Students' difficulties with eigenvalues and eigenvectors. An exploratory study

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Abstract. In this paper we study the problem of students' difficulties in interpreting the definition of eigenvalues and eigenvectors, extending the research of other scholars. We analyse the activity of small groups of students, during an optional specifically designed extra class, while trying to make sense of the definitions seen in class. Using Sfard's notions of process/object, we scrutinise students' speech and written productions to detect those aspects in the formal definition of eigenvectors and eigenvalues that might hinder their understanding. We consider also how commognitive conflicts present in students' discourse with their classmates may encourage difficulties in grasping the meaning of these mathematical concepts.

Keywords: Eigentheory, process/object duality, commognitive conflict, undergraduate mathematics education.

MSC 2020: 15A18, 97C70, 97D70, 97H60.

1. Introduction

In recent years, research in mathematics education has been taking an interest in the teaching of linear algebra, recognizing its importance not only in the purely mathematical sphere, but also as a basic subject for the study of STEM subjects, given the many applications this topic has in disciplines such as computer science or engineering. Early studies in the area have focused on the general difficulties university students usually experience in dealing with the "formalism" of linear algebra [5]. Some studies then investigated the teaching and learning processes of basic concepts of the discipline, such as linear dependence, determinant, span, bases, etc. Recently, several researchers have delved into investigating processes relative to more advanced topics, for which a coordination of basic concepts such as those mentioned above is necessary. One such example is the theory of

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eigenvalues which, in our opinion, deserves special attention for several reasons, as pointed out by other researchers who have already investigated educational aspects related to them (i.e. [22, 23]). Firstly, in accordance with what already mentioned regarding linear algebra in general, applications of eigentheory are widespread in STEM subjects, for example in the study of dynamical systems, or of rotating bodies or for modelling quantum mechanical systems [16]. Moreover, the understanding of eigenvalues and eigenvectors implies a good understanding and coordination of other linear algebra concepts such as linear dependence, linear transformation, determinant, kernel, etc. Hence, investigating its learning processes can let difficulties related to the aforementioned notions emerge; lastly, according to both literature and our experience, the topic seems to represent a major obstacle in the study of a subject which is in itself a hindrance for students. Although the literature on eigentheory is still scarce, it is evident how interest in the topic is growing in the research community. Most of the work on the topic is concentrated in the last fifteen years or so and concerns different institutional settings. Some investigate students' difficulties in dealing with such concepts, while others describe the implementation of innovative instructional pathways and their impact on different aspects of student learning - such as a better coordination of algebraic and visual/geometric aspects [2, 3, 8, 15]. Some other researchers [12, 24] have investigated how properly designed task sequences can assist students in understanding eigenvalues and eigenvectors.

In their article "Process-object difficulties in linear algebra: eigenvalues and eigenvectors", Stewart and Thomas [21] showed some difficulties related to the definitions of eigenvector and eigenvalue, faced by students when learning linear algebra. Two main problems related to learning these notions are highlighted by these authors: (i) in a linear algebra course, students are usually given the concept definition in words and then an easy formula to algebraically compute them is soon provided. "In this way the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic thrust" ([21], p. 186); (ii) another serious problem with the formula $f(v) = \lambda v$ for students is that the two sides of the equation are quite different processes, since on the left it is a matrix which is multiplied by a vector, and this is equated on the right side to a multiplication of a scalar by the same vector.

Both these difficulties are confirmed by a more recent study conducted by the first author of this paper [10]. Nevertheless, as highlighted also by Stewart and Thomas themselves, the role and genesis of these obstacles, particularly regarding the reading of the formula $f(v) = \lambda v$, requires further investigation. In this paper we will give a possible explanation for the difficulty in dealing with this formula, as emerged from the pilot study of the first author's PhD project, concerning eigentheory teaching and learning processes. This hypothesis is related to the way the formula $f(v) = \lambda v$ is read by students and to the way it, together with the traditional sequence for introducing eigenvalues and eigenvectors, supports what Sfard calls a "pseudostructural" conception of them, to the detriment of their conceptualization as objects.

