Prime-valent one-regular graphs of order $28p$

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Abstract. A graph is *one-regular* if its full automorphism group acts on its arcs regularly. In this paper, we classify connected one-regular graphs of prime valency and order $28p$ for each prime p, and prove that there is only one sporadic graph: the $\mathbb{Z}_7$-cover CQ_7 of the three dimensional hypercube Q_3 with valency 3.

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1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [22, 25] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An s-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X

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is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by $\mathbf{C}_n$ and $\mathbf{K}_n$ the cycle and the complete graph of order n, respectively. Denote by $\mathbf{D}_{2n}$ the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order n, that is, the cycle $\mathbf{C}_n$, which is s-regular with full automorphism group $\mathbf{D}_{2n}$. Let p be a prime. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [5] and [6], respectively. Wang [24] characterized the prime-valent s-transitive graphs of order $4p$. The classification of cubic and pentavalent s-transitive graphs of order $28p$ was given in [15] and [18], respectively. For heptavalent symmetric graphs of order $28p$, their characterizations can be obtained in [19].

For 2-valent case, the graph is a cycle, which is s-regular for any integer s, and for cubic case, s-transitivity always means s-regularity by Miller [9]. However, for the other prime-valent case, this is not true, see for example [11] for pentavalent case and [12] for heptavalent case. Thus, characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graph of order $28p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [17, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then, one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [6], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p, q)$ the unique subgroup of $\mathbb{Z}_p^*$ of order q. Define the graph $G(2p, q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p, q primes. Then, X is isomorphic to $\mathbf{K}_{2p}$ with $q = 2p-1$, $\mathbf{K}_{p,p}$ or $G(2p, q)$ with $q|(p-1)$. Furthermore, if $(p, q) \neq (11, 5)$ then $\mathrm{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p, q) = (11, 5)$ then $\mathrm{Aut}(G(2p, q)) = \mathrm{PGL}(2, 11)$.*

Next, by [24, Theorem 3.1], we have the characterization of prime valent symmetric graphs of order $4p$.

Proposition 2.3. *Let p and q be two primes with $q \geq 5$, and let X be a q-valent symmetric graph of order $4p$. Then, X is isomorphic to $\mathbf{K}_{4p}$ with $q = 4p-1$, $\mathbf{K}_{2p,2p} - 2p\mathbf{K}_2$ with $q = 2p-1$, or the quotient graph is isomorphic to $\mathbf{K}_{p,p}$ with $q = p$ or $\mathbf{K}_{2p}$ with $q = 2p-1$.*

The following proposition is about the prime-valent symmetric graphs of order $14p$ with p a prime, which is deduced from [20, Theorem 1.2].

Proposition 2.4. *Let p and q be two primes. If $q \geq 5$, then there is no q-valent symmetric graph of order $14p$ admitting a solvable arc-transitive automorphism group.*

The following proposition is the famous "N/C-Theorem", see for example [14, Chapter I, Theorem 4.5]).

Proposition 2.5. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

From [8, pp.12-14] and [23, Theorem 2], we can deduce the non-abelian simple groups whose orders have at most four different prime divisors.

Proposition 2.6. *Let p and q be two odd primes, and let G be a non-abelian simple group. If the order $|G|$ divides $2^2{\cdot}7{\cdot}p{\cdot}q$ with $p \geq 3$ and $q > 7$, then G is isomorphic to A_5, $\mathrm{PSL}(2, 13)$. If the order $|G|$ has at most three different prime divisors, then G is called K_3-simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2, 3, p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2, 17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(3, 3)$	$2^4 \cdot 3^3 \cdot 13$
$\mathrm{PSL}(2, 7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSU}(3, 3)$	$2^5 \cdot 3^3 \cdot 7$
$\mathrm{PSL}(2, 8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSU}(4, 2)$	$2^6 \cdot 3^4 \cdot 5$

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $28p$ for each prime p. Let q be a prime. In what follows, we always denote by X a connected q-valent one-regular graph of order $28p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then, the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 28pq$. Clearly, if $q = 2$, then $X \cong \mathbf{C}_{28p}$ with $A \cong \mathbf{D}_{56p}$, which is s-regular for any positive integer s. If $q = 3$, then by [15, Corollary 4.9] and [10, Theorem 5.1], we have that $X \cong CQ_7$, that is, the $\mathbb{Z}_7$-cover of three dimensional hypercube Q_3. If $q = 5$, then by [18, Theorem 3.1] and [21], there exists no pentavalent one-regular graph of order $28p$. Next we deal with the case $q = 7$.

