Application of new distance and entropy measures for probabilistic dual hesitant fuzzy sets in pattern recognition and VIKOR-based multi-criteria decision-making

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Abstract. Probabilistic dual hesitant fuzzy sets (PDHFSs) are constantly used to handle uncertainty problems. However, measures associated with PDHFSs have yet to be widely studied and may have limitations in application. In this paper, a weak equality is suggested to facilitate the conditional reflexivity of distance measures. Then, the consistency, probabilistic fuzziness and Hausdorff distances are defined to integrate different information in PDHFSs. Based on the above definitions, distance and similarity measures for PDHFSs that are not limited by the number of elements are presented. A simple example indicates that new distance measures can effectively differentiate between PDHFSs. Furthermore, a new entropy measure is built for PDHFSs by combining the sine function. Finally, the practicality and feasibility of development methods are demonstrated through two practical examples: pattern recognition and VIKOR-based multi-criteria decision-making (MCDM). Moreover, parameter analyses and comparative analyses indicate that these measures are superior to their existing counterparts.

Keywords: weak equality, distance measure, entropy measure, pattern recognition, multi-criteria decision-making, probabilistic dual hesitant fuzzy sets.

MSC 2020: 90B50.

1. Introduction

Solving fuzzy phenomena and uncertain events in real life is necessary as science and technology advance. Uncertainty theory brings usefulness and convenience to people's lives and is interconnected with many knowledge systems [17, 20,

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27]. Zadeh [36] explicitly and systematically proposed fuzzy sets (FSs) to solve ambiguity problems. Atanassov described the meanings of non-membership degree and hesitation degree, leading to the development of intuitionistic fuzzy sets (IFSs) [2]. Torra [29] changed the decision rule by hesitant fuzzy sets (HFSs), which permits decision makers(DMs) to take multiple potential values on interval $[0, 1]$. HFSs describe the hesitation of DMs in evaluation. Then, a series of aggregation operators for HFSs that could aggregate the values under each attribute were developed [30]. Alcantud [1] introduced a novel HFS with an inclusive ranking that obeys strict sorting in the stratified assessment. Further, Zhu et al. [41] proposed dual hesitant fuzzy sets (DHFSs), where FSs and IFSs were regarded as particular components of DHFSs. Some scholars systematically explained the operations and basic rules of DHFSs [40, 38, 26]. Xu and Zhou [33] considered the influence of DMs' knowledge level and subjective factors and introduced probability into HFSs to define the probability hesitation fuzzy sets (PHFSs). Krishankumaar [11] proposed a new ranking approach for PHFSs, which made the method more flexible by adding evidence theory. Hao [8] used PDHFSs to translate vague information more appropriately and then studied visualization techniques for PDHFSs to deal with risk evaluation problems. It is worth noting that PDHFSs comprehensively express the fuzziness and hesitancy of human thinking and the importance of different elements.

Measure methods were significant in dealing with uncertainty problems, such as decision analysis, pattern recognition and artificial intelligence [12, 23, 39, 24]. Xu and Xia [31] developed some distance measures to enrich the study of HFSs [32]. Considering DMs' preference, Tong and Yu [28] proposed a new distance measure for HFS to address the defects of measure in [31]. However, when constructing the above measure, assuming that the number of hesitant fuzzy elements isn't equal, DMs needed to increase elements to make them equal. This approach may violate the authenticity of data. Hussain and Yang [9] characterized the connection between HFSs and their complement to construct a new entropy measure. This measure relied on particular data values and had a relatively narrow range of applications. Yang and Hussain [34] complemented the existing measures of HFSs, simplifying the calculation. Farhadinia and Xu [6] extended the Euclidean distance between single-point sets to HFSs and defined a new distance measure. For the problem of unequal-length for elements, Chen et al. [3] described some new distance measures. Zhang [37] also constructed several distance and entropy measures suitable for pattern recognition. To a certain extent, these measures [3, 37] maintained the authenticity of data but did not have triangular inequality property. Su et al. [25] proposed a new entropy called membership entropy based on exponential functions. It provides an innovative method for measuring the uncertainty of PHFSs. In [16] and [15], the characteristics of PHFSs were comprehensively considered and a distance measure was developed. In addition, considering the influence of psychological changes when DMs encounter risk, a TODIM-based multi-criteria decision-making (MCDM) method was proposed [16]. Fang [5] proposed a series of entropy and cross-entropy with strong discrimination ability and wide application.

PDHFSs have apparent advantages in presenting fuzzy information, and their research value is evident. In [21], the equiprobability distance measure was also an available metric for PDHFSs, but it changed the original structure of PDHFSs and weakened the psychological factor of DMs. Garg and Kau [7] successfully extended distance measures and then brought rationality and loss of DMs into the aggregation operators, underlining the contribution of probability to PDHFSs. Ning et al. [18] proposed some distance measures for PDHFSs from the perspective of averaging and dispersion. However, the study [18] has a defect: the measures cannot distinguish different PDHFSs in exceptional cases, which violates conditional reflexivity. Ning et al. [19] also built some new distance measures that considered the discrete, continuous, ordered and disordered conditions of PDHFSs. Such distances are novel in form but do not satisfy triangular inequality. It is readily apparent from the above analyses that there are few studies and shortcomings on the relevant measures of PDHFS at present, and further studies are needed.

In order to enhance the conditional reflexivity of distance measures, we define weak equality. According to several properties of PDHFSs, distance measures with parameters are constructed. Moreover, a new entropy measure is proposed by combining the sine function. The proposed methods are applied in pattern recognition and VIKOR-based MCDM methods. The contributions of this paper are as follows:

(1) A weak equivalence between PDHFSs is defined to enhance the conditional reflexivity of distance measures.

(2) Assuming that all values in probabilistic dual hesitant fuzzy elements (PDHFEs) are in increasing order, we combine the consistency and probabilistic fuzziness of PDHFEs to obtain some distance measures with parameters. New distance measures are more discriminating between PDHFSs. They are suitable for the axioms of distance measure, especially the triangular inequality.

(3) A new entropy measure with an auxiliary function is presented, enriching the research of entropy measures in the PDHFS environment.

The rest of this paper is arranged as follows: Section 2 introduces the concepts related to with PDHFS and existing research. Section 3 proposes some new distance and similarity measures and discusses their properties. Section 4 defines the entropy measure of PDHFS in combination with the sine function. Section 5 applies the above distance, similarity and entropy measures to two examples. Finally, the conclusion and expectations are in Section 6.

2. Preliminaries

2.1 HFSs and PDHFSs

In this subsection, we introduce the concept of HFSs and PDHFSs and some existing outcomes. The notations used in this section are explained in Table 1.

Table 1: Notations interpretation I

Notations	Details	Notations	Details
$\hat{O}, M$	PDHFSs	$\hat{\partial}$	PDHFE
$\mathcal{G}(x)$	membership degree	$\kappa(x)$	non-membership degree
$p(x)$	nembership probabilistic	$q(x)$	non-nembership probabilistic
γ, η	possible elements	$\#\mathcal{G}, \#\kappa$	number of elements
$\lambda(\hat{\partial}), \lambda(\hat{O}(x))$	score function	$\ell(\hat{\partial}), \ell(\hat{O}(x))$	accuracy function
$\wp(\hat{\partial}), \wp(\hat{O}(x_i))$	membership means	$\Im(\hat{\partial}), \Im(\hat{O}(x_i))$	non-membership means
$\Phi(\hat{O}(x_i))$	membership standard deviation	$\Psi(\hat{O}(x_i))$	non-membership standard deviation
$\wp(x_i), \Im(x_i)$	means distance	$\Phi(x_i), \Psi(x_i)$	standard deviation distance

Let X be a universal set, $\mathcal{F} = \{\langle x, \hbar_{\mathcal{F}}(x)\rangle : x \in X\}$ is called an HFS, where $\hbar_{\mathcal{F}}(x) \in [0, 1]$ are hesitant fuzzy elements, representing the set of possible memberships of x of X [29].

Definition 1 (Hao et al. [8])**.** *defined PDHFS $\hat{O}$ on X as:*

$$\hat{O} = \{\langle \mathcal{G}(x)|p(x), \kappa(x)|q(x)\rangle \, | x \in X\}, \tag{1}$$

where $\mathcal{G}(x)$ and $\kappa(x)$ are the the membership and non-membership degrees of $x \in X$, respectively. The probabilities of $\mathcal{G}(x)$ and $\kappa(x)$ are $p(x)$ and $q(x)$ separately. Also, there is

$$\begin{gathered} \gamma \in [0,1], \eta \in [0,1], \gamma^+ + \eta^+ \in [0,1], \\ p_j \in [0,1], q_j \in [0,1], \sum_{j=1}^{\#\mathcal{G}} p_j = 1, \sum_{j=1}^{\#\kappa} q_j = 1, \end{gathered} \tag{2}$$

where γ, η are some possible elements, $\gamma \in \mathcal{G}(x), \eta \in \kappa(x)$, $\gamma^+ \in \mathcal{G}^+(x) = \bigcup_{\gamma \in \mathcal{G}(x)} \max\{\gamma\}$, $\eta^+ \in \kappa^+(x) = \bigcup_{\eta \in \kappa(x)} \max\{\eta\}$, $p_j \in p(x)$ and $q_j \in q(x)$. $\#\mathcal{G}$ and $\#\kappa$ are the total number of elements in $\mathcal{G}(x)|p(x)$ and $\kappa(x)|q(x))$.

For convenience, the pair $\hat{\partial} = \langle \mathcal{G}(x)|p(x), \kappa(x)|q(x))\rangle$ is called as PDHFE, denoted by $\hat{\partial} = \langle \mathcal{G}|p, \kappa|q\rangle$ ([8]).

