A note on hyperrings and hypermodules

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Abstract. The main purpose of this paper is to study the concept of the hyperring $(\mathbb{N}, \oplus, \cdot)$, where $m \oplus n = \{m+n, k \mid \min\{m, n\} + k = \max\{m, n\}\}$, for all $m, n \in \mathbb{N}$ and the operation $\cdot$ is the usual multiplication in $\mathbb{N}$. In particular, we prove that this hyperring $(\mathbb{N}, \oplus, \cdot)$ is isomorphic to Krasner's quotient hyperring $\frac{\mathbb{Z}}{G}$ in [10]. Moreover, we construct the hyperstructure $(\mathbb{N}_m, \oplus_m, \cdot)$, which is a class of examples of hypermodules and hyperrings.

Keywords: hyperring, hypermodule.

MSC 2020: 20N20, 16D80.

1. Introduction

Let H be a nonvoid set. A mapping from $H \times H$ into H is called a *composition* on H. A composition $\diamond$ on a set H is called *associative* if, for all $x, y, z \in H$, $x \diamond (y \diamond z) = (x \diamond y) \diamond z$, and is called *reproductive* if $x \diamond H = H \diamond x = H$, for all $x \in H$. The pair $(H, \diamond)$ is called *group* if H is a nonvoid set and $\diamond$ is an associative and reproductive composition on H. It follows from [16, Theorem 2

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and Theorem 3] that the pair $(H, \diamond)$ is a group if and only if H is a nonvoid set and $\diamond$ has the following properties:

(1) For all $x, y, z \in H$, $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ (**associative**).

(2) There exists $e \in H$ such that for all $x \in H$, $x \diamond e = e \diamond x = x$ (**existence of an identity**).

(3) For all $x \in H$, there exists y such that $x \diamond y = y \diamond x = e$ (**existence of an inverse**).

In [17], Marty, who is a French mathematician, extends a composition on a set H to a hypercomposition on a set H as follows. A mapping $\uplus : H \times H \longrightarrow P(H)$ is called a *hypercomposition* on a set H, where $P(H)$ is the power set of H. He calls $(H, \uplus)$ *hypergroup* if H is a nonvoid set and $\uplus$ is an associative and reproductive hypercomposition on H.The concept of hypergroups is an algebraic structure, and it is clear that the groups are an example of hypergroups. His French contemporaries continue his ideas to included additional algebraic structures, which they call hypercompositional structures. The nonvoid result of the hypercomposition in hypergroups and in all relevant structures such as hyperfields, hyperrings, hypermodules etc., is a consequence of the associative and reproductive laws ([16, Theorem 12]). Krasner introduces hyperfields, hyperrings, and hypermodules in his papers [10] and [11]. In the literature, the structure hyperring (respectively, hyperfield) is known as Krasner hyperring (respectively, Krasner hyperfield).

The main purpose of this paper is to develop the concept of the hyperring $(\mathbb{N}, \oplus, \cdot)$, where $m \oplus n = \{m + n, k \,|\, \min\{m, n\} + k = \max\{m, n\}\}$, for all $m, n \in \mathbb{N}$ and the operation $\cdot$ is the usual multiplication in $\mathbb{N}$. It follows that the hyperring $(\mathbb{N}, \oplus, \cdot)$ is a principal hyperideal domain. We prove that $(\mathbb{N}, \oplus, \cdot)$ is isomorphic to Krasner's quotient hyperring $\frac{\mathbb{Z}}{G}$ in [10]. Also, we construct the hyperstructure $(\mathbb{N}_m, \oplus_m, \cdot)$, which is a class of examples of hypermodules and hyperrings.

2. Preliminaries

This section briefly recalls the main concepts and results related to types of hyperrings and hypermodules. To better understand the topic, we start with some fundamental definitions of hypercompositional algebra presented in books [5, 7] and overview articles [12, 14, 15, 16, 18].

Let H be a nonvoid set and a mapping $+ : H \times H \longrightarrow \mathcal{P}(H)$ be a hypercomposition on H. Then, $(H, +)$ is said to be a *hypergroupoid*. Moreover, for any nonempty subsets X and Y of H, define

$$X + Y = \bigcup \{z \in x + y \,|\, x \in X \textit{ and } y \in Y\} = \bigcup_{(x,y) \in X \times Y} x + y.$$

We simply write $a+X$ and $X+a$ instead of $\{a\}+X$ and $X+\{a\}$, respectively, for any $a \in H$ and any nonvoid subset X of H. A hypergroupoid $(H,+)$ is said to be a

(1) *semihypergroup* if $+$ is an associative hypercomposition on H.

(2) *quasihypergroup* if $+$ is a reproductive hypercomposition on H.

A nonvoid subset S of a hypergroup $(H,+)$ is said to be a *subhypergroup* of H, if for every $a \in S$, $a+S=S=S+a$.

