GeoGebra applets and homework tasks to help engineering students with 3D visualization in a multivariable calculus course

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Abstract. In this article, we will examine, for 82 first-year Management Engineering students at Udine University, the difficulties of graphically representing regions in space, surface boundaries of these regions, and curve boundaries of these surfaces, and suggest GeoGebra Applets to overcome them. This included the design, and the analysis of the strategy used, and difficulties encountered in three unfamiliar tasks given as homework, two at the beginning and one during the course, and a non-anonymous Moodle questionnaire. The homework tasks were followed by the design and the use in class of ad-hoc GeoGebra Applets, left available to the students for personal use. The activities will be presented by specifying the aspects indicated in the DiGIMATH good practice guidelines. The potential and limitations of GeoGebra for these subjects will also be illustrated. Since the proposed object of study has not been studied a lot, our research is qualitative and exploratory, based on Duval's theory of registers of semiotic representation and the Visual-analytical model. Thematic analysis will be used to analyse the responses to the questionnaire.

Keywords: Register of semiotic representation theory, Visual-analytical model, Spatial visualization, Multivariable calculus, GeoGebra, Homework, Questionnaire.

MSC 2020: 97G40, 97I50, 97I60, 97D70, 97D40, 97U70.

1. Introduction

Representation of mathematical concepts is very important in the teaching and learning of mathematics. The ability to represent a mathematical situation in different forms is a very powerful tool in mathematics ([12]). It helps students to understand mathematical concepts, internalise the concepts, and establish connections between the concepts leading to conceptual understanding and creativity. Spatial skills have a vital significance in the acquisition of mathematical knowledge, particularly in the realms of geometry and calculus ([2]). Proficiency in spatial visualization, encompassing the manipulation of figures in three-dimensional space, recognition of patterns, projections, are essential attributes sought in promoting mathematical learning. Research shows that throughout development, there is a consistent, predictive, and strengthening relationship between spatial reasoning and mathematical achievement ([26]).

Particularly, for a comprehensive understanding of the fundamental concepts of calculus in multiple variables, students need spatial visualization abilities that enable them to grasp the relationships between surface transformations in space and the corresponding equations and inequalities ([23]).

Unfortunately Engineering students' incoming knowledge of analytical geometry in three-dimensional space for our experience is very scarce and is often limited to calculating the volumes of solids. During the Linear Algebra course, they begin to see planes and straight lines in space at the end of the course but, due to the lack of time, quadrics are rarely treated.

The purpose of this article (continuing a work started in [20]), is twofold: to investigate difficulties and strategies used by students in drawing and manipulating three-dimensional regions and in identifying their boundary surfaces and boundary curves of these surfaces; to design GeoGebra Applets to help the students to overcome these difficulties. They will be presented by specifying the "good practice" aspects indicated in the Italian Mathematical Union DiGIMATH group guidelines. The group, involving about 50 members from 25 Italian universities and two institutions, develops research projects aimed at a careful integration of technology in teaching and learning at university level (www.digimath.it). For the first purpose of this contribution, we will design and use three unfamiliar homework Tasks, two administered at the beginning and the other during the course, for the second the design of four GeoGebra Applets based on four exercises, linked to the previous three Tasks and presented during the class lessons, have been designed and tested. After the use of the Applets a questionnaire was also proposed to the students asking about the difficulties encountered and strategies used in solving the Tasks, and on the impressions of the use of the GeoGebra Applets. The research questions are the following: 1. What are the students' initial difficulties and strategies used in the representation and manipulation of three-dimensional regions involving quadrics? 2. What graphic perception do they have of the boundary surfaces of a region and boundary curves of a surface? 3. In what way is it possible to obtain improvements thanks to ad hoc GeoGebra Applets?

To design the Tasks and the Applets and to analyse the student productions Duval's theory of register of semiotic representation (TRSR) and the Visual-analytical model (VA) will be considered. Thematic analysis ([5]) will be used to analyse student response to the questionnaires. The potential and weaknesses of using GeoGebra will also be highlighted together with how to take advantage of or overcome them. In this article only the part concerning the drawing of boundary surfaces of a region and boundary curves of a surface, in preparation for Gauss and Stokes' Theorems, will be investigated. The study involved 82 Management Engineering students from the University of Udine, Italy, 56 of them responded to the questionnaire.

2. Literature review

Despite its great relevance for university mathematics courses, mathematics education research on students' understanding and difficulties regarding multivariable calculus and, particularly, vector calculus is still in its early stages. Some past studies have examined students' generalized knowledge of multivariate functions ([18]; [21]; [33]), domain and range ([7]), and derivative ([32]). Some research has touched on student understanding of multivariate integration ([17]; [1]), or on pedagogy related to multivariate integration ([22]).

Research of student understanding in 3D analytic geometry or linear algebra ([8]; [13]; [19]) has shown that, students tend to represent a line in the space by a single implicit linear equation instead of two, or an equation of a circular cylinder with no $z-$coordinate is interpreted as an equation of a circle. This difficulty can also be observed when analysed in the context of the intersection of surfaces. Manipulation of equations (representing surfaces) that yields a new equation is seen as a solution and it is misinterpreted as a curve of intersection instead of a new surface, or as the projection of that curve to the $xy-$plane if the equation has no $z-$coordinate ([4]). Henriques ([16]) in the multivariable integral context considered the volume computations of complex solids and researched students' geometrical and analytical strategies and the role of using Maple in overcoming students' difficulties. He found that students need more support in deriving boundaries of integration and relating them to geometrical representations. He suggested that students should work on activities to parametrize surfaces to produce screen images of volumes and relate these parametrizations to obtain limits of integration. Habre's study ([15]), examines students' difficulties associated with visualizing such surfaces and students' proficiency in using visual imagery when necessary.

The existence of free programs with versatile capabilities and interactive representations helps to improve the presentation of contents, allowing dynamic visualizations. Teaching software packages, such as GeoGebra, Maple, and Mathematica, support multiple representations that can simultaneously demonstrate a function in numeric, algebraic, and graphic models. This feature helps students to understand abstract concepts; therefore, this type of software is often applied in the teaching of calculus. Nobre et al. ([25]), Takaci et. al. ([29]), and Tatar and Zengin ([30]) explored students' learning outcomes in classes that applied GeoGebra to the teaching of calculus. GeoGebra is a software that allows to create different interactive applications that can be used as teaching tools in math classes.

Homework is one of students' opportunities to learn mathematics. Findings from research indicate that undergraduate calculus students spend more time doing homework than they do in class ([11]; [6]). Mathematicians view homework as a critical component of learning ([6]). Moreover, students report that homework and practice exercises are important for their learning, more than other forms of instruction such as lecture. However, undergraduate mathematics education researchers have only recently begun to study student learning from homework.

3. Theoretical frameworks

Depending on the field of study, visualization has been defined in diverse and multiple ways. A comprehensive and all-embracing definition, that draws together the various aspects of visualization, and serves our purpose in mathematics education, is by Arcavi ([2], p.217):

> *"Visualization is the ability, the process and the product of creation, interpretation, use of, and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings".*

In mathematics education, to "visualize" means to construct, create, or make connections between an external mathematical object or its representation (a diagram, a table, or a picture) and a mental or internal construct or image and apply analytical methods to develop and advance understanding. Interaction with the mental image can be through physical models, manipulatives, sketches, computer-based static outputs or animations such as simulations.

Comprehension in mathematics, according to Duval, requires the coordination of different registers of representation. For this coordination to be achieved, it is necessary to perform transformations of representations ([9]). Treating a concept within a certain representation (representational mode or register) and successfully converting between different registers of representation, is considered as a prerequisite for conceptual understanding.

Many students' difficulties can be described and explained by the lack of coordination of different registers of representation ([10]). Research provides examples ([3]; [24]) of students' difficulties in using the algebraic representation of lines and planes in space, and vice-versa, in recognizing straight lines and planes in space from their equations, and in converting between parametric and implicit viewpoints.

In Duval's analysis, understanding and learning mathematics require the comparison of similar semiotic representations. According to him, there are two different types of transformations of semiotic representations: treatments, which are transformations of representations that happen within the same representation register, and conversions which consist of changes of representation register without changing the object being denoted. He argues that these transformations are the source of many difficulties in learning mathematics, and that taking them into account helps in overcoming those difficulties: to compare similar representations and treatments within the same register in order to discriminate relevant values of the mathematical object so that the features that are mathematically relevant and cognitively significant are noticed, and to convert a representation from one register to another to dissociate the represented object and the content of the particular representation introduced so that the register does not remain compartmentalized.