2. Theoretical framework

2.1 The process/object duality

In different works, Sfard [9, 18] has drawn attention to the operational/structural dual nature of mathematical conceptions. This means that in all fields of mathematics, the same mathematical concepts, or to be more precise, their representations, may at times be interpreted as processes, and other times as objects. Although these two aspects seem to be conflicting, Sfard emphasizes the importance of considering them as complementary, meaning that processes and objects must be seen as two different sides of the same coin, and not as two completely distinct aspects of mathematical activities. Sfard has brought many examples to show how the learning process may follow a pattern which is similar to the way mathematical concepts have developed historically: a process (primary process) slowly becomes an entity in its own right through a process of reification whose outcome is a mathematical object. This new object-like entity serves then as an input for new processes (secondary processes) at a higher level. Let us show an example from the realm of linear algebra to make things clearer. If we consider the notion of determinant, this achieves, both in its historical development and in the learning process, an object-status just after having been considered and manipulated operationally. It is first introduced as the result of a computation, to then become, hopefully also to the students' eyes, a mathematical object that gives important information about a matrix or the linear transformation the matrix represents. The reified object "determinant" becomes then a useful tool to perform new procedures at a higher level, such as computing the inverse of a matrix.

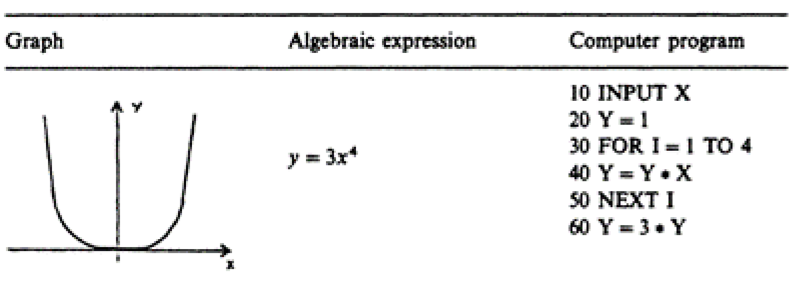

Figure 1: Different representations of a function, from (Sfard, 1991, p. 6)

As already mentioned, the dual nature of mathematics can be observed in the different representations used. Although "such property as structurality lies in the eyes of the beholder rather than in the symbols themselves" ([17], p. 5), some of these are more likely to be interpreted as structural, while others as operational. Sfard brings the example of different representations of a function (see Figure 1) stating that the computer program encourages an operational view while the graph supports a structural interpretation.

The algebraic expression can be easily interpreted in both ways and this may represent a benefit for the learning process, or an obstacle, as in the case we will present. It is important to notice, as Sfard points out, that this double interpretability of the algebraic representation also reflects the different ways the "=" sign can be read: "it can be regarded as a symbol of identity, or as a 'command' for executing the operations appearing at its right side" (p. 6).

The operational/structural duality of the equality sign as well as its misuse by primary and lower secondary school pupils has been widely discussed [1, 7, 13, 17]. However more recent studies (see for example [6]) have shown that an exclusively operational interpretation of the equal sign can persist in college students. Moreover, Das et al. [4] show how university students can struggle with the bidirectional flexibility in reading an equality, confirming a permanence well beyond elementary school of the difficulty with the structural interpretation of the equals sign.

2.2 Commognitive conflicts

It is crucial to remember that, according to Sfard's theory of commognition, thinking, and thus thinking mathematically, is a form of communication. Therefore, it is important to consider all the elements related to the development of mathematical discourse to comprehend phenomena such as the persistence of the operational view of mathematical concepts and the challenges associated with transitioning to a structural interpretation of them. In this context, one significant phenomenon to consider is what Sfard refers to as *commognitive conflict* [19]. When a new term is introduced in the discourse, or a new meaning of that term is presented, our impression of understanding it can be misleading. There is no guarantee that other participants in the discourse are using it in the same way, and this potential ambiguity exposes the interlocutors to the risks associated with the use of non-operationalized terms. Specifically, when one encounters a new discourse governed by metarules different from those with respect to which he or she has acted up to that point, a commognitive conflict may arise. This situation involves different participants in the discourse following different metarules. In such cases, the two discourses are compatible but incommensurable, meaning they do not share the same criteria for approving or disapproving the current narrative. According to Sfard, the resolution of these conflicts generates learning. However, this typically occurs when an 'oldcomer' in the mathematical discourse participates in the discussion and can provide meaning to the newcomer's discourse using shared metarules. Unfortunately, this does not always happen, and if a commognitive conflict remains unresolved, it can lead to a failure to achieve a structural interpretation of the mathematical object in question or to the development of a pseudostructural conception of it. Pseudostructural conceptions [14] are the conceptions which develop when the student, unable to think in the terms of abstract objects, uses symbols as things in themselves and, as a result, remains unaware of the relations between the secondary and primary processes. Consequently, they remain unaware of the relationships between secondary and primary processes.

