

Semihyperlattice regular equivalence relations on ordered semihypergroups

Yize Li

*College of Science
Northwest A&F University
Yangling, Shaanxi 712100
China
liyize@nwfau.edu.cn*

Xinyang Feng*

*College of Science
Northwest A&F University
Yangling, Shaanxi 712100
China
fxy1012@126.com*

Xing Gao

*School of Mathematics and Statistics
Lanzhou University
Lanzhou, Gansu 730000
China
and
Gansu Provincial Research Center for
Basic Disciplines of Mathematics and Statistics
Lanzhou, 730070
China
gaoxing@lzu.edu.cn*

Jingxiang Wu

*College of Science
Northwest A&F University
Yangling, Shaanxi 712100
China
jingxiang2024@126.com*

Abstract. In this paper, we explore the semihyperlattice regular equivalence relations on ordered semihypergroups. To begin with, we present the specific constructions of semihyperlattice regular equivalence relations. This effort is directed towards ensuring the preservation of the hyperalgebraic structure within the quotient hyperalgebra. Subsequently, we obtain a homomorphism theorem from ordered semihypergroups to ordered semihyperlattices. Finally, we discuss some related properties of congruence classes and principal pseudo-hyperfilters.

*. Corresponding author

Keywords: ordered semihypergroup, semihyperlattice regular equivalence relation, homomorphism, congruence class.

MSC 2020: 20N20.

1. Introduction

The advent of algebraic hyperstructures was pioneered by F. Marty in 1934 [1], signifying a profound expansion into realms such as hyperrings, hyperfields, and hyperlattices. Numerous scholars have explored various dimensions of semihypergroups, for instance, see [2, 3, 4, 5, 6, 7, 8, 9]. Notably, B. Davvaz has significantly propelled the development of semihypergroup since 2000, with a particular emphasis on the congruence theory. The integration of ordered semigroup algebra with hyperstructure theory was further enhanced by the efforts of D. Heidari and B. Davvaz in 2011 [10], culminating in the formulation of the concept of ordered semihypergroups, a domain replete with theoretical and practical possibilities.

A pivotal focus within this field has been the construction of strong regular equivalence relations on ordered semihypergroups by B. Davvaz using pseudo-order. In 2015, B. Davvaz introduced the pseudoorder to induce strong regular equivalence relations and quotient ordered semihypergroups into ordered semigroups, but in the process, the hyperstructure was lost [11]. Then, he raised an open question: Is there a regular equivalence relation ρ on an ordered semihypergroup $(S, \circ, \leq_S)$ for which S/ρ is an ordered semihypergroup? This question prompted subsequent research to find a solution. In 2016, Z. Gu addressed this issue using proper hyperideals [12], and in 2018, X.Y. Feng resolved it through the concept of the weak pseudoorder [13]. These contributions collectively solved the open problem posed by Davvaz, significantly advancing the understanding of regular equivalence relations in ordered semihypergroups and thereby enriching the field.

As we know, semilattice congruences plays an important role in the research of semigroup and ordered semigroup algebraic theory. In 2015, J. Tang et al. generalized the concept of filters to hyperfilters, by using this, he introduced the semilattice strong regular equivalence relations $\mathcal{N}$ on an ordered semihypergroup, resulting in $S/\mathcal{N}$ being an ordered semilattice. Therefore, this paper primarily investigates whether a semihyperlattice regular equivalence relation exists that ensures the quotient structure of any ordered semihypergroup forms an ordered semihyperlattice.

This paper explores the semihyperlattice regular equivalence relations (semihyperlattice congruences) on ordered semihypergroups in detail. After an introduction, in Section 2, we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper. In Section 3, we construct the semihyperlattice regular equivalence relation on ordered semihypergroups by using pseudo-hyperfilters, ensuring the preservation of the hyperalgebraic structure within the quotient hyperalgebra. In Section 4, we present a

homomorphism theorem from ordered semihypergroups to ordered semihyperlattices and we studies the relationship between congruence classes and principal pseudo-hyperfilters.

2. Preliminaries

For the sake of clarity and convenience, the essential definitions are provided first. A mapping $\circ : S \times S \rightarrow \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all non-empty subsets of S , is called a hyperoperation on S . A couple $(S, \circ)$ is called a hypergroupoid. For an element $x \in S$ and nonempty subsets $A, B \subseteq S$, the operations are denoted as $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$, and $A \circ x = A \circ \{x\}$. A hypergroupoid $(S, \circ)$ is called a semihypergroup if hyperoperation $\circ$ satisfies $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in S$.

