Operator products and algebraic spectral subspace preservers

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Abstract. Let $\mathcal{X}$ be an infinite-dimensional complex Banach space and $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. For $T \in \mathcal{B}(\mathcal{X})$, and a fixed nonzero complex scalar λ_0, we denote by $E_T(\{\lambda_0\})$, the algebraic spectral subspace of T associated with $\{\lambda_0\}$. In this paper, we characterize maps ϕ on $\mathcal{B}(\mathcal{X})$ for which whose ranges contain all operators of rank at most two (resp. at most four), and that satisfy $E_{TS}(\{\lambda_0\}) = E_{\phi(T)\phi(S)}(\{\lambda_0\})$ (resp. $E_{TST}(\{\lambda_0\}) = E_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\})$), for all $T, S \in \mathcal{B}(\mathcal{X})$.

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1. Introduction

Throughout this note, $\mathcal{X}$ will denote an infinite-dimensional complex Banach space, $\mathcal{X}^*$ the topological dual of $\mathcal{X}$, $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$ and $\mathbb{C}$ the field of complex numbers. For any $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, $x \otimes f$ stands for the operator of rank at most one defined by $(x \otimes f)(y) = f(y)x$ for every $y \in \mathcal{X}$. We denote by $span\{x\}$ the subspace spanned by x. The sets of all finite rank operators, all operators of rank at most n, all rank-one non-nilpotent operators, all rank-one idempotent operators are denoted, respectively, by $\mathcal{F}(\mathcal{X})$, $\mathcal{F}_n(\mathcal{X})$, $\mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ and $\mathcal{P}_1(\mathcal{X})$. Note that, $x \otimes f \in \mathcal{P}_1(\mathcal{X})$ if and only if $f(x) = 1$. For $T \in \mathcal{B}(\mathcal{X})$, the kernel and the range of T are denoted, respectively, by $\mathcal{N}(T)$ and $\mathcal{R}(T)$. We define the following set

$$\mathcal{F}_{1,\alpha}(\mathcal{X}) = \{x \otimes f : x \in \mathcal{X}, f \in \mathcal{X}^* \quad such \;\; that \quad f(x) = \alpha\},$$

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where $\alpha \in \mathbb{C} \setminus \{0\}$.

The algebraic core of T, denoted by $\mathcal{C}(T)$, is defined to be the greatest subspace M of $\mathcal{X}$ for which

$$TM = M.$$

Note that, $y \in \mathcal{C}(T)$ if and only if there exists a sequence $(y_n)_n \subset \mathcal{X}$ such that $y_0 = y$ and $Ty_{n+1} = y_n$, for all $n \in \mathbb{Z}_+$; see [1, Theorem 1.8].

For a vector $x_0 \in \mathcal{X}$, the local resolvent set of an operator $T \in \mathcal{B}(\mathcal{X})$ at x_0, denoted by $\rho_T(x_0)$, is defined as the union of all open subsets $U \subset \mathbb{C}$ for which there exists an analytic function $f : U \to \mathcal{X}$ such that $(T - \lambda I)f(\lambda) = x_0$, for all $\lambda \in U$. The subset $\sigma_T(x_0) = \mathbb{C} \setminus \rho_T(x_0)$ is the local spectrum of T at x_0.

For every subset Ω of $\mathbb{C}$, the local spectral subspace, $X_T(\Omega)$, is defined by

$$X_T(\Omega) = \{x \in \mathcal{X} : \sigma_T(x) \subset \Omega\}.$$

The algebraic spectral subspace of T associated with Ω, denoted by $E_T(\Omega)$, is defined as the algebraic sum of all subspace M of $\mathcal{X}$ with the property that

$$(T - \lambda I)M = M, \text{ for every } \lambda \in \mathbb{C} \setminus \Omega.$$

Evidently, $E_T(\Omega)$ is the largest subspace of $\mathcal{X}$ on which all the restrictions of $\lambda I - T$, $\lambda \in \mathbb{C} \setminus \Omega$, are surjective. In particular,

$$(T - \lambda I)E_T(\Omega) = E_T(\Omega), \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \Omega. \tag{1}$$

Note that, $E_T(\Omega) \subset \mathcal{C}(T - \lambda I)$, for all $\lambda \notin \Omega$, and for every $\lambda \in \Omega$ we have

$$\mathcal{N}(T - \lambda I) \subset E_T(\Omega) \tag{2}$$

(see, for instance, [1, 9]).

For $T \in \mathcal{B}(\mathcal{X})$, we denote $Lat(T)$ the lattice of T, that is, the set of all invariant subspaces of T. Note that, the subspace $E_T(\Omega)$ is an invariant subspace of T.

In recent decades, a considerable attention has been paid to the nonlinear preserver problems, which demand the characterization of maps between algebras that leave a given set, property or relation invariant without assuming in advance algebraic conditions such as linearity, additivity or multiplicity, but a weak algebraic condition is often imposed through the preserving property.; see for instance [5, 7, 8, 9, 10] and the reference cited there.

In [5], Dolinar, Du, Hou and Legiša characterized the form of maps preserving the lattice of product of operators. They showed that maps (not necessarily linear) $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ satisfy $\mathrm{Lat}(\phi(A)\phi(B)) = \mathrm{Lat}(AB)$ (resp. $\mathrm{Lat}(\phi(A)\phi(B)\phi(A)) = \mathrm{Lat}(ABA)$), for all $A, B \in \mathcal{B}(\mathcal{X})$, if and only if there is a map $\varphi : \mathcal{B}(\mathcal{X}) \to \mathbb{K}$ such that $\varphi(A) \neq 0$ if $A \neq 0$ and $\phi(A) = \varphi(A)A$, for all $A \in \mathcal{B}(\mathcal{X})$. These results have motivated several authors to study maps on Banach algebras which preserve invariant subspaces; see for instance [2, 3, 4, 11].

