Bi-endomorphism induces new types of derivations on BH-algebras

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Abstract. In this paper, we study the concepts of an (l, r) and an (r, l)-τ-derivation on a BH-algebra, which is induced by a left and a right bi-endomorphism and we provide

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important properties. In addition, the relationship among those derivations is also considered.

Keywords: BH-algebra, bi-endomorphism, derivation.

MSC 2020: 03G25.

1. Introduction

A new notion called a BH-algebra, a generalization of BCH/BCI/BCK-algebras, was introduced in 1998 in [13] by Jun, Roh and Kim. They defined the notions of ideals and boundedness in BH-algebras, showed that there is a maximal ideal in bounded BH-algebras, established the construction of the quotient BH-algebras via translation ideals, and obtained the fundamental theorem of homomorphisms for BH-algebras as a consequence. In 2000, Zhang, Jun and Roh [18] introduced a normal BH-algebra. They considered the branch in BH-algebra and investigated some related properties. A theoretical approach to the ideal structure in BH-algebras was established and introduced by Kim and Ahn [14] in 2012. The notions of a falling subalgebra, a falling ideal, a falling strong ideal, a falling n-fold strong ideal and a falling translation ideal of a BH-algebra are introduced. Some fundamental properties are investigated. Relations among a falling subalgebra, a falling ideal, a falling strong ideal, and a falling n-fold strong ideal are stated. A relation between a fuzzy subalgebra/ideal and a falling subalgebra/ideal is provided. In 2014, Abbass and Neamah [3] introduced an implicative ideal with respect to an element of a BH-algebra and some related properties are investigated. The relationships among this notion and other types of ideals of a BH-algebra are given. In 2015, Abbass and Mahdi [2] introduced the notion of a BH-algebra related to a B-algebra as a new class of a BH-algebra. Also, they introduced the notion of an ideal of a BH-algebra related to a B-algebra, and they stated and proved some theorems and examples which determine the relationships between these notions and some types of ideals of a BH-algebra. In 2018, Rajkumar and Kandara [17] introduced some results on derivations on rings and the generalization of BCK and BCI-algebras, and they defined (α, β) derivations on BH-algebras and investigate some important results. The notion of fuzzy BH-subalgebra of a BH-algebra with respect to a triangular norm, and they show that the Cartesian product of two fuzzy BH-subalgebras of a BH-algebra X with respect to a triangular norm is a fuzzy BH-subalgebra of the product of X with respect to the norm introduced by Anitha and Kandaraj [4] in 2019. In 2020, Ganesan and Kandaraj [9] introduced the notion of f-derivations on BH-algebras, the regular f-derivations, and the composition of f-derivations on BH-algebras, and they investigated interesting and elegant results. Next year, they introduced the notion of generalized derivations on BH-algebras and investigated simple, interesting, and elegant results, introduced the notion of t-derivations on BH-algebras and investigated simple, interesting, and elegant results; introduced the notion of generalized (g, h)-derivations on BH-algebras and investigated simple, interesting, and elegant results; and introduced the notion of composition of f-derivations on BH-algebras and investigated simple, interesting, and elegant results ([11, 6, 10, 7]).

In this paper, we study the concepts of an (l, r) and an (r, l)-τ-derivation on a BH-algebra, which is induced by a left and a right bi-endomorphism and provide important properties. In addition, the relationship among those derivations is also considered.

2. Preliminaries

First, we review some basic definitions and theorems required in our work.

Definition 2.1 ([13]). *Let X be a set with a binary operation $\circledast$ and a constant 0. Then $(X, \circledast, 0)$ is called a **BH-algebra** if it satisfies the following axioms:*

(BH1) $(\forall x \in X)(x \circledast x = 0)$,

(BH2) $(\forall x \in X)(x \circledast 0 = x)$,

(BH3) $(\forall x, y \in X)(x \circledast y = 0, y \circledast x = 0 \Rightarrow x = y)$.

If there is conditions (BH1), (BH2), and

(B) $(\forall x, y, z \in X)((x \circledast y) \circledast z = x \circledast (z \circledast (0 \circledast y)))$,

then ([16]) $(X, \circledast, 0)$ is called a B-algebra.

Example 2.1 ([13]). Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the Cayley table as follows:

$\circledast$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Then $(X, \circledast, 0)$ is a BH-algebra.

Definition 2.2 ([13]). *Let S be a nonempty subset of a BH-algebra X. Then S is called a **BH-subalgebra** of X if $x \circledast y \in S$ for all $x, y \in S$.*

By (BH1), we know that every subalgebra S in any BH-algebra X satisfies the condition $0 \in S$.

Example 2.2 ([13]). In Example 2.4, in X, the set $S_1 = \{0, 1, 2\}, S_2 = \{0, 1\}$, $S_3 = \{0, 1, 3\}$ and $S_4 = \{0, 2\}$ are BH-subalgebras of X, while $S_5 = \{0, 2, 3\}$ is not a BH-subalgebra of X.

