On semi-Hamilton groups and minimal non-semi-Hamilton groups

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Abstract. A subgroup H of a finite group G is said to be semipermutable in G if it is permutable with every subgroup K of G satisfying that $(|K|, |H|) = 1$. If every subgroup of G is semipermutable in G, then G is said to be a semi-Hamilton group. In this paper, the authors classify the non-semi-Hamilton groups whose proper subgroups are all semi-Hamilton groups.

Keywords: finite groups, semi-Hamilton groups, minimal non-semi-Hamilton groups, power automorphisms.

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1. Introduction

Given a group theoretical property $\mathcal{P}$, a $\mathcal{P}$-critical group or a minimal non-$\mathcal{P}$-group is a group which is not a $\mathcal{P}$-group but all of whose proper subgroups are $\mathcal{P}$-groups. There are many remarkable examples about minimal non-$\mathcal{P}$-groups: minimal non-abelian groups (Miller and Moreno [8]), minimal non-nilpotent groups (Schmidt), minimal non-supersoluble groups ([1]) and minimal non-p-nilpotent groups (Itô), minimal non-MSP-groups([4]) and minimal non-NSN-groups([5]).

In [10], Sastry classified the minimal non-PN-groups.

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Recall that a subgroup H is called quasinormal in a group G, if $HK = KH$ holds for every subgroup K of G, and a group G is called a $\mathcal{QN}$-group if every minimal subgroup of G is quasinormal in G (see [3]). Clearly, a $\mathcal{QN}$-group is a generalization of PN-groups.

In this paper, we consider a generalization of $\mathcal{QN}$-groups, which is called semi-Hamilton groups.

Definition 1.1. *A subgroup H of a group G is said to be s-semipermutable in G if it is permutable with every Sylow p-subgroup of G satisfying that $(p, |H|) = 1$. A subgroup H of a group is said to be semipermutable in G if it is permutable with every subgroup K of G satisfying that $(|K|, |H|) = 1$. If every subgroup of G is semipermutable in G, then G is said to be a semi-Hamilton group.*

By [11, Theorem 1], the following statements are equivalent:

(1) G is a semi-Hamilton group.

(2) every subgroup is semipermutable in G

(3) every Sylow subgroup is semipermutable in G.

(4) every subgroup with prime order is semipermutable in G.

In what follows we will use this result without any declarations.

Definition 1.2. *A group G is said to be a minimal non-semi-Hamilton group if all proper subgroups are all semi-Hamilton groups, but G itself is not a semi-Hamilton group.*

In this paper, we will investigate properties of semi-Hamilton groups, and give the structure of such groups in the first place. Next, by applying the structure of semi-Hamilton groups , we will give a classification of minimal non-semi-Hamilton groups.

Throughout this paper, only finite groups are considered and our notations are all standard. For example, we denote by $[A]P$ the semidirect product of A and P. C_n denotes a cyclic group of order n, and $\pi(G)$ denotes the set of all prime divisors of $|G|$. All unexplained notations can be found in [6] and [9].

2. Some preliminaries

In this section, we collect some lemmas which will be frequently used in the sequel.

Lemma 2.1 ([2, Theorem 7.47]). *Let G be a finite group. If every maximal subgroup of every Sylow subgroup of G is s-semipermutable in G, then G is supersolvable.*

Lemma 2.2 ([13, Lemma 3]). *Let G be a finite group. Then, G possesses a fixed-point-free power automorphism if and only if G is an abelian group of odd order.*

Lemma 2.3 ([12, Theorem 7.2.4]). *Suppose that p'-group H acts on a p-group G. If $\Omega(G)$ is H-reducible, then G is H-decomposable.*

Lemma 2.4 ([6, 8.4.6]). *Suppose that the action of A on an elementary abelian group G is coprime, and H is an A-invariant direct factor of G. Then, H has an A-invariant complement in G.*

Lemma 2.5 ([7]). *Suppose that p'-group H acts on a p-group G. Let*

$$\Omega(G) = \begin{cases} \Omega_1(G), & p > 2, \\ \Omega_2(G), & p = 2. \end{cases}$$

If H acts trivially on $\Omega(G)$, then H acts trivially on G as well.

