

Characterizing finite groups with mutually N -permutable products and smooth maximal subgroups

Mohamed. H. Abd-Ellatif

Department of Mathematics and Computer Science

Faculty of Science

Beni-Suef University 62511 Beni-Suef

Egypt

m.abdellatif86@yahoo.com

Abstract. Let $G = HK$ be a finite group, where H and K are proper subgroups of G . A group G is called a mutually N -permutable product of H and K if H permutes with every normal subgroup of K , and K permutes with every normal subgroup of H . In this paper, as a next step of some recently studies, we examine the structural properties of finite group G that is a mutually N -permutable product of two subgroups, with the additional assumption that all maximal subgroups of G are generalized smooth groups.

Keywords: generalized smooth groups, permutable subgroups, subgroup lattices.

MSC 2020: 20F22, 20E15.

1. Introduction

In this paper, only finite groups are considered. For a group G , let $\pi(G)$ stand for the set of primes dividing $|G|$, $L(G)$ the subgroup lattice of G and n the maximal length of $L(G)$.

Let $1 = G_0 < G_1 < G_2 < \dots < G_n = G$ be a maximal chain of subgroups of a group G . An interval $[G_{i+j}/G_j] = \{X \leq G : G_j \leq X \leq G_{i+j}\}$ is the set of all subgroups of G_{i+j} which contain G_j . A maximal chain is called smooth if any two intervals have the same length are isomorphic. If all maximal chains from any subgroup of G of prime order to G ($L < \dots < G$ where L is any subgroup of G of prime order) are smooth chains, then G is called a GS -group (a generalized smooth group) (see [10]). To clarify the concept of GS -groups more effectively, we present some examples:

- Let $G \cong Q_8$ (the quaternion group of order 8). It is well-known that Q_8 has a unique subgroup of prime order $\langle -1 \rangle$, and three maximal chains extending from $\langle -1 \rangle$ to Q_8 . Specifically, we have $\langle -1 \rangle < \langle i \rangle < Q_8$, $\langle -1 \rangle < \langle j \rangle < Q_8$ and $\langle -1 \rangle < \langle k \rangle < Q_8$. Each of these chains is smooth. Hence, G is a GS -group.
- Let $G \cong Z_{24}$ (cyclic group of order 24). The subgroup $\langle 12 \rangle$ has order 2, and the chain $\langle 12 \rangle < \langle 4 \rangle < \langle 2 \rangle < Z_{24}$ forms a maximal chain from $\langle 12 \rangle$ to Z_{24} . It is evident that $[Z_{24}/\langle 4 \rangle] = \{Z_{24}, \langle 2 \rangle, \langle 4 \rangle\}$ and $[\langle 2 \rangle/\langle 12 \rangle] =$

$\{\langle 2 \rangle, \langle 4 \rangle, \langle 6 \rangle, \langle 12 \rangle\}$. Clearly, the intervals $[Z_{24}/\langle 4 \rangle]$ and $[\langle 2 \rangle/\langle 12 \rangle]$ are not isomorphic, even though they have the same length. Thus, G is not a GS -group.

A group G is called a P -group if either G is an elementary abelian or $G = G_p G_q$, where G_p is an elementary abelian normal Sylow p -subgroup and G_q is a Sylow q -subgroup of order q which induces a non-trivial power automorphism on G_p , and $q \mid p - 1$ (see [16, p. 49]).

Let H and K be subgroups of a group G with $G = HK$. We say that, G is a mutually permutable product of H and K if H permutes with every subgroup of K and K permutes with every subgroup of H (see [6]), also G is called a mutually m -permutable product of H and K if H permutes with every maximal subgroup of K and K permutes with every maximal subgroup of H (see [5]). In this paper, we introduce the following concept:

Definition. Let $G = HK$ be a group with proper subgroups H and K . We say that, G is a mutually N -permutable product of H and K if H permutes with every normal subgroup of K and K permutes with every normal subgroup of H .

Many papers introduced the structure of a group whose maximal subgroups are GS -groups under suitable conditions (see [1]-[3] and [7]-[9]). In [3], the authors studied the structure of a group G which is a mutually permutable product or a mutually m -permutable product of two proper subgroups under the assumption that, all maximal subgroups of G are GS -groups. In this paper, as a next step, we replace these permutability conditions by a new one. More precisely, we prove the following result:

Main theorem. Assume that $G = HK$ is a mutually N -permutable product of its proper subgroups H and K . Suppose further that all maximal subgroups of G are GS -groups with $n \geq 4$. Then, one of the following holds:

- (i) $|G| = p_1 p_2 p_3 p_4$, where p_1, p_2, p_3 and p_4 are not necessarily distinct primes.
- (ii) G is a P -group.
- (iii) G is cyclic of square free order.
- (iv) $G = G_{p_1} A$, where $|G_{p_1}| = p_1$, $G_{p_1} \triangleleft G$ and A is cyclic of order $p_2 p_3 \dots p_m$ and operates faithfully on G_{p_1} where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, \dots, m$.
- (v) $G = G_p G_q$ where G_p is an elementary abelian normal Sylow p -subgroup of order p^2 , G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p .
- (vi) $G = G_p G_q$ where $G_p = P_1 P_2$ is an elementary abelian normal Sylow p -subgroup of order p^4 such that P_i ($i = 1, 2$) is a minimal normal subgroup

of G of order p^2 , G_q is a Sylow q -subgroup of order q and G_q operates irreducibly on P_i ($i = 1, 2$).

(vii) $G \cong L_2(11)$.