Definition 2.3 ([1]). *A BH-algebra X is called **medial** if $x \circledast (x \circledast y) = y$ for all $x, y \in X$.*

Definition 2.4 ([5]). *A BH-algebra X is called **associative** if $(x \divideontimes y) \divideontimes z = x \divideontimes (y \divideontimes z)$ for all $x, y, z \in X$.*

Theorem 2.1 ([3]). *Let X be an associative BH-algebra. Then*

(BH4) $(\forall x \in X)(0 \divideontimes x = x)$,

(BH5) $(\forall x, y \in X)(x \divideontimes y = y \divideontimes x)$,

(BH6) $(\forall x, y \in X)(x \divideontimes (x \divideontimes y) = y)$, *medial*

(BH7) $(\forall x, y, z \in X)((z \divideontimes x) \divideontimes (z \divideontimes y) = x \divideontimes y)$,

(BH8) $(\forall x, y \in X)(x \divideontimes y = 0 \Rightarrow x = y)$,

(BH9) $(\forall x, y \in X)((x \divideontimes (x \divideontimes y)) \divideontimes y = 0)$,

(BH10) $(\forall x, y, z \in X)((x \divideontimes y) \divideontimes z = (x \divideontimes z) \divideontimes y)$.

Definition 2.5. *A BH-algebra X corresponds to **BH11-13** if the following conditions are satisfied.*

(BH11) $(\forall x \in X)(0 \divideontimes (0 \divideontimes x) = x)$,

(BH12) $(\forall x, y, z \in X)(z \divideontimes x = z \divideontimes y \Rightarrow x = y)$,

(BH13) $(\forall x, y, z \in X)(x \divideontimes z = y \divideontimes z \Rightarrow x = y)$.

The concept of 0-commutativity of B-algebras is developed in [15].
For a associative BH-algebra X, we denote $x \wedge y = y \divideontimes (y \divideontimes x)$ for all $x, y \in X$.
We will get the following results as follows:

(BH14) $(\forall x \in X)(x \wedge 0 = x)$ (by (BH11)).

(BH15) $(\forall x \in X)(x \wedge x = x)$ if X is an associative BH-algebra (by (BH6)).

Definition 2.6 ([13]). *Let X be a BH-algebra and I be a subset of X. Then I is called a **BH-ideal** of X if*

(I1) $0 \in I$,

(I2) $(\forall x, y \in X)(x \divideontimes y \in I, y \in I \Rightarrow x \in I)$.

Example 2.3 ([13]). Let $X = \{0, 1, 2\}$ be a BH-algebra with the following table:

$\divideontimes$	0	1	2
0	0	0	1
1	1	0	2
2	2	2	0

We can easily show that $I = \{0, 1\}$ is a BH-ideal of X.

Example 2.4 ([13])**.** Let $X = \{0, 1, 2, 3\}$ be a BH-algebra with the Cayley table as follows:

$\circledast$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $(X, \circledast, 0)$ is a BH-algebra. The subset $S = \{0, 2\}$ is a BH-subalgebra of X, but it is not a BH-ideal of X because, for example, it is valid

$$1 \circledast 2 = 0 \in S, 2 \in S, \text{ but } 1 \notin S.$$

Definition 2.7 ([8])**.** *Let X be a BH-algebra. A self map $d : X \to X$ is called*

(1) *a **(left-right)-derivation (**(l, r)**-derivation)** of X if*

$$(\forall x, y \in X)(d(x \circledast y) = (d(x) \circledast y) \wedge (x \circledast d(y))),$$

(2) *a **(right-left)-derivation (**(r, l)**-derivation)** of X if*

$$(\forall x, y \in X)(d(x \circledast y) = (x \circledast d(y)) \wedge (d(x) \circledast y)),$$

(3) *a **derivation** of X if it is both an (l, r) and an (r, l)-derivation of X.*

Definition 2.8 ([8])**.** *Let X be a BH-algebra. A self map d of X is said to be **regular** if $d(0) = 0$.*

Definition 2.9 ([12])**.** *Let X be a BH-algebra and d, D be two self maps of X with the composition $\circ$, that is, $d \circ D : X \to X$ as $(d \circ D)(x) = d(D(x))$ for all $x \in X$.*

3. Main results

In this section, we introduce bi-endomorphisms on BH-algebras and prove some results of some results of bi-endomorphisms of a BH-algebra X and its derivations as follows:

Definition 3.1. *Let X be BH-algebra. A mapping $\tau : X \times X \to X$ is called*

(1) *a **left bi-endomorphism** on X if*

$$(\forall x, y, z \in X)(\tau(x \circledast y, z) = \tau(x, z) \circledast \tau(y, z)),$$

(2) *a **right bi-endomorphism** on X if*

$$(\forall x, y, z \in X)(\tau(x, y \circledast z) = \tau(x, y) \circledast \tau(x, z)),$$

(3) *a **bi-endomorphism** on X if it is a left and a right bi-endomorphism on X.*

Throughout this section, we assume that τ_l and τ_r are a left and a right bi-endomorphism of X, respectively.

