Codes from the incidence matrix of the essential graph

Ahmad Yousefian Darani*
Department of Mathematics and Applications
University of Mohaghegh Ardabili
P.O. Box 179, Ardabil
Iran
yousefian@uma.ac.ir

Fatemeh Soheilnia
Department of Mathematics
Faculty of Science
Imam Khomieini International University
P.O. Box: 34149-1-6818, Qazvin
Iran
f.soheilnia@edu.ikiu.ac.ir

Emel A. Ugurlu
Department of Mathematics
Marmara University
Istanbul
Turkey
emelakyugurlu@gmail.com

Abstract. Let R be a commutative ring and M be an R-module. The essential graph over M, denoted by $EG(M)$, is defined as a graph associated to M with vertex set $Z(M) \setminus \mathrm{Ann}_R(M)$, and a pair of distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_M(xy)$ is an essential submodule of M. In this paper, we investigate the linear codes with respect to the Hamming weight from incidence matrix of the essential graphs over M. If $\mathbb{Z}_n$ be the ring of integer module n, then $EG(\mathbb{Z}_n)$ is a linear code. Let p_1 and p_2 be distinct prime numbers. It is shown that if $n = p_1p_2$, then $C_2(EG(\mathbb{Z}_n)) = \big[(p_1-1)(p_2-1), p_1+p_2-2, \min\{p_1-1, p_2-1\}\big]_2$. Moreover if $n = p_1^{\alpha_1}p_2^{\alpha_2}$ with $\alpha_i \geq 1$ for $i = 1, 2$, then $C_2(EG(\mathbb{Z}_n)) = \big[|E|, |V|-1, \min\{p_1+1, p_2+1\}\big]_2$.

Keywords: linear code, incidence matrix, essential graph.
MSC 2020: 94B05, 05C50, 05C38.

1. Introduction

The study of graph theoretic methods and algebrice graph structures over linear codes has attracted a lot of attention in last decades and leads many authors to study and explore their properties, for instance see [1, 5, 6]. Initially, the concept of the zero-divisor graph was studied by I. Beck in [4]. Also, the concept of the zero-divisor graph was studied with a different set of vertices by Anderson et.al

∗. Corresponding author

in [2]. Let R be a commutative ring. The zero divisor graph of R, denoted by $\Gamma(R)$, is the graph with vertex set $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. The essential graph of R is a variation of the zero-divisor graph that changed the edge condition, which was studied and introduced in [8]. The essential graph of R is an undirected simple graph, denoted by $EG(R)$, with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_R(xy)$ is an essential ideal of R. Recently, the concept of the essential graph and its related topics on modules over commutative rings are studied by many authors (see [10, 11]).

First, we recall some basic properties of coding theory and graph theory which will be used in the sequel. Let $\Gamma(V, E)$ be a simple graph with the vertex set V or $V(\Gamma)$ and the edge set E or $E(\Gamma)$. The order of Γ is $|V|$ and the cardinality of the edge set denoted by $|E|$. A graph with no edge is called the null graph. For every $u, v \in V$, the distance between u and v is defined as the length of a shortest path from u to v and is denoted by $\mathrm{d}(u, v)$. We write $u \sim v$ if $\mathrm{d}(u, v) = 1$ and $u \not\sim v$ otherwise. The degree of a vertex u, denoted by $\deg(u)$, is the number of edges incident to u. Also, u is called an end-vertex if $\deg(u) = 1$. The minimum and maximum degree of the vertices of Γ are denoted by $\delta(\Gamma)$ and $\Delta(\Gamma)$, respectively. The graph is k-regular if all its vertices have degree k. Assume that u is a vertex of Γ. The open neighborhood of u is defined as $N(u) = \{v \in V(\Gamma) \ : \ \mathrm{d}(u, v) = 1\}$ and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$. The graph is connected if there is a path between any two distinct vertices. The diameter of Γ is $\mathrm{diam}(\Gamma) = \sup\{d(x, y)|x, y \in V(\Gamma)\}$, where $d(x, y)$ is the length of shortest path between x and y. The girth of Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle contained in the graph; otherwise $\mathrm{gr}(\Gamma) = \infty$. A complete graph is a graph in which each pair of vertices is connected by an edge and a complete graph with n vertices is denoted by K_n. A graph Γ is called bipartite if its vertex set can be partitioned into two parts V_1 and V_2 such that every edge has one end in V_1 and one in V_2. A complete bipartite graph is a bipartite graph $K_{m,n}$ which consist of an independent set of m vertices of V_1 completely joined to an independent set of n vertices V_2. A graph H is a subgraph of Γ whose vertex set and edge set are subsets of those of Γ. A dominating set of Γ is a subset D of $V(\Gamma)$ such that every vertex in $V(\Gamma) \setminus D$ is adjacent to some vertex in D. The domination number $\gamma(\Gamma)$ of Γ is the minimum cardinality of a dominating set.