Lemma 3.1. *Let $q = 7$. Then, there exists no heptavalent one-regular graph of order $28p$.*

Proof. Suppose that $p = 2$. Then, $|V(X)| = 56 = 8{\cdot}7$ and $|A| = 8{\cdot}7^2$. By [13, Theorem 3.1], there is no heptavalent symmetric graphs of order 56, and the statement holds for $p = 2$.

Suppose that $p \geq 3$. Then, by [19, Theorem 1.1], we have that $A \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p) \times \mathbb{Z}_2$ or $\mathrm{PGL}(2,p) \times \mathbb{Z}_2$. In particular, A is non-solvable. Note that $|A| = 2^2{\cdot}7^2{\cdot}p$. Thus, A has a non-solvable composition factor H isomorphic to a non-abelian K_3-simple group. By Proposition 2.6, $p = 3$ and $H \cong \mathrm{PSL}(2,7)$. However, $|H| = 2^3{\cdot}3{\cdot}7$, this is contrary to the fact that $|H| \big| 2^2{\cdot}7^2{\cdot}p$. □

Now, we treat with the case $q > 7$. Recall that $|A| = 28pq$, $A_v \cong \mathbb{Z}_q$ and $q > 7$. Let N be a minimal normal subgroup of A. We divide the proof into the following two cases: $p = q$ and $p \neq q$.

Lemma 3.2. *Let $p = q > 7$. Then, there exists no new graph.*

Proof. Suppose that $p = q$. Then, $|A| = 28p^2$ and $A_v \cong \mathbb{Z}_p$ with $p > 7$. Let P be a Sylow p-subgroup of A. Then, $|P| = p^2$. Note that $p = q > 7$. Thus, by Sylow Theorem, the number of Sylow p-subgroups of A is $kp+1 = |A : N_A(P)|$ for some integer k. Since $|A| = 28p^2$, we have that $(kp+1) \div 28$. It is easy to see that either $k = 0$ or $k = 1$ with $p = 13$. If $k = 1$, then P is normal in A. Since $|P| = p^2 > 7^2$, we have that $P_v \cong \mathbb{Z}_p$. However, P acting on $V(X)$ has 28 orbits, and hence P is semiregular by Proposition 2.1. This is contrary to the fact that $P_v \cong \mathbb{Z}_p$. Thus, $k = 1$ and $p = 13$. If A is non-solvable, then a composition factor is isomorphic to a non-abelian simple group and hence this composition factor has order dividing $|A| = 2^2{\cdot}7{\cdot}p^2$. By Proposition 2.6, there is no non-abelian simple K_3-group whose prime divisor does not have 3. It follows that A is solvable with $p = q = 13$ and P is not normal in A.

Let N be a minimal normal subgroup of A. Then, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_7$ or $\mathbb{Z}_p$ because A is solvable and has no normal Sylow p-subgroup by the above paragraph. By Proposition 2.1, N is semiregular and X_N is a q-valent symmetric graph of order $28p/|N|$. Note that there exists no regular graph of odd order and odd valency. Thus, $N \ncong \mathbb{Z}_2^2$.

Suppose that $N \cong \mathbb{Z}_2$. Then, X_N is a A/N-symmetric graph of order $2{\cdot}7{\cdot}13$ and valency 13. Since X_N has valency 13 and order $2{\cdot}7{\cdot}13$, we have that $13 \nmid (7-1)$. Note that A/N is solvable. By Proposition 2.4, there exists no symmetric graph of order 14 admitting a solvable arc-transitive automorphism group, a contradiction.

