1) by modulo 4, p, $p-1$ and $p+1$. Thus, many Diophantine equations of the form (1), which have no solutions, are demonstrated.

2. Non-existence of solutions by modulo 4

In this section, we investigate conditions on n modulo 4 for the non-existence of non-negative integer solutions to (1). First, we characterize the conditions for $n^z \equiv 0, 1, 2, 3 \pmod 4$.

Lemma 2.1. *Let n be a positive integer and z be a non-negative integer. Then,*

1. *$n^z \equiv 0 \pmod 4$ if and only if $n \equiv 0, 2 \pmod 4$, where $z \geq 2$;*
2. *$n^z \equiv 1 \pmod 4$ if and only if $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ and z is even, where $z \geq 1$;*
3. *$n^z \equiv 2 \pmod 4$ if and only if $n \equiv 2 \pmod 4$ and $z = 1$;*
4. *$n^z \equiv 3 \pmod 4$ if and only if $n \equiv 3 \pmod 4$ and z is odd.*

Proof. 1. Let $z \geq 2$. First, assume that $n^z \equiv 0 \pmod 4$. Suppose that $n \equiv 1, 3 \pmod 4$. Then, $n^z \equiv 1, 3 \pmod 4$, a contradiction. Thus, $n \equiv 0, 2 \pmod 4$. Conversely, it is easy to show that $n^z \equiv 0 \pmod 4$ if $n \equiv 0, 2 \pmod 4$ and $z \geq 2$.

2. Let $z \geq 1$. First, assume that $n^z \equiv 1 \pmod 4$. Then, $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$. If $n \equiv 3 \pmod 4$, then $n^z \equiv (-1)^z \pmod 4$ so that z is even. Therefore, $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$ and z is even. The converse is obtained obviously.

3. Assume that $n^z \equiv 2 \pmod 4$. Then, $n \equiv 2 \pmod 4$. Suppose that $z = 0$ or $z > 1$. Thus, $n^z \equiv 1 \pmod 4$ or $n^z \equiv 0 \pmod 4$, respectively. This contradicts to the assumption. Therefore, $n \equiv 2 \pmod 4$ and $z = 1$. It is obvious for the converse.

4. Assume that $n^z \equiv 3 \pmod 4$. Then, $n \equiv 3 \pmod 4$. Hence, $n^z \equiv (-1)^z \pmod 4$ so that z is odd. Therefore, $n \equiv 3 \pmod 4$ and z is odd. The converse is obtained obviously. $\square$

The following lemma gives the conditions that lead to $w^2 \equiv 2, 3 \pmod 4$ and contradict the existence of non-negative integer solutions of (1).

Lemma 2.2. *Let n be a positive integer. Then,* (1) *has no non-negative integer solution if*

1. $p \equiv 1 \pmod 4$ *and* $n^z \equiv 0, 1 \pmod 4$ *or*
2. $p \equiv 3 \pmod 4$, *x, y have same parity and* $n^z \equiv 0, 1 \pmod 4$ *or*
3. $p \equiv 3 \pmod 4$, *x, y have opposite parity and* $n^z \equiv 2, 3 \pmod 4$.

Proof. 1. Assume that $p \equiv 1 \pmod 4$ and $n^z \equiv 0, 1 \pmod 4$. Then, $p^x \equiv 1 \pmod 4$ and $p^y \equiv 1 \pmod 4$. Thus, $p^x + p^y + n^z \equiv 2, 3 \pmod 4$ so that $w^2 \equiv 2, 3 \pmod 4$. Therefore, (1) has no non-negative integer solution.

2. Assume that $p \equiv 3 \pmod 4$, x, y have same parity and $n^z \equiv 0, 1 \pmod 4$. Then, $p^x \equiv 1 \pmod 4$ and $p^y \equiv 1 \pmod 4$, where x, y are even or $p^x \equiv 3 \pmod 4$ and $p^y \equiv 3 \pmod 4$, where x, y are odd. Hence, $w^2 = p^x + p^y + n^z \equiv 2, 3 \pmod 4$. Therefore, (1) has no non-negative integer solution.

3. Assume that $p \equiv 3 \pmod 4$, x, y have opposite parity and $n^z \equiv 2, 3 \pmod 4$. Then, $p^x \equiv 1 \pmod 4$ and $p^y \equiv 3 \pmod 4$, where x is even and y is odd or $p^x \equiv 3 \pmod 4$ and $p^y \equiv 1 \pmod 4$, where x is odd and y is even. Hence, $w^2 = p^x + p^y + n^z \equiv 2, 3 \pmod 4$. Therefore, (1) has no non-negative integer solution. □

For $p \equiv 1 \pmod 4$, it is easy to show that (1) has no non-negative integer solution, where $z = 0$. Next, we study the case $z \geq 2$.