Definition 2 ([8]). *The complement definition of the PDHFE $\hat{\partial} = \langle \mathcal{G}|p, \kappa|q \rangle$ is as follows:*

$$\hat{\partial}^c = \bigcup_{\eta\in\kappa,\gamma\in\mathcal{G}} \langle \eta|q, \gamma|p \rangle,\ if\ \kappa \neq \emptyset \ and\ \mathcal{G} \neq \emptyset. \tag{3}$$

Definition 3 ([8]). *The score function of PDHFE $\hat{\partial}$ is:*

$$\lambda(\hat{\partial}) = \sum_{j=1,\gamma\in\mathcal{G}}^{\#\mathcal{G}} \gamma_j p_j - \sum_{j=1,\eta\in\kappa}^{\#\kappa} \eta_j q_j. \tag{4}$$

Definition 4 ([33]). *The accuracy function of PDHFE $\hat{\partial}$ is:*

$$\ell(\hat{\partial}) = \sum_{j=1,\gamma\in\mathcal{G}}^{\#\mathcal{G}} \gamma_j p_j + \sum_{j=1,\eta\in\kappa}^{\#\kappa} \eta_j q_j. \tag{5}$$

Let $\hat{\partial}_1, \hat{\partial}_2$ be two PDHFEs. The rules for comparing are as follows [33]:
If $\lambda(\hat{\partial}_1) > \lambda(\hat{\partial}_2)$, then $\hat{\partial}_1 > \hat{\partial}_2$. If $\lambda(\hat{\partial}_1) < \lambda(\hat{\partial}_2)$, then $\hat{\partial}_1 < \hat{\partial}_2$.
If $\lambda(\hat{\partial}_1) = \lambda(\hat{\partial}_2)$, there are the following rules:

(1) If $\ell(\hat{\partial}_1) > \ell(\hat{\partial}_2)$, then $\hat{\partial}_1 > \hat{\partial}_2$;

(2) If $\ell(\hat{\partial}_1) < \ell(\hat{\partial}_2)$, then $\hat{\partial}_1 < \hat{\partial}_2$;

(3) If $\ell(\hat{\partial}_1) = \ell(\hat{\partial}_2)$, then $\hat{\partial}_1 \sim \hat{\partial}_2$.

Definition 5 ([18]). *Let $\hat{\partial} = \langle \mathcal{G}|p, \kappa|q \rangle$ be a PDHFE on X . The means of membership and non-membership for $\hat{\partial}$ are:*

$$\wp(\hat{\partial}) = \sum_{j=1}^{\#\mathcal{G}} \gamma_j p_j,$$

$$\Im(\hat{\partial}) = \sum_{j=1}^{\#\kappa} \eta_j q_j.$$

2.2 An existing distance measure of PDHFSs

In such subsection, we review a distance measure and properties proposed by Ning et al [18]. A simple example explains that the measure has a disadvantage.

Definition 6 ([18]). *Let $\hat{\partial}_1$, $\hat{\partial}_2$ and $\hat{\partial}_3$ be three PDHFEs. Call d a distance measure if the following properties are satisfied:*

(D1) $0 \leq d(\hat{\partial}_1, \hat{\partial}_2) \leq 1$;

(D2) $d(\hat{\partial}_1, \hat{\partial}_2) = 0 \Leftrightarrow \hat{\partial}_1 = \hat{\partial}_2$;

(D3) $d(\hat{\partial}_1, \hat{\partial}_2) = d(\hat{\partial}_2, \hat{\partial}_1)$;

(D4) $d(\hat{\partial}_1, \hat{\partial}_3) \leq d(\hat{\partial}_1, \hat{\partial}_2) + d(\hat{\partial}_2, \hat{\partial}_3)$,

Definition 7. *Ning et al. [18] give a distance measure between PDHFSs $\hat{O}_1$ and $\hat{O}_2$.*

$$d_{gpnd}(\hat{O}_1, \hat{O}_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(a^{\frac{\wp^\lambda(x_i)+\Im^\lambda(x_i)}{2}} + b^{\frac{\Phi^\lambda(x_i)+\Psi^\lambda(x_i)}{2}} \right)\right]^{1/\lambda} \tag{6}$$

where $\lambda > 0$, $\lambda \in R$, $a, b \in [0,1], a+b=1$, $\wp(x_i), \Im(x_i)$ are means distances, $\Phi(x_i), \Psi(x_i)$ are standard deviation distances, $\wp(\hat{O}_m(x_i)), \Im(\hat{O}_m(x_i))$ $(m = 1,2)$ are membership and non-membership means, $\Phi(\hat{O}_m(x_i)), \Psi(\hat{O}_m(x_i))$ are membership and non-membership standard deviations, respectively.

$$\wp(x_i) = \left|\wp(\hat{O}_1(x_i)) - \wp(\hat{O}_2(x_i))\right| = \left|\sum_{j=1}^{\#\mathcal{G}_1}\gamma_{1j}p_{1j} - \sum_{t=1}^{\#\mathcal{G}_2}\gamma_{2t}p_{2t}\right|,$$

$$\Im(x_i) = \left|\Im(\hat{O}_1(x_i)) - \Im(\hat{O}_2(x_i))\right| = \left|\sum_{j=1}^{\#\kappa_1}\eta_{1j}q_{1j} - \sum_{t=1}^{\#\kappa_2}\eta_{2t}q_{2t}\right|,$$

$$\begin{aligned}\Phi(x_i) &= \left|\Phi(\hat{O}_1(x_i)) - \Phi(\hat{O}_2(x_i))\right| \\ &= \left| \left(\sum_{j=1}^{\#\mathcal{G}_1}\left(\gamma_{1j}p_{1j} - \wp(\hat{O}_1(x_i))\right)^2\right)^{1/2} - \left(\sum_{t=1}^{\#\mathcal{G}_2}\left(\gamma_{2t}p_{2t} - \wp(\hat{O}_2(x_i))\right)^2\right)^{1/2} \right|,\end{aligned}$$

$$\begin{aligned}\Psi(x_i) &= \left|\Psi(\hat{O}_1(x_i)) - \Psi(\hat{O}_2(x_i))\right| \\ &= \left| \left(\sum_{j=1}^{\#\kappa_1}\left(\eta_{1j}q_{1j} - \Im(\hat{O}_1(x_i))\right)^2\right)^{1/2} - \left(\sum_{t=1}^{\#\kappa_2}\left(\eta_{2t}q_{2t} - \Im(\hat{O}_2(x_i))\right)^2\right)^{1/2} \right|.\end{aligned}$$

This distance measure has applicability but violates the conditional reflexivity (D2) of Definition 6. We illustrate this with an example.

Example 1. Let $X = \{x_1\}$, and we give two PDHFSs
$\hat{O}_1 = \{\langle x_1, \{0.25|0.35, 0.35|\ 0.25, 0.4|0.4\}, \{0.45|0.6, 0.55|0.4\}\rangle\}$,
$\hat{O}_2 = \{\langle x_1, \{0.25|0.35, 0.35|0.25, 0.4|0.4\}, \{0.4|0.55, 0.6|0.45\}\rangle\}$ where $a = b = \frac{1}{2}$, $\lambda \in R$, and $\lambda \geq 1$. For $\hat{O}_1$, we have

$$\begin{aligned}\wp(\hat{O}_1(x_1)) &= 0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4 = 0.3350, \\ \Im(\hat{O}_1(x_1)) &= 0.45 \times 0.6 + 0.55 \times 0.4 = 0.4900, \\ \Phi(\hat{O}_1(x_1)) &= ((0.25 \times 0.35 - 0.3350)^2 + (0.35 \times 0.25 - 0.3350)^2 \\ &\quad + (0.4 \times 0.4 - 0.3350)^2)^{1/2} = 0.3913, \\ \Psi(\hat{O}_1(x_1)) &= \sqrt{(0.45 \times 0.6 - 0.4900)^2 + (0.55 \times 0.4 - 0.4900)^2} = 0.3483.\end{aligned}$$

For $\hat{O}_2$, we have

$$\wp(\hat{O}_2(x_1)) = 0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4 = 0.3350,$$
$$\Im(\hat{O}_2(x_1)) = 0.4 \times 0.55 + 0.6 \times 0.45 = 0.4900,$$
$$\Phi(\hat{O}_2(x_1)) = \sqrt{(0.25 \times 0.35 - 0.3350)^2 + (0.35 \times 0.25 - 0.3350)^2 + (0.4 \times 0.4 - 0.3350)^2} = 0.3913,$$
$$\Psi(\hat{O}_2(x_1)) = \sqrt{(0.4 \times 0.55 - 0.4900)^2 + (0.6 \times 0.45 - 0.4900)^2} = 0.3483.$$

Therefore, according to Eq. (6), we can have

$$d_{gpnd}(\hat{O}_1, \hat{O}_2) = \left(\frac{1}{2} \times \frac{|0.3350-0.3350|^\lambda + |0.4900-0.4900|^\lambda}{2} + \frac{1}{2} \times \frac{|0.3913-0.3913|^\lambda + |0.3483-0.3483|^\lambda}{2} \right)^{1/\lambda} = 0.$$

Moreover, score value $\lambdabar(\hat{O}_1(x_1)) = \wp(\hat{O}_1(x_1)) - \Im(\hat{O}_1(x_1)) = 0.3350 - 0.4900 = -0.1550$, accuracy value $\ell(\hat{O}_1(x_1)) = \wp(\hat{O}_1(x_1)) + \Im(\hat{O}_1(x_1)) = 0.3350 + 0.4900 = 0.8250$.

Similarly, $\lambdabar(\hat{O}_2(x_1)) = -0.1550$, $\ell(\hat{O}_2(x_1)) = 0.8250$.

Remark 1. Although $d_{gpnd}(\hat{O}_1, \hat{O}_2) = 0$ and $\hat{O}_1$, $\hat{O}_2$ have the same score and accuracy values, $\hat{O}_1$ and $\hat{O}_2$ are different PDHFSs. Therefore, $\hat{\partial}_1 = \hat{\partial}_2$ may be $\hat{\partial}_1 \sim \hat{\partial}_2$ in the conditional reflexivity (D2) of Definition 6. We need to propose new distance measures to improve the above defect.

3. Novel distance measures of PDHFSs

Since the conditions for equality of two PDHFSs are too strict, weak equality is constructed by relaxing these conditions appropriately. Then, we develop some novel notes and four-parameter generalized distance measures. Excellent properties of distance measures are discussed. Note that all values in PDHFEs are assumed to be in ascending order. The notations used in this section are explained in Table 2.

