An overview of hypercompositional algebra applications on graphs

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Abstract. Graphs are fundamental structures in mathematics and computer science for modeling relationships between objects. This paper studies three hypercompositional structures that are derived from graphs, namely the Path hyperoperation, Simple Path hyperoperation, and Ancestry hyperoperation. These hyperoperations capture complex relationships, offering a robust framework for analyzing intricate connections within graphs. We investigate their properties and provide detailed examples to illustrate their applications.

Keywords: hypercompositional algebras, graph theory, hyperstructures.

MSC 2020: 05C20, 05C25, 08Cxx.

1. Introduction

Graphs are ubiquitous in various fields such as computer science, transportation and communication, biology, and social sciences, serving as a crucial model for representing relationships among entities [1], [16]-[17], [29]. Traditionally, graph theory has mainly utilized adjacency matrices or doubly ranked graph operations to represent graphs and analyze their properties [14]-[15], [21]-[22]. This approach allows for the straightforward examination of graph structures and the relationships between vertices. Recently, the introduction of algebraic hyperstructures has provided a powerful alternative for representing and exploring the properties of graph structures. This approach takes advantage of the well-established relationship between hyperstructures and binary relations [9]-[11], [32]-[34], see also [2], [8], [23], [26]-[28], [30]-[31]. Additionally, the inherent symmetry with hyperstructures has been further explored in [3], [12]-[13], [24], [25].

The application of hypercompositional algebra in graph theory has demonstrated its robustness and convenience, providing new insights and methods for exploring complex graph characteristics [18], [19], [35]. By leveraging the rich algebraic framework of hyperstructures, researchers can uncover deeper connections and more intricate properties within graph theory, enhancing the analytical capabilities and expanding the scope of graph analysis.

This paper delves into three such hyperoperations: Path hyperoperation, Simple Path hyperoperation, and Ancestry hyperoperation, exploring their theoretical foundations and practical implications. We demonstrate that the characteristics of these hyperoperations are dictated by the structure of their underlying graphs and that these characteristics define the corresponding classes of graphs. This important connection between hyperoperations and graph properties reveals that we can identify and understand graphs by studying the properties of the related hyperoperations, and vice versa.

2. Preliminaries

A **Hypergroupoid** $(V, \star)$ consists of a nonempty set V and a hyperoperation

$$\star : V \times V \to \mathcal{P}(V),$$

where $\mathcal{P}(V)$ is the powerset of V. A hypergroupoid is called:

Nonpartial if $v \star w \neq \emptyset$ for all $v, w \in V$,

Degenerative if $v \star w = \emptyset$ for all $v, w \in V$,

Total if $v \star w = V$ for all $v, w \in V$.

Given a binary relation $R \subseteq V \times V$, Corsini's hyperoperation [4]-[7], is defined by the below mapping

$$(v, w) \mapsto v *_R w = \{z \in V \mid (v, z),\ (z, w) \in R\}.$$

A **directed graph** G is defined as a pair (V, E) where V is a set of vertices or nodes and $E \subseteq V \times V$ is a set of directed edges, represented as ordered pairs from the set V.

Example 1. Consider the directed graph G_1 of Figure 1 with set of vertices $V_1 = \{v, w, x, y, z\}$ and set of edges

$$E_1 = \{(v, w), (w, x), (w, y), (x, y), (y, z), (z, x)\}.$$

A subgraph of a directed graph $G = (V, E)$ is a graph $H = (V_H, E_H)$ where:

$V_H \subseteq V$ is a subset of the vertices of G,

$E_H \subseteq E$ is a subset of the edges of G such that every edge in E_H has both endpoints in V_H.

Example 2. A subgraph of the graph G_1 of Example 1 is the graph H_1 of figure 2 with set of vertices $V_{H_1} = \{v, w, x\}$ and set of edges $E_{H_1} = \{(v, w), (w, x)\}$.