Now, we list some examples to illustrate the importance of our main theorem:

- Let $G \cong L_2(13)$ (the projective special linear group). All maximal subgroups of G are GS -groups; however, G cannot be expressed as a mutually N -permutable of two subgroups. The same observation holds for $L_2(19)$ and $L_2(29)$. Actually, by the main theorem, A_5 and $L_2(11)$ are the only simple groups that can be expressed as a mutually N -permutable product of two subgroups, with all maximal subgroups being GS -groups.
- Let $G \cong S_5$ (the symmetric group of degree 5). Clearly, $S_5 = A_5 P_2$, where P_2 is a Sylow 2-subgroup of S_5 . Here, S_5 can be expressed as a mutually N -permutable product of A_5 and P_2 . However, S_4 is a maximal subgroup of S_5 , is not a GS -group.
- Let G be an abelian group of order $240 = 2^4 \cdot 3 \cdot 5$. Clearly, G can be expressed as a mutually N -permutable subgroup of any two maximal subgroups. However, G has a maximal subgroup of order 80 that is not GS -group.
- Let $G \cong Z_{210}$ (the cyclic group of order 210). In this case, all maximal subgroups of G are GS -groups and G can be expressed as a mutually N -permutable subgroup of any two maximal subgroups.

2. Preliminaries

Lemma 2.1 ([10], Main Theorem). *A group G is a GS -group if and only if one of the following holds:*

- (i) $|G| = p_1 p_2 p_3$, where p_1, p_2 , and p_3 are not necessarily distinct primes.
- (ii) G is cyclic of prime power order.
- (iii) G is a P -group.
- (iv) G is cyclic of square free order.
- (v) $G = G_p G_q$, where G_p is a minimal normal subgroup of order p^2 and G_q is cyclic of order q^2 such that G_q and $\Phi(G_q)$ operate irreducibly on G_p .
- (vi) $G = G_{p_1} A$, where $|G_{p_1}| = p_1$, $G_{p_1} \triangleleft G$ and A is cyclic of order $p_2 p_3 \dots p_m$ and operates faithfully on G_{p_1} where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, \dots, m$.
- (vii) $G \cong A_5$.

Lemma 2.2 ([3], Main Theorem). *Assume that G is a mutually m -permutable product of its proper subgroups H and K with $n \geq 4$. Assume further that all maximal subgroups of G are GS -groups. Then, G is one of the following.*

- (i) $|G| = p_1 p_2 p_3 p_4$, where p_1, p_2, p_3 and p_4 are not necessarily distinct primes.
- (ii) G is a P -group.
- (iii) G is cyclic of square free order.
- (iv) $G = G_{p_1} A$, where $|G_{p_1}| = p_1$, $G_{p_1} \triangleleft G$ and A is cyclic of order $p_2 p_3 \dots p_m$ and operates faithfully on G_{p_1} where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, \dots, m$.
- (v) $G = G_p G_q$ where G_p is an elementary abelian normal Sylow p -subgroup of order p^2 , G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p .
- (vi) $G = G_p G_q$ where $G_p = P_1 P_2$ is an elementary abelian normal Sylow p -subgroup of order p^4 such that P_i ($i = 1, 2$) is a minimal normal subgroup of G of order p^2 , G_q is a Sylow q -subgroup of order q and G_q operates irreducibly on P_i ($i = 1, 2$).

Lemma 2.3 ([3], Lemma 8). *Assume that G is a supersolvable group. Assume further that all maximal subgroups of G are GS -groups with $n \geq 4$. Then, G is one of the following.*

- (i) $|G| = p_1 p_2 p_3 p_4$, where p_1, p_2, p_3 and p_4 are not necessarily distinct primes.
- (ii) G is cyclic of prime power order.
- (iii) G is a P -group.
- (iv) G is cyclic of square free order.
- (v) $G = G_{p_1} A$, where $|G_{p_1}| = p_1$, $G_{p_1} \triangleleft G$ and A is cyclic of order $p_2 p_3 \dots p_m$ and operates faithfully on G_{p_1} where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, \dots, m$.

Corollary 2.1. *Assume that G is a supersolvable group with $|\pi(G)| = 2$. If all maximal subgroups of G are GS -groups with $n > 4$, then G is a nonabelian P -group.*

3. Results

In the following, for simplicity, we will concern that G satisfies structure:

- (a) if $G = G_p G_q$, where G_p is a minimal normal subgroup of order p^2 and G_q is cyclic of order q which operates irreducibly on G_p .
- (b) if $G = G_p G_q$, where G_p is a minimal normal subgroup of order p^2 and G_q is cyclic of order q^2 such that G_q and $\Phi(G_q)$ operate irreducibly on G_p .

Lemma 3.1. *Assume that G is a mutually N -permutable product of its proper subgroups H and K . Assume further that all maximal subgroups of G are GS -groups. Then, either G is solvable or $G \cong L_2(11)$.*

Proof. A group of odd order is solvable so let G be of even order. Firstly, let both H and K are supersolvable. By ([17]; Lemma 2.4), if G has a Hall $2'$ -subgroup, then G is not simple and hence by ([3]; Lemma 6), G is solvable. So, in the following paragraph, in order to prove that G is solvable, we need only to show that it has a Hall $2'$ -subgroup.