The notation for designs and codes is as in [3]. An incidence structure $\mathfrak{D} = (\mathfrak{P}, \mathfrak{B}, \mathfrak{J})$, with point set $\mathfrak{P}$, block set $\mathfrak{B}$ and incidence $\mathfrak{J}$ is a $t - (\nu, k, \lambda)$ design, if $|\mathfrak{P}| = v$, every block $B \in \mathfrak{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The code $C_F(\mathfrak{D})$ of the design $\mathfrak{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. If $\mathfrak{Q}$ is any subset of $\mathfrak{P}$, then we will denote the incidence vector of $\mathfrak{Q}$ by $n^{\mathfrak{Q}}$, and if $\mathfrak{Q} = \{P\}$ where $P \in \mathfrak{P}$, then we will write ν^P instead of $\nu^{\{P\}}$. Thus, $C_F(\mathfrak{D}) = \langle \nu^B | B \in \mathfrak{B} \rangle$, and is a subspace of $F^{\mathfrak{P}}$, the full vector space of functions from $\mathfrak{P}$ to F. For any $w \in F^{\mathfrak{P}}$ and

$P \in \mathfrak{P}$, $\omega(P)$ or ω_P will denote the value of ω at P. If $F = \mathbb{F}_p$, then we write $C_p(\mathfrak{D})$ for $C_F(\mathfrak{D})$. We refer the reader to [1] for undefined terms and conditions.

An incidence matrix $|V| \times |E|$ of Γ is matrix $G = [g]_{ij}$ with $g_{ij} = 1$ if the vertex u_i is on the edge e_j and $g_{ij} = 0$, otherwise. Let W and X be non-empty subsets of V with $W \cap X = 0$ and let $E(W, X)$ be a subset of E which is included one end of W and other end of X. Set $|E(W, X)| = q(W, X)$. An edge-cut of a connected graph Γ, denoted by $\lambda_e(\Gamma)$, is the set S with the property that $\Gamma \setminus S = (V, E \setminus S)$ is disconnected. Also, the edge-connectivity of Γ, denoted by $\lambda(\Gamma)$, is the minimum cardinality of $\lambda_e(\Gamma)$. In the other words, $\lambda(\Gamma) = \min_{\emptyset \neq W \subseteq V} q(W, V \setminus W)$. Clearly, for any connected graph Γ, we have $\lambda(\Gamma) \leq \delta(\Gamma)$, where $\delta(\Gamma)$ is the minimum degree of Γ. The graph Γ is said to be a supper-λ, whenever $\lambda(\Gamma) = \delta(\Gamma)$ and the only edge sets of cardinality $\lambda(\Gamma)$ whose removal disconnected Γ are the set of edges incident with a vertex of degree $\delta(\Gamma)$.

Let $\mathbb{F}_q$ be a finite field with cardinality q and let $A = \{0, 1, \cdots, q-1\}$ be an alphabet of order q. The elements of A are called symbols. A q-ary code C of length n and size $|C|$ is a non empty subset of A^n, and any element of C is called a codeword. A q-ary linear code C of length n is a subspace of the vector space $\mathbb{F}_q^n$ over $\mathbb{F}_q$. Let x and y be words of length n over an alphabet A. The Hamming distance from x to y, denoted by $d_H(x, y)$, is defined to be the number of places at which x and y differ. If $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$, then $d_H(x, y) = d_H(x_1, y_1) + \cdots + d_H(x_n, y_n)$, where x_i and y_i are words of length 1, and

$$d_H(x_i, y_i) = \begin{cases} 1, & \text{if } x_i \neq y_i \\ 0, & \text{if } x_i = y_i \end{cases}.$$