Suppose that $N \cong \mathbb{Z}_7$. Then, X_N is a A/N-symmetric graph of order $4p$ and valency 13. Since $|A/N| = 4{\cdot}13^2$, by Sylow Theorem we have that the Sylow p-subgroup of A/N is nomal in A/N, and by Proposition 2.5, we can easily deduce that A has a normal Sylow p-subgroup, a contradiction.

Suppose that $N \cong \mathbb{Z}_p$. Then, X_N is a A/N-symmetric graph of order 28. By [7], there are two symmetric graphs of order 28, their full automorphism groups are $S_{14} \times \mathbb{Z}_2$ and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$, respectively. However, these two groups have no subgroup of order $28{\cdot}13$, a contradiction. □

Lemma 3.3. *Let $p \neq q$ and $q > 7$. Then, there exists no new graph.*

Proof. Suppose that $p \neq q$. Then, $|A| = 28pq$ and $A_v \cong \mathbb{Z}_q$ with $p \neq q > 7$. Since $|A| = 28pq$ and $A_v \cong \mathbb{Z}_q$, we have that A_v is a Sylow q-subgroup of A. It follows that the Sylow q-subgroups of A cannot be normal in A. Let N be a minimal normal subgroup of A. Then, N is either a direct product of some isomorphic non-abelian simple groups or an elementary abelian r-group with $r = 2$, 7 or p.

Case 1. Assume that N is non-solvable.

Since $|A| = 2^2{\cdot}3{\cdot}7{\cdot}q$, we have that $|N|\big|2^2{\cdot}3{\cdot}7{\cdot}q$. By Proposition 2.6, we have that $N \cong \mathrm{A}_5$ or $\mathrm{PSL}(2,13)$.

Let $N \cong \mathrm{A}_5$. Then, $|N| = 2^2{\cdot}3{\cdot}5$. Since $|N|\big||A|$, we have that $p = 3$ and $q = 5$. This is contrary to our assumption that $q > 7$.

Let $N \cong \mathrm{PSL}(2,13)$. Then, $|N| = 2^2{\cdot}3{\cdot}7{\cdot}13$. Since $q > 7$, we have that $p = 3$ and $q = 13$. It follows that N is arc-transitive on X and $N_v \cong \mathbb{Z}_{13}$ is a Sylow 13-subgroup of N. Thus, X can be viewed as an orbital graph of N acting on N_v. By using the functions `CosetAction` and `OrbitalGraph` in Magma [3], up to graph isomorphism, there is only one connected orbital graph of valence 13 admitting N as an arc-transitive automorphism group. However, this orbital graph has full automorphism group isomorphic to $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$ and so is not one-regular, a contradiction.

Case 2. Assume that A has no non-solvable minimal normal subgroup.

Suppose that $p = 2$. Then, by [16, Theorem 3.3], there is no q-valent one-regular graph of order 56 with $q > 7$. In what follows, we may suppose that $p \geq 3$. Since N is solvable, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_7$, $\mathbb{Z}_7^2$ with $p = 7$ or $\mathbb{Z}_p$. By Proposition 2.1, N is semiregular and X_N is a q-valent symmetric graphs of order $28p/|N|$ with $A/N \lesssim \mathrm{Aut}(X_N)$. Clearly, $N \ncong \mathbb{Z}_2^2$ because there exists no regular graph of odd order and odd valency.

Let $N \cong \mathbb{Z}_p$. Then, X_N is a q-valent symmetric graph of order 28 with $q > 7$. By [7], there are two symmetric graphs of order 28 and valency q, their full automorphism groups are $S_{14} \times \mathbb{Z}_2$ and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$, respectively. By Proposition 2.1, A/N can be embedded in $S_{14} \times \mathbb{Z}_2$ or $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$. However, by Magma [3], both $S_{14} \times \mathbb{Z}_2$ and $\mathrm{PSL}(2,13) \times \mathbb{Z}_2$ have no subgroup of order $28{\cdot}13$, a contradiction.

Let $N \cong \mathbb{Z}_7^2$. Then, $p = 7$, $|A| = 2^2{\cdot}7^2{\cdot}q$ and X_N is a A/N-symmetric graph of order 4 and valency q. Note that $q > 7$. This is clearly impossible.