In [2], the authors characterized maps on $\mathcal{B}(\mathcal{X})$ preserving the commutant of the sum $A+B$, the product AB, the Jordan triple product ABA and the Jordan product $AB+BA$, for all $A, B \in \mathcal{B}(\mathcal{X})$.

In the context of local spectral subspace, we cite the results found in [3], in which the authors described surjective maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ satisfying $X_{\phi(A)\phi(B)}(\{\lambda\}) = X_{AB}(\{\lambda\})$ (resp. $X_{\phi(A)\phi(B)\phi(A)}(\{\lambda\}) = X_{ABA}(\{\lambda\}),$) for all $A, B \in \mathcal{B}(\mathcal{X})$ and $\lambda \in \mathbb{C}$.

For a fixed scalar $\lambda_0 \in \mathbb{C}$, in [4], Bouchangour and Jaatit find a similar result as in [3] by removing the surjectivity condition on ϕ for product or Jordan triple product of operators. They determine the form of all maps on $\mathcal{B}(\mathcal{X})$ which satisfying $X_{TS}(\{\lambda_0\}) = X_{\phi(T)\phi(S)}(\{\lambda_0\})$ (resp.$X_{TST}(\{\lambda_0\}) = X_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\})$), for all $T, S \in \mathcal{B}(\mathcal{X})$ with λ_0 is a fixed complex scalar.

In this paper, we continue our study of mappings that preserve invariant subspaces. We, therefore, propose to determine the forms of all surjective maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$(not necessarily additive or surjective) which preserve the algebraic spectral subspace of the product or Jordan triple product of operators associated with a fixed singleton $\{\lambda_0\}$ ($\lambda_0 \in \mathbb{C} \setminus \{0\}$).

The paper is organized as follows:
In the second section we give the basic properties of the algebraic spectral subspace and the lemmas necessary of the proofs of the main results.

In the third section, we describe maps ϕ on $\mathcal{B}(\mathcal{X})$, for which their ranges contain all operators of rank at most two, and that preserve the algebraic spectral subspace of product of operators associated with the singleton $\{\lambda_0\}$.

In the fourth section, we determine the structure of maps ϕ on $\mathcal{B}(\mathcal{X})$, for which their ranges contain all operators of rank at most four, and that preserve the algebraic spectral subspace of Jordan triple product of operators associated with the singleton $\{\lambda_0\}$. Note that certain ideas in our main results are inspired by [4].

2. Preliminaries

In this section, we state some useful lemmas needed for the proof of our main results. The first one summarizes some properties of the algebraic spectral subspace which will be used frequently.

Lemma 2.1. *Let $A, B \in \mathcal{B}(\mathcal{X})$ and λ_0 be a fixed nonzero complex scalar. The following statements hold:*

1. $E_{\alpha A}(\{\lambda\}) = E_A(\{\frac{\lambda}{\alpha}\})$, *for all* $\alpha \in \mathbb{C} \setminus \{0\}$ *and all* $\lambda \in \mathbb{C}$.
2. $E_I(\{1\}) = \mathcal{X}$.
3. *If* $E_{AR}(\{\lambda_0\}) = E_{BR}(\{\lambda_0\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, *then* $E_{AR}(\{\lambda\}) = E_{BR}(\{\lambda\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ *and all* $\lambda \in \mathbb{C} \setminus \{0\}$.
4. *If* $E_{RAR}(\{\lambda_0\}) = E_{RBR}(\{\lambda_0\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, *then* $E_{RAR}(\{\lambda\}) = E_{RBR}(\{\lambda\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ *and all* $\lambda \in \mathbb{C} \setminus \{0\}$.

Proof. 1. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C}$. For $\mu \in \mathbb{C}$ such that $\mu \neq \frac{\lambda}{\alpha}$, we have $\alpha\mu \neq \lambda$ and

$$(A - \mu I)E_{\alpha A}(\{\lambda\}) = (\alpha A - \alpha\mu I)E_{\alpha A}(\{\lambda\}) = E_{\alpha A}(\{\lambda\}).$$

The fact that $E_A(\{\frac{\lambda}{\alpha}\})$ is the largest subspace M of X for which $(A-\mu I)M = M$ for every $\mu \neq \frac{\lambda}{\alpha}$, implies that

$$E_{\alpha A}(\{\lambda\}) \subset E_A(\{\frac{\lambda}{\alpha}\}).$$

Conversely, let $\mu \in \mathbb{C}$ such that $\mu \neq \lambda$, we have $\frac{\mu}{\alpha} \neq \frac{\lambda}{\alpha}$ and

$$(\alpha A - \mu I)E_A(\{\frac{\lambda}{\alpha}\}) = \alpha(A - \frac{\mu}{\alpha}I)E_A(\{\frac{\lambda}{\alpha}\}) = \alpha E_A(\{\frac{\lambda}{\alpha}\}) = E_A(\{\frac{\lambda}{\alpha}\}).$$

Therefore,

$$E_A(\{\frac{\lambda}{\alpha}\}) \subset E_{\alpha A}(\{\lambda\}).$$

Thus,

$$E_{\alpha A}(\{\lambda\}) = E_A(\{\frac{\lambda}{\alpha}\}).$$

2. We know that $I - \mu I$ is invertible for all $\mu \neq 1$, so $(I - \mu I)\mathcal{X} = \mathcal{X}$, for all $\mu \neq 1$.