The second theoretical framework used is the visualization-analysis (VA) model proposed by Zazkis' et al. ([34]) to account for flexibility and links between acts of visualization and analysis or analytical thinking in mathematical performance. The model views visual and analytical reasoning as complementing each other in the solution of mathematical problems. The model proposes a series of switches between repeated acts of visualization V1, V2, V3, interspersed with acts of analysis A1, A2, A3 ... following a spiral of steps, each act of analysis leads to a better and richer visual representation followed by more sophisticated analyses.

4. Method

To answer the first two research questions on the difficulties and the strategies used in the representation and manipulation of a three-dimensional region, recognise boundary surfaces and curves, three unfamiliar tasks given as homework, two at the beginning and one during the course were designed and tested together with the answers to a non-anonymous questionnaire inserted as a quiz on the Moodle platform. For all three tasks (Task1, Task2 and Task3), their expected solution in terms of treatment and conversion is discussed in session 4.2 and in 5.1 all the students' solutions were analysed and the strategies used were catalogued; the most representative ones were chosen as examples. To answer the third and final question, four GeoGebra Applets were proposed during class lessons, in relation to four exercises (EX1, EX2, EX3 and EX4) connected with the three previous tasks. After that Task3 was proposed again with the request to draw it again and to write the parametric equations of the surfaces and their intersection curve involved. The Applets will be discussed in 5.2 and in 5.3 the improvements will be examined. We start to summarise the tasks:

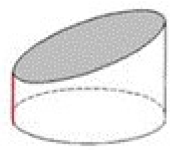
Task1. Draw the set

$$A = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}.$$

Moreover, represent graphically what is obtained, respectively, if $x = 0$ or $y = 0$ and $z = 0$ in the respective Cartesian planes.

This task concerns the representation, expressed in the form of inequalities, of a simple quarter of a cylinder with the $z-$axis cut with a plane parallel to the $y-$axis. For this cylinder we ask two thinks: to draw it and to identify and draw the sections on the fundamental Cartesian planes. The choice of the cylinder was made for its simplicity and the inclined plane to have non-trivial sections.

Task2. Imagine cutting a cylinder, obtained by rolling up a sheet of paper, with an oblique plane, as in the figure below. Suppose now cutting the shortest side dotted in red and unrolling it. Draw the figure that you obtain. Explain how you get it. Do you recognize a function you know on the top of the unrolled figure? If so which one?



Task2 requires, starting from the figure, this time of an entire cylinder, to recognise the function of the boundary curve of the unrolled figure. Task1 and Task2 would make the students reflect on the boundary surfaces of a three-dimensional region and the boundary curves of surfaces, topics that they would address later during the third part of the course which included scalar and vector integral calculus in multiple variables. After having introduced curves, with cartesian and parametric equations, and double integrals with the Gauss-Green formula, which relates this concepts, it was the turn of surface integrals with Stokes' Theorem (which relates integrals of surfaces with the curvilinear integrals of the boundary of the surface). Before this a review of the fundamental conic and quadrics (spheres, cylindroids, cones, paraboloids, hyperboloids, etc.) was given and visualised with the help of GeoGebra. The definition of surface has been given in implicit Cartesian form but above all in parametric form, showing notable examples: graph of two variable functions, surfaces of rotation such as the cylinder, the cone, the sphere and the torus. After that we wanted to test the students' ability to recognise graphically the boundary surfaces of a three-dimensional region and the boundary curve of each surface involved, proposing various tasks. For this article, we chose to analyse the following:

Task3. After drawing the solid between the surfaces $x^2 + y^2 - z = 0$ and $x + y - z + 1 = 0$, graphically identify the boundary surfaces. For each surface found, graphically identify the boundary curves.

This task concerns the graphic representation of the region between a paraboloid and an inclined plane both expressed, this time, in the form of an equality. In this case the two quadrics were known to the students so, that they could concentrate exclusively on the graphic perception of the boundary surfaces of the set and the boundary curves of the two surfaces which was required to be explained.

In summary the exercises and the applets were the following:

EX1. Given the surface $z = 1 - x^2 - y^2$ of parametric equation

$$\begin{cases} x = u \\ y = v \\ z = 1 - u^2 - v^2 \end{cases} \qquad \text{with } (u, v) \in T.$$

What happens for different choices of T?

This is not a real exercise and has the aim of exploring with GeoGebra the effects of the variation of the set T of parameters u and v with the use of Applet1 (https://www.geogebra.org/m/ebcrctxn).

EX2. Draw the set $D = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}$ and calculate the total external area.

The set D is the same of Task1, before moving on to the calculation of the area of the external surface of the region, all the surfaces involved were identified graphically using Applet2 (https://www.geogebra.org/m/nfszf9bk).

EX3. Given Σ the cylindrical surface $x^2 + y^2 = 1$ between $z = 0$ and $z = 1 + x$ calculate $\iint_\Sigma z dS$.

In addition to the calculation of the surface integral on a region like the roll of Task2, attention was then placed with Applet3
(https://www.geogebra.org/m/wqx2axhc) on the boundary curve of the surface, anticipating the use of Stokes' Theorem.

EX4. For

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 | y^2 + z^2 = 4, -1 \leq x \leq z + 1\}$$

oriented outward and the vector field $F = (x^2, z^2, y^2)$ verify Stokes' Theorem.

For the graphic representation of this exercise, the GeoGebra intersection command was used with Applet4 (https://www.geogebra.org/m/yve3ttgy).

The students' work on of the three tasks was examined together with the answers to the following three questions of a non-anonymous questionnaire inserted as a quiz on the Moodle platform:

Question1. Task1 required drawing a three-dimensional set and finding the sections in the fundamental planes $x = 0$, $y = 0$ and $z = 0$. Justify your solution, specify whether you used previous knowledge and, if so, where or in what situation you encountered them or, alternatively, what reasoning you used. If you did not do the exercise or part of it, explain why. In the lesson of 22/05, the area of the frontier surface of the set was calculated for the same set. Tell us your impressions of using the GeoGebra Applet https://www.geogebra.org/m/nfszf9bk in class on this exercise, if necessary, review it again.

Question2. Task2 required you to imagine what the result of unrolling a cylinder cut by an oblique plane was and whether a function was recognized on the upper part of the resulting plane figure. Explain how you arrived at the indicated result. If you did not do the exercise, explain why. In the lesson of 22/05, the analytical solution to the exercise was given, describe its effect. Tell us your impressions of using the GeoGebra Applet https://www.geogebra.org/m/wqx2axhc in class on this exercise. If necessary, review it.

Question3. Task3 required you to draw the three-dimensional region between a paraboloid and a plane. Tell us what difficulties you had in drawing it and in identifying the boundary surfaces of the set and the boundary curve of the surfaces.

For the questionnaires, thematic analysis was used to identify all the answers in reference to the three main requests of the questions, namely: the strategies, the difficulties and finally their perception of the use of GeoGebra and the Applets in particular. The answers to the first two requests will be analysed within the task results while the last one regarding GeoGebra and Applets will be analysed at the end of their description. Also, in this case, our interest was the treatments and conversions.

4.1 The research context

The research for this contribution was carried out during Mathematical Analysis 2 course in the academic year 23/24 and involved 82 Management Engineering students at the University of Udine, Italy, 56 responded to the questionnaire. Mathematical Analysis 2 is a second course of calculus (9 credits corresponding to 72 hours of lessons) given in the second semester of the first year. It is preceded, among others, by Mathematical Analysis 1 (12 credits), Linear Algebra (6 credits) and it is simultaneous with Physics 1 (9 credits). At the beginning of Mathematical Analysis 2 course, after having introduced the space $\mathbb{R}^3$ as a linear space with scalar (and Fourier series) and vector product, and as a normed space (open, closed sets...), an introduction was made on the various functions treated in the course: scalar-valued and vector-valued functions (curves, surfaces, fields). The initial focus on these functions concerned how to represent them graphically (level curves and surfaces, support for curves and surfaces). The role of the "derivative" is also presented: tangent plane, tangent line, normal vector, curl, and divergence respectively. After this first introduction, the course topics were divided into three macro-sectors, in the order: first and second order differential equations, systems of linear differential equations with simple stability studies; differential calculus in several variables with free and constrained extremes calculations, vector function and implicit functions theorems; the scalar and vector integral calculus with the Gauss-Green formula, Stokes and Divergence theorems. The students' preparation is constantly monitored through weekly homework tasks, students must photograph their solutions and send them to us via email. They are mandatory for participation to three partial tests on the three macro-topics described previously. The best performances of each task were selected and made available to all students as correction.