Note that, $d_H(x, y) = d_H(y, x)$. The minimum Hamming distance of a code C, denoted by $d_H(C)$, is defined by $d_H(C) = \min_{c_1, c_2 \in C}\{d_H(c_1, c_2) | c_1 \neq c_2\}$. Let c be a codeword of $\mathbb{F}_q$. The Hamming weight $wt_H(c)$ is defined by

$$wt_H(c) = \begin{cases} 0, & \text{if } c = 0 \\ 1, & \text{if } c \neq 0 \end{cases}.$$

Let c be a word in $\mathbb{F}_q^n$. The Hamming weight of c, denoted by $wt_H(c)$ (or simply $wt(c)$), is defined to be the number of non zero coordinates in c, that is $wt(c) = d(c, 0)$, where 0 is the zero vector. If $c \in F_n^q$ with $c = (c_1, c_2, \cdots, c_n)$, then $wt(c) = \sum_{i=1}^n w(c_i)$. For any two elements $x, y \in \mathbb{F}_q^n$, we have $d_H(x, y) = wt(x - y)$. The minimum weight of a code C is the smallest among all weights of the non zero codewords of C. For every q-ary liner code C, we have $d_H(C) = wt(C)$. The support of a vector v, denoted by $\text{Supp}(v)$, is the set of coordinate positions where the entry in v is non-zero. Obviously $|\text{Supp}(v)| = |wt(v)|$. A generator matrix G for a linear code C is a matrix whose rows form a basis for C and $C_p(G)$ is a code generated by the matrix G over a finite field F_q and dimension of the code $C_q(G)$ is the rank of the matrix G over the field F_q. A parity-check matrix for C is a generator matrix for $C^{\perp}$ where $C^{\perp} = \{v \in \mathbb{F}_q^n | uv = 0, \forall u \in C\}$ is the dual code of C. Here, all the codes are linear codes and the notation $[n,k,d]_q$ will be used for a q-ary code C of length n, dimension k and minimum distance d over the field F_q. For more details we refer the reader to [7].

Let M be an R-module. The annihilator, the set of zero-divisors, the set of associated prime ideals and the minimal set of associated prime ideals of M are denoted by $Z_R(M)$, $\mathrm{Ann}_R(M)$, $\mathrm{Ass}_R(M)$ and $\mathrm{Min}\,\mathrm{Ann}_R(M)$, respectively. Let M be an R-module. Then it is proved that $Z_R(M) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}(M)} \mathfrak{p}$. The submodule N of M is called essential, if $N \cap K \neq \{0\}$ for every non-zero submodule K of M. The radical of an ideal I over commutative ring R, denoted by $r(I)$, is defined to be the set of all elements a of R with $a^n \in I$ for some positive integer n. Then $r(I) = \bigcap_{\substack{\mathfrak{p}\in Spec(R)\\ \mathfrak{p}\supseteq I}} \mathfrak{p}$, see [9].

Throughout this paper all rings are commutative, all modules are unitary and all codes assumed to be linear codes. Here is a brief summary of the paper. In Theorem 2.2, for Noetherian R-module M with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$, we show that $C_2(EG(M)) = \big[|E|, |Z(M) \setminus \mathrm{Ann}_R(M)| - 1, \lambda(EG(M))\big]_2$, if $|\mathrm{Min}\,\mathrm{Ass}(M)| = 2$. In Theorem 2.3, it is shown that $C_2(EG(\mathbb{Z}_n)) = \big[(p_1 - 1)(p_2 - 1), p_1 + p_2 - 2, \min\{p_1 - 1, p_2 - 1\}\big]_2$, whenever $n = p_1p_2$ where p_1 and p_2 are distinct prime numbers. Also, in Theorem 2.4, it is proved that if $n = p_1^{\alpha_1}p_2^{\alpha_2}$ is an integer, where p_1, p_2 are prime numbers, α_1, α_2 are positive integers and $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$, then $C_2(EG(\mathbb{Z}_n)) = \big[|E|, |V| - 1, \min\{p_1 + 1, p_2 + 1\}\big]_2$.

2. Main results

Let R be a commutative ring and let M be an R-module. The essential graph of M, denoted by $EG(M)$, is a graph with vertex set $Z(M) \setminus \mathrm{Ann}_R(M)$ and two distinct vertices $x, y \in Z(M) \setminus \mathrm{Ann}_R(M)$ are adjacent if and only if $\mathrm{Ann}_M(xy)$ is an essential submodule of M, see [10].