Consider a semihypergroup $(S, \circ)$ and a relation ρ on S . For nonempty subsets $A, B \subseteq S$, we define

$$A\bar{\rho}B \Leftrightarrow (\forall a \in A, \exists b \in B) a\rho b \text{ and } (\forall b' \in B, \exists a' \in A) a'\rho b',$$

and

$$A\bar{\bar{\rho}}B \Leftrightarrow (\forall a \in A, \forall b \in B) a\rho b.$$

An equivalence relation ρ is classified as regular if

$$(\forall a, b, x \in S) a\rho b \Rightarrow a \circ x\bar{\rho}b \circ x \text{ and } x \circ a\bar{\rho}x \circ b,$$

and as strongly regular if

$$(\forall a, b, x \in S) a\rho b \Rightarrow a \circ x\bar{\bar{\rho}}b \circ x \text{ and } x \circ a\bar{\bar{\rho}}x \circ b.$$

For a semihypergroup $(S, \circ)$ with an equivalence relation ρ , the equivalence ρ -class containing a is denoted by $(a)_\rho$. As known from [11], if ρ is a regular equivalence relation on S , the quotient S/ρ forms a semihypergroup under the operation $(x)_\rho \star (y)_\rho = \{(z)_\rho \mid z \in x \circ y\}$. Furthermore, if ρ is strongly regular, then S/ρ constitutes a semigroup with the operation $(x)_\rho \star (y)_\rho = (z)_\rho$ for every $z \in x \circ y$.

An ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ accompanied by an order relation $\leq$ such that $a \leq b$ implies $ax \leq bx$ and $xa \leq xb$ for any $a, b, x \in S$. This concept extends to the hyper version as an ordered semihypergroup $(S, \circ, \leq)$, which is a semihypergroup $(S, \circ)$ with an order relation $\leq$ compatible with the hyperoperation $\circ$. That is, $A, B \in \mathcal{P}^*(S)$, $A \preceq B$ if and only if, for all $a \in A$, there exists $b \in B$ such that $a \leq b$. And $a \approx b$ means that a is not comparable to b . A nonempty subset A of an ordered semihypergroup S is a subsemihypergroup if $A \circ A \subseteq A$. A subsemihypergroup A of S is a hyperfilter if it satisfies: (1) for any $a, b \in S$, $(a \circ b) \cap A \neq \emptyset$ implies $a, b \in A$; (2) $a \in A$, $a \leq b \in S$ implies $b \in A$.

For two ordered semihypergroups $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$, a mapping $f : S \rightarrow T$ is called a normal homomorphism if it satisfies: (1) $f(x \circ y) = f(x) \diamond f(y)$

for all $x, y \in S$, where $f(A) = \{f(a) \mid a \in A\}$ for any nonempty subset A of S ;
(2) f is isotone, i.e., for any $x, y \in S$, $x \leq_S y$ implies $f(x) \leq_T f(y)$. Moreover, a bijective normal homomorphism f from S onto T is called an isomorphism if f satisfies $f(x) \leq_T f(y)$, then $x \leq_S y$ for any $x, y \in S$.

3. Ordered semihyperlattice regular equivalence relations

In this section, we define and study the ordered semihyperlattice regular equivalence relations of an ordered semihypergroup S . Especially, we construct a semihyperlattice regular equivalence relation on ordered semihypergroups in terms of the pseudo-hyperfilter, and discuss its related properties.

Definition 3.1. Let F be a subset of an ordered semihypergroup S . F is called a pseudo-hyperfilter S if it satisfies the following:

- (1) $(a \circ b) \cap F \neq \emptyset$ if and only if $a, b \in F$;
- (2) If $(a \circ b) \cap F \neq \emptyset$, then there exists $u \in (a \circ b) \cap F$ such that $\forall f \in F$, $u \leq f$ or $u \approx f$;
- (3) If $a \in F$ and $a \leq b \in S$ or $S \ni b \approx a$, then $b \in F$.

The following is an example of a pseudo hyperfilter on an ordered semihypergroup.