4.2 Solution of the tasks and expected reasoning

In Task1 the set

$$A = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}$$

is the internal part of a cylinder with the $z-$axis cut by a plane of equation $z - x = 1$ parallel to the $y-$axis. The choice of the cylinder was made both by thinking about the students' previous knowledge, everyone recognizes the cartesian equation in the plane of the circumference with centre the origin and radius 1, and by the simplicity of its sections in the various fundamental planes, obtainable by imagining the figure or by making simple projections. The only element of difficulty should have been the presence of the inclined plane $z-x = 1$. To answer the first request of the task, after the transposition into the three-dimensional space, the students need to convert from the analytic register to the geometric one but also to use their imagination to find the best perspective so that all the elements of the figure are recognizable. To see what happens if $x = 0$, $y = 0$ or $z = 0$, VA could be useful: they could go back to the analytic representation for the three cases, and again to the geometric representation to draw the projections in the planes xy, xz and yz. For example, if $x = 0$, they get in the $yz-$plane that $y, z \geq 0$, $y^2 \leq 1$, $z \leq 1$, i.e. a square in the graphical register. Finally, to see if the set represented was correct, it is expected that the students could reassemble the projections to reobtain the three-dimensional solid passing again to the geometric register in 3D. For the students this is a complex task, treatments and conversions are needed in VA sequences, as well as spatial orientation, rotation and translation of geometric figures using mental visualization to find the best view. As suggested by DiGIMAT "good practice", we not only want to identify the obstacles encountered by students in carrying out the task but also to see the strategies used to solve it. These latter will be the basis for the design of the Applet2.

The initially perceived figure of Task2 reveals a circular base, an elliptical top, and a curved surface that make the sides of the truncated cylinder. The solution, as seen in Figure 1, is a sinusoid shape.

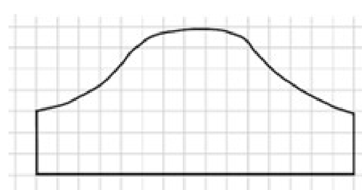


Figure 1: Student solution of Task2

Three possible solutions for the top were expected: a triangle, a semicircle and a sinusoid ([27] p.5). The intersection looks elliptical. One way to find the solution is, as suggested in the task test, to unfold the truncated cylinder and check its graph, but this has obvious limitations in accuracy. How do we show (analytically) that the third is the correct answer? In Stylianou ([28], p. 314) a mathematics educator proposed the following solution: Suppose that the cylinder has the equation $x^2 + y^2 = 1$ and the plane has the equation $z - x = 1$, passing to the polar coordinates of the base $(\cos(\theta), \sin(\theta))$, the height z has the trigonometric expression $z = 1 + \cos(\theta)$, i.e. a sinusoid. In this case, the previous analytical solution was not expected from the students, and it will be provided later in the course with GeoGebra Applet3. For this task, the treatment going from the 3D to the 2D representation and a conversion to the analytical/graphic recognition of the boundary will be analysed.

Task3 regards the graphical representation of the three-dimensional region delimited by the paraboloid of equation $z = x^2 + y^2$ and the oblique plane $x + y - z + 1 = 0$, both surfaces known to the students. In addition to choosing the Cartesian axes appropriately for a good graphic representation, the expected difficulty was identifying the intersection curve that delimited both the surface given by the plane and that given by the paraboloid. Task3 thus required sequential VA steps that could also be thought of as the achievement of 4 levels. Level1: the two surfaces are recognised and drawn separately. Level2: the two surfaces are drawn on the same graph and highlighted separately. Level3: there is an attempt to identify the boundary curve, but it is not correct. Level4: the intersection curve is correctly identified and consequently also the two surfaces. The latter case is more easily recognisable if a conversion from the geometric to the analytical register was done: by working simultaneously with both equations, the circumference of equation $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{3}{2}$ is obtained in the $xy-$plane, and then switching back to the geometric register. Reaching Level4, at least Level3, is desirable to obtain the correct parametric equations of the two surfaces and that of the boundary curve (but also for a possible triple integral calculation if the chord method is chosen). Also, for this task we will present how the students solved it, in particular which level was achieved, together with the strategies used and/or the difficulties encountered.

5. Results

The first paragraph begins with the analysis of students' solutions of Task1, Task2 and Task3 along with the strategies used. The most representative ones have been chosen as examples with the answers to the questionnaire. In the second paragraph, four GeoGebra Applets have been designed in relation to four exercises (EX1, EX2, EX3 and EX4) and linked to the three previous tasks. The answer to the questionnaire on their use will also be analyzed. Finally, in the last paragraph, the results of student work improvements after the use of GeoGebra Applets will be presented.

5.1 Tasks analysis

5.1.1 Task1

Before going into the detail of the analysis of the students' paper and pencil work on Task1, it is necessary to make two premises, both regarding previous results on the same students investigated. The first is about the students' perceptions of their initial knowledge, to the question: "Before the Analysis 2 course, what was your knowledge of three-dimensional space and where did it come from?", 23% wrote that it was absent and 35% basic, 25% wrote that it came from the Linear Algebra course and 12% from high school, 1% from technical drawing lessons and 5% did not answer. Therefore, more than half of the students declare to have very little knowledge and only 1/4 that such knowledge comes from the previous Linear Algebra course. The second regards the results, which are consistent with the research found in the literature, about planes and cylindroids: even 22% of the students were unable to recognize and draw $x = 1$ as a plane in the three-dimensional cartesian space but drew a straight line or even a point (this in four cases of 82), 16% did the same thing for the plane $x + y = 1$ and 24% with the cylindroid $y = -x^2 + 1$. Only 17% managed to do all these three tasks correctly. Therefore, as to Task1, it is not surprising that less than half of them, precisely 7%, managed to correctly draw the whole set and the various sections and 38% tried to draw a part of them, the remaining 62% did not perform the exercise. As we see in Table 1, of these 38% just 24% completed the first part of the task correctly. 12% at least recognized that $x^2 + y^2 = 1$ was a cylinder, but 6% drew a sphere and 3% a paraboloid, even 50% drew instead a circumference cut by a straight line in space. The remaining 5% did not draw or describe it in words.

Correct	**Circum. line**	**Cylinder**	**Sphere**	**Paraboloid**	**No draw**
24%	50%	12%	6%	3%	5%

Table 1: Results of first part of Task1

Finally, 26% drew all three sections correctly and 12% forgot the positivity conditions. These results confirm the initial difficulty of students to represent three-dimensional sets in space, partly due to the lack of previous knowledge but also, as we will see from the analysis of some students' work, of coordination between 2D and 3D semiotic registers.

Examining the students' works, two strategies were identified for Task1: starting from drawing the cylinder and the plane and from these identifying either analytically or geometrically which were the sections with the fundamental planes, in Figure 2 there are two examples; or starting first from the sections and then build the three-dimensional figure from those as in Figure 3.

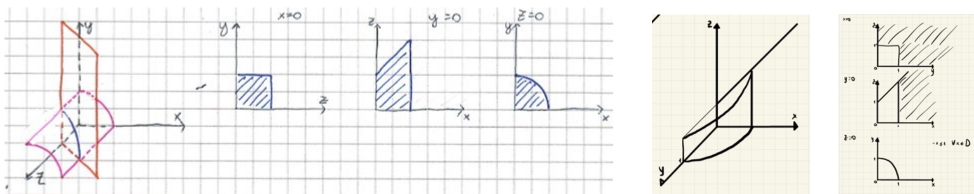


Figure 2: Same strategies but different choice of the axis 2a on the left and 2b on the right

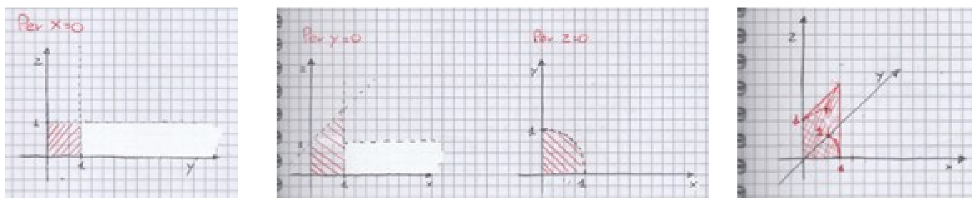


Figure 3: Second strategy

The confirmation of this procedure comes from the answers to Question1 of the questionnaire.

