Finite groups with $124p$ elements of the largest order

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Abstract. We study a finite group with $124p$ elements of the largest order, where p is a prime greater than 5 and not equal to 31, and prove that such group is either a solvable group, or has a normal series like $1 \leq H \leq K \leq G$ such that H is a nilpotent $\{2,3\}$-group, $G/H \leq Aut(S)$, where $S \cong L_2(7), L_2(8), U_3(3)$.

Keywords: group, element, order, solvable group.

MSC 2020: 20D10.

1. Introduction

In this paper, all groups considered are finite. Let G be a finite group, $\pi_e(G)$ denotes the set of orders of elements of G, $M_i(G)$ denotes the set of elements of order i in G. In the whole paper, k always denote the largest element order of G, and write $M_k(G)$ as $M(G)$. For a positive integer n, $\pi(n)$ denotes the set of prime divisors of n, and $\pi(G) = \pi(|G|)$. We use $P_r(G)$ to denote a Sylow r-subgroup, where $r \in \pi(G)$, and $\varphi(x)$ the an Euler function of x. If there is no confusion, $P_r(G)$ is abbreviated as P_r. Sometimes we write the order of a group in the bracket next to the group, for example $A_5(2^2 \cdot 3 \cdot 5)$ means that A_5 has order $2^2 \cdot 3 \cdot 5$.

J. G. Thompson proposed the following famous problem in 1987.

Thompson Problem. *Let G_1 and G_2 be two finite groups. If $|M_i(G_1)| = |M_i(G_2)|$ and G_1 is solvable, where $i = 1, 2, 3, \cdots$, is G_2 solvable?*

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Many mathematicians tried to prove Thompson Problem, but not so many beautiful results as a positive answer to this problem were obtained, it is worth to mention that Rulin Shen, Wujie Shi and Feng Tang in 2023 get a positive answer to Thompson Problem for a solvable group with the non-connected prime graph in [16], and Pawel Piwek in 2024 gives a counter example in [15]. Hence it is meaningful to study related topics to Thompson Problem. In the past three decades, several related studies were sponsored, one of them was to study if a finite group is solvable with a given number of elements of the largest order. In fact this topic does be related to Thompson Problem, since if we can prove that a finite group G is solvable with $|M_k(G)| = t$, then we can conclude that if G_1 and G_2 satisfy $|M_k(G_1)| = |M_k(G_2)| = t$, then both G_1 and G_2 are solvable, hence if G_1 and G_2 satisfy conditions of Thompson Problem and $|M_k(G_1)| = t$, then G_2 is solvable, Thompson Problem holds for $M_k(G_1) = t$. The first paper on this topic appeared in 1993, Yang studied finite groups with special number of elements of the largest order in [1], for example, the number is a prime, etc. From 1998 to 2004, Jiang, Du, Liu and Qian etc studied finite groups with the number of the largest order elements is less than 20, or equal to 32, 44, 52, $2p^2$, $2p^3$, $10m$, or $6p$ in [3] -[9], where p is a prime, and m is a positive integer coprime to 10. From 2005 to 2012, Yan, He and Chen studied finite groups with the number of elements of the largest order equal to 42, $68p(p > 7)$, $2p^4(p > 3)$, $52p(p > 5)$, $10p^m(p > 5$ and m is a natural number), $76p(p > 5)$, $4pq$, $10pq(p,\ q > 5)$, where p and q are prime, see [10] -[13], [20] - [24]. It is worth to mention that because the approaches used in previous articles are not effective for the number of elements of the largest order divided by 15, Chen and Shi studied a finite group with the number of elements of the largest order equal to 30 in [1]. In this paper, we continue this topic to study the finite group having $124p$ elements of the largest order, where p is a prime> 5 and not equal to 31. We shall prove the following theorem.

Main Theorem. *Suppose that G is a finite group with $124p$ elements of the largest order, where p is a prime > 5 and $p \neq 31$, then G is either a solvable group, or has a normal series like $1 \leq H \leq K \leq G$, such that H is a nilpotent $\{2,3\}$-group, $G/H \leq Aut(S)$, where $S \cong L_2(7), L_2(8), U_3(3)$.*

2. Preliminaries

At first we introduce some necessary lemmas.