The following theorem plays an important role in this paper, so we recall it.

Theorem 2.1 ([5, Theorem 1]). *Let $\Gamma = (V, E)$ be a connected graph and G be a $|V| \times |E|$ incidence matrix for Γ. Then the following statements hold:*

(*i*) $C_2(G) = \big[|E|, |V| - 1, \lambda(\Gamma)\big]_2$.

(*ii*) *If Γ is super-λ, then $C_2(G) = \big[|E|, |V| - 1, \delta(\Gamma)\big]_2$, and the minimum words are the rows of G of weight $\delta(\Gamma)$.*

Theorem 2.2. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. If $|\mathrm{Min}\,\mathrm{Ass}(M)| = 2$, then*

$$C_2(EG(M)) = \big[|E|, |Z(M) \setminus \mathrm{Ann}_R(M)| - 1, \lambda(EG(M))\big]_2.$$

Proof. Assume that c is a non zero codeword in $C_2(EG(M))$ and Min Ass(M) $= \{\mathfrak{p}_1, \mathfrak{p}_2\}$ where $\mathfrak{p}_i = \mathrm{Ann}_R(m_i)$ with $m_i \in M$ for $i = 1, 2$. Then we conclude that

$c = \sum_{v \in Z(M) \setminus \mathrm{Ann}_R(M)} \mathfrak{p}_i EG_v$, where EG_v is the row of $EG(M)$ corresponding to the vertex v, by [5, Theorem 1]. Let $e = uv$ be an edge of $EG(M)$. Thus, $EG_{u,e}$ and $EG_{v,e}$ are incidence where EG_e is the column corresponding to the edge e. Set $c_e = \mathfrak{p}_1 + \mathfrak{p}_2$, where c_e is a summation of prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We consider that $\mathrm{Supp}(c) = \{e = uv | \mathfrak{p}_i \neq \mathfrak{p}_j, \text{ for all } \mathfrak{p}_i, \mathfrak{p}_j \in \mathrm{Min\,Ass}(M)\}$. Now, suppose that $EG(M)_c$ is a graph with the set of edges $E \setminus \mathrm{Supp}(c)$. Hence, if a, b are adjacent in $EG(M)_c$, then $\mathfrak{p}_1 + \mathfrak{p}_2 = 0$ and so $\mathfrak{p}_1 = \mathfrak{p}_2$ and $|\mathrm{Min\,Ass}(M)| = 1$. Therefore, $EG(M)$ is a null graph or it has only one vertex, by [10, Theorem 3.2]. Then $EG(M)_c$ is a disconnected graph. Since otherwise, if $EG(M)_c$ is a connected graph, then $c = \mathfrak{p}_1 \sum EG_v = \mathfrak{p}_1(0) = 0$, which is a contradiction. Therefore, $\mathrm{Supp}(c)$ is the set that contained edge connectivity and hence, $wt(c) = |\mathrm{Supp}(c)| \geq \lambda(EG(M))$. Now, we assume that $\mathrm{Min\,Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, where $\mathfrak{p}_i = \mathrm{Ann}_R(m_i)$ with $m \in M$, for $i = 1, 2$. Let $u, v \in \mathfrak{p}_1 \setminus \{0\}$. If u and v are adjacent, then $\mathrm{Ann}_R(uv)$ is essential and so $\mathrm{Ann}(uv) \cap Rm \neq 0$ for some $m \in M$. Then there exist $r \in R$ such that $0 \neq rm \in \mathrm{Ann}(uv)$. Thus, $ruv \in \mathfrak{p}_2$. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = r(\mathrm{Ann}_R(M)) = 0$. Then $u \in \mathfrak{p}_2$ or $v \in \mathfrak{p}_2$ and so $u = 0$ or $v = 0$, which are contradiction. By similar argument it is shown that the previous result is satisfied for p_1. Therefore, the vertices of the graph are divided two parts. Assume that W and $V \setminus W$ are two parts which included vertices of the graph with $W \cap (V \setminus W) = \emptyset$. Now, let $S \subseteq E$ where S is the set of edge connectivity and $\lambda(EG(M))$ is the minimum number. Assume that c is a non zero codeword of $\lambda(EG(M))$. Clearly, if $uv \in W$ and so $\mathrm{Ann}(uv)$ is essential, then $\mathrm{Ann}(uv)$ is not essential in S. Now, it is enough to show that the edges of S are edge connectivity between two parts of the graph. Let $u \in \mathfrak{p}_1 \setminus \{0\}$ and $v \in \mathfrak{p}_2 \setminus \{0\}$. Then $uv \in \mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 = 0$ and so $uvM = 0$, that is, each element of $u \in \mathfrak{p}_1 \setminus \{0\}$ with element of $v \in \mathfrak{p}_2 \setminus \{0\}$ are adjacent. So we assume that $c = \sum_{vG22 \in W} EG_v = \sum_{v \in V} \mu_v EG_v$, where if $v \in W$, then $\mu_v = \mathfrak{p}_1$ and if $v \in V \setminus W$, then $\mu_2 = \mathfrak{p}_2$. Thus, $uv \in \mathrm{Supp}(c)$ if and only if $\mathfrak{p}_1 \neq \mathfrak{p}_2$ if and only if $uv \in S$. Hence, $wt(c) = |Suup(c)| = |S| = \lambda(EG(M))$. Therefore, the minimum Hamming weight of incidence matrix over the essential graph equals to the minimum cardinality of the edge connectivity $\lambda(EG(M))$. □

