The (p,q,r)-generations of the Mathieu sporadic simple group M_{23}

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Abstract. A finite group G is called (l,m,n)*-generated*, if it is a quotient group of the triangle group $T(l,m,n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1 \rangle$. In [25], Moori posed the question of finding all the (p,q,r) triples, where p, q and r are prime numbers, such that a non-abelian finite simple group G is a (p,q,r)-generated. In this paper we establish all the (p,q,r)-generations of the Mathieu sporadic simple group M_{23}. GAP [16] and the Atlas of finite group representations [30] are used in our computations.

Keywords: conjugacy classes, generation, simple groups, sporadic groups.

1. Introduction

Generations of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple groups (see Woldar [32] for details). Also Di Martino et al. [23] established a useful connection between generation of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups.

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Recently more attention was given to the generation of finite groups by conjugate elements. In his PhD Thesis [29], Ward considered generation of a simple group by conjugate involutions satisfying certain conditions. In this paper we are interested in the generation of the Mathieu sporadic simple group M_{23} by two elements of prime orders not necessary distinct such that the product is an element of a prime order.

A finite group G is said to be (l,m,n)-*generated*, if $G = \langle x,y\rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of X in G containing elements of order l. The same applies to $[y]$ and $[z]$. In this case G is also a quotient group of the triangular group $T(l,m,n)$ and, by definition of the triangular group, G is also $(\sigma(l),\sigma(m),\sigma(n))$-generated group for any $\sigma \in S_3$. Therefore we may assume that $l \le m \le n$. In a series of papers [18, 19, 20, 21, 22, 24, 25], Moori and Ganief established all possible (p,q,r)-generations, where p, q and r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22}. Ashrafi in [3, 4] did the same for the sporadic simple groups He and HN. Also Darafsheh and Ashrafi established in [11, 12, 13, 14], the (p,q,r)-generations of the sporadic simple groups Co_1, Ru, $O'N$ and Ly. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

In establishing the (p,q,r)-generations of the group M_{23}, we follow the methods used in [6], [7]and [8], and also methods used in the recent papers [1] and [2] by Ali, Ibrahim and Woldar. Note that, in general, if G is a $(2,2,n)$-generated group, then G is a dihedral group and therefore G is not simple. Also by [9], if G is a non-abelian (l,m,n)-generated group, then either $G \cong A_5$ or $\frac{1}{l}+\frac{1}{m}+\frac{1}{n} < 1$. Thus for our purpose of establishing the (p,q,r)-generations of $G = M_{23}$, the only cases we need to consider are when $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$. The result on the (p,q,r)-generations of M_{23} can be summarized in the following theorem.

Theorem 1.1. *The sporadic simple group M_{23} is generated by all the triples (p,q,r), p, q and r primes dividing $|M_{23}|$, except for the cases $(p,q,r) \in \{(2,3,r),(2,5,5),(2,5,7),(3,3,3),(3,3,5),(3,3,7)\}$, for all r.*

2. Preliminaries

Let G be a finite group and for $k \ge 3$, suppose $C_1, C_2, \ldots, C_k$ (not necessarily distinct) be conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \le i \le k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \ldots \times C_{k-1}$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \dots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\dots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{1}$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \dots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \dots, g_{k-1})$ satisfying

$$g_1 g_2 \dots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \dots, g_{k-1} \rangle . \tag{2}$$

Definition 2.1. *If* $\Delta_G^*(C_1, C_2, \dots, C_k) > 0$, *the group* G *is said to be* $(C_1, C_2, \dots, C_k)$***-generated***.

Furthermore if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, C_2, \dots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \dots, h_{k-1})$ such that

$$h_1 h_2 \dots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \dots, h_{k-1} \rangle \leq H. \tag{3}$$

The value of $\Sigma_H(C_1, C_2, \dots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \dots, c_k)$ of H-conjugacy classes $c_1, c_2, \dots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 2.1. *Let* G *be a finite group and* H *be a subgroup of* G *containing a fixed element* g *such that* $\gcd(o(g), [N_G(H){:}H]) = 1$. *Then the number* $h(g, H)$ *of conjugates of* H *containing* g *is* $\chi_H(g)$, *where* $\chi_H(g)$ *is the permutation character of* G *with action on the conjugates of* H. *In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \dots, x_m$ *are representatives of the* $N_G(H)$*-conjugacy classes fused to the* G*-class of* g.

Proof of Theorem 2.1. See Ganief and Moori [19, 22].

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta_G^*(C_1, C_2, \dots, C_k)$, namely $\Delta_G^*(C_1, C_2, \dots, C_k)$, where

$$\Delta_G^*(C_1, \dots, C_k) = \Delta_G(C_1, \dots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, \dots, C_k), \tag{4}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \dots, C_k$. Since we have all the maximal subgroups of the sporadic simple groups except for $G = \mathbb{M}$ the Monster group, it is possible to build a small subroutine in GAP to compute the values of $\Theta_G = \Theta_G(C_1, C_2, \dots, C_k)$ for any collection of conjugacy classes and a sporadic simple group.

Lemma 2.1, Theorems 2.2 and 2.3 are in some cases useful in establishing non-generation of finite groups.

Lemma 2.1. *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$ and therefore G is not $(C_1, C_2, \ldots, C_k)$-generated.*

Proof of Theorem 2.2. See [5].

Theorem 2.2 ([26])**.** *Let G be a transitive permutation group generated by permutations $g_1, g_2, \ldots, g_s$ acting on a set of n elements such that $g_1g_2 \ldots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

For the Mathieu sporadic simple group $G = M_{23}$ and from the Atlas of finite group representations [30] we have G acting on 23 points, so that $n = 23$ and since our generation is triangular, we have $s = 3$. Hence if G is (l, m, n)-generated, then $\sum c_i \leq 25$.

Theorem 2.3 ([27])**.** *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G with $g_1g_2 \ldots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([15]):

$$\begin{aligned} d_i &= \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \left\langle \chi\downarrow^G_{\langle g_i \rangle}, \mathbf{1}_{\langle g_i \rangle} \right\rangle \\ &= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \end{aligned} \tag{5}$$

Theorem 2.4 ([18])**.** *Let G be a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.*

Theorem 2.5 ([18])**.** *Let G be a finite group and let l, m and n be integers that are pairwise coprime. Then for any integer t coprime to n, we have*

$$\Delta(lx, mY, nZ) = \Delta(lX, mY, (nZ)^t).$$

Remark 2.1. Moreover, G is (lX, mY, nZ)-generated if and only if G is $(lX, mY, (nZ)^t)$-generated.

We see that $(7A)^{-1} = 7B$, $(11A)^{-1} = 11B$ and $(23A)^{-1} = 23B$ in M_{23}. As an application of Theorem 2.5, the group M_{23} is $(p, q, 7A)$-generated if and only if it is $(p, q, 7B)$-generated, is $(p, q, 11A)$-generated if and only if it is $(p, q, 11B)$-generated and it is also $(p, q, 23A)$-generated if and only if it is $(p, q, 23B)$-generated. Therefore, it is sufficient to check the $(p, q, 7A)$-, $(p, q, 11A)$- and $(p, q, 23A)$-generations of M_{23}.

3. The Mathieu sporadic simple group M_{23}

In this section we apply the results discussed in Section 2, to the group M_{23}. We determine all the (p, q, r)-generations of M_{23}, where p, q and r are primes dividing the order of M_{23}.