Theorem 2.1. *Let* $p \equiv 1 \pmod 4$ *and* $z \geq 2$. *Then,* (1) *has no non-negative integer solution if*

1. $n \equiv 0, 1, 2 \pmod 4$ *or*
2. $n \equiv 3 \pmod 4$ *and z is even.*

Proof. 1. Assune that $n \equiv 0, 1, 2 \pmod 4$. By Lemma 2.1 (1) and (2), we obtain that $n^z \equiv 0, 1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (1).

2. Assume that $n \equiv 3 \pmod 4$ and z is even. By Lemma 2.1 (2), we obtain that $n^z \equiv 1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (1). □

Corollary 2.1. *The Diophantine equation* $p^x + p^y + n^{2z+2} = w^2$ *has no non-negative integer solution, where* $p \equiv 1 \pmod 4$.

Proof. It is obvious that $2z + 2$ is even and $2z + 2 \geq 2$ for all non-negative integer z. By Theorem 2.1 (1) and (2), $p^x + p^y + n^{2z+2} = w^2$ has no non-negative integer solution. □

Furthermore, some conditions of n that (1) has no non-negative integer solution are investigated where $z \geq 0$.

Corollary 2.2. *The Diophantine equation* $p^x+p^y+n^z = w^2$ *has no non-negative integer solution, where* $p \equiv 1 \pmod 4$ *and* $n \equiv 0, 1 \pmod 4$*.*

Proof. Since $n \equiv 0, 1 \pmod 4$, we can conclude that $n^z \equiv 0, 1 \pmod 4$. By Lemma 2.2 (1), $p^x + p^y + n^z = w^2$ has no non-negative integer solution. □

Similarly, we obtain the non-existence of non-negative integer solutions for $p{\equiv}3 \pmod 4$ and x, y have same parity by Lemma 2.1 (1), (2) and Lemma 2.2 (2).

Theorem 2.2. *Let* $p \equiv 3 \pmod 4$*,* $z \geq 2$ *and* x, y *have same parity. Then,* (1) *has no non-negative integer solution if*

1. $n \equiv 0, 1, 2 \pmod 4$ *or*
2. $n \equiv 3 \pmod 4$ *and* z *is even.*

Proof. 1. Assune that $n \equiv 0, 1, 2 \pmod 4$. By Lemma 2.1 (1) and (2), we obtain that $n^z \equiv 0, 1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (2).

2. Assume that $n{\equiv}3 \pmod 4$ and z is even. By Lemma 2.1 (2), we obtain that $n^z{\equiv}1 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (2). □

Corollary 2.3. *The Diophantine equation* $p^{2x} + p^{2y} + n^{2z+2} = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$*.*

Proof. It is obvious that $2x$, $2y$ and $2z + 2$ are even. Then, $2x, 2y$ have same parity and $2z + 2 \geq 2$ for all non-negative integer z. By Theorem 2.2 (1) and (2), $p^{2x} + p^{2y} + n^{2z+2} = w^2$ has no non-negative integer solution. □

Corollary 2.4. *The Diophantine equation* $p^{2x+1} + p^{2y+1} + n^{2z+2} = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$*.*

Proof. It is clear that $2x + 1$, $2y + 1$ are odd and $2z + 2$ is even. Then, $2x + 1, 2y + 1$ have same parity and $2z + 2 \geq 2$ for all non-negative integer z. By Theorem 2.2 (1) and (2), $p^{2x+1} + p^{2y+1} + n^{2z+2} = w^2$ has no non-negative integer solution. □

Next, we can confirm the non-existence of non-negative integer solution of (1) where x, y have same parity and $z \geq 0$.

Corollary 2.5. *The Diophantine equation* $p^{2x} + p^{2y} + n^z = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$ *and* $n \equiv 0, 1 \pmod 4$*.*

Proof. It is obvious that $2x$, $2y$ have same parity and $n^z \equiv 0, 1 \pmod 4$. By Lemma 2.2 (2), $p^{2x} + p^{2y} + n^z = w^2$ has no non-negative integer solution. □

Corollary 2.6. *The Diophantine equation* $p^{2x+1} + p^{2y+1} + n^z = w^2$ *has no non-negative integer solution, where* $p \equiv 3 \pmod 4$ *and* $n \equiv 0, 1 \pmod 4$*.*

Proof. It is obvious that $2x+1$, $2y+1$ have same parity and $n^z \equiv 0, 1 \pmod 4$. By Lemma 2.2 (2), $p^{2x+1}+p^{2y+1}+n^z = w^2$ has no non-negative integer solution. $\square$

By Lemma 2.1 (3), (4) and Lemma 2.2 (3), we obtain the following theorem.

Theorem 2.3. *Let $p \equiv 3 \pmod 4$ and x, y have opposite parity. Then,* (1) *has no non-negative integer solution if*

1. *$n \equiv 2 \pmod 4$ and $z = 1$ or*
2. *$n \equiv 3 \pmod 4$ and z is odd.*

Proof. 1. Assune that $n \equiv 2 \pmod 4$ and $z = 1$. By Lemma 2.1 (3), we obtain that $n^z \equiv 2 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (3).