Since $E_I(\{1\})$ is the largest subspace M such that $(I - \mu I)M = M$, for all $\mu \neq 1$, it follows that $E_I(\{1\}) = \mathcal{X}$.

3. Let $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Using (1), we get

$$\begin{aligned} E_{AR}(\{\lambda\}) &= E_{A(\frac{\lambda_0}{\lambda}R)}(\{\lambda_0\}) \\ &= E_{B(\frac{\lambda_0}{\lambda}R)}(\{\lambda_0\}) \\ &= E_{BR}(\{\lambda\}). \end{aligned}$$

4. Fix $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, $\lambda \in \mathbb{C}^*$ and μ_0 such that $\mu_0^2 = \frac{\lambda_0}{\lambda}$. Using (1) once again, we have

$$\begin{aligned} E_{RAR}(\{\lambda\}) &= E_{\frac{\lambda_0}{\lambda}RAR}(\{\lambda_0\}) \\ &= E_{(\mu_0 R)A(\mu_0 R)}(\{\lambda_0\}) \\ &= E_{(\mu_0 R)B(\mu_0 R)}(\{\lambda_0\}) \\ &= E_{\frac{\lambda_0}{\lambda}RBR}(\{\lambda_0\}) \\ &= E_{RBR}(\{\lambda\}). \quad \square \end{aligned}$$

The next lemma gives an explicit formula for the algebraic spectral subspace of the operators that has at most rank one associated with a nonzero fixed complex scalar.

Lemma 2.2. *For $x \in \mathcal{X} \setminus \{0\}$, $f \in \mathcal{X}^*$ and $\lambda_0 \in \mathbb{C} \setminus \{0\}$, we have*

$$E_{x\otimes f}(\{\lambda_0\}) = \begin{cases} span\{x\}, & \text{if } f(x) = \lambda_0 \\ \{0\}, & \text{if } f(x) \neq \lambda_0. \end{cases}$$

Proof. Let λ_0 be a fixed nonzero complex scalar, $x \in \mathcal{X} \setminus \{0\}$ and $f \in \mathcal{X}^*$. Since $(x \otimes f)E_{x\otimes f}(\{\lambda_0\}) = E_{x\otimes f}(\{\lambda_0\})$, then

$$E_{x\otimes f}(\{\lambda_0\}) \subset \mathcal{C}(x \otimes f).$$

The description of the algebraic core in terms of sequences shows that $\mathcal{C}(x\otimes f) \subset span\{x\}$. Therefore,

$$E_{x\otimes f}(\{\lambda_0\}) \subset span\{x\}. \tag{3}$$

Now, we shall discuss two cases.

Case 1. $f(x) = \lambda_0$. Let us show first that $(x \otimes f - \mu I)span\{x\} = span\{x\}$, for all $\mu \neq \lambda_0$. Indeed, let $\mu \in \mathbb{C}$ such that $\mu \neq \lambda_0 = f(x)$. We have $(x \otimes f - \mu I)((f(x) - \mu)^{-1}x) = x$. Then, $x \in (x \otimes f - \mu I)span\{x\}$, which proves that $span\{x\} \subset (x \otimes f - \mu I)span\{x\}$. The reverse implication deserves to be added. Indeed, let $y \in span\{x\}$, then $y = \lambda x$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Hence, we have

$$(x \otimes f - \mu I)(y) = (x \otimes f - \mu I)(\lambda x) = \lambda(\lambda_0 - \mu)x \in span\{x\}.$$

Therefore, for all $\mu \in \mathbb{C} \setminus \{\lambda_0\}$, it follows that

$$(x \otimes f - \mu I)span\{x\} = span\{x\} \ .$$

Since $E_{x\otimes}(\{\lambda_0\})$ is the largest subspace M of $\mathcal{X}$ such that $(x\otimes f - \mu I)M = M$, for all $\mu \neq \lambda_0$, we conclude that

$$span\{x\} \subset E_{x\otimes f}(\{\lambda_0\}).$$

Together with the Eq.(3), we obtain:

$$E_{x\otimes f}(\{\lambda_0\}) = span\{x\}.$$

Case 2. $f(x) \neq \lambda_0$. Suppose that $E_{x\otimes f}(\{\lambda_0\}) = span\{x\}$. Then, for every $\mu \neq \lambda_0$, we have

$$(x \otimes f - \mu I)span\{x\} = span\{x\}.$$

In particular, for $\mu = f(x)$, we obtain

$$span\{x\} = (x \otimes f - f(x)I)span\{x\} = \{0\}$$

which gives a contradiction . By (3), we get that $E_{x\otimes f}(\{\lambda_0\}) = \{0\}$. $\square$

In the following lemma, we give an identity principle that we will use to prove our main results.

Lemma 2.3. *Let $T, S \in \mathcal{B}(\mathcal{X})$, and λ_0 be a fixed nonzero scalar in $\mathbb{C}$. The following statements are equivalent:*

1. $T = S$
2. $E_{TR}(\{\lambda\})) = E_{SR}(\{\lambda\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ *and all* $\lambda \in \mathbb{C} \setminus \{0\}$.
3. $E_{TR}(\{\lambda_0\})) = E_{SR}(\{\lambda_0\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$.
4. $E_{RTR}(\{\lambda\}) = E_{RSR}(\{\lambda\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ *and all* $\lambda \in \mathbb{C} \setminus \{0\}$.
5. $E_{RTR}(\{\lambda_0\}) = E_{RSR}(\{\lambda_0\})$, *for all* $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$.

