The influence of $IC\overline{s}$-subgroups on the structure of finite groups

Huajie Zheng
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang 471023,
China
huajie219@163.com

Yong Xu*
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang 471023,
China
xuy_2011@163.com

Songtao Guo
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang 471023,
China
gsongtao@gmail.com

Abstract. A subgroup H of a group G is said to be an $IC\overline{s}$-subgroup of G if the intersection of H and $[H,G]$ is contained in $H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the maximal s-semipermutable subgroup of G contained in H. Our main result here is the following. Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and E be a normal subgroup of a group G such that $G/E \in \mathfrak{F}$. Let $X = E$ or $X = F^*(E)$. If every non-trivial Sylow subgroup P of X has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$.

Keywords: $IC\overline{s}$-subgroup, p-nilpotent group, p-supersoluble group, saturated formation.

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1. Introduction

All groups considered in this paper are finite groups. Let G be a group. $\pi(G)$ denotes the set of all primes dividing $|G|$. $\mathfrak{U}$ denotes the class of all supersoluble groups. $Z_{\mathfrak{U}}(G)$ denotes the product of all normal subgroups N of G such that every chief factor of G below N has prime order. We use standard notation as in [2] and [5].

*. Corresponding author

Let H be a subgroup of G. It is well known that the normal closure H^G of H in G is the smallest normal subgroup of G containing H and $H^G = H[H,G]$, where $[H,G]$ is the commutator subgroup of H and G. It is an interesting question to research the relationship between $H \cap [H,G]$ and the structure of G. Recall that a subgroup H of G is said to be s-semipermutable in G if H permutes with every Sylow q-subgroup of G for every prime q not dividing $|H|$. In [12], the authors introduced the concept of an $IC\overline{s}$-subgroup of a group.

Definition 1.1. *Let H be a subgroup of G. Then, H is called an $IC\overline{s}$-subgroup of G if $H \cap [H,G] \leq H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the maximal s-semipermutable subgroup of G contained in H.*

The main result of [12] is as follows: Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that, $X = E$ or $X = F^*(E)$. If every cyclic subgroup of every noncyclic Sylow subgroup of X with order p and 4 (if $p = 2$) or every maximal subgroup of every Sylow subgroup of X is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$. The goal of the present paper is to generalize and extend the result mentioned above by proving the theorems below.

Theorem 1.1. *Let G be a group and $P \in Syl_p(G)$, where p is the smallest prime dividing $|G|$. Suppose that, there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then G is p-nilpotent.*

Theorem 1.2. *Let G be a group and $P \in Syl_p(G)$, where $p \in \pi(G)$. Suppose that, there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then G is p-supersoluble.*

Theorem 1.3. *Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and E be a normal subgroup of a group G such that $G/E \in \mathfrak{F}$. Let $X = E$ or $X = F^*(E)$. If every non-trivial Sylow subgroup P of X has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$.*

2. Preliminary results

Lemma 2.1 ([7, Lemma 2.2])**.** *Let G be a group. Suppose that, H is an s-semipermutable subgroup of G. Then:*

(1) *If $H \leq K \leq G$, then H is s-semipermutable in K.*

(2) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(3) *If $H \leq O_p(G)$, then H is s-permutable in G.*

(4) *Suppose that, H is a p-group for some prime $p \in \pi(G)$ and N is normal in G. Then, $H \cap N$ is also an s-semipermutable subgroup of G.*

Lemma 2.2 ([9, Lemma A]). *If H is an s-permutable subgroup of G and H is a p-group. Then, $O^p(G) \leq N_G(H)$.*

Lemma 2.3 ([12, Lemma 2.3]). *Let G be a group, $H \leq G$, $N \trianglelefteq G$. Suppose that, H is an $IC\overline{s}$-subgroup of G. Then*

(1) *If $H \leq K \leq G$, then H is an $IC\overline{s}$-subgroup of K.*

(2) *Let $N \leq H$. If H is a p-group for some prime $p \in \pi(G)$, then H/N is an $IC\overline{s}$-subgroup of G/N.*