Definition 8. *Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs. For any $x_i \in X$, score function $\lambdabar(\hat{O}_1(x_i)) = \lambdabar(\hat{O}_2(x_i))$, accuracy function $\ell(\hat{O}_1(x_i)) = \ell(\hat{O}_2(x_i))$ and $\gamma_{1j} = \gamma_{2j}$, $\eta_{1j} = \eta_{2j}$. Then we call $\hat{O}_1$ and $\hat{O}_2$ week equality, denoted by $\hat{O}_1 =^w \hat{O}_2$.*

Definition 9. *Let $X = \{x_1, x_2..., x_n\}$ be a fixed set, $\hat{\partial} = \langle \mathcal{G}|p, \kappa|q \rangle$ be a PDHFE on X.*

(1) *The consistencies of membership and non-membership degrees are defined as:*

$$\iota(\hat{\partial}) = \left| \frac{1}{\#\mathcal{G}} \right|$$

Table 2: Notations interpretation II

Notations	**Details**	**Notations**	**Details**
$\iota(\hat{\partial}), \iota(\hat{O}(x_i))$	consistencies of membership degree	$\zeta(\hat{\partial}), \zeta(\hat{O}(x_i))$	consistencies of non-membership degree
$\Upsilon(\hat{\partial}), \Upsilon(\hat{O}(x_i))$	probabilistic fuzziness of membership degree	$\Upsilon(\hat{\partial}), \Upsilon(\hat{O}(x_i))$	probabilistic fuzziness of non-membership degree
$\iota(x_i), \zeta(x_i)$	consistency distances	$\Upsilon(x_i), \Upsilon(x_i)$	probabilistic fuzziness distances
$\theta(x_i), \psi(x_i)$	Hausdorff distances	λ	distance parameter
a, b, c, e	parameters	ω_i	weight

$$\zeta(\hat{\partial}) = \left| \frac{1}{\#\kappa} \right|$$

Our subjective perception is that the more elements in PDHFE, the less consistent. For example, the maximum consistency is achieved when $\#\mathcal{G} = 1$ *or* $\#\kappa = 1$*. As the numbers* $\#\mathcal{G}$ *and* $\#\kappa$ *increase, the consistency of* $\iota(\hat{\partial})$ *and* $\zeta(\hat{\partial})$ *decreases.*

(2) *The probabilistic fuzziness for membership and non-membership degrees as:*

$$\Upsilon(\hat{\partial}) = \frac{1}{\#\mathcal{G}} \sum_{j=1}^{\#\mathcal{G}} (1 - 2\,|p_j - 0.5|)\gamma_j,$$

$$\Upsilon(\hat{\partial}) = \frac{1}{\#\kappa} \sum_{j=1}^{\#\kappa} (1 - 2\,|q_j - 0.5|)\eta_j,$$

when probability $p_j = 0.5$ *or* $q_j = 0.5$*, the opinions of DMs are most uncertain and have high ambiguity. On the contrary, the probabilistic fuzziness* $\Upsilon(\hat{\partial})$ *or* $\Upsilon(\hat{\partial})$ *decreases when probability moves away from* 0.5*.*

Definition 10. *Let* $\hat{O}_1 = \{\langle x_i, \mathcal{G}_1(x_i)|p_1(x_i),\ \kappa_1(x_i)|q_1(x_i)\rangle\, |x_i \in X\}$ *and* $\hat{O}_2 = \{\langle x_i, \mathcal{G}_2(x_i)|\ \ p_2(x_i), \kappa_2(x_i)|q_2(x_i)\rangle\, |x_i {\in} X\}$ *be two PDHFSs on* $X{=}\{x_1, x_2, ..., x_n\}$*. We give some definitions between* $\hat{O}_1$ *and* $\hat{O}_2$*:*

(1) *The consistency distances between* $\hat{O}_1$ *and* $\hat{O}_2$*:*

$$\iota(x_i) = \left| \iota(\hat{O}_1(x_i)) - \iota(\hat{O}_2(x_i)) \right| = \left| \frac{1}{\#\mathcal{G}_1} - \frac{1}{\#\mathcal{G}_2} \right|,$$

$$\zeta(x_i) = \left| \zeta(\hat{O}_1(x_i)) - \zeta(\hat{O}_2(x_i)) \right| = \left| \frac{1}{\#\kappa_1} - \frac{1}{\#\kappa_2} \right|.$$

(2) *The probabilistic fuzziness distances between $\hat{O}_1$ and $\hat{O}_2$:*

$$\begin{aligned}
\Tau(x_i) &= \left|\Tau(\hat{O}_1(x_i)) - \Tau(\hat{O}_2(x_i))\right| \\
&= \left|\frac{1}{\#\mathcal{G}_1}\sum_{j=1}^{\#\mathcal{G}_1}(1-2\left|p_{1j}-0.5\right|)\gamma_{1j} - \frac{1}{\#\mathcal{G}_2}\sum_{t=1}^{\#\mathcal{G}_2}(1-2\left|p_{2t}-0.5\right|)\gamma_{2t}\right|, \\
\Upsilon(x_i) &= \left|\Upsilon(\hat{O}_1(x_i)) - \Upsilon(\hat{O}_2(x_i))\right| \\
&= \left|\frac{1}{\#\kappa_1}\sum_{j=1}^{\#\kappa_1}(1-2\left|q_{1j}-0.5\right|)\eta_{1j} - \frac{1}{\#\kappa_2}\sum_{t=1}^{\#\kappa_2}(1-2\left|q_{2t}-0.5\right|)\eta_{2t}\right\}.
\end{aligned}$$

(3) *The Hausdorff distances between $\hat{O}_1$ and $\hat{O}_2$:*

$$\begin{aligned}
\theta(x_i) &= \theta\left(\hat{O}_1(x_i), \hat{O}_2(x_i)\right) \\
&= \max\{\max_{\gamma_{1j}\in\mathcal{G}_1}\min_{\gamma_{2t}\in\mathcal{G}_2}\left\|\gamma_{1j}-\gamma_{2t}\right\| p_{1j}p_{2t}, \\
&\qquad \max_{\gamma_{2t}\in\mathcal{G}_2}\min_{\gamma_{1j}\in\mathcal{G}_1}\left\|\gamma_{1j}-\gamma_{2t}\right\| p_{1j}p_{2t}\}, \\
\psi(x_i) &= \psi\left(\hat{O}_1(x_i), \hat{O}_2(x_i)\right) \\
&= \max\{\max_{\eta_{1j}\in\kappa_1}\min_{\eta_{2t}\in\kappa_2}\left\|\eta_{1j}-\eta_{2t}\right\| q_{1j}q_{2t}, \\
&\qquad \max_{\eta_{2t}\in\kappa_2}\min_{\eta_{1j}\in\kappa_1}\left\|\eta_{1j}-\eta_{2t}\right\| q_{1j}q_{2t}\}.
\end{aligned}$$

Note that, $\|*\|$ denotes Euclidean distance from one $\gamma_{1j}(\eta_{1j})$ of $\hat{O}_1(x_i)$ to another $\gamma_{2t}(\eta_{2t}) \in \hat{O}_2(x_i)$.

(4) *The mean distances between $\hat{O}_1$ and $\hat{O}_2$:*

$$\wp(x_i) = \left|\wp(\hat{O}_1(x_i)) - \wp(\hat{O}_2(x_i))\right| = \left|\sum_{j=1}^{\#\mathcal{G}_1}\gamma_{1j}p_{1j} - \sum_{t=1}^{\#\mathcal{G}_2}\gamma_{2t}p_{2t}\right|,$$

$$\Im(x_i) = \left|\Im(\hat{O}_1(x_i)) - \Im(\hat{O}_2(x_i))\right| = \left|\sum_{j=1}^{\#\kappa_1}\eta_{1j}q_{1j} - \sum_{t=1}^{\#\kappa_2}\eta_{2t}q_{2t}\right|.$$

We can define a generalized distance measure between PDHFSs based on the above results.

Definition 11. *Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs on X, the following four-parameter generalized distance between $\hat{O}_1$ and $\hat{O}_2$ is proposed:*

$$\begin{aligned}
d_{pgdh}(\hat{O}_1, \hat{O}_2) = \Bigg[\frac{1}{n}\sum_{i=1}^{n}\Bigg(& a\frac{\iota(x_i)+\zeta(x_i)}{2} \\
&+ b\frac{\Tau(x_i)+\Upsilon(x_i)}{2} + c\frac{\theta(x_i)+\psi(x_i)}{2} + e\frac{\wp(x_i)+\Im(x_i)}{2}\Bigg)^{\lambda}\Bigg]^{1/\lambda},
\end{aligned} \tag{7}$$

where $\lambda \geq 1$, $\lambda \in R$, $0 < a, b, c, e < 1$ *and* $a + b + c + e = 1$.

Definition 12. *Let* $\hat{O}_1$, $\hat{O}_2$ *and* $\hat{O}_3$ *be three PDHFSs.* d_{pgdh} *is a distance measure if* d_{pgdh} *satisfies the following conditions:*

(D1) $0 \leq d_{pgdh}(\hat{O}_1, \hat{O}_2) \leq 1$ *(boundedness);*

(D2) $d_{pgdh}(\hat{O}_1, \hat{O}_2) = 0 \Leftrightarrow \hat{O}_1 =^{w} \hat{O}_2$ *(conditional reflexivity);*

(D3) $d_{pgdh}(\hat{O}_1, \hat{O}_2) = d_{pgdh}(\hat{O}_2, \hat{O}_1)$ *(symmetric);*

(D4) $d_{pgdh}(\hat{O}_1, \hat{O}_3) \leq d_{pgdh}(\hat{O}_1, \hat{O}_2) + d_{pgdh}(\hat{O}_2, \hat{O}_3)$ *(triangle inequality).*

Theorem 1. d_{pgdh} *is a distance measure between PDHFSs and satisfies the four properties of Definition 12.*

Example 2. We use the data from Example 1, let $a = b = c = e = \frac{1}{4}$,λ=1, we have $\iota(x_1) = |\frac{1}{3} - \frac{1}{3}| = 0$, $\zeta(x_1) = |\frac{1}{2} - \frac{1}{2}| = 0$,

$$\Upsilon(x_1) = \left| \begin{array}{l} \frac{1}{3}\left(\begin{array}{l} (1-2\left|0.35-0.5\right|)\,0.25 \\ +(1-2\left|0.25-0.5\right|)\,0.35 \\ +(1-2\left|0.4-0.5\right|)\,0.4 \end{array} \right) \\ -\frac{1}{3}\left(\begin{array}{l} (1-2\left|0.35-0.5\right|)\,0.25 \\ +(1-2\left|0.25-0.5\right|)\,0.35 \\ +(1-2\left|0.4-0.5\right|)\,0.4 \end{array} \right) \end{array} \right| = 0,$$

$$\Upsilon(x_1) = \left| \begin{array}{l} \frac{1}{2}\left(\begin{array}{l} (1-2\left|0.6-0.5\right|)\,0.45 \\ +(1-2\left|0.4-0.5\right|)\,0.55 \end{array} \right) \\ -\frac{1}{2}\left(\begin{array}{l} (1-2\left|0.55-0.5\right|)\,0.4 \\ +(1-2\left|0.45-0.5\right|)\,0.6 \end{array} \right) \end{array} \right| = 0.0500,$$

$$\theta(x_1) = \left\{ \begin{array}{l} \max_{\gamma_{Mj} \in \mathcal{G}_M} \left\{ \min\{0,0,0\}, \min\{0,0,0\} \right\}, \\ \max_{\gamma_{Nj} \in \mathcal{G}_N} \left\{ \min\{0,0,0\}, \min\{0,0,0\} \right\} \end{array} \right\} = 0,$$

$$\psi(x_1) = \left\{ \begin{array}{l} \max_{\gamma_{Mj} \in h_M} \left\{ \min\{0.0165, 0.0405\}, \min\{0.033, 0.009\} \right\}, \\ \max_{\gamma_{Nj} \in h_N} \left\{ \min\{0.0165, 0.033\}, \min\{0.0405, 0.009\} \right\} \end{array} \right\} = 0.0165,$$

$$\wp(x_1) = \left| \begin{array}{l} (0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4) \\ -(0.25 \times 0.35 + 0.35 \times 0.25 + 0.4 \times 0.4) \end{array} \right| = 0,$$

$$\Im(x_1) = \left| (0.45 \times 0.6 + 0.55 \times 0.4) - (0.4 \times 0.55 + 0.6 \times 0.45) \right| = 0,$$