2. Assume that $n \equiv 3 \pmod 4$ and z is odd. By Lemma 2.1 (4), we obtain that $n^z \equiv 3 \pmod 4$. Thus, (1) has no non-negative integer solution by Lemma 2.2 (3). $\square$

Corollary 2.7. *The Diophantine equation $p^{2x+1} + p^{2y} + n^{2z+1} = w^2$ has no non-negative integer solution, where $p \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$.*

Proof. It is obvious that $2x + 1$, $2y$ have opposite parity and $2z + 1$ is odd. By Theorem 2.3 (2), $p^{2x+1} + p^{2y} + n^{2z+1} = w^2$ has no non-negative integer solution. $\square$

Corollary 2.8. *The Diophantine equation $p^{2x+1} + p^{2y} + n = w^2$ has no non-negative integer solution, where $p \equiv 3 \pmod 4$ and $n \equiv 2 \pmod 4$.*

Proof. It is obvious that $2x + 1$, $2y$ have opposite parity. By Theorem 2.3 (1), $p^{2x+1} + p^{2y} + n = w^2$ has no non-negative integer solution. $\square$

3. Non-existence of solutions by modulo p

In this section, we confirm that (1) has no positive integer solution by modulo p and Legendre symbol $\left(\frac{n}{p}\right)$. Moreover, we gather some forms of an odd prime number p in Legendre symbol $\left(\frac{q}{p}\right)$, $\left(\frac{2q}{p}\right)$ and $\left(\frac{p_1 p_2}{p}\right)$, where q, p_1 and p_2 are distinct prime numbers.

Theorem 3.1. *Let x and y be positive integers and z be an odd positive integer. If $\left(\frac{n}{p}\right) = -1$, then* (1) *has no positive integer solution.*

Proof. Assume that (1) has a positive integer solution. Since x and y are positive integers, we have $w^2 \equiv n^z \pmod p$. Then, $\left(\frac{n^z}{p}\right) = 1$. Since z is odd, we can conclude that $\left(\frac{n}{p}\right) = 1$. $\square$

By the above theorem, we found that an odd prime number p with $\left(\frac{n}{p}\right) = -1$ has an important role non-existence of positive integer solutions of (1). In [16], Tadee and Siraworakun investigated the forms of an odd prime number p in Legendre symbols $\left(\frac{q}{p}\right)$ and $\left(\frac{2q}{p}\right)$, where q is an odd prime number.

Theorem 3.2 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 1 \pmod 4$. Then,*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q + r^{S_1}q + r^{S_1} \pmod{2q} \\ -1 & \text{if } p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$ and r is a primitive root modulo q.

Theorem 3.3 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 3 \pmod 4$. Then,*

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 3q + 4n_0 r^{S_1} \pmod{4q} \text{ or} \\ & \quad\ p \equiv -3q + 4n_0 r^{S_2} \pmod{4q} \\ -1 & \text{if } p \equiv 3q + 4n_0 r^{S_2} \pmod{4q} \text{ or} \\ & \quad\ p \equiv -3q + 4n_0 r^{S_1} \pmod{4q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$, r is a primitive root modulo q and $n_0 = \frac{q+1}{4}$.

Theorem 3.4 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 1 \pmod 4$. Then,*

$$\left(\frac{2q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv q^2 + 8n_1 r^{S_1} \pmod{8q} \text{ or} \\ & \quad\ p \equiv -q^2 + 8n_1 r^{S_1} \pmod{8q} \text{ or} \\ & \quad\ p \equiv 3q^2 + 8n_1 r^{S_2} \pmod{8q} \text{ or} \\ & \quad\ p \equiv -3q^2 + 8n_1 r^{S_2} \pmod{8q} \\ -1 & \text{if } p \equiv q^2 + 8n_1 r^{S_2} \pmod{8q} \text{ or} \\ & \quad\ p \equiv -q^2 + 8n_1 r^{S_2} \pmod{8q} \text{ or} \\ & \quad\ p \equiv 3q^2 + 8n_1 r^{S_1} \pmod{8q} \text{ or} \\ & \quad\ p \equiv -3q^2 + 8n_1 r^{S_1} \pmod{8q} \end{cases},$$

where $S_1 \in \{2, 4, 6, \ldots, q-1\}$, $S_2 \in \{1, 3, 5, \ldots, q-2\}$, r is a primitive root modulo q and if $\frac{q-1}{4}$ is an even number, then $n_1 = \frac{-q+1}{8}$, and if otherwise, then $n_1 = \frac{3q+1}{8}$.