2a. *"I set first the constraint $x, y, z \geq 0$ to be able to simplify the work space and have to manage only the space of positive x,y,z instead of working with the whole space, after which I evaluated the other inequalities as equations by drawing then the figures obtained (plane and cylinder) evaluating where they intersected and then colouring the intersection in blue and reporting the projection on the zy, xz and xy planes"*

2b. *"The procedure was sequential: first of all, having x, y and z greater than or equal to 0 implies that we only work in one of the eight parts of the space, so, I only represented that; then I imagined the second condition as a cylinder, given that if we reasoned in two dimensions it would represent a circle; finally, I depicted the third condition, given that, if we reasoned in two dimensions, this would be a straight line, like a plane that cuts the previously depicted cylinder".*

We can first notice how both students followed the order indicated in the analytic description of the set. In either case the sections are already visible from their very good three-dimensional drawing. There were students' VA transitions from geometric to analytical and again to geometric register in order to have a precise drawing of the sections as we expected. In fact, for example, the writing "$-x \leq 1$ $\forall x \in D$" of 2b in Figure 2, where D is the disk, presupposes that the analytical calculation was done. The difference between the two drawings is the choice of the position of the three cartesian axes which is fundamental, in fact the choice of representation of 2a was more effective. Moreover, those who chose the xyz sequence for the axes, as they are used to do from high school, obtained an unclear drawing, as can be seen in Figure 3.

Moreover, the student in Figure 3 used the second strategy as can be seen from his work but also from what he wrote in the questionnaire: *"I started drawing the sections: For $x = 0$: Through the first constraint I know that $y \geq 0$ and $z \geq 0$, so I have excluded all quadrants except the first, then from the second constraint I know that $x^2 + y^2 \leq 1$ and that y is a quantity greater than zero, so, the maximum value that y can reach is 1. I did the same reasoning for the third constraint, and I get $z \leq 1$. [...] I drew the respective section on each plane and then I connected them"*. In this answer the student also described all the steps he took to obtain the sections that are those expected.

For drawing the sections, there were two cases in which despite the request that, for example $x = 0$, a three-dimensional set was drawn which is, in this case, a parallelepiped as in the case of Figure 4 (indicated with a red row).

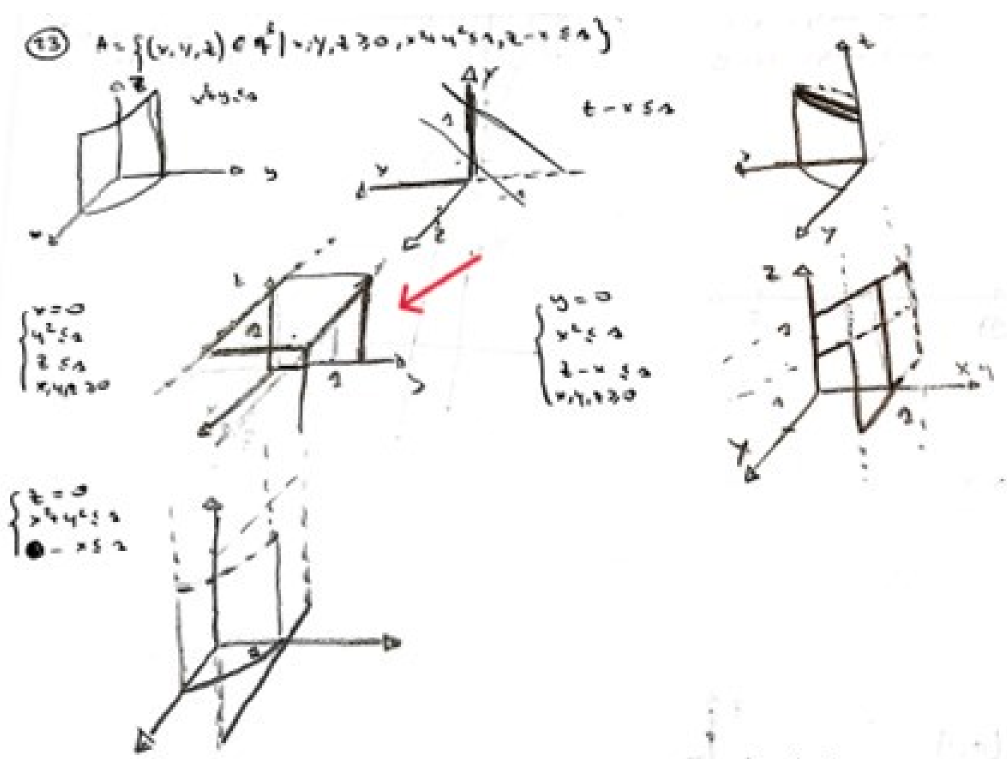


Figure 4: Three dimensional sections

As an explanation of the used method the student wrote in the questionnaire: *"First, I drew the two figures separately and then I joined them by placing them* *in the cartesian space to see the various intersections. I found it easier to draw them in steps so, as not to get confused once I joined them all together. The correction of the exercise helped me to understand that I had made mistakes in drawing the figure and thanks to it I thought better about the exercises proposed in the future"* The student noticed the error but did not explain the reason.

We can deduce that the transition from plane to space and vice versa is not so, immediate for students. In these two cases, they continued to think in 3D even when it was explicitly written to make drawings in Cartesian planes. Similarly, the transition from 2D to 3D is not immediate, in fact, the most frequent error was to draw the circle instead of the cylinder and the straight line instead of the plane even though cylindroids and planes were recognized and drawn correctly in the exercises of the initial premise to Task1. In Figure 5 there is an example.

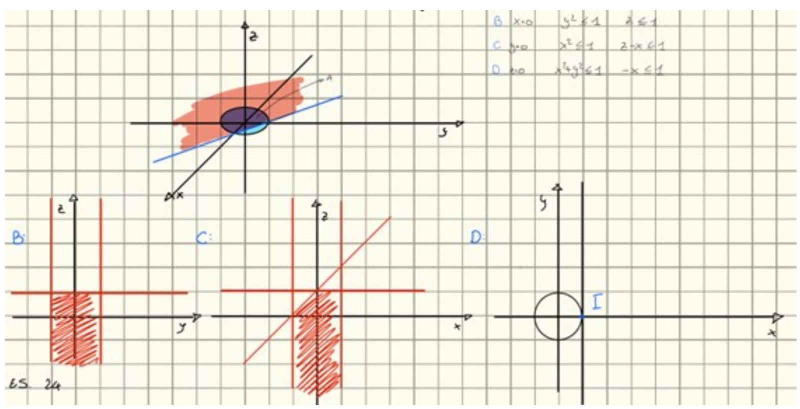


Figure 5: Erroneous cylinder

This student wrote in the questionnaire: *"To complete the exercise I started by creating three Cartesian planes, the first $y - z$ cancelling the variable x, the second $x - z$ cancelling the variable y and finally the third $x - y$ cancelling the variable z. By cancelling the variables in turn in the starting equations, three graphs of three surfaces were obtained, in order a plane between 3 lines, a plane between 4 lines and finally a circumference. After having drawn the three graphs I transferred the results on a three-dimensional space; looking at the correction given to me days later I noticed that my mistake was to have transferred the circumference as in the section and not as a cylinder"*. In this case the student, as he reported, used the second strategy but not having recognized that he was dealing with a cut cylinder he was unable to obtain the correct set. It is noted in several cases, as in this one, that the request for positivity especially in the sections is often neglected or forgotten. This error was not even highlighted by the student, most likely because he considered it irrelevant.