The group M_{23} is a simple group of order $10200960 = 2^7 \times 3^2 \times 5 \times 7 \times 11 \times 23$. By the Atlas of finite groups [10], the group M_{23} has exactly 17 conjugacy classes of its elements and 7 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$K_1 = M_{22}$ $\quad$ $K_2 = L_3(4){:}2_2$ $\quad$ $K_3 = 2^4{:}A_7$ $\quad$ $K_4 = A_8$
$K_5 = M_{11}$ $\quad$ $K_6 = 2^4{:}(3 \times A_5){:}2$ $\quad$ $K_7 = 23{:}11$

In this section we let $G = M_{23}$. From the electronic Atlas of finite group representations [30], we can see that M_{23} has a permutation representation on 23. Generators g_1 and g_2 can be taken as follows $g_1 = (1, 2)(3, 4)(7, 8)(9, 10)(13, 14)(15, 16)(19, 20)(21, 22)$, $g_2 = (1, 16, 11, 3)(2, 9, 21, 12)(4, 5, 8, 23)(6, 22, 14, 18)(13, 20)(15, 17)$, with $o(g_1) = 2$, $o(g_2) = 4$ and $o(g_1g_2) = 23$.

In Table 1, we list the values of the cyclic structure for each conjugacy of G which containing elements of prime order together with the values of both c_i and d_i obtained from Ree and Scotts theorems, respectively.

Table 2 gives all the values of $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ for nX classes of prime order for the G with $\dim(\mathbb{V}) = 7$. This table will be referred to when we are proving non-generation of a triple for the group G.

In Table 3 we list the representatives of classes of the maximal subgroups together with the orbits lengths of M_{23} on these groups and the permutation characters except for the smallest maximal subgroup of M_{23}.

Table 4 gives us partial fusion maps of classes of maximal subgroups into the classes of M_{23}. These will be used in our computations.

Table 1: Cycle structures of conjugacy classes of M_{23}

nX	Cycle Structure	c_i	d_i
$1A$	1^{23}	23	0
$2A$	$1^7 2^8$	15	8
$3A$	$1^5 3^6$	11	12
$4A$	$1^3 2^2 4^4$	9	14
$5A$	$1^3 5^4$	7	16
$6A$	$1^1 2^2 3^2 6^2$	7	16
$7A$	$1^2 7^3$	5	18
$7B$	$1^2 7^3$	5	18
$8A$	$1^1 2^1 4^1 8^2$	5	18
$11A$	$1^1 11^2$	3	20
$11B$	$1^1 11^2$	3	20
$14A$	$2^1 7^1 14^1$	3	20
$14B$	$2^1 7^1 14^1$	3	20
$15A$	$3^1 5^1 15^1$	3	20
$15B$	$3^1 5^1 15^1$	3	20
$23A$	23^1	1	22
$23B$	23^1	1	22

Table 2: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 22$.

nX	$2A$	$3A$	$5A$	$7A$	$7B$	$11A$	$11B$	$23A$	$23B$
Cycle Structure	$1^7\ 2^8$	$1^5\ 3^6$	$1^3\ 5^4$	$1^2\ 7^3$	$1^2\ 7^3$	$1^1\ 11^2$	$1^1\ 11^2$	23^1	23^1
c_i	15	11	7	5	5	3	3	1	1
d_{nX}	8	12	16	18	18	20	20	22	22

Table 3: Maximal subgroups of M_{23}

Maximal Subgroup	Order	Orbit Lengths	Character
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	[1,22]	$1a + 22a$
$L_3(4){:}2_2$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[2,21]	$1a + 22a + 230a$
$2^4{:}A_7$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[7,16]	$1a + 22a + 230a$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	[8,15]	$1a + 22a + 230a + 253$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[11,12]	$1a + 22a + 230a + 1035a$
$2^4{:}(3 \times A_5){:}2$	$2^7 \cdot 3^2 \cdot 5$	[3,20]	$1a + 22a + 230a + 253a + 1035a$
23:11	$11 \cdot 23$	[23]	

Table 4: The partial fusion maps into M_{23}

M_{22}-class	2a	3a	5a	7a	7b	11a	11b					
$\rightarrow M_{23}$	2A	3A	5A	7A	7B	11A	11B					
h				2	2	1	1					
$L_3(4){:}2_2$-class	2a	2b	3a	5a	7a	7b						
$\rightarrow M_{23}$	2A	2A	3A	5A	7A	7B						
h					1	1						
$2^4{:}A_7$-class	2a	2b	3a	3b	5a	7a	7b					
$\rightarrow M_{23}$	2A	2A	3A	3A	5A	7A	7B					
h						1	1					
A_8-class	2a	2b	3a	3b	5a	7a	7b					
$\rightarrow M_{23}$	2A	2A	3A	3A	5A	7A	7B					
h						2	2					
M_{11}-class	2a	3a	5a	11a	11b							
$\rightarrow M_{23}$	2A	3A	5A	11A	11B							
h				1	1							
$2^4{:}(3 \times A_5){:}2$-class	2a	2b	2c	3a	3b	3c	5a					
$\rightarrow M_{23}$	2A	2A	2A	3A	3A	3A	5A					
h							1					
23:11-class	11a	11b	11c	11d	11e	11f	11g	11h	11i	11j	23a	23b
$\rightarrow M_{23}$	11A	11B	11A	11A	11A	11B	11B	11B	11A	11B	23A	23B
h	1	1	1	1	1	1	1	1	1	1	1	1

4. The $(2, q, r)$-generations of M_{23}

Let pX, $p \in \{2, 3, 5, 7, 11, 23\}$, $X \in \{A, B\}$ be a conjugacy class of $G = M_{23}$ and c_i be the number of disjoint cycles in a representative of pX. For M_{23} with three disjoint cycles, and acting on $n = 23$ points, we get $n(s-2) + 2 = 23 + 2 = 25$. Also G is not $(2A, 2A, pX)$-generated, for if G is $(2A, 2A, pX)$-generated, then G is a dihedral group and thus is not simple. Also we know that if G is (l, m, n)-generated with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \geq 1$ and G is simple, then $G \cong A_5$, but $G \cong M_{23}$ and $M_{23} \ncong A_5$. Hence if G is (p, q, r)-generated, then we must have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

Moreover if G is $(2A, 3A, rX)$-generated, then we must have $r > 6$ but we show in Theorem 4.1 below that in our case G is not $(2A, 3A, rX)$-generated for all r.

Now, the $(2, q, r)$-generations of M_{23} comprises the cases $(2, 3, r)$-, $(2, 5, r)$-, $(2, 7, r)$-, $(2, 11, r)$- and $(2, 23, r)$- generations.

4.1 $(2, 3, r)$-generations

Proposition 4.1. *G is not $(2A, 3A, r)$-generated for all r.*

Proof. The condition $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} < 1$ shows that $r > 6$. Therefore we have to consider the cases $(2A, 3A, 7X)$, $(2A, 3A, 11X)$ and $(2A, 3A, 23X)$ for all $X \in \{A, B\}$. Theorem 1.1 of [28] implies that G is not a Hurwitz group and hence G is not a $(2A, 3A, 7X)$-generated for $X \in \{A, B\}$. Generally, if G is $(2A, 3A, r)$-generated group, then we must have $c_{2A}+c_{3A}+c_p \leq 25$. From Table 1 we see that $c_{2A} + c_{3A} + c_r = 15 + 11 + c_p > 25$ for $p \in \{7A, 7B, 11A, 11B, 23A, 23B\}$. Now, using Ree's Theorem [26], it follows that G is not $(2A, 3A, r)$-generated. □

Remark 4.1. The above results can be deduced by Scott's Theorem [27], as from Table 2 we can see that $d_{2A} + d_{3A} + d_{nX} = 8 + 12 + d_{nX} < 2 \times 22$ for $nX \in \{7A, 7B, 11A, 11B, 23A, 23B\}$.