Proof. Obviously, by Lemma 2.1, we have (1) $\Rightarrow$ (2) $\iff$ (3) and (1) $\Rightarrow$ (4) $\iff$ (5). Thus, it remains to show that the implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are true.

(2) $\Rightarrow$ (1). Suppose that (2) holds, and let x be a nonzero vector in $\mathcal{X}$.

Let $f \in \mathcal{X}^*$ such that $f(x) \neq 0$. For $R = x \otimes f$, we have $TR = Tx \otimes f$ and $SR = Sx \otimes f$. Suppose first that $f(Tx) \neq 0$. By Lemma 2.2, we get that

$$span\{Tx\} = E_{TR}(\{f(Tx)\}) = E_{SR}(\{f(Tx)\}).$$

This implies that $f(Tx) = f(Sx)$.

If $f(Tx) = 0$. Suppose, on the contrary, that $f(Sx) \neq 0$. Using Lemma 2.2 once again, we obtain

$$span\{Sx\} = E_{SR}(\{f(Sx)\}) = E_{TR}(\{f(Sx)\}) = \{0\}.$$

This contradiction entails that $f(Tx) = f(Sx) = 0$. Therefore, $f(Tx) = f(Sx)$, for all $f \in \mathcal{X}^*$ for which $f(x) \neq 0$.

Now, if $f(x) = 0$, one can find a functional $g \in \mathcal{X}^*$ such that $g(x) \neq 0$. By applying what has been shown above to g and $f + g$, we get that $(f + g)(Tx) = (f + g)(Sx)$ and $g(Tx) = g(Sx)$, which proves that $f(Tx) = f(Sx)$, for all $f \in \mathcal{X}^*$. Thus, by Hahn-Banach theorem we have $T = S$.

(4) $\Rightarrow$ (1) Is done by the same reasoning. $\square$

The following lemma gives a characterization of rank-one operators in term of the algebraic spectral subspace.

Lemma 2.4. *Let λ_0 be a nonzero fixed scalar of $\mathbb{C}$, and R be a nonzero operator of $\mathcal{B}(\mathcal{X})$. The following statements are equivalent:*

1. *R is a rank-one operator.*
2. $dim E_{TR}(\{\lambda_0\}) \leq 1$, *for all* $T \in \mathcal{B}(\mathcal{X})$ *of rank at most two.*
3. $dim E_{TRT}(\{\lambda_0\}) \leq 1$, *for all* $T \in \mathcal{B}(\mathcal{X})$ *of rank at most four.*

Proof. It is easy to see that 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) are evident. Thus, it remains to show that 2) $\Rightarrow$ 1) and 3) $\Rightarrow$ 1) hold.

Suppose that, R has rank at least two, and let y_1 and y_2 be two linearly independent vectors in the range of R. Let x_1 and x_2 be two vectors of $\mathcal{X}$ such that $Rx_1 = y_1$ and $Rx_2 = y_2$, and note that x_1 and x_2 are linearly independent too. We show that $\dim E_{TR}(\{\lambda_0\}) \geq 2$ for some $T \in \mathcal{F}_2(X)$. Let $f_i \in X^*(i = 1, 2)$ such that $f_i(y_j) = \delta_{ij}$ (δ_{ij} is a Kronecker delta) for $i, j = 1, 2$. Set $T = \lambda_0 x_1 \otimes f_1 + \lambda_0 x_2 \otimes f_2$ and note that $TRx_1 = \lambda_0 x_1$ and $TRx_2 = \lambda_0 x_2$. Which implies that $x_1, x_2 \in \mathcal{N}(TR - \lambda_0 I)$. By (2), we conclude that $span\{x_1, x_2\} \subset \mathcal{N}(TR - \lambda_0 I) \subset E_{TR}(\{\lambda_0\})$. Thus,

$$\dim E_{TR}(\{\lambda_0\}) \geq 2.$$

Consequently, 2) $\Rightarrow$ 1) is established.

Now, we show that $\dim E_{TRT}(\{\lambda_0\}) \geq 2$ for some $T \in \mathcal{F}_4(\mathcal{X})$. Since $\mathcal{X}$ is an infinite dimension space, one can choose y_3 and y_4 in $\mathcal{X}$ such that y_1, y_2, y_3 and y_4 are linearly independent. Let $f_{i,j} \in \mathcal{X}^*(i, j = 1, 2, 3, 4)$ such that $f_i(y_j) = \delta_{i,j}$ for $i, j = 1, 2, 3, 4$. Consider $T \in \mathcal{B}(\mathcal{X})$ such that

$$T = x_1 \otimes f_3 + x_2 \otimes f_4 + \lambda_0 y_3 \otimes f_1 + \lambda_0 y_4 \otimes f_2.$$

We have $TRTy_3 = \lambda_0 y_3$ and $TRTy_4 = \lambda_0 y_4$. This gives $y_3, y_4 \in \mathcal{N}(TRT - \lambda_0 I)$. The Eq.(2), once more, ensures that $span\{y_3, y_3\} \subset E_{TRT}(\{\lambda_0\})$. Thus, $\dim E_{TRT}(\{\lambda_0\}) \geq 2$, which end the proof. $\square$

We close this section with the following lemma, which we will use for the proof of our main theorems.