Theorem 3.5 ([16]). *Let p and q be distinct odd prime numbers with $q \equiv 3 \pmod 4$. Then,*

$$\left(\frac{2q}{p}\right)=\begin{cases}1 & \text{if } p\equiv q^2+32n_0n_1r^{S_1} \pmod{8q} \text{ or}\\ & \phantom{\text{if }} p\equiv -q^2+32n_0n_1r^{S_2} \pmod{8q} \text{ or}\\ & \phantom{\text{if }} p\equiv 3q^2+32n_0n_1r^{S_1} \pmod{8q} \text{ or}\\ & \phantom{\text{if }} p\equiv -3q^2+32n_0n_1r^{S_2} \pmod{8q}\\ -1 & \text{if } p\equiv q^2+32n_0n_1r^{S_2} \pmod{8q} \text{ or}\\ & \phantom{\text{if }} p\equiv -q^2+32n_0n_1r^{S_1} \pmod{8q} \text{ or}\\ & \phantom{\text{if }} p\equiv 3q^2+32n_0n_1r^{S_2} \pmod{8q} \text{ or}\\ & \phantom{\text{if }} p\equiv -3q^2+32n_0n_1r^{S_1} \pmod{8q}\end{cases},$$

where $S_1\in\{2,4,6,\dots,q-1\}$*,* $S_2\in\{1,3,5,\dots,q-2\}$*,* r *is a primitive root modulo* q*,* $n_0=\frac{q+1}{4}$ *and if* $\frac{q-3}{4}$ *is an even number, then* $n_1=\frac{5q+1}{8}$*, and if otherwise, then* $n_1=\frac{q+1}{8}$*.*

Moreover, Siraworakun, Wannaphan and Seesod gave forms of an odd prime number p in Legendre symbol $\left(\frac{p_1p_2}{p}\right)$, where p_1 and p_2 are distinct prime numbers in [13].

Theorem 3.6 ([13])**.** *Let* p_1,p_2 *and* p *be distinct odd prime numbers with* $p_1\equiv 1 \pmod 4$ *and* $p_2\equiv 1 \pmod 4$*. Then,*

$$\left(\frac{p_1p_2}{p}\right)=\begin{cases}1 & \text{if } p\equiv p_1p_2+2(n_1r_1^{S_1}p_2+n_2r_2^{S_2}p_1) \pmod{2p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv p_1p_2+2(n_1r_1^{T_1}p_2+n_2r_2^{T_2}p_1) \pmod{2p_1p_2}\\ -1 & \text{if } p\equiv p_1p_2+2(n_1r_1^{S_1}p_2+n_2r_2^{T_2}p_1) \pmod{2p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv p_1p_2+2(n_1r_1^{T_1}p_2+n_2r_2^{S_2}p_1) \pmod{2p_1p_2}\end{cases},$$

where

$$S_1\in\{2,4,6,\dots,p_1-1\}, S_2\in\{2,4,6,\dots,p_2-1\},$$
$$T_1\in\{1,3,5,\dots,p_1-2\}, T_2\in\{1,3,5,\dots,p_2-2\}$$

and r_1,r_2 *are primitive roots modulo* p_1 *and* p_2*, respectively and* n_1,n_2 *are integers with* $2p_2n_1\equiv 1 \pmod{p_1}$ *and* $2p_1n_2\equiv 1 \pmod{p_2}$*.*

Theorem 3.7 ([13])**.** *Let* p_1,p_2 *and* p *be distinct odd prime numbers with* $p_1\equiv 1 \pmod 4$ *and* $p_2\equiv 3 \pmod 4$*. Then,*

$$\left(\frac{p_1p_2}{p}\right)=\begin{cases}1 & \text{if } p\equiv -p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv -p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2}\\ -1 & \text{if } p\equiv -p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv -p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or}\\ & \phantom{\text{if }} p\equiv p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2}\end{cases},$$

where

$$S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\},$$
$$T_1 \in \{1,3,5,\dots,p_1-2\}, T_2 \in \{1,3,5,\dots,p_2-2\}$$

and r_1, r_2 are primitive roots modulo p_1 and p_2, respectively and n_0, n_3, n_4 are integers with $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$.

Theorem 3.8 ([13])**.** *Let p_1, p_2 and p be distinct odd prime numbers with $p_1 \equiv 3 \pmod 4$ and $p_2 \equiv 3 \pmod 4$. Then,*

$$\left(\frac{p_1p_2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \\ -1 & \text{if } p \equiv p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1) \pmod{4p_1p_2} \text{ or} \\ & \quad p \equiv -p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2} \end{cases},$$

where

$$S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\},$$
$$T_1 \in \{1,3,5,\dots,p_1-2\}, T_2 \in \{1,3,5,\dots,p_2-2\}$$

and r_1, r_2 are primitive roots modulo p_1 and p_2, respectively and m_0, n_0, n_3, n_4 are integers with $m_0 = \frac{p_1+1}{4}$, $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$.