A **path** P of length n from a node u_1 to a node u_n in a graph $G = (V, E)$ is a sequence of edges

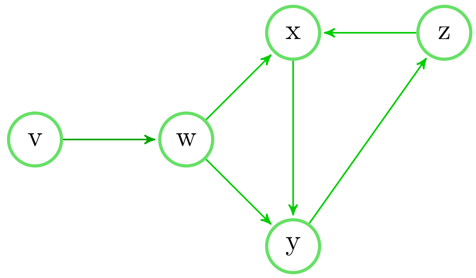

Figure 1: A directed graph G_1

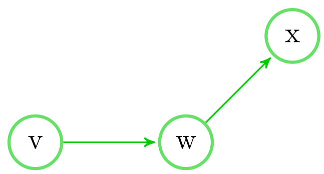

Figure 2: A subgraph H_1 of the graph G_1

$$(u_1, u_2), (u_2, u_3), \ldots, (u_{n-1}, u_n),$$

where each edge $(u_i, u_{i+1}) \in E$. Based on this definition, we say that the nodes $u_1, u_2, \ldots, u_n$ lie on the path P and the same for all the edges in the above sequence. We also note that nodes and edges may appear more than once in a path. If all nodes appear at most once in a path then the path is called **simple**. A **cycle** is a path that starts and ends at the same node. The empty sequence ϵ is a cycle from u to itself, containing only u. The set of all nodes that lie in a path from u_1 to u_n is denoted

$$path(u_1, u_n) = \{x \mid x \text{ lies on a path from } u_1 \text{ to } u_n\}.$$

Example 3. Considering the graph G_1 of Example 1 we see that the sequence $(v, w), (w, x), (x, y)$ is a path from v to y, only the empty path ϵ exists from w to w while there are infinite paths from x to x and there are infinite paths from w to z. Hence we have

$$path(v, w) = \{v, w\},\ path(w, v) = \emptyset,\ path(w, y) = \{w, x, y, z\},$$

and

$$path(w, w) = \{w\},\ path(x, x) = \{x, y, z\}.$$

A directed graph G is called **strongly connected** if, for any two nodes u_1 and u_2, there exists at least one path from u_1 to u_2 and vice versa.

Example 4. The graph S of figure 3 with set of vertices $V_S = \{a, b, c, d\}$ and set of edges

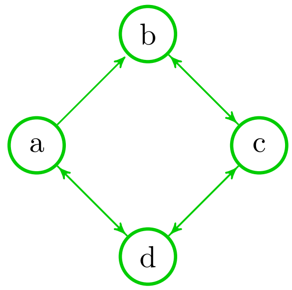

Figure 3: A strongly connected graph S

$$E_S = \{(a, b), (a, d), (d, a), (b, c), (c, b), (c, d), (d, c)\}$$

is strongly connected.

A strongly connected subgraph of a graph G is called **strongly connected component** of G.

Example 5. Considering the graph G_1 of Example 1, we can define a subgraph of it H, depicted in Figure 4, with set of vertices $V_H = \{x, y, z\}$ and set of edges $E_H = \{(x, y), (y, z), (z, x)\}$.

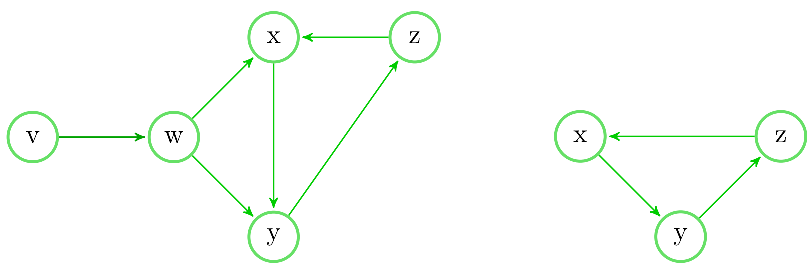

Figure 4: The graph G_1 of Example 1 and a strongly connected subgraph H

3. Hyperoperations on graphs

In this section we are going to introduce three main hyperoperations on graphs. We start with the path hyperoperation $\star_G$ which is defined for a given directed graph G as a mapping that associates a given pair of graph nodes (vertices) to the set composed of all graph nodes that lie on a directed path between the given nodes. This definition extends a well-known hyperoperation introduced by Corsini, where given a relation R on V and u, v elements of V, their product with respect to the Corsini operation is properly included in $u \star_G v$. More formally, given a graph $G = (V, E)$, the path hyperoperation is a mapping

$$\star_G : V \times V \to \mathcal{P}(V)$$

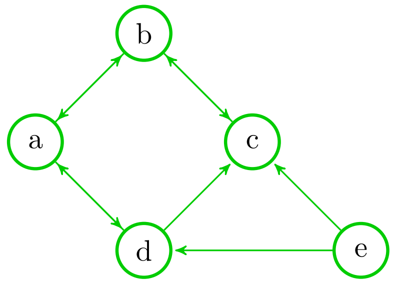

Figure 5: The directed graph G_2 of Example 7

defined by

$$v \star_G w = \{u \in V \mid u \text{ lies on a path between } v \text{ and } w\}.$$

The hyperstructure $(V, \star_G)$ is called the path hypergroupoid corresponding to G.

