On the p-biharmonic maps, conformal deformation and the warped product

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Abstract. In this paper we present some constructions of the p-biharmonic maps by conformal deformation, we characterize a p-biharmonicity of the first projection and we give many examples of p-biharmonic maps.
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1. Introduction

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between two Riemannian manifolds. A map ϕ is called p-harmonic if it is a critical point of the p-energy functional

$$E_p(\phi) = \frac{1}{p}\int_D |d\phi|^p dv_g, \qquad p \geq 1,$$

for every compact domain $D \subset M$ and is characterized by the vanishing of the p-tension field

$$\tau_p(\phi) = |d\phi|^{p-2}\left(\tau(\phi) + (p-2)\, d\phi\left(grad \ln |d\phi|\right)\right) = 0,$$

where

$$\tau(\phi) = Tr_g \nabla d\phi$$

is the tension field of ϕ. The p-bi-energy of $\phi : (M^m, g) \longrightarrow (N^n, h)$ is defined by

$$E_{p,2}(\phi) = \frac{1}{p}\int_D |\tau(\phi)|^p dv_g.$$

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Then, the map ϕ is called p-biharmonic map if it is a critical point of the p-bi-energy functional, the first variation formula for the p-bi-energy shows that the Euler-Lagrange equation for p-bi-energy is

$$\tau_{p,2}(\phi) = -\frac{2}{p}\Delta\left(|\tau(\phi)|^{p-2}\tau(\phi)\right) - \frac{2}{p}|\tau(\phi)|^{p-2} Tr_g R^N(\tau(\phi), d\phi)\, d\phi = 0.$$

$\tau_{p,2}(\phi)$ is called the p-bi-tension field of the map ϕ. A simple calculation gives

$$\begin{aligned}\Delta\left(|\tau(\phi)|^{p-2}\tau(\phi)\right) &= |\tau(\phi)|^{p-2} Tr_g\left(\nabla^{\phi}\right)^2 \tau(\phi)\\ &+ \frac{(p-2)}{2}|\tau(\phi)|^{p-4}\Delta\left(|\tau(\phi)|^2\right)\tau(\phi)\\ &+ \frac{(p-2)(p-4)}{4}|\tau(\phi)|^{p-6}\left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi)\\ &+ (p-2)|\tau(\phi)|^{p-4}\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi),\end{aligned}$$

then, the p-bi-tension field of the map ϕ is given by

$$\begin{aligned}\tau_{p,2}(\phi) = &\frac{2}{p}|\tau(\phi)|^{p-2}\tau_2(\phi) - \frac{p-2}{p}|\tau(\phi)|^{p-4}\Delta\left(|\tau(\phi)|^2\right)\tau(\phi)\\ &- \frac{(p-2)(p-4)}{2p}|\tau(\phi)|^{p-6}\left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi)\\ &- \frac{2(p-2)}{p}|\tau(\phi)|^{p-4}\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = 0,\end{aligned}\tag{1}$$

where $\tau_2(\phi)$ is the bi-tension field of ϕ defined by

$$\tau_2(\phi) = -Tr_g\left(\nabla^{\phi}\right)^2\tau(\phi) - Tr_g R^N((\phi), d\phi)\, d\phi = 0.$$

The map ϕ is p-biharmonic if and only if

$$\begin{aligned}&|\tau(\phi)|^4\tau_2(\phi) - (p-2)|\tau(\phi)|^2\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi)\\ &- \frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2\tau(\phi)\\ &- \frac{(p-2)}{2}|\tau(\phi)|^2\Delta\left(|\tau(\phi)|^2\right)\tau(\phi) = 0.\end{aligned}\tag{2}$$

The construction of harmonic maps and biharmonic maps has been the subject of several papers. In [1], [3] and [9], the authors present some methods for constructing biharmonic maps by conformally deformation of the metrics and they give several examples of biharmonic non-harmonic maps. In [4], the authors give some properties of the f-biharmonic maps and they characterize the p-biharmonicity of some particular cases. The authors in [5] investigate p-biharmonic maps from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. In [6], the authors introduce an intrinsic version of the p-biharmonic energy functional for maps and they prove, by means of the direct method, existence of minimizers of the p-bi-energy within the corresponding intrinsic Sobolev space. This paper is composed of two sections, in the first and by conformal deformation of the metric g, we give a necessary and sufficient condition of the p-biarmonicity for $Id : \left(M^m, \widetilde{g} = e^{2\gamma} g\right) \to (M^m, g)$ and $Id : (M^m, g) \to \left(M^m, \widetilde{g} = e^{2\gamma} g\right)$ ans we construct some examples of p-biharmonic maps.In the last section, we present other examples where we study the p-biharmonicity of some smooth maps.

2. The conformal deformation and the p-biharmonic maps

Let (M^m, g) be a smooth manifold and let $\widetilde{g} = e^{2\gamma} g$ be a metric conformally equivalent to g, $\gamma \in C^{\infty}(M)$. The relation between $\widetilde{\nabla}$ and ∇ is given by the following equation (see [2])

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(\gamma)Y + Y(\gamma)X - g(X,Y)\operatorname{grad}\gamma, \tag{3}$$

where ∇ et $\widetilde{\nabla}$ are respectively the connections on M associated with g and $\widetilde{g}$. Let us choose$\{e_i\}_{i=1}^m$ to be an orthonormal frame on (M^m, g), then an orthonormal frame on $\left(M^m, \widetilde{g} = e^{2\gamma} g\right)$ is given by $\{\widetilde{e_i} = e^{-\gamma} e_i\}_{i=1}^m$. We have

$$\widetilde{\nabla}_{e_i} e_i = \nabla_{e_i} e_i - (m-2)\operatorname{grad}\gamma$$

and

$$\widetilde{\nabla}_{\widetilde{e_i}} \widetilde{e_i} = e^{-2\gamma}\left(\nabla_{e_i} e_i - (m-1)\operatorname{grad}\gamma\right).$$

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map, we have

$$\widetilde{\tau}(\phi) = e^{-2\gamma}\left(\tau(\phi) + (m-2)d\phi(\operatorname{grad}\gamma)\right),$$

where $\widetilde{\tau}(\phi)$ denotes the tension field of the map ϕ with respect to $\widetilde{g}$. In a first result, we will study the p-biharmonicity of $Id : \left(M^m, \widetilde{g} = e^{2\gamma} g\right) \longrightarrow (M^m, g)$ where $m \neq 2$. We obtain the following result

Theorem 1. *The identity map $Id : \left(M^m, \widetilde{g} = e^{2\gamma} g\right) \longrightarrow (M^m, g)\,(m \neq 2)$ is p-biharmonic if and only if*

$$\begin{aligned}
&|\operatorname{grad}\gamma|^4 \operatorname{grad}\Delta\gamma + \frac{(m-4p+2)}{2}|\operatorname{grad}\gamma|^4 \operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\\
&+ \frac{(p-2)(m-4p+2)}{2}|\operatorname{grad}\gamma|^2 d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\operatorname{grad}\gamma\\
&+ \frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right|^2 \operatorname{grad}\gamma\\
&+ (p-2)|\operatorname{grad}\gamma|^2 \nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)} \operatorname{grad}\gamma
\end{aligned} \tag{4}$$

$$
\begin{aligned}
&+\frac{(p-2)}{2}\left|\operatorname{grad}\gamma\right|^{2}\Delta\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\operatorname{grad}\gamma\\
&-2\left(p-1\right)\left(m-2p\right)\left|\operatorname{grad}\gamma\right|^{6}\operatorname{grad}\gamma\\
&-2\left(p-1\right)\left|\operatorname{grad}\gamma\right|^{4}\left(\Delta\gamma\right)\operatorname{grad}\gamma\\
&+2\left|\operatorname{grad}\gamma\right|^{4}Ricci\left(\operatorname{grad}\gamma\right)=0.
\end{aligned}
$$

