The MPBT inverse of a complex matrix based on the Hartwig-Spindelböck decomposition

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Abstract. Let A be a square complex matrix. A new generalized inverse of A is introduced by using the Moore-Penrose inverse and B-T inverse of A, named the MPBT inverse of A. The formula of the MPBT inverse can be got by using the Hartwig-Spindelböck decomposition of A. A relationship between the MPBT inverse and the inverse along two given matrix is investigated.

Keywords: Moore-Penrose inverse, B-T inverse, Core-EP decomposition, Hartwig-Spindelböck decomposition.

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1. Introduction

Let $\mathbb{C}$ be the complex field. The symbol $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices over $\mathbb{C}$. Let $A \in \mathbb{C}^{m\times n}$. The symbol A^* denotes the conjugate transpose of A. The notations $\mathcal{R}(A) = \{y \in \mathbb{C}^m : y = Ax, x \in \mathbb{C}^n\}$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^n : Ax = 0\}$ will be used in the sequel. The smallest positive integer k such that $\mathrm{rank}\,(A^k) = \mathrm{rank}(A^{k+1})$ is called the index of $A \in \mathbb{C}^{n\times n}$ and denoted by $\mathrm{ind}(A)$.

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Let $A \in \mathbb{C}^{m\times n}$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies

$$AXA = A,\ XAX = X,\ (AX)^* = AX \text{ and } (XA)^* = XA,$$

then X is called the Moore-Penrose inverse of A ([16, 20]) and denoted by $X = A^{\dagger}$. If $AXA = A$ and $(AX)^* = AX$ holds, then X is called a $\{1,3\}$-inverse of A and the set of all $\{1,3\}$-inverse of A is denoted by $A\{1,3\}$. If $XAX = X$ and $(XA)^* = XA$ holds, then X is called a $\{2,4\}$-inverse of A and the set of all $\{2,4\}$-inverse of A is denoted by $A\{2,4\}$. Let $A, X \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. The definition of the Drazin inverse is as follows:

$$AXA = A,\ XA^{k+1} = A^k \text{ and } AX = XA,$$

then X is called the Drazin inverse of A. It is unique and denoted by A^D ([9]). If $\operatorname{ind}(A) \leqslant 1$, X is called the group inverse of A and denoted by $A^{\#}$.

The core inverse for a complex matrix were introduced by Baksalary and Trenkler [5]. Let $A \in \mathbb{C}^{n\times n}$. A matrix $X \in \mathbb{C}^{n\times n}$ is called a core inverse of A, if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where P_A is the orthogonal projector onto $\mathcal{R}(A)$. And if such a matrix exists, then it is unique (and denoted by $A^{\#\!\!\!\bigcirc}$). Baksalary and Trenkler gave several characterizations of the core inverse by using the decomposition of Hartwig and Spindelböck [12]. Manjunatha Prasad and Mohana [17] introduced the core-EP inverse of matrix [17, Definition 3.1]. Let $A \in \mathbb{C}^{n\times n}$. If there exists $X \in \mathbb{C}^{n\times n}$ such that $XAX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, then X is called the core-EP inverse of A. If such inverse exists, then it is unique and denoted by $A^{\textcircled{\dagger}}$. The weak group inverse of a complex matrix was introduced by Wang and Chen in [25], which is the unique matrix X suck that $AX^2 = X$ and $AX = A^{\textcircled{\dagger}}A$ and denoted by $X = A^{\textcircled{w}}$. The CMP inverse for a complex matrix was introduced by Mehdipour and Salemi [18]. Let $A \in \mathbb{C}^{n\times n}$. Mehdipour and Salemi [18] introduced the CMP inverse of A by using the core part A_1 of A and the Moore-Penrose inverse $A^{\dagger}$ of A. The *CMP inverse* of A is a matrix $X \in \mathbb{C}^{n\times n}$ such that the following equations hold:

$$XAX = X,\ AXA = A_1,\ AX = A_1A^{\dagger} \text{ and } XA = A^{\dagger}A_1.$$

Such matrix X is unique and denoted by $A^{c,\dagger}$. The CMP inverse can be regarded as a tool to study the core part of the core-nilpotent decomposition of a matrix. The concept of the MPCEP-inverse of a Hilbert space operators was initially introduced by Chen, Mosić and Xu [8] and this concept was expanded on quaternion matrices by Kyrchei, Mosić and Stanimirović [13, 14]. Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If there exists a matrix $X \in \mathbb{C}^{n\times n}$ such that

$$XAX = X,\ AX = AA^{\textcircled{\dagger}} \text{ and } XA = A^{\dagger}AA^{\textcircled{\dagger}}A$$

then, X is called the MPCEP-inverse of A and denoted by $A^{\dagger,\textcircled{\dagger}}$. The MPWC inverse of A was introduced by Liu, Miu and Jin [15] by using the weak group inverse of A and denoted by A°. Moreover, several characterizations of different generalized inverses along the core parts of three matrix decompositions can be found in [7].