4.1.1 $(2, 5, r)$-generations

The condition $\frac{1}{2}+\frac{1}{5}+\frac{1}{r} < 1$ shows that $r > \frac{10}{3}$. Thus we have to consider the cases $(2A, 5A, 5A)$, $(2A, 5A, 7X)$, $(2A, 5A, 11X)$ and $(2A, 5A, 23X)$ for $X \in \{A, B\}$.

Proposition 4.2. *The group G is neither $(2A, 5A, 5A)$- nor $(2A, 5A, 7X)$-generated for $X \in \{A, B\}$.*

Proof. If G is a $(2A, 5A, 5A)$-generated group, then we must have $c_{2A} + c_{5A} + c_{5A} \leq 25$. From Table 1 we see that $c_{2A} + c_{5A} + c_{5A} = 15 + 7 + 7 = 29 > 25$. Now, using Ree's Theorem, it follows that G is not $(2A, 5A, 5A)$-generated.

By the same Table 1 we see that $c_{2A} + c_{5A} + c_{7A} = 15 + 7 + 5 = 27 > 25$. Again by Ree's Theorem, it follows that G is not $(2A, 5A, 7A)$-generated. Since the same holds for $(2A, 5A, 7B)$, it follows that G is not $(2A, 5A, 7X)$-generated for $X \in \{A, B\}$ and the proof is complete. □

Proposition 4.3. *The group G is $(2A, 5A, 11X)$-generated for $X \in \{A, B\}$.*

Proof. By Table 4, we see that K_1, K_5 and K_7 are the maximal subgroups having elements of order 11.

The intersection of the conjugacy classes these three maximal subgroups do not contain elements of order 11. Considering all various pairwise intersections of the conjugacy classes for these three maximal subgroups, we found that the only candidate having elements of order 11 is isomorphic to the group $PSL_2(11)$.

The maximal subgroup K_7 will not have any contributions because it does not contain elements of orders 2 and 5. We obtained that $\sum_{K_1}(2a, 5a, 11b) =$

176, $\sum_{K_5}(2a,5a,11b)=33$ and $\sum_{PSL_2(11)}(2a,5x,11b)=\Delta_{PSL_2(11)}(2a,5a,11b)+\Delta_{PSL_2(11)}(2a,5b,11b)=11+11=22$. By [17, 31], we have $h(11A,K_1)=h(11A,K_5)=h(11A,PSL_2(11))=1$. Since by Table 5 we have $\Delta_G(2A,5A,11A)=253$, we then obtain that $\Delta^*_G(2A,5A,11A)\geq\Delta_G(2A,5A,11A)-\sum_{K_1}(2a,5a,11b)-\sum_{K_5}(2a,5a,11b)+\sum_{PSL_2(11)}(2a,5x,11b)=253-176-33+22=66>0$. Hence G is $(2A,5A,11A)$-generated. Since the same holds for $(2A,5A,11B)$ (see Remark 2.1), it follows that G is $(2A,5A,11X)$-generated, for all $X\in\{A,B\}$. □

Proposition 4.4. *The group G is $(2A,5A,23X)$-generated for $X\in\{A,B\}$.*

Proof. By Table 4, we see the maximal subgroup K_7 is the only one have elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 2 and 5.

Since by Table 5, we have $\Delta_G(2A,5A,23A)=138$, we then deduce that $\Delta^*_G(2A,5A,23A)=\Delta_G(2A,5A,23A)=138>0$. Thus G is $(2A,5A,23A)$-generated. Since the same holds for $(2A,5A,23B)$, it follows that G is a $(2A,5A,23X)$-generated group, for $X\in\{A,B\}$. □

4.1.2 $(2,7,r)$-generations

We check for the generation of G through the triples $(2A,7X,7Y)$, $(2A,7X,11Y)$ and $(2A,7X,23Y)$ for all $X,Y\in\{A,B\}$.

Proposition 4.5. *The group G is $(2A,7X,7Y)$-generated for all $X,Y\in\{A,B\}$.*

Proof. By Table 4 we see that the maximal subgroups of G whose orders are divisible by 7 are K_1, K_2, K_3 and K_4.

The intersection of conjugacy classes from these four maximal subgroups do not contain elements of order 7. The intersection of the conjugacy classes from any three maximal subgroups do not contain elements of order 7. Considering all various intersections of the conjugacy classes for pairwise of these three maximal subgroups, we noticed that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ are the only ones having elements of order 7.

The group $PSL_3(2)$ has its relevant structure constant zero and as such it will not have any contributions. We obtained that $\sum_{PSL_3(4)}(2a,7a,7a)=42$, $\sum_{A_7}(2a,7a,7a)=7$, $\sum_{2^3:PSL_3(2)}(2a,7b,7b)=7$ and $h(7A,PSL_3(4))=h(7A,A_7)=h(7A,2^3{:}PSL_3(2))=2$. For the contributing maximal subgroups, we have $\sum_{K_1}(2a,7b,7b)=147$, $\sum_{K_2}(2x,7b,7b)=\Delta_{K_2}(2a,7b,7b)+\Delta_{K_2}(2b,7b,7b)=0+42=42$, $\sum_{K_3}(2a,7a,7a)=7$, $\sum_{K_4}(2x,7b,7b)=\Delta_{K_4}(2a,7b,7b)+\Delta_{K_4}(2b,7b,7b)=14+28=42$ and found that $h(7A,K_2)=h(7A,K_3)=1$ and $h(7A,K_1)=h(7A,K_4)=2$. Since by Table 5 we have $\Delta_G(2A,7A,7A)=301$, we then obtain that $\Delta^*_G(2A,7A,7A)\geq\Delta_G(2A,7A,7A)-2\cdot\sum_{K_1}(2a,7b,7b)-\sum_{K_2}(2x,7b,7b)-\sum_{K_3}(2a,7a,7a)-2\cdot\sum_{K_4}(2x,7b,7b)+2\cdot\sum_{PSL_3(4)}(2a,7b,7b)+2\cdot 2\cdot\sum_{A_7}(2a,7a,7a)+2\cdot 2\cdot\sum_{2^3:PSL_3(2)}(2a,7b,7b)=301-2(147)-42-7-2(42)+2(42)+2(2)(7)+2(2)(7)=14>0$ and it follows that $(2A,7A,7A)$ is a generating triple for G. Since the same holds for $(2A, 7B, 7B)$, it follows that the group G is $(2A, 7X, 7X)$-generated, for all $X \in \{A, B\}$.

We now investigate the $(2A, 7A, 7B)$- generations for G. From the intersections, we noticed that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ will all contribute here. We obtained that $\sum_{PSL_3(4)}(2a, 7a, 7b) = 63$, $\sum_{A_7}(2a, 7a, 7b) = 28$, $\sum_{2^3{:}PSL_3(2)}(2a, 7b, 7a) = 14$, $\sum_{PSL_3(2)}(2a, 7a, 7b) = 7$ and $h(7B, PSL_3(4)) = h(7B, A_7) = h(7B, 2^3{:}PSL_3(2)) = h(7B, PSL_3(2)) = 2$.