A hypergroup $(H,+)$ is said to be *canonical hypergroup* if

(1) for every $a,b \in H$, $a+b=b+a$, that is, it is commutative;

(2) There exists a unique $0 \in H$ such that for each $a \in H$ there exists a unique element a' in H, denoted by $-a$, such that $0 \in a+(-a)$;

(3) for every $a,b,c \in H$, if $c \in a+b$, then $a \in c+(-b) := c-b$.

As it is proved in [13], if $(H,+)$ is a canonical hypergroup, then $a+0=a$, for all $a \in H$.

Let $(R,+,.)$ be a hypercompositional structure. $(R,+,.)$ is said to be a *(Krasner) hyperring* if

(1) $(R,+)$ is a canonical hypergroup;

(2) $(R,.)$ is a semigroup with a bilaterally absorbing element 0, i.e.,

  (a) $a.b \in R$, for all a, $b \in R$;
  (b) $a.(b.c)=(a.b).c$, for all a, b, $c \in R$;
  (c) $a.0=0.a=0$, for all $a \in R$;

(3) The multiplication distributes over the addition on both sides.

If in addition:

(4) $a.b=b.a$, for all a, $b \in R$,

then R is said to be a *commutative hyperring*. If $(R,+,.)$ contains an element 1_R such that

(5) $a=a.1_R=1_R.a$ for every $a \in R$,

then R is said to be a hyperring with identity.

Let $(R,+,\cdot)$ be a hyperring and I be a nonvoid subset of R. I is called a *left hyperideal* (respectively, *right hyperideal*) of R provided $(I,+)$ is a subhypergroup and $r.a \in I$ (respectively, $a.r \in I$), for all $a \in I$ and $r \in R$. I is said to be *hyperideal* of R if it is both a right and a left hyperideal of R.

A left Krasner hypermodule over a hyperring R with identity is a canonical hypergroup $(M, +)$ together with a map $R \times M \longrightarrow M$ such that to every (r, m), where $r \in R$ and $m \in M$, there corresponds a uniquely determined element $rm \in M$ and the following conditions are satisfied:

(1) $r(m_1 + m_2) = rm_1 + rm_2$;

(2) $(r + s)m = rm + sm$;

(3) $(r.s)m = r(sm)$;

(4) $1_R m = m$ and $r0_M = 0_R m = 0_M$,

for any m, m_1, $m_2 \in M$ and r, $s \in R$.

Throughout this paper, for a simple explanation, when we say hypermodule, we mean the left Krasner hypermodule. A nonvoid subset N of an R-hypermodule M is called a *subhypermodule* of M, denoted by $N \leq M$ if N is an R-hypermodule under the same hyperoperations of M. It is clear that M and $\{0_M\}$ are *trivial subhypermodules* of M. It is known that a non-empty subset N of an R-hypermodule M is a subhypermodule of M if and only if $a - b \subseteq M$ and $ra \in M$, for all a, $b \in M$ and $r \in R$.

Let R be a hyperring. It follows from [3, Lemma 3.1] that R is an R-hypermodule. Then, a nonvoid subset I of R is a left hyperideal of R if and only if it is a subhypermodule of the hypermodule ${}_R R$.

Let M be a hypermodule over a hyperring R and K be a subhypermodule of M. Consider the set $\frac{M}{K} = \{\, a + K \mid a \in M \,\}$. Then, $\frac{M}{K}$ is a hypermodule over the hyperring R under the hyperoperation $+ : \frac{M}{K} \times \frac{M}{K} \longrightarrow \mathcal{P}(\frac{M}{K})$ and the external operation $\cdot : R \times \frac{M}{K} \longrightarrow \frac{M}{K}$ via $(a+K)+(a'+K) = \{\, b+K \mid b \in a+a' \,\}$ and $r \cdot (a+K) = ra+K$ for every $a, a', b \in M$ and $r \in R$. The hypermodule $\frac{M}{K}$ is called the *quotient hypermodule* of the hypermodule M.

Let M and N be R-hypermodules. A single-valued function $f : M \longrightarrow N$ is called *normal homomorphism* (or briefly, homomorphism) if

(1) $f(m_1 +_M m_2) = f(m_1) +_N f(m_2)$, for all m_1, $m_2 \in M$

(2) $f(rm) = rf(m)$, for all $r \in R$ and $m \in M$.

We denote by $Hom_R(M, N)$ the family of all homomorphisms form M to N.

3. Hyperrings and hypermodules

Let $\mathbb{Z}$ denote the set of all integers. Let $+$ and $\cdot$ denote the usual addition and multiplication. Then, it is well known that the structure $(\mathbb{Z}, +, \cdot)$ is a principal ideal domain. Let $\mathbb{N}$ denote the set of all non-negative integers. However, under the same operations, the structure $(\mathbb{N}, +, \cdot)$ does not have the structure of a ring. We will now construct the hyperring structure on the set $\mathbb{N}$ with the help of the same operations. Then, using this hypercompositional structure, we will give new structures of hyperrings and hypermodules. Note that we will use the following structure of the hyperring we give in this section freely in this article without reference.