2. Some matrix inverses based on the generalized inverses of ΣK

Every matrix $A \in \mathbb{C}^{n\times n}$ of rank r can be represented in the form

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \sigma_1 I_{r_1} \oplus \sigma_2 I_{r_2} \oplus \cdots \oplus \sigma_t I_{r_t}$ is the diagonal matrix of the nonzero singular values of A, where $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ satisfy

$$KK^* + LL^* = I_r.$$

The decomposition in (1) is known as the Hartwig-Spindelböck decomposition [12].

The B-T inverse of A was introduced by Baksalary and Trenkler [6, Definition 1], which is the Moore-Penrose of $A^2A^\dagger$ and denoted by $A^\diamond$. One can see that the B-T inverse of A is an outer inverse of A ([6, Corollary]). The B-T inverse can be characterized by the Moore-Penorse inverse of ΣK ([6, Lemma 2]), that is the B-T inverse of A is

$$A^\diamond = U \begin{bmatrix} (\Sigma K)^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2}$$

Lemma 2.1 ([12, Corollary 6(a)])**.** *Let $A \in \mathbb{C}^{n\times n}$ be represented as in* (1)*. Then A is group invertible if and only if K is nonsingular.*

Lemma 2.2 ([4, p2799 (1.4)])**.** *Let $A \in \mathbb{C}^{n\times n}$ be represented as in* (1)*. Then*

$$A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*. \tag{3}$$

By Lemma 2.1 and Lemma 2.2, we have the following proposition.

Proposition 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be represented as in* (1) *and be group invertible. Then*

$$A^\dagger = U \begin{bmatrix} (K^*K)(\Sigma K)^{-1} & 0 \\ (L^*K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^*. \tag{4}$$

Proof. By Lemma 2.2, we have

$$A^\dagger = U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^*. \tag{5}$$

The condition A is group invertible gives K is nonsingular by Lemma 2.1, thus

$$\begin{aligned} A^\dagger &= U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^* = U \begin{bmatrix} K^*KK^{-1}\Sigma^{-1} & 0 \\ L^*KK^{-1}\Sigma^{-1} & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (K^*K)(K^{-1}\Sigma^{-1}) & 0 \\ (L^*K)(K^{-1}\Sigma^{-1}) & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (K^*K)(\Sigma K)^{-1} & 0 \\ (L^*K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^*. \end{aligned} \tag{6}$$

□

There are three generalizations of the Proposition 2.1, that is, the CMP inverse, MPWC inverse and MPCEP inverse.

Lemma 2.3 ([18, p.3 (7)]). *Let* $A \in \mathbb{C}^{n\times n}$ *be represented as in* (1) *with* $\operatorname{ind}(A) = k$. *Then*

$$A^{c,\dagger} = U \begin{bmatrix} (K^*K)(\Sigma K)^D & 0 \\ (L^*K)(\Sigma K)^D & 0 \end{bmatrix} U^*. \tag{7}$$

Lemma 2.4 ([15, Lemma 3.1]). *Let* $A \in \mathbb{C}^{n\times n}$ *be represented as in* (1) *with* $\operatorname{ind}(A) = k$. *Then*

$$A^{\circ} = U \begin{bmatrix} (K^*K)(\Sigma K)^{ⓦ} & 0 \\ (L^*K)(\Sigma K)^{ⓦ} & 0 \end{bmatrix} U^*. \tag{8}$$

Lemma 2.5 ([30, Theorem 3.2]). *Let* $A \in \mathbb{C}^{n\times n}$ *be represented as in* (1) *with* $\operatorname{ind}(A) = k$. *Then*

$$A^{\dagger,\textcircled{\dagger}} = U \begin{bmatrix} (K^*K)(\Sigma K)^{\textcircled{\dagger}} & 0 \\ (L^*K)(\Sigma K)^{\textcircled{\dagger}} & 0 \end{bmatrix} U^*. \tag{9}$$

Lemma 2.6 ([28, Lemma 3.3]). *Let* $A \in \mathbb{C}^{n\times n}$ *be represented as in* (1). *Then,* $\operatorname{ind}(A) = k$ *if and only if* $\operatorname{ind}(\Sigma K) = k - 1$.

In Proposition 2.1, Lemma 2.3 and Lemma 2.4, we present that the Moore-Penrose inverse, CMP inverse and MPWC inverse can relate the inverse, Drazin inverse and weak core inverse of ΣK, respectively. In general, the group inverse and Moore-Penrose inverse are two classical generalized inverses, there are two matrix inverses can relate the group inverse and Moore-Penrose inverse of ΣK, respectively. For the matrix inverse relate the group inverse of ΣK, we can prove this matrix inverse is a very special inverse, which can be showed as follows:

Let

$$X = U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^*, \tag{10}$$

which says that ΣK is group invertible, then $\operatorname{ind}(\Sigma K) = 0$ by Σ and K are nonsingular, thus $\operatorname{ind}(A) = 1$ by Lemma 2.6. Then, (10) can be written as

$$\begin{aligned} U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^* &= U \begin{bmatrix} (K^*K)(\Sigma K)^{-1} & 0 \\ (L^*K)(\Sigma K)^{-1} & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (K^*K)(K^{-1}\Sigma^{-1}) & 0 \\ (L^*K)(K^{-1}\Sigma^{-1}) & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} K^*KK^{-1}\Sigma^{-1} & 0 \\ L^*KK^{-1}\Sigma^{-1} & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix} U^* \\ &= A^{\dagger}. \end{aligned} \tag{11}$$