Let $x \in X$. By (BH2), we have

$$\tau_l(x, x) \divideontimes 0 = \tau_l(x, x) = \tau_l(x \divideontimes 0, x) = \tau_l(x, x) \divideontimes \tau_l(0, x)$$

and

$$\tau_r(x, x) \divideontimes 0 = \tau_r(x, x) = \tau_r(x, x \divideontimes 0) = \tau_r(x, x) \divideontimes \tau_r(x, 0).$$

For an associative BH-algebra X, we have (BH12) which is proved by (BH6). Thus, we get

(BH16) $(\forall x \in X)(\tau_l(0, x) = 0)$,

(BH17) $(\forall x \in X)(\tau_r(x, 0) = 0)$.

Example 3.1. In Example 2.4, we define a mapping $\tau_1 : X \times X \to X$ by

$$\tau_1(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(0, 0), (1, 0), (3, 0)\} \\ 0, & \text{otherwise.} \end{cases}$$

Then τ_1 is a left bi-endomorphism on X.

Definition 3.2. *Let X be BH-algebra. A mapping $\tau : X \times X \to X$ is said to be **symmetric** if $\tau(x, y) = \tau(y, x)$ for all $x, y \in X$.*

Remark 3.1. Every symmetric left (right) bi-endomorphism on a BH-algebra is a bi-endomorphism.

For any BH-algebra X, define a mapping $0 : X \times X \to X$ by $0(x, x) = 0$ for all $x \in X$, which is a bi-endomorphism on X.

Definition 3.3. *Let τ be bi-endomorphism on a BH-algebra X. A BH-ideal I of X is called a τ**-ideal** of X if $\tau(x, y) \in I$ for all $x, y \in I$.*

Example 3.2. Let $X = \{0, 1, 2\}$ be as in Example 2.3. Then $I = \{0, 1\}$ is a BH-ideal of X. If we define the self map $\tau_1 : X \times X \to X$ in the following way $(\forall x, y \in X)(\tau_1(x, y) \in I)$, then τ_1 is a bi-endomorphism on X and I is a τ_1-ideal of X.

Next, we generalize derivations on a BH-algebra X with a left (right) bi-endomorphism $\tau : X \times X \to X$ from the concept of Definition 2.7.

Definition 3.4. *Let X be a BH-algebra. Let $\tau : X \times X \to X$ be a left (right) bi-endomorphism on X. A self map d_τ on X is called*

(1) *an* (l,r)-***τ-derivation*** *of* X *if*

$$(\forall x, y \in X)(d_\tau(x \divideontimes y) = (d_\tau(x) \divideontimes \tau(x,y)) \wedge (\tau(x,y) \divideontimes d_\tau(y))),$$

(2) *an* (r,l)-***τ-derivation*** *of* X *if*

$$(\forall x, y \in X)(d_\tau(x \divideontimes y) = (\tau(x,y) \divideontimes d_\tau(y)) \wedge (d_\tau(x) \divideontimes \tau(x,y))),$$

(3) *a* ***τ-derivation*** *of* X *if it is both an* (l,r)- *and an* (r,l)-*τ-derivation of* X.

Example 3.3. Consider the following BH-algebra X with the Cayley table.

$\divideontimes$	0	1	2
0	0	0	0
1	1	0	1
2	2	1	0

We define $\tau = 0$ and $d_\tau : X \to X$ by $d_\tau(x) = 0$. Then τ is a bi-endomorphism on X and d_τ is an (l,r)-τ-derivation of X.

Definition 3.5. *Let d be a self map on a BH-algebra X. A BH-ideal I of X is said to be* d-***invariant*** *if $d(I) \subseteq I$.*

Example 3.4. In Example 2.3, we have $I = \{0,1\}$ is a BH-ideal of X. From Example 3.3, we have d_τ is an (l,r)-τ-derivation of X. It is obvious that $d_\tau(x) \in I$ for all $x \in I$, that is, $d_\tau(I) \subseteq I$. Hence, I is d_τ-invariant.

Theorem 3.1. *Let d_{τ_l} be an (l,r)-τ_l-derivation on an associative BH-algebra X. Then d_{τ_l} is regular if and only if $d_{\tau_l}(0 \divideontimes x) = 0$ for all $x \in X$ with condition* $(BH12)$.