Let $N \cong \mathbb{Z}_7$. Then, X_N is a A/N-symmetric graph of order $4p$ and valency q. By Proposition 2.3, $X_N \cong \mathbf{K}_{4p}$ with $q = 4p-1$, $\mathbf{K}_{2p,2p} - 2p\mathbf{K}_2$ with $q = 2p-1$, or A/N has a normal subgroup $M/N \cong \mathbb{Z}_2$ such that $X_M \cong \mathbf{K}_{p,p}$ with $q = p$ or $\mathbf{K}_{2p}$ with $q = 2p-1$. By our assumption with $p \neq q$, we have that $X_M \ncong \mathbf{K}_{p,p}$.

Assume that $X_N \cong \mathbf{K}_{4p}$ with $q = 4p-1$. Then, $A/N \lesssim S_{4p}$ and $|A/N| = 4pq$. If A/N is non-solvable, then A/N has a composition factor isomorphic to a non-abelian K_3-simple group. By Proposition 2.6, this is impossible because $q > 7$. Thus, A/N is solvable. Since $|A/N| = 4pq$ with $q = 4p-1$, we have that A/N is 2-transitive on $V(X)$. It follows that the vertex stabilizer of A/N is isomorphic to $\mathbb{Z}_q$ and normalizes the subgroup of A/N and order $4p$. By Burnside's Theorem [4, p.192, Theorem IX], A/N is affine. This forces that A/N has a unique minimal normal elementary abelian subgroup and hence $4p = r^k$ for some prime r. This is impossible because $p \geq 3$.

Assume that $X_N \cong \mathbf{K}_{2p,2p} - 2p\mathbf{K}_2$ with $q = 2p-1$. Then, $A/N \lesssim S_{2p} \times \mathbb{Z}_2$. Since $|A/N| = 4pq$ and X_N is a bipartite graph, A/N has a subgroup B/N of index 2, which acting on each bipartition of X_N is 2-transitive. Clearly, $|B/N| = 2pq$ and so B/N is solvable. By Burnside's Theorem, B/N is affine. Similar arguments as the above paragraph, we can deduce that a contradiction because $4p$ can not be a prime power with $p \geq 3$.

Assume that $X_M \cong \mathbf{K}_{2p}$ with $q = 2p-1$. Then, $A/M \lesssim \S_{2p}$ and A/M is 2-transitive on $V(X_M)$. Since $|A/M| = 2pq$, we have that A/M is solvable. By Burnside's Theorem, A/N is affine. Similarly, this is impossible because $2p$ cannot be a prime power with $p \geq 3$.

Let $N \cong \mathbb{Z}_2$. Then, X_N is a A/N-symmetric graph of order $2{\cdot}7{\cdot}p$ and valency q. If $p = 7$, then $|A/N| = 2{\cdot}7^2$. By Sylow Theorem, the Sylow p-subgroup of A/N must be normal, and by Proposition 2.5, A has a normal Sylow p-subgroup P of order 7^2. Thus, X_P is a symmetric graph of order 4 and valency $q > 7$. Clearly, this is impossible. If $p \neq 7$, then X_N has a square free order. By Proposition 2.4, there is no symmetric graph of order $2{\cdot}7{\cdot}p$ admitting a solvable arc-transitive automorphism group, a contradiction. □

Combining the above arguments with the cases $q = 2, 3$ and 5, and by Lemmas 3.1-3.3, we have the following result.

Theorem 3.1. *Let p, q be two primes and let X be a connected q-valent one-regular graph of order $28p$. Then $X \cong CQ_7$ with $p = 7$ and valency 3.*

4. Conclusion

As is known to all, arc-transitive graphs have much higher symmetries and much larger full automorphism groups, and one-regular graphs have smallest full automorphism groups such that the graphs are arc-transitive. Thus, the characterization and classification of one-regular graphs not only reveal the local action but also global action of the full automorphism group acting on vertices and arcs. In the paper, we classify the one-regular graphs of order $28p$ and prime valency for each prime p, and prove that there is only one sporadic such graph. As a natural continuation, could we find and classify one-regular graphs of order $28p$ with more general valencies.