In a similar way as in (11), for the core inverse of A, we have

$$U\begin{bmatrix}(K^*K)(\Sigma K)^{\oplus} & 0\\ (L^*K)(\Sigma K)^{\oplus} & 0\end{bmatrix}U^* = U\begin{bmatrix}(K^*K)(\Sigma K)^{-1} & 0\\ (L^*K)(\Sigma K)^{-1} & 0\end{bmatrix}U^* = U\begin{bmatrix}K^*\Sigma^{-1} & 0\\ L^*\Sigma^{-1} & 0\end{bmatrix}U^* = A^{\dagger}. \tag{12}$$

Lemma 2.7. *Let $A \in \mathbb{C}^{n\times n}$ and be group invertible. If A has the Hartwig-Spindelböck decomposition as given in*(1), *then*

$$A^{\dagger} = U\begin{bmatrix}(K^*K)(\Sigma K)^{-1} & 0\\ (L^*K)(\Sigma K)^{-1} & 0\end{bmatrix}U^* = U\begin{bmatrix}(K^*K)(\Sigma K)^{\#} & 0\\ (L^*K)(\Sigma K)^{\#} & 0\end{bmatrix}U^* = U\begin{bmatrix}(K^*K)(\Sigma K)^{\oplus} & 0\\ (L^*K)(\Sigma K)^{\oplus} & 0\end{bmatrix}U^*. \tag{13}$$

Proof. It is trivial by Proposition 2.1, equalities (11) and (12). □

Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = 2$ and has the Hartwig-Spindelböck decomposition as given in (1). Then, $\operatorname{ind}(A) = 2$ if and only if $\operatorname{ind}(\Sigma K) = 1$ by Lemma 2.6. In the following example, we will show that $U\begin{bmatrix}(K^*K)(\Sigma K)^{\#} & 0\\ (L^*K)(\Sigma K)^{\#} & 0\end{bmatrix}U^*$ is different from the $U\begin{bmatrix}(K^*K)(\Sigma K)^{\oplus} & 0\\ (L^*K)(\Sigma K)^{\oplus} & 0\end{bmatrix}U^*$.

Example 2.1. Let $A = \begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 1 & 1 & \frac{1}{2}\end{bmatrix} \in \mathbb{C}^{3\times 3}$, then it is easy to check that $\operatorname{ind}(A) = 2$. The Singular Value Decomposition of A is

$$A = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^*,$$

where unitary maytix $U = \begin{bmatrix}-\frac{519}{1025} & \frac{1210}{1593} & -\frac{881}{2158}\\ -\frac{452}{703} & -\frac{197}{304} & -\frac{881}{2158}\\ -\frac{739}{1286} & \frac{270}{4841} & \frac{881}{1079}\end{bmatrix}$, $\Sigma = \begin{bmatrix}\frac{715}{274} & 0\\ 0 & \frac{683}{1029}\end{bmatrix}$ and unitary maytix $V = \begin{bmatrix}-\frac{519}{1025} & \frac{1210}{1593} & -\frac{881}{2158}\\ -\frac{452}{703} & -\frac{197}{304} & -\frac{881}{2158}\\ -\frac{739}{1286} & \frac{270}{4841} & \frac{881}{1079}\end{bmatrix}$. By

$$V^*U = \begin{bmatrix}\frac{242}{251} & -\frac{118}{1261} & \frac{410}{1651}\\ \frac{153}{619} & -\frac{244}{10171} & -\frac{402}{415}\\ -\frac{327}{3385} & -\frac{1064}{1069} & 0\end{bmatrix} \triangleq \begin{bmatrix}K & L\\ M & N\end{bmatrix},$$

where $K = \begin{bmatrix} \frac{242}{251} & -\frac{118}{1261} \\ \frac{153}{619} & -\frac{244}{10171} \end{bmatrix}, L = \begin{bmatrix} \frac{410}{1651} \\ -\frac{402}{415} \end{bmatrix}, M = \begin{bmatrix} -\frac{327}{3385} & -\frac{1064}{1069} \end{bmatrix}, N = 0.$
It is easy to check that

$$(\Sigma K)^{\#} = \begin{bmatrix} \frac{316}{785} & -\frac{247}{6322} \\ \frac{178}{6781} & -\frac{61}{23943} \end{bmatrix} \text{ and } (\Sigma K)^{\textcircled{\#}} = \begin{bmatrix} \frac{2775}{6967} & \frac{421}{16209} \\ \frac{421}{16209} & \frac{54}{31883} \end{bmatrix}.$$

Thus,

$$U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^* = \begin{bmatrix} \frac{195}{1463} & \frac{933}{6998} & \frac{323}{2423} \\ \frac{933}{6998} & \frac{181}{1357} & \frac{447}{3352} \\ \frac{323}{2423} & \frac{447}{3352} & \frac{2319}{17393} \end{bmatrix};$$
$$U \begin{bmatrix} (K^*K)(\Sigma K)^{\textcircled{\#}} & 0 \\ (L^*K)(\Sigma K)^{\textcircled{\#}} & 0 \end{bmatrix} U^* = \begin{bmatrix} \frac{8}{77} & \frac{12}{77} & \frac{10}{77} \\ \frac{12621}{121477} & \frac{12}{77} & \frac{10}{77} \\ \frac{8}{77} & \frac{12}{77} & \frac{10}{77} \end{bmatrix},$$

which says that the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^*$ is different from the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\textcircled{\#}} & 0 \\ (L^*K)(\Sigma K)^{\textcircled{\#}} & 0 \end{bmatrix} U^*$.

Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}\,(A) = 2$ and has the Hartwig-Spindelböck decomposition as given in (1). In the following proposition, we will show that if ΣK is an EP matrix, then the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^*$ coincides with the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\textcircled{\#}} & 0 \\ (L^*K)(\Sigma K)^{\textcircled{\#}} & 0 \end{bmatrix} U^*$.

Proposition 2.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}\,(A) = 2$ and has the Hartwig-Spindelböck decomposition as given in* (1)*. If ΣK is an EP matrix, then the matrix inverse $U \begin{bmatrix} (K^*K)(\Sigma K)^{\#} & 0 \\ (L^*K)(\Sigma K)^{\#} & 0 \end{bmatrix} U^*$ coincides with the matrix inverse*

$$U \begin{bmatrix} (K^*K)(\Sigma K)^{\textcircled{\#}} & 0 \\ (L^*K)(\Sigma K)^{\textcircled{\#}} & 0 \end{bmatrix} U^*.$$

Proof. It is trivial by [22, Theorem 3.1]. □

Let $A \in \mathbb{C}^{n\times n}$. A generalized inverse relate the Moore-Penrose inverse of ΣK was introduced by using the the Moore-Penrose inverse and B-T inverse of A, named the MPBT inverse of A. Let $A \in \mathbb{C}^{n\times n}$ be represented as in (1). We will prove that the formula of the MPBT inverse of A is

$$X = U \begin{bmatrix} (K^*K)(\Sigma K)^{\dagger} & 0 \\ (L^*K)(\Sigma K)^{\dagger} & 0 \end{bmatrix} U^*. \tag{14}$$

Note that the formula in (14) can be found in Theorem 3.2.

3. The matrix inverse based on the Moore-Penrose inverse and B-T inverse

Motivated by the definition of the MPWC inverse [15], we introduce the MPBT inverse by using the Moore-Penrose inverse and B-T inverse.

Let $A_1^\diamond = AA^\diamond A$, where $A^\diamond$ is the B-T inverse of A.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$. The matrix $X = A^\dagger A_1^\diamond A^\dagger$ is the unique matrix that satisfies the following system of equations*

$$XAX = X, \ \ XA = A^\dagger A_1^\diamond \text{ and } AX = A_1^\diamond A^\dagger. \tag{15}$$

Proof. Let $X = A^\dagger A_1^\diamond A^\dagger$. Note that $A^\diamond AA^\diamond = A^\diamond$ by [6, Corollary 1]. Then

$$\begin{aligned}XAX &= A^\dagger A_1^\diamond A^\dagger AA^\dagger A_1^\diamond A^\dagger = A^\dagger A_1^\diamond A^\dagger A_1^\diamond A^\dagger = A^\dagger AA^\diamond AA^\dagger AA^\diamond AA^\dagger\\ &= A^\dagger AA^\diamond AA^\diamond AA^\dagger = A^\dagger AA^\diamond AA^\dagger = A^\dagger A_1^\diamond A^\dagger = X;\\ XA &= A^\dagger A_1^\diamond A^\dagger A = A^\dagger AA^\diamond AA^\dagger A = A^\dagger AA^\diamond A = A^\dagger A_1^\diamond;\\ AX &= AA^\dagger A_1^\diamond A^\dagger = AA^\dagger AA^\diamond AA^\dagger = AA^\diamond AA^\dagger = A_1^\diamond A^\dagger,\end{aligned}$$

which says that X is a solution of system (15). Let X_1 and X_2 are two candidates of system (15), then

$$X_1 = X_1AX_1 = X_1A_1^\diamond A^\dagger = X_1AX_2 = A^\dagger A_1^\diamond X_2 = X_2AX_2 = X_2,$$

thus X is unique. □

Definition 3.1. *Let $A \in \mathbb{C}^{n\times n}$. The solution of the system* (15) *is called the Moore-Penrose B-T inverse, the MPBT inverse is short for the Moore-Penrose B-T inverse, denoted by $A^{\dagger,\diamond}$.*