(3) *If H is a p-group and N is a p'-group for some prime $p \in \pi(G)$, then HN/N is an $IC\overline{s}$-subgroup of G/N.*

In the following two lemmas, we collect some results related to weakly τ-embedded subgroups. Recall that a subgroup H of G is said to be τ-permutable (τ-quasinormal) in G if H permutes with all Sylow q-subgroups Q of G such that $(q, |H|) = 1$ and $(|H|, |Q^G|) \neq 1$. A subgroup H of G is said to be weakly τ-embedded in G if there exists a normal subgroup T of G such that HT is s-permutable in G and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of H which are τ-permutable (τ-quasinormal) in G. Obviously, $IC\overline{s}$-subgroups are weakly τ-embedded subgroups.

Lemma 2.4 ([8, Theorem 2.1]). *Let p be a prime dividing the order of a group G. Assume that all maximal subgroups of every Sylow p-subgroup of G are weakly τ-embedded in G. Then, either G is a group whose Sylow p-subgroups are of order p or G is a p-supersoluble group.*

Lemma 2.5 ([8, Theorem 2.2]). *Assume that p is a prime dividing the order of a group G. If every cyclic subgroup of G of order p or 4 (if $p = 2$) is weakly τ-embedded in G, then G is p-supersoluble.*

Lemma 2.6. *Let G be a group with an abelian Sylow 2-subgroup, and assume that any subgroup of G with order 2 is weakly τ-embedded in G. Then, G is 2-nilpotent.*

Proof. Assume that the lemma is false and choose G to be a counterexample of the smallest order. Let L be a proper subgroup of G. By the subgroup heritability of weakly τ-embedding, any subgroup of L with order 2 is weakly τ-embedded in L. Hence, L is 2-nilpotent by the minimality of G. It follows that G is minimal non-2-nilpotent. Then, G has an elementary abelian Sylow 2-subgroup P, and P is a minimal normal subgroup of G. Then, let $H = \langle x \rangle$ be a subgroup of P with order 2. Since H is weakly τ-embedded in G, there is a normal subgroup T of G such that HT is s-permutable in G and $H \cap T \leq H_{\tau G}$.

Then, $P \cap HT = H(P \cap T)$ is s-permutable in G and thus normal in G(since P is abelian). If $P \cap T = 1$, it follows that $P = H$, which implies that G is 2-nilpotent, a contradiction. Then, $P \leq T$, and so, $H = H_{\tau G}$. If Q is a non-trivial Sylow subgroup of G different from P, then it follows that HQ is a subgroup of G. Since HQ is nilpotent, Q centralizes H. Then, H is normal in G, a contradiction. □

Lemma 2.7 ([1, Theorem 2.1.6]). *Let G be a p-supersoluble group. Then, the derived subgroup G' of G is p-nilpotent. In particular, if $O_{p'}(G) = 1$, then G has a unique Sylow p-subgroup.*

Lemma 2.8 ([5, VI, 4.10]). *Assume that A and B are two subgroups of a group G and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either A or B is contained in a proper normal subgroup of G.*

Lemma 2.9 ([11, Lemma 2.6]). *Let p be a prime dividing the order of G and P a normal p-subgroup of G. Assume that there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi_s$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.10 ([4, Lemma 3.3]). *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups. Suppose that, E is a normal subgroup of G such that $G/E \in \mathfrak{F}$. If $E \leq Z_{\mathfrak{U}}(G)$, then $G \in \mathfrak{F}$. In particular, if E is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.11 ([10, Theorem B]). *Let $\mathfrak{F}$ be a formation and E a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

3. Proofs of the main theorems

Proof of Theorem 1.1. Assume that the result is false. Let G be a counterexample with minimal order. Obviously, $|P| \geqslant p^2$ since $1 < |D| < |P|$.

(1) $|D| > p$ and $|P : D| > p$.

Assume that $|D| = p$ or $|P : D| = p$. Then, by Lemma 2.5 and Lemma 2.6 or Lemma 2.4, G is p-supersoluble. Since p is the smallest prime dividing $|G|$, we have that G is p-nilpotent, a contradiction.

(2) $O_{p'}(G) = 1$.

It follows from Lemma 2.3(3).