Let, without loss of generality, that $2 \mid |H|$. As H is supersolvable, it has a normal 2-complement $H_{2'}$. If K is of odd order, then by hypothesis $G_{2'} = H_{2'}K$ is a subgroup of G . So, let both H and K are of even order. Also, K has a normal 2-complement $K_{2'}$ as it is supersolvable. By our hypothesis, $HK_{2'}$ and $H_{2'}K$ are subgroups of G . If $HK_{2'}$ or $H_{2'}K$ has a cyclic Sylow 2-subgroup then it has a normal 2-complement which implies that $G_{2'} = H_{2'}K_{2'}$ is a subgroup of G . Clearly, if H or K is 2-group, then $G_{2'} = K_{2'}$ or $G_{2'} = H_{2'}$ respectively. Now, we can say that, H and K are supersolvable GS -groups with non cyclic Sylow 2-subgroups and $|H|$ and $|K|$ are divided at least by two different primes. According to Lemma 2.1, $|H| = 2^2q$ and $|K| = 2^2r$ where q and r are primes. If $q = r$, then $|\pi(G)| = 2$ and hence G is solvable. So, let $q \neq r$. Say G_2 be the Sylow 2-subgroup of G , if $|G_2| > 2^2$, then by hypothesis HK_r is a proper subgroup of G but not a GS -group, a contradiction. Therefore, $[G : H] = r$ and $[G : K] = q$ and hence by ([4]; Lemma 10), G is solvable. Now, assume that H is not supersolvable. According to Lemma 2.1, either $H \cong A_5$ or $H = H_p H_q$ and satisfies structure (a) or (b). If both H and K are isomorphic to A_5 , then by [13], either $G \cong A_5 \times A_5$ or $G \cong A_6$. Since there is no a GS -group contains A_5 , we have $G \not\cong A_5 \times A_5$. Also A_6 has a subgroup isomorphic to S_4 which is not a GS -group, a contradiction. Thus, we can assume that $K \not\cong A_5$. According to Lemma 2.1, K would be a solvable group. It follows that, K has a minimal normal subgroup L , say, which is elementary abelian. By hypothesis HL is a subgroup of G . We handel the following cases:

1. $H \not\leq HL \not\leq G$. If H satisfies structure (b) or $H \cong A_5$, then H would be a maximal subgroup of G as there is no a GS -group contains it. Hence H satisfies structure (a). By hypothesis HL is a GS -group. Applying Lemma 2.1, $HL \cong A_5$ which implies that $H \cong A_4$ and $|L| = 5$ is a Sylow 5-subgroup of A_5 . Hence $H_2 \triangleleft H$ where H_2 is a Sylow 2-subgroup of H and $L \not\leq K$ as $HL \not\leq G$. By hypothesis, H_2K is a proper GS -subgroup of G . Applying Lemma 2.1, H_2K is of order 2^25^2 and satisfies structure (b). This implies that A_5 has a subgroup of order 2^25 , a contradiction.
2. $G = HL$. Hence H is of prime power index. If G is not simple then by ([3]; Lemma 6), G is solvable. So, let G be a simple group. By ([14]; Theorem 1), $H \cong A_5$ and $G \cong L_2(11)$. Note that, If $H \cong A_4$ and $G \cong A_5$, then by hypothesis G has a subgroup of order 20, a contradiction.

3. $H = HL$. Hence $L \leq H \cap K$ for each minimal normal subgroup of K . As the previous case if H is of prime index, we are done. So, let $[G : H]$ is divided at least by two different primes and consequently, $|\pi(K)| \geq 2$. Assume that $|\pi(K)| \geq 3$. Applying Lemma 2.1, K is of square free order and hence has a Sylow tower property. Say $K = K_{r_1}K_{r_2}\dots K_{r_m}$, where K_{r_i} ($i = 1, 2, \dots, m$) is a Sylow r_i -subgroup of K with $r_i > r_{i+1}$. Since each minimal normal subgroup of K is a subgroup of H , we get $K_{r_1} < H$. By hypothesis, $HK_{r_1}K_{r_2} = HK_{r_2}$ is a subgroup of G . If $H \not\leq HK_{r_2} \leq G$, then similar as the previous cases we are done. So, let $H = HK_{r_2}$. If $|\pi(K)| = 3$, then H is of prime index, a contradiction. Thus, $|\pi(K)| > 3$. But in this case, $HK_{r_1}K_{r_2}K_{r_3} = HK_{r_3}$ is a proper subgroup of G but not a GS -group, a contradiction. Thus, $|\pi(K)| = 2$. Since $L < H$ and $[G : H]$ is not prime power, L can't be a Sylow subgroup of K . Say $K = K_{r_1}K_{r_2}$ (it's not necessary that $r_1 > r_2$) and $L \not\leq K_{r_1}$. If $|K_{r_1}| > r_1^2$, then by Lemma 2.1, K is a nonabelian P -group and hence every subgroup of K_{r_1} is a subgroup of H , a contradiction. Thus, $|K_{r_1}| = r_1^2$. Since $L \not\leq K_{r_1}$ and K is a GS -group, we get by Lemma 2.1, K is a supersolvable group of order $r_1^2 r_2$. Clearly, either K_{r_1} or K_{r_2} is a normal subgroup of K and hence a subgroup of H which contradicts with our assumption that $[G : H]$ is divided by at least two different primes. By this final contradiction our proof is completed. $\square$

Lemma 3.2. *Assume that G is a solvable group with all maximal subgroups of G are GS -groups. If $|\pi(G)| \neq 2$, then G is supersolvable or $|G| = p^2qr$ where p, q and r are distinct primes.*

Proof. As we know if G is a p -group, then G is supersolvable. And by our hypothesis $|\pi(G)| \neq 2$. So, $|\pi(G)| \geq 3$. Firstly, let $|\pi(G)| > 3$. Since G is solvable, then there exist for each prime $p_i \in \pi(G)$, a maximal subgroup M_i such that $[G : M_i] = p_i^e$ ($e \geq 1$). By hypothesis, M_i is a GS -group. Applying Lemma 2.1, M_i is of square free order $\forall i$. It follows that G itself is of square free order and consequently G is supersolvable.