Corollary 2.1. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) = \mathrm{Ann}_R(M)$. If $|\mathrm{Min\,Ass}(M)| = 2$ and $EG(M)$ is a super-λ, then*

$$C_2(EG(M)) = \big[|E|, |Z(M) \setminus \mathrm{Ann}_R(M)| - 1, \delta(EG(M))\big]_2.$$

Proof. It follows from Theorems 2.2 and [12, Theorem 2.4]. □

Corollary 2.2. *Let M be a Noetherian R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$ and let $EG(M)$ be k-regular graph. Then $C_2(EG(M))$ is a binary code with minimum weight k.*

Proof. Assume that M is a Noetherian R-module with $r(\mathrm{Ann}_R(M)) \neq \mathrm{Ann}_R(M)$. Then $diam(EG(M)) \leq 2$ and $gr(EG(M)) \in \{3, \infty\}$, by [10, Theorem 2.6]. Since

$EG(M)$ is k-regular, it is a complete graph. Moreover, G is an incidence matrix $n \times \frac{nk}{2}$. Thus, $C_2(EG(M))$ is a binary code with minimum weight k, by [5, Corollary 1]. □

Example 2.1. Consider the ring $\mathbb{Z}_8$. It is clear that $\mathrm{Ass}(\mathbb{Z}_8) = \{2\mathbb{Z}_8\}$ and $Z(\mathbb{Z}_8)^* = \mathrm{Nil}(\mathbb{Z}_8) = 2\mathbb{Z}_8$. Thus, $EG(\mathbb{Z}_8)$ is a complete graph with 3 vertices, see Figure 1. Moreover, the incidence matrix G is:

$$G = \begin{array}{c} \\ v_1{=}2 \\ v_2{=}4 \\ v_3{=}6 \end{array} \begin{array}{c} \begin{matrix} e_2 & e_3 & e_1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3} \end{array}$$

Hence the minimum weight is 2 and so $C_2(EG(\mathbb{Z}_8))$ is a binary code.

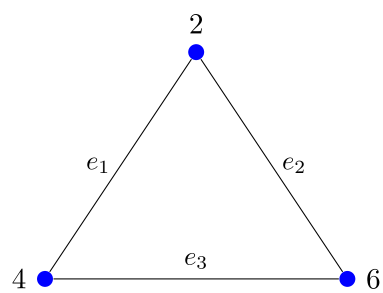

Figure 1: The essential graph $EG(\mathbb{Z}_8)$ with minimum weight 2.