(3) Let L be a proper normal subgroup of G and $L_p \in Syl_p(L)$. If $|L_p| > |D|$, then L is p-nilpotent.

It follows from Lemma 2.3(1).

(4) Let K be a proper normal subgroup of G. Then, $K \leq P$.

If $PK < G$, then PK is p-nilpotent by the hypothesis and Lemma 2.3(1) and so K is p-nilpotent. Hence, $K \leq P$ by (2). If $PK = G$, then $G/K = PK/K \cong P/P \cap K$ is a p-group. Let M/K be a maximal subgroup of G/K. Clearly, $M \trianglelefteq G$, $|G : M| = p$ and $M \cap P$ is a maximal subgroup of P. By (1) and (3), we have M is p-nilpotent. Hence, $K \leq M \leq P$ by (2).

(5) $G = O^p(G)$.

If $O^p(G) < G$, then $O^p(G) \leq P$ by (4). Hence, $G = P$, a contradiction.

(6) G is not a non-abelian simple group.

Assume that G is a non-abelian simple group. Let H be a subgroup of P with order $|D|$ and Q a Sylow q-subgroup of G for some $q \in \pi(G)$ with $q \neq p$. Then, $H \cap [H,G] \leq H_{\overline{s}G}$. If $[H,G] = G$, then $H = H_{\overline{s}G}$. So $HQ^g = Q^gH$ holds for any $g \in G$. This is contrary to the simplicity of G by Lemma 2.8. If $[H,G] = 1$, then $H \leq Z(G) = 1$, so $|D| = 1$, a contradiction.

(7) Let N be a minimal normal subgroup of G. Then, $|N| < |D|$.

By (4) and (6), we have $N \leq P$. Assume that $|N| \geq |D|$. Let H be a subgroup of N with order $|D|$. Then, $H \cap [H,G] \leq H_{\overline{s}G}$. If $[H,G] = 1$, then $H \leq Z(G)$ and so $N = H \leq Z(G)$. It follows that $|N| = |D| = p$, this is contrary to (1). Hence, $[H,G] \neq 1$. Note that $H[H,G] = H^G \leq N$, so $[H,G] = N$. It follows that $H = H \cap N = H \cap [H,G] \leq H_{\overline{s}G}$. Then, $H = H_{\overline{s}G} \leq N \leq O_p(G)$ and so $G = O^p(G) \leq N_G(H)$ by Lemma 2.1(3), Lemma 2.2 and (5). This implies that $H = N$. Let U/N be a normal subgroup of P/N with order p. Since N is non-cyclic, U is non-cyclic, there exists a maximal subgroup H_1 of U such that $U = NH_1$. Obviously, $|H_1| = |N| = |D|$, and so $H_1 \cap [H_1,G] \leq (H_1)_{\overline{s}G}$. It is easy to see that $N \cap H_1 \neq 1$ and $[N \cap H_1, G] \neq 1$, so $1 < [N \cap H_1, G] \leq [N,G] \leq N$. It follows that $N = [N,G] = [N \cap H_1, G] \leq [H_1,G]$ and so $H_1 \cap N \leq H_1 \cap [H_1,G] \leq (H_1)_{\overline{s}G}$. Hence, $H_1 \cap N = (H_1)_{\overline{s}G} \cap N$ is s-permutable in G by Lemma 2.1(3)-(4). Further, $G = O^p(G) \leq N_G(H_1 \cap N)$ by Lemma 2.2 and (5). This implies $H_1 \cap N \trianglelefteq G$ and $H_1 \cap N = N$ for the minimal normality of N, a contradiction.

(8) Let N be a minimal normal subgroup of G. Then, G/N is p-nilpotent.

By (7), $|N| < |D|$. If $p > 2$ or $p = 2$ and P/N is an abelian 2-group or $p = 2$ and $|D/N| > 2$, then G/N satisfies the hypothesis of the theorem by Lemma 2.3(2), so G/N is p-nilpotent by the minimal choice of G. Now suppose that $p = 2$ and P/N is not abelian and $|D/N| = 2$. Then, $|D| = 2|N|$. Obviously, every subgroup of P/N with order 2 is an $IC\overline{s}$-subgroup of G/N. Let U/N be a cyclic subgroup of P/N with order 4. We will prove that U/N is an $IC\overline{s}$-subgroup of G/N.