Now, let $|\pi(G)| = 3$. Then, $G = G_{p_1}G_{p_2}G_{p_3}$ where G_{p_i} is a Sylow p_i -subgroup of G . Solvability of G implies that $G_{p_1}G_{p_2}$ is a proper subgroup of G . Let $G_{p_1}G_{p_2}$ is not a maximal subgroup of G . Then, there exist a subgroup M , say, of G such that $G_{p_1}G_{p_2} < M < G$. By hypothesis, M is a GS -group. Applying Lemma 2.1, M is of square free order. It follows that $|G_{p_i}| = p_i$ ($i = 1, 2$). As $n \geq 4$, $|G_{p_3}| \geq p_3^2$. Clearly, if $|G_{p_3}| = p_3^2$, then $|G| = p_1p_2p_3^2$ and we are done. So, assume that $|G_{p_3}| > p_3^2$. By applying Lemma 2.1, $G_{p_3}G_{p_i}$ is a nonabelian P -group with $p_3 > p_i$ ($i = 1, 2$). So, every subgroup of G_{p_3} is normal in G . This implies that $G_{p_3}^*G_{p_1}G_{p_2}$ is a proper subgroup of G , where $G_{p_3}^*$ is a maximal subgroup of G_{p_3} . But $G_{p_3}^*G_{p_1}G_{p_2}$ can't be a GS -group, a contradiction. Thus, assume that, $G_{p_i}G_{p_j}$ is a maximal subgroup of $G \forall i, j \in \{1, 2, 3\}$ and $i \neq j$.

Since G is solvable, it has a minimal normal subgroup L , say, which is elementary abelian. Without loss of generality let $L \leq G_{p_1}$. Maximality of $G_{p_2}G_{p_3}$ in G implies that $L = G_{p_1}$. Also by the maximality of $G_{p_1}G_{p_i}$ ($i = 2, 3$) in G , G_{p_2} and G_{p_3} must be maximal subgroups in $G_{p_2}G_{p_3}$. By hypothesis, $G_{p_2}G_{p_3}$ is a GS -group. Then, either $|G_{p_2}G_{p_3}| = p_2p_3$ or $G_{p_2}G_{p_3}$ satisfies structure (a). Let $|G_{p_2}G_{p_3}| = p_2p_3$. If $|L| > p_1^2$, then $G_{p_1}G_{p_i}$ ($i = 2, 3$) is a nonabelian P -group and hence every subgroup of L is normal in G , a contradiction. Thus, $|L| = p_1^2$ and hence $|G| = p_1^2p_2p_3$. Now, let $|G_{p_2}G_{p_3}| = p_2^2p_3$ and satisfies structure (a). Once again $|G| = p_1p_2^2p_3$ if $|L| = p_1$. If else, LG_{p_2} is a proper subgroup of G with order divided by $p_1^2p_2^2$. Applying Lemma 2.1, LG_{p_2} satisfies structure (b) with G_{p_2} is cyclic. It follows that $G_{p_2}G_{p_3}$ is supersolvable which contradicts with G_{p_3} is a maximal subgroup of $G_{p_2}G_{p_3}$. By this final contradiction our proof is completed. $\square$

Lemma 3.3. *Assume that $G = HK$ is a mutually N -permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS -groups with $n > 4$. If H is a nonabelian P -group so as G .*

Proof. Let $H = H_pH_q$ be of order p^jq ($j \geq 1$). If $|G_q| = q$, then by ([18], Corollary 1.10, p. 6), G is supersolvable and hence by Corollary 2.1, G is a nonabelian P -group. So, let $|G_q| \geq q^2$.

Let $|H_p| \geq p^2$. By hypothesis, $H_pK \leq G$. Assume that $G = H_pK$. Choose H_p^* be a maximal subgroup of H_p such that H_p^*K is a proper subgroup of G . Since $[G : H_p^*K] = p$, $|G_q| \geq q^2$ and $n > 4$, we get by Lemma 2.1 that H_p^*K satisfies structure (b) which contradicts with H is a nonabelian P -group. Thus, H_pK is a proper subgroup of G . Applying Lemma 2.1, H_pK is either a nonabelian P -group with $p > q$ or satisfies structure (b). Then, $H_p \triangleleft H_pK$ and since $H_p \triangleleft H$, we get $H_p \triangleleft G$. If H_pK is a nonabelian P -group, then H_pG_q is a proper subgroup of G of order p^jq^2 ($j \geq 2$) and by applying Lemma 2.1, H_pG_q satisfies structure (b) which contradicts with H is a nonabelian P -group. Also if H_pK satisfies structure (b), then $H_pG_q^*$ is a proper subgroup of G for each maximal subgroup G_q^* of G_q and by applying Lemma 2.1, $H_pG_q^*$ satisfies structure (b), once again we get a contradiction with H is a nonabelian P -group. This final contradiction shows that $|H_p| = p$. If $|\pi(K)| = 1$, then $|K| \geq q^3$ as $|G_q| \geq q^2$ and $n > 4$. By hypothesis, HK^* is a proper subgroup of G for some maximal subgroup K^* of K . As $pq^3 \mid |HK^*|$ with $p > q$, HK^* can't be a GS -group, a contradiction. Thus, $|\pi(K)| = 2$. Applying Lemma 2.1, we have the following:

1. K satisfies structure (b). If $H \cap K = 1$, then by hypothesis H_pK is a proper subgroup of G but not a GS -group, a contradiction. Thus, either $H_p < K$ or $H_q < K$. Assume that $H_q < K$. Then, $|G| = p^3q^2$ with $p > q$. By ([15], Theorem 6.5.5, p. 147), G has a normal Sylow subgroup. If $G_p \triangleleft G$, then by Lemma 2.1, $G_pG_q^*$ is a nonabelian P -group which contradicts with K

satisfies structure (b). Thus, $G_q \triangleleft G$. Choose G_p^* be a maximal subgroup of G_p such that $H_p < G_p^*$. Hence $G_p^*G_q$ is a proper subgroup of G of order p^2q^2 and $H < G_p^*G_q$. Clearly, $G_p^*G_q$ can't be a GS -group, a contradiction. Now, let $H_p < K$. If $K_q \triangleleft K$, then by hypothesis HK_q is a proper subgroup of G of order pq^3 ($p > q$). Applying Lemma 2.1, HK_q is not a GS -group, a contradiction. Thus, $K_p \triangleleft K$. By hypothesis, K_pH is a proper subgroup of G with $|K_pH| = p^2q$. By ([15], Theorem 6.2.11, p. 138), $K_p \triangleleft K_pH$ and consequently, $K_p \triangleleft G$. Choose G_q^* be a maximal subgroup of G_q with $H_q < G_q^*$. Then, $K_pG_q^*$ is a proper subgroup of G of order p^2q^2 which contains H . By Lemma 2.1, $K_pG_q^*$ is not a GS -group, a contradiction.