Proof of Theorem 1. By equation (2), the identity map

$$Id:\left(M^{m},\widetilde{g}=e^{2\gamma}g\right)\longrightarrow\left(M^{m},g\right)$$

is p-biharmonic if and only if

$$
\begin{aligned}
&\left|\widetilde{\tau}\left(Id\right)\right|^{4}\widetilde{\tau}_{2}\left(Id\right)-\left(p-2\right)\left|\widetilde{\tau}\left(Id\right)\right|^{2}\nabla_{\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}(Id)\right|^{2}\right)}\widetilde{\tau}\left(Id\right)\\
&-\frac{(p-2)(p-4)}{4}\left|\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\right|_{\widetilde{g}}^{2}\widetilde{\tau}\left(Id\right)\\
&-\frac{(p-2)}{2}\left|\widetilde{\tau}\left(Id\right)\right|^{2}\widetilde{\Delta}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\widetilde{\tau}\left(Id\right)=0,
\end{aligned}
\tag{5}
$$

where

$$\widetilde{\tau}\left(Id\right)=\left(m-2\right)e^{-2\gamma}\operatorname{grad}\gamma.$$

A rigorous calculation gives us the following formulas

$$\left|\widetilde{\tau}\left(Id\right)\right|^{2}=\left(m-2\right)^{2}e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2},$$

$$
\begin{aligned}
\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)&=\left(m-2\right)^{2}\widetilde{\operatorname{grad}}\left(e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2}\right)\\
&=\left(m-2\right)^{2}\widetilde{e_{i}}\left(e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2}\right)\widetilde{e_{i}}\\
&=\left(m-2\right)^{2}e^{-2\gamma}e_{i}\left(e^{-4\gamma}\left|\operatorname{grad}\gamma\right|^{2}\right)e_{i}\\
&=\left(m-2\right)^{2}e^{-4\gamma}e^{-2\gamma}e_{i}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)e_{i}\\
&\quad+\left(m-2\right)^{2}e^{-2\gamma}\left|\operatorname{grad}\gamma\right|^{2}e_{i}\left(e^{-4\gamma}\right)e_{i}\\
&=\left(m-2\right)^{2}e^{-6\gamma}\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\\
&\quad-4\left(m-2\right)^{2}e^{-6\gamma}\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\right|_{\widetilde{g}}^{2}=\widetilde{g}\left(\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right),\widetilde{\operatorname{grad}}\left(\left|\widetilde{\tau}\left(Id\right)\right|^{2}\right)\right)\\
&=\left(m-2\right)^{4}e^{-10\gamma}g\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right),\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right)\\
&+16\left(m-2\right)^{4}e^{-10\gamma}g\left(\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma,\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma\right)\\
&-8\left(m-2\right)^{4}e^{-10\gamma}g\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right),\left|\operatorname{grad}\gamma\right|^{2}\operatorname{grad}\gamma\right)\\
&=\left(m-2\right)^{4}e^{-10\gamma}\left|\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right|^{2}+16\left(m-2\right)^{4}e^{-10\gamma}\left|\operatorname{grad}\gamma\right|^{6}\\
&-8\left(m-2\right)^{4}e^{-10\gamma}\left|\operatorname{grad}\gamma\right|^{2}d\gamma\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^{2}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\tau}_2(Id) = &-(m-2)\,e^{-4\gamma}\left\{\operatorname{grad}\Delta\gamma+\frac{(m-6)}{2}\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right\}\\
&+2\,(m-2)\,e^{-4\gamma}\left((\Delta\gamma)+(m-4)\,|\operatorname{grad}\gamma|^2\right)\operatorname{grad}\gamma\\
&-2\,(m-2)\,e^{-4\gamma}Ricci\,(\operatorname{grad}\gamma)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{\widetilde{\operatorname{grad}}\left(|\widetilde{\tau}(Id)|^2\right)}\widetilde{\tau}\,(Id) &= (m-2)^3\,e^{-6\gamma}\nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)}e^{-2\gamma}\operatorname{grad}\gamma\\
&\quad-4\,(m-2)^3\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^2\,\nabla_{\operatorname{grad}\gamma}e^{-2\gamma}\operatorname{grad}\gamma,\\
&=(m-2)^3\,e^{-8\gamma}\nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)}\operatorname{grad}\gamma\\
&\quad+8\,(m-2)^3\,e^{-8\gamma}\,|\operatorname{grad}\gamma|^4\operatorname{grad}\gamma\\
&\quad-2\,(m-2)^3\,e^{-8\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\operatorname{grad}\gamma\\
&\quad-2\,(m-2)^3\,e^{-8\gamma}\,|\operatorname{grad}\gamma|^2\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)
\end{aligned}
$$

Finally, for the term $\widetilde{\Delta}(|\widetilde{\tau}(Id)|^2)$, we have

$$
\begin{aligned}
&\widetilde{\Delta}\left(|\widetilde{\tau}\,(Id)|^2\right)=(m-2)^2\,\widetilde{\Delta}\left(e^{-4\gamma}\,|\operatorname{grad}\gamma|^2\right)\\
&=(m-2)^2\,e^{-4\gamma}\widetilde{\Delta}\left(|\operatorname{grad}\gamma|^2\right)+(m-2)^2\,|\operatorname{grad}\gamma|^2\,\widetilde{\Delta}\left(e^{-4\gamma}\right)\\
&+2\,(m-2)^2\,\widetilde{g}\left(\widetilde{\operatorname{grad}}\left(|\operatorname{grad}\gamma|^2\right),\widetilde{\operatorname{grad}}\left(e^{-4\gamma}\right)\right)\\
&=(m-2)^2\,e^{-6\gamma}\left(\Delta\left(|\operatorname{grad}\gamma|^2\right)+(m-2)\,d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\right)\\
&+(m-2)^2\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\left(\Delta\left(e^{-4\gamma}\right)+(m-2)\,d\gamma\left(\operatorname{grad}\left(e^{-4\gamma}\right)\right)\right)\\
&-8\,(m-2)^2\,e^{-6\gamma}g\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right),\operatorname{grad}\gamma\right)\\
&=(m-2)^2\,e^{-6\gamma}\Delta\left(|\operatorname{grad}\gamma|^2\right)+(m-2)^3\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\\
&+(m-2)^2\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\,\Delta\left(e^{-4\gamma}\right)\\
&+(m-2)^3\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\,d\gamma\left(\operatorname{grad}\left(e^{-4\gamma}\right)\right)\\
&-8\,(m-2)^2\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\\
&=(m-2)^2\,e^{-6\gamma}\Delta\left(|\operatorname{grad}\gamma|^2\right)-4\,(m-2)^3\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^4\\
&+(m-2)^2\,e^{-2\gamma}\,|\operatorname{grad}\gamma|^2\left(-4e^{-4\gamma}\,(\Delta\gamma)+16e^{-4\gamma}\,|\operatorname{grad}\gamma|^2\right)\\
&+(m-2)^2\,(m-10)\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\\
&=(m-2)^2\,e^{-6\gamma}\Delta\left(|\operatorname{grad}\gamma|^2\right)-4\,(m-2)^3\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^4\\
&-4\,(m-2)^2\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^2\,(\Delta\gamma)+16\,(m-2)^2\,e^{-6\gamma}\,|\operatorname{grad}\gamma|^4\\
&+(m-2)^2\,(m-10)\,e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right),
\end{aligned}
$$

which gives us

$$\begin{aligned}\widetilde{\Delta}\left(\left|\widetilde{\tau}\left(Id\right)\right|^2\right) &= \left(m-2\right)^2 e^{-6\gamma}\Delta\left(\left|\operatorname{grad}\gamma\right|^2\right) - 4\left(m-2\right)^2 e^{-6\gamma}\left|\operatorname{grad}\gamma\right|^2\left(\Delta\gamma\right)\\ &+ \left(m-2\right)^2\left(m-10\right)e^{-6\gamma}d\gamma\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^2\right)\right)\\ &- 4\left(m-2\right)^2\left(m-6\right)e^{-6\gamma}\left|\operatorname{grad}\gamma\right|^4.\end{aligned}$$