The maximal subgroup K_3 will not have any contributions because its relevant structure constant is zero. For the contributing maximal subgroups, we have $\sum_{K_1}(2a, 7b, 7a) = 224$, $\sum_{K_2}(2x, 7b, 7a) = \Delta_{K_2}(2a, 7b, 7a) + \Delta_{K_2}(2b, 7b, 7a) = 0 + 63 = 63$, $\sum_{K_4}(2x, 7b, 7a) = \Delta_{K_4}(2a, 7b, 7a) + \Delta_{K_4}(2b, 7b, 7a) = 21 + 42 = 63$ and found that $h(7B, K_2) = 1$ and $h(7B, K_1) = h(7B, K_4) = 2$. Since by Table 5 we have $\Delta_G(2A, 7A, 7B) = 462$, we then obtain that $\Delta^*_G(2A, 7A, 7B) = \Delta_G(2A, 7A, 7B) - 2 \cdot \sum_{K_1}(2a, 7b, 7a) - \sum_{K_2}(2x, 7b, 7a) - 2 \cdot \sum_{K_4}(2x, 7b, 7a) + 2 \cdot \sum_{PSL_3(4)}(2a, 7a, 7b) + 2 \cdot \sum_{A_7}(2a, 7b, 7a) + 2 \cdot \sum_{2^3{:}PSL_3(2)}(2a, 7b, 7a) + 2 \cdot \sum_{PSL_3(2)}(2a, 7b, 7a) = 462 - 2(224) - 63 - 2(21) + 2(63) + 2(2)(28) + 2(2)(14) + 2(7) = 217 > 0$. Therefore G is $(2A, 7A, 7B)$-generated. □

Proposition 4.6. *The group G is $(2A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we see that $PSL_2(11)$ is the only group having elements of order 11. This group $PSL_2(11)$ will not have any contributions because it does not contain elements of order 7. With regard to maximal subgroups having elements of order 11, by Table 4 we see that the maximal subgroup K_1 of G is the only one whose order is divisible by 7 and 11. We obtained that $\sum_{K_1}(2a, 7x, 11y) = 176$ and $h(11Z, K_1) = 1$ for $Z \in \{A, B\}$. By Table 5 we have $\Delta_G(2A, 7X, 11Y) = 308$ so that $\Delta^*_G(2A, 7X, 11Y) \geq \Delta_G(2A, 7X, 11Y) - \sum_{K_1}(2a, 7x, 11y) = 308 - 176 = 132 > 0$, implies that G is $(2A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.7. *The group G is $(2A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup K_7 does not have elements of order 7. By Table 5 we have $\Delta_G(2A, 7X, 23Y) = 184$. Since there are no contributions from any of the maximal subgroups of G, we then have $\Delta^*_G(2A, 7X, 23Y) = \Delta_G(2A, 7X, 23Y) = 184 > 0$, proving that G is $(2A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

4.1.3 $(2, 11, r)$-generations

Also here we check for the generation of G through the triples $(2A, 11A, 11A)$-, $(2A, 11A, 11B)$-, $(2A, 11A, 23A)$-, $(2A, 11A, 23B)$-, $(2A, 11B, 11B)$-, $(2A, 11B, 23A)$- and $(2A, 11B, 23B)$-generation. For this we have the following theorems:

Proposition 4.8. *The group G is $(2A, 11X, 11Y)$-generated for $X, Y \in \{A, B\}$.*

Proof. Looking at the discussions in Proposition 4.3 for the intersections, we see that the group $PSL_2(11)$ may be involved when proving $(2A, 11X, 11Y)$-generations. By Table 4 we see that the maximal subgroups of G containing elements of orders 2 and 11 are K_1 and K_5. The groups K_1, K_5 and $PSL_2(11)$ have elements of orders 2 and 11. We obtained that $\sum_{K_1}(2a, 11x, 11x) = 99$, $\sum_{K_5}(2a, 11x, 11x) = 11$ and $\sum_{PSL_2(11)}(2a, 11x, 11x) = 11$. We found that $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table 5 we have $\Delta_G(2A, 11X, 11X) = 341$, so that $\Delta^*_G(2A, 11X, 11X) \geq \Delta_G(2A, 11X, 11X) - \sum_{K_1}(2a, 11x, 11x) - \sum_{K_5}(2a, 11x, 11x) + \sum_{PSL_2(11)}(2a, 11x, 11x) = 341 - 147 - 11 + 11 = 194 > 0$, proving that G is $(2A, 11X, 11X)$-generated for $X \in \{A, B\}$.

Finally, we show that G is $(2A, 11A, 11B)$-generated. We obtained that $\sum_{K_1}(2a, 11b, 11a) = 132$ and $\sum_{K_5}(2a, 11b, 11a) = 11$.The group $PSL_2(11)$ will not have any contributions because its relevant structure constant is zero. Since by Table 5 we have $\Delta_G(2A, 11A, 11B) = 341$, so that $\Delta^*_G(2A, 11A, 11B) = \Delta_G(2A, 11A, 11B) - \sum_{K_1}(2a, 11b, 11a) - \sum_{K_5}(2a, 11b, 11a) = 341 - 224 - 11 = 106 > 0$, implies that G is $(2A, 11A, 11B)$-generated. We conclude that G is $(2A, 11Y, 11Z)$-generated for all $Y, Z \in \{A, B\}$. □

Proposition 4.9. *The group G is $(2A, 11X, 23Y)$-generated for $X, Y \in \{A, B\}$.*

Proof. By Table 4 we see the K_7 is the only maximal subgroup of G containing elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 2. By Table 5 we have $\Delta_G(2A, 11X, 23Y) = 391$, so that $\Delta^*(2A, 11X, 23Y) = \Delta_G(2A, 11X, 23Y) = 391 > 0$. Hence the group G is $(2A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

4.1.4 $(2, 23, r)$-generations

In here we check for the generation of G through the triples $(2A, 23A, 23A)$, $(2A, 23A, 23B)$ and $(2A, 23B, 23B)$. For these we have the following theorems:

Proposition 4.10. *The group G is $(2A, 23X, 23Y)$-generated for $X, Y \in \{A, B\}$.*

Proof. By Table 4 we see the K_7 is the only maximal subgroup of G containing elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 2. By Table 5 we have $\Delta_G(2A, 23X, 23X) = 161$ and $\Delta_G(2A, 23A, 23B) = 230$ for $X \in \{A, B\}$. Since there is no contributing group here, we then obtain that $\Delta^*_G(2A, 23X, 23X) = \Delta_G(2A, 23X, 23X) = 161 > 0$ and $\Delta^*_G(2A, 23A, 23B) = \Delta_G(2A, 23A, 23B) = 230 > 0$ for all $X \in \{A, B\}$. Hence, the group G is a $(2A, 23X, 23Y)$-generated for $X, Y \in \{A, B\}$. □

4.2 The $(3, q, r)$-generations

The condition $\frac{1}{3} + \frac{1}{3} + \frac{1}{r} < 1$ shows that $r > 3$. We then handle all the possible $(3, q, r)$-generations, namely $(3A, 3A, 5A)$-, $(3A, 3A, 7X)$-, $(3A, 3A, 11X)$-

, $(3A, 3A, 23X)$-, $(3A, 5A, 5A)$-, $(3A, 5A, 7X)$-, $(3A, 5A, 11X)$-, $(3A, 5A, 23X)$-, $(3A, 7X, 7Y)$-, $(3A, 7X, 11Y)$-, $(3A, 7X, 23Y)$-, $(3A, 11X, 11Y)$-, $(3A, 11X, 23Y)$- and $(3A, 23X, 23Y)$-generations in this section.