2. K is a nonabelian P -group. Then, $H_q \neq K_q$ as $|G_q| \geq q^2$. By hypothesis, K_pH is a proper subgroup of G . Applying Lemma 2.1, K_pH is a nonabelian P -group. It follows that every subgroup of K_p is normal in K_pH and hence normal in G . By similar, every subgroup of H_p is normal in G . Therefore, every subgroup of G_p is normal in G and hence $G_p^*G_q$ is a proper subgroup of G but not a GS -group, a contradiction.
3. $|K| = p^2q$. Clearly, $H \cap K = 1$ as $n > 4$. By ([15], Theorem 6.2.11, p. 138), $K_p \triangleleft K$. Our hypothesis and Lemma 2.1 implies that K_pH and H_pK are nonabelian P -groups. Hence every subgroup of G_p is normal in G . Once again $G_p^*G_q$ is a proper subgroup of G but not a GS -group, a contradiction.
4. $|K| = pq^2$. Clearly, $H \cap K = 1$ as $n > 4$. If $K_q \triangleleft K$, then HK_q is a proper subgroup of G but not a GS -group, a contradiction. Thus, $K_p \triangleleft K$ and hence K has a subgroup of order pq . Since $H_p \triangleleft H$, we get by hypothesis that H_pK is a proper subgroup of G of order p^2q^2 . But it is not a GS -group, a contradiction. $\square$

Lemma 3.4. *Assume that $G = HK$ is a mutually N -permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS -groups with $n > 4$. If H satisfies structure (b), then $G = G_pG_q$ where G_p is an elementary abelian normal Sylow p -subgroup of order p^2 , G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p .*

Proof. Let $H = H_pH_q$, where H_p is a minimal normal subgroup of order p^2 and H_q is cyclic of order q^2 such that H_q and $\Phi(H_q)$ operate irreducibly on H_p . By hypothesis, $H_pK \leq G$. Firstly, let $G = H_pK$. Then, K has a cyclic subgroup of order q^2 . If $|\pi(K)| = 2$, then by Lemma 2.1, either K is supersolvable of order pq^2 or K satisfies structure (b). Since $H_p \triangleleft H$ and $H_p \triangleleft G_p$, we get $H_p \triangleleft G$. Then, $H_pK_pK_q^*$ is a proper subgroup of G and by Lemma 2.1, $H_pK_pK_q^*$ is a nonabelian P -group which contradicts with H satisfies structure (b). Thus, $K = G_q$. By Lemma 2.1, K is either cyclic or nonabelian of order q^3 . Solvability of G implies that, it has a minimal normal subgroup L , say, which is elementary abelian. Let $L \leq G_q$. Since G_q is either cyclic or nonabelian of

order q^3 , we have $|L| = q^2$ or $|L| = q$. If $|L| = q^2$ (or $|L| = q$), then $H_p L$ (or $H_p H_q^* L$), respectively, is a proper subgroup of G but not a GS -group, a contradiction. Thus, $L \leq G_p = H_p$. Clearly, if L is a proper subgroup of H_p , we get a contradiction with the structure of H . So, $L = H_p$ and hence $H_p K^*$ is a proper subgroup of G for every maximal subgroup K^* of K . Applying Lemma 2.1, $H_p K^*$ satisfies structure (b). Therefore, every subgroup of K is cyclic of order q^2 and hence by ([12], Satz 8.2, p. 310), either K is cyclic or $K \cong Q_8$ (quaternion group of order 8). If $K \cong Q_8$, then by ([18], Corollary 1.10, p. 6), $H_p \Phi(H_q)$ is supersolvable which contradicts with $\Phi(H_q)$ operates irreducibly on H_p . Thus, K is cyclic and every subgroup of K operates irreducibly on H_p and we are done.

Now, assume that $H_p K$ is a proper subgroup of G . Let $|\pi(H_p K)| = 1$. Clearly, $G_p = H_p K$ and $G_q = H_q$. We will study the structure of $N_G(G_q)$. If $N_G(G_q) = G_q$, then by ([12], Hauptsatz 2.6, p. 419), $G_p \triangleleft G$ and hence $G_p G_q^*$ is a proper subgroup of G . By Lemma 2.1, $G_p G_q^*$ is a nonabelian P -group which contradicts with H satisfies structure (b). Also $N_G(G_q) \cap H_p = 1$ as G_q operates irreducibly on H_p . Thus, $N_G(G_q)$ is a proper subgroup of G with $p^2 \mid [G : N_G(G_q)]$ and $|N_G(G_q)| = p^i q^2$ ($i \geq 1$). Applying Lemma 2.1, $N_G(G_q)$ must be of order pq^2 . Hence $N_G(G_q)$ has a maximal subgroup N^* , say, of order pq . As $H_p \triangleleft H$ and $H_p \triangleleft G_p$, we have $H_p \triangleleft G$ and hence $H_p N^*$ is a proper subgroup of G and we get the same previous contradiction. This final contradiction implies that $|\pi(H_p K)| = 2$. By Lemma 2.1, either $H_p K$ is a nonabelian P -group ($p > q$) or $|H_p K| = p^2 q$ or $H_p K$ satisfies structure (b).