Example 3.1. Let $S = \{a, b, c, d, e\}$ with the operation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
d	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$
e	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c, d\}$	$\{a, b, c, d, e\}$

$$\begin{aligned} \leq := & \{(a, a), (a, c), (a, d), (a, e), \\ & (b, b), (b, c), (b, d), (b, e), \\ & (c, c), (c, d), (c, e), \\ & (d, d), (d, e), \\ & (e, e)\}. \end{aligned}$$

The associativity of $\circ$ is easily verified, and it is also straightforward to confirm that $\leq$ is a partial order that satisfies the compatibility condition. Therefore, we can clearly verified that S is an ordered semihypergroup.

We proceed to prove that $T = \{d, e\}$ is a pseudo-hyperfilter on S .

(1) For $d, e \in T$, we have $d \circ e = e \circ d = \{a, b, c, d\}$. Thus,

$$(d \circ e) \cap T = \{a, b, c, d\} \cap T = \{d\} \neq \emptyset.$$

Moreover,

$$(d \circ d) \cap T = \{a, b, c, d\} \cap T = \{d\} \neq \emptyset$$

and

$$(e \circ e) \cap T = \{a, b, c, d, e\} \cap T = \{d, e\} \neq \emptyset.$$

The condition (1) in Definition 3.1 is satisfied.

(2) By (1), we have

$$(d \circ e) \cap T = (e \circ d) \cap T = (d \circ d) \cap T = \{d\} \neq \emptyset$$

and

$$(e \circ e) \cap T = \{d, e\} \neq \emptyset.$$

Moreover, $d \leq d, d \leq e$. Thus, for any $x, y \in T$, there exists $d \in (x \circ y) \cap T$ such that for all $t \in T, d \leq t$. The condition (2) in Definition 3.1 is satisfied.

(3) For $d, e \in T$, since $d \leq e$ and $e \in \{d, e\}$, the condition (3) in Definition 3.1 is satisfied.

Hence, $\{d, e\}$ is a pseudo-hyperfilter on S .

For any $a \in S$, we denote the principal pseudo-hyperfilter generated by element a as $W(a)$, that is, $W(a)$ is the smallest pseudo-hyperfilter that include a . Let $\mathcal{W} := \{(x, y) \mid W(x) = W(y)\}$. Then, we can obtain the following result.

Lemma 3.1. *Let S be an ordered semihypergroup, the relation $\mathcal{W} := \{(x, y) \in S \times S \mid W(x) = W(y)\}$ is a regular equivalence relation on S .*

Proof. Clearly, $\mathcal{W}$ is an equivalence relation on S . Assume that $x\mathcal{W}y$, then $W(x) = W(y)$. For all $z \in S$, and for any $a \in x \circ z$, we have $(x \circ z) \cap W(a) \neq \emptyset$. Thus, $x, z \in W(a)$ implies $W(x) \subseteq W(a)$, hence $W(y) \subseteq W(a)$. Therefore, $y, z \in W(a)$. Furthermore, there exists $b \in (y \circ z) \cap W(a)$, such that for any $w \in W(a)$, either $b \leq w$ or $b \approx w$.

If $b \leq a$, and therefore $a \in W(b) \Rightarrow W(a) \subseteq W(b)$. Since $b \in W(a)$, we have $W(b) \subseteq W(a)$. That is, there exists $b \in y \circ z$ such that $W(a) = W(b)$.

If $b \approx a$, then $a \in W(b) \Rightarrow W(a) \subseteq W(b)$. Since $b \in W(a)$, we have $W(b) \subseteq W(a)$. That is, there exists $b \in y \circ z$ such that $W(a) = W(b)$.

Similarly, for all $b' \in y \circ z$, there exists $a' \in x \circ z$ satisfying $b'\mathcal{W}a'$. The same holds for $z \circ x$ and $z \circ y$. Therefore, $\mathcal{W}$ is a regular equivalence relation on S . $\square$

Definition 3.2. *Let L be a semihypergroup. Then, L is called a semihyperlattice if it satisfies the following conditions for all $x, y \in L$:*

(1) $x \in x \circ x$;

(2) $x \circ y = y \circ x$.

In the study of semihyperlattices, the introduction of an order relation is a crucial and indispensable step. Therefore, we are committed to introducing an appropriate order relation on semihyperlattices, thus establishing the concept of the ordered semihyperlattices.