Theorem 2.3. *Let $n = p_1p_2$, where p_1 and p_2 are distinct prime numbers. Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then $EG(\mathbb{Z}_n)$ is a linear code and*

$$C_2(EG(\mathbb{Z}_n)) = \big[(p_1 - 1)(p_2 - 1), p_1 + p_2 - 2, \min\{p_1 - 1, p_2 - 1\}\big]_2.$$

Proof. Assume that $n = p_1p_2$, where p_1 and p_2 are distinct prime numbers. Then $Z(\mathbb{Z}_n) = p_1\mathbb{Z}_n \cup p_2\mathbb{Z}_n$ and $\mathrm{Nil}(\mathbb{Z}_n) = \{0\}$. It is easy to see that $|Z(\mathbb{Z}_n)^*| = p_1 + p_2 - 1$. Then in view of [10, Theorem 3.7], $EG(\mathbb{Z}_n)$ is a bipartite complete graph. We divided the vertices with two parts:

$$V_1 = \{\mathrm{Ann}(a) |\, \mathrm{Ann}(a) = p\mathbb{Z}/q\mathbb{Z}, a = p_1k_i, p|q, (p_1, k_i) = 1, 1 \leq i \leq n\};$$
$$V_2 = \{\mathrm{Ann}(b) |\, \mathrm{Ann}(b) = p\mathbb{Z}/q\mathbb{Z}, a = p_2s_i, p|q, (p_2, s_i) = 1, 1 \leq i \leq n\}.$$

Now, it is enough to find an incidence matrix of $EG(\mathbb{Z}_n)$. Without loss of generality we may assume that $p_1 < p_2$. Then the incidence block matrix of $EG(\mathbb{Z}_n)$ is

$$G = \left[\begin{array}{c:c:c:c} I_{p_2-1} & I_{p_2-1} & \cdots & I_{p_2-1} \\ \hdashline 1 & 0 & \cdots & 0 \\ \hdashline 0 & 1 & \cdots & 0 \\ \hdashline \vdots & \vdots & \vdots & \vdots \\ \hdashline 0 & 0 & \cdots & 1 \end{array}\right]_{|V|\times|E|}$$

where I_{p_2-1} is the $(p_2-1)\times(p_2-1)$, $1 = [1,1,\cdots,1]_{1\times(p_2-1)}$ and $0 = [0,\cdots,0]_{1\times(p_2-1)}$. Therefore, by Theorem 2.2 and [12, Theorem 3.3], we get that $EG(\mathbb{Z}_n)$ is a code with minimum distance $\lambda(EG(\mathbb{Z}_n))$ and so we conclude that $C_2(EG(\mathbb{Z}_n)) = \big[(p_1-1)(p_2-1), p_1+p_2-2, \min\{p_1-1, p_2-1\}\big]_2$. □

Example 2.2. Consider the ring $\mathbb{Z}_{15}$. It is clear that $\mathrm{Ass}(\mathbb{Z}_{15}) = \{3\mathbb{Z}_{15}, 5\mathbb{Z}_{15}\}$ and $\mathrm{Nil}(\mathbb{Z}_{15}) = 3\mathbb{Z}_{15}\cap 5\mathbb{Z}_{15} = 0$. In view of [10, Theorem 3.7] and [11, Theorem 2.3], we get that $EG(\mathbb{Z}_{15}) = K_{2,4}$ is a complete bipartite graph with 6 vertices and $D = \{3,5\}$ is a dominating set for $EG(\mathbb{Z}_{15})$, see Figure 2. In this case

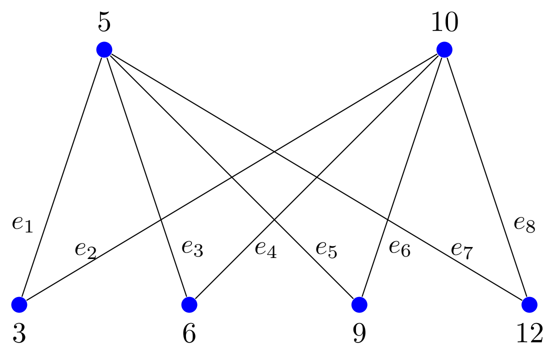

Figure 2: The essential graph $EG(\mathbb{Z}_{15})$ with linear code $\big[8,4,2\big]_2$.

$$G = \begin{array}{r} \\ v_1{=}3 \\ v_2{=}6 \\ v_3{=}9 \\ v_4{=}12 \\ v_5{=}5 \\ v_6{=}10 \end{array} \left[\begin{array}{cccc|cccc} e_1 & e_3 & e_5 & e_7 & e_2 & e_4 & e_6 & e_8 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]_{6\times 8}$$

is an incidence matrix for $EG(\mathbb{Z}_{15})$. Then the minimum distance of the code generated by the matrix G is 2 by [12, Theorem 3.3]. Therefore, $C_2(\mathbb{Z}_{15}) = \big[8,4,2\big]_2$ by Theorem 2.3.