Firstly, we claim that $|N| > 2$. If $|N| = 2$, then $|D| = 4$. By the hypothesis, all subgroups of P with order 4 are $IC\overline{s}$-subgroups of G. Clearly, N is an $IC\overline{s}$-subgroup of G with order 2. Assume that there is a subgroup $\langle x \rangle$ of P with order 2 such that $\langle x \rangle \neq N$. Then, $T = \langle x \rangle N$ is an elementary abelian 2-group with order 4. If $\langle x \rangle \cap [\langle x \rangle, G] = 1$, obviously, $\langle x \rangle$ is an $IC\overline{s}$-subgroup of G. If $\langle x \rangle \cap [\langle x \rangle, G] = \langle x \rangle$, then $\langle x \rangle = \langle x \rangle \cap [\langle x \rangle, G] \leq T \cap [T,G] \leq T_{\overline{s}G}$. Note that $N \leq T_{\overline{s}G}$, hence $T = T_{\overline{s}G}$. Let $Q \in Syl_q(G)$, where $q \neq 2$. Since $NQ \trianglelefteq TQ$ and NQ is 2-nilpotent, we have $Q \trianglelefteq TQ$ and so $\langle x \rangle Q$ is a subgroup of G. This implies that $\langle x \rangle = \langle x \rangle_{\overline{s}G}$. Hence, $\langle x \rangle$ is an $IC\overline{s}$-subgroup of G. We have proved that every subgroup of P with order 2 and 4 is an $IC\overline{s}$-subgroup of G. Therefore, G is 2-supersoluble by Lemma 2.5 and so G is 2-nilpotent, a contradiction. Hence, $|N| > 2$ and $|D| > 4$.

Suppose that, $N \leq \Phi(U)$, then U is cyclic and N is cyclic, a contradiction.

Hence, $N \nleq \Phi(U)$. Then, there exists a maximal subgroup U_1 of U such that $U = NU_1$. Obviously, $|U_1| = |D|$. Then, $U_1 \cap [U_1, G] \leq (U_1)_{\overline{s}G}$. It is easy to see that $N \cap U_1 \neq 1$ and $[N \cap U_1, G] \neq 1$, then $1 < [N \cap U_1, G] \leq [N, G] \leq N$. It follows that $N = [N, G] = [N \cap U_1, G] \leq [U_1, G]$. So $U \cap [U, G] = NU_1 \cap [NU_1, G] = NU_1 \cap [U_1, G]^N [N, G] = NU_1 \cap [U_1, G] = N(U_1 \cap [U_1, G]) \leq N(U_1)_{\overline{s}G} \leq (NU_1)_{\overline{s}G} = U_{\overline{s}G}$. This shows that U is an $IC\overline{s}$-subgroup of G and so U/N is an $IC\overline{s}$-subgroup of G/N. Hence, G/N is 2-nilpotent by Lemma 2.5.

(9) The final contradiction.

By (8), let K/N be the normal p-complement of G/N. Then, G/K is a p-group. On the other hand, $K \leq P$ by (4). Hence, G is a p-group, the final contradiction.

This completes the proof.

Proof of Theorem 1.2. If $p = 2$, then G is 2-nilpotent by Theorem 1.1. Hence, the theorem holds. Now we consider the case when p is an odd prime.

Assume that the result is false. Let G be a counterexample with minimal order. Obviously, $|P| \geqslant p^2$ since $1 < |D| < |P|$.

(1) $|D| > p$ and $|P : D| > p$.

It follows from Lemma 2.5 and Lemma 2.4.

(2) $O_{p'}(G) = 1$.

It follows from Lemma 2.3(3).

(3) If N is a minimal normal subgroup of G contained in P, then $|N| \leq |D|$.