Construction. Let $\mathbb{N}$ denote the set of all non-negative integers. Let $+$ and $\cdot$ denote the usual addition and multiplication in $\mathbb{N}$. Define the hypercomposition "$\oplus$" on $\mathbb{N}$ as follows: for any $m, n \in \mathbb{N}$

$$m \oplus n = \{m+n,\, k \mid \min\{m, n\} + k = \max\{m, n\} \text{ for some } k \in \mathbb{N}\}.$$

It is clear that $(\mathbb{N}, \oplus)$ is the hypergroupoid.

(1) Let us, now, show that $(\mathbb{N}, \oplus)$ is the canonical hypergroup. Since $\mathbb{N}$ is well-ordered, we get $m \oplus n = n \oplus m$ and so we can always choose $m \leq n$ whenever $(m, n) \in \mathbb{N} \times \mathbb{N}$.
Let $m, n, p \in \mathbb{N}$. Since $\mathbb{N}$ is well-ordered, we will assume that $m \leq n \leq p$ without restriction of generality. Therefore, $\min\{m, n\} = m$, $\max\{m, n\} = n$, $\min\{n, p\} = n$ and $\max\{n, p\} = p$. It follows that $n = k_1 + m$ and $p = k_2 + n$, for some elements $k_1, k_2 \in \mathbb{N}$. Now,

$$\begin{aligned} m \oplus (n \oplus p) &= m \oplus \{n+p,\, k_2\} \\ &= m \oplus (n+p) \cup m \oplus k_2 \end{aligned}$$

Case 1. Let $m \leq k_2$. Then, $k_2 = k_3 + m$. So we can write $p = n + k_2 = (n+m) + k_3$. Now

$$\begin{aligned} m \oplus (n \oplus p) &= m \oplus \{n+p,\, k_2\} \\ &= m \oplus (n+p) \cup m \oplus k_2 \\ &= \{m+(n+p),\, m+k_1+k_2\} \cup \{m+k_2,\, k_3\} \\ &= \{m+(n+p),\, m+k_1+k_2,\, m+k_2,\, k_3\} \\ &= \{(m+n)+p,\, k_3\} \cup \{m+k_1+k_2,\, m+k_2\} \\ &= \{(m+n)+p,\, k_3\} \cup \{p+k_1,\, m+k_2\} \\ &= (m+n) \oplus p \cup k_1 \oplus p \\ &= \{m+n,\, k_1\} \oplus p = (m \oplus n) \oplus p \end{aligned}$$

Case 2. Let $k_2 \leq m$. Then, we can write $m = k_2 + k_4$, for some element $k_4 \in \mathbb{N}$. Thus,

$$\begin{aligned} m \oplus (n \oplus p) &= m \oplus \{n+p,\, k_2\} \\ &= m \oplus (n+p) \cup m \oplus k_2 \\ &= \{m+(n+p),\, m+k_1+k_2\} \cup \{m+k_2,\, k_4\} \\ &= \{(m+n)+p,\, m+2k_1+k_2,\, m+k_2,\, k_4\} \\ &= \{(m+n)+p,\, k_1+p,\, k_4,\, m+k_2\} \\ &= \{(m+n)+p,\, k_4\} \cup \{k_1+p,\, m+k_2\} \\ &= (m+n) \oplus p \cup (k_1 \oplus p) \\ &= (m \oplus n) \oplus p. \end{aligned}$$

Let $m \in \mathbb{N}$. Then, $m \oplus m = \{m+m, 0\}$ and so $0 \in m \oplus m$. It means that $-m := m$.

For any elements $m, n, p \in \mathbb{N}$, let $m \in n \oplus p = p \oplus n$. Then, there exists an element $k \in \mathbb{N}$ such that $p = n + k$. It follows that $m \in n \oplus p = \{n+p, k\}$ and so $m = n + p$ or $m = k$. Therefore, $n \in m \oplus p$. Hence, $(\mathbb{N}, \oplus)$ is the canonical hypergroup.

(2) It is obvious that $(\mathbb{N}, \cdot)$ is a commutative monoid and $n \cdot 0 = 0$, for all $n \in \mathbb{N}$, where the operation $\cdot$ is the multiplication in $\mathbb{N}$.

(3) Let $m, n, p \in \mathbb{N}$ and $n \leq p$. There, exists an element $k \in \mathbb{N}$ with $n+k = p$. Now

$$\begin{aligned} m \cdot (n \oplus p) = m \cdot \{n+p, k\} &= \{m \cdot (n+p), m \cdot k\} \\ &= \{m \cdot n + m \cdot p, m \cdot k\} \\ &= m \cdot n \oplus m \cdot p \end{aligned}$$

Hence, the structure $(\mathbb{N}, \oplus, \cdot)$ is a hyperdomain.

We will use these conventions $mn = m \cdot n$ for any elements $m, n \in \mathbb{N}$ and the hyperring $(\mathbb{N}, \oplus, \cdot)$ as the hyperring $\mathbb{N}$.