Example 3.1. In general, the MPBT inverse is different from the MPWC inverse, CMP inverse, MPCEP inverse and Moore-Penrose inverse. Let $A = \begin{bmatrix} 0 & 0 & 2 & 1\\ 0 & 0 & -1 & 0\\ 0 & 3 & 2 & 3\\ 0 & 3 & 3 & 3\end{bmatrix} \in \mathbb{C}^{4\times 4}$. Then

$$A^{\dagger,\diamond} = \begin{bmatrix} 0 & 0 & 0 & 0\\ -1 & -\frac{3}{62} & \frac{15}{62} & \frac{9}{31}\\ 0 & -\frac{1}{62} & \frac{5}{62} & \frac{3}{31}\\ 1 & \frac{1}{31} & -\frac{5}{31} & -\frac{6}{31}\end{bmatrix},$$

however

$$A^{\diamond}=\begin{bmatrix}0&0&0&0\\0&-\frac{163}{13545}&-\frac{10}{387}&-\frac{187}{13545}\\0&\frac{163}{2709}&\frac{50}{387}&\frac{187}{2709}\\0&\frac{326}{4515}&\frac{20}{129}&\frac{374}{4515}\end{bmatrix},\ A^{c,\dagger}=\begin{bmatrix}0&0&0&0\\0&-\frac{1}{75}&-\frac{2}{75}&-\frac{1}{75}\\0&\frac{1}{15}&\frac{2}{15}&\frac{1}{15}\\0&\frac{2}{25}&\frac{4}{25}&\frac{2}{25}\end{bmatrix},$$

$$A^{\dagger,\textcircled{\dagger}}=\begin{bmatrix}0&0&0&0\\-\frac{8}{903}&\frac{5}{1806}&-\frac{25}{1806}&-\frac{5}{301}\\\frac{40}{903}&-\frac{25}{1806}&\frac{125}{1806}&\frac{25}{301}\\\frac{16}{301}&-\frac{5}{301}&\frac{25}{301}&\frac{30}{301}\end{bmatrix},\ A^{\dagger}=\begin{bmatrix}0&0&0&0\\-1&-\frac{7}{9}&-\frac{2}{9}&\frac{5}{9}\\0&-\frac{2}{3}&-\frac{1}{3}&\frac{1}{3}\\1&\frac{4}{3}&\frac{2}{3}&-\frac{2}{3}\end{bmatrix}.$$

Theorem 3.2. *Let $A\in\mathbb{C}^{n\times n}$. If A has the Hartwig-Spindelböck decomposition as given in* (1)*, then*

$$A^{\dagger,\diamond}=U\begin{bmatrix}(K^*K)(\Sigma K)^{\dagger}&0\\(L^*K)(\Sigma K)^{\dagger}&0\end{bmatrix}U^*. \tag{16}$$

Proof. By Theorem 3.1, we have $A^{\dagger}A_1^{\diamond}A^{\dagger}$ is the MPBT inverse of A, thus

$$\begin{aligned}A^{\dagger}A_1^{\diamond}A^{\dagger}=&A^{\dagger}AA^{\diamond}AA^{\dagger}\\=&U\begin{bmatrix}K^*\Sigma^{-1}&0\\L^*\Sigma^{-1}&0\end{bmatrix}U^*U\begin{bmatrix}\Sigma K&\Sigma L\\0&0\end{bmatrix}U^*U\begin{bmatrix}(\Sigma K)^{\dagger}&0\\0&0\end{bmatrix}U^*\\&U\begin{bmatrix}\Sigma K&\Sigma L\\0&0\end{bmatrix}U^*U\begin{bmatrix}K^*\Sigma^{-1}&0\\L^*\Sigma^{-1}&0\end{bmatrix}U^*\\=&U\begin{bmatrix}(K^*K)(\Sigma K)^{\dagger}&0\\(L^*K)(\Sigma K)^{\dagger}&0\end{bmatrix}U^*.\end{aligned}$$

□

Let $A,B,C\in\mathbb{C}^{n\times n}$. We say that $Y\in\mathbb{C}^{n\times n}$ is a (B,C)-inverse of A if we have $YAB=B$, $CAY=C$, $\mathcal{N}(C)\subseteq\mathcal{N}(Y)$ and $\mathcal{R}(Y)\subseteq\mathcal{R}(B)$. If such Y exists, then it is unique (see, [1, Definition 4.1] and [21, Definition 1.2]), we also call the (B,C)-inverse of A is the inverse of A along B and C. Note that the (B,C)-inverse was introduced in the setting of semigroups [10]. The (B,C)-inverse of A will be denoted by $A^{\|(B,C)}$. Note that Bapat et al. investigated an outer inverse in [2, Theorem 5] is exactly the same as the (y,x)-inverse, where x and y are element in a semigroup. In [23], Rao and Mitra showed that $A^{\|(B,C)}=B(CAB)^{-}C$, where $(CAB)^{-}$ stands for arbitrary inner inverse of CAB.

Theorem 3.3. *Let $A\in\mathbb{C}^{n\times n}$. The MPBT inverse of A is the $(A^{\dagger}A_1^{\diamond},A_1^{\diamond}A^{\dagger})$-inverse of A.*

Proof. Let $Y=A^{\dagger}A_1^{\diamond}A^{\dagger}$ be the MPBT inverse of A. Then

$$\begin{aligned}YA(A^{\dagger}A_1^{\diamond})&=A^{\dagger}A_1^{\diamond}A^{\dagger}AA^{\dagger}A_1^{\diamond}=A^{\dagger}A_1^{\diamond}A^{\dagger}A_1^{\diamond}=A^{\dagger}AA^{\diamond}AA^{\dagger}AA^{\diamond}A\\&=A^{\dagger}AA^{\diamond}AA^{\diamond}A=A^{\dagger}AA^{\diamond}A=A^{\dagger}A_1^{\diamond};\\(A_1^{\diamond}A^{\dagger})AY&=A_1^{\diamond}A^{\dagger}AA^{\dagger}A_1^{\diamond}A^{\dagger}=A_1^{\diamond}A^{\dagger}A_1^{\diamond}A^{\dagger}=AA^{\diamond}AA^{\dagger}AA^{\diamond}AA^{\dagger}\\&=AA^{\diamond}AA^{\diamond}AA^{\dagger}=AA^{\diamond}AA^{\dagger}=A_1^{\diamond}A^{\dagger}.\end{aligned}$$