According to Eq. (7), the distance $d_{pgdh}(\hat{O}_1, \hat{O}_2)$ is

$$d_{pgdh}(\hat{O}_1, \hat{O}_2) = \frac{1}{4} \times \frac{0 + 0.0500}{2} + \frac{1}{4} \times \frac{0+0}{2} + \frac{1}{4} \times \frac{0 + 0.0165}{2} + \frac{1}{4} \times \frac{0+0}{2}$$
$$= 0.0083.$$

Remark 2. It is clear that $\hat{O}_1$ and $\hat{O}_2$ are distinct PDHFSs, which is confirmed by Example 2. Compared with the distance measure in [18], the new generalized distance measure d_{pgdh} can distinguish PDHFSs better.

If we consider the weight of $x_i \in X$ $(i = 1, 2, ..., n)$, a generalized weighted distance measure with preferences for PDHFS as:

$$d_{gwpdh}(\hat{O}_1, \hat{O}_2) = \Bigg[\sum_{i=1}^{n} \omega_i \Bigg(a \frac{\iota(x_i) + \zeta(x_i)}{2} + b \frac{\Tau(x_i) + \Upsilon(x_i)}{2} + c \frac{\theta(x_i) + \psi(x_i)}{2} + e \frac{\wp(x_i) + \Im(x_i)}{2} \Bigg)^{\lambda} \Bigg]^{1/\lambda}, \tag{8}$$

where $\lambda \geq 1$, ω_i is the weight of $x_i \in X$, with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

Remark 3. If $\omega_i = \frac{1}{n}$, then d_{gwpdh} can be simplified to d_{pgdh}.

Definition 13. *Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs on X. The generalized weighted similarity measure is defined as:*

$$S_{gwpdh}(\hat{O}_1, \hat{O}_2) = 1 - d_{gwpdh}(\hat{O}_1, \hat{O}_2). \tag{9}$$

Definition 14. *Let $\hat{O}_1$ and $\hat{O}_2$ be two PDHFSs, if $S_{gwpdh}(\hat{O}_1, \hat{O}_2)$ satisfies three properties as follows:*

(S1) $0 \leq S_{gwpdh}(\hat{O}_1, \hat{O}_2) \leq 1$ *(boundedness);*

(S2) $S_{gwpdh}(\hat{O}_1, \hat{O}_2) = 1 \Leftrightarrow \hat{O}_1 =^{w} \hat{O}_2$ *(conditional reflexivity);*

(S3) $S_{gwpdh}(\hat{O}_1, \hat{O}_2) = S_{gwpdh}(\hat{O}_2, \hat{O}_1)$ *(symmetric),*

then we call $S_{gwpdh}(\hat{O}_1, \hat{O}_2)$ the similarity measure for PDHFSs.

Theorem 2. *S_{gwpdh} is a similarity measure of PDHFS and satisfies the three properties in Definition 14.*

4. Entropy measure of PDHFS

Entropy is crucial in measuring the uncertainty of PDHFSs. This section proposes a new entropy measure, and the properties are discussed.

Let $\hat{\partial} = (\mathcal{G}|p, \kappa|q)$ be a PDHFE. [18] defines the mean of hesitant degree.

$$\Theta(\hat{\partial}) = 1 - \wp(\hat{\partial}) - \Im(\hat{\partial}) = 1 - \sum_{j=1}^{\#\mathcal{G}} \gamma_j p_j - \sum_{j=1}^{\#\kappa} \eta_j q_j.$$

Definition 15. *Let $\hat{\partial} = (\mathcal{G}|p, \kappa|q)$, $\hat{\partial}_1 = (\mathcal{G}_1|p_1, \kappa_1|q_1)$ and $\hat{\partial}_2 = (\mathcal{G}_2|p_2, \kappa_2|q_2)$ be three PDHFEs. A mapping $\Xi : PDHFEs(X) \to [0, 1]$, claiming that Ξ is an entropy and satisfies the following properties:*

(E1) $\Xi(\hat{\partial}) = 0$ *(minimum)* $\Leftrightarrow$ $\hat{\partial} = (\{0\,|1\}, \{1\,|1\})$ *or* $\hat{\partial} = (\{1\,|1\}, \{0\,|1\})$*;*

(E2) $\Xi(\hat{\partial}) = 1$ *(maximum)* $\Leftrightarrow$ $\wp(\hat{\partial}) = \Im(\hat{\partial})$*;*

(E3) $\Xi(\hat{\partial}) = \Xi(\hat{\partial}^c)$;

(E4) $\Xi(\hat{\partial}_2) \geq \Xi(\hat{\partial}_1)$, *if* $\wp(\hat{\partial}_2) \geq \Im(\hat{\partial}_2)$, *there have* $\wp(\hat{\partial}_1) \geq \wp(\hat{\partial}_2)$, $\Im(\hat{\partial}_2) \geq \Im(\hat{\partial}_1)$; *or* $\wp(\hat{\partial}_2) \leq \Im(\hat{\partial}_2)$, *there have* $\wp(\hat{\partial}_1) \leq \wp(\hat{\partial}_2)$, $\Im(\hat{\partial}_2) \leq \Im(\hat{\partial}_1)$.

Definition 16. *Let $\hat{\partial}$ be a PDHFE, we give the following definition of entropy:*

$$\Xi(\hat{\partial}) = 1 - \sin\left|\frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{2(1+\Theta(\hat{\partial}))}\pi\right|. \tag{10}$$

Theorem 3. *Ξ is an entropy measure for PDHFEs and satisfies the four axioms in Definition 15.*

Naturally, let $\hat{O} = \{\langle x_i, \mathcal{G}(x_i)|p(x_i), \kappa(x_i)|q(x_i) \ \ |x_i \in X\rangle\}$ be a PDHFS on X. The entropy $\Xi(\hat{O})$ for PDHFSs is as follows:

$$\Xi(\hat{O}) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \sin\left|\frac{\wp(\hat{O}(x_i)) - \Im(\hat{O}(x_i))}{2(1+\Theta(\hat{O}(x_i)))}\pi\right|\right). \tag{11}$$

5. Applications and comparisons

Solid theoretical support is involved in solving practical problems. In Subsection 5.1, an example of wood recognition demonstrates the proposed method's effectiveness. In Subsection 5.2, VIKOR-based multi-criteria decision-making technology is developed and applied to the example of a multinational company launching new products. The practicability of the proposed method is explained in parameter analysis and comparative analysis. The studied contributions are summarized in Subsection 5.3.

5.1 Application of the proposed method in pattern recognition

Wood recognition is taken as an example to display the application of new distance measures. The new entropy is used as a technique to derive attribute weights. Moreover, the developed method is compared with the method in [18]. The steps of pattern recognition are as follows:

Step 1. Obtain a PDHFS matrix $L = (\mathcal{L}_{ij})_{n\times m}$ from the evaluation of DMs, then standardize the PDHFS matrix.

$$\mathcal{L}_i = \begin{cases} \langle \mathcal{G}|p, \kappa|q\rangle, & for\ any\ benefit\ criterion\ C_i, \\ \langle \kappa|q, \mathcal{G}|p\rangle, & for\ any\ cost\ criterion\ C_i. \end{cases}$$

Step 2. Calculate the attribute weights $\omega(x_i)$ by Eq. (12).

$$\omega(C_i) = \frac{1 - \Xi(C_i)}{n - \sum_{i=1}^{n}\Xi(C_i)}, \tag{12}$$

where

$$\Xi(C_i) = \frac{1}{m}\sum_{j=1}^{m} \Xi(\mathcal{L}_j(C_i)),$$

$\Xi(\mathcal{L}_j(C_i)), (i = 1, 2, ..., n), (j = 1, 2, ..., m)$ can be determined by Eq. (10).
Step 3. Calculate the distance d_{gwpdh} between PDHFSs by Eq. (8).
Step 4. Sort the patterns based on calculation results. The shorter the distance, the closer the pattern.

Example 3. When enterprises use building wood, they to identify the classification of wood to maximize production benefits. The main characteristics of wood include: C_1: sound insulation performance, C_2: processability and C_3: corrosion resistance. There are three standard classifications of wood: $\hat{O}_j (j = 1, 2, 3)$. Now, the classification of wood N needs to be recognized. The PDHFS information for $\hat{O}_j (j = 1, 2, 3)$ and M is provided in Table 3.