Proposition 3.1. $\mathbb{N}$ *is a principle hyperideal domain.*

Proof. Firstly, note that $a\mathbb{N}$ is a hyperideal of $\mathbb{N}$, for all $a \in \mathbb{N}$. Let I be a hyperideal of the hyperring $\mathbb{N}$. If $I = \{0\}$, then $I = 0\mathbb{N}$. Assume that $I \neq \{0\}$. With the help of the principle of well-ordering, we can show that I contains a smallest positive integer, say $a \in I$. We claim that $I = \{an \,|\, n \in \mathbb{N}\} = a\mathbb{N}$. Clearly, $a\mathbb{N} \subseteq I$. Let $b \in I$. Therefore, we can write $b = aq + r$, $0 \leq r < a$, for some elements $q, r \in \mathbb{N}$. Now, $b \oplus aq = \{b + aq, r\} \subseteq I$ and so $r \in I$. Since $0 \leq r < a$ and a is the smallest positive integer of I, we get $r = 0$. It implies that $b = aq \in a\mathbb{N}$. Hence, $I = a\mathbb{N}$. □

Proposition 3.2. *Let I be a non-zero hyperideal of $\mathbb{N}$. Then, I contains a non-zero hyperideal K of $\mathbb{N}$ with $I \neq K$.*

Proof. By Proposition 3.1, we can write $I = a\mathbb{N}$, for some element $a \in I$. Let $0 \neq m \in \mathbb{N}$. Now, we consider the hyperideal $M = (ma)\mathbb{N}$. Then, K is a hyperideal of $\mathbb{N}$ and $I \neq K$. This completes the proof. □

Theorem 3.1. *Let I be a non-trivial hyperideal of $\mathbb{N}$. Then, the following statements are equivalent:*

(1) *I is a maximal hyperideal of $\mathbb{N}$.*

(2) *I is a prime hyperideal of $\mathbb{N}$.*

(3) *There exists a prime positive integer $\mathfrak{p} \in \mathbb{N}$ such that $\mathfrak{p}\mathbb{N} = I$*

Proof. (1) $\Rightarrow$ (2). By [7, Proposition 3.3.7].

(2) $\Rightarrow$ (3). By Proposition 3.1, we can write $I = \mathfrak{p}\mathbb{N}$, for some element $\mathfrak{p} \in \mathbb{N}$. Let $a, b \in \mathbb{N}$ be such that $\mathfrak{p} = ab$. Since I is a prime hyperideal of $\mathbb{N}$, it follows from [7, Lemma 3.3.6] that $a \in I$ or $b \in I$. Therefore, either $\mathfrak{p}|a$ or $\mathfrak{p}|b$. Thus, $\mathfrak{p}$ is a prime element of $\mathbb{N}$.

(3) $\Rightarrow$ (1). Let M be a hyperideal of $\mathbb{N}$ such that $I \subseteq M \subset \mathbb{N}$. Again applying Proposition 3.1, there exists an element $a \in M$ with $M = a\mathbb{N}$. Then, $\mathfrak{p} = \mathfrak{p}1 \in \mathfrak{p}\mathbb{N} \subseteq M = a\mathbb{N}$ and so $ab = \mathfrak{p}$, for some $b \in M$. Since $\mathfrak{p}$ is prime, $\mathfrak{p} = a$ or $\mathfrak{p} = b$. Thus, $I = M$. $\square$

Remark 3.1. Krasner gave a method for the construction of hyperrings (see [10, Theorem]). Let $(S, +, .)$ be a commutative ring with unity and $(G, .)$ be a subgroup of the monoid $(S, .)$. Then, $\{aG\}_{a \in S}$ is a partition of S and so this partition defines an equivalence relation on S as follows:

$$\text{“}a \sim b \Longleftrightarrow aG = bG\text{”}.$$

Let $\frac{S}{G}$ be the set of all equivalence classes aG. Define

$$aG + bG = \{cG \mid c = ax + by \text{ for some } x, y \in G\} \subseteq P^*(\frac{S}{G})$$

and

$$aG \cdot bG = abG.$$

Then, $(\frac{S}{G}, +, .)$ is a commutative hyperring. In particular, if $(S, +, .)$ is a field, then $(\frac{S}{G}, +, .)$ is a hyperfield. Krasner calls the hyperring $\frac{S}{G}$ the quotient hyperring of S by G.

Now, we shall show that the hyperring $\mathbb{N}$ is isomorphic to one of Krasner's quotient hyperrings.