Lemma 2.1. *Suppose that G is a finite group and has n cyclic subgroups of order l, then $|M_l(G)| = n\varphi(l)$. In particular, $|M(G)| = n\varphi(k)$. If $n = 1$, then G is supersolvable. If k is an odd prime number, then $n \equiv 1(\mod k)$.*

Proof. The conclusion is evident from Lemma 2.2 and Theorem 1 in [25] , as well as Theorem 2.1 in [19]. □

Lemma 2.2 ([4]). *If $x \in G$ and $|x| = k$, $M(G) \subseteq C_G(x)$, then $\pi_e(C_G(x)) = \pi_e(\langle x\rangle)$, $C_G(x) = \langle M(G)\rangle$ and $C_G(x) \trianglelefteq G$.*

Lemma 2.3 ([19]). *If $|G| = 2n$ and n is an odd number, then G is solvable.*

For a positive integer n, a finite group is called a K_n-group if its order contains exactly n distinct primes.

Lemma 2.4 ([14]). *Suppose that G is a simple K_3-group, then G is exactly isomorphic to one of the following simple groups:*

$A_5(2^2 \cdot 3 \cdot 5)$, $A_6(2^3 \cdot 3^2 \cdot 5)$, $L_2(7)(2^2 \cdot 3 \cdot 7)$, $L_2(8)(2^3 \cdot 3^2 \cdot 7)$, $L_2(17)(2^4 \cdot 3^2 \cdot 17)$, $L_3(3)(2^4 \cdot 3^4 \cdot 13)$, $U_3(3)(2^5 \cdot 3^3 \cdot 7)$, $U_4(2)(2^6 \cdot 3^4 \cdot 5)$.

Corollary 2.1. *If $|\pi(G)| = 3$ and $\pi(G) \neq \{2,\ 3,\ 5\}$, $\{2,\ 3,\ 7\}$, $\{2,\ 3,\ 13\}$, $\{2,\ 3,\ 17\}$, then G is solvable.*

Lemma 2.5 ([17]). *Let G be a simple K_4-group with $3 \notin \pi(G)$, then $G \cong S_z(8)(2^5 \cdot 5 \cdot 7 \cdot 13)$ or $G \cong S_z(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$.*

Corollary 2.2. *If G is a K_4-group, and $(15,\ |G|) = 1$, then G is solvable.*

Lemma 2.6 ([17]). *Suppose that G is a simple K_4-group. Then, G can only be isomorphic to one of the following groups:*

(1) $A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7)$, $A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7)$, $A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7)$;

(2) $M_{11}(2^4 \cdot 3^2 \cdot 5 \cdot 11)$, $M_{12}(2^6 \cdot 3^3 \cdot 5 \cdot 11)$, $J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7)$;

(3) $L_2(16)(2^4 \cdot 3 \cdot 5 \cdot 17)$, $L_2(25)(2^3 \cdot 3 \cdot 5^2 \cdot 13)$, $L_2(49)(2^4 \cdot 3 \cdot 5^2 \cdot 7^2)$, $L_2(81)(2^4 \cdot 3^4 \cdot 5 \cdot 41)$, $L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7)$, $L_3(5)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$, $L_3(7)(2^5 \cdot 3^2 \cdot 7^3 \cdot 19)$, $L_3(8)(2^9 \cdot 3^2 \cdot 7^2 \cdot 73)$, $L_3(17)(2^9 \cdot 3^2 \cdot 17^3 \cdot 307)$, $L_4(3)(2^7 \cdot 3^6 \cdot 5 \cdot 13)$, $S_4(4)(2^8 \cdot 3^2 \cdot 5^2 \cdot 17)$, $S_4(5)(2^6 \cdot 3^2 \cdot 5^4 \cdot 13)$, $S_4(7)(2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4)$, $S_4(9)(2^8 \cdot 3^8 \cdot 5^2 \cdot 41)$, $S_6(2)(2^9 \cdot 3^4 \cdot 5 \cdot 7)$, $O_8^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7)$, $G_2(3)(2^6 \cdot 3^6 \cdot 7 \cdot 13)$, $U_3(4)(2^6 \cdot 3 \cdot 5^2 \cdot 13)$, $U_3(5)(2^4 \cdot 3^2 \cdot 5^3 \cdot 7)$, $U_3(7)(2^7 \cdot 3 \cdot 7^3 \cdot 43)$, $U_3(8)(2^9 \cdot 3^4 \cdot 7 \cdot 19)$, $U_3(9)(2^5 \cdot 3^6 \cdot 5^2 \cdot 73)$, $U_4(3)(2^7 \cdot 3^5 \cdot 5 \cdot 7)$, $U_5(2)(2^{10} \cdot 3^5 \cdot 5 \cdot 11)$, $S_z(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$, $S_z(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$, ${}^2D_4(2)(2^{12} \cdot 3^4 \cdot 7^2 \cdot 13)$, ${}^2F_4(2)(2^{11} \cdot 3^3 \cdot 5^2 \cdot 13)$.