Table 3: The PDHFSs information

	C_1	C_2	C_3
$\hat{O}_1$	$\begin{pmatrix}\{0.2\|0.7, 0.4\|0.3\}, \\ \{0.4\|0.6, 0.5\|0.4\}\end{pmatrix}$	$\begin{pmatrix}\{0.55\|0.8, 0.65\|0.2\}, \\ \{0.2\|0.5, 0.1\|0.5\}\end{pmatrix}$	$\begin{pmatrix}\{0.1\|0.45, 0.3\|0.55\}, \\ \{0.5\|0.5, 0.65\|0.5\}\end{pmatrix}$
$\hat{O}_2$	$\begin{pmatrix}\{0.6\|0.2, 0.7\|0.8\}, \\ \{0.2\|0.2, 0.3\|0.8\}\end{pmatrix}$	$\begin{pmatrix}\{0.7\|1\}, \\ \{0.2\|0.8, 0.3\|0.2\}\end{pmatrix}$	$\begin{pmatrix}\{0.1\|0.4, 0.4\|0.6\}, \\ \{0.4\|0.7, 0.6\|0.3\}\end{pmatrix}$
$\hat{O}_3$	$\begin{pmatrix}\{0.6\|1\}, \\ \{0.1\|0.4, 0.3\|0.6\}\end{pmatrix}$	$\begin{pmatrix}\{0.5\|0.3, 0.6\|0.7\}, \\ \{0.3\|1\}\end{pmatrix}$	$\begin{pmatrix}\{0.2\|0.3, 0.4\|0.3, 0.5\|0.4\}, \\ \{0.4\|0.5, 0.5\|0.5\}\end{pmatrix}$
M	$\begin{pmatrix}\{0.2\|0.6, 0.3\|0.4\}, \\ \{0.4\|0.5, 0.5\|0.5\}\end{pmatrix}$	$\begin{pmatrix}\{0.5\|0.4, 0.6\|0.6\}, \\ \{0.1\|0.45, 0.2\|0.55\}\end{pmatrix}$	$\begin{pmatrix}\{0.2\|0.4, 0.3\|0.6\}, \\ \{0.6\|0.7, 0.65\|0.3\}\end{pmatrix}$

Step 1. All attributes are beneficial, so there is no need to be standardized.

Step 2. According to Eqs. (12) and (5.1), we have

$$\Xi(C_1) = 0.6601,\ \Xi(C_2) = 0.6838,\ \Xi(C_2) = 0.6381,$$
$$\omega(C_1) = 0.3339,\ \omega(C_2) = 0.3106,\ \omega(C_1) = 0.3555.$$

Step 3. The distance between $\hat{O}_j (j = 1, 2, 3)$ and M is calculated according to Eq. (8), as shown in Table 4, where $a = b = c = e = \frac{1}{4}$.

Step 4. When $\lambda \in [1, 20]$, all the sorting results are $d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$.

Table 4 and Fig. 1 show that the distance between $\hat{O}_j (j = 1, 2, 3)$ and M increases as λ increases from 1 to 20. This result is that M belongs to $\hat{O}_1$. Furthermore, the ranking order remains the same, that is, $d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$.

Table 4: The ranking for different values of λ

	$d(\hat{O}_1, M)$	$d(\hat{O}_2, M)$	$d(\hat{O}_3, M)$	Ranking
$\lambda = 1$	0.0545	0.1142	0.1450	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 2$	0.0602	0.1262	0.1586	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 6$	0.0747	0.1435	0.1944	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 10$	0.0799	0.1482	0.2084	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 15$	0.0827	0.1508	0.2161	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$
$\lambda = 20$	0.0841	0.1524	0.2201	$d(\hat{O}_1, M) < d(\hat{O}_2, M) < d(\hat{O}_3, M)$

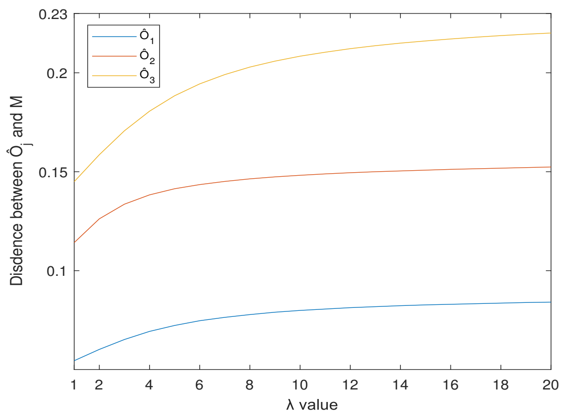


Figure 1: Distance between $\hat{O}_j$ and M with $\lambda \in [1, 20]$

Remark 4. The results using Eq. (6) [18] also show that M belongs to $\hat{O}_1$, but the ranking is $d(\hat{O}_1, M) < d(\hat{O}_3, M) < d(\hat{O}_2, M)$. Such a difference can be attributed to our consideration of weak equality. The new measure can identify different PDHFSs better so that the data information expression is comprehensive and valid.

5.2 Application in VIKOR-based multi-criteria decision-making

5.2.1 VIKOR-based multi-criteria decision-making

VIKOR (Serbian name: Vise Kriterijumski Optimizacioni Resenje) is generalized from the ensemble function L_p-metric [35]. The compromise solution obtained by VIKOR considers the situation in which conflicting attributes cannot be satisfied simultaneously, and the decision is made through mutual concessions. The classical TOPSIS technique only considers the distances from alternatives to positive/negative ideal solutions, while VIKOR maximizes group utility between attributes and minimizes individual regret utility. VIKOR technology is extended to fuzzy environments, such as intuitional fuzzy VIKOR [4], hesitant fuzzy VIKOR [13], hesitant fuzzy linguistic VIKOR [14], etc. Ren et al. [22] proposed a decision technique of integrated VIKOR and the Analytic Hierarchy Process (AHP) for PDHFSs. Based on this, we develop a VIKOR-based MCDM technique. The detailed steps are as follows:

Step 1. Get the MCDM matrix $R = (\rho_{ij})_{n\times m}$. The alternatives $M_j(j = 1, 2, ..., m)$ are evaluated by DMs, where M_j have n attributes.

Step 2. Determine the ρ_i^* (best evaluation information) and ρ_i^- (worst evaluation information) for attributes C_i adopting Eqs. (4-5) and standardize.

$$\rho_i^* = \begin{cases} \max_{j=1,2,..,m}\{\rho_{ij}\}, & \text{for any benefit criterion } C_i \\ \min_{j=1,2,..,m}\{\rho_{ij}\}, & \text{for any cost criterion } C_i \end{cases}$$

$$\rho_i^- = \begin{cases} \min_{j=1,2,..,m}\{\rho_{ij}\}, & \text{for any benefit criterion } C_i \\ \max_{j=1,2,..,m}\{\rho_{ij}\}, & \text{for any cost criterion } C_i \end{cases}$$

Step 3. Calculate the maximum group utility measure G_j and the individual regret measure I_j by Eqs. (13) and (14), where ω_i is the relevant weight of attributes C_i. $d\left(\rho_i^*, \rho_{ij}\right)$ and $d\left(\rho_i^*, \rho_i^-\right)$ is given by Eq. (7).

$$G_j = \sum_{i=1}^{n} \omega_i d\left(\rho_i^*, \rho_{ij}\right) / d\left(\rho_i^*, \rho_i^-\right), \tag{13}$$

$$I_j = \max_i \left[\omega_i d\left(\rho_i^*, \rho_{ij}\right) / d\left(\rho_i^*, \rho_i^-\right)\right], \tag{14}$$

where $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

Step 4. Calculate V_j by Eq. (15).

$$V_j = \theta \frac{(G_j - G^*)}{(G^- - G^*)} + (1-\theta) \frac{(I_j - I^*)}{(I^- - I^*)}, \tag{15}$$

where $G^- = \max_j G_j$, $G^* = \min_j G_j$, $I^- = \max_j I_j$ and $I^* = \min_j I_j$. θ is a parameter, let $\theta = 0.5$.

Step 5. Sort the G_j, I_j and V_j in ascending order.

Step 6. $M^{(1)}$ is an optimal solution if it satisfies the following three conditions:

Condition 1: $M^{(1)}$ has the first minimum in $V_j(j = 1, 2, ...m)$.

Condition 2: $V\left(M^{(2)}\right) - V\left(M^{(1)}\right) \geq \frac{1}{m-1}$, where $M^{(2)}$ has the second minimum in V_j.

Condition 3: $M^{(1)}$ has the first minimum in G_j or I_j.

If conditions 2 and 3 cannot be met simultaneously, go to step 7.

Step 7. If the Condition 2 isn't met. $M^{(\S)}$ satisfies $V\left(M^{(\S)}\right) - V\left(M^{(1)}\right) < \frac{1}{m-1}$, $M^{(1)}$ is an optimal solution, and $M^{(2)}, M^{(3)}, ..., M^{(\S)}$ are compromise solutions. If the Condition 3 isn't met. $M^{(1)}$ is an optimal solution, and $M^{(2)}$ is a compromise solution.

5.2.2 Launching new products by a multinational corporation

Example 4. [[7]]A multinational company wants to launch new products according to consumer buying behaviour. Thus, experts from three countries visited the available information about four products $M_j (j = 1, 2, 3, 4)$ based on consumer purchasing behavior. There are four attributes of consumer buying behaviour: C_1: suitable cultural environment, C_2: conform to global trends, C_3: suitable weather and C_4: excellent after-sales service. The evaluation information of experts is shown in Table 5. Let weights $\omega = (0.4385, 0.1986, 0.1815, 0.1814)$, $\lambda = 1$, and $a = b = c = e = \frac{1}{4}$.