Theorem 3.2. *Let $(\mathbb{Z}, +,)$ denote the ring of integers and $H = \{-1, 1\}$. Then, the hyperring $(\frac{\mathbb{Z}}{H}, +, .)$ is isomorphic to the hyperring $(\mathbb{N}, \oplus, \cdot)$.*

Proof. Define $f : \mathbb{N} \longrightarrow \frac{\mathbb{Z}}{H}$ by $f(n) = \overline{n} = \{-n, n\}$, for all $n \in \mathbb{N}$. Let $n, m \in \mathbb{N}$ with $n + k = m$. Now,

$$\begin{aligned} f(n \oplus m) &= \bigcup_{r \in n \oplus m} \{f(r)\} \\ &= \{f(n+m), f(k)\} \\ &= \{\overline{n+m}, \overline{k}\} \\ &= \{\overline{n+m}, \overline{n-m}\} \\ &= f(n) + f(m) \end{aligned}$$

and $f(nm) = \overline{nm} = \overline{n}.\overline{m} = f(n)f(m)$, which implies that f is a homomorphism of hyperrings. Clearly, f is surjective. Let $n \in Ker(f)$. It follows that $f(n) = \overline{n} = 0$ and so $n = 0$. It means that f is injective. Hence, the hyperring $(\frac{\mathbb{Z}}{H}, +, .)$ is isomorphic to the hyperring $(\mathbb{N}, \oplus, \cdot)$ as required. $\square$

Following [19], we construct the fractional hyperrings of the hyperring $\mathbb{N}$. Let S be a multiplicatively closed subset $\mathbb{N}$ such that $0 \notin S$. The relation on the set $\mathbb{N} \times S$ defined by

$$\text{“}(a,\, b) \equiv (c,\, d) \Longleftrightarrow \text{there exists } u \in S \text{ such that } u(ad) = u(bc)\text{”}.$$

This is an equivalence relation on the set $\mathbb{N} \times S$. The equivalence class of $(a,\, b)$ is denoted by $\frac{a}{b}$ and the set of all equivalence classes is denoted by $S^{-1}\mathbb{N}$. Define the hyperoperation $+$ and the operation $.$ on $S^{-1}\mathbb{N}$ as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad \oplus bc}{bd} = \{\frac{e}{bd} | e \in ad \oplus bc\}$$

and

$$\frac{a}{b} . \frac{c}{d} = \frac{ac}{bd}$$

for all $\frac{a}{b},\, \frac{c}{d} \in S^{-1}\mathbb{N}$. It follows [8, Theorem 3.1] that $(S^{-1}\mathbb{N},\, +,\, .)$ is a hyperring. Here, $\frac{0}{b} := 0$ is the scalar identity element of $(S^{-1}\mathbb{N},\, +)$. Moreover, for all $\frac{a}{b} \in S^{-1}\mathbb{N}$, $\frac{a}{b} . \frac{1}{1} = \frac{a1}{b} = \frac{a}{b}$ and so $\frac{1}{1} := 1$ is an identity element of the hyperring $(S^{-1}\mathbb{N},\, +,\, .)$. Hence, $(S^{-1}\mathbb{N},\, +,\, .)$ is a commutative hyperdomain.

Corollary 3.1. *Let $S = \mathbb{N} \backslash \{0\}$. Then, $(S^{-1}\mathbb{N},\, +,\, .)$ is a hyperfield.*

Proof. Let $0 \neq \frac{a}{b} \in S^{-1}\mathbb{N}$. Therefore, $\frac{a}{b} . \frac{b}{a} = 1$. It means that $(S^{-1}\mathbb{N},\, +,\, .)$ is a hyperfield. $\square$

Observe from Theorem 3.1 that $S^{-1}\mathbb{N} = \{\frac{a}{b} | a,\, b \in \mathbb{N} \textit{ and } b \neq 0\} = \mathbb{Q}^{\geq 0}$. Therefore, $(\mathbb{Q}^{\geq 0},\, +,\, .)$ is a hyperfield, where $\frac{a}{b} + \frac{c}{d} = \frac{ad \oplus bc}{bd}$ and $\frac{a}{b} . \frac{c}{d} = \frac{ad}{bc}$, for all $\frac{a}{b},\, \frac{c}{d} \in \mathbb{Q}^{\geq 0}$.

Let R be a non-zero hyperring with identity. Recall from [9] that R is *local* if R has the only left maximal hyperideal. Now, we give the following example. Later we shall give other examples of such hyperrings (see Proposition 3.3)

Example 3.1. Let $\mathfrak{p}$ be a prime element of $\mathbb{N}$ and $\mathcal{P} = \mathfrak{p}\mathbb{N}$. Then, by [8, Theorem 3.6-(ii)], $\mathcal{P}_{\mathfrak{p}}$ is the only maximal hyperideal of $\mathcal{P}^{-1}\mathbb{N}$. Therefore, $\mathcal{P}^{-1}\mathbb{N}$ is a local hyperring.

Let R be a hyperring and M be an R-hypermodule. Following [4], M is said to be *divisible* if for every $r \in R$ which is not a zero divisor and every $m \in M$, there exists $m' \in M$ such that $rm' = m$.