In two cases a sphere, instead of a cylinder, was drawn thought as the generalization in space of the circle, even though in one case the sections were reported correctly. Furthermore, in the sections, at least in a couple of cases, the presence of x^2 in $x^2 \leq 1$ made one think of the parabola.

Those who did not execute Task1 justified themselves by saying that at the time of the test they did not have the necessary knowledge to answer or that they had never encountered similar requests during high school. In fact, they wrote for example in Question1 of the questionnaire: *"I had a lot of difficulty especially in imagining the required solid, since I have never had to deal with similar exercises, nor with Cartesian space"*, *"I didn't do this exercise because I just couldn't figure out how to graphically represent multiple figures in the same space. Having done little or nothing about three-dimensional spaces in high school, penalized me a lot in this homework"*. Task1 was a challenge for the students given their limited prior knowledge of quadrics but also of simple planes and Cartesian space in general. These difficulties, as we will see, were taken into consideration when the Applet2 were designed.

5.1.2 Task2

It was not difficult for the student to recognise that the plane cut the cylinder in an ellipse (just few said circumference). Five possible solutions for the net of the unrolled cylinder were identified in the students' homework like those expected and reported in the literature: the correct one of a "Sinusoid", the two similar ones of a "Parabola" and "Circumference", a "Cusp", and a pentagon with a point at the top, that we called the "Tip".

Sinusoid	**Circumference**	**Parabola**	**Cusp**	**Tip**
16%	8%	9%	17%	36%

Table 2: Results of Task 2

As we see in Table 2, just 16% correctly draw the unrolled cylinder and recognize that there is a sinusoid at the top, 17% a parabola or a circumference, 17% think that it is a cusp (naming the hyperbola) and even 36% draw a tip with straight sides and name the absolute value as a function on the top.

In Figure 6. below we have reported three representative examples of students work with their respective useful reasoning and observations. The choice was also made with regard to the explanations reported directly on the paper that highlight on the one hand the strategies used and on the other the reasoning to individualize the function.

In fact, they wrote:

a. *"I tried to hypothesize the shape of the figure in my mind. I drew it and cut out a sheet of paper with that shape. I put the paper in the shape of*

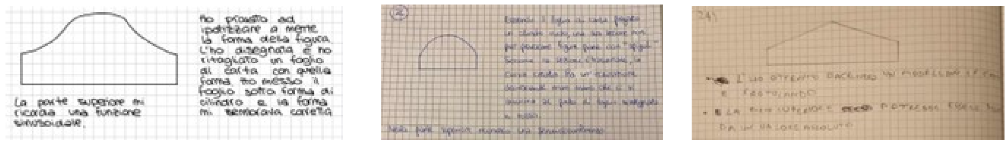


Figure 6: a. Sinusoid, b. Circumference, c. Tip

a cylinder and the shape seemed correct to me. The top reminds me a sinusoid".

She used one of the most frequent strategies, as we will see shortly, namely the empirical test followed by a subsequent verification of the correctness of the result.

b. *"Since the folded sheet of paper is an empty cylinder, its section cannot produce flat shapes with "edges". Since the section is transversal, the curve has a decreasing inclination as it approaches the cutting point dotted in red. In the upper part I recognize a semicircle".*

Despite the error, the student recognized that the shape could not be flat but must increase and decrease as one approaches the cut. He arrived at the analytical solution even if in a primitive form, which we used to design Applet3.

c. *"I got this by making a roll of paper and unrolling it. The top could be defined by an absolute value".*

6c wrote in Question2 of the questionnaire: *"I got it by making a paper model and unrolling it. The top could be an absolute value. I rolled up a sheet of paper and cut with scissors, but in doing so, I crushed the cylinder, the result was wrong. The correction surprised me somewhat, but I understood the error almost immediately".* The figure offered in Task2 may have been misleading, especially for those who have poor skills in three-dimensional representations. In fact, once it is observed that the required figure must be symmetrical with respect to the highest part of the cut cylinder, one can respond by looking at the front part and obtaining a cusp (or hyperbola) or the rear part obtaining a semicircle (or parabola), or perpendicularly obtaining a segment. In any case, the students somehow had to analytically rework the result obtained and talk about functions and conic sections. 58% have at least verbally explained which function they recognized on the upper part (sinusoid or Gaussian, parabola or circumference or ellipse, hyperbola, absolute value). Moreover, Task2 warned the students that the situation was not so trivial and that a hasty evaluation not supported by empirical evidence, which must also contain a counterevidence, or by an analytical solution could lead to making a mistake. In fact, a student wrote in the questionnaire: *"This was the most stimulating exercise because even if the question seems easy, it is necessary to pay attention. To solve it I* *tried to imagine what the figure would be like in the case of an elliptical cylinder more or less "squashed" compared to the normal cylinder; I also verified the result obtained experimentally by building a rudimentary cylinder and actually imagining the progress of the experiment".*

Let's now analyse the students' answers to the second part of Question2 of the questionnaire about the procedures used and the difficulties encountered in solving Task2. As to the procedures, two were identified: the empirical one reproducing the object (32%) and the other using imagination (34%). Moreover, 9% first tried to imagine it and then reconstruct it to see if it was right, while 13% did not give a real explanation but only reported that it was difficult or reported the correct result, 12% did not do the exercise. The most successful procedure was the empirical one of those who arrived at the correct answer, 57% had reconstructed the object and 29% used both their imagination and reconstruction, none of them got there just by imagining it. This result is interesting because it suggests that a future use of physical 3D models, or inviting students to build them, could be an effective strategy. It is interesting that 14% tried to provide a reasoned explanation of how the figure could be. Not much has been written about the difficulties, we just report one: *"In this case I had some difficulty even after seeing the solution"*. In most of the answers, the students referred to the use of GeoGebra and Apple3 as we will see later.

5.1.3 Task3

For Task3 the results of the reached levels 1, 2, 3 and 4 are summarized in Table 3. As we see, just 7% gave no response, nobody confused the paraboloid

No Answer	**Wrong**	**Level1**	**Level2**	**Level3**	**Level4**
7%	0%	9%	31%	34%	19%

Table 3: Results of Task 3

with another quadric, just 19% reached "Level4". But we couldn't expect a better result as we haven't yet got to the heart of the topic. The fact that 34% have already achieved Level3 is a good result: it will not be difficult for these students to take a step forward and reach Level4. Three representative examples of student (paper and pencil) homework of Level2, 3 and 4, with the answers to Question3 of the questionnaire, are the following:

a. *"The greatest difficulty was identifying the lines in which the two three-dimensional figures intersect, since their overlapping in the graph on the paper is not clear. Tools like GeoGebra are essential to truly understand the three-dimensional structure of interlocking solids since it is not easy to recognize them on the two-dimensional sheet".*

As she also stated, she was only able to represent the two quadrics, reaching Level2, but cannot put them together and identify the boundary curve. To

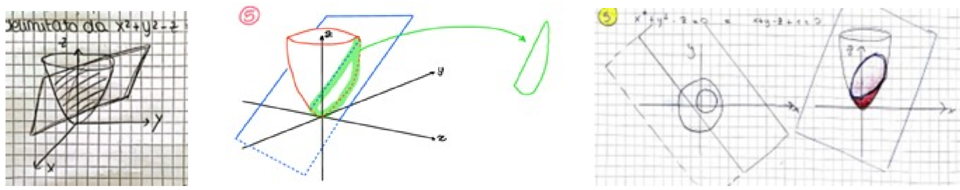


Figure 7: a. Level2, b. Level3, c. Level4

find out how the two superficies intersect it is necessary to find the intersection curve. In case 7b Level 3 is reached, and the student wrote:

b. *"The difficulties I encountered were how to draw the boundary curves of the involved surfaces"*

In this case the two surfaces, the plane and the paraboloid, have been identified and were also represented in two different colours, red and blue, but they are unbounded. Furthermore, the boundary curve of the surfaces is perceived as the contour of the graphic representation, the green one, and not as the intersection curve between the superficies. The same error was found in 9% of the students. Only 4% put the two surfaces in a system to identify the intersection curve, which however was not explicitly requested in the exercise. Instead, in case c. the student was able to identify both the two surfaces and the boundary curve and wrote:

c. *In this exercise I did not find any particular difficulties also because I relied on the formulas of the paraboloid present in the slides and therefore I managed more or less to understand and mentally imagine the figure.*

When we asked to explain better how he had obtained the graph, he wrote: *"To get to the result, first of all I recognized the basic equation of the plane* $ax+by+cz+d=0$ *and the basic equation of the paraboloid, which is* $x^2+y^2=z$*. Subsequently I tried to mentally imagine a three-dimensional plane so, that I could position the two figures. As for the paraboloid, I had no problems, while for the plane I used GeoGebra only to identify its position in space. Subsequently I set up the drawing and highlighted the intersections both for the top view and for the front view. As for the three-dimensional graphic representations of the boundary surfaces of a set and boundary curves of surfaces in general, let's say that I had an initial difficulty in conceptually understanding what I was going to do. In Task1 I had no idea what I had to do".*

The student used a sequential procedure to solve the task and GeoGebra to figure out how the plane was tilted, thus determining the choice of positioning the Cartesian axes. Furthermore, the answer confirms that the two concepts of the boundary surface of a set and even more that of the boundary curve of a surface are not immediately perceptible, even though intuition and the literal meaning of the used terms should help. Moreover, as the student wrote, the two surfaces must fit together to give rise to the curve. These observations will be taken into account to build GeoGebra Applet4.