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References

[1] N. Biggs, *Algebraic graph theory,* Second ed., Cambridge University Press, Cambridge, 1993.

[2] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, Elsevier Science Ltd, New York, 1976.

[3] W. Bosma, C. Cannon, C. Playoust, *The MAGMA algebra system I: The user language,* J. Symbolic Comput., 24 (1997), 235-265.

[4] W. Burnside, *Theory of groups of finite order*, Cambridge University Press, Cambridge, 1897.

[5] C.Y. Chao, *On the classification of symmetric graphs with a prime number of vertices,* Trans. Amer. Math. Soc., 158 (1971), 247-256.

[6] Y. Cheng, J. Oxley, *On the weakly symmetric graphs of order twice a prime*, J. Combin. Theory Ser. B, 42 (1987), 196-211.

[7] M.D.E. Conder, *A complete list of all connected symmetric graphs of order* 2 *to* 30*,* `https://www.math.auckland.ac.nz/ conder/symmetricgraphs-orderupto30.txt`.

[8] H.J. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A Wilson, *Atlas of finite group*, Clarendon Press, Oxford, 1985.

[9] D.Ž. Djoković, G.L. Miller, *Regular groups of automorphisms of cubic graphs,* J. Combin. Theory Ser. B, 29 (1980), 195-230.

[10] Y.Q. Feng, J.H. Kwak, K.S. Wang, *Classifying cubic symmetric graphs of order* $8p$ *or* $8p^2$*,* European J. Combin., 26 (2005), 1033-1052.

[11] S.T. Guo, Y.Q. Feng, *A note on pentavalent s-transitive graphs,* Discrete Math., 312 (2012), 2214-2216.

[12] S.T. Guo, Y.T. Li, X.H. Hua, (G, s)*-transitive graphs of valency* 7*,* Algebra Colloq., 23 (2016), 493-500.

[13] S.T. Guo, Y. Xu, G.Y. Chen, *Heptavalent symmetric graphs of order* $8p$*,* Ital. J. Pure Appl. Math., 43 (2020), 37-46.

[14] B. Huppert, *Eudiche gruppen I*, Springer-Verlag, Berlin, 1967.

[15] A. Imani, N. Mehdipoor, A.A. Talebi, *On application of linear algebra in classification cubic s-regular graphs of order* $28p$, Algebra Discrete Math., 25 (2018), 56-72.

[16] D.X. Li, *Prime-valent one-regular graphs of order* $8p$*,* Ital. J. Pure Appl. Math., 44 (2020), 449-353.

[17] P. Lorimer, *Vertex-transitive graphs: Symmetric graphs of prime valency*, J. Graph Theory, 8 (1984), 55-68.

[18] J. Pan, B. Lou, C. Liu, *Arc-transitive pentavalent graphs of order* $4pq$, Electron. J. Combin., 20 (2013), #36.

[19] J. Pan, B. Ling, S. Ding, *On symmetric graphs of order four times an odd square-free integer and valency seven,* Discrete Math., 340 (2017), 2071-2078.

[20] J. Pan, B. Ling, S. Ding, *On prime-valent symmetric graphs of square-free order*, Ars Math. Contemp., 15 (2018), 53-65.

[21] P. Potočnik, *Pentavalent arc-transitive graphs on up to 500 vertices which admit an arc-transitive group* G *with the vertex stabiliser* G_v *acting faithfully on the neighbourhood of* v *and being solvable,* `https://www.fmf.uni-lj.si/ potocnik/work_datoteke/AT5-Census.mag`.

[22] D.J. Robinson, *A Course in the theory of groups*, Springer-Verlag, New York, 1982.

[23] W.J. Shi, *On simple* K_4*-groups* (in Chinese), Chinese Science Bull, 36 (1991), 1281-1283.

[24] X. Wang, *Symmetric graphs of order* $4p$ *of valency prime,* 2015 Intl. Sym. Comp. Inform., 1 (2015), 1583-1590.

[25] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964.

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