Proof. Suppose that d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned}
d_{\tau_l}(0 \divideontimes x) &= (d_{\tau_l}(0) \divideontimes \tau_l(0,x)) \wedge (\tau_l(0,x) \divideontimes d_{\tau_l}(x)) \\
\text{(BH16)} \quad &= (0 \divideontimes 0) \wedge (0 \divideontimes d_{\tau_l}(x)) \\
\text{(BH1)} \quad &= 0 \wedge (0 \divideontimes d_{\tau_l}(x)) \\
&= (0 \divideontimes d_{\tau_l}(x)) \divideontimes ((0 \divideontimes d_{\tau_l}(x)) \divideontimes 0) \\
\text{(BH2)} \quad &= (0 \divideontimes d_{\tau_l}(x)) \divideontimes (0 \divideontimes d_{\tau_l}(x)) \\
\text{(BH1)} \quad &= 0.
\end{aligned}$$

Conversely, suppose that $d_{\tau_l}(0 \divideontimes x) = 0$ for all $x \in X$. Then

$$\begin{aligned}
0 &= d_{\tau_l}(0 \divideontimes 0) \\
&= (d_{\tau_l}(0) \divideontimes \tau_l(0,0)) \wedge (\tau_l(0,0) \divideontimes d_{\tau_l}(0)) \\
\text{(BH16)} \quad &= (d_{\tau_l}(0) \divideontimes 0) \wedge (0 \divideontimes d_{\tau_l}(0)) \\
\text{(BH2)} \quad &= d_{\tau_l}(0) \wedge (0 \divideontimes d_{\tau_l}(0)) \\
&= (0 \divideontimes d_{\tau_l}(0)) \divideontimes ((0 \divideontimes d_{\tau_l}(0)) \divideontimes d_{\tau_l}(0)).
\end{aligned}$$

By (BH8) and (BH2), we get $(0 \circledast d_{\tau_l}(0)) \circledast 0 = 0 \circledast d_{\tau_l}(0) = (0 \circledast d_{\tau_l}(0)) \circledast d_{\tau_l}(0)$. Using (BH12), we have $d_{\tau_l}(0) = 0$. Hence, d_{τ_l} is regular. $\square$

Theorem 3.2. *Let d_{τ_r} be an (r,l)-τ_r-derivation on an associative BH-algebra X. Then d_{τ_r} is regular and $\tau_r(x,0) = 0$ for all $x \in X$ if and only if $d_{\tau_r}(x \circledast 0) = 0$ for all $x \in X$.*

Proof. Suppose that d_{τ_r} is regular and $\tau_r(0,x) = 0$ for all $x \in X$. Let $x \in X$. Then

$$\begin{aligned}
& d_{\tau_r}(0 \circledast x) = (d_{\tau_r}(0) \circledast \tau_r(0,x)) \wedge (\tau_r(0,x) \circledast d_{\tau_r}(x)) \\
& \quad = (0 \circledast 0) \wedge (0 \circledast d_{\tau_r}(x)) \\
\text{(BH1)} \quad & \quad = 0 \wedge (0 \circledast d_{\tau_r}(x)) \\
& \quad = (0 \circledast d_{\tau_r}(x)) \circledast ((0 \circledast d_{\tau_r}(x)) \circledast 0) \\
\text{(BH2)} \quad & \quad = (0 \circledast d_{\tau_r}(x)) \circledast (0 \circledast d_{\tau_r}(x)) \\
\text{(BH1)} \quad & \quad = 0.
\end{aligned}$$

Conversely, suppose that $d_{\tau_r}(0 \circledast x) = 0$ for all $x \in X$. By (BH1), we have $d_{\tau_r}(0) = d_{\tau_r}(0 \circledast 0) = 0$. Thus, d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned}
0 &= d_{\tau_r}(0 \circledast x) \\
&= (d_{\tau_r}(0) \circledast \tau_r(0,x)) \wedge (\tau_r(0,x) \circledast d_{\tau_r}(x)) \\
&= (0 \circledast \tau_r(0,x)) \wedge (\tau_r(0,x) \circledast d_{\tau_r}(x)) \\
&= (\tau_r(0,x) \circledast d_{\tau_r}(x)) \circledast ((\tau_r(0,x) \circledast d_{\tau_r}(x)) \circledast (0 \circledast \tau_r(0,x))).
\end{aligned}$$

By (BH8) and (BH2), we get $(\tau_r(0,x) \circledast d_{\tau_r}(x)) \circledast 0 = \tau_r(0,x) \circledast d_{\tau_r}(x) = (\tau_r(0,x) \circledast d_{\tau_r}(x)) \circledast (0 \circledast \tau_r(0,x))$. Using (BH12), we have $0 \circledast \tau_r(0,x) = 0$. Using (BH8) again, we obtain $\tau_r(0,x) = 0$. $\square$

Next, we give a necessary and sufficient condition for (r,l)-τ_l-derivation and (r,l)-τ_r-derivation to be regular with some properties.

Theorem 3.3. *Let d_{τ_l} be an (l,r)-τ_l-derivation on an associative BH-algebra X. Then d_{τ_l} is regular and $\tau_l(0,x) = 0$ for all $x \in X$ if and only if $d_{\tau_l}(x) = 0$ for all $x \in X$.*

Proof. Suppose that d_{τ_l} is regular and $\tau_l(x,0) = 0$ for all $x \in X$. Let $x \in X$. Then

$$\begin{aligned}
\text{(BH2)} \quad & d_{\tau_l}(x) = d_{\tau_l}(x \circledast 0) \\
& \quad = (\tau_l(x,0) \circledast d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \circledast \tau_l(x,0)) \\
& \quad = (0 \circledast 0) \wedge (d_{\tau_l}(x) \circledast 0) \\
\text{(BH2)} \quad & \quad = 0 \wedge d_{\tau_l}(x) \\
& \quad = d_{\tau_l}(x) \circledast (d_{\tau_l}(x) \circledast 0) \\
\text{(BH2)} \quad & \quad = d_{\tau_l}(x) \circledast d_{\tau_l}(x) \\
\text{(BH1)} \quad & \quad = 0.
\end{aligned}$$