Regarding the answers to Question3 on the difficulties encountered in Task3: for 11% they were in identifying the intersection between the paraboloid and the plane, for 20% in highlighting the surfaces and boundary curves, 9% said they did not know what they had to highlight, 27% gave generic expositions. The remaining 34% answered that they had no difficulty: *"I had no difficulty because I used the sections to better understand what the resulting figure could be. I had more difficulty identifying the surfaces and boundary curves, since I based myself on the fact that the surfaces were the areas, and the curves were the boundary lines"*. Another wrote: *"I had no difficulty in carrying out this exercise nor any particular impressions on the correction"*. We can also add that 6% used multiple views like in Figure 7c. As for the difficulties reported in the questionnaire, the main ones were those of drawing the inclined plane and finding where the two surfaces meet. Some students (19%) helped themselves with the graphic representation of some sections.

5.2 Design of interactive GeoGebra Applets

To accompany the students in resolving the difficulties encountered in solving the Tasks and therefore to facilitate the analytical step of identifying the most appropriate parametric equations of surfaces and curves, four GeoGebra Applets were designed and used in class during the lessons as starting point for subsequent integral calculus. In this article we are interested only in the graphical representations of the three-dimensional regions involved in the exercises and therefore the Applets concern only these. Let's start examining the Applets in detail, strategies for overcoming some of GeoGebra's limitations in representing surfaces will also be presented.

5.2.1 Applet1

The first exercise was an investigative one to show the students, during the lesson, the effect of changing the set of parameters in the parametric surface equation. These will help students achieve a "Level4" vision of the mathematical concept "surface". In fact, as we have seen from the students' solutions to Task3, it was not clear that the plane and paraboloid surfaces cannot be unbounded and that one surface limits the other after the intersection between the two is considered.

Ex 1. Given the surface $z = 1 - x^2 - y^2$ of parametric equation

$$\begin{cases} x = u \\ y = v \\ z = 1 - u^2 - v^2 \end{cases} \qquad \text{with } (u, v) \in T.$$

What happens for different choices of T?

We showed the students Applet1 (https://www.geogebra.org/m/ebcrctxn, Figure 8): using different colours and starting with the graph of the function $c(x, y) = 1 - u^2 - v^2$ in GeoGebra command, the entire paraboloid was obtained (brown), reconnecting to what had already been studied in the second part of the course. We remembered that the GeoGebra command "Surface" in general needs that the two parameters vary in a rectangle. Then with the surface command, for T with black border being the square $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$, we obtained, as we see in Figure 8a, the violet part of the previous paraboloid, but this time it was bounded and contained four tips given by the limitations of the parameters.

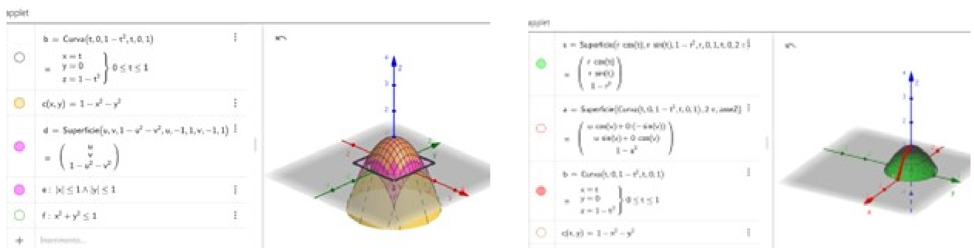


Figure 8: a. On the left the entire paraboloid and with T a square. b. On the right the surface using polar coordinate and solid of revolution

If, however, we are only interested in the part of the paraboloid with $z \geq 0$, we can modify the parametric equation of the surface by resorting, for example, to the change of variables into polar coordinates (Figure 8.b green), in this way we transform the circle $x^2 + y^2 \leq 1$ into the rectangle $0 \leq t \leq \frac{\pi}{2}$ and $0 \leq r \leq 1$ and the parametric equation becomes

$$\begin{cases} x = r\cos(t) \\ y = r\sin(t) \\ z = 1 - r^2 \end{cases} \qquad \text{with } 0 \leq t \leq \frac{\pi}{2} \text{ and } 0 \leq r \leq 1.$$

To the students we pointed out that, with this choice of x and y, z become $1 - r^2$. It was also shown that the same result was reached by GeoGebra using the rotate command with respect to the z axis of a parabolic arc. This applet requires VA: several registries change from algebraic to geometric and then back to algebraic due to the need to consider different parts of the paraboloid. It was also emphasized that in practical applications surfaces are bounded and therefore it is important to correctly identify the limiting set T of the parameters.

5.2.2 Applet2

After this first exploration we proposed the calculus of the area of the same set given in Task1 (EX2). Applet2's aim was just to facilitate the student's visualization of the various surfaces, to find the most appropriate parametric equations and to analyse what changes were needed for GeoGebra to work correctly.

EX2. Draw the set

$$D = \{(x, y, z) \in \mathbb{R}^3 | x, y, z \geq 0, x^2 + y^2 \leq 1, z - x \leq 1\}$$

and calculate the total external area.

This exercise involved identifying five surfaces. Three were easy to obtain, by putting $x = 0$, $y = 0$, $z = 0$ on the fundamental planes, we got a square, a trapezoid and a quarter of a circle, respectively. The remaining two were part of the cylinder and part of the oblique plane. The first problem to deal with these was using the GeoGebra Surface command which requires the two parameters to be defined in a rectangle as we said before. This problem has been solved in an easy way by exploiting the change to polar coordinates, as for Applet1, for the quarter circle to be transformed into a rectangle as follows:

$$\begin{cases} x = u \\ y = v \\ z = 0 \end{cases} \quad u, v \geq 0; x^2 + y^2 \leq 1;$$

became

$$\begin{cases} x = u\cos(v) \\ y = u\sin(v) \\ z = 0 \end{cases} \quad 0 \leq u \leq \frac{\pi}{2}; 0 \leq v \leq 1.$$

The other problem consisted in the limitations of one parameter being a function of the other, for example for the trapezium. In this case the problem was solved by rescaling the parameter v in the following way:

$$\begin{cases} x = u \\ y = 0 \\ z = v \end{cases}$$

$0 \leq u \leq 1$; $0 \leq v \leq 1 + u$; so,

$$\begin{cases} x = u \\ y = 0 \\ z = (1 + u)v \end{cases} \quad 0 \leq u \leq 1; 0 \leq v \leq 1.$$

With these two tricks, even used at the same time, we can obtain the following parametric representation of the portion of cylinder:

$$\begin{cases} x = \cos(u) \\ y = \sin(u) \\ z = v(1 + \cos(u)) \end{cases} \quad 0 \leq u \leq \frac{\pi}{2}; 0 \leq v \leq 1. \tag{1}$$

This exercise was also an occasion to resume a common error made by the students with the change into polar coordinates of wanting to transform every think into rectangle.