Substituting these formulas into equation (5), we deduce that

$$Id:\left(M^m,\widetilde{g}=e^{2\gamma}g\right)\longrightarrow\left(M^m,g\right)$$

is p-biharmonic if and only if

$$\begin{aligned}&\left|\operatorname{grad}\gamma\right|^4\operatorname{grad}\Delta\gamma + \frac{\left(m-4p+2\right)}{2}\left|\operatorname{grad}\gamma\right|^4\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^2\right)\\ &+\frac{\left(p-2\right)\left(m-4p+2\right)}{2}\left|\operatorname{grad}\gamma\right|^2 d\gamma\left(\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^2\right)\right)\operatorname{grad}\gamma\\ &+\frac{\left(p-2\right)\left(p-4\right)}{4}\left|\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^2\right)\right|^2\operatorname{grad}\gamma\\ &+\left(p-2\right)\left|\operatorname{grad}\gamma\right|^2\nabla_{\operatorname{grad}\left(\left|\operatorname{grad}\gamma\right|^2\right)}\operatorname{grad}\gamma\\ &+\frac{\left(p-2\right)}{2}\left|\operatorname{grad}\gamma\right|^2\Delta\left(\left|\operatorname{grad}\gamma\right|^2\right)\operatorname{grad}\gamma\\ &-2\left(p-1\right)\left(m-2p\right)\left|\operatorname{grad}\gamma\right|^6\operatorname{grad}\gamma\\ &-2\left(p-1\right)\left|\operatorname{grad}\gamma\right|^4\left(\Delta\gamma\right)\operatorname{grad}\gamma\\ &+2\left|\operatorname{grad}\gamma\right|^4 Ricci\left(\operatorname{grad}\gamma\right)=0.\end{aligned}$$

Example 1. Let $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$, $(m>2)$ the identity map defined by

$$Id\left(t,x_2,\ldots,x_m\right)=\left(t,x_2,\ldots,x_m\right),$$

where g is the Euclidean metric given by

$$g=dt^2+dx_2^2+.....+dx_m^2.$$

Let $\widetilde{g}=e^{2\gamma}g$ where the function γ depends only on t. By Theorem 1 and by using the fact that $p\geq 2$, we deduce that $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$ is p-biharmonic if and only if $\beta=\gamma'$ satisfies the following differential equation

$$\begin{aligned}&\left(p-1\right)\beta\beta''+\left(p-1\right)\left(p-2\right)\left(\beta'\right)^2+\left(m-4p\right)\left(p-1\right)\beta^2\beta'\\ &-2\left(p-1\right)\left(m-2p\right)\beta^4=0.\end{aligned}$$

We deduce the particular solutions of the form $\beta=\frac{a}{t}$ where $a\neq 0$. We obtain

$$2\left(m-2p\right)a^2+\left(m-4p\right)a-p=0.$$

There are two solutions of this type:

1. If $p=\frac{m}{2}$, $\ m\geq 4$, then $a=-\frac{1}{2}$ and $\gamma(t)=\ln\frac{1}{\sqrt{t}}$, we obtain the metric (up to a constant multiple) $\widetilde{g}=\frac{1}{t}g$. On substituting $u=2\sqrt{t}$, this takes the form
$$\widetilde{g}=du^2+\frac{4}{u^2}dx_2^2+\frac{4}{u^2}dx_3^2+\ldots+\frac{4}{u^2}dx_m^2.$$
So, we conclude that the identity map
$$Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$$
is a p-biharmonic map, where $p=\frac{m}{2}$.

2. If $p\neq\frac{m}{2}$, we obtain $a=\frac{p}{m-2p}$ or $a=-\frac{1}{2}$ then $\widetilde{g}=t^{\frac{2p}{m-2p}}g$ or $\widetilde{g}=\frac{1}{t}g$. In this cases, the identity map $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$ is a p-biharmonic map.

Remark 1. If we consider $Id:(\mathbb{R}^m,g)\longrightarrow(\mathbb{R}^m,g)$ $(m\neq 2)$ where we suppose that the function γ is radial $(\gamma=\gamma(r),\ \ r=|x|,\ \ x\in\mathbb{R}^m)$, then the p-biharmonicity of $Id:\left(\mathbb{R}^m,\widetilde{g}=e^{2\gamma}g\right)\longrightarrow(\mathbb{R}^m,g)$ is equivalent to an ordinary differential equation:

$$\begin{aligned}&(p-1)\,\beta\beta''+(p-2)\,(p-1)\left(\beta'\right)^2+\frac{(p-1)\,(m-1)}{r}\beta\beta'\\&+(p-1)\,(m-4p)\,\beta^2\beta'-\frac{2\,(p-1)\,(m-1)}{r}\beta^3\\&-\frac{(m-1)}{r^2}\beta^2-2\,(p-1)\,(m-2p)\,\beta^4=0,\end{aligned}\tag{6}$$

where $\beta=\gamma'$.

Example 2. If we consider $Id:(\mathbb{R}^m\setminus\{0\},g)\longrightarrow(\mathbb{R}^m\setminus\{0\},g)$ $(m\neq 2)$ where we suppose $\beta=\frac{a}{r}$, $a\in\mathbb{R}^*$. By equation (9), we deduce that
$$Id:\left(\mathbb{R}^m\setminus\{0\},\widetilde{g}=e^{2\gamma}g\right)\longrightarrow(\mathbb{R}^m\setminus\{0\},g)$$
is p-biharmonic if and only if
$$2\,(p-1)\,(m-2p)\,a^2+(p-1)\,(3m-4p-2)\,a+p\,(m-p)=0.$$
For the resolution of this last equation, we will present several cases ($p\geq 2$ and $m\geq 3$).

- *First case:* $p=m, m\geq 3$, we obtain $a=-\frac{m+2}{2m}$, then
$$\gamma=\ln r^{-\frac{m+2}{2m}}=\ln\frac{1}{r^{\frac{m+2}{2m}}}$$
and
$$\widetilde{g}=\frac{1}{r^{\frac{m+2}{m}}}g.$$

- *Second case:* $p=\frac{m}{2}, m\geq 4$, then $a=-\frac{m^2}{2(m-2)^2}$, which gives us
$$\gamma=\ln\frac{1}{r^{\frac{m^2}{2(m-2)^2}}}$$
and
$$\widetilde{g}=\frac{1}{r^{\frac{m^2}{(m-2)^2}}}g.$$

- *Third case:* $p{=}\frac{3m-2}{4}, m{\geq}4$, then
$$a{=}\pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2+4m-4}}{(m-2)}$$
and
$$\gamma{=}\ln r^{\pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2+4m-4}}{(m-2)}}.$$
It follows that $\widetilde{g}=r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}}g$ or $\widetilde{g}=\frac{1}{r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}}}g$.

- *Fourth case:* $p\neq\frac{m}{2}$, then $\widetilde{g}=r^{2a}g$, where $a=\frac{1}{4(m-2p)}(4p-3m+2-\sqrt{H(m,p)})$ or
$$a=\frac{1}{4(m-2p)}(4p-3m+2+\sqrt{H(m,p)})$$
and
$$H(m,p)=\frac{12m-12p+m^2p+12mp-9m^2-4}{p-1}.$$

Now, let's look the p-biharmonicity of the identity map
$$Id:(M^m,g)\longrightarrow\left(M^m,\widetilde{g}=e^{2\gamma}g\right),$$
where $m\neq 2$.