In classifying the finite semi-Hamilton groups, we need the structure of the minimal non-supersoluble groups, it has been given by Adolfo Ballester-Bolinches and Ramon Esteban-Romero in [1] , We list it as the following Lemma:

Lemma 2.6 ([1, Theorem 10]). *The minimal non-supersoluble groups are exactly the groups of the following types:*

(1) *$G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q.*

(2) *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, q is a prime, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and z centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

(3) *$G = [P]Q$, where $P = \langle a_0, a_1, ..., a_{q-1} \rangle$ is an elementary abelian p-group of order p^q, $Q = \langle z \rangle$ is cyclic of order q^r, with q a prime such that q^f is the highest power of q dividing $p - 1$ and $r > f \geq 1$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q - 1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f-th root of unity modulo p.*

(4) *$G = [P]Q$, where $P = \langle a_0, a_1 \rangle$ is an extraspecial group of order p^3 and exponent p, $Q = \langle z \rangle$ is cyclic of order 2^r, with 2^f the largest power of 2 dividing $p - 1$ and $r > f \geq 1$. Define $a_1 = a_0^z$ and $a_1^z = a_0^i x$, where $x \in \langle [a_0, a_1] \rangle$ and i is a primitive 2^f-th root of unity modulo p.*

(5) *$G = [P]E$, where E is a 2-group with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a quaternion group of order 8 and P is an irreducible module for E with kernel F over the field of p elements of dimension 2, where $4|p-1$.*

(6) *$G = [P]E$, where E is a 2-group with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a quaternion group of order 8, P is an extraspecial group of order p^3 and exponent p, where $4|p-1$, and $P/\Phi(P)$ is an irreducible module for E with kernel F over the field of p elements.*

(7) *$G = [P]E$, where E is a q-group (for a prime q) with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a group $G_{II}(q,m,1)$, P is an irreducible E-module of dimension q over the field of p elements with kernel F, and q^m divides $p-1$.*

(8) *$G = [P]E$,where E is a 2-group with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to a group $G_{II}(2,m,1)$, P is an extraspecial group of order p^3 and exponent p such that $P/\Phi(P)$ is an irreducible E-module of dimension 2 over the field of p elements with kernel F, and 2^m divides $p-1$.*

(9) *$G = [P]E$, where E is a q-group (for a prime q) with a normal subgroup F such that $F \leq \Phi(E)$ and E/F is isomorphic to an extraspecial group of order q^3 and exponent q, with q odd, P is an irreducible E-module over the field of p elements with kernel F and dimension q, and q divides $p-1$.*

(10) *$G = [P]MC$, where C is a cyclic subgroup of order r^{s+t}, with r a prime number and s and t integers such that $s \geq 1$ and $t \geq 0$, normalising a Sylow q-subgroup M of G, $M/\Phi(M)$ is an irreducible C-module over the field of q elements, q a prime, with kernel the subgroup D of order r^t of C, and P is an irreducible MC-module over the field of p elements, where q and r^s divide $p-1$. In this case, $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, coincides with $\Phi(M)$ and centralises P.*

(11) *$G = [P]MC$, where C is a cyclic subgroup of order 2^{s+t}, with s and t integers such that $s \geq 1$ and $t \geq 0$, normalising a Sylow q-subgroup M of G, q a prime, $M/\Phi(M)$ is an irreducible C-module over the field of q elements, with kernel the subgroup D of order 2^t of C, and P is an extraspecial group of order p^3 and exponent p such that $P/\Phi(P)$ is an irreducible MC-module over the field of p elements, where q and 2^s divide $p-1$. In this case, $\Phi(G)_{p'}$, the Hall p'-subgroup of $\Phi(G)$, is equal to $\Phi(M) \times D$ and centralises P.*

According to [1, Theorem 1], the notations $G_{II}(q,m,1)$ and $G_{II}(2,m,1)$ above are the following groups:

$$G_{II}(q,m,n) = \langle a,b | a^{q^m} = b^{q^n} = 1, a^b = a^{1+q^{m-1}} \rangle,$$

where q is a prime number, $m \geq 2, n \geq 1$.

3. Semi-Hamilton groups

In this section, we will classify the finite semi-Hamilton groups. We need some Lemmas before we proceed with our proof.