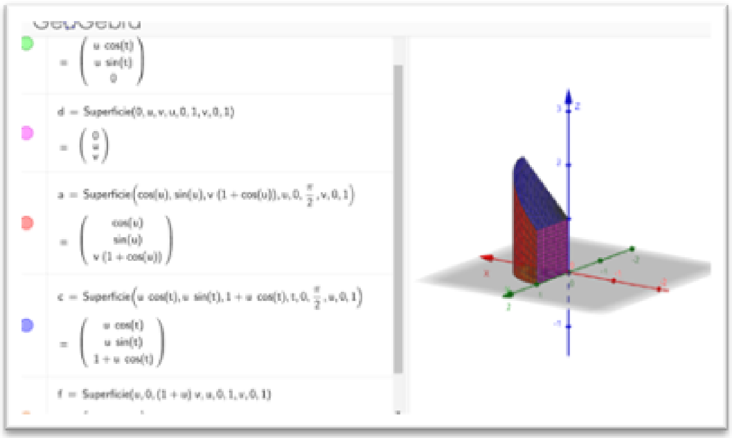


Figure 9: GeoGebra Applet2

Proposing EX2 and the connected Applet2 (https://www.geogebra.org/m/nfszf9bk, see Figure 9) had the advantage not only to give a graphic representation of what the boundary of the set was, which the students have already seen in Task1, but also to show the complex meaning of the parameters whose variation profoundly affects the surface appearance. Moreover, the possibility of using different colours for the various surfaces makes visualization easier. This Applet was designed to allow the students to think back to Task1 and correct their errors. By turning the figure all the single parts are shown, making it easier to recognise all the sections that are obtained by intersecting with the fundamental planes. Here too, VA with a sequential change of registry was necessary: analytical-geometric-analytic-geometric, from the analytical representation of the set through inequalities we moved on to the geometric representation of all the parts that make up the boundary, including the sections obtained by intersecting the fundamental planes, whose geometric representation again required the passage to the analytical register including the calculation of the parametric representation of all the various parts. We would still end up with the analytical and geometric register with the calculation of the surface integrals through double integrals.

5.2.3 Applet3

The next exercise (EX3) required calculating a surface integral on a cylinder cut by two planes as in the previous case. This time, however, it was simpler as there were no requests to stay on the first octant, which was proposed with the aim of strengthening concentration on only the cylindrical surface and its two boundary curves, thus anticipating Stokes' theorem. It also allowed the students to recall Task2.

EX3. Given Σ the cylindrical surface $x^2 + y^2 = 1$ between $z = 0$ and $z = 1 + x$ calculate

$$\iint_{\Sigma} z dS.$$

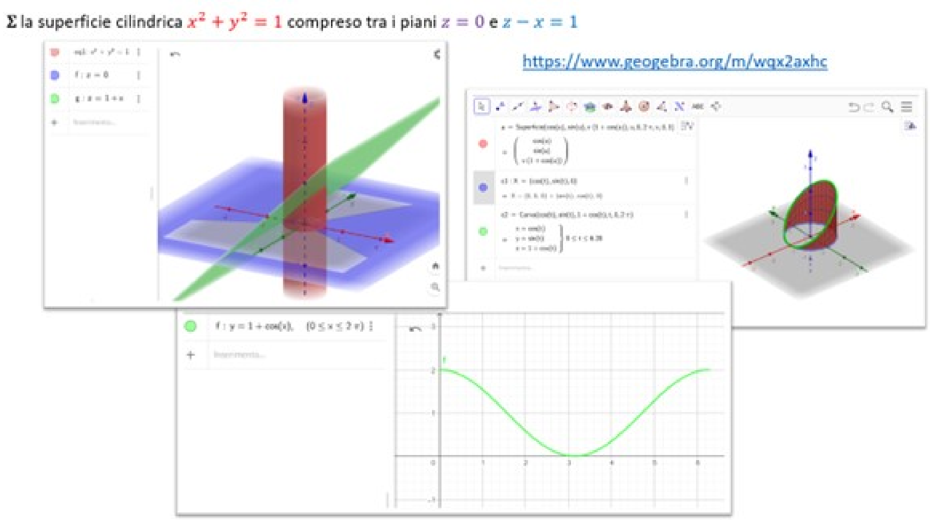


Figure 10: A slide of the lesson

In Figure 10 above, a slide of the lesson, there are the two ways to represent the set using GeoGebra by the overlap of the cartesian equations on the left and using the parametric equations of the surface and of the boundary curves on the right. At the bottom there is also the graphic in green of the unrolled cylinder cut vertically on the higher part of Task2. We took advantage of this example, to observe, using Applet3 (https://www.geogebra.org/m/wqx2axhc), what the border curve of the surfaces were and how their parametric equations resulted directly from the parametric equation (1) of the surface of EX2 (but now $0 \leq u \leq 2\pi$) just fixing the parameter u once 0 and once 1. This went into the direction of what the students have reported: it is difficult to see how two surfaces fit together.

In contrast to Task2, this time the cartesian equation of the cylinder $x^2+y^2 = 1$ is known just as that of the oblique plane $z = 1 + x$. Thus, the students were able to see firsthand that, using the parametric equation in polar form of the circumference in the xy plane, so $z = 1 + \cos(u)$ with $0 \leq u \leq 2\pi$, justifying that, on the upper part of the unrolled cylinder, a sinusoid is obtained and not a parabola or circumference or a cusp or an absolute value type function as many students have indicated. The analytical solution was highly appreciated by the students. Also, in this one, the possibility of using different colours, (method also used by some students in solving the tasks), for the various surfaces and choosing the same colour for a surface and its boundary curve made visualization easier. It could also be shown that the same curve was simultaneously the boundary of the paraboloid with the plane that delimited it, or the boundary of the plane with the paraboloid that delimited it, the students did not recognize this when they completed Task3. It was also possible from the parametric equations of the boundary curves to reproduce the parametric equation of the surface. Even in this case, to explain the topic, it was necessary to make several changes of register like in the previous case.

5.2.4 Applet4

The last exercise (EX4) was about the verification of Stokes Theorem.

EX4. For $\Sigma = \{(x, y, z) \in \mathbb{R}^3 | y^2 + z^2 = 4, -1 \leq x \leq z + 1\}$ oriented outward and the vector field $F = (x^2, z^2, y^2)$ verify Stokes' Theorem.

Also, in this case, we had a cylinder cut by two planes but this time the strategy for the corresponding adopted Applet4 (https://www.geogebra.org/m/yve3ttgy) was different. In fact, it became more difficult to represent the surface in parametric form with GeoGebra, so, we used the "intersection" command which showed the two boundary curves graphically. The advantage was that GeoGebra directly gave the parametric equations of the curves included the choice of the directions, which is fundamental for applying Stokes' theorem. The use of sliders permitted to see how the points move on the curves, making everything more dynamic (see Figure 11). We would like to point out that the intersection command of GeoGebra is limited only to specific cases such as the intersection of a quadric with a plane and to some intersections between quadrics.

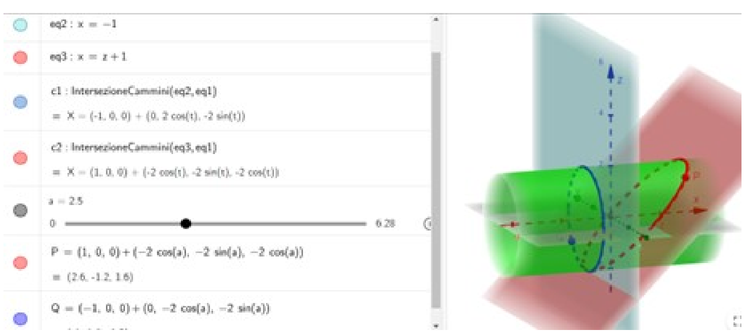


Figure 11: Cylinder cut by two planes

Also, in this case it was necessary to pass from the analytic register to the geometric one with the graph and again move to the algebraic register to find the intersection curves and check with GeoGebra that the two results coincide.

5.3 Effects of Applets use on the students

After the interventions and with the support of the GeoGebra applets, Task3 was represented to the students asking them to find the respective parametric equations. Of the 48 students participating in the revised Task3, as we see in Table 4, none was at Level1, just 4% at Level2 and as many as 71% at Level4, while before that was 19%.