Theorem 2. *The identity map* $Id:(M^m,g)\longrightarrow\left(M^m,\widetilde{g}=e^{2\gamma}g\right)$ *($m\neq 2$) is p-biharmonic if and only if*
$$\begin{aligned}
&|\operatorname{grad}\gamma|^4\operatorname{grad}(\Delta\gamma)+(p-2)^2|\operatorname{grad}\gamma|^2 d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right)\operatorname{grad}\gamma\\
&+\frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right|^2\operatorname{grad}\gamma+(p-4)|\operatorname{grad}\gamma|^4(\Delta\gamma)\operatorname{grad}\gamma\\
&-\left(m+2p-p^2-2\right)|\operatorname{grad}\gamma|^6\operatorname{grad}\gamma\\
&-\frac{(m-4p+2)}{2}|\operatorname{grad}\gamma|^4\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\\
&+(p-2)|\operatorname{grad}\gamma|^2\nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)}\operatorname{grad}\gamma\\
&+\frac{(p-2)}{2}|\operatorname{grad}\gamma|^2\Delta\left(|\operatorname{grad}\gamma|^2\right)\operatorname{grad}\gamma\\
&+2|\operatorname{grad}\gamma|^4\mathit{Ricci}(\operatorname{grad}\gamma)=0.
\end{aligned}\tag{7}$$

Proof of Theorem 2. By equation (2), the identity map

$$Id : (M^m, g) \longrightarrow \left(M^m, \widetilde{g} = e^{2\gamma} g\right)$$

is p-biharmonic if and only if

$$\begin{aligned} &|\widetilde{\tau}(Id)|^4 \widetilde{\tau}_2(Id) - \frac{(p-2)(p-4)}{4} \left|\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)\right|^2 \widetilde{\tau}(Id) \\ &- (p-2)|\widetilde{\tau}(Id)|^2 \widetilde{\nabla}_{\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)} \widetilde{\tau}(Id) \\ &- \frac{(p-2)}{2} |\widetilde{\tau}(Id)|^2 \Delta\left(|\widetilde{\tau}(Id)|^2\right) \widetilde{\tau}(Id) = 0, \end{aligned} \tag{8}$$

where

$$\widetilde{\tau}(Id) = (2-m) \operatorname{grad} \gamma.$$

A long calculation gives us

$$\widetilde{\tau}(Id) = (2-m) \operatorname{grad} \gamma,$$

$$|\tau(Id)|^2 = (2-m)^2 e^{2\gamma} |\operatorname{grad} \gamma|^2,$$

$$\begin{aligned} \operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right) &= (2-m)^2 e^{2\gamma} \operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right) \\ &\quad + 2(2-m)^2 e^{2\gamma} |\operatorname{grad}\gamma|^2 \operatorname{grad}\gamma, \end{aligned}$$

$$\begin{aligned} &\left|\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)\right|^2 \\ &= (2-m)^4 e^{4\gamma} \left|\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right|^2 + 4(2-m)^4 e^{4\gamma} |\operatorname{grad}\gamma|^6 \\ &+ 4(2-m)^4 e^{4\gamma} |\operatorname{grad}\gamma|^2 d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right), \end{aligned}$$

$$\begin{aligned} &\Delta\left(|\widetilde{\tau}(Id)|^2\right) \\ &= (2-m)^2 e^{2\gamma} \Delta\left(|\operatorname{grad}\gamma|^2\right) + 4(2-m)^2 e^{2\gamma} d\gamma\left(\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)\right) \\ &+ 2(2-m)^2 e^{2\gamma} |\operatorname{grad}\gamma|^2 (\Delta\gamma) + 4(2-m)^2 e^{2\gamma} |\operatorname{grad}\gamma|^4 \end{aligned}$$

$$\begin{aligned} &\widetilde{\nabla}_{\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)} \widetilde{\tau}(Id) \\ &= (2-m)^3 e^{2\gamma} \nabla_{\operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right)} \operatorname{grad}\gamma + 2(2-m)^3 e^{2\gamma} |\operatorname{grad}\gamma|^4 \operatorname{grad}\gamma \\ &+ 2(2-m)^3 e^{2\gamma} |\operatorname{grad}\gamma|^2 \operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right) \end{aligned}$$

and

$$\begin{aligned} \widetilde{\tau}_2(Id) = &-(2-m)\operatorname{grad}(\Delta\gamma) + \frac{(2-m)(m-6)}{2} \operatorname{grad}\left(|\operatorname{grad}\gamma|^2\right) \\ &+ 2(2-m)(\Delta\gamma)\operatorname{grad}\gamma - (2-m)^2\left(|\operatorname{grad}\gamma|^2 \operatorname{grad}\gamma\right) \\ &- 2(2-m)\, Ricci(\operatorname{grad}\gamma). \end{aligned}$$

Substituting these obtained formulas into the equation (8), we conclude that $Id:(M^m,g)\longrightarrow\left(M^m,\widetilde{g}=e^{2\gamma}g\right)$ is p-biharmonic if and only if

$$\begin{aligned}
&|\widetilde{\tau}(Id)|^4\,\widetilde{\tau}_2(Id)-\frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)\right|^2\widetilde{\tau}(Id)\\
&-(p-2)\,|\widetilde{\tau}(Id)|^2\,\widetilde{\nabla}_{\operatorname{grad}\left(|\widetilde{\tau}(Id)|^2\right)}\widetilde{\tau}(Id)\\
&-\frac{(p-2)}{2}\,|\widetilde{\tau}(Id)|^2\,\Delta\left(|\widetilde{\tau}(Id)|^2\right)\widetilde{\tau}(Id)=0,
\end{aligned}$$

Example 3. Let $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$, $(m>2)$ the identity map defined by

$$Id\,(t,x_2,\ldots,x_m)=(t,x_2,\ldots,x_m)\,,$$

where g is the Euclidean metric given by

$$g=dt^2+dx_2^2+.....+dx_m^2.$$

Let $\widetilde{g}=e^{2\gamma}g$ where the function γ depends only on t. By Theorem 2 and by using the fact that $p\geq 2$, we deduce that $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)$ is p-biharmonic if and only if $\beta=\gamma'$ satisfies the following differential equation

$$\begin{aligned}
&(p-1)\,\beta\beta''+(p-2)\,(p-1)\left(\beta'\right)^2\\
&+\left(2p^2-3p-m+2\right)\beta^2\beta'\\
&-\left(m+2p-p^2-2\right)\beta^4=0.
\end{aligned}$$

We deduce the particular solutions of the form $\beta=\frac{a}{t}$ where $a\neq 0$. We obtain

$$\left(m+2p-p^2-2\right)a^2+\left(2p^2-3p-m+2\right)a-p\,(p-1)=0.$$

There are two solutions of this type:

1. If $m+2p-p^2-2=0$, then $p=\sqrt{m-1}+1$, $m\geq 3$ and $a=1$. It follows that and $\gamma(t)=\ln t$, we obtain the metric $\widetilde{g}=t^2g$. So, we conclude that the identity map $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)$ is a p-biharmonic map, where $p=\sqrt{m-1}+1$. For example, if $m=5$, we obtain $p=3$

2. If $m+2p-p^2-2\neq 0$, we obtain $a=\frac{p-p^2}{m+2p-p^2-2}$ or $a=1$ then

$$\widetilde{g}=t^{\frac{2\left(p-p^2\right)}{m+2p-p^2-2}}g\text{ or }\widetilde{g}=t^2g.$$

   In this cases, the identity map $Id:\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},g\right)\to\left(\mathbb{R}_+^*\times\mathbb{R}^{m-1},\widetilde{g}\right)$ is a p-biharmonic map. For example, if $p=m=4$, we obtain $a=1$ or $a=2$ and if $p=m=3$, we obtain $a=1$ or $a=3$.

Remark 2. If we consider $Id : (\mathbb{R}^m, g) \longrightarrow (\mathbb{R}^m, g)$ $(m \neq 2)$ where we suppose that the function γ is radial $(\gamma = \gamma(r), \quad r = |x|, \quad x \in \mathbb{R}^m)$, then the p-biharmonicity of $Id : (\mathbb{R}^m, g) \longrightarrow \left(\mathbb{R}^m, \widetilde{g} = e^{2\gamma} g\right)$ is equivalent to an ordinary differential equation:

$$
\begin{aligned}
&(p-1)\,\beta\beta'' + \left(2p^2 - 3p - m + 2\right)\beta^2\beta' + \frac{(p-1)(m-1)}{r}\beta\beta' \\
&+ (p-2)(p-1)\left(\beta'\right)^2 - \frac{(m-1)}{r^2}\beta^2 + \frac{(p-4)(m-1)}{r}\beta^3 \\
&- \left(m + 2p - p^2 - 2\right)\beta^4 = 0
\end{aligned}
\tag{9}
$$

where $\beta = \gamma'$.