Example 3.2. Define $\cdot : \mathbb{N} \times \mathbb{Q}^{\geq 0} \longrightarrow \mathbb{Q}^{\geq 0}$ via $n\frac{a}{b} = \frac{na}{b}$, for all $n \in \mathbb{N}$ and $\frac{a}{b} \in \mathbb{Q}^{\geq 0}$. Let $m,\, n \in \mathbb{N}$ and $\frac{a}{b},\, \frac{c}{d} \in \mathbb{Q}^{\geq 0}$. Now

(1) $(m \oplus n)\frac{a}{b} = \bigcup_{r \in m \oplus n} r\frac{a}{b} = m\frac{a}{b} + n\frac{a}{b}$;

(2) $m(\frac{a}{b} + \frac{c}{d}) = m(\frac{ad \oplus bc}{bd}) = m\frac{a}{b} + m\frac{c}{d}$;

(3) $mn(\frac{a}{b}) = \frac{(mn)a}{b} = \frac{m(na)}{b} = m\frac{na}{b} = m(n\frac{a}{b})$;

(4) $1\frac{a}{b} = \frac{1a}{b} = \frac{a}{b}$.

Thus, $(\mathbb{Q}^{\geq 0}, +)$ is a $\mathbb{N}$-hypermodule. To show that $(\mathbb{Q}^{\geq 0}, +)$ is a divisible, let $n \in \mathbb{N} \backslash \{0\}$ and $\frac{a}{b} \in \mathbb{Q}^{\geq 0}$. Then, $\frac{a}{b} = n\frac{a}{nb}$, which implies that $\mathbb{Q}^{\geq 0}$ is divisible.

Let $m > 1$. Define the relation "$\equiv$" on $\mathbb{N}$ by for all $x, y \in \mathbb{N}$

$$\text{“}x \equiv y \Longleftrightarrow m|k, \text{ where } \min\{x, y\} + k = \max\{x, y\}\text{”}.$$

It can be seen that "$\equiv$" is an equivalence relation on $\mathbb{N}$. Let $\mathbb{N}_m = \{\overline{x} \,|\, x \in \mathbb{N}\}$, where $\overline{x} = \{0 + x, m + x, 2m + x, ...\} = \{nm + x \,|\, n \in \mathbb{N}\}$. Let $0 \leq x < y < m$. Suppose that $\overline{x} = \overline{y}$. Then, $y \in \overline{x}$ and so $m|k$, $x + k = y$, for some $k \in \mathbb{N}$. This is a contradiction since $0 < k < m$. Hence, the equivalence classes $\overline{0}, \overline{1}, ..., \overline{m-1}$ are distinct. Let $\overline{x}$ be any element of $\mathbb{N}_m$. By the division algorithm, $x = mq + r$, for some elements q and r such that $0 \leq r < m$. Since $m|mq$, we obtain that $\overline{r} = \overline{x}$. Hence, $\mathbb{N}_m = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{m-1}\}$.

Theorem 3.3. *Let $m > 1$. Define "$\oplus_m$" on $\mathbb{N}_m$ by*

$$\overline{x} \oplus_m \overline{y} = \{\overline{x+y}, \overline{k} | \min\{x, y\} + k = \max\{x, y\}\},$$

for all $\overline{x}, \overline{y} \in \mathbb{N}_m$. Then

(1) $(\mathbb{N}_m, \oplus_m)$ is a canonical hypergroup with scalar identity $\overline{0}$.

(2) $(\mathbb{N}_m, \oplus_m, .)$ is a commutative and unitary hyperring, where "." is the usual multiplication.

(3) $(\mathbb{N}^*_m, .)$ is a group, where $\mathbb{N}^*_m = \{\overline{x} \in \mathbb{N}_m \,|\, (x, m) = 1\}$.

(4) $(\mathbb{N}_m, \oplus_m, .)$ is a hyperfield if and only if m is prime.

(5) there exists a isomorphism of hyperrings $f : \frac{\mathbb{N}}{\mathbb{N}_m} \longrightarrow \mathbb{N}_m$.

(6) The canonical hypergroup $(\mathbb{N}_m, \oplus_m)$ is a $\mathbb{N}$-hypermodule.

Proof. (1), (2) and (3) are straightforward.

(4) $(\Rightarrow)$ Let $m = ab$, where $1 \leq a < b < m$. Then, $\overline{a}, \overline{b} \in \mathbb{N}_m$ and so $\overline{a}.\overline{b} = \overline{ab} = \overline{0}$, a contradiction.

$(\Leftarrow)$ Let $\overline{a} \in \mathbb{N}^*_m$. Then, $(a, m) = 1$ and so, we get $1 = ax + my$, for some $x, y \in \mathbb{N}$. It follows that $\overline{1} = \overline{ax + my} = \overline{ax} = \overline{a}.\overline{x}$. Hence, $\mathbb{N}^*_m = \mathbb{N}_m \backslash \{\overline{0}\}$. By (3), $(\mathbb{N}_m, \oplus_m, .)$ is a hyperfield.