Theorem 2.4. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}$, where p_1, p_2 are prime numbers, α_1, α_2 are positive integers and $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$. Then $EG(\mathbb{Z}_n)$ is a linear code and*

$$C_2(EG(\mathbb{Z}_n)) = \big[|E|, |V|-1, \min\{p_1+1, p_2+1\}\big]_2.$$

Proof. Assume that $n = p_1^{\alpha_1}p_2^{\alpha_2}$, where p_i is prime number and α_i is positive integer with $\alpha_i \geq 1$, for $i = 1,2$. If $\alpha_1 = \alpha_2 = 1$, then it follows from Theorem 2.3. Without loss of generality we may assume that $\alpha_1 \geq 2$, $\alpha_2 \geq 1$ and $p_1 < p_2$. It is clear that $Z(\mathbb{Z}_n) = p_1\mathbb{Z}_n \cup p_2\mathbb{Z}_n$, $\mathrm{Ass}(\mathbb{Z}_n) = \{p_1\mathbb{Z}_n, p_2\mathbb{Z}_n\}$,

$\mathrm{Nil}(\mathbb{Z}_n) = p_1\mathbb{Z}_n \cap p_2\mathbb{Z}_n$ and $|Z(\mathbb{Z}_n)^*| = p_1^{\alpha_1 - 1}p_2^{\alpha_2-1}(p_1 + p_2 - 1) - 1$. Now, we divided the vertices by the following subsets:

$$\begin{aligned} V_1 &= \{\mathrm{Ann}(a) |\, \mathrm{Ann}(a) = p\mathbb{Z}/q\mathbb{Z}, a = k_i p_1^{\alpha_1}, p|q, (k_i, p) = 1, 1 \le i \le n\},\\ V_2 &= \{\mathrm{Ann}(b) |\, \mathrm{Ann}(b) = p\mathbb{Z}/q\mathbb{Z}, b = s_i p_2^{\alpha_2}, p|q, (s_i, p) = 1, 1 \le i \le n\},\\ V_3 &= \{\mathrm{Ann}(c) | c = s_i p_1 p_2, (p_1, p_2) = 1, 1 \le i \le n\}. \end{aligned}$$

where $|V_1| = p_1^{\alpha_1-1}(p_1 - 1)$, $|V_2| = p_2^{\alpha_2-1}(p_2 - 1)$ and $|V_3| = 1$. Then all edges of the graph is $|E| = |V_1||V_2| + |V_1||V_3| + |V_2||V_3|$. In the other words,

$$|E| = \frac{(|V| - 1) - (|V|_2 - 1)}{2} - 1,$$

where $|V|$ is the number of vertices. Since $E(V_1)$ is adjacent with $E(V_2)$ and $E(V_3)$ is adjacent with others, we define the incidence matrix of $EG(\mathbb{Z}_n)$ as follows:

$$G = \begin{array}{c} \\ V_1 \\ V_2 \\ \\ V_3 \end{array} \begin{array}{c} \begin{array}{ccc} E(V_{ij}) & E(V_{ij}) & E(V_{ij}) \end{array} \\ \left[\begin{array}{c|c|c} I & I & 0 \\ \hline 1 & 0 & I \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]_{|V| \times |E|} \end{array}$$

where I is a $p_1^{\alpha_1} \times p_1^{\alpha_1}$ identity matrix and 0 is a zero block matrix $p_1^{\alpha_1} \times p_2^{\alpha_2}$ in the first row. Also, $1 = [1, 1, \cdots, 1]_{1\times p_1^{\alpha_1}}$ and $0 = [0, \cdots, 0]_{1\times p_1^{\alpha_1}}$, in other rows. In the last column, I is a $p_2^{\alpha_2} \times p_2^{\alpha_2}$ identity matrix and $1 = [1, 1, \cdots, 1]_{1\times p_2^{\alpha_2}}$. Therefore, by Theorem 2.2 and [12, Theorem 3.3], $C_2(EG(\mathbb{Z}_n)) = [n, k, d]_2$, where $n = |E(EG(\mathbb{Z}_n))|$, $k = |V(EG(\mathbb{Z}_n))| - 1$ and $d = \lambda(EG(\mathbb{Z}_n)) = \min_{1\le i\le 2}\{p_i + 1\}$. $\square$