3. Research questions and methodology

In autumn 2021, as part of the pilot study for the first author's PhD thesis on eigentheory teaching and learning processes, 67 first-year engineering students participated in an experimental activity. This activity took place during tutoring or extra hours a few days after the introduction of eigenvalue and eigenvector notions in a standard lecture by the linear algebra course's teacher.

During this two-hour activity, students were organized into groups of three. The first part involved a collective re-reading of the lecture notes and other study resources to reorganize newly encountered notions and deepen their understanding. To encourage a true comprehension of eigenvectors and eigenvalues, rather than mere repetition of the teacher's or textbook definitions, a guiding question was posed: "How would you explain what eigenvectors and eigenvalues are to someone who only knows basic notions of linear algebra such as linearity, dependence, basis, etc.?" Following this, four problems were presented. Written protocols from all groups were collected, and eight groups were audio and video recorded throughout the activity. A multimodal semiotic analysis [14] was conducted to detect moments where the simultaneous use of different semiotic resources (e.g., utterances, inscriptions, gestures) provided interesting insights into students' understanding of the concepts.

For a more detailed description of the methodology used and some further results, refer to [11]. In this discussion, our focus is on presenting two specific segments of activity from two small groups, highlighting difficulties in reading the formula "$f(v) = \lambda v$" and various interpretations of the equal sign within it.

Using the described theoretical framework, we address the following research questions: (Q1) In what ways is a commognitive conflict likely to arise in a discourse on eigenvectors, when newcomers participate in that specific discourse? (Q2) What aspects in the formulation of the equation "$f(v) = \lambda v$" may hinder the development of a structural view of eigenvectors and eigenvalues?

To answer these questions, we present excerpts from discussions between students in two of the observed groups, showcasing their utterances, gestures and drawings produced and used in the discourse. Our analysis seeks elements suggesting the presence of a commognitive conflict, where participants use the same terms or visual mediators but according to different discursive rules. We aim to detect if and how these conflicts produce or reinforce a pseudostructural conception of eigenvectors in, at least, one participant in the discourse. Being our focus on how students interpret the commonly provided definition of eigenvectors and eigenvalues, $f(v) = \lambda v$, we do so examining the segment of the video recording where students respond to the leading question mentioned earlier.

We start presenting and describing the chosen extracts and then we analyze them in order to show the elements that help us answer our research questions.

4. Results

4.1 Description of discourse in Group 1

From now on, we will call the three students in the considered group, Antonio, Bruno and Claudio. Right after having read the assignment, the students agree to start with reading the examples given by the teacher in the lecture:

1. Bruno: We must look at the examples

2. Claudio: Yes, exactly, the definition here is useless.

Their choice of trying to understand the examples, instead of interpreting the definition, seems effective. Indeed, it seems that the chosen example, eigenvectors and eigenvalues of a geometrical transformation in $\mathbb{R}^2$, helps them understand what these actually are.

resume Bruno: Then, basically, when is it that they form… when they overlap?

resume Claudio: When the vector remains itself, so when nothing changes.

resume Bruno: When it is rescaled times λ.

resume Claudio: Mmh…

resume Bruno: Yes, because f of v is equal to λ times v (he points to the formula "$f(v) = \lambda v$" written on his lecture notes) so this means that the image of the vector is the vector rescaled times λ.

resume Claudio: Yes!

resume Bruno: So it means that it remains on the same line basically, when the two end on the same line (while pronouncing this sentence he performs the gesture in Figure 2), so what they do is to increase or decrease.