For any $u \in \mathcal{N}(A_1^\diamond A^\dagger)$, then we have $Yu = A^\dagger A_1^\diamond A^\dagger u = A^\dagger(A_1^\diamond A^\dagger u) = 0$, which gives $\mathcal{N}(A_1^\diamond A^\dagger) \subseteq \mathcal{N}(Y)$. $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A_1^\diamond)$ is trivial. Thus, the $(A^\dagger A_1^\diamond, A_1^\diamond A^\dagger)$-inverse of A by the definition of the (B, C)-inverse. □

In the following proposition, we will give some properties of the MPBT inverse and the matrix $A_1^\diamond$.

Proposition 3.1. *Let $A \in \mathbb{C}^{n\times n}$, $A_1^\diamond = AA^\diamond A$ and $A^{\dagger,\diamond}$ be the MPBT inverse of A, where $A^\diamond$ is the B-T inverse of A. If A has the Hartwig-Spindelböck decomposition as given in*(1)*, then*

(1) $AA^{\dagger,\diamond} = AA^\diamond = A_1^\diamond A^\dagger = U\begin{bmatrix}(\Sigma K)(\Sigma K)^\dagger & 0\\ 0 & 0\end{bmatrix}U^*$;

(2) $A^{\dagger,\diamond}A = A^\dagger A_1^\diamond = U\begin{bmatrix}K^*K & K^*K(\Sigma K)^\dagger\Sigma L\\ L^*K & L^*K(\Sigma K)^\dagger\Sigma L\end{bmatrix}U^*$;

(3) $A_1^\diamond A^\dagger = (A_1^\diamond A^\dagger)^2 = (A_1^\diamond A^\dagger)^*$;

(4) $A^\dagger A_1^\diamond = (A^\dagger A_1^\diamond)^2$;

(5) $A_1^\diamond A^\dagger A_1^\diamond = A_1^\diamond$;

(6) $\mathcal{R}(A^\dagger A_1^\diamond) = \mathcal{R}(A^\dagger AA^\diamond) = \mathcal{R}(A^\dagger A^2(A^\dagger)^2)$;

(7) $\mathcal{N}(A_1^\diamond A^\dagger) = \mathcal{N}(A^\diamond AA^\dagger) = \mathcal{N}(AA^\diamond)$.

Proof. (1). By Theorem 3.2 and equality (2), note that $KK^* + LL^* = I_r$, then we have

$$\begin{aligned}
AA^{\dagger,\diamond} &= U\begin{bmatrix}\Sigma K & \Sigma L\\ 0 & 0\end{bmatrix}U^*U\begin{bmatrix}(K^*K)(\Sigma K)^\dagger & 0\\ (L^*K)(\Sigma K)^\dagger & 0\end{bmatrix}U^*\\
&= U\begin{bmatrix}\Sigma K(K^*K)(\Sigma K)^\dagger + \Sigma L(L^*K)(\Sigma K)^\dagger & 0\\ 0 & 0\end{bmatrix}U^*\\
&= U\begin{bmatrix}\Sigma(KK^* + LL^*)K(\Sigma K)^\dagger & 0\\ 0 & 0\end{bmatrix}U^*\\
&= U\begin{bmatrix}(\Sigma K)(\Sigma K)^\dagger & 0\\ 0 & 0\end{bmatrix}U^*;\\
AA^\diamond &= U\begin{bmatrix}\Sigma K & \Sigma L\\ 0 & 0\end{bmatrix}U^*U\begin{bmatrix}(\Sigma K)^\dagger & 0\\ 0 & 0\end{bmatrix}U^* = U\begin{bmatrix}(\Sigma K)(\Sigma K)^\dagger & 0\\ 0 & 0\end{bmatrix}U^*.
\end{aligned}$$

Since $AA^{\dagger,\diamond} = AA^\dagger A_1^\diamond A^\dagger = AA^\dagger AA^\diamond AA^\dagger AA^\diamond AA^\dagger = A^\dagger A_1^\diamond$, so

$$A_1^\diamond A^\dagger = U\begin{bmatrix}(\Sigma K)(\Sigma K)^\dagger & 0\\ 0 & 0\end{bmatrix}U^*.$$