(4) *$L_2(r)$, where r is a prime satisfying the equation: $r^2 - 1 = 2^a \cdot 3^b \cdot u^c$ for positive integers $a \geq 1$, $b \geq 1$, $c \geq 1$ and a prime $u > 3$.*

(5) *$L_2(2^m)$, where m satisfies the equation:*

$$\begin{cases} 2^m - 1 = u, \\ 2^m + 1 = 3t^b \end{cases}$$

where $m \geq 1$, $t > 3$, $b \geq 1$, u and t are primes.

(6) *$L_2(3^m)$, where m satisfies the equation:*

$$\begin{cases} 3^m - 1 = 2u^c, \\ 3^m + 1 = 4t \end{cases}$$

or

$$\begin{cases} 3^m - 1 = 2u, \\ 3^m + 1 = 4t^b \end{cases}$$

where $b \geq 1$, $c \geq 1$, u and t are primes.

Definition 1 ([23]). *Let k be the largest element order of G, consider the action of G on the set of all cyclic subgroups of order k by conjugation, let M_1, M_2, $\cdots$, M_t be all conjugacy classes of this action, then the lengthes of conjugacy classes are uniquely determined. Take $\langle x_i \rangle \in M_i$, then $|M_i| = |G : N_G(\langle x_i \rangle)|$, $1 \leqslant i \leqslant t$, are independent of the choice of $\langle x_i \rangle$. For convenience, let $m_i = |M_i|$, we call the array $(m_1, m_2, \cdots, m_t)$ the m-type of G and denote it as $m(G) = (m_1, m_2, \cdots, m_t)$, and we say that G is of m-type if $m(G) = (m_1, m_2, \cdots, m_t)$.*

Lemma 2.7. *Let G be of m-type, $m(G) = (m_1, m_2, \cdots, m_t)$, $n = m_1 + m_2 + \cdots + m_t$ the number of all cyclic groups of order k. For any i, if $q \in \pi(m_i)$, then q divides $\frac{|G|}{m_i} \times n \times \varphi(k)$. Further $\pi(G) \subseteq \pi(n) \cup \pi(\varphi(k)) \cup \pi(k))$.*

Proof. Reduction to absurdity. Assume there exists some i, without loss of generality, say $i = 1$, and a prime $q \in \pi(m_1)$ such that q does not divide $\frac{|G|}{m_1} \times n \times \varphi(k)$, so $q \nmid n$ and $q \nmid \varphi(k)$, thus there exists some m_j $(j > 1)$ not divided by q for n is a sum of m_i, say $q \nmid m_2$. Using notations in Definition 1, we get the following equations:

$$m_1 \cdot |N_G(\langle x_1 \rangle) : C_G(x_1)| \, |C_G(x_1)| = |G| = m_2 \cdot |N_G(\langle x_2 \rangle) : C_G(x_2)| \, |C_G(x_2)| \, .$$