Definition 3.3. *Let L be an ordered semihypergroup. Then, L is called an ordered semihyperlattice if it is also a semihyperlattice.*

Remark 3.1. The ordered semihypergroup S in Example 3.1 is an instance of an ordered semihyperlattice.

Furthermore, we proceed to present the concept of ordered semihyperlattice regular equivalence relations.

Definition 3.4. *Let S be an ordered semihypergroup. The relation σ on S is called an ordered semihyperlattice regular equivalence relation if it satisfies the following two conditions:*

- (1) σ is a regular equivalence relation on S ;
- (2) the quotient structure S/σ constitutes an ordered semihyperlattice.

Let S be an ordered semihypergroup, for each $a \in S$, the $\mathcal{W}$ -class containing a is denoted by $(a)_{\mathcal{W}}$.

Theorem 3.1. *Let S be an ordered semihypergroup. Then, $\mathcal{W}$ is an ordered semihyperlattice regular equivalence relation on S .*

Proof. $\mathcal{W}$ is a regular equivalence relation by Lemma 3.1. The conclusion is then proven in two steps.

Step 1. $S/\mathcal{W}$ is an ordered semihypergroup. It is known that $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_{\mathcal{W}})$ is an ordered semihypergroup, where $\star_{\mathcal{W}}$ and $\preceq_{\mathcal{W}}$ are defined respectively as follows:

$$\begin{aligned} (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}} &:= \{(z)_{\mathcal{W}} \in S/\mathcal{W} \mid z \in x \circ y\}, \\ \preceq_{\mathcal{W}} &:= \{((x)_{\mathcal{W}}, (y)_{\mathcal{W}}) \in S/\mathcal{W} \times S/\mathcal{W} \mid (x)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}\}. \end{aligned}$$

Let $\pi : S \rightarrow S/\mathcal{W}$ be the mapping defined by $\pi(x) = (x)_{\mathcal{W}}$ for all $x \in S$. Let $x, y \in S$, $x \leq y$. Then, $y \in W(x)$. For any $z \in x \circ y$, we have $W(z) \cap (x \circ y) \neq \emptyset$, and thus $x \in W(z)$. Then, $(x)_{\mathcal{W}} \subseteq W(x) \subseteq W(z) \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}$. Therefore, $(x)_{\mathcal{W}} \preceq_{\mathcal{W}} (y)_{\mathcal{W}}$, that is, $\pi(x) \preceq_{\mathcal{W}} \pi(y)$.

Step 2. $S/\mathcal{W}$ is a semihyperlattice. Take any $(x)_{\mathcal{W}} \in S/\mathcal{W}$. Since $x \in W(x)$, we have $(x \circ x) \cap W(x) \neq \emptyset$. Therefore, there exists $y \in (x \circ x) \cap W(x)$ such that $y \leq w$ or $y \approx w$ for all $w \in W(x)$, which further implies $y \leq x$ or $y \approx x$. Consequently, $x \in W(y)$ leads to $W(x) = W(y)$, then $(x)_{\mathcal{W}} = (y)_{\mathcal{W}} \in (x \circ x)_{\mathcal{W}} = (x)_{\mathcal{W}} \star_{\mathcal{W}} (x)_{\mathcal{W}}$. Moreover, let $x, y \in S$. Take any $(a)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}$, then there exists

$z \in x \circ y$, such that $(z)_W = (a)_W$. Hence, $(x \circ y) \cap W(a) = (x \circ y) \cap W(z) \neq \emptyset$, then $x, y \in W(a)$. Then, there exists $b \in (y \circ x) \cap W(a) \neq \emptyset$, such that $b \leq a$ or $b \approx a$. Therefore, $a \in W(b)$, so $W(a) \subseteq W(b)$, then $W(a) = W(b)$. Hence, $(a)_W = (b)_W \in (y \circ x)_W = (y)_W \star_W (x)_W$. Thus $(x)_W \star_W (y)_W \subseteq (y)_W \star_W (x)_W$. Similarly, take any $(b)_W \in (y)_W \star_W (x)_W$, we obtain $(y)_W \star_W (x)_W \subseteq (x)_W \star_W (y)_W$. Thus, $(x)_W \star_W (y)_W = (y)_W \star_W (x)_W$, then S/W is a semihyperlattice with the hyperoperation $\star_W$. $\square$

Next, we proceed to introduce a more general example, extending beyond the specific instance of the ordered semihypergroup discussed earlier.