(5) Consider the map $\Phi : \mathbb{N} \longrightarrow \mathbb{N}_m$ via $\Phi(x) = \overline{x}$, for all $x \in \mathbb{N}$. Let $x, y \in \mathbb{N}$. Assume that $x + k = y$, for some $k \in \mathbb{N}$. Now,

$$\Phi(x \oplus y) = \bigcup_{r \in x \oplus y} \{\Phi(r)\} = \{\Phi(x+y), \Phi(k)\} = \{\overline{x+y}, \overline{k}\} = \overline{x} \oplus_m \overline{y} = \Phi(x) \oplus_m \Phi(y)$$

and $\Phi(xy) = \overline{xy} = \overline{x}.\overline{y} = \Phi(x)\Phi(y)$. Thus, Φ is a homomorphism of hyperrings. It is clear that Φ is surjective with $Ker(\Phi) = m\mathbb{N}$. Thus, we obtain that $\frac{\mathbb{N}}{\mathbb{N}_m} \cong \mathbb{N}_m$.

(6) Define the map $\cdot : \mathbb{N} \times \mathbb{N}_m \longrightarrow \mathbb{N}_m$ via $n \cdot \overline{x} = \overline{nx}$, for all $n \in \mathbb{N}$ and for all $\overline{x} \in \mathbb{N}_m$. According to the map, it can be checked that N_m is a $\mathbb{N}$-hypermodule. $\square$

The next result gives examples of local hypermodules.

Proposition 3.3. *Let $\mathfrak{p}$ be a prime positive integer. Then, $(\mathbb{N}_{\mathfrak{p}^k}, \oplus, .)$ is a local hyperring, for all $k > 0$.*

Proof. Let $k > 0$. Using Theorem 3.3 (5), we deduce that $\Phi(\mathfrak{p}\mathbb{N})$ is the only maximal hyperideal of the hyperring $(\mathbb{N}_{\mathfrak{p}^k}, \oplus, .)$. Hence, $(\mathbb{N}_{\mathfrak{p}^k}, \oplus, .)$ is a local hyperring. $\square$

Note that the condition "prime positive integer" in the above proposition is necessary. Let's take the following example to see this.

Example 3.3. Given the the hyperring $\mathbb{N}_6$. Using Theorem 3.3, we obtain the following tables:

$\oplus_6$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$
$\overline{1}$	$\{\overline{1}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{0}, \overline{4}\}$
$\overline{2}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{0}, \overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{3}$	$\{\overline{3}\}$	$\{\overline{2}, \overline{4}\}$	$\{\overline{1}, \overline{5}\}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$
$\overline{4}$	$\{\overline{4}\}$	$\{\overline{3}, \overline{5}\}$	$\{\overline{2}\}$	$\{\overline{1}\}$	$\{\overline{0}, \overline{2}\}$	$\{\overline{1}, \overline{3}\}$
$\overline{5}$	$\{\overline{5}\}$	$\{\overline{0}, \overline{4}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{2}\}$	$\{\overline{1}, \overline{3}\}$	$\{\overline{0}, \overline{4}\}$

and

.	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{0}$	$\overline{2}$	$\overline{4}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{2}$	$\overline{0}$	$\overline{4}$	$\overline{2}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Thus, the only maximal hyperideals of the hyperring $\mathbb{N}_6$ are $I_1 = \{\overline{0}, \overline{3}\}$ and $I_2 = \{\overline{0}, \overline{2}, \overline{4}\}$. Also, we have $\mathbb{N}_6 = I_1 \oplus_6 I_2$. It follows that every hyperideal of $\mathbb{N}_6$ is a direct summand of $\mathbb{N}_6$. Hence, the hyperring $\mathbb{N}_6$ is not local.

For hyperstructures the example we will give below is an analogue of $\mathbb{Z}_{\mathfrak{p}^\infty}$, which has a very important place in classical algebra.

Example 3.4. For a prime positive integer $\mathfrak{p} \in \mathbb{N}$, consider the following set:

$$\mathbb{N}[\frac{1}{\mathfrak{p}}] = \{\frac{m}{\mathfrak{p}^s} \in \mathbb{Q}^{\geq 0} \mid m,\, s \in \mathbb{N} \textit{ and } \mathfrak{p} \textit{ is prime}\}.$$

Let $x = \frac{m}{\mathfrak{p}^s}$, $y = \frac{n}{\mathfrak{p}^k} \in \mathbb{N}[\frac{1}{\mathfrak{p}}]$. Then, $x+y = \frac{m}{\mathfrak{p}^s} + \frac{n}{\mathfrak{p}^k} = \frac{m\mathfrak{p}^k \oplus n\mathfrak{p}^k}{\mathfrak{p}^{s+k}} \in \mathbb{N}[\frac{1}{\mathfrak{p}}]$. Thus, $\mathbb{N}[\frac{1}{\mathfrak{p}}]$ is a canonical subhypergroup of $\mathbb{Q}^{\geq 0}$. Now, we consider the quotient canonical hypergroup $\frac{\mathbb{N}[\frac{1}{\mathfrak{p}}]}{\mathbb{N}}$ of the canonical hypergroup $\mathbb{N}[\frac{1}{\mathfrak{p}}]$ by $\mathbb{N}$. Put $\mathbb{N}_{\mathfrak{p}^\infty} := \frac{\mathbb{N}[\frac{1}{\mathfrak{p}}]}{\mathbb{N}}$.