We argue that $H_p \triangleleft G$ and $|G_q| = q^3$. Firstly, let $H_p K$ be a nonabelian P -group. Then, $H_p \triangleleft H_p K$ and hence $H_p \triangleleft G$. If $K_q < H_q$ then H has a subgroup of order pq which contradicts with $\Phi(H_q)$ operates irreducibly on H_p . Thus, $|G_q| = q^3$. Now, let $|H_p K| = p^2 q$. Clearly $|G_q| = q^3$ as $n > 4$. If $q < p$, then $H_p \triangleleft H_p K$ and hence $H_p \triangleleft G$. So, let $q > p$. By ([15], Theorem 6.5.5, p. 147), G has a normal Sylow subgroup. If $G_q \triangleleft G$, then $G_q G_p^*$ is a proper subgroup of G but not a GS -group, a contradiction. Thus, $G_p = H_p \triangleleft G$. Finally, let $H_p K$ satisfies structure (b). Once again $H_p \triangleleft H_p K$ and hence $H_p \triangleleft G$. If $|G_q| > q^3$, then $H_p G_q^*$ is a proper subgroup of G but not a GS -group, a contradiction. Since $n > 4$, we get $|G_q| = q^3$. Therefore, our argument is done. Hence $H_p G_q^*$ is a proper subgroup of G of order $p^2 q^2$. Applying Lemma 2.1, $H_p G_q^*$ satisfies structure (b). Hence every subgroup of G_q is cyclic. Since G_q is not cyclic as $G_q = H_q K_q$ is a factorized group, we have $G_q \cong Q_8$. Once again, by ([18], Corollary 1.10, p. 6), $H_p \Phi(G_q^*)$ is a supersolvable group which contradicts with $H_p G_q^*$ satisfies structure (b). $\square$

Lemma 3.5. *Assume that $G = HK$ is a mutually N -permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS -groups with $n > 4$. If $|H| = p^2 q$, then either G is a nonabelian P -group or $G = G_p G_q$ where $G_p = P_1 P_2$ is an elementary abelian normal Sylow p -subgroup of order p^4 such that P_i ($i = 1, 2$) is a minimal normal*

subgroup of G of order p^2 , G_q is a Sylow q -subgroup of order q and G_q operates irreducibly on P_i ($i = 1, 2$).

Proof. Let H be abelian. Then, H is a maximal subgroup of G as there is no a GS -group contains it. Solvability of G implies that H is of prime power index. Firstly, assume that $[G : H] = q^e$ ($e \geq 2$ as $n > 4$). Then, $G_p = H_p$ and $|G_q| \geq q^3$. Maximality of H in G implies that H_p is not normal in G . Therefore, $N_G(G_p) = C_G(G_p) = H$ and hence $G_q \triangleleft G$. It follows that $G_p^*G_q$ is a proper subgroup of G but not a GS -group, a contradiction. Thus, $[G : H] = p^e$ ($e \geq 2$). Then, $G_q = H_q$ and $|G_p| \geq p^4$. By Lemma 2.1, G_p is either cyclic or elementary abelian. Clearly, if G_p is cyclic, then G is supersolvable and we get a contradiction with maximality of H . Thus, G_p is elementary abelian. Since $H_p^* \triangleleft H$ and $H_p^* \triangleleft G_p$, we get $H_p^* \triangleleft G$. By ([15], Theorem 9.3.7, p. 225), H_p^* is complemented in G but this complement can't be a GS -group, a contradiction. This final contradiction shows that either H_p or H_q is not normal subgroup of H . We have the following two cases:

Case (i). $H_q \triangleleft H$ and $H_p \not\triangleleft H$. Then, H is supersolvable and hence $q > p$. Let $|G_p| \geq p^3$. If $|G_p| = p^3$, then $|G_q| \geq q^2$ as $n > 4$ and hence $[G : H]$ is divided by pq . Solvability of G implies that, H is not a maximal subgroup of G . Then, G has a maximal subgroup contains H but it is not a GS -group, a contradiction. Thus, $|G_p| \geq p^4$. Since $q > p$, we get G_p is a maximal subgroup of G as there is no a GS -group contains it. Hence either $N_G(G_p) = G_p$ or $G_p \triangleleft G$. If $G_p \triangleleft G$, then by ([11], Theorem 4.5, p. 253), $N_G(H_q)/C_G(H_q)$ is q -subgroup. But $H_q \triangleleft H$. Then, $H_p < C_G(H_q)$ and hence H is abelian, a contradiction. Thus, $N_G(G_p) = G_p$. Since G_p is abelian, we have by ([12], Hauptsatz 2.6, p. 419), $G_q \triangleleft G$ and hence $G_qG_p^*$ is a proper subgroup of G but not a GS -group, a contradiction. This final contradiction shows that $G_p = H_p$ and hence $|G_q| \geq q^3$ as $n > 4$.