(2). By Theorem 3.2, we have

$$\begin{aligned}
A^{\dagger,\diamond}A &= U\begin{bmatrix}(K^*K)(\Sigma K)^\dagger & 0\\ (L^*K)(\Sigma K)^\dagger & 0\end{bmatrix}U^*U\begin{bmatrix}\Sigma K & \Sigma L\\ 0 & 0\end{bmatrix}U^*\\
&= U\begin{bmatrix}(K^*K)(\Sigma K)^\dagger\Sigma K & (K^*K)(\Sigma K)^\dagger\Sigma L\\ (L^*K)(\Sigma K)^\dagger\Sigma K & (L^*K)(\Sigma K)^\dagger\Sigma L\end{bmatrix}U^*\\
&= U\begin{bmatrix}(K^*\Sigma^{-1}\Sigma K)(\Sigma K)^\dagger\Sigma K & (K^*K)(\Sigma K)^\dagger\Sigma L\\ (L^*\Sigma^{-1}\Sigma K)(\Sigma K)^\dagger\Sigma K & (L^*K)(\Sigma K)^\dagger\Sigma L\end{bmatrix}U^*\\
&= U\begin{bmatrix}(K^*\Sigma^{-1})\Sigma K(\Sigma K)^\dagger\Sigma K & (K^*K)(\Sigma K)^\dagger\Sigma L\\ (L^*\Sigma^{-1})\Sigma K(\Sigma K)^\dagger\Sigma K & (L^*K)(\Sigma K)^\dagger\Sigma L\end{bmatrix}U^*\\
&= U\begin{bmatrix}(K^*\Sigma^{-1})\Sigma K & (K^*K)(\Sigma K)^\dagger\Sigma L\\ (L^*\Sigma^{-1})\Sigma K & (L^*K)(\Sigma K)^\dagger\Sigma L\end{bmatrix}U^*\\
&= U\begin{bmatrix}K^*K & K^*K(\Sigma K)^\dagger\Sigma L\\ L^*K & L^*K(\Sigma K)^\dagger\Sigma L\end{bmatrix}U^*.
\end{aligned}$$

Since $A^{\dagger,\diamond}A = A^\dagger A_1^\diamond A^\dagger A = A^\dagger AA^\diamond AA^\dagger A = A^\dagger AA^\diamond A = A_1^\diamond A^\dagger$, so

$$A^\dagger A_1^\diamond = U\begin{bmatrix}K^*K & K^*K(\Sigma K)^\dagger\Sigma L\\ L^*K & L^*K(\Sigma K)^\dagger\Sigma L\end{bmatrix}U^*.$$

(3) is trivial by item (1).

(4). By the definition of the MPBT inverse, we have $A^{\dagger,\diamond}AA^{\dagger,\diamond} = A^{\dagger,\diamond}$, then by item (2), we have

$$(A^\dagger A_1^\diamond)^2 = (A^{\dagger,\diamond}A)^2 = A^{\dagger,\diamond}AA^{\dagger,\diamond}A = A^\dagger A_1^\diamond.$$

(5). $A_1^\diamond A^\dagger A_1^\diamond = AA^\diamond AA^\dagger AA^\diamond A = AA^\diamond A = A_1^\diamond$.

(6). The equality $\mathcal{R}(A^\dagger A_1^\diamond) = \mathcal{R}(A^\dagger AA^\diamond)$ holds by $A_1^\diamond = AA^\diamond A$ and $A^\diamond$ is an outer inverse of A. Moreover,

$$\begin{aligned}
\mathcal{R}(A^\dagger AA^\diamond) &= \mathcal{R}\left(A^\dagger A(A^2A^\dagger)^\dagger\right) = \mathcal{R}\left(A^\dagger A(A^2A^\dagger)^*\right) = \mathcal{R}\left(A^\dagger A(AA^\dagger)^*A^*\right)\\
&= \mathcal{R}\left(A^\dagger A^2A^\dagger A^*\right) = \mathcal{R}\left(A^\dagger A^2A^\dagger A^\dagger\right) = \mathcal{R}\left(A^\dagger A^2(A^\dagger)^2\right).
\end{aligned}$$

(7) is trivial by item (1) and $A^\diamond$ is an outer inverse of A. □

Lemma 3.1 ([3, Remark 2.2 (i)]). *Let $A, B, C, U, V \in \mathbb{C}^{n\times n}$. If $\mathcal{R}(B) = \mathcal{R}(U)$ and $\mathcal{N}(C) = \mathcal{N}(V)$, then A is (B, C)-invertible if and only if A is (U, V)-invertible. In this case, we have $A^{\|(B,C)} = A^{\|(U,V)}$.*

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$. The MPBT inverse of A is the $(A^\dagger AA^\diamond, A^\diamond AA^\dagger)$-inverse of A.*

Proof. It is obvious by Theorem 3.3, Proposition 3.1 and Lemma 3.1. □

Proposition 3.2. *Let $A \in \mathbb{C}^{n\times n}$, then $A_1^\diamond \in A^\dagger\{2, 4\}$.*

Proof. It is trivial by items (3) and (5) in Proposition 3.1. □

In the next example, we will shows that $A^{\dagger}A_1^{\diamond}$ in Proposition 3.1 is not Hermitian and $A^{\dagger}A_1^{\diamond}A^{\dagger} \neq A^{\dagger}$ in general, that is $A_1^{\diamond} \notin A^{\dagger}\{1,3\}$.