For every $s = 1, 2, ...$, let $c_s = \frac{1}{\mathfrak{p}^s} + \mathbb{N} \in \mathbb{N}_{\mathfrak{p}^\infty}$. Then

$$\mathfrak{p}c_1 = 0;\ \mathfrak{p}c_2 = c_1;\ ...,\ \mathfrak{p}c_{s+1} = c_s.$$

Therefore, the set $\{c_1, c_2, ..., c_s, ...\}$ generates the canonical hypergroup $\mathbb{N}_{\mathfrak{p}^\infty}$. Let H be any proper canonical subhypergroup of $\mathbb{N}_{\mathfrak{p}^\infty}$. Put $n = sup\{k \mid c_k \in H\}$. If $n = \infty$, then there exists $s \in \mathbb{N}$ such that $k > s$ and $c_k \in H$ for every $a = \frac{m}{\mathfrak{p}^s} + \mathbb{N} \in \mathbb{N}_{\mathfrak{p}^\infty}$. It follows that $a = m\mathfrak{p}^t(\frac{1}{\mathfrak{p}^k} + \mathbb{N}) = m\mathfrak{p}^t c_k + \mathbb{N}$, where $s+t = k$, for some $t \in \mathbb{N}$. So $H = \mathbb{N}_{\mathfrak{p}^\infty}$. This is a contradiction. Thus, $n < \infty$. Next, we show that $H =< c_n >$. Clearly, $< c_n >\subseteq H$. Let $a = \frac{m}{\mathfrak{p}^s} + \mathbb{N} \in H$. We can take $(m, \mathfrak{p}^s) = 1$ without losing generality. Then, we can write $mu + \mathfrak{p}^s v = 1$, for some elements $u, v \in \mathbb{N}$. Thus, $au = \frac{mu}{\mathfrak{p}^s} + \mathbb{N} = \frac{1}{\mathfrak{p}^s} + \mathbb{N} = c_s$ and then $c_s = au \in H$. From this choice of n, we obtain that $s \leq n$. Therefore, $a = m\mathfrak{p}^t c_n \in< c_n >$, where $s + t = n$. It means that $H =< c_n >$. Also, it can be seen that $< c_n >\cong \mathbb{N}_{\mathfrak{p}^n}$. Hence, $\mathbb{N}_{\mathfrak{p}^\infty} = \bigcup_{n\in\mathbb{N}} \mathbb{N}_{\mathfrak{p}^n}$.

Define $\cdot : \mathbb{N} \times \mathbb{N}_{\mathfrak{p}^\infty} \longrightarrow \mathbb{N}_{\mathfrak{p}^\infty}$ by $n \cdot (\frac{m}{\mathfrak{p}^s} + \mathbb{N}) = \frac{nm}{\mathfrak{p}^s} + \mathbb{N}$, for all $n \in \mathbb{N}$ and $\frac{m}{\mathfrak{p}^s} + \mathbb{N} \in \mathbb{N}_{\mathfrak{p}^\infty}$. Thus, it is easily seen that $\mathbb{N}_{\mathfrak{p}^\infty}$ is a normal injective $\mathbb{N}$-hypermodule.

In [6], an R-hypermodule M is said to be *simple* if $RM \neq 0$ and M has no subhypermodules other than $\{0_M\}$ and M. It is shown in [6, Lemma 3.9] that an R-hypermodule M is simple if and only if it is isomorphic to $\frac{R}{I}$, for some maximal left hyperideal I of R. Using this fact and Theorem 3.1, we deduce that a simple $\mathbb{N}$-hypermodule M is of the form $\frac{\mathbb{N}}{\mathfrak{p}\mathbb{N}} \cong \mathbb{N}_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime positive integer. Then, we have:

Corollary 3.2. *Every simple $\mathbb{N}$-hypermodule can be embedded the normal injective $\mathbb{N}$-hypermodule $\mathbb{N}_{\mathfrak{p}^\infty}$, for some prime positive integer $\mathfrak{p} \in \mathbb{N}$.*

4. Conclusions

In this study, we constructed the hyperring structure on the set $\mathbb{N}$ with the help of the usual operations. Thanks to this construction, we obtain very useful classes of hypermodules and hyperrings. These classes are a resource for researchers working in this category of hypermodules.

Acknowledgments

The third author gratefully acknowledge the support they have received from TUBITAK (The Scientific and Technological Research Council of Turkey) with Grant No. 123F236. Also, the authors thank the referee for his/her careful reading of the paper, which has greatly helped improve its presentation.

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Accepted: January 28, 2025