Now, we combine Theorem 3.1 with Theorem 3.2-3.8 to prove Theorem 3.9 - 3.15. In addition, many examples of (1), that have no positive integer solution, are demonstrated in Corollary 3.1 - 3.7.

Theorem 3.9. *Let x and y be positive integers and z be an odd positive integer. If p and q are distinct odd prime numbers with the following conditions:*

1. *$q \equiv 1 \pmod 4$ and*
2. *$p \equiv q + r^{S_2}q + r^{S_2} \pmod{2q}$,*

where $S_2 \in \{1,3,5,\dots,q-2\}$ and r is a primitive root modulo q, then the Diophantine equation $p^x + p^y + q^z = w^2$ has no positive integer solution.

Proof. By Theorem 3.2, $\left(\frac{q}{p}\right) = -1$. Thus, $p^x + p^y + q^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.1. *The Diophantine equation $p^x + p^y + 5^{2z+1} = w^2$ has no positive integer solution, where $p \equiv \pm 3 \pmod{10}$.*

Proof. It is obvious that $2z+1$ is odd. Let $q=5$ and $r=2$. Then, $q \equiv 1 \pmod 4$ and r is a primitive root modulo q. Since $p \equiv \pm 3 \pmod{10}$, we have $p \equiv q + r^{S_2} q + r^{S_2} \pmod{2q}$, where $S_2 \in \{1,3,5,\dots,q-2\}$. By Theorem 3.9, $p^x + p^y + 5^{2z+1} = w^2$ has no positive integer solution. □

Theorem 3.10. *Let x and y be positive integers and z be an odd positive integer. If p and q are distinct odd prime numbers with the following conditions:*

1. *$q \equiv 3 \pmod 4$ and*
2. *$p \equiv 3q + 4n_0 r^{S_2},\ -3q + 4n_0 r^{S_1} \pmod{4q}$,*

where $S_1 \in \{2,4,6,\dots,q-1\}$, $S_2 \in \{1,3,5,\dots,q-2\}$, r is a primitive root modulo q and $n_0 = \frac{q+1}{4}$, then the Diophantine equation $p^x + p^y + q^z = w^2$ has no positive integer solution.

Proof. By Theorem 3.3, $\left(\frac{q}{p}\right) = -1$. Thus, $p^x + p^y + q^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.2. *The Diophantine equation $p^x + p^y + 3^{2z+1} = w^2$ has no positive integer solution, where $p \equiv \pm 5 \pmod{12}$.*

Proof. It is obvious that $2z+1$ is odd. Let $q=3$, $r=2$ and $n_0=1$. Then, $q \equiv 3 \pmod 4$, r is a primitive root modulo q and $n_0 = \frac{q+1}{4}$. Since $p \equiv \pm 5 \pmod{12}$, we have $p \equiv 3q + 4n_0 r^{S_2},\ -3q + 4n_0 r^{S_1} \pmod{4q}$, where $S_1 \in \{2,4,6,\dots,q-1\}$ and $S_2 \in \{1,3,5,\dots,q-2\}$. By Theorem 3.10, $p^x + p^y + 3^{2z+1} = w^2$ has no positive integer solution. □

Theorem 3.11. *Let x and y be positive integers and z be an odd positive integer. If p and q are distinct odd prime numbers with the following conditions:*

1. *$q \equiv 1 \pmod 4$ and*
2. *$p \equiv\ q^2 + 8n_1 r^{S_2},\ -q^2 + 8n_1 r^{S_2},\ 3q^2 + 8n_1 r^{S_1},\ -3q^2 + 8n_1 r^{S_1} \pmod{8q}$,*

where $S_1 \in \{2,4,6,\dots,q-1\}$, $S_2 \in \{1,3,5,\dots,q-2\}$, r is a primitive root modulo q and if $\frac{q-1}{4}$ is an even number, then $n_1 = \frac{-q+1}{8}$, and if otherwise, then $n_1 = \frac{3q+1}{8}$, then the Diophantine equation $p^x + p^y + (2q)^z = w^2$ has no positive integer solution.

Proof. By Theorem 3.4, $\left(\frac{2q}{p}\right) = -1$. Thus, $p^x + p^y + (2q)^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.3. *The Diophantine equation $p^x + p^y + 10^{2z+1} = w^2$ has no positive integer solution, where $p \equiv \pm 7, \pm 11, \pm 17, \pm 19 \pmod{40}$.*

Proof. It is obvious that $2z+1$ is odd. Let $q=5$, $r=2$ and $n_1=2$. Then, $q \equiv 1 \pmod 4$, r is a primitive root modulo q and $n_1 = \frac{3q+1}{8}$. Since $p \equiv \pm 7, \pm 11, \pm 17, \pm 19 \pmod{40}$, we have $p \equiv \ q^2 + 8n_1 r^{S_2},\ -q^2 + 8n_1 r^{S_2},\ 3q^2 + 8n_1 r^{S_1},\ -3q^2 + 8n_1 r^{S_1} \pmod{8q}$, where $S_1 \in \{2,4,6,\dots,q-1\}$ and $S_2 \in \{1,3,5,\dots,q-2\}$. By Theorem 3.11, $p^x + p^y + 10^{2z+1} = w^2$ has no positive integer solution. □