Example 2.3. Consider the ring $\mathbb{Z}_{12}$. It is clear that $\mathrm{Ass}(\mathbb{Z}_{12}) = \{2\mathbb{Z}_{12}, 3\mathbb{Z}_{12}\}$, $\mathrm{Nil}(\mathbb{Z}_{12}) = 2\mathbb{Z}_{12} \cap 3\mathbb{Z}_{12} = 6\mathbb{Z}_{12}$ and $D = \{6\}$ is a dominating set for $EG(\mathbb{Z}_{12})$, see Figure 3. Then

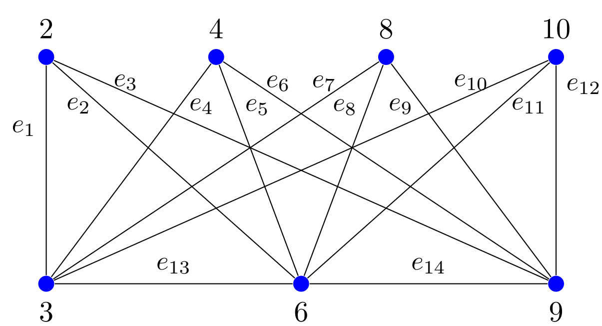

Figure 3: The essential graph $EG(\mathbb{Z}_{12})$ with linear code $[14, 6, 3]_2$.

$$G = \begin{array}{c} \\ v_1{=}2 \\ v_2{=}4 \\ v_3{=}8 \\ v_4{=}10 \\ v_5{=}3 \\ v_6{=}9 \\ v_7{=}6 \end{array} \begin{array}{cccc:cccc:cccc:cc} e_1 & e_4 & e_7 & e_{10} & e_2 & e_5 & e_8 & e_{11} & e_3 & e_6 & e_9 & e_{12} & e_{13} & e_{14} \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}_{7\times 14}$$

is an incidence matrix of $EG(\mathbb{Z}_{12})$. Consequently, the minimum distance of the code generated by the matrix G is 3 by [12, Theorem 3.3]. Therefore, $C_2(EG(\mathbb{Z}_n)) = \left[14, 6, 3\right]_2$ by Theorem 2.4.

References

[1] N. Annamalai, C. Durairajan, *Codes from the incidence matrices of a zero-divisor graph*, arXiv preprint arXiv: 2011.01602, 2020.

[2] D. F. Anderson, P.S. Livingston, *The zero-divisor graph of a commutative ring*, Journal of Algebra, 217 (1999), 434-447.

[3] E. F. Assmus, J.D. Key, *Designs and their codes*, Cambridge Tracts in Mathematics, 103. Cambridge University Press, Cambridge, 1992 (second printing with corrections, 1993).

[4] I. Beck, *Coloring of commutative rings*, Journal of Algebra, 116 (1988), 208-226.

[5] P. Dankelmann, J.D. Key, B.G. Rodrigues, *Codes from incidence matrices of graphs*, Designs, Codes and Cryptography, 68 (2013), 373–393.

[6] S.El. Rouayheb, C.N. Georghiades, *Graph theoretic methods in coding theory*, Classical, Semi-Classical and Quantum Noise. Springer, New York, NY, 2012, 53-62.

[7] S. Ling, Ch. Xing, *Coding theory, a first course,* Cambridge University Press, 2004.

[8] M. J. Nikmehr, R. Nikandish, M. Bakhtyiari, *On the essential graph of a commutative ring*, Journal of algebra Algebr. Appl., 16 (2017), 1750132.

[9] R. Y. Sharp, *Steps in commutative algebra,* Second edition, Cambridge University Press, Cambridge, 2000.

[10] F. Soheilnia, Sh. Payrovi, A. Behtoei, *A generalization of the essential graph for modules over commutative rings*, International Electronic Journal of Algebra, 29 (2021), 211-222.

[11] F. Soheilnia, Sh. Payrovi, A. Behtoei, *Perfectness of the essential graph for modules over commutative rings*, AUT. Journal of Mathematics and Computing, preprint.

[12] F. Soheilnia, A.Y. Darani, *Connectivity of the essential graph for modules over commutative rings*, preprint.

Accepted: February 20, 2022