Example 6. For the graph G_1 of Example 1 the related path hyperoperation is given in the following table.

$\star_{G_1}$	v	w	x	y	z
v	$\{v\}$	$\{v, w\}$	$\{v, w, x, y, z\}$	$\{v, w, x, y, z\}$	$\{v, w, x, y, z\}$
w	$\emptyset$	$\{w\}$	$\{w, x, y, z\}$	$\{w, x, y, z\}$	$\{w, x, y, z\}$
x	$\emptyset$	$\emptyset$	$\{x, y, z\}$	$\{x, y, z\}$	$\{x, y, z\}$
y	$\emptyset$	$\emptyset$	$\{x, y, z\}$	$\{x, y, z\}$	$\{x, y, z\}$
z	$\emptyset$	$\emptyset$	$\{x, y, z\}$	$\{x, y, z\}$	$\{x, y, z\}$

We are now ready to illustrate the relationship between the path hyperoperation and some well known graph properties cf. [20]. The first result illustrates the relationship of the path hyperoperation with the existence of strongly connected components of graphs.

Theorem 1. *For any graph $G = (V, E)$ and nodes $u_1, u_2 \in V$, the following conditions are equivalent*

i) $u_1 \star_G u_2 \neq \emptyset$ *and* $u_2 \star_G u_1 \neq \emptyset$.

ii) There is a strongly connected component of G that includes the nodes u_1 and u_2.

Example 7. The path hyperoperation of the graph G_2 depicted in Figure 5 is given below.

$\star_{G_2}$	a	b	c	d	e
a	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
b	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
c	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
d	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\emptyset$
e	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{a,b,c,d,e\}$	$\{e\}$

A strongly connected component of G_2 is the graph H_2 depicted in Figure 6. It is easy to check that the conditions of Theorem 1 are satisfied for the nodes a, b, c, d of G_2.

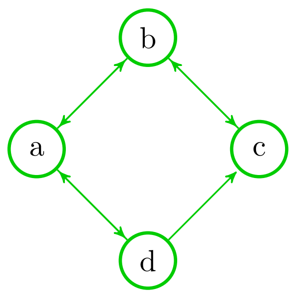

Figure 6: A connected component of the graph G_2

The next result characterizes strong connectivity cf. [18].

Theorem 2. *For any graph $G = (V, E)$, the below conditions are equivalent*

i) The graph G is strongly connected.

ii) The corresponding hypergroupoid $(V, \star_G)$ is nonpartial.

iii) The hyperoperation $\star_G$ is total.

Example 8. It is straightforward to check that the below path hyperoperation table of the strongly connected graph G_3 of Figure 7 satisfies conditions $ii)$ and

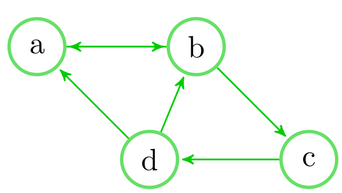

Figure 7: The Strongly connected graph G_3 of Example 8

iii) of Theorem 2.

$\star_G$	a	b	c	d
a	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
b	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
c	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$
d	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$

The existence of cycles inside a graph can be also investigated by examining the properties of the related path hyperoperation cf. [19].

Theorem 3. *Given a graph $G = (V, E)$ and nodes $u_1, u_2 \in V$, the following conditions are equivalent.*

i) *It holds $u_1 \star_G u_2 = u_2 \star_G u_1 \neq \emptyset$.*

ii) *There exists a cycle in G that includes both nodes u_1 and u_2.*

Example 9. By examining graph G_1 of Example 1 we can identify a cycle going through the nodes x, y, z. We can verify the validity of Theorem 3 by checking the corresponding path hyperoperation table of G_1 presented in Example 6.

Commutativity of the path hyperoperation is related with strongly connected graph components as it is illustrated in the next theorem cf. [20].

Theorem 4. *The below conditions are equivalent for a given graph $G = (V, R)$.*

i) *The path hyperoperation $\star_G$ is commutative.*

ii) *G can be obtained as the union of disjoint strongly connected graphs.*

Associativity of the path hyperoperation can be obtained as a corollary of the following theorem cf. [19].