4.2.1 $(3, 3, r)$-generations

Proposition 4.11. *The group G is neither $(3A, 3A, 5A)$- nor $(3A, 3A, 7X)$-generated group for $X \in \{A, B\}$.*

Proof. By Table 2, the group G acts on a 22-dimensional irreducible complex module $\mathbb{V}$. By Scott's Theorem applied to this module and using the Atlas of finite groups, we see that $d_{3A} = \dim(\mathbb{V}/C_{\mathbb{V}}(3A)) = \frac{2(22-4)}{3} = 12$, $d_{5A} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) = \frac{4(22-2)}{5} = 16$ and $d_{7A} = d_{7B} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) = \frac{6(22-1)}{7} = 18$. For the case $(3A, 3A, 5A)$, we get $d_{3A} + d_{3A} + d_{5A} = 2 \times 12 + 16 = 40 < 44$ showing that G is not $(3A, 3A, 5A)$-generated. We also get that $d_{3A}+d_{3A}+d_{7X} = 2 \times 12 + 16 = 42 < 44$ for $X \in \{A, B\}$ and by Scott's Theorem G is not $(3A, 3A, 7X)$-generated for all $X \in \{A, B\}$ and the proof is complete. □

Proposition 4.12. *The group G is $(3A, 3A, 11X)$-generated for $X \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we notice that the subgroups of G involved here are K_1, K_5 and $PSL_2(11)$ because both subgroups have their elements of respective orders 3 and 11 which fuse to the elements $3A$ and $11A$ (or $11B$) of the group G. We obtained that $\sum_{K_1}(3a, 3a, 11b) = 209$, $\sum_{K_5}(3a, 3a, 11b) = 11$ and $\sum_{PSL_2}(11)(3a, 3a, 11b) = 11$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table 6 we have $\Delta_G(3A, 3A, 11A) = 275$, we then obtain that $\Delta^*_G(3A, 3A, 11A) \geq \Delta_G(3A, 3A, 11A) - \sum_{K_1}(3a, 3a, 11b) - \sum_{K_5}(3a, 3a, 11b)+\sum_{PSL_2(11)}(3a, 3a, 11b) = 275-209-11+11 = 66 > 0$, proving that G is $(3A, 3A, 11A)$-generated. Since the same holds for $(3A, 3A, 11B)$, it follows that G is $(3A, 3A, 11X)$-generated, for all $X \in \{A, B\}$. □

Proposition 4.13. *The group G is a $(3A, 3A, 23X)$-generated for $X \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table 6 we have that $\Delta_G(3A, 3A, 23X) = 138$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 3A, 23X) = \Delta_G(3A, 3A, 23X) = 138 > 0$, so that G is $(3A, 3A, 23X)$-generated for $X \in \{A, B\}$. □

4.2.2 $(3, 5, r)$-generations

Proposition 4.14. *The group G is $(3A, 5A, 5A)$-generated.*

Proof. Looking at Table 4 we see that all the maximal subgroups of G have elements of order 5 except for the seventh maximal subgroup. Let T be the set of all maximal subgroups of G except the seventh one. We look at various intersections of conjugacy classes for these maximal subgroups. We have the following:

- The groups arising from the intersections of conjugacy classes for any 4, 5 or 6 maximal subgroups in T do not contain elements of order 5.

- The group arising from intersections of the conjugacy classes for any three maximal subgroups in T having elements of orders 3 and 5 is S_5 (2-copies). We obtained that $\sum_{S_5}(3a,5a,5a) = 10$ and $h(5A,S_5) = 3$.

- The groups arising from intersections of the conjugacy classes for any two maximal subgroups in T having elements of orders 3 and 5 are $2^4{:}S_5$ (3-copies), $PSL_3(4)$, A_7 (2-copies), $2^4{:}A_6$, $PSL_2(11)$, $A_6{:}2$, A_5 and S_5 (2-copies). We obtained that $\sum_{2^4:S_5}(3a,5a,5a) = 160$, $\sum_{PSL_3(4)}(3a,5x,5y) = \Delta_{PSL_3(4)}(3a,5a,5a) + \Delta_{PSL_3(4)}(3a,5a,5b) + \Delta_{PSL_3(4)}(3a,5b,5b) = 445 + 445 + 445 = 1335$, $\sum_{A_7}(3x,5a,5a) = \Delta_{A_7}(3a,5a,5a) + \Delta_{A_7}(3b,5a,5a) = 20+60 = 80$, $\sum_{2^4:A_6}(3x,5y,5z) = \Delta_{2^4:A_6}(3a,5a,5a) + \Delta_{2^4:A_6}(3a,5a,5b) + \Delta_{2^4:A_6}(3a,5b,5b) + \Delta_{2^4:A_6}(3b,5a,5a) + \Delta_{2^4:A_6}(3b,5a,5b) + \Delta_{2^4:A_6}(3b,5b,5b) = 80+160+80+20+40+20 = 400$, $\sum_{PSL_2(11)}(3a,5x,5y) = \Delta_{PSL_2(11)}(3a,5a,5a) + \Delta_{PSL_2(11)}(3a,5a,5b) + \Delta_{PSL_2(11)}(3a,5b,5b) = 20+20+20 = 60$, $\sum_{A_6:2}(3a,5a,5a) = 30$, $\sum_{A_5}(3a,5x,5y) = \Delta_{A_5}(3a,5a,5a) + \Delta_{A_5}(3a,5a,5b) + \Delta_{A_5}(3a,5b,5b) = 5+5+5 = 15$ and $\sum_{S_5}(3a,5a,5a) = 10$. We found that the value of h for each of these eight groups is 3.

By Table 6 we have $\Delta_G(3A,5A,5A) = 6550$. We obtained that $\sum_{K_1}(3a,5a,5a) = 2800$, $\sum_{K_2}(3a,5a,5a) = 910$, $\sum_{K_3}(3x,5a,5a) = \Delta_{K_3}(3a,5a,5a) + \Delta_{K_3}(3b,5a,5a) = 320+240 = 560$, $\sum_{K_4}(3x,5a,5a) = \Delta_{K_4}(3a,5a,5a) + \Delta_{K_4}(3b,5a,5a) = 25+135 = 160$, $\sum_{K_5}(3a,5a,5a) = 80$, $\sum_{K_6}(3x,5a,5a) = \Delta_{K_6}(3a,5a,5a) + \Delta_{K_6}(3b,5a,5a) + \Delta_{K_6}(3c,5a,5a) = 0+0+160 = 160$. The value of h for each maximal subgroup is 3 except for K_4 and K_6. The value of h is 1 for each of these maximal subgroups K_4 and K_6. It follows that $\Delta^*_G(3A,5A,5A) \geq \Delta_G(3A,5A,5A) - 3\cdot\sum_{K_1}(3a,5a,5a) - 3\cdot\sum_{K_2}(3a,5a,5a) - 3\cdot\sum_{K_3}(3x,5a,5a) - \sum_{K_4}(3x,5a,5a) - 3\cdot\sum_{K_5}(3a,5a,5a) - \sum_{K_6}(3x,5a,5a) - 2\cdot 3\cdot\sum_{S_5}(3a,5a,5a) + 3\cdot 3\cdot\sum_{2^4:S_5}(3a,5a,5a) + 3\cdot\sum_{PSL_3(4)}(3a,5x,5y) + 2\cdot 3\cdot\sum_{A_7}(3x,5a,5a) + 3\cdot\sum_{2^4:A_6}(3x,5y,5z) + 3\cdot\sum_{PSL_2(11)}(3a,5x,5y) + 3\cdot\sum_{A_6:2}(3a,5a,5a) + 3\cdot\sum_{A_5}(3a,5x,5y) + 2\cdot 3\cdot\sum_{S_5}(3a,5a,5a) = 6550 - 3(2800) - 3(910) - 3(560) - 1(160) - 3(80) - 1(160) - 2(3)(10) + 3(3)(160) + 3(1335) + 2(3)(80) + 3(400) + 3(60) + 3(30) + 3(15) + 2(3)(10) = 620 > 0$. It follows that the group G is $(3A,5A,5A)$-generated. $\square$