Figure 2: The gesture produced by Bruno (line 9)

They also re-read another example, an algebraic one, and also in this case it seems that they have grasped the idea of what it means for a vector to be an eigenvector. Nevertheless, a first doubt emerges, expressed by Claudio:

resume Claudio: So, does the eigenvector always coincide with the linear vector?

resume Bruno: Yes, sure, the one that you must have, isn't it? It is the vector of which you take the image.

resume Claudio: I don't know...

This exchange of lines is very brief, and the two students do not dig deeper in this miscomprehension, even if it is clear that Claudio is not satisfied with Bruno's answer. Nevertheless, by analyzing the whole activity, this perplexity emerges at different times and seems to be related to the fact that in the formula "$f(v) = \lambda v$", v appears both as the argument of the function and as the vector which is multiplied by λ. After having, almost, convinced themselves of this characterization of eigenvectors and eigenvalues, Bruno tries to write down a definition and he shares it with his group classmates:

resume Bruno: Is it good? If I say that the eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue.

resume Antonio: Can you repeat?

resume Bruno: The eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue.

The three of them agree on this definition. By reading this, it might seem that they have indeed understood what eigenvectors and eigenvalues are. This is probably right for Bruno, and this can be inferred by the way he solves the exercises in the second part of the activity. Nevertheless, the following part of their discussion makes clear that Antonio has misinterpreted the definition given by Bruno.

resume Antonio: But, I don't understand, this e_1 (they are now looking at an example where e_1 is an eigenvector for the given function f) would be v which finds v? Bruno and Claudio ask him to explain it with clearer words.

resume Antonio: I mean, f of v is λ times v? So in this case...

resume Claudio: It is a vector rescaled by its image.

resume Bruno: Because v is 3 times e_1.

resume Claudio: Basically, the image of e_1 is e_1 rescaled by the eigenvalue.

resume Antonio: Ok!

Are Bruno, Claudio and Antonio's discourses really aligned? When they say "f of v is λ times v" or when they read "$f(v) = \lambda v$", are they really interpreting it in the same way?

Let us move a little forward, when the three students are solving the problems. The first one consists of finding the eigenvectors and eigenvalues related to the symmetry with respect to the line $y = x$, in $\mathbb{R}^2$. In this problem-solving phase, students' conceptions of the meaning of "=" appear more evidently. For example, after having realized that all vectors lying on the line $y = x$ are eigenvectors, they cannot find the related eigenvalue. The researcher leading the activity intervenes to help them, by asking which is the image of a generic vector $(3, 3)$. Claudio answers "It is λ times $(3, 3)$". He seems convinced of this statement because in the following lines he adds "It is $(3, 3)$ rescaled for whatever eigenvalue—, and then again restates "[the image of $(3, 3)$ is] $(3, 3)$ times λ—. A few lines later, Antonio asks:

resume Antonio: if we had the vector $(3, 3)$, and the vector $(6, 6)$ [as its image], is it wrong to say that it is the vector times λ that is 2?

4.2 Interpretation of Antonio and Claudio's statements

Antonio and Claudio's confusion in solving this problem seems to be related to the way they interpret "$f(v) = \lambda v$", as well as the literal definition given by Bruno at the beginning of the activity, namely "The eigenvector is a vector whose image is defined by the vector itself scaled by the eigenvalue". It seems that they are interpreting "=" in the formula as a defining equal sign. This means that they do not recognize the equality in the formula as the fact that the vector obtained computing v's image is equal to the vector obtained multiplying v by λ. They seem to be reading the formula as "the image of v is given by λ times v". Bruno's definition unconsciously helps Antonio and Claudio build this interpretation of the formula. When he says "is defined" he is improperly using this verb and it seems that he uses it to intend "is equal to". Nevertheless, the use of the verb "is defined" likely reinforces Antonio and Claudio's interpretation of "=" as a defining equal sign. This way of reading the equality is evident in the fact that, when I ask what is the image of $(3, 3)$, instead of calculating its actual image and checking if it actually is equal to a multiple of $(3, 3)$ itself, Claudio immediately answers "λ times $(3, 3)$", as if to calculate a vector's image one had to multiply it times some eigenvalue. Also Antonio seems to think that one is allowed to arbitrarily consider a vector's multiple as its image, without actually applying the transformation considered. This would also shed light on the perplexity shown before by these two students. When Claudio asks "does the eigenvector always coincides with the linear vector?", he seems to be confused by the fact that v appears in both the left and the right side of the formula as if he did not know which one to take to compute the eigenvalue. The same doubt seems to emerge for Antonio when he states that the eigenvector "would be v which finds v".