Theorem 5. *Given a graph $G = (V, E)$ and nodes $v, w, u \in V$ it holds*

$$(v \star_G w) \star_G u = v \star_G (w \star_G u) = v \star_G w \star_G u.$$

Corollary 1. *The path hyperoperation is associative*

Given a graph G, the **simple path hyperoperation** $\star_G^s$ maps a pair of nodes of the graph G to the set that includes all nodes lying on a directed simple path between the two given nodes. It is evident that the simple path hyperoperation between two nodes of a graph results in a set that is always included in the path hyperoperation between the same nodes. To formally introduce it, given a graph $G = (V, E)$, the simple path hyperoperation is a mapping

$$\star_G^s : V \times V \to \mathcal{P}(V)$$

defined by

$$u_1 \star_G^s u_2 = \{u \in V \mid u \text{ lies on a simple path between } u_1 \text{ and } u_2\}.$$

The hyperstructure $(V, \star_G^s)$ is called the simple path hypergroupoid corresponding to G.

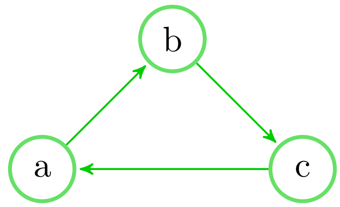

Figure 8: The graph G_4 of Example 10

Example 10. For the graph G_4 of Figure 8, the table of the simple path hyperoperation is given below.

$\star_G^s$	a	b	c
a	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$
b	$\{a,b,c\}$	$\{a,b,c\}$	$\{b,c\}$
c	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

It is clear that $\star_G^s$ is nonpartial and not total. Hence nonpartiality and totality are not equivalent for the simple path hyperoperation as opposed to the case for the path hyperoperation as it is described in Theorem 2.

Discrete graphs can be also characterized by the path and the simple path hyperoperations as follows.

Proposition 1. *For a graph $G = (V, E)$, the following conditions are equivalent*

i) The graph G is discrete.

ii) The path hyperoperation $\star_G$ organizes V into a weakly degenerative hypergroupoid.

iii) The simple path hyperoperation $\star_G^s$ organizes V into of a weakly degenerative hypergroupoid.

The third hyperoperation we will introduce is the **ancestry hyperoperation**, which assigns any two nodes u_1 and u_2 of a graph G to the set of all the nodes of G that have paths going to u_1 and u_2. Formally we have the following definition, given a graph $G = (V, E)$, the ancestry hyperoperation is a mapping

$$\star_G^c : V \times V \to \mathcal{P}(V)$$

defined by

$$u_1 \star_G^c u_2 = \{u \in V \mid \mathrm{path}(u, u_1) \neq \emptyset \text{ and } \mathrm{path}(u, u_2) \neq \emptyset\}.$$

The hyperstructure $(V, \star_G^c)$ is called the ancestry hypergroupoid corresponding to G.

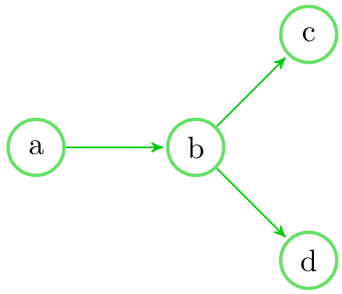

Figure 9: The graph G_5 of Example 11

Example 11. Consider the graph G_5 of Figure 9. In the below tables of the path and the ancestry hyperoperation we can see that the path hyperoperation is partial and non-commutative as opposed to the ancestry hyperoperation which is non-partial and commutative.

$\star_G$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,d\}$
b	$\emptyset$	$\{b\}$	$\{b,c\}$	$\{b,d\}$
c	$\emptyset$	$\emptyset$	$\{c\}$	$\emptyset$
d	$\emptyset$	$\emptyset$	$\emptyset$	$\{d\}$

$\star_G^c$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$
d	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,d\}$

The ancestry hyperoperation of the previous example was commutative and in the next theorem we see that this is a property that holds in general [20].

Theorem 6. *The ancestry hyperoperation $\star_G^c$ is commutative and associative.*

4. Conclusion

By investigating hyperstructures derived from graphs, we obtain a powerful framework to analyze complex relationships within graphs. We explored three key hyperoperations: the path hyperoperation, which maps two vertices to the set of all vertices on paths between them; the simple path hyper-operation, which is similar but only considers simple paths between nodes; and the ancestry hyper-operation, which maps two nodes to the set of their common ancestors, defined by the paths leading to these vertices.

These hyperoperations provide new insights and tools for analyzing the intricate structures of graphs. Future research can extend these concepts to more complex graph structures, such as weighted and dynamic graphs. Additionally, the introduced hyperoperations can be applied to network analysis, aiding in the development of efficient algorithms for large-scale graphs.

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Accepted: December 2, 2024