Example 3.2. Let $S = \{a, b, c, d, e, f\}$ with the operation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
d	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$
e	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c, d\}$	$\{a, b, c, d, e\}$	$\{a, b, c, d, e\}$
f	$\{a\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c, d\}$	$\{a, b, c, d, e\}$	$\{a, b, c, d, e, f\}$

$$\leq := \{(a, a), (a, c), (a, d), (a, e), (a, f), (b, b), (b, e), (b, f), \\ (c, c), (c, d), (c, f), (d, d), (d, f), (e, e), (f, f)\}.$$

- (1) $(S, \circ, \leq)$ is an ordered semihypergroup. By verifying the associativity of the hyperoperation $\circ$ and the compatibility of the partial order $\leq$ on S , we find that $(S, \circ, \leq)$ still forms an ordered semihypergroup.
- (2) $(S/W, \star_W)$ is a semihyperlattice. It is easy to derive from the Definition 3.1 of pseudo-hyperfilters that $W(a) = W(b) = S, W(c) = W(d) = W(e) = \{c, d, e, f\}, W(f) = \{e, f\}$. Hence, $(a)_W = (b)_W = \{a, b\}, (c)_W = (d)_W = (e)_W = \{c, d, e\}, (f)_W = \{f\}$. Consequently, We can immediately derive the hyperoperation $\star_W$ and the order $\preceq_W$ for as follows:

$\star_W$	$(a)_W$	$(c)_W$	$(f)_W$
$(a)_W$	$\{(a)_W\}$	$\{(a)_W\}$	$\{(a)_W\}$
$(c)_W$	$\{(a)_W\}$	$\{(a)_W, (c)_W\}$	$\{(a)_W, (c)_W\}$
$(f)_W$	$\{(a)_W\}$	$\{(a)_W, (c)_W\}$	$\{(a)_W, (c)_W, (f)_W\}$

$$\preceq_W := \{((a)_W, (a)_W), ((a)_W, (c)_W), ((a)_W, (f)_W), \\ ((c)_W, (c)_W), ((c)_W, (f)_W), \\ ((f)_W, (f)_W)\}.$$

It is easy to check that the table above meets the requirements of Definition 3.2. Therefore, S/W with respect to $\star_W$ forms a semihyperlattice.

- (3) $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_{\mathcal{W}})$ is an ordered semihyperlattice. It is easy to verify that the order $\preceq_{\mathcal{W}}$ defined by Theorem 3.1 is compatible with $\star_{\mathcal{W}}$, thus $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_{\mathcal{W}})$ constitutes an ordered semihyperlattice.

Hereafter, we provide an equivalent characterization of $\mathcal{W}$ through the introduction of the concept of a -maximal.

Definition 3.5. Let A be a pseudo-hyperfilter of an ordered semihypergroup S and $a \in S$. A pseudo-hyperfilter A is called a -maximal if A is maximal in the set of all pseudo-hyperfilters not containing a .

We define $\mathcal{M}(a)$ to encompass all a -maximal pseudo-hyperfilters within the semihypergroup S . It is conceivable that for some elements a in S , $\mathcal{M}(a)$ may be void, expressed as $\mathcal{M}(a) = \emptyset$. For instance, the case where S includes a supreme element e , in such scenarios, $\mathcal{M}(e)$ is invariably void, indicated by $\mathcal{M}(e) = \emptyset$.

Theorem 3.2. Let S be an ordered semihypergroup. Then, $x\mathcal{W}y$ if and only if $\mathcal{M}(x) = \mathcal{M}(y)$ for all $x, y \in S$.

Proof. $(\Rightarrow)$ Assume $x\mathcal{W}y$, which implies $W(x) = W(y)$. Consider any A from $\mathcal{M}(x)$. If $y \in A$, it would suggest $W(y) \subseteq A$, leading to the inclusion $x \in A$, which contradicts the definition of A . Therefore, $y \notin A$, and consequently, $A \in \mathcal{M}(y)$. Contrariwise, if $A \notin \mathcal{M}(y)$, there must exist a y -maximal pseudo-hyperfilter A' in S with $A \subset A'$, implying $x \in A'$ and subsequently $y \in A'$, which is untenable. Thus, we establish that $\mathcal{M}(x) \subseteq \mathcal{M}(y)$. Following a similar argument, we can also deduce $\mathcal{M}(y) \subseteq \mathcal{M}(x)$. Then, $\mathcal{M}(y) = \mathcal{M}(x)$.