Noticing q does not divide $\frac{|G|}{m_1} = |N_G(< x_1 >)|$, we see $q \nmid |C_G(< x_1 >)|$. Now, $q \nmid \varphi(k)$ yields q does not divide $|N_G(\langle x_2 \rangle)/C_G(\langle x_2 \rangle)|$, hence q divides $|C_G(\langle x_2 \rangle)|$ for $q \nmid m_2$ and $q \big| m_1$. Because x_1 and x_2 are two elements of the largest order in G, so are in $C_G(x_1)$ and $C_G(x_2)$ respectively, it forces $\pi(C_G(x_1)) = \pi(|x_1|) = \pi(|x_2|) = \pi(C_G(x_2))$, a contradiction to $q \big| |C_G(x_2)|$ and $q \nmid |C_G(x_1)|$. The first part of the lemma is proved, i.e., $\pi(m_i) \subseteq \pi(\frac{|G|}{m_i}) \cup \pi(n) \cup \pi(\varphi(k))$. Now, the second part of the lemma follows from the first part of the lemma and $|G| = m_i \times \frac{|G|}{m_i}$. □

Corollary 2.3. *Suppose that G is of m-type and $m(G) = (m_1, m_2, \cdots, m_t)$, $n = m_1 + m_2 + \cdots + m_t$ the number of all cyclic groups of order k, if $\pi(n) \subseteq \{2, r\}$, $\pi(\varphi(k)) \subseteq \{2, p\}$, $\pi(k) \subseteq \{2, 3, q\}$, then G is a $\{2, 3, p, q\}$-group or $\{2, 3, p, q, r\}$-group.*

Lemma 2.8 ([1]). *Let G be a finite group with $|M(G)|$ elements of the largest order k. If k is a prime, then $k \big| (|M(G)| + 1)$.*

Lemma 2.9 ([1]). *Let G be a finite group with $|M(G)|$ elements of the largest order k. Then, there exists a positive integer α such that $|G| \big| |M(G)| \cdot k^{\alpha}$.*

Lemma 2.10. *Let x be an element of the largest order in G, and r a prime dividing $|x|$, P_r the Sylow r-subgroup of $C_G(x)$. If $P_r \trianglelefteq C_G(x)$, then $P_r \trianglelefteq N_G(\langle x\rangle)$, and $|G : N_G(P_r)|$ divides $|G : N_G(\langle x\rangle)|$. Further, if $mr+1 \nmid |G : N_G(\langle x\rangle)|$ for any positive integer m, then $P_r \trianglelefteq G$.*

Proof. The lemma follows by trivial calculation and Sylow Theorem. □

Definition 2. *Let G be a finite group, the prime graph $\Gamma(G)$ of G is a graph with the set of vertices $V(G) = \pi(G)$. Let p, q be in $V(G)$, there exists an edge connecting p and q if and only if G has an element of order pq.*

Lemma 2.11 ([18]). *Let G be a finite group whose prime graph has more than one component, then one of the following holds:*

1. G is a Frobenius group or 2-Frobenius group;

2. G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group and $|G/K|\big||Out(K/H)|$.

3. Conclusions

Now, we prove the Main Theorem and give a positive answer to Thompson Problem for a finite group with $124p$ elements of the largest order, where p is a prime > 5 and $p \neq 31$.

Proof of Main Theorem. Take an element $x \in G$ of the largest order k, then $|N_G(\langle x\rangle) : C_G(x)|\,\big|\varphi(k)$. Let n be the number of cyclic subgroups of order k in G. The following formula is always true

$$|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(x)| \cdot |C_G(x)|. \tag{3.1}$$

By Lemma 1, if $|M(G)| = 124p$, where p is prime> 5 and not equal to 31, then all possibilities for n, $\varphi(k)$ and k can be determined as the following Table 1.

Table 1: Values of n, $\varphi(k)$ and k

n	$\varphi(k)$	k
1	$124p$	k with $\varphi(k) = 124p$
2	$62p$	q or $2q$, $q = 62p+1$ is a prime
31	$4p$	(1) q or $2q$, $q = 4p+1$ is a prime; (2) $3q$ or $4q$ or $6q$, $q = 2p+1$ is a prime
62	$2p$	q or $2q$, $q = 2p+1$ is a prime
$31p$	4	5, 8, 10, 12
$62p$	2	3, 4, 6
$124p$	1	2

In the following, we shall prove that G is solvable case by case upon what n is.

Case 1. If $n = 1$, then G is a supersolvable group by Lemma 1.