Example 4. If we consider $Id : (\mathbb{R}^m \setminus \{0\}, g) \longrightarrow (\mathbb{R}^m \setminus \{0\}, g)$ $(m \neq 2)$ where we suppose $\beta = \frac{a}{r}$, $a \in \mathbb{R}^*$. By equation (9), we deduce that $Id : (\mathbb{R}^m \setminus \{0\}, g) \longrightarrow \left(\mathbb{R}^m \setminus \{0\}, \widetilde{g} = e^{2\gamma} g\right)$ is p-biharmonic if and only if

$$\left(m + 2p - p^2 - 2\right)a^2 + \left(3m - 2p - mp + 2p^2 - 2\right)a + p(m-p) = 0.$$

For the resolution of this last equation, we will present several cases ($p \geq 2$ and $m \geq 3$).

- *First case:* $p = m, \quad m \geq 3$, we obtain $a = \frac{m+2}{m-2}$, then

$$\gamma = \ln r^{\frac{m+2}{m-2}}$$

  and

$$\widetilde{g} = r^{\frac{2(m+2)}{m-2}} g.$$

- *Second case:* $p = \sqrt{m-1} + 1, \quad m \geq 3$, then $a = \frac{m-2}{m-2-4\sqrt{m-1}}$, which gives us

$$\widetilde{g} = r^{\frac{2(m-2)}{m-2-4\sqrt{m-1}}} g.$$

  For example, if $m = 10$, we obtain $p = 4$ and $a = -4$

- *Third case:* $p = \frac{3m-2}{4}, \quad m \geq 4$, then

$$a = \pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2 + 4m - 4}}{(m-2)}$$

  and

$$\gamma = \ln r^{\pm\frac{\sqrt{3}}{6}\frac{\sqrt{3m^2+4m-4}}{(m-2)}}.$$

  It follows that

$$\widetilde{g} = r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}} g$$

  or

$$\widetilde{g} = \frac{1}{r^{\frac{\sqrt{3}\sqrt{3m^2+4m-4}}{3(m-2)}}} g.$$

- *Fourth case:* $p \neq \sqrt{m-1}+1$, then

$$\widetilde{g} = r^{2a} g,$$

where

$$a = \frac{1}{-2m-4p+2p^2+4}\left(3m-2p-mp+2p^2-2+A(m,p)\right)$$

or

$$a = -\frac{1}{-2m-4p+2p^2+4}\left(-3m+2p+mp-2p^2+2+A(m,p)\right)$$

and

$$A(m,p) = \sqrt{-12m+8p+12mp^2-10m^2p+m^2p^2+9m^2-12p^2+4}.$$

Note that, if $m=1$, we obtain the following equation:

$$(p-1)\,a^2-(2p-1)\,a+p=0.$$

The solutions for this equation are $a=1$ and $a=\frac{p}{p-1}, \quad p \geq 2$.

In a following section, we present other examples where we give the p-biharmonicity condition of some maps.

3. Other examples

Example 5. Let the map $\phi : \left(\mathbb{R}^3, g_{\mathbb{R}^3}\right) \longrightarrow \left(\mathbb{R}^2, g_{\mathbb{R}^2}\right)$ defined by

$$\phi(x_1,x_2,x_3) = \left(\sqrt{x_1^2+x_2^2}, x_3\right).$$

Using cylindrical coordinates (r,θ,x_3) in $\mathbb{R}^3$ and the standard Euclidean coordinates in $\mathbb{R}^2$, the map ϕ is written in the form

$$\phi(r\cos\theta, r\sin\theta, x_3) = (y_1,y_2),$$

where $y_1 = r = \sqrt{x_1^2+x_2^2}$ et $y_2 = x_3$. The metrics on $\mathbb{R}^3$ and $\mathbb{R}^2$ have respectively the expressions

$$g_{\mathbb{R}^3} = dr^2 + r^2 d\theta^2 + dx_3^2$$

and

$$g_{\mathbb{R}^2} = dy_1^2 + dy_2^2.$$

A rigorous calculation give

$$\tau(\phi) = \frac{1}{r}\frac{\partial}{\partial y_1}, \qquad |\tau(\phi)|^2 = \frac{1}{r^2}, \qquad \tau_2(\phi) = -\frac{3}{r^3}\frac{\partial}{\partial y_1},$$

$$\operatorname{grad}\left(|\tau(\phi)|^2\right) = -\frac{2}{r^3}\frac{\partial}{\partial r}, \qquad \left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2 = \frac{4}{r^6}$$

and

$$\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = \frac{2}{r^5}\frac{\partial}{\partial y_1}, \qquad \Delta\left(|\tau(\phi)|^2\right) = \frac{4}{r^4}.$$

Then, we conclude the the map $\phi : \left(\mathbb{R}^3, g_{\mathbb{R}^3}\right) \longrightarrow \left(\mathbb{R}^2, g_{\mathbb{R}^2}\right)$ is p-biharmonic if and only if $p = 1$.

Example 6. We consider the inversion $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{0\}$ $(n \geq 3)$ defined by $\phi(x) = \frac{x}{|x|^2}$. Note that this map is p-harmonic if and only if $p = n$. A long calculation gives us

$$\tau(\phi) = \frac{2(n-2)}{r^3}\frac{\partial}{\partial r}, \quad |\tau(\phi)|^2 = \frac{4(n-2)^2}{r^6}, \quad \tau_2(\phi) = \frac{8(n-2)(n-4)}{r^5}\frac{\partial}{\partial r},$$

$$\operatorname{grad}\left(|\tau(\phi)|^2\right) = -\frac{24(n-2)^2}{r^7}\frac{\partial}{\partial r}, \quad \left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2 = \frac{576(n-2)^4}{r^{14}}$$

and

$$\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = \frac{144(n-2)^3}{r^{11}}\frac{\partial}{\partial r}, \quad \Delta\left(|\tau(\phi)|^2\right) = -\frac{24(n-2)^2(n-8)}{r^8},$$

where $r = |x|$, $x \in \mathbb{R}^n$. Then, the inversion $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{0\}$ is p-biharmonic non-p-harmonic if and only if $p = \frac{n+2}{3}$. A class of solutions is given by the case where $n = 3k + 1$, $\quad k \in \mathbb{N}^*$, which gives us $p = k + 1$. For example, if we take $n = 7$, we find $p = 3$.

Example 7. Let $\phi : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R} \times \mathbb{S}^{n-1}$ given in polar coordinates by

$$\phi(r\theta) = (\ln r, \theta), \qquad r > 0, \qquad \theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n.$$

The same calculation method gives us

$$\tau(\phi) = \frac{(n-2)}{r^2}\frac{\partial}{\partial r}, \quad |\tau(\phi)|^2 = \frac{(n-2)^2}{r^4}, \quad \tau_2(\phi) = \frac{2(n-2)(n-4)}{r^4}\frac{\partial}{\partial r},$$

$$\operatorname{grad}\left(|\tau(\phi)|^2\right) = -\frac{4(n-2)^2}{r^5}\frac{\partial}{\partial r}, \quad \left|\operatorname{grad}\left(|\tau(\phi)|^2\right)\right|^2 = \frac{16(n-2)^4}{r^{10}}$$

and

$$\nabla^{\phi}_{\operatorname{grad}\left(|\tau(\phi)|^2\right)}\tau(\phi) = \frac{8(n-2)^3}{r^8}\frac{\partial}{\partial r}, \quad \Delta\left(|\tau(\phi)|^2\right) = -\frac{4(n-2)^2(n-6)}{r^6}.$$

We conclude that ϕ is p-biharmonic if and only if $p = \frac{n}{2}$, $\quad n \geq 3$ or $p = 1$.

4. The warped product and the p-biharmonic maps

Let (M^m, g) and (N^n, h) two Riemannian manifolds and let $f \in C^\infty(M)$ be a positive function. The warped product $M \times_f N$ is the product manifolds $M \times N$ endowed with the Riemannian metric G_f defined, for $X, Y \in \Gamma(T(M \times N))$, by

$$G_f(X, Y) = g(d\pi(X), d\pi(Y)) + (f \circ \pi)^2 h(d\eta(X), d\eta(Y)),$$

where $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ are respectively the first and the second projection. The function f is called the warping function function of the warped product. Let $X, Y \in \Gamma(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$. The Levi-Civita connection $\widetilde{\nabla}$ of the warped product $M \times_f N$ is given by

$$\begin{aligned}\widetilde{\nabla}_X Y = {} & \nabla_X Y + X_1(\ln f)(0, Y_2) + Y_1(\ln f)(0, X_2)\\ & - f^2 h(X_2, Y_2)(\operatorname{grad}\ln f, 0).\end{aligned} \tag{10}$$

There are several methods to give the relation between the curvature tensor fields of G_f and G, we are going to prove one of these methods.