$(\Leftarrow)$ If $\mathcal{M}(x) = \mathcal{M}(y)$, it follows that $x \in W(y)$. Suppose the contrary, that is, there is a x -maximal pseudo-hyperfilter A for which $W(y) \subseteq A$. This would necessitate $y \in A$, contradicting the presumption that A is part of $\mathcal{M}(y)$. By symmetry, it can be shown that $y \in W(x)$, leading to the conclusion $W(x) = W(y)$, hence $x\mathcal{W}y$. $\square$

4. The $\mathcal{W}$ -classes of ordered semihypergroup

In this section, we first establish the homomorphism theorem from ordered semihypergroups to ordered semihyperlattices, and then investigate some properties of the $\mathcal{W}$ -classes of an ordered semihypergroup S and describe the relationship between $(a)_{\mathcal{W}}$ and the principal pseudo-hyperfilter $W(a)$ generated by a .

Lemma 4.1. Let S be an ordered semihypergroup, and $\mathcal{W}$ be an ordered semihyperlattice regular equivalence relation on S , then the mapping $\pi : S \rightarrow S/\mathcal{W}$ given by $\pi(x) = (x)_{\mathcal{W}}$ is an epimorphism.

Proof. Straightforward. $\square$

Theorem 4.1. Let $(S, \circ, \leq_S)$ be an ordered semihypergroup, $(T, \diamond, \leq_T)$ be an ordered semihyperlattice. $\varphi : S \rightarrow T$ is a homomorphism. Then, if $\mathcal{W}$ defined

by $\mathcal{W} = \{(x, y) \mid W(x) = W(y)\}$ and $\mathcal{W} \subseteq \ker \varphi$, there exists the unique homomorphism $f : S/\mathcal{W} \rightarrow T \mid (\alpha)_{\mathcal{W}} \mapsto \varphi(\alpha)$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \pi \downarrow & \nearrow f & \\ S/\mathcal{W} & & \end{array}$$

commutes. Moreover, $\text{Im}(\varphi) = \text{Im}(f)$.

Proof. (1) f is well defined. Indeed, if $(x)_{\mathcal{W}} = (y)_{\mathcal{W}}$, then $(x, y) \in \mathcal{W} \subseteq \ker \varphi$, hence $\varphi(x) = \varphi(y)$. (2) f is a homomorphism and $\varphi = f \circ \pi$. In fact, By Lemma 4.1, there exists an ordered relation $\preceq_{\mathcal{W}}$ on the quotient semihyperlattice $(S/\mathcal{W}, \star_{\mathcal{W}})$ such that $(S/\mathcal{W}, \star_{\mathcal{W}}, \preceq_{\mathcal{W}})$ is an ordered semihyperlattice and the mapping π is a homomorphism. Moreover, $(x)_{\mathcal{W}} \preceq_{\mathcal{W}} (y)_{\mathcal{W}}$ means that $(x)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}} = (x \circ y)_{\mathcal{W}}$. Then, there exists $z \in x \circ y$, such that $(x)_{\mathcal{W}} = (z)_{\mathcal{W}}$. Thus, $(x, z) \in \mathcal{W} \subseteq \ker \varphi$, then we have

$$\varphi(x) = \varphi(z) \in \{\varphi(t) \mid t \in x \circ y\} = \varphi(x) \diamond \varphi(y).$$

Hence, $\varphi(x) \leq_T \varphi(y)$, that is, $f((x)_{\mathcal{W}}) \leq_T f((y)_{\mathcal{W}})$

Also, let $(x)_{\mathcal{W}}, (y)_{\mathcal{W}} \in S/\mathcal{W}$. Since φ is a homomorphism from S to T , we have

$$\begin{aligned} f((x)_{\mathcal{W}}) \diamond f((y)_{\mathcal{W}}) &= \varphi(x) \diamond \varphi(y) \\ &= \{\varphi(z) \mid z \in x \circ y\} \\ &= \{f((z)_{\mathcal{W}}) \mid (z)_{\mathcal{W}} \in (x)_{\mathcal{W}} \star_{\mathcal{W}} (y)_{\mathcal{W}}\}. \end{aligned}$$

In addition, for any $\alpha \in S$, $(f \circ \pi)(\alpha) = f((\alpha)_{\mathcal{W}}) = \varphi(\alpha)$, hence $\varphi = f \circ \pi$.