Proposition 4.15. *The group G is $(3A,5A,7X)$-generated for $X \in \{A,B\}$.*

Proof. As in Proposition 4.5, the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here.The groups $2^3{:}PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(3a,5x,7b) = \Delta_{PSL_3(4)}(3a,5a,7b) +$

$\Delta_{PSL_3(4)}(3a, 5b, 7b) = 441 + 441 = 882$, $\sum_{A_7}(3x, 5a, 7b) = \Delta_{A_7}(3a, 5a, 7b) + \Delta_{A_7}(3b, 5a, 7b) = 56 + 7 = 63$ and $h(7A, PSL_3(4)) = h(7A, A_7) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $3A$, $5A$, $7A$ classes of G. We obtained that $\sum_{K_1}(3a, 5a, 7b) = 2464$, $\sum_{K_2}(3a, 5a, 7b) = 882$, $\sum_{K_3}(3x, 5a, 7a) = \Delta_{K_3}(3a, 5a, 7a) + \Delta_{K_3}(3b, 5a, 7a) = 112 + 224 = 336$, $\sum_{K_4}(3x, 5a, 7b) = \Delta_{K_4}(3a, 5a, 7b) + \Delta_{K_4}(3b, 5a, 7b) = 77 + 7 = 84$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 6 we have $\Delta_G(3A, 5A, 7A) = 5124$, we then obtain that $\Delta^*_G(3A, 5A, 7A) \geq \Delta_G(3A, 5A, 7A) - 2 \cdot \sum_{K_1}(3a, 5a, 7b) - \sum_{K_2}(3a, 5a, 7b) - \sum_{K_3}(3x, 5a, 7a) - 2 \cdot \sum_{K_4}(3a, 5a, 7b) + 2 \cdot \sum_{PSL_3(4)}(3a, 5x, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(3x, 5a, 7b) = 5124 - 2(2464) - 882 - 336 - 2(84) + 2(882) + 2(2)(63) = 826 > 0$. Therefore, the group G is $(3A, 5A, 7A)$-generated. Since the same holds for $(3A, 5A, 7B)$, it follows that the group G is $(3A, 5A, 7X)$-generated for $X \in \{A, B\}$. $\square$

Proposition 4.16. *The group G is $(3A, 5A, 11X)$-generated for $X \in \{A, B\}$.*

Proof. By Table 4 we see that the maximal subgroups of G containing elements of orders 3 and 11 are K_1 and K_5. The group $PSL_2(11)$contains elements of orders 3, 5 and 11. We obtained that $\sum_{K_1}(3a, 5a, 11b) = 2112$, $\sum_{K_5}(3a, 5a, 11a) = 99$ and $\sum_{PSL_2(11)}(3a, 5x, 11b) = \Delta_{PSL_2(11)}(3a, 5a, 11b) + \Delta_{PSL_2(11)}(3a, 5b, 11b) = 22+22 = 44$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. Since by Table 6 we have $\Delta_G(3A, 5A, 11A) = 4136$, we then have $\Delta^*_G(3A, 5A, 11A) \geq \Delta_G(3A, 5A, 11A) - \sum_{K_1}(3a, 5a, 11b) - \sum_{K_5}(3a, 5a, 11b) + \sum_{PSL_2(11)}(3a, 5x, 11b) = 4136 - 2112 - 99 + 44 = 1969 > 0$, so that G is $(3A, 5A, 11A)$-generated. Since the same holds for $(3A, 5A, 11B)$, it follows that the group G is $(3A, 5A, 11X)$-generated for $X \in \{A, B\}$. $\square$

Proposition 4.17. *The group G is $(3A, 5A, 23X)$-generated group for $X \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table 6 we have that $\Delta_G(3A, 5A, 23X) = 2438$. Since there is no contributing group, we then obtain that $\Delta^*_G(3A, 5A, 23X) = \Delta_G(3A, 5A, 23X) = 2438 > 0$, so that G is $(3A, 5A, 23X)$-generated for $X \in \{A, B\}$. $\square$

4.2.3 $(3, 7, r)$-generations

In this subsection we discuss the case $(3, 7, r)$-generations. It follows that we will end up with 11 cases, namely $(3A, 7A, 7A)$-, $(3A, 7A, 7B)$-, $(3A, 7A, 11A)$-, $(3A, 7A, 11B)$-, $(3A, 7A, 23A)$-, $(3A, 7A, 23B)$-, $(3A, 7B, 7B)$-, $(3A, 7B, 11A)$-, $(3A, 7B, 11B)$-, $(3A, 7B, 23A)$ and $(3A, 7B, 23B)$-generation.

Proposition 4.18. *The group G is $(3A, 7X, 7Y)$-generated for all $X, Y \in \{A, B\}$*

Proof. As in Proposition 4.5, the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ have elements of order 7. We obtained that $\sum_{PSL_3(4)}(3a, 7b, 7b) = 357$, $\sum_{A_7}(3x, 7b, 7b) = \Delta_{A_7}(3a, 7b, 7b) + \Delta_{A_7}(3b, 7b, 7b) = 56+21 = 77$, $\sum_{2^3:PSL_3(2)}(3a, 7b, 7b) = 28$, $\sum_{PSL_3(2)}(3a, 7a, 7a) = 7$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3{:}PSL_3(2)) = h(7A, PSL_3(2)) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $3A$, $7A$ classes of G. We obtained that $\sum_{K_1}(3a, 7b, 7b) = 1792$, $\sum_{K_2}(3a, 7b, 7b) = 357$, $\sum_{K_3}(3x, 7a, 7a) = \Delta_{K_3}(3a, 7a, 7a) + \Delta_{K_3}(3b, 7a, 7a) = 168+126 = 294$, $\sum_{K_4}(3x, 7b, 7b) = \Delta_{K_4}(3a, 7b, 7b) + \Delta_{K_4}(3b, 7b, 7b) = 147+21 = 168$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 6 we have $\Delta_G(3A, 7A, 7A) = 4886$, we then obtain that $\Delta^*_G(3A, 7A, 7A) \geq \Delta_G(3A, 7A, 7A) - 2 \cdot \sum_{K_1}(3a, 7b, 7b) - \sum_{K_2}(3a, 7b, 7b) - \sum_{K_3}(3x, 7a, 7a) - 2 \cdot \sum_{K_4}(3a, 7b, 7b) + 2 \cdot \sum_{PSL_3(4)}(3a, 7b, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(3x, 7b, 7b) + 2 \cdot 2 \cdot \sum_{2^3:PSL_3(2)}(3a, 7b, 7b) + 2 \cdot \sum_{PSL_3(2)}(3a, 7a, 7a) = 4886 - 2(1792) - 357 - 394 - 2(168) + 2(357) + 2(2)(77) + 2(2)(28) + 2(7) = 1363 > 0$. Therefore, the group G is $(3A, 7A, 7A)$-generated. Since the same holds for $(3A, 7B, 7B)$, it follows that the group G is $(3A, 7X, 7X)$-generated for $X \in \{A, B\}$.