Assume that $|N| > |D|$. Let H be a subgroup of N with order $|D|$ such that $H \trianglelefteq P$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. It is easy to see that $[H, G] \neq 1$ and $[H, G] = N$. It follows that $H = H \cap N = H \cap [H, G] \leq H_{\overline{s}G}$. Then, $H = H_{\overline{s}G} < N \leq O_p(G)$ and so $O^p(G) \leq N_G(H)$ by Lemma 2.1(3) and Lemma 2.2. Since $H \trianglelefteq P$, we have $H \trianglelefteq G$ and so $|H| = |D| = 1$, a contradiction.

(4) If N is a minimal normal subgroup of G contained in P, then G/N is p-supersoluble.

By (3), $|N| \leq |D|$. If $|N| < |D|$, then G/N satisfies the hypothesis of the theorem by Lemma 2.3(2), so G/N is p-supersoluble by the minimal choice of G.

If $|N| = |D|$. Now we claim that every subgroup of P/N with order p is an $IC\overline{s}$-subgroup of G/N. Let A/N be a subgroup of P/N with order p. By (1), N is non-cyclic, so A is non-cyclic. Hence, there exists a maximal subgroup T of A such that $A = TN$. Obviously, $|T| = |N| = |D|$. Then, $T \cap [T, G] \leq T_{\overline{s}G}$. It is easy to see that $N \cap T \neq 1$ and $[N \cap T, G] \neq 1$, then $1 < [N \cap T, G] \leq [N, G] \leq N$. It follows that $N = [N, G] = [N \cap T, G] \leq [T, G]$. So $A \cap [A, G] = TN \cap [TN, G] = TN \cap [T, G]^N [N, G] = TN \cap [T, G] = (T \cap [T, G])N \leq T_{\overline{s}G} N \leq (TN)_{\overline{s}G} = A_{\overline{s}G}$. This shows that A is an $IC\overline{s}$-subgroup of G and so A/N is an $IC\overline{s}$-subgroup of G/N. Hence, G/N is p-supersoluble by Lemma 2.5.

(5) $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in P. Then, $N \leq O_p(G)$. By (4), it is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Moreover, $\Phi(G) = 1$. Hence, $O_p(G)$ is an elementary abelian p-group, and G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $N = O_p(G)$. By (4), obviously, G is p-soluble. Hence, N is the unique minimal normal subgroup of G by (2). By (3), $|N| \leq |D|$.

If $|N| < |D|$. Let $M_p = M \cap P$. Then, $P = NM_p$. Obviously, $M_p \neq 1$ and $|N| > p$. Let P_1 be a maximal subgroup of P containing M_p. Then, $P = NP_1$ and $1 < N \cap P_1 < N$. Let H be a subgroup of P_1 containing $N \cap P_1$ such that $|H| = |D|$ and $H \trianglelefteq P$. Then, $H \cap N = P_1 \cap N \neq 1$. By the hypothesis, $H \cap [H, G] \leq H_{\overline{s}G}$. Obviously, $[H, G] \neq 1$. Hence, $H \cap N \leq H \cap [H, G] \leq H_{\overline{s}G}$. It follows that $H \cap N = H_{\overline{s}G} \cap N$ and so $O^p(G) \leq N_G(H \cap N)$ by Lemma 2.1(3)-(4) and Lemma 2.2. Since $H \cap N \trianglelefteq P$, we have $H \cap N \trianglelefteq G$ and so $H \cap N = N$ by the minimality of N, then $N \leq H \leq P_1$, a contradiction.

If $|N| = |D|$. Let T/N be a normal subgroup of P/N with order p. Then, we can write $T = N\langle x\rangle$, where $x^p \in N$, but $x \notin N$. Assume that $\Phi(T) = N$. Then, T is cyclic, so is N. It follows that $|N| = p$, a contradiction. Hence, $\Phi(T) < N$. Since $T \trianglelefteq P$, we have $\Phi(T) \trianglelefteq P$. Hence, we can choose a maximal subgroup N_1 of N containing $\Phi(T)$ such that $N_1 \trianglelefteq P$. Let $H = N_1\langle x\rangle$. Since $x^p \in \Phi(T) \leq N_1$, we have $|H| = |N| = |D|$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. Hence, we can obtain $N_1 = H \cap N \trianglelefteq G$ by a similar discussion as in the process of proving $|N| < |D|$. Hence, $N_1 = 1$ and $|N| = p$, a contradiction.