Furthermore, we claim that f is the unique homomorphism from $S/\mathcal{W}$ to T . Let $\tilde{f}$ is a homomorphism from $S/\mathcal{W}$ to T such that $\varphi = \tilde{f} \circ \pi$. Therefore,

$$f((x)_{\mathcal{W}}) = \varphi(x) = (\tilde{f} \circ \pi)(x) = \tilde{f}((x)_{\mathcal{W}}),$$

that is, $f = \tilde{f}$. Finally, we have $\text{Im}(f) = \{f((x)_{\mathcal{W}}) \mid x \in S\} = \{\varphi(x) \mid x \in S\} = \text{Im}(\varphi)$. $\square$

Based on the homomorphism Theorem 4.1, the subsequent discussion will focus on the study of congruence classes and their properties. Looking ahead, our future research will aim to explore how the properties of quotient semilattices and $\mathcal{W}$ -classes can reflect the intrinsic structure of the original ordered semihypergroups.

Lemma 4.2. $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$ if and only if $W(a) = W(b)$.

Proof. ($\Rightarrow$) For any $a, b \in S$, Assume $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$. Then, $a \in (b)_{\mathcal{W}}$ implies $a\mathcal{W}b$, and thus $W(a) = W(b)$.

($\Leftarrow$) Assume $W(a) = W(b)$. Then, $a \in (b)_{\mathcal{W}}$ and $b \in (a)_{\mathcal{W}}$. For any $c \in (a)_{\mathcal{W}}$, given $c\mathcal{W}a$ and $a\mathcal{W}b$, it follows that $c\mathcal{W}b$, implying $c \in (b)_{\mathcal{W}}$. Therefore, $(a)_{\mathcal{W}} \subseteq (b)_{\mathcal{W}}$. Similarly, $(b)_{\mathcal{W}} \subseteq (a)_{\mathcal{W}}$, hence $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$. $\square$

Remark 4.1. Since $(a)_{\mathcal{W}}$ and $(b)_{\mathcal{W}}$ are congruence classes of the ordered semi-hypergroup S under the ordered semihyperlattice equivalence relation $\mathcal{W}$, $(a)_{\mathcal{W}}$ and $(b)_{\mathcal{W}}$ either equal or disjoint. Thus, from $(a)_{\mathcal{W}} \subseteq (b)_{\mathcal{W}}$, one can directly conclude $(a)_{\mathcal{W}} = (b)_{\mathcal{W}}$.

Theorem 4.2. $(a)_{\mathcal{W}} \preceq_{\mathcal{W}} (b)_{\mathcal{W}}$ if and only if $W(b) \subseteq W(a)$.

Proof. ($\Rightarrow$) Assume $(a)_{\mathcal{W}} \preceq_{\mathcal{W}} (b)_{\mathcal{W}}$. This is equivalent to $(a)_{\mathcal{W}} \in (a)_{\mathcal{W}} \star_{\mathcal{W}} (b)_{\mathcal{W}} = \{(x)_{\mathcal{W}} \mid x \in a \circ b\}$, implying there exists some $t \in a \circ b$ such that $(a)_{\mathcal{W}} = (t)_{\mathcal{W}}$. Given $t \in a \circ b$, then $(a \circ b) \cap W(t) \neq \emptyset$, leading to $a, b \in W(t)$. Hence $W(b) \subseteq W(t) = W(a)$.