Table 5: The comprehensive PDHFSs matrix

	C_1	C_2	C_3	C_4
M_1	$\left(\begin{array}{c}\left\{\begin{array}{l}0.2\|0.1333, 0.3\|0.3667,\\ 0.5\|0.1667, 0.75\|0.3333\end{array}\right\}\\ \{0.2\|0.3333, 0.4\|0.6667\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.2\|0.3333, 0.45\|0.14,\\ 0.5\|0.3333, 0.6\|0.1934\end{array}\right\}\\ \left\{\begin{array}{l}0.2\|0.3, 0.3\|0.2,\\ 0.5\|0.1667, 0.7\|0.3333\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.2\|0.3333, 0.3\|0.3334,\\ 0.9\|0.3333,\end{array}\right\}\\ \left\{\begin{array}{l}0.1\|0.3333, 0.4\|0.2667,\\ 0.6\|0.4\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\{0.6\|0.9, 0.7\|0.1\},\\ \{0.25\|0.3333, 0.3\|0.6667\}\end{array}\right)$
M_2	$\left(\begin{array}{c}\left\{\begin{array}{l}0.2\|0.3333, 0.6\|0.2334,\\ 0.8\|0.4333\end{array}\right\},\\ \{0.1\|0.6667, 0.7\|0.3333\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2\|0.5, 0.3\|0.5\},\\ \left\{\begin{array}{l}0.15\|0.1667, 0.2\|0.1667,\\ 0.6\|0.3333, 0.7\|0.3333\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.2\|0.3334, 0.6\|0.3333,\\ 0.9\|0.3333\end{array}\right\},\\ \left\{\begin{array}{l}0.1\|0.6667, 0.2\|0.1667,\\ 0.6\|0.1666\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\{0.2\|0.4333, 0.3\|0.5667\},\\ \left\{\begin{array}{l}0.5\|0.1333, 0.6\|0.5333,\\ 0.8\|0.3333\end{array}\right\}\end{array}\right)$
M_3	$\left(\begin{array}{c}\left\{\begin{array}{l}0.05\|0.2334, 0.2\|0.1,\\ 0.4\|0.1333, 0.5\|0.2,\\ 0.9\|0.3333\end{array}\right\},\\ \{0.1\|0.3333, 0.5\|0.6667\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.45\|0.3333, 0.5\|0.3333,\\ 0.6\|0.3334\end{array}\right\},\\ \left\{\begin{array}{l}0.1\|0.1666, 0.2\|0.1667,\\ 0.5\|0.6667\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\{0.6\|0.3333, 0.8\|0.6667\},\\ \left\{\begin{array}{l}0.1\|0.1, 0.15\|0.3333,\\ 0.2\|0.5666\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.1\|0.3333, 0.12\|0.3333,\\ 0.2\|0.3334\end{array}\right\},\\ \left\{\begin{array}{l}0.6\|0.2333, 0.7\|0.3,\\ 0.8\|0.4667\end{array}\right\}\end{array}\right)$
M_4	$\left(\begin{array}{c}\left\{\begin{array}{l}0.3\|0.2333, 0.4\|0.4,\\ 0.5\|0.3667\end{array}\right\},\\ \left\{\begin{array}{l}0.2\|0.1667, 0.3\|0.5,\\ 0.4\|0.2, 0.5\|0.1333\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.1\|0.3334, 0.2\|0.1333,\\ 0.5\|0.5333\end{array}\right\},\\ \left\{\begin{array}{l}0.2\|0.1, 0.3\|0.2667,\\ 0.4\|0.3, 0.8\|0.3333\end{array}\right\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.3\|0.6667, 0.4\|0.0333,\\ 0.5\|0.3\end{array}\right\},\\ \{0.3\|0.6667, 0.65\|0.3333\}\end{array}\right)$	$\left(\begin{array}{c}\left\{\begin{array}{l}0.35\|0.3334, 0.4\|0.3333,\\ 0.5\|0.3333\end{array}\right\},\\ \left\{\begin{array}{l}0.2\|0.1, 0.4\|0.2334,\\ 0.6\|0.6666\end{array}\right\}\end{array}\right)$

Step 1. The MCDM matrix $R = (\rho_{ij})_{n \times m}$ is represented in Table 5.
Step 2. All attributes are beneficial. According to Eqs. (4) and (5), we have

$$
\rho_i^* = \left(\begin{array}{c}
\left\langle \begin{array}{l} \{0.2|0.3333, 0.6|0.2334, 0.8|0.4333\}, \\ \{0.1|0.6667, 0.7|0.3333\} \end{array} \right\rangle \\
\left\langle \begin{array}{l} \{0.45|0.3333, 0.5|0.3333, 0.6|0.3334\}, \\ \{0.1|0.1666, 0.2|0.1667, 0.5|0.6667\} \end{array} \right\rangle, \\
\left\langle \begin{array}{l} \{0.6|0.3333, 0.8|0.6667\}, \\ \{0.1|0.1, 0.15|0.3333, 0.2|0.5666\} \end{array} \right\rangle, \\
\langle \{0.6|0.9, 0.7|0.1\}, \{0.25|0.3333, 0.3|0.6667\} \rangle
\end{array}\right)
$$

$$
\rho_i^- = \left(\begin{array}{c}
\left\langle \begin{array}{l} \{0.3|0.2333, 0.4|0.4, 0.5|0.3667\}, \\ \{0.2|0.1667, 0.3|0.5, 0.4|0.2, 0.5|0.1333\} \end{array} \right\rangle, \\
\left\langle \begin{array}{l} \{0.2|0.5, 0.3|0.5\} \\ , \left\{ \begin{array}{l} 0.15|0.1667, 0.2|0.1667, \\ 0.6|0.3333, 0.7|0.3333 \end{array} \right\} \end{array} \right\rangle \\
\left\langle \begin{array}{l} \{0.3|0.6667, 0.4|0.0333, 0.5|0.3\}, \\ \{0.3|0.6667, 0.65|0.3333\} \end{array} \right\rangle, \\
\left\langle \begin{array}{l} \{0.1|0.3333, 0.12|0.3333, 0.2|0.3334\}, \\ \{0.6|0.2333, 0.7|0.3, 0.8|0.4667\} \end{array} \right\rangle
\end{array}\right).
$$

Step 3. Compute the values of $d\left(\rho_i^*, \rho_i^-\right)$ and $d\left(\rho_i^*, \rho_{ij}\right)$ according to Eq. (7). The values of G_j and I_j are derived by Eqs. (13) and (14), and the full results are presented in Table 6.

Take product M_1 as an example, there is

$$\begin{aligned} d\left(\rho_1^*, \rho_1^-\right) =& \frac{\left(\left|\frac{1}{3}-\frac{1}{3}\right|+\left|\frac{1}{2}-\frac{1}{4}\right|\right)}{8}+\frac{(|0.3689-0.3022|+|0.2666-0.1650|)}{8} \\ &+\frac{(0.0477+0.0667)}{8}+\frac{(|0.5533-0.4133|+|0.3000-0.3300|)}{8} \\ &= 0.0878. \end{aligned}$$

Similarly,

$$\begin{aligned} &d\left(\rho_2^*, \rho_2^-\right)=0.1123, \quad d\left(\rho_3^*, \rho_3^-\right)=0.1708, \quad d\left(\rho_4^*, \rho_4^-\right)=0.2231, \\ &d\left(\rho_1^*, \rho_{11}\right)=0.0876, \quad d\left(\rho_2^*, \rho_{21}\right)=0.1481, \quad d\left(\rho_3^*, \rho_{31}\right)=0.1772, \\ &d\left(\rho_4^*, \rho_{41}\right)=0. \end{aligned}$$

We can get

$$\begin{aligned} G_1 &= \sum_{i=1}^{4} \omega_i d\left(\rho_i^*, \rho_{i1}\right) / d\left(\rho_i^*, \rho_i^-\right) \\ &= 0.4385 \times \frac{0.0876}{0.0878}+0.1986 \times \frac{0.1481}{0.1123}+0.1815 \times \frac{0.1772}{0.1708}+0.1814 \times \frac{0}{0.2231} \\ &= 0.8875, \\ I_1 &= \max_i \left[\omega_i d\left(\rho_i^*, \rho_{i1}\right) / d\left(\rho_i^*, \rho_i^-\right)\right]=0.4372. \end{aligned}$$

Step 4. Calculate the values of V_j according to Eq. (15), where $G^-=\max_j G_j=1.1237$, $G^*=\min_j G_j=0.4775$, $I^-=\max_j I_j=0.5168$, $I^*=\min_j I_j=0.1980$. So,

$$\begin{aligned} V_1 &= \theta \frac{\left(G_1-G^*\right)}{\left(G^- - G^*\right)}+(1-\theta) \frac{\left(I_1-I^*\right)}{\left(I^- - I^*\right)} \\ &= 0.5 \times \frac{0.8875-0.4775}{1.1237-0.4775}+0.5 \times \frac{0.6924-0.1980}{0.5168-0.1980} \\ &= 0.6924. \end{aligned}$$

Similarly, $V_2=0, V_3=0.4321, V_4=1$.

Step 5. Sort G_j, I_j and V_j: $G_2<G_3<G_1<G_4$, $I_2<I_3<I_1<I_4$ and $V_2<V_3<V_1<V_4$.

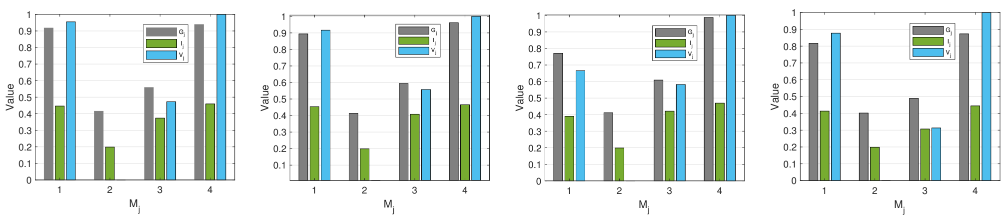
Step 6. According to the ordering in Step 5, M_2 has the first minimum in V_j, $V\left(M_3\right)-V\left(M_2\right)=0.4321 \geq \frac{1}{4-1}=0.3333$. M_2 has again the minimum in G_j and I_j. Thus, the optimal product is M_2.

Table 6: The distance of $\rho_i^*, \rho_i^-, \rho_{ij}$

	C_1	C_2	C_3	C_4	G_j	I_j
	$\omega_1 = 0.4385$	$\omega_2 = 0.1986$	$\omega_3 = 0.1815$	$\omega_4 = 0.1814$		
$d(\rho_i^*, \rho_{i1})$	0.0876	0.1481	0.1772	0	0.8875	0.4372
$d(\rho_i^*, \rho_{i2})$	0	0.1123	0.0644	0.1772	0.4775	0.1980
$d(\rho_i^*, \rho_{i3})$	0.0711	0	0	0.0912	0.6203	0.4031
$d(\rho_i^*, \rho_{i4})$	0.0912	0.0854	0.1951	0.1297	1.1237	0.5168
$d(\rho_i^*, \rho_i^-)$	0.0878	0.1123	0.1708	0.2231	–	–

5.2.3 Parameter analysis

We analyze the influence of distance parameters (a, b, c, e) on decision. First, set (a, b, c, e) to $(0.2, 0.3, 0.3, 0.2)$, $(0.2, 0.4, 0.2, 0.2)$, $(0.3, 0.5, 0.1, 0.1)$ and $(0.4, 0.1, 0.1, 0.4)$ randomly without changing the other parameters. The results are shown in Table 7.



(a) $(0.2, 0.3, 0.3, 0.2)$ (b) $(0.2, 0.4, 0.2, 0.2)$ (c) $(0.3, 0.5, 0.1, 0.1)$ (d) $(0.4, 0.1, 0.1, 0.4)$

Figure 2: Values of M_j under parameters (a, b, c, e)

Table 7 and Fig.2 illustrate that the optimal product is M_2. However, when $(a, b, c, e) = (0.4, 0.1, 0.1, 0.4)$, the optimal product is M_2, followed by M_3. Suppose the company has the strength to support both products. It may launch M_2 and M_3. From the actual situation, consumer demand for products may differ due to cultural differences between countries. Therefore, the results obtained in this paper are reasonable. Furthermore, the proposed method can set appropriate parameters according to the actual situations, with stability and practicality.