Lemma 2.5. *Let T and S be two non-scalar operators in $\mathcal{B}(\mathcal{X})$:*

1. *If $TP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$ implies $SP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$, for all $P \in \mathcal{P}_1(\mathcal{X})$, then $S = \lambda I + (1 - \lambda I)T$ for some $\lambda \in \mathbb{C} \setminus \{1\}$.*
2. *If $PTP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$ implies $PSP \in \mathcal{P}(\mathcal{X}) \setminus \{0\}$, for all $P \in \mathcal{P}_1(\mathcal{X})$, then $S = \lambda I + (1 - \lambda I)T$ for some $\lambda \in \mathbb{C} \setminus \{1\}$. Where $\mathcal{P}(\mathcal{X})$ denotes the set of idempotent operators on $\mathcal{X}$.*

Proof. See [6, Proposition 2.3] and [12, Proposition 3.3]. $\square$

3. Maps preserving the algebraic spectral subspace of product of operators

The following theorem is our main result in this section.

Theorem 3.1. *Let λ_0 be a fixed nonzero scalar in $\mathbb{C}$, and $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ be a map such that its range contains all operators of rank at most two. Then, ϕ satisfies*

$$E_{TS}(\{\lambda_0\}) = E_{\phi(T)\phi(S)}(\{\lambda_0\}), \qquad (T, S \in \mathcal{B}(\mathcal{X})), \tag{4}$$

if and only if there exists $\alpha \in \mathbb{C}$*, with* $\alpha^2 = 1$*, such that* $\phi(T) = \alpha T$*, for all* $T \in \mathcal{B}(\mathcal{X})$*.*

Proof. The "if" part is easily verified by using Lemma 2.1, so we only need to prove the " only if " part. Indeed, assume that ϕ is a map from $\mathcal{B}(\mathcal{X})$ into itself such that its range contains all operators of rank at most two and satisfies the Eq.(4). We fix a nonzero scalar z_0 in $\mathbb{C}$ such that $z_0^2 = \lambda_0$. We divide the proof into different propositions.

Proposition 3.1. *ϕ is injective and $\phi(0) = 0$.*

Proof. Let $T, S \in \mathcal{B}(\mathcal{X})$ such that $\phi(T) = \phi(S)$. For any $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{TR}(\{\lambda_0\}) &= E_{\phi(T)\phi(R)}(\{\lambda_0\}) \\ &= E_{\phi(S)\phi(R)}(\{\lambda_0\}) \\ &= E_{SR}(\{\lambda_0\}). \end{aligned}$$

By Lemma 2.3 we conclude that $T = S$, and ϕ is injective.

Now, let us show that $\phi(0) = 0$. Indeed, for any $T \in \mathcal{B}(\mathcal{X})$, we have $\{0\} = E_{0T}(\{\lambda_0\}) = E_{\phi(0)\phi(T)}(\{\lambda_0\})$ and $\{0\} = E_{0\phi(T)}(\{\lambda_0\})$. Then,

$$E_{\phi(0)\phi(T)}(\{\lambda_0\}) = E_{0\phi(T)}(\{\lambda_0\}),$$

for all $T \in \mathcal{B}(\mathcal{X})$.

It follows from the fact that $\mathcal{F}_2(\mathcal{X}) \subset \phi(\mathcal{B}(\mathcal{X}))$ that $E_{0R}(\{\lambda_0\}) = E_{\phi(0)R}(\{\lambda_0\})$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$. Lemma 2.3 implies that $\phi(0) = 0$, as desired. □

Proposition 3.2. *ϕ preserves rank-one operators and $\phi(R) = \alpha_R R$ For every $R \in \mathcal{F}_{1,z_0}(\mathcal{X})$, where $\alpha_R \in \mathbb{C} \setminus \{0\}$.*

Proof. First, let $R = x \otimes f$ be a rank-one operator, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. By Proposition 3.1, we have $\phi(R) \neq 0$. For every $T \in \mathcal{B}(\mathcal{X})$, we see that

$$\dim E_{TR}(\{\lambda_0\}) = \dim E_{\phi(T)\phi(R)}(\{\lambda_0\}) \leq 1.$$

The fact that $\mathcal{F}_2(\mathcal{X}) \subset \phi(\mathcal{B}(\mathcal{X}))$ implies that $\dim E_{S\phi(R)}(\{\lambda_0\}) \leq 1$, for all $S \in \mathcal{F}_2(\mathcal{X})$. Therefore, by Lemma 2.4, $\phi(R)$ is an operator of rank one.

Next, let $R = x \otimes f$ be a rank-one operator in $\mathcal{F}_{1,z_0}(\mathcal{X})$ $(x \in \mathcal{X}, f \in \mathcal{X}^*)$. Thus, $\phi(R)$ is a rank-one operator, say $\phi(x \otimes f) = y \otimes g$, where $y \in \mathcal{X}$ and $g \in \mathcal{X}^*$. By hypothesis and Lemma 2.2, we have

$$span\{x\} = E_{f(x)x\otimes f}(\{\lambda_0\}) = E_{(x\otimes f)(x\otimes f)}(\{\lambda_0\}) = E_{g(y)y\otimes g}(\{\lambda_0\}).$$

This implies that $g(y)^2 = \lambda_0$ and $span\{x\} = span\{y\}$. Without loss of generality, we may and shall assume that $x = y$, and therefore $\phi(x \otimes f) = x \otimes g$.