Theorem 3.12. *Let x and y be positive integers and z be an odd positive integer. If p and q are distinct odd prime numbers with the following conditions:*

1. $q \equiv 3 \pmod 4$ *and*
2. $p \equiv \ q^2+32n_0n_1r^{S_2},\ -q^2+32n_0n_1r^{S_1},\ 3q^2+32n_0n_1r^{S_2},\ -3q^2+32n_0n_1r^{S_1} \pmod{8q}$,

where $S_1 \in \{2,4,6,\dots,q-1\}$, $S_2 \in \{1,3,5,\dots,q-2\}$, r is a primitive root modulo q, $n_0 = \frac{q+1}{4}$ and if $\frac{q-3}{4}$ is an even number, then $n_1 = \frac{5q+1}{8}$, and if otherwise, then $n_1 = \frac{q+1}{8}$, then the Diophantine equation $p^x + p^y + (2q)^z = w^2$ has no positive integer solution.

Proof. By Theorem 3.5, $\left(\frac{2q}{p}\right) = -1$. Thus, $p^x+p^y+(2q)^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.4. *The Diophantine equation $p^x + p^y + 6^{2z+1} = w^2$ has no positive integer solution, where $p \equiv \pm 7, \pm 11, \pmod{24}$.*

Proof. It is obvious that $2z+1$ is odd. Let $q=3$, $r=2$, $n_0=1$ and $n_1=2$. Then, $q \equiv 3 \pmod 4$, r is a primitive root modulo q, $n_0 = \frac{q+1}{4}$ and $n_1 = \frac{5q+1}{8}$. Since $p \equiv \pm 7, \pm 11, \pmod{24}$, we have $p \equiv \ q^2 + 32n_0n_1r^{S_2},\ -q^2 + 32n_0n_1r^{S_1},\ 3q^2 + 32n_0n_1r^{S_2},\ -3q^2 + 32n_0n_1r^{S_1} \pmod{8q}$, where $S_1 \in \{2,4,6,\dots,q-1\}$ and $S_2 \in \{1,3,5,\dots,q-2\}$. By Theorem 3.12, $p^x+p^y+6^{2z+1} = w^2$ has no positive integer solution. □

Theorem 3.13. *Let x and y be positive integers and z be an odd positive integer. If p_1, p_2 and p are distinct odd prime numbers with the following conditions:*

1. $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 1 \pmod 4$ *and*
2. $p \equiv p_1p_2+2(n_1r_1^{S_1}p_2+n_2r_2^{T_2}p_1),\ p_1p_2+2(n_1r_1^{T_1}p_2+n_2r_2^{S_2}p_1) \pmod{2p_1p_2}$,

where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}$, $T_2 \in \{1,3,5,\dots,p_2-2\}$ and r_1, r_2 are primitive roots modulo p_1 and p_2, respectively and n_1, n_2 are integers with $2p_2n_1 \equiv 1 \pmod{p_1}$ and $2p_1n_2 \equiv 1 \pmod{p_2}$, then the Diophantine equation $p^x + p^y + (p_1p_2)^z = w^2$ has no positive integer solution.

Proof. By Theorem 3.6, $\left(\frac{p_1p_2}{p}\right) = -1$. Thus, $p^x + p^y + (p_1p_2)^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.5. *The Diophantine equation* $p^x+p^y+65^{2z+1}=w^2$ *has no positive integer solution, where* $p \equiv \pm 3, \pm 11, \pm 17, \pm 19, \pm 21, \pm 23, \pm 27, \pm 31, \pm 41, \pm 43, \pm 53, \pm 59 \pmod{130}$.

Proof. It is obvious that $2z+1$ is odd. Let $p_1=5$, $p_2=13$, $r_1=2$, $r_2=2$, $n_1=1$ and $n_2=4$. Then, $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 1 \pmod 4$, r_1, r_2 are primitive roots modulo p_1 and p_2, respectively, $2p_2n_1 \equiv 1 \pmod{p_1}$ and $2p_1n_2 \equiv 1 \pmod{p_2}$. Since $p \equiv \pm 3, \pm 11, \pm 17, \pm 19, \pm 21, \pm 23, \pm 27, \pm 31, \pm 41, \pm 43, \pm 53, \pm 59 \pmod{130}$, we have $p \equiv p_1p_2+2(n_1r_1^{S_1}p_2+n_2r_2^{T_2}p_1),\ p_1p_2+2(n_1r_1^{T_1}p_2+n_2r_2^{S_2}p_1) \pmod{2p_1p_2}$, where $S_1 \in \{2,4,6,\ldots,p_1-1\}, S_2 \in \{2,4,6,\ldots,p_2-1\}$, $T_1 \in \{1,3,5,\ldots,p_1-2\}$ and $T_2 \in \{1,3,5,\ldots,p_2-2\}$. By Theorem 3.13, $p^x+p^y+65^{2z+1}=w^2$ has no positive integer solution. $\square$