Conversely, suppose that $d_{\tau_l}(x) = 0$ for all $x \in X$. Then d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned}
& 0 = d_{\tau_l}(x) \\
\text{(BH2)} \quad & = d_{\tau_l}(x \circledast 0) \\
& = (\tau_l(x,0) \circledast d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \circledast \tau_l(x,0)) \\
& = (\tau_l(x,0) \circledast 0) \wedge (0 \circledast \tau_l(x,0)) \\
\text{(BH2)} \quad & = \tau_l(x,0) \wedge (0 \circledast \tau_l(x,0)) \\
& = (0 \circledast \tau_l(x,0)) \circledast ((0 \circledast \tau_l(x,0)) \circledast \tau_l(x,0)).
\end{aligned}$$

By (BH8) and (BH2), we get $(0 \circledast \tau_l(x,0)) \circledast 0 = 0 \circledast \tau_l(x,0) = (0 \circledast \tau_l(x,0)) \circledast \tau_l(x,0)$. Using (BH12), we have $\tau_l(x,0) = 0$. □

Theorem 3.4. *Let d_{τ_r} be an (r,l)-τ_r-derivation on an associative BH-algebra X. Then d_{τ_r} is regular if and only if $d_{\tau_r}(x) = 0$ for all $x \in X$.*

Proof. Suppose that d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned}
\text{(BH2)} \quad & d_{\tau_r}(x) = d_{\tau_r}(x \circledast 0) \\
& = (\tau_r(x,0) \circledast d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \circledast \tau_r(x,0)) \\
\text{(BH15)} \quad & = (0 \circledast 0) \wedge (d_{\tau_r}(x) \circledast 0) \\
\text{(BH2)} \quad & = 0 \wedge d_{\tau_r}(x) \\
& = d_{\tau_r}(x) \circledast (d_{\tau_r}(x) \circledast 0) \\
\text{(BH2)} \quad & = d_{\tau_r}(x) \circledast d_{\tau_r}(x) \\
\text{(BH1)} \quad & = 0.
\end{aligned}$$

The converse shows that d_{τ_r} is regular. □

By (BH16), we define an (l,r)-τ_l-derivation d_{τ_l} (or (r,l)-τ_l-derivation) on a BH-algebra X by $d_{\tau_l}(x) = \tau_l(x,0)$ for all $x \in X$, and obtain the following proposition.

Proposition 3.1. *Let d_{τ_l} be an (l,r)-τ_l-derivation on an associative BH-algebra X. If $d_{\tau_l}(x) = \tau_l(0,x)$ for all $x \in X$, then d_{τ_l} is the zero self map on X and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x,0)$ for all $x \in X$. By (BH16), we have $d_{\tau_l}(0) = \tau_l(0,0) = 0$, that is, d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned}
\text{(BH2)} \quad & d_{\tau_l}(x) = d_{\tau_l}(x \circledast 0) \\
& = (d_{\tau_l}(x) \circledast \tau_l(x,0)) \wedge (\tau_l(x,0) \circledast d_{\tau_l}(0)) \\
& = (d_{\tau_l}(x) \circledast d_{\tau_l}(x)) \wedge (d_{\tau_l}(x) \circledast d_{\tau_l}(0)) \\
\text{(BH1)} \quad & = 0 \wedge (d_{\tau_l}(x) \circledast d_{\tau_l}(0)) \\
& = (d_{\tau_l}(x) \circledast d_{\tau_l}(0)) \circledast ((d_{\tau_l}(x) \circledast d_{\tau_l}(0)) \circledast 0) \\
\text{(BH2)} \quad & = (d_{\tau_l}(x) \circledast d_{\tau_l}(0)) \circledast (d_{\tau_l}(x) \circledast d_{\tau_l}(0)) \\
\text{(BH1)} \quad & = 0.
\end{aligned}$$

Hence, d_{τ_l} is the zero self map on X. $\square$

Proposition 3.2. *Let d_{τ_l} be an (l,r)-τ_l-derivation on an associative BH-algebra X. If $d_{\tau_l}(x) = \tau_l(0,x)$ for all $x \in X$, then d_{τ_l} is the zero self map on X and d_{τ_l} is regular.*