Lemma 3.1. *Let $G = PQ$, where $P \in Syl_p(G), Q \in Syl_q(G), p < q$. Then, G is a semi-Hamilton-group if and only if one of the following statements is true:*

(1) *G is a nilpotent group.*

(2) *$G = [Q]P$, $P = \langle a, C_P(Q)\rangle$, and a induces a fixed-point-free power automorphism of Q.*

Proof. Let y be any element in P, for any $u \in Q$, we have $\langle y\rangle\langle u\rangle = \langle u\rangle\langle y\rangle$, then $\langle u\rangle \trianglelefteq \langle u, y\rangle$, which implies that y induces a power automorphism of Q.

Choose $v \in Z(Q)$ of order q and $z \in C_P(\langle v\rangle)$, then $v^z = v$. Let $w \in Q$ of order q, and $w^z = w^i$, then $(vw)^z = vw^i$. Since z induces a power automorphism of Q, there is a natural number j such that $(vw)^j = (vw)^z = vw^i$. Thus, we have $1 \equiv j \equiv i(mod\ q)$, that is z acts trivially on w. Hence, z acts trivially on $\Omega_1(Q)$. Now, by Lemma 2.5 we know that z acts trivially on Q. Therefore, we get that $C_P(\langle v\rangle) \leq C_P(Q)$. On the other hand, $C_P(Q) \leq C_P(\langle v\rangle)$, hence $C_P(Q) = C_P(\langle v\rangle)$.

Let $T = C_P(\langle v\rangle)$. If $T = P$, then G is nilpotent. If $T \neq P$, then we have $P/C_P(Q) = P/T \lesssim Aut(\langle v\rangle \lesssim C_{q-1}$ by the so called N/C-Theorem. Thus there exists an element $a \in P$, such that $P = \langle a, C_P(Q)\rangle$. If there exists an element $x \in Q$ satisfying that $a \in C_P(x)$, then there exists an element $w \in Q$ of order q satisfying that $w^a = w$. Since a induces a power automorphism of Q, there exists a natural number k such that $(vw)^k = (vw)^a = v^i w$, where i is also a natural number. Thus, we have $1 \equiv k \equiv i(mod\ q)$, that is a acts trivially on v. Hence, $a \in C_P(v) = C_P(Q)$, a contradiction. Hence, we get that a acts fixed-point-freely on Q. It is easy to check that G is a semi-Hamilton-group in above two cases.

The proof is completed. □

Lemma 3.2. *Suppose that $G = PHQ$, where $P \in Syl_p(G), Q \in Syl_q(G)$, H is a Hall subgroup of G, p is the smallest prime divisor of $|G|$, $P = \langle a, C_P(H)\rangle$, a induces a fixed-point-free power automorphism of H. Then, G is a semi-Hamilton-group if and only if one of the following statements is true:*

(1) *If P acts trivially on Q, then $G = (PH) \times Q$.*

(2) *If P acts non-trivially on Q, then $G = [HQ]P$, $C_P(HQ) = C_P(H)$, a induces a fixed-point-free power automorphism of HQ.*

Proof. The necessity of the theorem is obvious, so we only need to prove the sufficiency.

Suppose that P acts trivially on Q. If the result is not true, then there is a Sylow subgroup $R \in Syl_r(H)$ (without loss of generality, we let $r < q$ here) such that R acts on Q non-trivially, then by Lemma 3.1, there exist three elements $x \in P$, $y \in R$, and $u \in Q$ of order q and integer number i, j, k such that $y^x = y^i$, $u^y = u^j$, $y^i u = (yu)^x = (yu)^k$.

Let $o(y) = r^m$, then $i \not\equiv 1(mod\ r)$, $j^{r^m} \equiv 1(mod\ q)$, and the index of j modulo q is a power of r. By calculations, we have $(yu)^k = y^k u^{\frac{j^k-1}{j-1}}$, then $i \equiv k(mod\ r^m)$, and $1 \equiv \frac{j^k-1}{j-1}(mod\ q)$. Thus, $j^{k-1} \equiv 1(mod\ q)$. Since the index of j modulo q is a power of r, we have $k \equiv 1(mod\ r)$. Now, by $k \equiv i(mod\ r^m)$ we can obtain that $i \equiv 1(mod\ r)$, a contradiction since $i \not\equiv 1(mod\ r)$. Thus, if P acts trivially on Q, then H must acts trivially on Q too. Hence, we obtain that $G = (PH) \times Q$ in this case.