Proposition 1. *The relation between the curvature tensor fields of G_f and G is given by*

$$\begin{aligned}\widetilde{R}(X, Y)Z = {} & R(X, Y)Z - f^2 h(Y_2, Z_2)(\nabla_{X_1}\operatorname{grad}\ln f, 0)\\ & + f^2 h(X_2, Z_2)(\nabla_{Y_1}\operatorname{grad}\ln f, 0)\\ & - f^2 h(Y_2, Z_2) X_1(\ln f)(\operatorname{grad}\ln f, 0)\\ & + f^2 h(X_2, Z_2) Y_1(\ln f)(\operatorname{grad}\ln f, 0)\\ & - f^2 h(Y_2, Z_2)|\operatorname{grad}\ln f|^2(0, X_2)\\ & + f^2 h(X_2, Z_2)|\operatorname{grad}\ln f|^2(0, Y_2)\\ & - g(\nabla_{Y_1}\operatorname{grad}\ln f, Z_1)(0, X_2)\\ & + g(\nabla_{X_1}\operatorname{grad}\ln f, Z_1)(0, Y_2)\\ & + X_1(\ln f) Z_1(\ln f)(0, Y_2)\\ & - Y_1(\ln f) Z_1(\ln f)(0, X_2)\end{aligned} \tag{11}$$

for all $X, Y, Z \in \Gamma(T(M \times N))$.

Proof of Proposition 1. By definition, we have

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z.$$

Using equation (10), we obtain

$$\begin{aligned}\widetilde{\nabla}_Y Z = {} & \nabla_Y Z + Y_1(\ln f)(0, Z_2) + Z_1(\ln f)(0, Y_2)\\ & - f^2 h(Y_2, Z_2)(\operatorname{grad}\ln f, 0),\end{aligned}$$

then

$$\begin{aligned}\widetilde{\nabla}_X\widetilde{\nabla}_Y Z &= \widetilde{\nabla}_X\nabla_Y Z + \widetilde{\nabla}_X Y_1(\ln f)(0,Z_2) + \widetilde{\nabla}_X Z_1(\ln f)(0,Y_2)\\ &\quad - \widetilde{\nabla}_X f^2 h(Y_2,Z_2)(\operatorname{grad}\ln f,0).\end{aligned}$$

A long calculation gives us

$$\begin{aligned}\widetilde{\nabla}_X\nabla_Y Z &= \nabla_X\nabla_Y Z + X_1(\ln f)(0,\nabla_{Y_2}Z_2) + (\nabla_{Y_1}Z_1)(\ln f)(0,X_2)\\ &\quad - f^2 h(X_2,\nabla_{Y_2}Z_2)(\operatorname{grad}\ln f,0),\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_X Y_1(\ln f)(0,Z_2) &= X_1(\ln f)\,Y_1(\ln f)(0,Z_2) + X_1(Y_1(\ln f))(0,Z_2)\\ &\quad - f^2 h(X_2,Z_2)\,Y_1(\ln f)(\operatorname{grad}\ln f,0)\\ &\quad + Y_1(\ln f)(0,\nabla_{X_2}Z_2),\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_X Z_1(\ln f)(0,Y_2) &= X_1(\ln f)\,Z_1(\ln f)(0,Y_2) + X_1(Z_1(\ln f))(0,Y_2)\\ &\quad - f^2 h(X_2,Y_2)\,Z_1(\ln f)(\operatorname{grad}\ln f,0)\\ &\quad + Z_1(\ln f)(0,\nabla_{X_2}Y_2)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\nabla}_X f^2 h(Y_2,Z_2)(\operatorname{grad}\ln f,0) &= f^2 h(Y_2,Z_2)(\nabla_{X_1}\operatorname{grad}\ln f,0)\\ &\quad + 2f^2 X_1(\ln f)\,h(Y_2,Z_2)(\operatorname{grad}\ln f,0)\\ &\quad + f^2 h(Y_2,Z_2)\,|\operatorname{grad}\ln f|^2(0,X_2)\\ &\quad + f^2 h(\nabla_{X_2}Y_2,Z_2)(\operatorname{grad}\ln f,0)\\ &\quad + f^2 h(Y_2,\nabla_{X_2}Z_2)(\operatorname{grad}\ln f,0).\end{aligned}$$

It follows that

$$\begin{aligned}&\widetilde{\nabla}_X\widetilde{\nabla}_Y Z\\ &= \nabla_X\nabla_Y Z - f^2 h(X_2,\nabla_{Y_2}Z_2)(\operatorname{grad}\ln f,0) - f^2 h(Y_2,Z_2)(\nabla_{X_1}\operatorname{grad}\ln f,0)\\ &- f^2 h(X_2,Y_2)\,Z_1(\ln f)(\operatorname{grad}\ln f,0) - f^2 h(\nabla_{X_2}Y_2,Z_2)(\operatorname{grad}\ln f,0)\\ &- 2f^2 X_1(\ln f)\,h(Y_2,Z_2)(\operatorname{grad}\ln f,0) - f^2 h(Y_2,\nabla_{X_2}Z_2)(\operatorname{grad}\ln f,0)\\ &- f^2 h(X_2,Z_2)\,Y_1(\ln f)(\operatorname{grad}\ln f,0) - f^2 h(Y_2,Z_2)\,|\operatorname{grad}\ln f|^2(0,X_2)\\ &+ (\nabla_{Y_1}Z_1)(\ln f)(0,X_2) + X_1(Z_1(\ln f))(0,Y_2) + X_1(\ln f)\,Z_1(\ln f)(0,Y_2)\\ &+ X_1(\ln f)\,Y_1(\ln f)(0,Z_2) + X_1(Y_1(\ln f))(0,Z_2) + X_1(\ln f)(0,\nabla_{Y_2}Z_2)\\ &+ Y_1(\ln f)(0,\nabla_{X_2}Z_2) + Z_1(\ln f)(0,\nabla_{X_2}Y_2).\end{aligned}$$

A similar calculation gives

$$
\begin{aligned}
&\widetilde{\nabla}_Y \widetilde{\nabla}_X Z \\
&= \nabla_Y \nabla_X Z - f^2 h\left(Y_2, \nabla_{X_2} Z_2\right)(\operatorname{grad}\ln f, 0) - f^2 h\left(X_2, Z_2\right)\left(\nabla_{Y_1} \operatorname{grad}\ln f, 0\right) \\
&- f^2 h\left(X_2, Y_2\right) Z_1(\ln f)(\operatorname{grad}\ln f, 0) - f^2 h\left(\nabla_{Y_2} X_2, Z_2\right)(\operatorname{grad}\ln f, 0) \\
&- 2f^2 Y_1(\ln f)\, h\left(X_2, Z_2\right)(\operatorname{grad}\ln f, 0) - f^2 h\left(X_2, \nabla_{Y_2} Z_2\right)(\operatorname{grad}\ln f, 0) \\
&- f^2 h\left(Y_2, Z_2\right) X_1(\ln f)(\operatorname{grad}\ln f, 0) - f^2 h\left(X_2, Z_2\right)|\operatorname{grad}\ln f|^2 (0, Y_2) \\
&+ \left(\nabla_{X_1} Z_1\right)(\ln f)(0, Y_2) + Y_1\left(Z_1(\ln f)\right)(0, X_2) + Y_1(\ln f) Z_1(\ln f)(0, X_2) \\
&+ Y_1(\ln f) X_1(\ln f)(0, Z_2) + Y_1\left(X_1(\ln f)\right)(0, Z_2) + Y_1(\ln f)\left(0, \nabla_{X_2} Z_2\right) \\
&+ X_1(\ln f)\left(0, \nabla_{Y_2} Z_2\right) + Z_1(\ln f)\left(0, \nabla_{Y_2} X_2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\nabla}_{[X,Y]} Z = {} & \nabla_{[X,Y]} Z + \left(\left[X_1, Y_1\right]\right)(\ln f)(0, Z_2) + Z_1(\ln f)\left(0, \left[X_2, Y_2\right]\right) \\
& - f^2 h\left(X_2, \left[X_2, Y_2\right]\right)(\operatorname{grad}\ln f, 0),
\end{aligned}
$$