By hypothesis $H_qK \leq G$. Let $G = H_qK$. Then, K is a GS -group with $p^2q^2 \mid |K|$ and by Lemma 2.1, K satisfies structure (b). By ([15], Theorem 6.5.5, p. 147), G has a normal Sylow subgroup. Since $G_p = H_p$ is not normal in H , we get $G_q \triangleleft G$ and hence $G_qG_p^*$ is a proper subgroup of G of order q^3p . Applying Lemma 2.1, $G_qG_p^*$ is a nonabelian P -group which contradicts with K satisfies structure (b). Thus, H_qK is a proper subgroup of G with $[G : H_qK] = p^e$ ($e = 1, 2$). If $[G : H_qK] = p^2$, then $G_q = H_qK$. By Lemma 2.1, K is either cyclic, elementary abelian or nonabelian of order q^3 and hence each maximal subgroup of K is normal in K . By hypothesis, $HK^* \leq G$ for each maximal subgroup K^* of K with $p^2q^2 \mid |HK^*|$. Choose K^* such that HK^* be a proper subgroup of G . Applying Lemma 2.1, HK^* can't be a GS -group, a contradiction. Thus, $[G : H_qK] = p$ and by Lemma 2.1, H_qK is a nonabelian P -group which implies that $H_q \triangleleft G$. By ([15], Theorem 9.3.7, p. 225), H_q has a complement U , say. Applying Lemma 2.1, U satisfies structure (b) which contradicts with H_qK is a nonabelian P -group.

Case (ii). $H_p \triangleleft H$ and $H_q \ntriangleleft H$. By hypothesis $H_p K \leq G$. Assume first that, $G = H_p K$.

Let $|G_q| = q$. Then, $|G_p| \geq p^4$ as $n > 4$. By Lemma 2.1, K is a nonabelian P -group or $|K| = p^2 q$. If K is a nonabelian P -group, then by Lemma 3.3, G is a nonabelian P -group and we are done. So, we can assume that, $|K| = p^2 q$ with $H_p \cap K_p = 1$ and both of H and K are not nonabelian P -groups. Since $H_p \triangleleft H$ and $H_p \triangleleft G_p$, we get $H_p \triangleleft G$. If K has a subgroup K^* , say, of order pq , then $H_p K^*$ is a proper subgroup of G which contains H . Applying Lemma 2.1, $H_p K^*$ is a nonabelian P -group, a contradiction. Thus, G_q operates irreducibly on H_p and K_p and we are done.

Now, let $|G_q| = q^2$. Since $G = H_p K$, we have $pq^2 \mid |K|$. By Lemma 2.1, either $|K| = pq^2$ or K satisfies structure (b). Assume that $K = K_p K_q$ satisfies structure (b). If $K_p \triangleleft K$, then by hypothesis, HK_p is a proper subgroup of G . Applying Lemma 2.1, HK_p is a nonabelian P -group which contradicts with $\Phi(K_q)$ operates irreducibly on K_p . If $K_q \triangleleft K$, then K_q is elementary abelian. By hypothesis, HK_q is a proper subgroup of G and by Lemma 2.1, $H_p K_q$ satisfies structure (b). But $H_p \triangleleft H$ which follows that K_q is cyclic, a contradiction. Thus, $|K| = pq^2$. It follows that $H \cap K = H_q$ and hence $[G : H] = pq$. Solvability of G implies that, H is not maximal subgroup of G . Then, G has a maximal subgroup G^* , say, contains H . Applying Lemma 2.1, G^* is a nonabelian P -group or satisfies structure (b). If G^* is a nonabelian P -group, then by Lemma 3.3, G is a nonabelian P -group, a contradiction. Thus, $G^* = H_p K_q$ satisfies structure (b) and hence $H_p \triangleleft G$ and K_q is cyclic. By ([15], Theorem 13.3.1, p. 383), K is supersolvable and consequently has a maximal subgroup K^* of order pq . By Lemma 2.1, $H_p K^*$ is a nonabelian P -group which contradicts with G^* satisfies structure (b).

Finally, let $|G_q| \geq q^3$. If $G_q \not\leq K$, then by Lemma 2.1, K is a nonabelian P -group and by Lemma 3.3, G is a nonabelian P -group, a contradiction. Thus, $G_q = K$. Choose K^* be a maximal subgroup of K which contains H_q . By hypothesis, HK^* is a proper subgroup of G . Applying Lemma 2.1, $HK^* = H_p G_q^*$ satisfies structure (b). Hence G_q^* is cyclic of order q^2 and consequently K is either cyclic or nonabelian with $|K| = q^3$. Clearly if K is cyclic, then our result holds and we are done. So, let K be nonabelian. Since G is solvable, it has a minimal normal subgroup L , say. Let $L \leq G_q$. If $L = G_q$, then LH_p^* is a proper subgroup of but not a GS -group, a contradiction. Also we get a contradiction if $L < G_q$ as HL is a proper subgroup of but not a GS -group. Thus, $L \leq H_p$. If $L < H_p$, then HK^* has a subgroup of order pq , a contradiction with structure (b). Therefore, $L = H_p$ and hence all maximal subgroups of K are cyclic. By ([12], Satz 8.2, p. 310), $K \cong Q_8$. But by ([18], Corollary 1.10, p. 6), $H_p \Phi(G_q^*)$ is supersolvable which contradicts with $\Phi(G_q^*)$ operates irreducibly on H_p .

Now, assume that, $H_p K$ is a proper subgroup of G . If $H_p K = G_p$, then $|G_q| = q$ and $|G_p| \geq p^4$ as $n > 4$. Applying Lemma 2.1, G_p is either cyclic or elementary abelian. If G_p is cyclic, then by ([15], Theorem 13.3.1, p. 383), G is