Suppose that P acts non-trivially on Q. We need only to prove that for any $R \in Syl_r(H)$, $C_P(R) = C_P(Q)$ holds. If $z \in C_P(R)$, and $z \notin C_P(Q)$, then by the proof of above paragraph, we know that $PRQ = (PQ) \times R$, a contradiction since a induces a fixed-point-free power automorphism of H. If $z \in C_P(Q)$, and $z \notin C_P(R)$, then we have $PRQ = (PR) \times Q$, a contradiction since a acts on Q non-trivially.

The proof is completed. □

Now, we can prove our main result of this section, this result is a sketchy description of the structure of semi-Hamilton-groups.

Theorem 3.1. *Let G be a finite group. Then, G is a semi-Hamilton-group if and only if $G = G_1 \times G_2 \times \ldots \times G_n$, where G_i are Sylow subgroups of G or Hall subgroups of G satisfying that $G_i = [H_i]P_i$, $P_i \in Syl_{p_i}(G_i)$, where H_i are Hall subgroups of G_i, and p_i is the smallest prime divisor of $|G_i|$, $P_i = \langle a_i, C_{P_i}(H_i)\rangle$, and a_i induces a fixed-point-free power automorphism of H_i.*

Proof. Let $\mathscr{B} = \{P_i \in Syl_{p_i}(G) | 1 \le i \le n, p_1 < p_2 < \ldots < p_n\}$ be a Sylow system of G. By Lemma 2.1, G is supersolvable. Then, P_i acts on P_j conjugately if $i < j$.

If P_1 acts trivially on every $P_i (i \ge 2)$, then we may let $G_1 = P_1$, and $S = P_2 P_3 \ldots P_n$. In this case we have that $G = G_1 \times S$.

Suppose that P_1 acts non-trivially on some elements of $\mathscr{B}$ (for instance $Q_1, Q_2, \ldots, Q_r$) and acts some elements of $\mathscr{B}$ (for instance $R_1, R_2, \ldots, R_t$). In this case we can let $H_1 = Q_1 Q_2 \ldots Q_r$, $G_1 = P_1 H_1$, and $S = R_1 R_2 \ldots R_t$. Then, by Lemma 3.1 and Lemma 3.2, we have that $G = G_1 \times S$.

If $S \neq 1$, then we can repeat the way we used above. Thus, we get that $G = G_1 \times G_2 \times \ldots \times G_n$, where G_i are Sylow subgroups of G or Hall subgroups of G satisfying that $G_i = H_i P_i$, $P_i \in Syl_{p_i}(G_i)$, where H_i are Hall subgroups of G_i, and p_i is the smallest prime divisor of $|G_i|$. The rest can be obtained by Lemma 3.2 easily. The proof is completed. □

4. Minimal non-semi-Hamilton groups

In this section, we will give a structure of finite minimal non-semi-Hamilton groups. Our proof will be divided into several parts.

Proposition 4.1. *Suppose that G is a finite group which is not supersolvable. Then G is a minimal non-semi-Hamilton group if and only if one of the following statements is true:*

(1) *$G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q \rangle$ in Q.*

(2) *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, q is a prime, $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and z centralises $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$.*

(3) *$G = [P]Q$, where $P = \langle a_0, a_1, ..., a_{q-1} \rangle$ is an elementary abelian p-group of order p^q, $Q = \langle z \rangle$ is cyclic of order q^r, with q a prime such that q^f is the highest power of q dividing $p-1$ and $r > f \geq 1$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f-th root of unity modulo p.*

Proof. By the hypothesis and Lemma 2.1, every proper subgroup of G is supersolvable. Then, G is minimal non-supersolvable, and hence G is isomorphic to one of groups listed in Lemma 2.6.

It easy to check that the groups of type (1) and (3) in Lemma 2.6 are minimal non-semi-Hamilton groups.