which leads us to the following formula

$$
\begin{aligned}
\widetilde{R}(X, Y) Z = {} & R(X, Y) Z - f^2 h\left(Y_2, Z_2\right)\left(\nabla_{X_1} \operatorname{grad}\ln f, 0\right) \\
& + f^2 h\left(X_2, Z_2\right)\left(\nabla_{Y_1} \operatorname{grad}\ln f, 0\right) \\
& - f^2 h\left(Y_2, Z_2\right) X_1(\ln f)(\operatorname{grad}\ln f, 0) \\
& + f^2 h\left(X_2, Z_2\right) Y_1(\ln f)(\operatorname{grad}\ln f, 0) \\
& - f^2 h\left(Y_2, Z_2\right)|\operatorname{grad}\ln f|^2 (0, X_2) \\
& + f^2 h\left(X_2, Z_2\right)|\operatorname{grad}\ln f|^2 (0, Y_2) \\
& - g\left(\nabla_{Y_1} \operatorname{grad}\ln f, Z_1\right)(0, X_2) \\
& + g\left(\nabla_{X_1} \operatorname{grad}\ln f, Z_1\right)(0, Y_2) \\
& + X_1(\ln f) Z_1(\ln f)(0, Y_2) \\
& - Y_1(\ln f) Z_1(\ln f)(0, X_2)
\end{aligned}
$$

As consequence, we obtain

Proposition 2.

$$
\begin{aligned}
\widetilde{Ricci}(X) = {} & \left(Ricci\left(X_1\right), 0\right) - n\left(\nabla_{X_1} \operatorname{grad}\ln f, 0\right) - n X_1(\ln f)(\operatorname{grad}\ln f, 0) \\
& + \frac{1}{f^2}\left(0, Ricci\left(X_2\right)\right) - (\Delta \ln f)(0, X_2) - n|\operatorname{grad}\ln f|^2 (0, X_2),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{Ric}(X, Y) = {} & Ric\left(X_1, Y_1\right) + Ric\left(X_2, Y_2\right) - n g\left(\nabla_{X_1} \operatorname{grad}\ln f, Y_1\right) \\
& - n X_1(\ln f) Y_1(\ln f) - f^2 (\Delta \ln f)\, h\left(X_2, Y_2\right) \\
& - n f^2 |\operatorname{grad}\ln f|^2 h\left(X_2, Y_2\right)
\end{aligned}
$$

and

$$S_{G_f} = S_g + \frac{1}{f^2} S_h - 2n\left(\Delta \ln f\right) - n\left(n+1\right)\left|\operatorname{grad}\ln f\right|^2 .$$

Using the fact that

$$\Delta\left(f^k\right) = k f^k \left(\Delta \ln f + k\left|\operatorname{grad}\ln f\right|^2\right),$$

we obtain the following result. If $S_g = S_h = 0$, then

$$S_{G_f} = 0 \textit{ if and only if the function } f^{\frac{n+1}{2}} \textit{ is harmonic.}$$

In the first, we study the p-biharmonicity of the first projection, we obtain the following result:

Theorem 3. *The first projection $\pi : M^m \times_f N^n \longrightarrow M^m$ is p-biharmonic if and only if*

$$\begin{aligned}
&\left|\operatorname{grad}\ln f\right|^4 \operatorname{grad}\left(\Delta \ln f\right) + \frac{(p-2)}{2}\left|\operatorname{grad}\ln f\right|^2 \Delta\left(\left|\operatorname{grad}\ln f\right|^2\right) \operatorname{grad}\ln f \\
&+ \frac{n}{2}\left|\operatorname{grad}\ln f\right|^4 \operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right) + 2\left|\operatorname{grad}\ln f\right|^4 \mathit{Ricci}\left(\operatorname{grad}\ln f\right) \\
&+ (p-2)\left|\operatorname{grad}\ln f\right|^2 \nabla_{\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right)} \operatorname{grad}\ln f \\
&+ \frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right)\right|^2 \operatorname{grad}\ln f = 0.
\end{aligned} \tag{12}$$

Proof of Theorem 3. By equation (2), the first projection

$$\pi : M^m \times_f N^n \longrightarrow M^m$$

is p-biharmonic if and only if

$$\begin{aligned}
&\left|\widetilde{\tau}(\pi)\right|^4 \widetilde{\tau}_2(\pi) - (p-2)\left|\widetilde{\tau}(\pi)\right|^2 \nabla^\pi_{\operatorname{grad}\left(\left|\widetilde{\tau}(\pi)\right|^2\right)} \widetilde{\tau}(\pi) \\
&- \frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(\left|\widetilde{\tau}(\pi)\right|^2\right)\right|^2 \widetilde{\tau}(\pi) \\
&- \frac{(p-2)}{2}\left|\widetilde{\tau}(\pi)\right|^2 \Delta\left(\left|\widetilde{\tau}(\pi)\right|^2\right) \widetilde{\tau}(\pi) = 0.
\end{aligned}$$

A long and rigorous calculation gives us the following formulas:

$$\widetilde{\tau}(\pi) = n\left(\operatorname{grad}\ln f\right)\circ\pi, \qquad \left|\widetilde{\tau}(\pi)\right|^2 = n^2\left|\operatorname{grad}\ln f\right|^2,$$

$$\Delta\left(\left|\widetilde{\tau}(\pi)\right|^2\right) = n^2\Delta\left(\left|\operatorname{grad}\ln f\right|^2\right), \qquad \left|\widetilde{\tau}(\pi)\right|^4 = n^4\left|\operatorname{grad}\ln f\right|^4,$$

$$\operatorname{grad}\left(\left|\widetilde{\tau}(\pi)\right|^2\right) = n^2 \operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right)\circ\pi,$$

$$\left|\operatorname{grad}\left(\left|\widetilde{\tau}(\pi)\right|^2\right)\right|^2 = n^4\left|\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right)\right|^2,$$

$$\nabla^{\pi}_{\operatorname{grad}\left(|\widetilde{\tau}(\pi)|^2\right)}\widetilde{\tau}(\pi) = n^3\left(\nabla_{\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right)}\operatorname{grad}\ln f\right)\circ\pi$$

and

$$\widetilde{\tau}_2(\pi) = -n\left(\operatorname{grad}(\Delta\ln f) + 2\operatorname{Ricci}(\operatorname{grad}\ln f) + \frac{n}{2}\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right)\right)\circ\pi.$$

Then, we deduce that the first projection π is p-biharmonic if and only if

$$\begin{aligned}
&|\operatorname{grad}\ln f|^4\operatorname{grad}(\Delta\ln f) + \frac{(p-2)}{2}|\operatorname{grad}\ln f|^2\,\Delta\left(|\operatorname{grad}\ln f|^2\right)\operatorname{grad}\ln f\\
&+\frac{n}{2}|\operatorname{grad}\ln f|^4\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right) + 2|\operatorname{grad}\ln f|^4\operatorname{Ricci}(\operatorname{grad}\ln f)\\
&+(p-2)|\operatorname{grad}\ln f|^2\,\nabla_{\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right)}\operatorname{grad}\ln f\\
&+\frac{(p-2)(p-4)}{4}\left|\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right)\right|^2\operatorname{grad}\ln f = 0.
\end{aligned}$$

Example 8. We consider the first projection

$$\pi : \mathbb{R}^m\setminus\{0\}\times_f N^n \longrightarrow \mathbb{R}^m\setminus\{0\}$$

defined by

$$\pi\left(x = (t, x_2, \ldots, x_2), y\right) = (t, x_2, \ldots, x_2),$$

where we suppose that the function f depends only on the t. By the Theorem 1, we deduce that π is p-biharmonic if and only if the function $\beta(t) = (\ln f(t))'$ satisfies the following differential equation

$$(p-1)\beta\beta'' + (p-1)(p-2)\left(\beta'\right)^2 + n\beta^2\beta' = 0.$$

To solve this last equation, we will treat two cases:

1. Looking for particular solutions of type $\beta(t) = \frac{a}{t}\,(a\in\mathbb{R}^*)$, then π is p-biharmonic if and only if $a = \frac{p^2-p}{n}$. In this case, we obtain $f(t) = Ct^{\frac{p^2-p}{n}}, C>0$.
2. $\beta = a\,(a\in\mathbb{R}^*)$ is a trivial solution of the differential equation. We obtain $f(t) = Ce^{at}$, $C>0$.