4.3 Description of discourse in Group 2

The second observed group is also formed by three students, Davide, Emma, and Flavio. The discussion predominantly involves Davide and Emma, with Flavio primarily listening and contributing minimally. Once they initiate the task, Davide and Emma deliberate on how to approach answering the guiding question. Emma suggests starting "by saying what it [an eigenvalue] is", while Davide insists on the importance of starting with an example to make it clear what it is. He draws a cartesian plane and provides an example that he recalls the lecturer presenting in class:

1. Davide: There was a rotation. That if this goes this way (drawing a random vector in the first quadrant of the cartesian plane, and showing with an arrows that it rotates anticlockwise, as visible in Figure 3), the other one goes there (drawing a second vector on the second quadrant and show with an arrow that it rotates clockwise – Figure 3). And so, obviously, these two don't have the same direction, but for some specific coordinates, they do have the same direction (drawing a vector lying on the y-axis). For α equal to specific values, these two overlap or we could say that one v is equal to v (and writes $1v = v$).

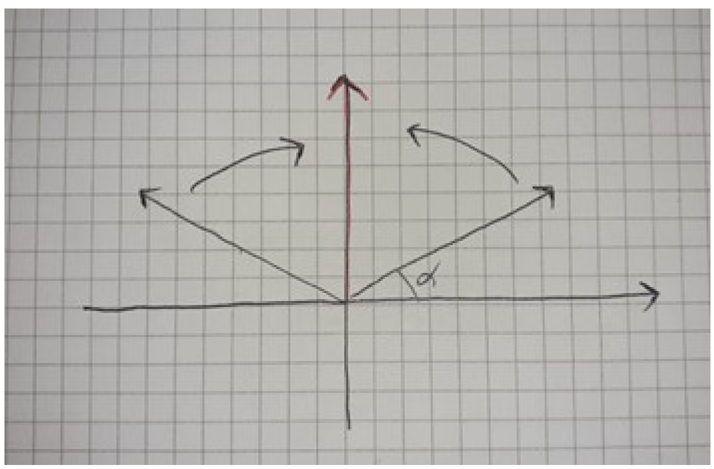

Figure 3: Davide's drawing

Davide talks about a "rotation" but what he is really talking about and representing is in reality a symmetry. The thing that probably confuses him and makes him talk about "rotation" is that he immediately represents a vector and its image, speaking of them as two different vectors. After that, he "rotates" the first vector and consequently the second. What he is doing with this apparent rotation (and probably doing in imitation of what he has seen the teacher do in class) is exploring the various positions that the starting vector and consequently its image can take according to the transformation involved, that is, its symmetry with respect to the y-axis. He then points out that "for specific values of α", by which he means the vector of the same (arbitrarily chosen) length lying on the y-axis, the two vectors (vector and its image) overlap.

At this point, Davide attempts to provide additional examples, but they appear to confuse the other two members of the small group. The first author, who is present during the activity, tries to redirect the focus to the initial example given. She emphasizes that they are exploring a specific transformation; one that, when given a vector, yields its image symmetrical with respect to the y-axis. She then proceeds to inquire about the eigenvalues and eigenvectors of this particular transformation.

resume Flavio: So, I would say that the eigenvalues are 1, because it brings the vector into the multiple of itself and so it is a multiple of 1.

resume Davide: But how can you explain that?

resume Emma: Well... it is all those times in which the vectors overlap, they coincide.

resume Davide: But it is not that they always coincide. You must explain the generic concept of eigenvalue.

resume Emma: But you made that example!

resume Davide: Yes but I would make the example saying, in all these cases, when the direction is the same, if we make the ratio between the lengths of the two vectors, the result is 1. [...]

resume Davide: In this case it is 1.

resume Emma: And this is the scalar that multiplied to the vector...

resume Davide: But if this example was like this...