Now, let us prove that f and g are linearly dependent. Indeed, assume, by the way of contradiction, that f and g are linearly independent, and let z be a nonzero vector in $\mathcal{X}$ such that $f(z) = z_0$ and $g(z) = 0$. From what was shown above, there exists $g_{z,f}$ a linear functional on $\mathcal{X}$ such that $\phi(z \otimes f) = z \otimes g_{z,f}$. Note that, $(x \otimes f)(z \otimes f) = f(z)x \otimes f$ and $(x \otimes g)(z \otimes g_{z,f}) = 0$. By hypothesis and Lemma 2.2, we obtain

$$span\{x\} = E_{(x\otimes f)(z\otimes f)}(\{\lambda_0\}) = E_{(x\otimes g)(z\otimes g_{z,f})}(\{\lambda_0\}) = \{0\}.$$

This contradiction shows that there is a nonzero scalar $\alpha_R \in \mathbb{C}$, such that $\phi(R) = \alpha_R R$. $\square$

Proposition 3.3. $\phi((z_0 I)) = \alpha(z_0 I)$, *where* α *be a nonzero scalar of* $\mathbb{C}$ *such that* $\alpha^2 = 1$.

Proof. Assuming that $\phi(z_0 I)$ and $z_0 I$ are linearly independent implies that there exists a nonzero vector x such that $\phi(z_0 I)x$ and $z_0 x$ are linearly independent. Let $f \in \mathcal{X}^*$ such that $f(\phi(z_0 I)x) = 0$ and $f(z_0 x) = z_0^2 = \lambda_0$. For $R = x \otimes f \in \mathcal{F}_{1,z_0}(\mathcal{X})$, we have from Proposition 3.2 $\phi(R) = \alpha_R R$, where $\alpha_R \in \mathbb{C} \setminus \{0\}$. By hypothesis and Lemma 2.2, we arrive at

$$span\{x\} = E_{(z_0 I)R}\{\lambda_0\} = E_{\alpha_R \phi(z_0 I)R}(\{\lambda_0\}) = E_{\alpha_R \phi(z_0 I)x\otimes f}(\{\lambda_0\}) = \{0\}.$$

This contradiction tells us that $\phi((z_0 I)) = \alpha(z_0 I)$, where $\alpha \in \mathbb{C} \setminus \{0\}$.

On the other hand, we have

$$\begin{aligned} \mathcal{X} &= E_I(\{1\}) \\ &= E_{(z_0 I)(z_0 I)}(\{\lambda_0\}) \\ &= E_{\alpha^2 \lambda_0 I}(\{\lambda_0\}) \\ &= E_{\alpha^2 I}(\{1\}). \end{aligned}$$

Then, by the Eq.(1), $(\alpha^2 I - \mu I)\mathcal{X} = \mathcal{X}$, for all $\mu \in \mathbb{C} \setminus \{1\}$, which forces that $\alpha^2 = 1$. $\square$

Proof of Theorem 3.1. First, let us prove that $\phi(R) = \alpha R$, for all $R \in \mathcal{F}_{1,z_0}(\mathcal{X})$. Let R be a rank- one operator in $\mathcal{F}_{1,z_0}(\mathcal{X})$, say $R = x \otimes f$, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ such that $f(x) = z_0$. Note that, by Proposition 3.2, we have $\phi(R) = \alpha_R R$. Since

$$\begin{aligned} span\{x\} &= E_{(z_0 I)(x\otimes f)}(\{\lambda_0\}) \\ &= E_{(z_0 I)R}(\{\lambda_0\}) \\ &= E_{\phi(z_0 I)\phi(R)}(\{\lambda_0\}) \\ &= E_{\alpha\alpha_R (z_0 I)R}(\{\lambda_0\}). \end{aligned}$$

Then, $f(\alpha\alpha_R z_0 x) = \lambda_0$. Thus, $\alpha\alpha_R = 1$, which implies that α_R does not depend on the operator R and we may write α instead of α_R. Therefore,

$$\phi(R) = \alpha R.$$

Now, let us prove that ϕ takes the desired form. Note that, for every $P \in \mathcal{P}_1(\mathcal{X})(z_0P \in \mathcal{F}_{1,z_0}(\mathcal{X}))$. Then, $\phi(z_0P) = \alpha(z_0P)$.

On the other hand, let $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$. For every $P \in \mathcal{P}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{RP}(\{1\}) &= E_{(z_0R)(z_0P}(\{\lambda_0\}) \\ &= E_{\phi(z_0R)\phi(z_0P)}(\{\lambda_0\}) \\ &= E_{\alpha z_0\phi(z_0R)P}(\{\lambda_0\}) \\ &= E_{\frac{\alpha}{z_0}\phi(z_0R)P}(\{1\}). \end{aligned}$$

It follows by Lemma 2.2 that

$$RP \in \mathcal{P}(\mathcal{X}) \setminus \{0\} \Rightarrow \frac{1}{\alpha z_0}\phi(z_0R)P \in \mathcal{P}(\mathcal{X}) \setminus \{0\},$$

for all $P \in \mathcal{P}_1(\mathcal{X})$. By Proposition 3.2 once again, $\phi(z_0R)$ is a rank-one operator, then R and $\frac{1}{\alpha z_0}\phi(z_0R)$ are non-scalar operators. Lemma 2.5 implies that

$$\frac{1}{\alpha z_0}\phi(z_0R) = \lambda_R I + (1-\lambda_R)R,$$

for some $\lambda_R \in \mathbb{C} \setminus \{1\}$. Since $\phi(z_0R)$ has rank one, then $\lambda_R I$ has rank at most two, which implies that $\lambda_R = 0$. Thus, $\phi(z_0R) = \alpha z_0 R$. Consequently,

$$\phi(R) = \alpha R, \quad \text{for all} \quad R \in \mathcal{F}_1(X) \setminus \mathcal{N}_1(\mathcal{X}).$$

Finally, let $T \in \mathcal{B}(\mathcal{X})$. For every $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, we have

$$E_{TR}(\{\lambda_0\}) = E_{\alpha\phi(T)R}(\{\lambda_0\}).$$

Lemma 2.3 implies that $\alpha\phi(T) = T$. Therefore, $\phi(T) = \alpha T$, for all $T \in \mathcal{B}(\mathcal{X})$. The proof is complete. $\square$

4. Maps preserving the algebraic spectral subspace of Jordan triple product of operators

As in the previous section, we determine, using the same approach, the form of a map ϕ on $\mathcal{B}(\mathcal{X})$ that preserves the algebraic spectral subspace of Jordan triple product of operators. Our main result in this section is the following theorem.