Theorem 3.14. *Let x and y be positive integers and z be an odd positive integer. If p_1, p_2 and p are distinct odd prime numbers with the following conditions:*

1. $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 3 \pmod 4$ *and*
2. $p \equiv -p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{S_2}p_1),\ p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{T_2}p_1),$
   $-p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{T_2}p_1),\ p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{S_2}p_1)$
   $\pmod{4p_1p_2}$,

where $S_1 \in \{2,4,6,\ldots,p_1-1\}, S_2 \in \{2,4,6,\ldots,p_2-1\}$, $T_1 \in \{1,3,5,\ldots,p_1-2\}, T_2 \in \{1,3,5,\ldots,p_2-2\}$ and r_1, r_2 are primitive roots modulo p_1 and p_2, respectively and and n_0, n_3, n_4 are integers with $n_0=\frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$, then the Diophantine equation $p^x+p^y+(p_1p_2)^z=w^2$ has no positive integer solution.

Proof. By Theorem 3.7, $\left(\frac{p_1p_2}{p}\right)=-1$. Thus, $p^x+p^y+(p_1p_2)^z=w^2$ has no positive integer solution by Theorem 3.1. $\square$

Corollary 3.6. *The Diophantine equation* $p^x+p^y+15^{2z+1}=w^2$ *has no positive integer solution, where* $p \equiv \pm 13, \pm 19, \pm 23, \pm 29 \pmod{60}$.

Proof. It is obvious that $2z+1$ is odd. Let $p_1=5$, $p_2=3$, $r_1=2$, $r_2=2$, $n_0=1$, $n_3=3$ and $n_4=2$. Then, $p_1 \equiv 1 \pmod 4$, $p_2 \equiv 3 \pmod 4$, r_1, r_2 are primitive roots modulo p_1 and p_2, respectively, $n_0=\frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$. Since $p \equiv \pm 13, \pm 19, \pm 23, \pm 29 \pmod{60}$, we have $p \equiv -p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{S_2}p_1),\ p_1p_2+4(n_3r_1^{T_1}p_2+4n_0n_4r_2^{T_2}p_1),\ -p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{T_2}p_1),\ p_1p_2+4(n_3r_1^{S_1}p_2+4n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2}$, where $S_1 \in \{2,4,6,\ldots,p_1-1\}, S_2 \in \{2,4,6,\ldots,p_2-1\}$, $T_1 \in \{1,3,5,\ldots,p_1-2\}$, and $T_2 \in \{1,3,5,\ldots,p_2-2\}$. By Theorem 3.14, $p^x+p^y+15^{2z+1}=w^2$ has no positive integer solution. $\square$

Theorem 3.15. *Let x and y be positive integers and z be an odd positive integer. If p_1, p_2 and p are distinct odd prime numbers with the following conditions:*

1. $p_1 \equiv 3 \pmod 4$, $p_2 \equiv 3 \pmod 4$ *and*
2. $p \equiv p_1p_2+16(m_0n_3r_1^{T_1}p_2+n_0n_4r_2^{S_2}p_1),\ -p_1p_2+16(m_0n_3r_1^{S_1}p_2+n_0n_4r_2^{T_2}p_1),\ p_1p_2+16(m_0n_3r_1^{S_1}p_2+n_0n_4r_2^{T_2}p_1),\ -p_1p_2+16(m_0n_3r_1^{T_1}p_2+n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2}$,

where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}, T_2 \in \{1,3,5,\dots,p_2-2\}$ *and* r_1, r_2 *are primitive roots modulo* p_1 *and* p_2, *respectively and and* m_0, n_0, n_3, n_4 *are integers with* $m_0 = \frac{p_1+1}{4}$, $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ *and* $4p_1n_4 \equiv 1 \pmod{p_2}$, *then the Diophantine equation* $p^x + p^y + (p_1p_2)^z = w^2$ *has no positive integer solution.*

Proof. By Theorem 3.8, $\left(\frac{p_1p_2}{p}\right) = -1$. Thus, $p^x + p^y + (p_1p_2)^z = w^2$ has no positive integer solution by Theorem 3.1. □

Corollary 3.7. *The Diophantine equation* $p^x+p^y+21^{2z+1} = w^2$ *has no positive integer solution, where* $p \equiv \pm11, \pm13, \pm19, \pm23, \pm29, \pm31 \pmod{84}$.