Suppose that the groups of type (2) are minimal non-semi-Hamilton groups. Then, we claim that $|Q| = q$. Otherwise $\langle P, z^q \rangle$ is a Hamilton-group, and $P/\Phi(P)$ faithfully, z^q acts on $\Phi(P)$ trivially, which contradicts with Lemma 3.1. Hence, we get that $|Q| = q$ here. On the other hand, if G is a group of type (2) in Lemma 2.6 with $|Q| = q$, we can easily check that G is a minimal non-semi-Hamilton group.

Suppose that G is groups of type (4). Then, z^2 will induces a power automorphism of $\langle a_0 \rangle$, since $\langle P, z^2 \rangle$ is a semi-Hamilton group. On the other hand, by Lemma 2.6, $a_0^{z^2} = a_0^i x$, which implies that $x = 1$. Hence, we have $a_0^{z^2} = a_0^i$, and therefore we get $[a_0, a_1]^{z^2} = [a_0, a_1]^{i^2}$, which contradicts with Lemma 3.1. Thus, G cannot be a minimal non-semi-Hamilton group.

Suppose that G is groups of type (5). If G is a minimal non-semi-Hamilton group, then every element of E induces a power automorphism of P, and hence E acts decomposably on P, a contradiction.

By the same argument we can obtain that groups of type (6) -(11) of Lemma 2.6 are not minimal non-semi-Hamilton groups.

The proof is completed. □

Proposition 4.2. *Suppose that G is a finite supersolvable group. If G is a minimal non-semi-Hamilton group, then $|\pi(G)| \leq 3$.*

Proof. Assume that $|\pi(G)| > 3$. Let $\{P_1, P_2, \ldots, P_n\}(n \geq 4)$ be a Sylow system of G, and p_1 be the smallest prime divisor of $|G|$. If P_1 acts on each $P_i(i > 1)$ trivially, then $G = P_1 \times (P_2P_3 \ldots P_n)$, and hence G itself is a semi-Hamilton group, a contradiction. If P_1 acts on every $P_i(i > 1)$ non-trivially, then for any i, j, then $P_1P_iP_j$ is a semi-Hamilton group. By Lemma 3.2, $G = [P_2P_3 \ldots P_n]P_1$, $P_1 = \langle a, C_{P_1}(P_2)\rangle$, $C_{P_1}(P_2P_3 \ldots P_n) = C_{P_1}(P_2)$, and a induces a fixed-point-free power automorphism of $P_2P_3 \ldots P_n$. That is, G itself is a semi-Hamilton group, a contradiction. Choose a Sylow system $\{P_0, Q_1, \ldots, Q_m, R_{m+1}, \ldots, R_n\}$ of G such that P_0 acts on Q_i non-trivially, and acts on R_j trivially, where $i \in \{1, \ldots, m\}, j \in \{m+1, \ldots, n\}$. Then, for any i, j, $P_1Q_iR_j$ is a semi-Hamilton group. By Lemma 3.2, $G = (P_0Q_1 \ldots Q_m) \times (R_{m+1} \ldots R_n)$, which means that G itself is a semi-Hamilton group also, a contradiction. Hence, $|\pi(G)| \leq 3$. The proof is completed. □

The following Proposition classifies supersolvable minimal non-semi-Hamilton groups which having three prime divisors.

Proposition 4.3. *Suppose that G is a finite supersolvable group and $\pi(G) = \{p, q, r\}, p < q < r$. Then, G is a minimal non-semi-Hamilton group if and only if one of the following statements is true:*

(1) $G = \langle u, v, w | u^p = 1, v^{q^m} = 1, w^r = 1, v^u = v^i, w^u = w, w^v = w^j, i \not\equiv 1(mod\ q), i^p \equiv 1(mod\ q^m), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.

(2) $G = \langle u, v, w | u^p = 1, v^{q^n} = 1, w^r = 1, v^u = v, w^u = w^i, w^v = w^j, i \not\equiv 1(mod\ r), i^p \equiv 1(mod\ r), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.

Proof. The necessity of the theorem is obvious, so we only need to prove the sufficiency.