Theorem 4.1. *Let λ_0 be a fixed nonzero scalar of $\mathbb{C}$, and $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ be a map such that its range contains all operators of rank at most four. Then, ϕ satisfies*

$$E_{TST}(\{\lambda_0\}) = E_{\phi(T)\phi(S)\phi(T)}(\{\lambda_0\}), \qquad (T, S \in \mathcal{B}(\mathcal{X})), \tag{5}$$

if and only if there exists $\alpha \in \mathbb{C}$, with $\alpha^3 = 1$, such that $\phi(T) = \alpha T$, for all $T \in \mathcal{B}(\mathcal{X})$.

We only need to show that the "only if" part holds. Let ϕ be a map such that its range contains all operator of rank at most four and satisfies (5). Just as in the proof of Theorem 3.1, we fix a nonzero scalar $\mu_0 \in \mathbb{C}$ such that $\mu_0^3 = \lambda_0$. We break the proof into several propositions.

Proposition 4.1. $\phi(0) = 0$ *and* ϕ *is injective.*

Proof. For every $T \in \mathcal{B}(\mathcal{X})$, we have

$$\{0\} = E_{T0T}(\{\lambda_0\}) = E_{\phi(T)\phi(0)\phi(T)}(\{\lambda_0\}) = E_{\phi(T)0\phi(T)}(\{\lambda_0\}).$$

It follows from the assumption on range of ϕ that

$$E_{R0R}(\{\lambda_0\}) = E_{R\phi(0)R}(\{\lambda_0\}), \text{ for all } \ R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X}).$$

Thus, by Lemma 2.3, $\phi(0) = 0$.

Now, let $A, B \in \mathcal{B}(\mathcal{X})$ such that $\phi(A) = \phi(B)$. For every $T \in \mathcal{F}_1(\mathcal{X})\setminus\mathcal{N}_1(\mathcal{X})$, we have

$$\begin{aligned} E_{TAT}(\{\lambda_0\}) &= E_{\phi(T)\phi(A)\phi(T)}(\lambda_0) \\ &= E_{\phi(T)\phi(B)\phi(T)}(\{\lambda_0\}) \\ &= E_{TBT}(\{\lambda_0\}). \end{aligned}$$

It follows, from Lemma 2.3, that $A = B$, thus ϕ is injective, as desired. □

Proposition 4.2. *For every* $R \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$*, there exists a nonzero scalar* $\alpha_R \in \mathbb{C}$ *such that* $\phi(R) = \alpha_R R$.

Proof. Let $R = x \otimes f$ be a rank-one operator, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ such that $f(x) = \mu_0$. Note that, $\phi(R) \neq 0$, since Proposition 4.1. Firstly, for every $T \in \mathcal{B}(\mathcal{X})$, we have

$$\dim E_{TRT}(\{\lambda_0\}) = \dim E_{S\phi(R)S}(\{\lambda_0\}) = \dim E_{\phi(T)\phi(R)\phi(T)}(\{\lambda_0\}) \leq 1.$$

Since $\mathcal{F}_4(\mathcal{X}) \subset \phi(\mathcal{B}(\mathcal{X}))$, then $\dim E_{S\phi(R)S}(\{\lambda_0\}) \leq 1$, for all $S \in \mathcal{F}_4(\mathcal{X})$. Therefore, by Lemma 2.4, $\phi(R)$ is a rank-one operator, say $\phi(R) = y \otimes g$, where $y \in \mathcal{X}$ and $g \in \mathcal{X}^*$.

Now, we show that there exists an $\alpha_{x,f} \in \mathbb{C}$ such that $\phi(x \otimes f) = \alpha_{x,f}(x \otimes f)$. Indeed, by hypothesis and Lemma 2.1, we have

$$span\{x\} = E_{R^3}(\{\lambda_0\}) = E_{\phi(R)^3} = E_{g(y)^2 y\otimes g}(\{\lambda_0\}).$$

This implies that $g(y) \neq 0$ and $span\{y\} = span\{x\}$. Assume without loss of generality that $x = y$. Thus, it remains to show that f and g are linearly dependent. Suppose, for the sake of contradiction, that f and g are linearly independent and let $z \in \mathcal{X}$ such that $f(z) = \mu_0$ and $g(z) = 0$. Then, there exists an $g_{z,f} \in \mathcal{X}^*$ such that $\phi(z \otimes f) = z \otimes g_{z,f}$. Note that, $(x \otimes f)(z \otimes f)(x \otimes f) = \mu_0^2 x \otimes f$ and $(y \otimes g)(z \otimes g_{z,f})(y \otimes g) = 0$. By hypothesis, we have

$$\begin{aligned} span\{x\} &= E_{(x\otimes f)(z\otimes f)(x\otimes f)}(\{\lambda_0\}) \\ &= E_{(y\otimes g)(z\otimes g_{z,f})(y\otimes g)}(\{\lambda_0\}) \\ &= \{0\}. \end{aligned}$$

This contradiction shows that f and g are linearly dependent, then $\phi(x \otimes f) = \alpha_{x,f} x \otimes f$, where $\alpha_{x,f}$ is a scalar in $\mathbb{C}$, as desired. □

Proposition 4.3. $\phi(\mu_0 I) = \alpha(\mu_0 I)$, *where* $\alpha \in \mathbb{C}$ *is such that* $\alpha^3 = 1$.