Proof. It is obvious that $2z+1$ is odd. Let $p_1 = 3$, $p_2 = 7$, $r_1 = 2$, $r_2 = 3$, $m_0 = 1$, $n_0 = 2$, $n_3 = 1$ and $n_4 = 3$. Then, $p_1 \equiv 3 \pmod 4$, $p_2 \equiv 3 \pmod 4$, r_1, r_2 are primitive roots modulo p_1 and p_2, respectively, $m_0 = \frac{p_1+1}{4}$, $n_0 = \frac{p_2+1}{4}$, $4p_2n_3 \equiv 1 \pmod{p_1}$ and $4p_1n_4 \equiv 1 \pmod{p_2}$. Since $p \equiv \pm11, \pm13, \pm19, \pm23, \pm29, \pm31 \pmod{84}$, we have

$$p \equiv p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1), \quad -p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1),$$
$$p_1p_2 + 16(m_0n_3r_1^{S_1}p_2 + n_0n_4r_2^{T_2}p_1),$$
$$-p_1p_2 + 16(m_0n_3r_1^{T_1}p_2 + n_0n_4r_2^{S_2}p_1) \pmod{4p_1p_2},$$

where $S_1 \in \{2,4,6,\dots,p_1-1\}, S_2 \in \{2,4,6,\dots,p_2-1\}$, $T_1 \in \{1,3,5,\dots,p_1-2\}$ and $T_2 \in \{1,3,5,\dots,p_2-2\}$ By Theorem 3.15, $p^x + p^y + 21^{2z+1} = w^2$ has no positive integer solution. □

4. Non-existence of solutions by modulo p-1 and p+1

In the last section, we are interested in exploring some conditions of n by modulo $p-1$ and $p+1$ that (1) has no positve integer solution.

Theorem 4.1. *Let* $n \equiv 0 \pmod{p-1}$. *If* $p \equiv 1 \pmod 4$, *then* (1) *has no positive integer solution.*

Proof. Assume that (1) has a positive integer solution. Then, $w^2 \equiv 2 \pmod{p-1}$. There exists an integer k such that $w^2 = (p-1)k+2 = 2((\frac{p-1}{2})k+1)$. Thus, $\frac{p-1}{2}$ is odd. Therefore, $p \equiv 3 \pmod 4$. □

Corollary 4.1. *The Diophantine equation* $p^x + p^y + ((p-1)k)^z = w^2$ *has no positive integer solution, where* $p \equiv 1 \pmod 4$ *and* k *is a positive integer.*

Proof. Since $(p-1)k \equiv 0 \pmod{p-1}$ and $p \equiv 1 \pmod 4$, we can conclude that $p^x + p^y + ((p-1)k)^z = w^2$ has no positive integer solution by Theorem 4.1. $\square$

Theorem 4.2. *Let $n \equiv 0 \pmod{p+1}$ and x, y have the same parity. If $p \equiv 3 \pmod 4$, then (1) has no positive integer solution.*

Proof. Assume that (1) has a positive integer solution. Then, $w^2 = p^x + p^y + n^z \equiv (-1)^x + (-1)^y + 0 \pmod{p+1}$. Since x, y have same parity, we obtain that $w^2 \equiv \pm 2 \pmod{p+1}$. So, there exists an integer k such that $w^2 = (p+1)k \pm 2 = 2((\frac{p+1}{2})k \pm 1)$. Thus, $\frac{p+1}{2}$ is odd. Therefore, $p \equiv 1 \pmod 4$. $\square$

Corollary 4.2. *The Diophantine equation $p^{2x} + p^{2y} + ((p+1)k)^z = w^2$ has no positive integer solution, where $p \equiv 3 \pmod 4$ and k is a positive integer.*

Proof. It is obvious that $2x$, $2y$ have same parity. Since $(p+1)k \equiv 0 \pmod{p+1}$ and $p \equiv 3 \pmod 4$, we can conclude that $p^{2x} + p^{2y} + ((p+1)k)^z = w^2$ has no positive integer solution by Theorem 4.2. $\square$

Corollary 4.3. *The Diophantine equation $p^{2x+1} + p^{2y+1} + ((p+1)k)^z = w^2$ has no positive integer solution, where $p \equiv 3 \pmod 4$ and k is a positive integer.*

Proof. It is obvious that $2x+1$, $2y+1$ have same parity. Since $(p+1)k \equiv 0 \pmod{p+1}$ and $p \equiv 3 \pmod 4$, we can conclude that $p^{2x} + p^{2y} + ((p+1)k)^z = w^2$ has no positive integer solution by Theorem 4.2. $\square$

5. Conclusion

We have studied the Diophantine equations $p^x + p^y + n^z = w^2$, where p is an odd prime number and n is a positive integer. Various conditions are provided to confirm that the Diophantine equations $p^x + p^y + n^z = w^2$ has no non-negative or positive integer solution. Moreover, we obtain numerous examples form all corollaries in this article.

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