Proof. Suppose that $d_{\tau_l}(x) = \tau_l(x,0)$ for all $x \in X$. By (BH16), we have $d_{\tau_l}(0) = \tau_l(0,0) = 0$, that is, d_{τ_l} is regular. Let $x \in X$. Then

$$\begin{aligned} & d_{\tau_l}(0 \circledast x) = (\tau_l(0,x) \circledast d_{\tau_l}(x)) \wedge (d_{\tau_l}(0) \circledast \tau_l(0,x)) \\ \text{(BH16)} \quad & \qquad = (0 \circledast d_{\tau_l}(x)) \wedge (0 \circledast 0) \\ \text{(BH1)} \quad & \qquad = (0 \circledast d_{\tau_l}(x)) \wedge 0 \\ \text{(BH11)} \quad & \qquad = 0 \circledast d_{\tau_l}(x). \end{aligned}$$

Hence, d_{τ_l} is the zero self map on X. $\square$

By (BH17), we define an (l,r)-τ_l-derivation d_{τ_l} (or (r,l)-τ_l-derivation) on a BH-algebra X by $d_{\tau_r}(x) = \tau_r(0,x)$ for all $x \in X$, and obtain the following proposition.

Proposition 3.3. *Let d_{τ_r} be an (l,r)-τ_r-derivation on an associative BH-algebra X. If $d_{\tau_r}(x) = \tau_r(x,0)$ for all $x \in X$, then $d_{\tau_r}(0 \circledast x) = 0 \circledast d_{\tau_r}(x)$ for all $x \in X$ and d_{τ_r} is regular with condition (BH11).*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0,x)$ for all $x \in X$. By (BH17), we have $d_{\tau_r}(0) = \tau_r(0,0) = 0$, that is, d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned} & d_{\tau_r}(0 \circledast x) = (d_{\tau_r}(0) \circledast \tau_r(0,x)) \wedge (\tau_r(0,x) \circledast d_{\tau_r}(x)) \\ & \qquad = (d_{\tau_r}(0) \circledast d_{\tau_r}(x)) \wedge (d_{\tau_r}(x) \circledast d_{\tau_r}(x)) \\ \text{(BH1)} \quad & \qquad = (d_{\tau_r}(0) \circledast d_{\tau_r}(x)) \wedge 0 \\ \text{(BH11)} \quad & \qquad = d_{\tau_r}(0) \circledast d_{\tau_r}(x) \\ & \qquad = 0 \circledast d_{\tau_r}(x). \end{aligned}$$

$\square$

Proposition 3.4. *Let d_{τ_r} be an (r,l)-τ_r-derivation on an associative BH-algebra X. If $d_{\tau_r}(x) = \tau_r(x,0)$ for all $x \in X$, then d_{τ_r} is the zero self map on X and d_{τ_r} is regular.*

Proof. Suppose that $d_{\tau_r}(x) = \tau_r(0,x)$ for all $x \in X$. By (BH17), we have $d_{\tau_r}(0) = \tau_r(0,0) = 0$, that is, d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned} & d_{\tau_r}(x) = d_{\tau_r}(x \circledast 0) \\ & \qquad = (\tau_r(x,0) \circledast d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \circledast \tau_r(x,0)) \\ \text{(BH17)} \quad & \qquad = (0 \circledast 0) \wedge (d_{\tau_r}(x) \circledast 0) \\ \text{(BH2)} \quad & \qquad = 0 \wedge d_{\tau_r}(x) \\ & \qquad = d_{\tau_r}(x) \circledast (d_{\tau_r}(x) \circledast 0) \\ \text{(BH2)} \quad & \qquad = d_{\tau_r}(x) \circledast d_{\tau_r}(x) \\ \text{(BH1)} \quad & \qquad = 0. \end{aligned}$$

Hence, d_{τ_r} is the zero self map on X. □

Theorem 3.5. *Let d_{τ_l} be a regular (r,l)-τ_l-derivation on associative BH-algebra X. Every τ_l-ideal of X is d_{τ_l}-invariant.*

Proof. Let X be an element in a τ_l-ideal I of X. Then

$$\begin{aligned}
&\text{(BH2)} && d_{\tau_l}(x) \divideontimes 0 = d_{\tau_l}(x)\\
&\text{(BH2)} && = d_{\tau_l}(x \divideontimes 0)\\
& && = (\tau_l(x,0) \divideontimes d_{\tau_l}(0)) \wedge (d_{\tau_l}(x) \divideontimes \tau_l(x,0))\\
& && = (\tau_l(x,0) \divideontimes 0) \wedge (d_{\tau_l}(x) \divideontimes \tau_l(x,0))\\
&\text{(BH2)} && = \tau_l(x,0) \wedge (d_{\tau_l}(x) \divideontimes \tau_l(x,0))\\
& && = (d_{\tau_l}(x) \divideontimes \tau_l(x,0)) \divideontimes ((d_{\tau_l}(x) \divideontimes \tau_l(x,0)) \divideontimes \tau_l(x,0))\\
&\text{(BH7)} && = d_{\tau_l}(x) \divideontimes (d_{\tau_l}(x) \divideontimes \tau_l(x,0)).
\end{aligned}$$

By (BH12), we get $0 = d_{\tau_l}(x) \divideontimes \tau_l(x,0)$. Since I is a τ_l-ideal of X and by (BH8), we have $d_{\tau_l}(x) = \tau_l(x,0) \in I$. Hence, I is d_{τ_l}-invariant. □