Case 2. If $n = 2$, then by Table 1, $\varphi(k) = 62p$, and k equals q or $2q$, where $q = 62p+1$ is a prime. Hence $C_G(a)$ is a $\{2, q\}$-group or q-group and $N_G(\langle a\rangle)/C_G(a)$ is a $\{2,\ 31,\ p\}$- group. By Lemma 4 and Corollary 3, we conclude that $C_G(a)$ and $N_G(\langle a\rangle)/C_G(a)$ are solvable, so $N_G(\langle a\rangle)$ is solvable. Moreover, $|G : N_G(\langle a\rangle)| \leq 2$, which implies that G is solvable.

Case 3. If $n = 31$, then $\varphi(k) = 4p$, either $q = 4p + 1$ is a prime and k equals one of q and $2q$, or $q = 2p + 1$ is a prime and k equals one of $3q$, $4q$ and $6q$.

Subcase 3.1. If $q = 4p + 1$ is a prime and k equals one of q and $2q$, then $\pi(\varphi(k)) = \{2, p\}$, and $\pi(k) \subseteq \{2, q\}$, thus G is a $\{2, p, q\}$-group or $\{2, 31, p, q\}$-group by Corollary 3.

While G is a $\{2, p, q\}$-group. Since both p and $q = 4p + 1$ are primes, the order of any simple K_3-group does not contain such kind of prime factors, so G is solvable.

While G is a $\{2,\ 31,\ p,\ q\}$-group, we have $(15,\ |G|) = 1$, so G is solvable by Corollary 2.

Subcase 3.2. If $q = 2p + 1$ is a prime and $k = 4q$, then $\pi(\varphi(k)) = \{2,\ p\}$, and $\pi(k) \subseteq \{2,\ q\}$, thus G is $\{2,\ p,\ q\}$-group or $\{2,\ 31,\ p,\ q\}$-group. By the same reasoning as Subcase 3.1, G is solvable.

Subcase 3.3. If $q = 2p + 1$ is a prime and $k = 3q$ or $6q$, this case is much complicated, we divide the proof into two steps.

3.3.1 To prove that G is solvable if $p > 13$, $q = 2p + 1$ is prime and $k = 3q$.

In this case, G is a $\{2,\ 3,\ p,\ q\}$-group or $\{2,\ 3,\ 31,\ p,\ q\}$-group. Let a be an element of the largest order in G, then $C_G(a)$ is a $\{3,\ q\}$-group. Suppose that $|C_G(a)| = 3^u \cdot q^v$. Obviously, $C_G(a)$ has no element of order 9 or q^2. If $u \geq 4$, then there are at least $(3^4 - 1)(q - 1) = 160p$ elements of order 3q in $C_G(a)$, a contradiction to $|M(G)| = 124p$. Therefore, $u \leq 3$. If $v > 1$, we get a contradiction by similar reason. Hence $v = 1$, and $|C_G(a)| = 3^u \cdot q$, where $u \leq 3$. Let Q be a Sylow q-group of $C_G(a)$. Obviously, $Q \trianglelefteq C_G(a)$, then $Q \trianglelefteq N_G(\langle a\rangle)$. Since $n = 31$, we have $|G : N_G(\langle a\rangle)| \leq 31$. Notice $q > 31$ in this case, we come to $Q \trianglelefteq G$. Let P_3 be the Sylow $p-$subgroup of G, and $|P_3| = 3^\beta$, then it follows from $|P_3| = |G|_3 = |G : N_G(\langle a\rangle)|_3 \times |C_G(a)|_3$ that $\beta \leq 6$.

While G is a $\{2,\ 3,\ p,\ q\}$-group, as G/Q is a $\{2,\ 3,\ p\}$-group, both p and $q = 2p+1$ are primes and $p > 13$, so G/Q has no section isomorphic to a simple K_3-group, G/Q is solvable, so is G.

While G is a $\{2,\ 3,\ 31,\ p,\ q\}$-group, G/Q is a $\{2,\ 3,\ 31,\ p\}$-group. Let $\overline{G} = G/Q$, then $|\overline{G}| = |G/Q| = 2^\alpha \cdot 3^\beta \cdot 31 \cdot p^\gamma$, where $\alpha \leq 2$, $\beta \leq 3$, $\gamma \leq 1$. By Lemma 3, we may assume that $\alpha = 2$, $|\overline{G}| = 2^2 \cdot 3^\beta \cdot 31 \cdot p^\gamma$, where $\beta \leq 3$, $\gamma \leq 1$.