Davide replicates the previous illustration, this time doubling the length of the image vector. He endeavours to illustrate to his classmates that with the vectors configured in this manner, as they both converge on the y-axis (reiterating the concept from earlier that the vectors rotate towards the y-axis), they share the same direction. However, the crucial distinction lies in the ratio between their lengths, which is now 2 instead of 1. After reflecting on this example, the group tries to synthesize their thoughts.

resume Davide: I would say so: for all those case where a vector is image of itself, this function..

resume Emma: But this is a specific case.

resume Davide: Yes, but I started with the example of this function and then I extend it to all the functions.

resume Emma: And how can you extend it?

resume Davide: It is a concept that I am trying to say. For a generic function...

resume Emma: So, any function that has this ratio, I mean, this symmetry...

resume Davide: Exactly! But not for a symmetry. I am

resume Emma: In general

resume Davide: For any function where the image of a vector is equal to a multiple of the vector itself, that is a number that is multiplied for the vector itself that in this case (referring to the example in Figure 3) is 1 and in this other -1 (referring to the same example but to the vectors lying on the x-axis), then we will call that value "eigenvalue".

resume Emma: And the vectors "eigenvectors".

resume Davide: And all the vectors that respect this criterion "eigenvectors".

resume Emma: So, eigenvalue and eigenvectors are... the eigenvalue, in this case -1, always works for those functions that go into the same direction.

resume Davide: Yes, if you talk about the plane.

4.4 Interpretation of Davide and Emma's interaction

The analysis of this excerpt, while rooted in results, may not be entirely straightforward, and the forthcoming interpretation is acknowledged to be somewhat subjective. Nevertheless, it appears that a commognitive conflict is arising here as well. Upon examining the entire sequence, it becomes apparent that Davide has grasped the concepts of eigenvectors and eigenvalues. However, it seems that the discourse between Davide and Emma is incommensurable. In other words, due to the different representations employed and the terminology used to describe them, it appears that at certain points in their conversation, the two students are using the same terms to refer to different concepts.

Davide's drawing (Figure 3) represents a crucial element in the discussion. In his representation, the distinction that one of the two vectors is independent, while the other is dependent— with its image dictated by the prescribed transformation (symmetry with respect to the y-axis)— is not immediately evident. Specifically, his discourse involves rotation, and he mentions that, for certain values of λ, the two vectors overlap. The commognitive conflict here seems to emerge in two ways. Firstly, the fact that Davide uses the term 'overlap' initially makes Emma think that in order to be eigenvectors, the vectors must overlap. Davide recognizes the presence of this misconception by Emma and attempts to resolve it by providing an example in which the two vectors have twice the length of one another. However, in this case, the dependence between the vector and its image is even more tenuous. In fact, it appears that in this instance, Davide himself is constructing the function arbitrarily, ensuring that one of the two vectors is twice the length of the other. In our opinion, this may further reinforce the notion that eigenvalues exist for functions that are defined by taking a multiple of the initial vector as its image. Furthermore, the fact that it is not evident that the dependence between the two vectors is dictated by the transformation in question, seems to make Emma think that what for Davide are different vector-image configurations for the same transformation, are instead different transformations. In fact, later in the conversation, she explicitly states: "any function that has this ratio ..." and "the eigenvalue [...] always works for those functions that go into the same direction". It seems, therefore, that for Emma, there are functions for which all vectors have the property of having their image aligned, and in those cases, "the eigenvalue works"—that is, probably, it exists. However, she does not grasp the fact that for every transformation, there can be some vectors with this property. The way in which she and Davide use terms like "exists" or "any" in the dialogue certainly supports this commognitive conflict.