Theorem 3.6. *Let d_{τ_r} be a regular (r,l)-τ_r-derivation on an associative BH-algebra X. Every BH-ideal of X is d_{τ_r}-invariant. In particular, every τ_r-ideal of X is d_{τ_r}-invariant.*

Proof. Let x be an element in a BH-ideal I on X. Then

$$\begin{aligned}
&\text{(BH2)} && d_{\tau_r}(x) = d_{\tau_r}(x \divideontimes 0)\\
& && = (\tau_r(x,0) \divideontimes d_{\tau_r}(0)) \wedge (d_{\tau_r}(x) \divideontimes \tau_r(x,0))\\
&\text{(BH17)} && = (0 \divideontimes 0) \wedge (d_{\tau_r}(x) \divideontimes 0)\\
&\text{(BH2)} && = 0 \wedge d_{\tau_r}(x)\\
& && = d_{\tau_r}(x) \divideontimes (d_{\tau_r}(x) \divideontimes 0)\\
&\text{(BH2)} && = d_{\tau_r}(x) \divideontimes d_{\tau_r}(x)\\
&\text{(BH1)} && = 0 \in I.
\end{aligned}$$

Hence, I is d_{τ_r}-invariant. □

Next, we focus on the composition of τ_l and τ_r-derivations.

Theorem 3.7. *Let d_{τ_r} and D_{τ_r} be (l,r)-τ_r-derivations on an associative BH-algebra X. If $\tau_r(x,y) = y$ for all $x, y \in X$, then $d_{\tau_r} \circ D_{\tau_r}$ is also an (l,r)-τ_r-derivation on X.*

Proof. Suppose that $\tau_r(x,y) = y$ for all $x, y \in X$. Then

$$
\begin{aligned}
& (d_{\tau_r} \circ D_{\tau_r})(x \divideontimes y) \\
& = d_{\tau_r}(D_{\tau_r}(x \divideontimes y)) \\
& = d_{\tau_r}((D_{\tau_r}(x) \divideontimes \tau_r(x,y)) \wedge (\tau_r(x,y) \divideontimes D_{\tau_r}(y))) \\
& = d_{\tau_r}((D_{\tau_r}(x) \divideontimes y) \wedge (y \divideontimes D_{\tau_r}(y))) \\
\text{(BH15)} \quad & = d_{\tau_r}(D_{\tau_r}(x) \divideontimes y) \\
& = (d_{\tau_r}(D_{\tau_r}(x)) \divideontimes \tau_r(D_{\tau_r}(x), y)) \wedge (\tau_r(D_{\tau_r}(x), y) \divideontimes d_{\tau_r}(y)) \\
\text{(BH15)} \quad & = d_{\tau_r}(D_{\tau_r}(x)) \divideontimes \tau_r(D_{\tau_r}(x), y) \\
& = d_{\tau_r}(D_{\tau_r}(x)) \divideontimes y \\
& = d_{\tau_r}(D_{\tau_r}(x)) \divideontimes \tau_r(x,y) \\
\text{(BH15)} \quad & = (d_{\tau_r}(D_{\tau_r}(x)) \divideontimes \tau_r(x,y)) \wedge (\tau_r(x,y) \divideontimes d_{\tau_r}(D_{\tau_r}(y)) \\
& = ((d_{\tau_r} \circ D_{\tau_r})(x) \divideontimes \tau_r(x,y)) \wedge (\tau_r(x,y) \divideontimes (d_{\tau_r} \circ D_{\tau_r})(y)).
\end{aligned}
$$

Hence, $d_{\tau_r} \circ D_{\tau_r}$ is an (l,r)-τ_l-derivation on X. □

Theorem 3.8. *Let d_{τ_l} and D_{τ_l} be (r,l)-τ_l-derivations on an associative BH-algebra X. If $\tau_l(x,y) = x$ for all $x, y \in X$, then $d_{\tau_l} \circ D_{\tau_l}$ is also an (r,l)-τ_l-derivation on X.*

Proof. Suppose that $\tau_r(x,y) = x$ for all $x, y \in X$. Then

$$
\begin{aligned}
& (d_{\tau_l} \circ D_{\tau_l})(x \divideontimes y) \\
& = d_{\tau_l}(D_{\tau_l}(x \divideontimes y)) \\
& = d_{\tau_l}((\tau_l(x,y) \divideontimes D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) \divideontimes \tau_l(x,y))) \\
& = d_{\tau_l}((x \divideontimes D_{\tau_l}(y)) \wedge (D_{\tau_l}(x) \divideontimes x)) \\
\text{(BH15)} \quad & = d_{\tau_l}(x \divideontimes D_{\tau_l}(y)) \\
& = (\tau_l(x, D_{\tau_l}(y)) \divideontimes d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(x) \divideontimes \tau_l(x, D_{\tau_l}(y))) \\
\text{(BH15)} \quad & = \tau_l(x, D_{\tau_l}(y)) \divideontimes d_{\tau_l}(D_{\tau_l}(y)) \\
& = x \divideontimes d_{\tau_l}(D_{\tau_l}(y)) \\
& = \tau_l(x,y) \divideontimes d_{\tau_l}(D_{\tau_l}(y)) \\
\text{(BH15)} \quad & = (\tau_l(x,y) \divideontimes d_{\tau_l}(D_{\tau_l}(y))) \wedge (d_{\tau_l}(D_{\tau_l}(x)) \divideontimes \tau_l(x,y)) \\
& = (\tau_l(x,y) \divideontimes (d_{\tau_l} \circ D_{\tau_l})(y)) \wedge ((d_{\tau_l} \circ D_{\tau_l})(x) \divideontimes \tau_l(x,y)).
\end{aligned}
$$