1) If $\gamma = 0$, then $|\overline{G}| = 2^2 \cdot 3^\beta \cdot 31$. By Corollary 1, $\overline{G}$ is solvable, so is G.

2) If $\gamma = 1$, then $|\overline{G}| = |G/Q| = 2^2 \cdot 3^\beta \cdot 31 \cdot p$.

Clearly, no section of $\overline{G}$ is isomorphic to a simple K_3-group. Suppose $\overline{G}$ is non-solvable, then there exists a section of $\overline{G}$, say $\overline{W}/\overline{S}$, isomorphic to a simple K_4-group, and $|\overline{W}/\overline{S}| = 2^2 \cdot 3^t \cdot 31 \cdot p$, where $t \leq \beta \leq 3$. Moreover, $\overline{W}/\overline{S}$ is isomorphic to one of K_4-groups listed in (4), (5) and (6) in Lemma 6.

(a) Assume $\overline{W}/\overline{S} \cong L_2(r)$, where r is a prime such that $r^2-1=2^a\cdot 3^b\cdot v^c$, where $a\geq 1,\ 3\geq b\geq 1,\ c\geq 1$, and $v>3$ is a prime. It is easy to see $r\neq 31$, and then $r=p$. Since $(r^2-1)/(2,r-1)$ divides $|L_2(r)|$, we have v^c divides $|\overline{W}/\overline{S}|$, which concludes $v=31$ and $c=1$. Comparing orders of Sylow 2−subgroups of G and $L_2(r)$, we get $a=3$. Solving equations $p^2-1=2^3\cdot 3^b\cdot 31$ for $1\leq b\leq 3$ respectively, we find that there is no prime p satisfying the equation, a contradiction. Therefore $\overline{W}/\overline{S}\ncong L_2(r)$.

(b) Assume $\overline{W}/\overline{S}\cong L_2(2^m)$, and m satisfies the system of equations:

$$\begin{cases}2^m-1=u,\\ 2^m+1=3t^b\end{cases}$$

where $m\geq 1,\ t\geq 3,\ b\geq 1,\ u$ and t are primes. Since $|L_2(2^m)|=2^m\cdot(2^m+1)(2^m-1)=2^m\cdot 3t^b\cdot u$, we get $u\neq 31$, $t=31$, $u=p$, $b=1$, which is impossible by checking the system of equations, so $\overline{W}/\overline{S}\ncong L_2(2^m)$.

(c) Assume $\overline{W}/\overline{S}\cong L_2(3^m)$, and m satisfies the system of equations:

$$\begin{cases}3^m-1=2u^c,\\ 3^m+1=4t\end{cases}$$

or

$$\begin{cases}3^m-1=2u,\\ 3^m+1=4t^b\end{cases}$$

where $b\geq 1,\ c\geq 1,\ u$ and t are primes, we can prove $\overline{W}/\overline{S}\ncong L_2(3^m)$ by the same reasoning as (b).

From (a), (b) and (c), G has no section is isomorphic to a simple K_4-group, which concludes that G is solvable.

3.3.2 To prove that G is solvable if $p>13$, $q=2p+1$ is a prime and $k=6q$.

Clearly, $n=31$, $\pi(\varphi(k))=\{2,p\}$, $\pi(k)=\{2,\ 3,\ q\}$, and G is a $\{2,\ 3,p,\ q\}$-group or $\{2,\ 3,\ 31,\ p,\ q\}$-group.

If G is a $\{2,\ 3,\ p,\ q\}$-group, we can prove that G is solvable by the same argument as $k=3q$.

If G is a $\{2,\ 3,\ 31,\ p,\ q\}$-group, then $C_G(a)$ is a $\{2,\ 3,\ q\}$-group. Let $|C_G(a)|=2^\delta\cdot 3^\epsilon\cdot q^\varepsilon$, where $\delta\leq 5,\epsilon\leq 3,\varepsilon\leq 1$. It is easy to see that $C_G(a)$ has no element of order 9 or q^2. Let Q be a Sylow q-subgroup of $C_G(a)$, we assert $Q\trianglelefteq G$. In fact, it is enough to show $Q\trianglelefteq C_G(a)$, then by the same reason as Step 3.3.1 we get $Q\trianglelefteq G$. Otherwise, $C_G(a)$ has at least $q+1$ Sylow q-subgroups, this means that $C_G(a)$ has $3(q^2-1)$ elements of order $6q$ since $p>13$, more than $124p$, a contradiction. Hence $Q\trianglelefteq G$. Suppose that $\overline{G}=G/Q$, then $|\overline{G}|=2^\alpha\cdot 3^\beta\cdot 31\cdot p^\gamma$, where $\alpha\leq 7$, $\beta\leq 3$, $\gamma\leq 1$.