5.2.4 Comparative analysis

Our method is compared with three methods: the probabilistic dual hesitant fuzzy ordered weighted Einstein averaging (PDHFOWEA) operator [7], the probabilistic dual hesitant fuzzy weighted average (PDHFWA) operator in [8], and distance measure d_{gpnd} in [18]. The results are given in Table 8.

The PDHFWA operator [8] calculates that the best product is M_2, consistent with our result. The PDHFOWEA operator [7] represents that M_3 is

Table 7: The ranking result with different parameter a, b, c, e

	G_j	I_j	V_j	Ranking
$(a, b, c, e) = (0.2, 0.3, 0.3, 0.2)$				
M_1	0.9177	0.4464	0.9551	
M_2	0.4152	0.1986	0	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_3	0.5585	0.3736	0.4725	
M_4	0.9390	0.4593	1	
$(a, b, c, e) = (0.2, 0.4, 0.2, 0.2)$				
M_1	0.8940	0.4537	0.9165	
M_2	0.4132	0.1986	0	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_3	0.5940	0.4080	0.5574	
M_4	0.9616	0.4654	1	
$(a, b, c, e) = (0.3, 0.5, 0.1, 0.1)$				
M_1	0.7708	0.3903	0.6657	
M_2	0.4113	0.1986	0	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_3	0.6090	0.4212	0.5822	
M_4	0.9872	0.4697	1	
$(a, b, c, e) = (0.4, 0.1, 0.1, 0.4)$				
M_1	0.8167	0.4135	0.8772	
M_2	0.4019	0.1986	0	$M_2 \succ M_3 \succ M_1 \succ M_4$
M_3	0.4893	0.3070	0.3132	
M_4	0.8730	0.4445	1	

Table 8: Comparison of existing methods

References	**Method**	**Ranking**
[7]	PDHFOWEA	$M_3 \succ M_1 \succ M_2 \succ M_4$
[18]	d_{gpnd}	$M_1 \succ M_2 \succ M_3 \succ M_4$
[8]	PDHFWA	$M_2 \succ M_3 \succ M_1 \succ M_4$
Our devised method	d_{pgdh}	$M_2 \succ M_3 \succ M_1 \succ M_4$

the best product. M_1 is the optimal product of d_{gpnd} [18]. Different MCDM methods have their advantages in the fuzzy environment, respectively. Our proposed method considers the importance of overall balance and extreme data, but PDHFOWEA and PDHFWA operators pay more attention to the overall balance. d_{gpnd} [18] only considers the mean and standard deviation of PDHFSs. However, our proposed method considers the influence of consistency and probabilistic fuzziness of PDHFSs, expressing the hidden data information. With the support of weak equality, the measurement results are more reasonable. Therefore, compared with existing methods, the novel method is comprehensive and superior in processing MCDM problems.

5.3 The contribution to previous studies

This paper's theoretical results and applications are relatively outstanding compared to previous research.

Theoretically, weak equality is a new definition between similarity and equality. Secondly, the proposed distance measures are taken as a combination of consistency, probabilistic fuzziness, Hausdorff distance and mean. They are an extension of partially existing research. Moreover, the proposed distance measures satisfy triangular inequality, and this property is only found in a few results. New entropy is relatively simple to calculate and provides a tool for determining attribute weights.

For applications, the distance measures are suitable for dealing with unequal-length data. They also have good applicability when compared with existing studies. Weak equality relaxes the constraint on set equality, allowing for a broader application. These comparisons of several examples show that our approach is practical.

6. Conclusion

The different PDHFSs are perfectly equal at zero distance. It is challenging to guarantee this with existing distance measures. Therefore, we defined weak equality. Considering some properties of PDHFSs, new distance measures were proposed and performed well in distinguishing data. We described a similarity measure in conjunction with the correlation between distance and similarity measures. In addition, we developed a new entropy measure and applied it to finding attribute weights. Finally, a pattern recognition method with distance and entropy measures was developed. Moreover, we combined traditional VIKOR methods with the PDHFS environment to construct a new MCDM technology. Through comparative analysis and parameter analysis, we verified the practicality of our proposed methods.

In the future, we will continue to study the weak equality problem of PDHFSs to achieve complete equality. This question will involve investigating more advanced techniques to improve the accuracy of our method. Additionally, we plan to explore the potential of applying distance measures of PDHFSs to other fields, such as stock risk analysis and medical diagnostics.

Appendix A. The proof of Theorem 1

Lemma 1 ([10]). *Let* $(\xi_1, \xi_2, ..., \xi_z), (\eta_1, \eta_2, ..., \eta_z) \in R^z (i = 1, 2, ..., z)$*, and* $1 \leq \lambda < +\infty$*, then*

$$\left(\sum_{i=1}^{z} |\xi_i + \eta_i|^\lambda\right)^{1/\lambda} \leq \left(\sum_{i=1}^{z} |\xi_i|^\lambda\right)^{1/\lambda} + \left(\sum_{i=1}^{z} |\eta_i|^\lambda\right)^{1/\lambda}.$$

Lemma 2 ([34])**.** *Let $\hat{O}_1$, $\hat{O}_2$ and $\hat{O}_3$ be three HFSs, then*

$$\begin{aligned}&\max\left\{\max_{\gamma_{1j}\in\mathcal{G}_1}\min_{\gamma_{2t}\in\mathcal{G}_2}\|\gamma_{1j}-\gamma_{2t}\|,\max_{\gamma_{2t}\in\mathcal{G}_2}\min_{\gamma_{1j}\in\mathcal{G}_1}\|\gamma_{1j}-\gamma_{2t}\|\right\}\\&+\max\left\{\max_{\gamma_{2t}\in\mathcal{G}_2}\min_{\gamma_{3l}\in\mathcal{G}_3}\|\gamma_{2t}-\gamma_{3l}\|,\max_{\gamma_{3l}\in\mathcal{G}_3}\min_{\gamma_{2t}\in\mathcal{G}_2}\|\gamma_{2t}-\gamma_{3l}\|\right\}\\&\geq\max\left\{\max_{\gamma_{1j}\in\mathcal{G}_1}\min_{\gamma_{3l}\in\mathcal{G}_3}\|\gamma_{1j}-\gamma_{3l}\|,\max_{\gamma_{3l}\in\mathcal{G}_3}\min_{\gamma_{1j}\in\mathcal{G}_1}\|\gamma_{1j}-\gamma_{3l}\|\right\}.\end{aligned}$$

The proof of Theorem 1 is as follows:

Proof. (D1). *Using Eq. (2), know that $\gamma, p, \eta, q \in [0,1]$. It is easily obtained $0 \leq \iota(x_i), \zeta(x_i), \mathrm{T}(x_i), \Upsilon(x_i), \theta(x_i), \psi(x_i), \wp(x_i), \Im(x_i) \leq 1$. Besides, $a+b+c+e=1$, and $\lambda \geq 1$. So , $0 \leq d_{pgdh}(\hat{O}_1, \hat{O}_2) \leq 1$*

(D2). *If $d_{pgdh}(\hat{O}_1, \hat{O}_2) = 0$, from Eq. (7), for any $x_i \in X$, we have $\iota(x_i)$=0, $\zeta(x_i) = 0$, $\mathrm{T}(x_i)$=0, $\Upsilon(x_i)$= 0, $\theta(x_i) = 0$, $\psi(x_i) = 0$, $\wp(x_i) = 0$ and $\Im(x_i) = 0$, then*

$$\begin{cases}\dfrac{1}{\#\mathcal{G}_1}=\dfrac{1}{\#\mathcal{G}_2}\\\dfrac{1}{\#\kappa_1}=\dfrac{1}{\#\kappa_2}\end{cases},\quad\begin{cases}\displaystyle\sum_{j=1}^{\#\mathcal{G}_1}\gamma_{1j}p_{1j}=\sum_{t=1}^{\#\mathcal{G}_2}\gamma_{2t}p_{2t}\\\displaystyle\sum_{j=1}^{\#\kappa_1}\eta_{1j}q_{1j}=\sum_{t=1}^{\#\kappa_2}\eta_{2t}q_{2t}\end{cases},$$

$\max_{\gamma_{1j}\in\mathcal{G}_1}\min_{\gamma_{2t}\in\mathcal{G}_2}\|\gamma_{1j}-\gamma_{2t}\|p_{1j}p_{2t}=0$ *and*

$$\max_{\gamma_{2t}\in\mathcal{G}_2}\min_{\gamma_{1j}\in\mathcal{G}_1}\|\gamma_{1j}-\gamma_{2t}\|p_{1j}p_{2t}=0,$$

where the PDHFEs are sorted in ascending order in this article, thus $\|\gamma_{1j}-\gamma_{2j}\| = 0$, $(j=t)$. Similarly, $\|\eta_{1j}-\eta_{2j}\|=0$, we can get

$$\begin{cases}\#\mathcal{G}_1=\#\mathcal{G}_2\\\#\kappa_1=\#\kappa_2\end{cases},\quad\begin{cases}\gamma_{1j}=\gamma_{2j}\\\eta_{1j}=\eta_{2j}\end{cases}.$$

Besides, for any $x_i \in X$, score function $\lambda\!\!\!^{-}(\hat{O}_1(x_i)) = \lambda\!\!\!^{-}(\hat{O}_2(x_i))$, accuracy function $\ell(\hat{O}_1(x_i)) = \ell(\hat{O}_2(x_i))$, according to Definition 8, we have $\hat{O}_1 =^w \hat{O}_2$. If $\hat{O}_1 =^w \hat{O}_2$, then $d_{pgdh}(\hat{O}_1, \hat{O}_2) = 0$.

(D3). *Straightforward.*

(D4). *By Lemma 2 , we can easily get*

$$\begin{aligned}&\max\left\{\begin{array}{l}\max_{\gamma_{1j}\in\mathcal{G}_1}\min_{\gamma_{2t}\in\mathcal{G}_2}\|\gamma_{1j}-\gamma_{2t}\|p_{1j}p_{2t},\\\max_{\gamma_{2t}\in\mathcal{G}_2}\min_{\gamma_{1j}\in\mathcal{G}_1}\|\gamma_{1j}-\gamma_{2t}\|p_{1j}p_{2t}\end{array}\right\}\\&+\max\left\{\begin{array}{l}\max_{\gamma_{2t}\in\mathcal{G}_2}\min_{\gamma_{3l}\in\mathcal{G}_3}\|\gamma_{2t}-\gamma_{3l}\|p_{2t}p_{3l},\\\max_{\gamma_{3l}\in\mathcal{G}_3}\min_{\gamma_{2t}\in\mathcal{G}_2}\|\gamma_{2t}-\gamma_{3l}\|p_{2t}p_{3l}\end{array}\right\}\\&\geq\max\left\{\begin{array}{l}\max_{\gamma_{1j}\in\mathcal{G}_1}\min_{\gamma_{3l}\in\mathcal{G}_3}\|\gamma_{1j}-\gamma_{3l}\|p_{1j}p_{3l},\\\max_{\gamma_{3l}\in\mathcal{G}_3}\min_{\gamma_{1j}\in\mathcal{G}_1}\|\gamma_{1j}-\gamma_{3l}\|p_{1j}p_{3l}\end{array}\right\}\end{aligned},$$

then $\theta_{\hat{O}_1\hat{O}_2}(x_i) + \theta_{\hat{O}_2\hat{O}_3}(x_i) \geq \theta_{\hat{O}_1\hat{O}_3}(x_i)$. Similarly, $\psi_{\hat{O}_1\hat{O}_2}(x_i) + \psi_{\hat{O}_2\hat{O}_3}(x_i) \geq \psi_{\hat{O}_1\hat{O}_3}(x_i)$.