Suppose that $\{P \in Syl_p(G), Q \in Syl_q(G), R \in Syl_r(G)\}$ is a Sylow system of G. Since PQ, PR are all semi-Hamilton groups, $QR \trianglelefteq G$. If QR is nilpotent, then $Q \trianglelefteq G$ and $R \trianglelefteq G$. Hence, G itself is a semi-Hamilton group, since every subgroup with prime order is semipermutable in G, a contradiction. Therefore, Q acts on R non-trivially.

If P is not cyclic, then we can choose two maximal subgroups P_1, P_2 of P. Assume that there exists one $P_i(i = 1$ or $2)$ acts on Q or R non-trivially, then QR is nilpotent by Lemma 3.2, a contradiction. Hence, both P_1 and P_2 acts on Q and R trivially, which means that $G = P \times (QR)$, that is, G itself is a semi-Hamilton group, a contradiction. Thus, P is cyclic.

If Q is not cyclic, then we can choose two maximal subgroups Q_1, Q_2 of Q, and there exists at least one Q_i(named Q_1 here), such that Q_1 acts on R non-trivially. By hypothesis and Lemma 3.2, $PQ_1R = P \times (Q_1R)$. Hence, P acts on Q trivially by Lemma 3.1 since PQ is a semi-Hamilton group. Thus, we get that $G = P \times (QR)$, which implies that G itself is a semi-Hamilton group, a contradiction. Thus, Q is cyclic.

Now, we claim that $|R| = r$. Let R_1 be a proper subgroup of R, then Q acts on R_1 trivially by Lemma 3.1 since PQ is a semi-Hamilton group. Moreover, we can obtain that P acts on QR_1 trivially by Lemma 3.2 since PQ is a semi-Hamilton group. Thus, we get P acts on R trivially by Lemma 3.1. Hence, $G = P \times (QR)$, that is G itself is a semi-Hamilton group, a contradiction. Thus, we get that $|R| = r$.

Let Q_1 be the maximal subgroup of Q, then Q_1R is nilpotent by Lemma 3.2 since PQ_1R is a semi-Hamilton group. Let P_1 be the maximal subgroup of P, then $P_1QR = P_1 \times (QR)$ by Lemma 3.2 since P_1QR is a semi-Hamilton group.

Let $P = \langle u\rangle$, $Q = \langle v\rangle$, and $R = \langle w\rangle$. If P acts on R trivially, then P must act non-trivially on Q. Otherwise we should obtain that $G = P \times (QR)$, which implies that G itself is a semi-Hamilton group, a contradiction. Hence, we get that:

$$G = \langle u, v, w | u^{p^n} = 1, v^{q^m} = 1, w^r = 1, v^u = v^i, w^u = w, w^v = w^j, i \not\equiv 1(mod\ q), \\ i^p \equiv 1(mod\ q^m), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle.$$

If P acts on R non-trivially, then $PQ/\Phi(Q)$ induces an automorphism of R of order pq, and hence P acts on Q trivially. Thus, we get that:

$$G = \langle u, v, w | u^{p^n} = 1, v^{q^n} = 1, w^r = 1, v^u = v, w^u = w^i, w^v = w^j, i \not\equiv 1(mod\ r), \\ i^p \equiv 1(mod\ r), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle. \qquad \square$$

The following Proposition classifies supersolvable minimal non-semi-Hamilton groups which order having just two prime divisors.

Proposition 4.4. *Suppose that G is a finite supersolvable group and $\pi(G) = \{p, q\}, p < q$. Then, G is a minimal non-semi-Hamilton group if and only if $G = \langle u, v_1, v_2 | u^{p^m} = v_1^q = v_2^q = 1,\ v_1^u = v_1^i, v_2^u = v_2^j, v_1v_2 = v_2v_1,\ i \not\equiv j(mod\ q), i^p \equiv j^p(mod\ q), i^{p^m} \equiv j^{p^m} \equiv 1(mod\ q)\rangle$.*

Proof. The necessity of the theorem is obvious, so we only need to prove the sufficiency.