While $\gamma=0$, $|\overline{G}|=2^\alpha\cdot 3^\beta\cdot 31$, then $\overline{G}$ is a $\{2,3,31\}$-group. By Corollary 1, $\overline{G}$ is solvable.

While $\gamma = 1$, $|\overline{G}| = 2^\alpha \cdot 3^\beta \cdot 31 \cdot p$. Due to the absence of 31 and p in the order of a simple K_3-group, if $\overline{G}$ is non-solvable, then it should have a section isomorphic to a simple K_4-group, by the same reasoning as $k = 3q$, we can prove that this is impossible, so $\overline{G}$ is solvable, and then G is solvable.

3.3.3 To prove G is solvable if $p = 7,\ 11,\ 13$.

(1) If $p = 7$, then $\varphi(k) = 28$, $k = 29$ or 58, $n = 31$ by Table 1, so $\pi(\varphi(k)) = \{2, 7\}$, $\pi(k) \subseteq \{2, 29\}$. Hence G is a $\{2, 7, 29\}$-group or $\{2,\ 7,\ 29,\ 31\}$-group, so $(15,\ |G|) = 1$, and then G is solvable.

(2) If $p = 11$, then $\varphi(k) = 44$, $k = 3q,\ 4q,\ 6q$ by Table 1, where $q = 23$.

(a) If $k = 4q = 4 \times 23$, then $n = 31$, $\pi(\varphi(k)) = \{2,\ 11\}$, $\pi(k) = \{2,\ 23\}$ and G is a $\{2,\ 11,\ 23\}$-group or $\{2,\ 11,\ 23,\ 31\}$-group. Hence $(15,\ |G|) = 1$, then G is solvable by Corollary 1.

(b) If $k = 3q = 3 \times 23$, then $n = 31$, $\varphi(k) = 44$, and G is a $\{2,\ 3,\ 11,\ 23\}$-group or $\{2,\ 3,\ 11,\ 23,\ 31\}$-group.

Assume that G is a $\{2,\ 3,\ 11,\ 23\}$-group. Since the absence of 11 and 23 in the order of a simple K_3-group, G has no section isomorphic to a simple K_3-group. Hence if G is non-solvable, then there exists a section of G isomorphic to a simple K_4-group. By the same reasoning as the case of $k = 3q$ and $p > 13$, we come to a contradiction, so G is solvable.

Now, G is a $\{2,\ 3,\ 11,\ 23,\ 31\}$-group. In this case, 31 divides $|G|$, so it follows by Lemma 7 and Formula 3.1 that there exists an element a of the largest order such that $|G : N_G(\langle a \rangle)| = 31$. Then, $C_G(a)$ is a $\{3,\ 23\}$-group, and there is no element of order 9 or 23^2 in $C_G(a)$. Suppose that $|C_G(a)| = 3^u \cdot 23^v$, then $3^u \cdot 23,\ 3 \cdot 23^v \le 124p = 124 \times 11$, which implies that $u \le 3$, $v \le 1$. By Formula 3.1 we get that $|G|$ divides $31 \cdot 4 \cdot 11 \cdot 3^u \cdot 23$, thus $|G| = 2^\alpha \cdot 3^\beta \cdot 11^\gamma \cdot 23 \cdot 31$ with $\alpha \le 2$, $\beta \le 3$, $\gamma \le 1$. Let Q be a Sylow 23-subgroup of $C_G(a)$, which is also a Sylow subgroup of G. Obviously, $Q \trianglelefteq C_G(\langle a \rangle)$, and then $Q \trianglelefteq N_G(\langle a \rangle)$. Hence the number of Sylow q-subgroups of G is $23t + 1$, it divides $|G : N_G(\langle a \rangle)| = 31$, this forces $23t + 1 = 1$, hence $Q \trianglelefteq G$. Now, $\overline{G} = G/Q$ is a $\{2,\ 3,\ 11,\ 31\}$-group of order $2^\alpha \cdot 3^\beta \cdot 11^\gamma \cdot 31$, where $\alpha \le 2$, $\beta \le 3$, $\gamma \le 1$. By the same reasoning as $k = 3q$, $p > 13$, $\overline{G}$ has no section isomorphic to a simple K_3-group or K_4-group, so $\overline{G}$ is solvable, and then G is solvable.