supersolvable and by Corollary 2.1, G is a nonabelian P -group, a contradiction. Thus, G_p is elementary abelian. Clearly, K has a maximal subgroup K^* such that HK^* is a proper subgroup of G with $|HK^*| = p^j q$ ($j \geq 3$). Applying Lemma 2.1, HK^* is a nonabelian P -group and hence H is a nonabelian P -group. By Lemma 3.3, G is a nonabelian P -group and we are done. So, let $|\pi(H_p K)| = 2$ and hence $|G_q| \geq q^2$. By Lemma 2.1, $H_p K$ is either a nonabelian P -group or satisfies structure (b). Since $p^2 q^2 \mid |G|$, G can't be a nonabelian P -group and hence if $H_p K$ is a nonabelian P -group, then we get a contradiction with Lemma 3.3. Thus, $H_p K$ satisfies structure (b). Since $H_p K$ is a proper subgroup of G with $[G : H_p K] = q$, we have $|G_q| = q^3$. Let $H_p \triangleleft H_p K$. Then, $H_p \triangleleft G$ and hence $H_p G_q^*$ is a proper subgroup of G for each maximal subgroup G_q^* of G_q . Applying Lemma 2.1, $H_p G_q^*$ satisfies structure (b). Therefore, all maximal subgroups of G_q are cyclic. Similar as we show in the last paragraph of the proof of Lemma 3.4, $G_q \cong Q_8$. This implies that $H_p K_q^*$ is a supersolvable group which contradicts with K_q^* operates irreducibly on H_p . Now, let $K_q \triangleleft H_p K$. Then, $K_q \triangleleft G$. Since H_p is cyclic, we get H is supersolvable and hence has a subgroup H^* of order pq . By applying Lemma 2.1, $K_q H^*$ is a nonabelian P -group which contradicts with $H_p K$ satisfies structure (b). $\square$

Lemma 3.6. *Assume that $G = HK$ is a mutually N -permutable product of its proper subgroups H and K with $|\pi(G)| = 2$. Suppose further that all maximal subgroups of G are GS -groups with $n \geq 4$. Then, one of the following holds:*

- (i) $|G| = p^2 q^2$ or $p^3 q$, where p and q are distinct primes.
- (ii) G is a nonabelian P -group.
- (iii) $G = G_p G_q$ where G_p is an elementary abelian normal Sylow p -subgroup of order p^2 , G_q is cyclic of order q^3 and every subgroup of G_q operates irreducibly on G_p .
- (iv) $G = G_p G_q$ where $G_p = P_1 P_2$ is an elementary abelian normal Sylow p -subgroup of order p^4 such that P_i ($i = 1, 2$) is a minimal normal subgroup of G of order p^2 , G_q is a Sylow q -subgroup of order q and G_q operates irreducibly on P_i ($i = 1, 2$).

Proof. If both H and K are nilpotent, then G is a mutually m -permutable product of H and K and by Lemma 2.2, we are done. So, let H be not nilpotent. Applying Lemma 2.1, either H is a nonabelian P -group, H satisfies structure (b) or $|H| = p^2 q$. Clearly, if $n = 4$, then either $|G| = p^2 q^2$ or $p^3 q$ and (i) holds. So, let $n > 4$ and hence by the previous three Lemmas, we are done. $\square$

Proof of the main theorem

It is a direct result from Lemma 3.1, Lemma 3.2, Lemma 2.3 and Lemma 3.6.

Conclusion

One of the most important objectives in group theory is to explore the structure of groups through certain properties of their subgroups. This paper deepens that concept by linking the properties of smooth chains and mutually N -permutable subgroups, aiming to determine the structure of a group. To extend this work, we can replace our hypothesis with weaker ones. For instance, we hope to determine the structure of a group G that is mutually N -permutable of two generalized smooth subgroups.

References

- [1] M.H. Abd-Ellatif, *Finite groups with some generalized smooth maximal subgroups*, Sao Paulo J. Math. Sci., 18 (2024), 149-158.
- [2] M.H. Abd-Ellatif, *Notes on influence of certain permutable subgroups on finite smooth groups*, Int. J. Group Theory, 14 (2025), 253-262.
- [3] M.H. Abd-Ellatif, A.M. Elkholy, *On mutually m -permutable product of GS -groups*, Asian-Eur. J. Math., 14 (2021).
- [4] M. Asaad, *On the solvability, supersolvability and nilpotency of finite groups*, Annales Univ. Sci., Budapest, Sectio., 16 (1973), 115-124.
- [5] A. Ballester-Bolinchés, J. Cossey, M.C. Pedraza-Aguilera, *On the products of finite supersolvable groups*, Comm. Algebra, 29 (2001), 3145-3152.
- [6] A. Carocca, *p -supersolvability of factorized finite groups*, Hokkaido Math. J., 21 (1992), 395-403.
- [7] A.M. Elkholy, M.H. Abd-Ellatif, *Finite groups with certain S -permutable and GS -maximal subgroups*, Algebra Colloq., 27 (2020), 661-668.
- [8] A.M. Elkholy, M.H. Abd-Ellatif, S.H. El-sherif, *Influence of S -permutable GS -subgroups on finite groups*, C. R. Acad. Bulg. Sci., 72 (2019), 853-860.
- [9] A.M. Elkholy, A. Heliel, *Influence of certain permutable subgroups on finite smooth groups*, Acta Math. Sinica, 27 (2011), 1547-1556.
- [10] A.M. Elkholy, *On generalized smooth groups*, Forum Math., 18 (2006), 99-105.
- [11] D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.
- [12] B. Huppert, *Endliche gruppen I*, Springer-Verlag, Berlin, 1967.
- [13] O. Kegel, H. Luneberg, *Über die kleine reidemeister bedingungen*, Arch. Math.(Basel), 14 (1963), 7-10.

- [14] Robert M. Guralnick, *Subgroups of prime power index in a simple group*, J. of Algebra, 81 (1983), 304-311.
- [15] W.R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.
- [16] R. Schmidt, *Subgroup lattices of groups*, Walter de Gruyter, Berlin-New York, 1994.
- [17] Lifang Wang, Yanming Wang, *On mutually sm-permutable products of finite groups*, Int. J. Algebra, 29 (2011), 1413-1419.
- [18] M. Weinstein (editor), *Between nilpotent and solvable*, Polygonal Publishing House, Passaic, 1982.

Accepted: December 11, 2024