($\Leftarrow$) Assume $W(b) \subseteq W(a)$. Considering $a, b \in W(a)$, therefore, $(a \circ b) \cap W(a) \neq \emptyset$ and there exists some $t \in (a \circ b) \cap W(a)$, such that $t \leq a$ or $t \approx a$. In either case, we have $W(a) \subseteq W(t)$, and since $t \in W(a)$, it follows that $W(t) \subseteq W(a)$, thus $W(a) = W(t)$. By Lemma 4.2, we obtain $(a)_{\mathcal{W}} = (t)_{\mathcal{W}}$. Lastly, as $t \in a \circ b$, it follows that $(a)_{\mathcal{W}} = (t)_{\mathcal{W}} \in \{(x)_{\mathcal{W}} \mid x \in a \circ b\} = (a)_{\mathcal{W}} \star_{\mathcal{W}} (b)_{\mathcal{W}}$, which means $(a)_{\mathcal{W}} \preceq_{\mathcal{W}} (b)_{\mathcal{W}}$. $\square$

Conclusion

In this paper, we introduced the concept of pseudo-hyperfilters in ordered semi-hypergroups and by using pseudo-hyperfilters, we constructed the semihyperlattice regular equivalence relation $\mathcal{W}$ which enabled us to establish a homomorphism theorem from ordered semihypergroups to ordered semihyperlattices, preserving the hyperstructure in the quotient process.

This work advances existing theories in two significant aspects. First, it extends the framework of ordered semihypergroup theory by overcoming the limitation where ordered semihypergroups could previously only be mapped to ordered semilattices via semilattice strong regular equivalence relations, resulting in the loss of hyperstructures. By retaining the hyperstructure, our results generalize the classical theory of ordered semigroups. Second, the study enriches the broader field of algebraic hyperstructures by providing a new perspective on semihyperlattice regular equivalence relations and their induced quotient hyperstructures.

Future research could focus on leveraging the properties of quotient semihyperlattices to investigate the internal structure of the original ordered semihypergroups. This approach has the potential to reveal deeper connections between the external quotient hyperstructures and the intrinsic algebraic properties.

Acknowledgments

This work was supported by the Natural Science Project of Shaanxi Province (No. 2022JQ-035 and No. 2022JQ-040) and the National Innovation and Entrepreneurship Training Program for College Students (No. 2024014609E).

References

- [1] F. Marty, *Sur une generalization de la notion de groupe*, Proc. 8th Congress Mathematiciens Scandenaves, Stockholm, (1934), 45-49.
- [2] S. M. Anvariye, S. Mirvakili, O. Kazancı and B. Davvaz, *Algebraic hyperstructures of soft sets associated to semihypergroups*, Southeast Asian Bull. Math., 35 (2011), 911-925.
- [3] B. Davvaz, *Some results on congruences on semihypergroups*, Bull. Malays. Math. Sci. Soc., 23 (2000), 53-58.
- [4] B. Davvaz, N. S. Poursalavati, *Semihypergroups and S-hypersystems*, Pure Math. Appl., 11 (2000), 43-49.
- [5] M. D. Salvo, D. Freni and G. L. Faro, *Fully simple semihypergroups*, J. Algebra, 399 (2014), 358-377.
- [6] D. Fasino, D. Freni, *Existence of proper semihypergroups of type U on the right*, Discrete Math., 307 (2007), 2826-2836.
- [7] K. Hila, B. Davvaz and K. Naka, *On quasi-hyperideals in semihypergroups*, Comm. Algebra, 39 (2011), 4183-4194.
- [8] V. Leoreanu, *About the simplifiable cyclic semihypergroups*, Ital. J. Pure Appl. Math., 7 (2000), 69-76.
- [9] N. Kehayopulu, M. Tsingelis, *Pseudoorder in ordered semigroups*, Semigroup Forum, 50 (1995), 389-392.
- [10] D. Heidari, B. Davvaz, *On ordered hyperstructures*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 73 (2011), 85-96.
- [11] B. Davvaz, P. Corsini and T. Changphas, *Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder*, European J. Combin., 44 (2015), 208-217.
- [12] Z. Gu, X. Tang, *Ordered regular equivalence relations on ordered semihypergroups*, J. Algebra, 450 (2016), 384-397.
- [13] X. Y. Feng, J. Tang, and Y. F. Luo, *Regular equivalence relations on ordered $\star$ -semihypergroups*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 80 (2018), 135-144.

- [14] T. Changphas, B. Davvaz, *Hyperideal theory in ordered semihypergroups*, in: International Congress on Algebraic Hyperstructures and Its Applications, Xanthi, Greece, (2014), 51-54.
- [15] J. Tang, B. Davvaz and Y. F. Luo, *Hyperfilters and fuzzy hyperfilters of ordered semihypergroups*, J. Intell. Fuzzy Systems, 29 (2015), 75-84.

Accepted: January 22, 2025