Proof. Suppose, by the way of contradiction, that there exists a non zero vector $x \in \mathcal{X}$ such that $\phi(\mu_0 I)x$ and $(\mu_0 I)x$ are linearly independent. Let $f \in \mathcal{X}^*$ such that $f(\phi(\mu_0 I)x) = 0$ and $f((\mu_0 I)x) = \mu_0^2$. For $R = x \otimes f$, it is easy to see that $R \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$. By Proposition 4.2, we have $\phi(R) = \alpha_R R$ where $\alpha_R \in \mathbb{C} \setminus \{0\}$. Then, from Lemma 2.2, we have

$$\begin{aligned} span\{x\} &= E_{R(\mu_0 I)R}(\{\lambda_0\}) \\ &= E_{\alpha_R^2 R\phi(\mu_0 I)R}(\{\lambda_0\}) \\ &= \{0\}. \end{aligned}$$

This gives a contradiction. Thus, $\phi(\mu_0 I) = \alpha(\mu_0 I)$ where $\alpha \in \mathbb{C} \setminus \{0\}$.

On the other hand, since

$$\begin{aligned} \mathcal{X} &= E_I(\{1\}) \\ &= E_{(\mu_0 I)(\mu_0 I)(\mu_0 I)}(\{\lambda_0\}) \\ &= E_{\alpha^3\mu_0^3 I}(\{\lambda_0\}) \\ &= E_{\alpha^3 I}(\{1\}). \end{aligned}$$

By using (1), we conclude that $(\alpha^3 I - \mu I)\mathcal{X} = \mathcal{X}$, for all $\mu \in \mathbb{C} \setminus \{1\}$, which implies that $\alpha^3 = 1$. □

Proof of Theorem 4.1. First, let us show that $\phi(R) = \alpha R$, for all $R \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$. Let $R = x \otimes f \in \mathcal{F}_{1,\mu_0}(\mathcal{X})$, where $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. Note that, $\phi(R) = \alpha_R R$ with $\alpha_R \in \mathbb{C} \setminus \{0\}$. Using Proposition 4.2, we obtain

$$span\{x\} = E_{(\mu_0 I)R(\mu_0 I)}(\{\lambda_0\}) = E_{\alpha^2\alpha_R(\mu_0 I)R(\mu_0 I)}(\{\lambda_0\}).$$

Lemma 2.2 implies that $f(\alpha^2\alpha_R\mu_0^2 x) = \lambda_0$. Thus, $\alpha^2\alpha_R = 1$, which implies that α_R does not depend on the operator R, and we my write α instead of α_R. Therefore,

$$\phi(R) = \alpha R.$$

Next, we show that $\phi(R) = \alpha R$, for all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$. Note that, for every $P \in \mathcal{P}_1(\mathcal{X})$ $(\mu_0 P \in \mathcal{F}_{1,\mu_0}(\mathcal{X}))$, we have

$$\phi(\mu_0 P) = \alpha(\mu_0 P).$$

Let $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$ be a rank one non-nilpotent operator. For every $P \in \mathcal{P}_1(\mathcal{X})$, we have

$$\begin{array}{rcl} E_{PRP}(\{1\}) & = & E_{(\mu_0 P)(\mu_0 R)(\mu_0 P)}(\{\lambda_0\}) \\ & = & E_{\alpha^2(\mu_0 P)\phi(\mu_0 R)(\mu_0 P)}(\{\lambda_0\}) \\ & = & E_{P(\frac{1}{\alpha\mu_0}\phi(\mu_0 R))P}(\{1\}). \end{array}$$

It follows from Lemma 2.2 that

$$PRP \in \mathcal{P}(\mathcal{X}) \setminus \{0\} \Rightarrow P(\frac{1}{\alpha\mu_0}\phi(\mu_0 R))P \in \mathcal{P}(\mathcal{X}) \setminus \{0\},$$

for all $P \in \mathcal{P}_1(\mathcal{X})$. Since R is a rank-one operator(non-scalar operator), by what has been shown in Proposition 4.2, $\frac{1}{\alpha\mu_0}\phi(\mu_0 R)$ is also a rank-one operator. Therefore, from lemma 2.5, there exists an $\lambda_R \in \mathbb{C} \setminus \{1\}$ such that

$$\frac{1}{\alpha\mu_0}\phi(\mu_0 R) = \lambda_R I + (1 - \lambda_R)R.$$

Since $\phi(\mu_0 R)$ has rank one, it follows that $\lambda_R = 0$. This implies that

$$\phi(\mu_0 R) = \alpha\mu_0 R.$$

Therefore,

$$\phi(R) = \phi(\mu_0(\frac{1}{\mu_0}R)) = \alpha R.$$

Finally, let $T \in \mathcal{B}(\mathcal{X})$. For all $R \in \mathcal{F}_1(\mathcal{X}) \setminus \mathcal{N}_1(\mathcal{X})$, we have

$$\begin{array}{rcl} E_{RTR}(\{\lambda_0\}) & = & E_{\phi(R)\phi(T)\phi(R)}(\{\lambda_0\}) \\ & = & E_{\alpha^2 R\phi(T)R}(\{\lambda_0\}) \\ & = & E_{R(\frac{1}{\alpha}\phi(T))R}(\{\lambda_0\}). \end{array}$$

This implies, by Lemma 2.3, that $\phi(T) = \alpha T$, and the proof is complete.

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