(6) Let A be a minimal normal subgroup of G. Then, A is non-p-supersoluble.

If A is p-supersoluble, then $A_p \trianglelefteq A$ by (2) and Lemma 2.7, where $A_p \in Syl_p(A)$. So $A_p \trianglelefteq G$, but this is contrary to (5).

(7) G is a non-abelian simple group.

Suppose that, G is not a simple group. Let A be a minimal normal subgroup of G. Then, $A < G$. If $|A_p| > |D|$, it easily follows that A is p-supersoluble by Lemma 2.3(1), this is contrary to (6). If $|A_p| \leq |D|$, we can pick a subgroup P_1 of P such that $A_p = A \cap P \leq P_1$ and $|P_1| = p|D|$. Then, P_1 is a Sylow p-subgroup of P_1A. Since every maximal subgroup of P_1 is an $IC\overline{s}$-subgroup of G by the hypothesis, we have every maximal subgroup of P_1 is an $IC\overline{s}$-subgroup of P_1A by Lemma 2.3(1), so P_1A is p-supersoluble by Lemma 2.4. Therefore, A is p-supersoluble, this is contrary to (6) again.

(8) The final contradiction.

Let H be a subgroup of P with $|H| = |D|$. Then, $H \cap [H, G] \leq H_{\overline{s}G}$. By (1), (6) and (7), we have $1 \neq [H, G] = G$, $H = H \cap G = H \cap [H, G] \leq H_{\overline{s}G}$ and $H = H_{\overline{s}G}$. Let Q be a Sylow q-subgroup of G for some $q \in \pi(G)$ with $q \neq p$. Then, $HQ^g = Q^gH$ for any $g \in G$. Since G is a simple group, so $G = HQ$ by Lemma 2.8, the final contradiction.

This completes the proof.

Corollary 3.1. *Let P be a normal p-subgroup of G. Suppose that, there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\overline{s}$-subgroup of G, then $P \le Z_{\mathfrak{U}}(G)$.*

Proof. Let $H \le P$ such that H is an $IC\overline{s}$-subgroup of G. Then, $H \cap [H, G] \le H_{\overline{s}G}$. By Lemma 2.1(3), $H_{\overline{s}G}$ is equal to H_{sG}, i.e. the subgroup of H generated by all subgroups of H which are s-permutable in G. Thus, H is an $IC\Phi_s$-subgroup of G (see, [11, Definition 1.1]). It follows that any subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi_s$-subgroup of G. Then, $P \le Z_{\mathfrak{U}}(G)$ by Lemma 2.9. This completes the proof. □

Proof of Theorem 1.3. We first prove that the theorem is true if $X = E$. Suppose that, this is not the case, and let (G, E) be a counterexample with $|G| + |E|$ minimal.

By the hypothesis and Theorem 1.2, we have E is supersoluble. Let $P \in Syl_p(E)$, where p is the largest prime divisor of $|E|$. Then, $P \trianglelefteq E$ and so $P \trianglelefteq G$. Since $(G/P)/(E/P) \cong G/E \in \mathfrak{F}$ and $(G/P, E/P)$ satisfies the hypothesis of the theorem, we have $G/P \in \mathfrak{F}$. Moreover, $P \le Z_{\mathfrak{U}}(G)$ by Corollary 3.1. Hence, $G \in \mathfrak{F}$ by Lemma 2.10, and this contradiction completes the proof for the case $X = E$.

Now we prove that the theorem holds for $X = F^*(E)$.

By the hypothesis and Theorem 1.2, we have $F^*(E)$ is supersoluble. Hence, $F(E) = F^*(E)$. Let P be a Sylow p-subgroup of $F(E)$. Then, $P \trianglelefteq G$. By Corollary 3.1, $P \le Z_{\mathfrak{U}}(G)$. It follows that $F(E) \le Z_{\mathfrak{U}}(G)$. Thus we have $G \in \mathfrak{F}$ by Lemma 2.10 and Lemma 2.11.

This completes the proof.

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