Example 3.2. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$. It is easy to check that

$$A^{\dagger} = \begin{bmatrix} \frac{4}{21} & \frac{1}{21} & \frac{5}{21} \\ \frac{3}{14} & -\frac{1}{14} & \frac{1}{7} \\ -\frac{11}{42} & \frac{13}{42} & \frac{1}{21} \end{bmatrix},$$

thus $A^{\dagger}A_1^{\diamond} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{3}{28} & \frac{1}{28} & \frac{3}{28} \\ \frac{15}{28} & \frac{5}{28} & \frac{15}{28} \end{bmatrix}$, which says $A^{\dagger}A_1^{\diamond} \neq (A^{\dagger}A_1^{\diamond})^*$ and $A^{\dagger}A_1^{\diamond}A^{\dagger} = \begin{bmatrix} 0 & \frac{1}{7} & \frac{1}{7} \\ 0 & \frac{1}{28} & \frac{1}{28} \\ 0 & \frac{5}{28} & \frac{5}{28} \end{bmatrix} \neq A^{\dagger}$.

In the following two tables, we will collect some facts on the MPBT inverse, MPD inverse, MPCEP inverse, MPWC inverse and MPCEP inverse.

Table 1: Formulae of the MPBT, MPD, CMP, MPWC and MPCEP inverses

generalized inverses	formulae	sources
MPBT inverse	$U\begin{bmatrix} (K^*K)(\Sigma K)^{\dagger} & 0 \\ (L^*K)(\Sigma K)^{\dagger} & 0 \end{bmatrix}U^*$	Theorem 3.2
MPD inverse	$U\begin{bmatrix} (K^*K)(\Sigma K)^D & K^*K((\Sigma K)^D)^2\Sigma L \\ (L^*K)(\Sigma K)^D & L^*K((\Sigma K)^D)^2\Sigma L \end{bmatrix}U^*$	[19, Remark 2.9]
CMP inverse	$U\begin{bmatrix} (K^*K)(\Sigma K)^D & 0 \\ (L^*K)(\Sigma K)^D & 0 \end{bmatrix}U^*$	Lemma 2.3
MPWC inverse	$U\begin{bmatrix} (K^*K)(\Sigma K)^{ⓦ} & 0 \\ (L^*K)(\Sigma K)^{ⓦ} & 0 \end{bmatrix}U^*$	Lemma 2.4
MPCEP inverse	$U\begin{bmatrix} (K^*K)(\Sigma K)^{\textcircled{\dagger}} & 0 \\ (L^*K)(\Sigma K)^{\textcircled{\dagger}} & 0 \end{bmatrix}U^*$	Lemma 2.5

Table 2: The MPBT, MPD, CMP, MPWC and MPCEP inverses

generalized inverses	column parts	null parts	sources
MPBT inverse	$A^{\dagger}AA^{\diamond}$	$A^{\diamond}AA^{\dagger}$	Theorem 3.4
MPD inverse	$A^{\dagger}A^k$	A^k	[11, Theorem 3.2]
CMP inverse	$A^{\dagger}A^k$	$A^kA^{\dagger}$	[11, Theorem 3.2]
MPWC inverse	$A^{\dagger}A^k$	$(A^k)^*A^2A^{\dagger}$	[15, Theorem 2.7]
MPCEP inverse	$A^{\dagger}A^k$	$(A^k)^*$	[29, Theorem 3.11]

Theorem 3.5. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}\,(A) = k$*. We have*

(1) *If* $\mathcal{N}\left((A^k)^*\right) = \mathcal{N}\left((A^k)^* A^2 A^\dagger\right)$*, then* $A^\circ = A^{\dagger,\textcircled{\dagger}}$*;*

(2) *If* $\mathcal{N}\left((A^k)^* A^2 A^\dagger\right) = \mathcal{N}(A^k A^\dagger)$*, then* $A^\circ = A^{c,\dagger}$*.*

Proof. It is trivial by Lemma 3.1 and Table 1. □

Corollary 3.1. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}\,(A) = 2$*, then* $A^\circ = A^{c,\dagger}$*.*

Proof. For $\mathrm{ind}\,(A)$=2, we have $\mathcal{N}\left((A^2)^* A^2 A^\dagger\right) = \mathcal{N}\left(A^2 (A^2)^\dagger A^2 A^\dagger\right) = \mathcal{N}(A^2 A^\dagger)$, then, the proof is finished by Theorem 3.5. □

4. Conclusions

For a given complex matrix with a given index, one can get that the computation of the MPBT inverse by using the Hartwig-Spindelböck decomposition of this matrix. The future perspectives for research are proposed:

Part 1. The MPBT inverse is one of the useful tools to investigate the Hartwig-Spindelböck decomposition of a complex matrix.

Part 2. The rank properties of a given matrix, such as $\mathrm{rank}\,(AA^{\dagger,\diamond} - A^{\dagger,\diamond}A)$.

Part 3. The relationships between different generalized inverses relate the generalized inverses of ΣK in Hartwig-Spindelböck decomposition.

Data availability

All relevant data are within the paper.

Author Contributions

Writing-original draft preparation, Sanzhang Xu; writing-review and editing, Qingyuan Xu and Sanzhang Xu; methodology, Sanzhang Xu and Nan Zhou; supervision, Qingyuan Xu and Nan Zhou.

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