Example 9. In this example, we suppose that the function f is radial $(f = f(r))$. A rigorous calculus give us

- $\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right) = 2\beta\beta'\frac{\partial}{\partial r}$,
- $\operatorname{grad}\Delta\ln f = \left(\beta'' + \frac{(m-1)}{r}\beta' - \frac{(m-1)}{r^2}\beta\right)\frac{\partial}{\partial r}$,
- $\Delta\left(|\operatorname{grad}\ln f|^2\right) = 2\beta\beta'' + 2\left(\beta'\right)^2 + \frac{2(m-1)}{r}\beta\beta'$,

- $\nabla_{\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right)} \operatorname{grad}\ln f = 2\beta\left(\beta'\right)^2 \frac{\partial}{\partial r}$,

where $\beta(r) = (\ln f(r))'$. Then, by Theorem 3, we deduce that the first projection

$$\pi : \mathbb{R}^m \setminus \{0\} \times_f N^n \longrightarrow \mathbb{R}^m \setminus \{0\}$$

defined by $\pi(x, y) = x$ is p-biharmonic if and only if the function β satisfies the following differential equation

$$\begin{aligned}&(p-1)\beta\beta'' + (p-1)(p-2)\left(\beta'\right)^2 + n\beta^2\beta' \\ &+ \frac{(p-1)(m-1)}{r}\beta\beta' - \frac{(m-1)}{r^2}\beta^2 = 0.\end{aligned}$$

Looking for particular solutions of type $\beta = \frac{a}{r}$ $(a \in \mathbb{R}^*)$, then π is p-biharmonic if and only if

$$a = \frac{p(p-m)}{n}.$$

We obtain

$$f(r) = Cr^{\frac{p(p-m)}{n}}, \quad C > 0$$

and the first projection π is p-biharmonic.

Now, we consider two Riemannian manifolds (M^m, g) and (N^n, h) and we will study the p-biharmonicity of the inclusion map $i_{x_0} : N \longrightarrow (M \times_f N, G_f)$ defined by $i_{x_0}(y) = (x_0, y)$. Note that for this map, we have $di_{x_0}(Y) = (0, Y)$ for any $Y \in \Gamma(TN)$. In the case where $f \in C^\infty(M)$, we obtain (see [4])

Proposition 3 ([4]). *The tension and bitension field of the inclusion map $i_{x_0} : N \longrightarrow (M \times_f N, G_f)$ where $f \in C^\infty(M)$ are given by*

$$\widetilde{\tau}(i_{x_0}) = -nf^2(\operatorname{grad}\ln f, 0) \circ i_{x_0}$$

and

$$\widetilde{\tau}_2(i_{x_0}) = -\frac{n^2}{2} f^4 \left(\left(\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right), 0\right) + 4|\operatorname{grad}\ln f|^2(\operatorname{grad}\ln f, 0)\right) \circ i_{x_0}.$$

Then, i_{x_0} is biharmonic if and only if

$$\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right) + 4|\operatorname{grad}\ln f|^2 \operatorname{grad}\ln f = 0.$$

Remark 3. Based on the following two equations:

$$Tr_h\left(\widetilde{\nabla}^{i_{x_0}}\right)^2(\operatorname{grad}\ln f, 0) \circ i_{x_0} = -nf^2|\operatorname{grad}\ln f|^2(\operatorname{grad}\ln f, 0) \circ i_{x_0}$$

and

$$\begin{aligned}Tr_h\widetilde{R}\left(\widetilde{\tau}(i_{x_0}), di_{x_0}\right)di_{x_0} = &\frac{n^2}{2} f^4 \left(\operatorname{grad}\left(|\operatorname{grad}\ln f|^2\right), 0\right) \circ i_{x_0} \\ &+ n^2 f^4 |\operatorname{grad}\ln f|^2 (\operatorname{grad}\ln f, 0) \circ i_{x_0},\end{aligned}$$

we deduce that the p-bitension field is given by

$$\begin{aligned}\widetilde{\tau}_{p,2}\left(i_{x_0}\right) = & -\frac{1}{p}n^p f^{2p}\left|\operatorname{grad}\ln f\right|^{p-2}\left(\operatorname{grad}\left(\left|\operatorname{grad}\ln f\right|^2\right),0\right)\circ i_{x_0} \\ & -\frac{4}{p}n^p f^{2p}\left|\operatorname{grad}\ln f\right|^p\left(\operatorname{grad}\ln f,0\right)\circ i_{x_0}.\end{aligned}$$

Then, we conclude that

The inclusion map i_{x_0} is p-biharmonic if and only if it is biharmonic.

Proposition 4. *For the inclusion map $i_{x_0} : N \longrightarrow (M \times_f N, G_f)$, we have*

$$\widetilde{S}_2\left(i_{x_0}\right) = -\frac{n\left(n-4\right)}{2}f^4\left|\operatorname{grad}\ln f\right|^2 h\circ i_{x_0}.$$

Proof of Proposition 4. By definition, we have

$$\begin{aligned}\widetilde{S}_2\left(i_{x_0}\right)(X,Y) = & \left(\frac{1}{2}\left|\widetilde{\tau}\left(i_{x_0}\right)\right|^2 + \operatorname{Tr}_h G_g\left(\widetilde{\nabla}\widetilde{\tau}\left(i_{x_0}\right), di_{x_0}\right)\right)h\left(X,Y\right) \\ & - G_f\left(\widetilde{\nabla}_X\widetilde{\tau}\left(i_{x_0}\right), di_{x_0}\left(Y\right)\right) - G_f\left(\widetilde{\nabla}_Y\widetilde{\tau}\left(i_{x_0}\right), di_{x_0}\left(X\right)\right).\end{aligned}$$

Using the fact that $\widetilde{\tau}\left(i_{x_0}\right) = -nf^2\left(\operatorname{grad}\ln f, 0\right)\circ i_{x_0}$, a rigorous calculation gives

$$\left|\widetilde{\tau}\left(i_{x_0}\right)\right|^2 = n^2 f^4\left|\operatorname{grad}\ln f\right|^2,$$

$$Tr_h G_f\left(\widetilde{\nabla}\widetilde{\tau}\left(i_{x_0}\right), d\phi\right) = -n^2 f^4\left|\operatorname{grad}\ln f\right|^2,$$

$$G_f\left(\widetilde{\nabla}_X\widetilde{\tau}\left(i_{x_0}\right), di_{x_0}\left(Y\right)\right) = -nf^4\left|\operatorname{grad}\ln f\right|^2 h\left(X,Y\right)$$

and

$$G_f\left(\widetilde{\nabla}_Y\widetilde{\tau}\left(i_{x_0}\right), di_{x_0}\left(X\right)\right) = -nf^4\left|\operatorname{grad}\ln f\right|^2 h\left(X,Y\right).$$

It follows that

$$\widetilde{S}_2\left(i_{x_0}\right)(X,Y) = -\frac{n\left(n-4\right)}{2}f^4\left|\operatorname{grad}\ln f\right|^2 h\left(X,Y\right).$$

Remark 4. Based on the expression of $\widetilde{S}_2\left(i_{x_0}\right)$, we conclude that

$$\widetilde{S}_2\left(i_{x_0}\right) = 0 \text{ if and only if } n = 4.$$

5. Conclusion

In this paper, we have presented some constructions of p-biharmonic maps by conformal deformation and by using the warped product, we have studied the p-biharmonicity in some particular cases. The results obtained have allowed us to construct new examples of p-biharmonic maps. This paper is a generalization of the results obtained for biharmonic maps and our next objective is to study this class of maps by using the doubly warped product manifolds.

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