Hence, $d_{\tau_l} \circ D_{\tau_l}$ is an (r,l)-τ_l-derivation on X. □

Proposition 3.5. *Let d_{τ_r} be an (r,l)-τ_r-derivation and D_{τ_r} be an (l,r)-τ_r-derivation on an associative BH-algebra X. Then*

(1) $(\forall x \in X)((d_{\tau_r} \circ D_{\tau_r})(x) = 0)$ *if d_{τ_r} is regular,*

(2) $(\forall x \in X)((D_{\tau_r} \circ d_{\tau_r})(x) = D_{\tau_r}(0) \divideontimes \tau_r(0, d_{\tau_r}(0)))$,

(3) $(\forall x \in X)((d_{\tau_r} \circ D_{\tau_r})(x) = 0 = (D_{\tau_r} \circ d_{\tau_r})(x))$ *if* d_{τ_r} *and* D_{τ_r} *are regular.*

Proof. (1) Suppose that d_{τ_r} is regular. Let $x \in X$. Then

$$\begin{aligned}
&\text{(BH2)} & (d_{\tau_r} \circ D_{\tau_r})(x) &= (d_{\tau_r} \circ D_{\tau_r})(x \ast 0) \\
& & &= d_{\tau_r}(D_{\tau_r}(x \ast 0)) \\
&\text{(BH15)} & &= d_{\tau_r}(D_{\tau_r}(x) \ast \tau_r(x, 0)) \\
&\text{(BH17)} & &= d_{\tau_r}(D_{\tau_r}(x) \ast 0) \\
&\text{(BH15)} & &= \tau_r(D_{\tau_r}(x), 0) \ast d_{\tau_r}(0) \\
&\text{(BH17)} & &= 0 \ast 0 \\
&\text{(BH1)} & &= 0.
\end{aligned}$$

(2) Let $x \in X$. Then

$$\begin{aligned}
&\text{(BH2)} & (D_{\tau_r} \circ d_{\tau_r})(x) &= (D_{\tau_r} \circ d_{\tau_r})(x \ast 0) \\
& & &= D_{\tau_r}(d_{\tau_r}(x \ast 0)) \\
&\text{(BH15)} & &= D_{\tau_r}(\tau_r(x, 0) \ast d_{\tau_r}(0)) \\
&\text{(BH17)} & &= D_{\tau_r}(0 \ast d_{\tau_r}(0)) \\
&\text{(BH15)} & &= D_{\tau_r}(0) \ast \tau_r(0, d_{\tau_r}(0)).
\end{aligned}$$

(3) It is straightforward by (1) and (2). □

Let $\mathrm{Der}_\tau(X)$ be the set of all τ-derivations on an associative BH-algebra X. For $d_\tau, D_\tau \in \mathrm{Der}_\tau(X)$, we define the binary operation $\curlywedge$ on $\mathrm{Der}_\tau(X)$ as follows:

$$(\forall x \in X)((d_\tau \curlywedge D_\tau)(x) = d_\tau(x) \wedge D_\tau(x)).$$

Indeed, let $x \in X$. Then

$$\begin{aligned}
& & (d_\tau \curlywedge D_\tau)(x) &= d_\tau(x) \wedge D_\tau(x) \\
&\text{(BH15)} & &= d_\tau(x).
\end{aligned}$$

Hence, $d_\tau \wedge D_\tau = d_\tau$ for all $d_\tau, D_\tau \in \mathrm{Der}_\tau(X)$.

4. Conclusion and discussion

In this paper, we have introduced the concepts of an (l, r) and an (r, l)-τ-derivation on a BH-algebra, which is induced by a left and a right bi-endomorphism, and provided important properties. The study found that the composition of (l, r) and (r, l)-τ-derivations is also an (l, r) and an (r, l)-τ-derivation on a 0-commutative BH-algebra, respectively. In addition, we can show that the set of all τ-derivations on a 0-commutative BH-algebra X is a left zero semigroup.

Finally, studying bi-endomorphisms in other algebras (d/BF/BG-algebras) is an interesting open problem.

Acknowledgment

This research was supported by the University of Phayao and the Thailand Science Research and Innovation Fund (Fundamental Fund 2025).

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Accepted: May 22, 2024