Suppose that $P \in Syl_p(G), Q \in Syl_q(G)$. Then, $Q \trianglelefteq G$. If P is not cyclic, choose two different maximal subgroups P_1, P_2 of P, then P_1Q, P_2Q are all semi-Hamilton groups. Hence, for any $Q_1 \leq Q$ we have $Q_1 \trianglelefteq P_1Q_1$ and $Q_1 \trianglelefteq P_2Q_1$, therefore we get $PQ_1 = Q_1P$, which implies that Q_1 is semipermutable in G. On the other hand, each p-subgroup of G is clearly semipermutable in G. Hence, every subgroup with prime order is semipermu- table in G, and thus G itself is a semi-Hamilton group, a contradiction. Thus, P is cyclic.

Now, we have the following conclusions:

(1) Q is a 2-generator group.

Obviously Q cannot be cyclic. Since G is supersolvable, P acts reducibly on $Q/\Phi(Q)$, hence P acts decomposably on $Q/\Phi(Q)$ by Lemma 2.3. Thus, there exist two P-invariant proper subgroups Q_1, Q_2 of Q satisfying $Q = Q_1Q_2$. By hypothesis, both PQ_1 and PQ_2 are semi-Hamilton groups. Let $P = \langle u\rangle$, hence there exists a minimal generating system $\{v_1, v_2, \ldots, v_n\}$ of Q such that $v_i^u = v_i^{m_i}$, where $i = 1, 2, \ldots, n$, and m_1 is a natural number. If $n > 2$, we will prove that $m_i \neq 1$ for each i. Without loss of generality we suppose that $m_1 = 1$, then by Lemma 3.1, u acts trivially on $\langle v_1, v_i\rangle$ for any $i \neq 1$, which means that G is nilpotent, a contradiction. Thus, we get u induces a fixed-point-free power automorphism of $\langle v_i, v_j\rangle$ for any $i \neq j$. Therefore, $v_iv_j = v_jv_i$, that is Q is abelian. Without loss of generality, let $\mathrm{o}(v_i) = q^{t_i}$, where $i = 1, 2, \ldots, n$, and v_1 be the element of maximal order in $\{v_1, v_2, \ldots, v_n\}$then for any $i \neq 1$, there exists a natural number k_i such that $(v_1v_i)^u = v_1^{m_1}v_i^{m_i} = (v_1v_i)^{k_i} = v_1^{k_i}v_i^{k_i}$. Hence, $m_1 \equiv k_i(mod\ q^{t_1}), m_i \equiv k_i(mod\ q^{t_i})$, and thus $m_1 \equiv m_i(mod\ q^{t_i})$, that is $v_i^u = v_i^{m_i} = v_i^{m_1}$, which means that u induces a fixed-point-free power automorphism of Q. Hence, G itself is a semi-Hamilton group, a contradiction. Thus, we have already proved that Q is a 2-generator group.

(2) $\Phi(Q) = 1$. In this case Q is an elementary abelian q-group of type (q, q).

Assume that $\Phi(Q) \neq 1$ and let $v \in \Phi(Q)$ be an element of order q and $v^u = v^m$, where m is a natural number. Then, there exists a natural number k such that $(v_1v)^u = v_1^{m_1}v^m = (v_1v)^k = v_1^kv^k$, $m_1 \equiv k(mod\ q^{t_1}), m \equiv k(mod\ q)$, $m_1 \equiv m(mod\ q)$. Hence, $(v_1\Phi(Q))^u = v_1^m\Phi(Q) = (v_1\Phi(Q))^m$. By the same argument we have $(v_2\Phi(Q))^u = (v_2\Phi(Q))^m$, which means that u induces a fixed-point-free power automorphism of $Q/\Phi(Q)$. Thus, we get that u acts trivially on every maximal subgroup of Q , and u induces P-invariant power automorphisms of proper subgroups of Q. Therefore, u induces a fixed-point-free power automorphism of Q, which implies that G itself is a semi-Hamilton group, a contradiction. Thus, Q is an elementary abelian q-group of type (q, q). By above discussion we get that:

$$\begin{aligned}G = \langle u, v_1, v_2 | u^{p^m} = v_1^q = v_2^q = 1, v_1^u = v_1^i, v_2^u = v_2^j, v_1v_2 = v_2v_1,\\ i \not\equiv j(mod\ q), i^p \equiv j^p(mod\ q), i^{p^m} \equiv j^{p^m} \equiv 1(mod\ q)\rangle.\end{aligned} \qquad \square$$