We now prove that G is $(3A, 7A, 7B)$-generated.

We obtained that $\sum_{PSL_3(4)}(3a, 7b, 7a) = 357$, $\sum_{A_7}(3x, 7b, 7b) = \Delta_{A_7}(3a, 7b, 7b) + \Delta_{A_7}(3b, 7b, 7b) = 28+14 = 32$, $\sum_{2^3:PSL_3(2)}(3a, 7b, 7b) = 28$, $\sum_{PSL_3(2)}(3a, 7a, 7b) = 7$, $\sum_{K_1}(3a, 7b, 7a) = 1792$, $\sum_{K_2}(3a, 7b, 7a) = 357$, $\sum_{K_3}(3x, 7a, 7b) = \Delta_{K_3}(3a, 7a, 7b) + \Delta_{K_3}(3b, 7a, 7b) = 112 + 70 = 182$, $\sum_{K_4}(3x, 7b, 7a) = \Delta_{K_4}(3a, 7b, 7a) + \Delta_{K_4}(3b, 7b, 7a) = 21 + 147 = 168$, $\sum_{PSL_3(4)}(3a, 7b, 7a) = 357$, $\sum_{A_7}(3x, 7b, 7a) = \Delta_{A_7}(3a, 7b, 7a) + \Delta_{A_7}(3b, 7b, 7a) = 56+21 = 77$, $\sum_{2^3:PSL_3(2)}(3a, 7b, 7a) = 28$ and $\sum_{PSL_3(2)}(3a, 7a, 7b)$. Since by the same Table 6 we have $\Delta_G(3A, 7A, 7B) = 4886$, so that $\Delta^*_G(3A, 7A, 7B) \geq \Delta_G(3A, 7A, 7B) - 2 \cdot \sum_{K_1}(3a, 7b, 7a) - \sum_{K_2}(3a, 7b, 7a) - \sum_{K_3}(3x, 7a, 7b) - 2 \cdot \sum_{K_4}(3x, 7a, 7b) + 2 \cdot \sum_{PSL_3(4)}(3a, 7b, 7a) + 2 \cdot 2 \cdot \sum_{A_7}(3x, 7b, 7a) + 2 \cdot 2 \cdot \sum_{2^3:PSL_3(2)}(3a, 7b, 7a) + 2 \cdot \sum_{PSL_3(2)}(3a, 7a, 7b) = 4886 - 2(1792) - 357 - 182 - 2(168) + 2(357) + 2(2)(42) + 2(2)(28) + 2(7) = 1435 > 0$. This proves that G is $(3A, 7A, 7B)$-generated group. □

Proposition 4.19. *The group G is $(3A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The group $PSL_2(11)$ will not have any contributions because it does not have elements of order 7. Looking at Table 4, we see that K_1 is the only maximal subgroup of G having elements of orders 3, 7 and 11. We obtained that $\sum_{K_1}(3a, 7x, 11y) = 1760$ and $h(11X, K_1) = 1$ for $X \in \{A, B\}$. By Table 6 we have $\Delta_G(3A, 7X, 11Y) = 4136$. We obtained that $\Delta^*_G(3A, 7X, 11Y) \geq \Delta_G(3A, 7X, 11Y) - \sum_{K_1}(3a, 7x, 11y) = 4136 - 1760 = 2376$ and so that the group G becomes is $(3A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.20. *The group G is $(3A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 3 and 7. By Table 6 we have that $\Delta_G(3A, 7X, 23Y) = 3312$. Since there is no contributing group, we then obtain that $\Delta^*_G(3A, 7X, 23Y) = \Delta_G(3A, 7X, 23Y) = 3312 > 0$, so that the group G is $(3A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

4.2.4 $(3, 11, r)$-generations

In this subsection we discuss the case $(3, 11, r)$-generations.

It follows that we will end up with 7 cases, namely $(3A, 11A, 11A)$-, $(3A, 11A, 11B)$-,$(3A, 11A, 23A)$-, $(3A, 11A, 23B)$-, $(3A, 11B, 11B)$-, $(3A, 11B, 23A)$-, $(3A, 11B, 23B)$-generation.

Proposition 4.21. *The group G is $(3A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The maximal subgroup K_7 will not have any contributions because it does not have elements of order 3. We obtained that $\sum_{K_1}(3a, 11b, 11b) = 1320$, $\sum_{K_5}(3a, 11b, 11b) = 22$ and $\sum_{PSL_2(11)}(3a, 11b, 11b) = 0$. The value of h for each group is 1. Since by Table 6 we have $\Delta_G(3A, 11A, 11A) = 5126$, it follows that $\Delta^*_G(3A, 11A, 11A) \geq \Delta_G(3A, 11A, 11A) - \sum_{K_1}(3a, 11b, 11b) - \sum_{K_5}(3a, 11b, 11b) + \sum_{PSL_2(11)}(3a, 11b, 11b) = 5126 - 1320 - 22 + 0 = 3784 > 0$. Therefore, the group G is $(3A, 11A, 11A)$-generated. Since the same holds for $(3A, 11B, 11B)$,the group G becomes $(3A, 11X, 11X)$-generated for $X \in \{A, B\}$.

We now prove that G is $(3A, 11A, 11B)$-generated.

We obtained that $\sum_{K_1}(3a, 11a, 11b) = 1276$, $\sum_{K_5}(3a, 11a, 11b) = 77$ and $\sum_{PSL_2(11)}(3a, 11a, 11b) = 22$. By the same Table 6 we have $\Delta_G(3A, 11A, 11B) = 5379$. Then, we obtain that $\Delta^*_G(3A, 11A, 11B) \geq \Delta_G(3A, 11A, 11B) - \sum_{K_1}(3a, 11b, 11a) - \sum_{K_5}(3a, 11b, 11a) + \sum_{PSL_2(11)}(3a, 11b, 11a) = 5379 - 1276 - 77 + 22 = 4048 > 0$, proving that G is $(3A, 11A, 11B)$-generated. Hence, the group G is $(3A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.22. *The group G is $(3A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table 6 we have that $\Delta_G(3A, 11X, 23Y) = 5129$. Since there is no contributing group, we then obtain that $\Delta^*(3A, 11X, 23Y) = \Delta_G(3A, 11X, 23Y) = 5129 > 0$, so that the group G is $(3A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

4.2.5 $(3,23,r)$-generations

In this subsection we discuss the case $(3,23,r)$-generations. It follows that we will end up with 3 cases, namely $(3A,23A,23A)$-, $(3A,23A,23B)$-, $(3A,23B,23B)$-generation which will be handled in the following Proposition 4.23.