(c) Let $k = 6q$. Since $\varphi(6q) = \varphi(3q)$, we can use the same approach as the case $k = 3q$ to prove that G is solvable.

(3) If $p = 13$, $\varphi(k) = 4p$, then $k = q$ or $2q$, where $q = 4p + 1$ is a prime, $n = 31$, $\pi(\varphi(k)) = \{2,\ 13\}$, $\pi(k) \subseteq \{2,\ 53\}$. Hence G is a $\{2,\ 13,\ 53\}$-group or $\{2,\ 13,\ 31,\ 53\}$-group. Hence $(15,\ |G|) = 1$, which concludes that G is solvable by Corollary 2.

Case 4. Assume $n = 62$ and $\varphi(k) = 2p$. By Table 1, we have $k = q$ or $2q$, where $q = 2p + 1$ is a prime. Clearly, $\pi(n) = \{2, 31\}$, $\pi(\varphi(k)) = \{2, p\}$, $\pi(k) \subseteq \{2,\ q\}$. Then, G is a $\{2,\ p,\ q\}$-group or $\{2,\ 31,\ p,\ q\}$-group.

If G is a $\{2,\ p,\ q\}$-group, then G is solvable by Corollary 1.

If G is a $\{2,\ 31,\ p,\ q\}$-group, then $(15,\ |G|) = 1$, and G is solvable by Corollary 2.

Case 5. Assume $n = 31p$ and $\varphi(k) = 4$. By Table 1, k is one of $5,\ 8,\ 10$ and 12, then $\pi(n) = \{31,\ p\}$, $\pi(\varphi(k)) = \{2\}$. If $k = 5$ or 8, then $|\pi(G)| \leq 2$, so G is solvable.

If $k = 10$, then $31 \notin \pi(G)$. By Lemma 9, $\pi(G) \subseteq \{2,\ 5,\ 31,\ p\}$, then $\pi(G) \subseteq \{2,\ 5,\ 7\}$. By Corollary 1, G is solvable.

If $k = 12$, then $31 \notin \pi(G)$. By Lemma 9, $\pi(G) \subseteq \{2,\ 3,\ 31,\ p\}$, then G is a $\{2,\ 3,\ p\}$-group, where $p = 7,\ 11$. If $p = 11$, then G is solvable by Corollary 1. If $p = 7$, then G is a $\{2,\ 3,\ 7\}$-group.

By Lemma 9, $7 \parallel |G|$. Suppose that $|G| = 2^\alpha \cdot 3^\beta \cdot 7$, where $\alpha \geq 2,\ \beta \geq 1$. If G is non-solvable, then by Lemma 11, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong L_2(7)$ or $L_2(8)$ or $U_3(3)$ and $|G/K|\big||Out(K/H)|$, H is a nilpotent $\{2,3\}$-group.

Case 6. Assume $n = 62p$, then $\varphi(k) = 2$. By Table 1, we have $k = 3,\ 4,\ 6$, then G is a $\{2,\ 3\}$-group, and G is solvable.

Case 7. If $n = 124p$, then $\varphi(k) = 1$. Hence $k = 2$, and G is a 2-group. Clearly, G has $|G| - 1$ elements of order 2, and $|G| - 1$ cannot be equal to $124p$, a contradiction.

Now, Main Theorem follows from cases 1 to 7. □

4. Remark

It is meaningful to study if the non-solvable group in the Main Theorem could be eleminated.

5. Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grant No.12071376). The authors are grateful to the reviewers for their valuable suggestions to revise the paper.

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Accepted: May 21, 2024