From Proposition 4.1, Proposition 4.2, Proposition 4.3 and Proposition 4.4 we can obtain immediately the classification of finite minimal non-semi-Hamilton groups:

Theorem 4.1. *Suppose that G is a finite group. Then, G is a minimal non-semi-Hamilton group if and only if one of the following statements is true:*

(1) *$G = [P]Q$, where $Q = \langle z\rangle$ is cyclic of order $q^r > 1$, with q a prime not dividing $p-1$, and P is an irreducible Q-module over the field of p elements with kernel $\langle z^q\rangle$ in Q.*

(2) *$G = [P]Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, q is a prime, $Q = \langle z\rangle$ is cyclic of order q, z induces*

an automorphism in P *such that* $P/\Phi(P)$ *is a faithful and irreducible* Q*-module, and* z *centralises* $\Phi(P)$*. Furthermore,* $|P/\Phi(P)| = p^{2m}$ *and* $|P'| \leq p^m$*.*

(3) $G = [P]Q$*, where* $P = \langle a_0, a_1, \ldots, a_{q-1}\rangle$ *is an elementary abelian* p*-group of order* p^q*,* $Q = \langle z\rangle$ *is cyclic of order* q^r*, with* q *a prime such that* q^f *is the highest power of* q *dividing* $p-1$ *and* $r > f \geq 1$*. Define* $a_j^z = a_{j+1}$ *for* $0 \leq j < q-1$ *and* $a_{q-1}^z = a_0^i$*, where* i *is a primitive* q^f*-th root of unity modulo* p*.*

(4) $G = \langle u, v, w | u^p = 1, v^{q^m} = 1, w^r = 1, v^u = v^i, w^u = w, w^v = w^j, i \not\equiv 1(mod\ q), i^p \equiv 1(mod\ q^m), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.

(5) $G = \langle u, v, w | u^p = 1, v^{q^n} = 1, w^r = 1, v^u = v, w^u = w^i, w^v = w^j, i \not\equiv 1(mod\ r), i^p \equiv 1(mod\ r), j \not\equiv 1(mod\ r), j^q \equiv 1(mod\ r)\rangle$.

(6) $G = \langle u, v_1, v_2 | u^{p^m} = v_1^q = v_2^q = 1,\ v_1^u = u^i, v_2^u = v_2^j, v_1v_2 = v_2v_1,\ i \not\equiv j(mod\ q), i^p \equiv j^p(mod\ q), i^{p^m} \equiv j^{p^m} \equiv 1(mod\ q)\rangle$.

Example. Let $A = \langle a_1\rangle \times \langle a_2\rangle$ and $B = \langle b\rangle$, where $o(a_1) = o(a_2) = 2$ and $o(b) = 9$. Define the group $G = A \rtimes B = (\langle a_1\rangle \times \langle a_2\rangle) \rtimes \langle b\rangle$ with $a_1^b = a_2$ and $a_2^b = a_1a_2$. Clearly, $\langle b^3\rangle = Z(G)$. Therefore, the only proper subgroups of G with prime factors greater than 2 are $\langle a_1, b^3\rangle = \langle a_1\rangle \times \langle b^3\rangle$, $\langle a_2, b^3\rangle = \langle a_2\rangle \times \langle b^3\rangle$, and $\langle a_1a_2, b^3\rangle = \langle a_1a_2\rangle \times \langle b^3\rangle$. All of these are semi-Hamilton groups, while G itself is not a semi-Hamilton group. Thus, G is a minimal non-semi-Hamilton group.

Conclusion

In group theory, subgroups and quotient groups play crucial roles due to their generally simpler structure compared to the original group. Analyzing the properties of the original group through its subgroups and quotient groups is a common and effective method. For finite groups, mathematical induction is especially useful for this "small to large" approach, and when combined with proof by contradiction, it leads to the effective method of minimal counterexamples. This paper aims to clarify the properties of "minimal counterexamples" for Semi-Hamilton Groups under specific conditions and provides classifications for Semi-Hamilton Groups and Minimal Non-semi-Hamilton Groups. These properties not only have intrinsic significance but also offer a powerful tool for studying related groups.

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