Level1	**Level2**	**Level3**	**Level4**
0%	4%	25%	71%

Table 4: Results of Revised Task3

Analysing the improvements, as we see in Table 5, we can add that no one remained at level2, 15% went from Level2 to Level4 and 25% from Level3 to Level4 (missing 2% who went from Level1 to Level2 and another 2% who went from not having solved the exercise to Level2).

Level3/Level3	**Level4/Level4**	**Level2/Level4**	**Level3/Level4**
23%	31%	15%	25%

Table 5: Results of improvements of revised Task3

In Figure 12 there is a representative example of the procedure carried out, respectively, before and after the intervention.

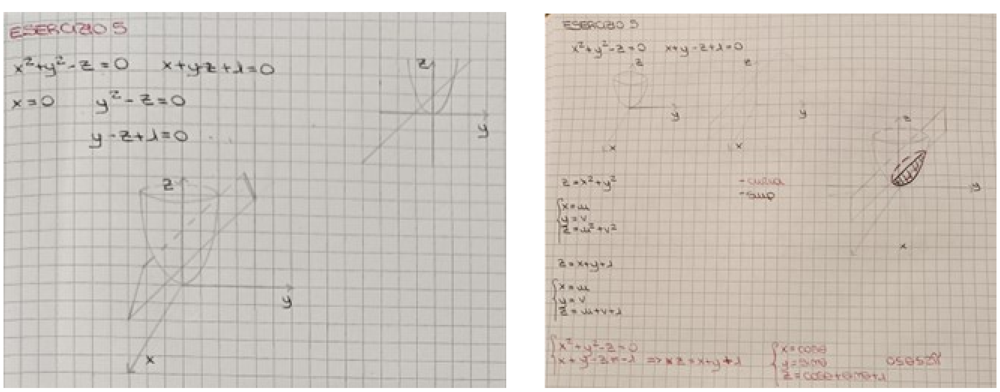


Figure 12: Task3 first (left) and after (right) the intervention

As we can see this student passed from Level2 to Level4 for the graphic representation. The evolution was lighted by the student's comment on this Task in Question3 of the questionnaire: *"I didn't find any difficulty in drawing the three-dimensional set as I understood the figure. As regards the second request, I didn't really understand what I should have highlighted, however, after the week's explanations I understood how I should have done it"*. The execution of Task3 is not yet complete due to the lack of the set in which the parameters vary for the parametric equations of the surfaces, but it is justified by the fact that the lessons on solving this problem with the use of the Applet took place only a few days before the delivery of the revised Task3. Of the 48 students who returned the test, 54% wrote the parametric equation of the paraboloid but of these only 19% explicitly wrote the set T of variation of the parameters, 90% wrote the parametric equation of the plane but again only 23% of them made T explicit and finally 42% found the analytical representation of the intersection curve of which 25% wrote its parametric equation, 50% found the cartesian equation and finally wrote the corresponding system.

Of these 48 students answering Question1 of the questionnaire, 46% explicitly underlined the importance of GeoGebra and of these 50% referred to Applet2 used in class. Instead, to Question2 of the questionnaire 34% expressed opinions on the use of GeoGebra and of these 74% made explicit reference to Applet3. In all cases they expressed that their use was appreciated. It is interesting to note that of the latter 45% explicitly declared their interest in the analytical solution of the problem contained in Applet3. They wrote for Question2 of the questionnaire: *"To obtain the "unrolled" section of the cylinder I imagined this process in my mind. I tried to sketch out what I thought was the result and then rewind it and verify that the initial figure was formed again. The analytical confirmation is comforting, it is not necessary to be able to carry out a process of imagination, sometimes not obvious, to arrive at the solution, but the calculations help us to confirm or not our "initial theory". The use of the applet certainly provides us with a clear and useful representation to confirm or not our initial hypotheses"*; *"In this exercise I must say that I had difficulty understanding how once cut perpendicularly as indicated in the text it could unroll, once I saw the correction I must say that it was still not clear to me how it was possible to obtain that figure. These doubts were clarified for me during the lesson of 05/22, as an analytical solution was clearer to me than perhaps a qualitative graphic solution"*; *"I had done that exercise incorrectly because I had not considered that a tip could not appear in the upper part of the graph. But as soon as I saw the correction I immediately understood my mistake and it helped me to see the 3D figures more clearly"*.

6. Discussion and conclusions

The results of our study were based on two sources of data: three paper and pencil homework tasks (Task1, Task2 and Task3) and the answers to a questionnaire. The findings confirm the initial difficulties of the students in drawing sets, expressed through inequalities, in the Cartesian space both for their previous knowledge and for the difficulty in manipulating three-dimensional objects. Furthermore, this study shows that the concepts of boundary surfaces of a set and boundary curves of a surface, indispensable in vector integral calculus, are difficult for students to understand. To overcome these problems, unusual Tasks and GeoGebra Applets were designed and tested.

Students have difficulties with vector calculus as it involves concepts and problems that require students to think in terms of three-dimensional space and to visualize objects such as curves, surfaces, and volumes. Our Engineering students' incoming knowledge of analytical geometry in three-dimensional space is very poor in fact, out of 82 students, 23% said that it was null and 31% basic. During the Linear Algebra course, they begin to see planes and straight lines in space but, due to lack of time, quadrics are rarely treated. In fact, only 25% of the surveyed students stated that their prior knowledge of three-dimensional Cartesian space was due to the previous Linear Algebra course. The results of Tasks1 and 2 confirm this. The first request of Task1 to draw the set in space was undoubtedly unfamiliar to the students and this perhaps also discouraged them from continuing with the exercise, but in order to draw the sections in the various planes it was sufficient to rewrite the set in the three situations $x = 0$, $y = 0$ and $z = 0$ and draw them in the Cartesian planes. When faced with a new problem, students often give up on tackling it, in fact 63% did not do the exercise because of their poor knowledge of three-dimensional representations, as they wrote in the questionnaire. Moreover, the transition from 2D to 3D is not immediate as we noted in the most common error of drawing a circle and a straight line instead of the cylinder and the plane. But in the same way the transition from 3D to 2D is not obvious as we could see in the three-dimensional representation of the sections. Task2 had, above all, the aim of anticipating the notion of the boundary of a surface and making students appreciate the power of the analytical solution, which removes all doubts about any false concept. However, it was possible to observe also in those who had operationally tried to reconstruct the object a lack of critical evaluation of the obtained result. As regards Task3, after having already had to deal with different types of quadrics in the second part of the course, the students had no problems in graphically representing the paraboloid and the plane. The difficulty, as they stated in the questionnaire, too, was to put them together identifying the intersection curve. These difficulties are probably and mainly due to the poor habit of students to see mathematical objects and problems from different perspectives, and to their tendency to focus their attention on univocal and mechanical methods of treatment. Only a few students immediately managed the analytical calculus of the intersection curve; this number considerably increased after the presentation of the GeoGebra Applets in class. The notion of boundary of a region and boundary of a surface are not as intuitive as one would expect, only 19% were able to correctly draw the boundary surfaces of the sets and the boundary curve of the surfaces in Task3. In 34% of the cases, the latter was confused with the contour of the representation on the sheet, which is necessarily flat (see Figure

7b). But after the use of the GeoGebra Applets, the 48 students who returned the same Task3, but with the request to write the parametric equations of the involved surfaces and curves, had improved from both the analytical and the graphic point of view. Even if there was still some difficulty in describing the set T of variability of the parameters of the two surfaces.

To conclude, we can say that for the teaching of scalar or vector multivariable integral calculus to be graphically effective, it is necessary to consider the students' difficulties in managing three-dimensional representations as could be noted in the results of the three Tasks. To do this, the use of dynamic 3D visualization tools such as GeoGebra could prove to be very effective. The use of GeoGebra facilitates not only the transition from the analytical to the graphic register but also the transition from two different analytical registers, in fact a surface could be expressed in Cartesian or parametric form and even the parametric one is not unique. It is difficult for students to identify a set where the parameters vary and, in the case of multiple intersecting surfaces, to understand where one ends and the other begins. Furthermore, the GeoGebra command intersection between two quadrics can provide them, in at least some cases, "for free", the parametric equation of the resulting curve ([31], [14]).

There are some limitations to the research: the students rarely included all the steps used in their homework and rarely commented on what they were doing even when explicitly asked; the answers to the questionnaires were generally very short. Moreover, too much time had passed between the beginning of the course and the treatment of vector integral calculus.

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