5. Conclusions

In the two excerpts, recurring elements are noticeable. Despite their different manifestations, it appears that in both instances, a participant in the discourse fails to grasp a fundamental aspect of the definition of eigenvalues and eigenvectors: a precise mapping is provided, and for that specific mapping, vectors may exist whose images are multiples of themselves. Instead, it seems that these students (Antonio in the first example and seemingly Emma in the second) are under the impression that the function itself is defined in such a way that each vector has, as its image, a multiple of itself. This misconception is underpinned by a commognitive conflict, wherein the same sentences or representations are employed and interpreted differently by various participants in the discourse based on distinct metarules - in the sense of [19]. In the first example, the conflict arises from the use of the term "is defined" while in the second example, it appears to stem from how the example application is represented and described. Both conflicts are closely tied to the classical formulation of the definition of eigenvalues and eigenvectors. For instance, the definition provided by the teacher of one of the involved courses, which can be taken as a model for the classically given definition, is: "Let $f : V \to V$ be an endomorphism on the vectorial space V. We say that a vector $v \in V$, $v \neq 0_V$ is an eigenvector of f if there exists a value $\lambda \in K$ (the field of scalars), called its eigenvalue, such that $f(v) = \lambda v$ stands". In this formulation, two elements are likely to cause confusion. Firstly, the expression $f(v) = \lambda v$ might be misinterpreted as implying that $f(v)$ is defined as $f(v) = \lambda v$, leading to a potential reinforcement of an interpretation as that by Antonio. Secondly, the term 'exists' appears to be associated with λ. The way it is formulated assumes, almost taken for granted, the existence of vectors with such a property for that particular application. Consequently, the emphasis is placed on the value λ rather than on what happens to v, considering the specific application. This tendency is observed in both groups, mirroring Antonio's and Emma's interpretations of their peers' examples. In these cases, there seems to be a tendency to overlook the crucial point that a vector's image is determined by the given application: only instances where the image is a multiple of the initial vector qualify this as an eigenvector. This interpretation also stems from a challenge in understanding the inherent logical dependencies in the classical definition through the formulation $f(v) = \lambda v$ (or equivalently $Av = \lambda v$). A correct reading of it would be that there are some linear applications f, for which $\exists v\, s.t. f(v) = \lambda v$ for some $\lambda \in K$ [1]. Contrary to the intended interpretation, students like Antonio or Emma tend to interpret it as: there exist linear applications f such that $\forall v$, $\exists \lambda[(v, \lambda v) \in f \longleftrightarrow fv = \lambda v]$. In our experience, we have observed this interpretation in many first-year STEM students taking linear algebra courses.

What seems to happen in the two groups is that somehow the definition they have already seen in class guides them in their reasoning. However, the meaning of the definition had remained obscure to them. In their reasoning, they start from an example, which would have the great potential of being able to start them from an operational interpretation of eigenvalues and eigenvectors. However, they have the bias of having already seen the structural definition, which they somehow want to make sense of at the same time as exploring the example. The fact that in both groups, there are those who have actually understood what eigenvalues and eigenvectors are but use representations and terms that are interpreted by their peers according to different metarules, generates commognitive conflicts. As a result, a pseudostructural conception of the concepts involved seems to be reinforced.

These results confirm students' difficulties in dealing with the formula $f(v) = \lambda v$ identified also by Stewart and Thomas [21]. Moreover we have tried to delve into the reasons behind such difficulties, showing how also the way students engage in discourses about these concepts might reveal and reinforce some misconceptions. It seems that, in their attempts at giving an operational meaning to eigenvalues and eigenvectors, through the analysis of examples, students are strongly biased by the fact that they have already encountered their definition. In this way, the passage between process and object is not smooth and results in reaching a pseudostructural conception of eigenvalues and eigenvectors. One potential didactical implication is the importance of starting with an operational interpretation of mathematical concepts, as suggested by Sfard, even at the advanced level of university instruction. Specifically, for eigentheory, instead of immediately presenting the definition of eigenvectors and eigenvalues, it might be beneficial to begin with the exploration of examples. Introduce a linear transformation (in the form of a geometrically known transformation or as a matrix) and ask students to determine if there are vectors whose image

1. Equivalently, in prenex normal form [20], the formula would be: $\exists f \exists v \exists \lambda (f(v) = \lambda v)$.

is a multiple of themselves. Only then could these vectors be designated as eigenvectors, with the ratio between the image and the vector representing the corresponding eigenvalue. This approach can enable students to initiate their understanding from exploring a process before transforming it into a mathematical object. Furthermore, it might reveal to students the inherent logical dependence in the formal definition through concrete examples.

Following this pilot study, we have indeed formulated the subsequent phase of the research project, taking into account these considerations. Specifically, emphasis has been placed on the utilization of dynamic digital representations of linear applications, such as videos or an explorable GeoGebra applet. This approach aims to support the initial process-like conceptualization of eigenvectors, preventing misconceptions likely to arise in a static drawing, as exemplified by Davide in the second example.

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