On the other hand, using Lemma 1 and Eq. (7), we have $d_{pgdh}(Q_1, Q_3) \leq d_{pgdh}(Q_1, Q_2) + d_{pgdh}(Q_2, Q_3)$. □

Appendix B. The proof of Theorem 2

Proof. This is similar to the proof of property (D1) - (D3) of Theorem 1. □

Appendix C. The proof of Theorem 3

Proof. Since Eq. (2), we have $\wp(\hat{\partial}), \Im(\hat{\partial}) \in [0,1]$, thus $\Theta(\hat{\partial}) \in [0,1]$. Again, $\left|\wp(\hat{\partial}) - \Im(\hat{\partial})\right| \leq \left|1 + \Theta(\hat{\partial})\right|$, then $0 \leq \left|\frac{\wp(\hat{\partial})-\Im(\hat{\partial})}{1+\Theta(\hat{\partial})}\right| \leq 1, 0 \leq \left|\frac{\wp(\hat{\partial})-\Im(\hat{\partial})}{2\left(1+\Theta(\hat{\partial})\right)}\pi\right| \leq \frac{\pi}{2}$, thus $0 \leq \Xi(\hat{\partial}) \leq 1$.

(E1) If $\Xi(\hat{\partial}) = 0$, then $\sin\left|\frac{\wp(\hat{\partial})-\Im(\hat{\partial})}{2(1+\Theta(\hat{\partial}))}\pi\right|$=1 $\Leftrightarrow \left|\frac{\wp(\hat{\partial})-\Im(\hat{\partial})}{1+\Theta(\hat{\partial})}\right|$=1, we can get $\wp(\hat{\partial}) - \Im(\hat{\partial}) = 1 + \Theta(\hat{\partial})$ or $\Im(\hat{\partial}) - \wp(\hat{\partial}) = 1 + \Theta(\hat{\partial})$. Because $\Theta(\hat{\partial}) = 1 - \wp(\hat{\partial}) - \Im(\hat{\partial}), \wp(\hat{\partial}), \Im(\hat{\partial}), \Theta(\hat{\partial}) \in [0,1]$, then $\wp(\hat{\partial}) - \Im(\hat{\partial}) = 2 - \wp(\hat{\partial}) - \Im(\hat{\partial})$ or $\Im(\hat{\partial}) - \wp(\hat{\partial}) = 2 - \wp(\hat{\partial}) - \Im(\hat{\partial})$. We have $\wp(\hat{\partial}) = 1, \Im(\hat{\partial})$=0 or $\wp(\hat{\partial}) = 0, \Im(\hat{\partial})$= 1. Therefore, $\Xi(\hat{\partial}) = 0 \Leftrightarrow \hat{\partial} = (\{0\,|1\}, \{1\,|1\})$ or $\hat{\partial} = (\{1\,|1\}, \{0\,|1\})$.

(E2) If $\Xi(\hat{\partial}) = 1$, then $\sin\left|\frac{\wp(\hat{\partial})-\Im(\hat{\partial})}{2(1+\Theta(\hat{\partial}))}\pi\right| = 0 \Leftrightarrow \left|\wp(\hat{\partial}) - \Im(\hat{\partial})\right|$=0 $\Leftrightarrow \wp(\hat{\partial}) = \Im(\hat{\partial})$. Thus, $\Xi(\hat{\partial}) = 1 \Leftrightarrow \wp(\hat{\partial}) = \Im(\hat{\partial})$.

(E3) By Definition (1-2), then the following conclusions hold: $\wp(\hat{\partial}) = \Im(\hat{\partial}^c)$, $\Im(\hat{\partial}) = \wp(\hat{\partial}^c)$, $\Theta(\hat{\partial}) = \Theta(\hat{\partial}^c)$, which means that

$$\Xi(\hat{\partial}^c) = 1 - \sin\left|\frac{\Im(\hat{\partial}^c) - \wp(\hat{\partial}^c)}{2(1 + \Theta(\hat{\partial}^c))}\pi\right| = 1 - \sin\left|\frac{\wp(\hat{\partial}) - \Im(\hat{\partial})}{2(1 + \Theta(\hat{\partial}))}\pi\right| = \Xi(\hat{\partial}).$$

(E4) Let $\wp(\hat{\partial}_2) \geq \Im(\hat{\partial}_2)$, there have $\wp(\hat{\partial}_1) \geq \wp(\hat{\partial}_2)$, $\Im(\hat{\partial}_2) \geq \Im(\hat{\partial}_1)$, then $\wp(\hat{\partial}_1) \geq \Im(\hat{\partial}_1)$. Here we give a suppose. If $\left|\frac{\wp(\hat{\partial}_1)-\Im(\hat{\partial}_1)}{1+\Theta(\hat{\partial}_1)}\right| < \left|\frac{\wp(\hat{\partial}_2)-\Im(\hat{\partial}_2)}{1+\Theta(\hat{\partial}_2)}\right|$, then

$$\begin{aligned}&\left[2 - \wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)\right]\left[\wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)\right] < \left[2 - \wp(\hat{\partial}_1) - \Im(\hat{\partial}_1)\right]\left[\wp(\hat{\partial}_2) - \Im(\hat{\partial}_2)\right]\\ &\Leftrightarrow \wp(\hat{\partial}_2)\Im(\hat{\partial}_1) - \wp(\hat{\partial}_1)\Im(\hat{\partial}_2) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) - \Im(\hat{\partial}_1) + \Im(\hat{\partial}_2) < 0.\end{aligned}$$

We have

$$\left[\wp(\hat{\partial}_2) - 1\right]\Im(\hat{\partial}_1) + \left[1 - \wp(\hat{\partial}_1)\right]\Im(\hat{\partial}_2) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) < 0. \tag{16}$$

Moreover $1 - \wp(\hat{\partial}_1) \geq 0$, $\Im(\hat{\partial}_2) \geq \Im(\hat{\partial}_1)$, then

$$\begin{aligned}&\left[\wp(\hat{\partial}_2) - 1\right]\Im(\hat{\partial}_1) + \left[1 - \wp(\hat{\partial}_1)\right]\Im(\hat{\partial}_2) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2)\\ &\geq \left[\wp(\hat{\partial}_2) - 1\right]\Im(\hat{\partial}_1) + \left[1 - \wp(\hat{\partial}_1)\right]\Im(\hat{\partial}_1) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2)\\ &= \Im(\hat{\partial}_1)\wp(\hat{\partial}_2) - \Im(\hat{\partial}_1)\wp(\hat{\partial}_1) + \wp(\hat{\partial}_1) - \wp(\hat{\partial}_2) = \left[\wp(\hat{\partial}_1) - \wp(\hat{\partial}_2)\right]\left[1 - \Im(\hat{\partial}_1)\right] \geq 0.\end{aligned}$$

This is contradictory to Eq. (16). Therefore,

$$\left|\frac{\wp(\hat{\partial}_1)-\Im(\hat{\partial}_1)}{1+\Theta(\hat{\partial}_1)}\right| \geq \left|\frac{\wp(\hat{\partial}_2)-\Im(\hat{\partial}_2)}{1+\Theta(\hat{\partial}_2)}\right|.$$

When $\wp(\hat{\partial}_2) \leq \Im(\hat{\partial}_2)$, there is $\wp(\hat{\partial}_1) \leq \wp(\hat{\partial}_2)$, $\Im(\hat{\partial}_2) \leq \Im(\hat{\partial}_1)$, similarly

$$\left|\frac{\wp(\hat{\partial}_1)-\Im(\hat{\partial}_1)}{1+\Theta(\hat{\partial}_1)}\right| \geq \left|\frac{\wp(\hat{\partial}_2)-\Im(\hat{\partial}_2)}{1+\Theta(\hat{\partial}_2)}\right|,$$

then $0 \leq \left|\frac{\wp(\hat{\partial}_2)-\Im(\hat{\partial}_2)}{2\left(1+\Theta(\hat{\partial}_2)\right)}\pi\right| \leq \left|\frac{\wp(\hat{\partial}_1)-\Im(\hat{\partial}_1)}{2\left(1+\Theta(\hat{\partial}_1)\right)}\pi\right| \leq \frac{\pi}{2}$. Form the monotonicity of the sine function on the interval $\left[0, \frac{\pi}{2}\right]$, we can easily obtain

$$\sin\left|\frac{\wp(\hat{\partial}_2)-\Im(\hat{\partial}_2)}{2\left(1+\Theta(\hat{\partial}_2)\right)}\pi\right| \leq \sin\left|\frac{\wp(\hat{\partial}_1)-\Im(\hat{\partial}_1)}{2\left(1+\Theta(\hat{\partial}_1)\right)}\pi\right|,$$

then $1-\sin\left|\frac{\wp(\hat{\partial}_2)-\Im(\hat{\partial}_2)}{2\left(1+\Theta(\hat{\partial}_2)\right)}\pi\right| \geq 1-\sin\left|\frac{\wp(\hat{\partial}_1)-\Im(\hat{\partial}_1)}{2\left(1+\Theta(\hat{\partial}_1)\right)}\pi\right|$. So, $\Xi(\hat{\partial}_2) \geq \Xi(\hat{\partial}_1)$. □

Acknowledgments

This work is supported by the Institute of Meteorological Big Data-Digital Fujian, the Fujian Key Laboratory of Data Science and Statistics, the National Natural Science Foundation Project of China (No. 11871259) and the Natural Science Foundation of Fujian Province (No. 2020J02043, No. 2020J01801, No. 2021J01984).

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Accepted: February 2, 2024