Proposition 4.23. *The group G is $(3A,23X,23Y)$-generated for all $X,Y \in \{A,B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 3. By Table 6 we have $\Delta_G(3A,23A,23A) = 3082$. Since there is no contributing group, we then obtain that $\Delta^*(3A,23X,23X) = \Delta_G(3A,23A,23A) = 3082 > 0$, so that the group G is $(3A,23A,23A)$-generated. Since the same holds for $(3A,23B,23B)$, the group G will be $(3A,23B,23B)$-generated. Similarly, $\Delta^*(3A,23A,23B) = \Delta_G(3A,23A,23B) = 2714 > 0$, so that the group G becomes $(3A,23A,23B)$-generated. $\square$

4.3 Other results

In this section we handle all the remaining cases, namely the $(5,q,r)$-, $(7,q,r)$-, $(11,q,r)$- and $(23,q,r)$-generations.

4.3.1 $(5,5,r)$-generations

In this subsection we discuss the case $(5,5,r)$-generations. It follows that we will end up with 5 cases, namely $(5A,5A,5A)$-, $(5A,5A,11A)$-, $(5A,5A,11B)$-, $(5A,5A,23A)$-, $(5A,5A,23B)$-generation.

Proposition 4.24. *The group G is $(5A,5A,5A)$-generated.*

Proof. From Table 4 we see that all the maximal subgroups of G have elements of order 5 except for the seventh maximal subgroup. Let T be the set of all maximal subgroups of G except the seventh one. We look at various intersections of conjugacy classes for these maximal subgroups. We have the following:

- The groups arising from the intersections of conjugacy classes for any 4, 5 or 6 maximal subgroups in T do not contain elements of order 5.
- The groups arising from intersections of the conjugacy classes for any three maximal subgroups in T having elements of order 5 are S_5 (2-copies), D_{10} and 5:4. We obtained that $\sum_{S_5}(5a,5a,5a) = 8$, $\sum_{D_{10}}(5x,5y,5z) = \Delta_{D_{10}}(5a,5a,5a) + \Delta_{D_{10}}(5a,5a,5b) + \Delta_{D_{10}}(5a,5b,5b) + \Delta_{D_{10}}(5b,5b,5b) = 0+1+1+0 = 2$ and $\sum_{5:4}(5a,5a,5a) = 3$. We found that the value of h for each of these three groups is 3.
- The groups arising from intersections of the conjugacy classes for any two maximal subgroups in T having elements of order 5 are $2^4{:}S_5$ (3-copies), $PSL_3(4)$, A_7 (2-copies), $2^4{:}A_6$, $PSL_2(11)$, $A_6{:}2$, A_5 and S_5 (2-copies).

We obtained that $\sum_{2^4:S_5}(5a,5a,5a) = 128$, $\sum_{PSL_3(4)}(3a,5x,5y) = \Delta_{PSL_3(4)}(5a,5a,5a) + \Delta_{PSL_3(4)}(5a,5a,5b) + \Delta_{PSL_3(4)}(5a,5b,5b) + \Delta_{PSL_3(4)}(5b,5b,5b) = 845 + 781 + 781 + 845 = 3252$, $\sum_{A_7}(a,5a,5a) = \Delta_{A_7}(5a,5a,5a) = 108$, $\sum_{2^4:A_6}(5x,5y,5z) = \Delta_{2^4:A_6}(5a,5a,5a)+\Delta_{2^4:A_6}(5a,5a,5b)+\Delta_{2^4:A_6}(5a,5b,5b)+\Delta_{2^4:A_6}(5b,5b,5b) = 320+176+176+320 = 992$, $\sum_{PSL_2(11)}(5x,5y,5z) = \Delta_{PSL_2(11)}(5a,5a,5a) + \Delta_{PSL_2(11)}(5a,5a,5b) + \Delta_{PSL_2(11)}(5a,5b,5b) + \Delta_{PSL_2(11)}(5b,5b,5b) = 20 + 31 + 31 + 20 = 102$, $\sum_{A_6:2}(5a,5a,5a) = 53$, $\sum_{A_5}(5x,5y,5z) = \Delta_{A_5}(5a,5a,5a)+\Delta_{A_5}(5a,5a,5b)+\Delta_{A_5}(5a,5b,5b)+\Delta_{A_5}(5b,5b,5b) = 5+1+1+5 = 12$ and $\sum_{S_5}(5a,5a,5a) = 8$. We found that the value of h for each of these eight groups is 3.

By Table 7 we have $\Delta_G(5A,5A,5A) = 61058$. We obtained that $\sum_{K_1}(5a,5a,5a) = 18368$, $\sum_{K_2}(5a,5a,5a) = 3188$, $\sum_{K_3}(5a,5a,5a) = 1728$, $\sum_{K_4}(5a,5a,5a) = 173$, $\sum_{K_5}(5a,5a,5a) = 378$, $\sum_{K_6}(5a,5a,5a) = 128$ The value of h for each maximal subgroup is 3 except for K_4 and K_6. The value of h is 1 for each of these maximal subgroups K_4 and K_6. It follows that $\Delta^*_G(5A,5A,5A) \geq \Delta_G(5A,5A,5A) - 3\cdot\sum_{K_1}(5a,5a,5a) - 3\cdot\sum_{K_2}(5a,5a,5a) - 3\cdot\sum_{K_3}(5a,5a,5a) - \sum_{K_4}(5a,5a,5a) - 3\cdot\sum_{K_5}(5a,5a,5a) - \sum_{K_6}(5a,5a,5a) - 2\cdot 3\cdot\sum_{S_5}(5a,5a,5a) - 3\cdot\sum_{D_{10}}(5x,5y,5z) - 3\cdot\sum_{5:4}(5a,5a,5a) + 3\cdot 3\cdot\sum_{2^4:S_5}(5a,5a,5a) + 3\cdot\sum_{PSL_3(4)}(5x,5y,5z) + 2\cdot 3\cdot\sum_{A_7}(5a,5a,5a) + 3\cdot\sum_{2^4:A_6}(5x,5y,5z) + 3\cdot\sum_{PSL_2(11)}(5x,5y,5z) + 3\cdot\sum_{A_6:2}(5a,5a,5a) + 3\cdot\sum_{A_5}(5x,5y,5z) + 2\cdot 3\cdot\sum_{S_5}(5a,5a,5a) = 61058 - 3(18368) - 3(3188) - 3(1728) - 1(173) - 3(378) - 1(128) - 2(3)(11) - 3(2) - 3(3) + 3(3)(128) + 3(3252) + 2(3)(108) + 3(992) + 3(102) + 3(53) + 3(12) + 2(3)(8) = 6499 > 0$. It follows that the group G is $(5A,5A,5A)$-generated. $\square$

Proposition 4.25. *The group G is $(5A,5A,7X)$-generated for $X \in \{A,B\}$.*

Proof. As in Proposition 4.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here.The groups $2^3{:}PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(5x,5y,7b) = \Delta_{PSL_3(4)}(5a,5a,7b) + \Delta_{PSL_3(4)}(5a,5b,7b) + \Delta_{PSL_3(4)}(5b,5b,7b) = 819 + 819 + 819 = 2457$, $\sum_{A_7}(5a,5a,7b) = 84$ and $h(7A,PSL_3(4)) = h(7A,A_7) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $5A$, $7A$ classes of G. We obtained that $\sum_{K_1}(5a,5a,7b) = 17920$, $\sum_{K_2}(5a,5a,7b) = 3276$, $\sum_{K_3}(5a,5a,7a) = 1344$, $\sum_{K_4}(5a,5a,7b) = 91$. We found that $h(7A,K_1) = h(7A,K_4) = 2$ and $h(7A,K_2) = h(7A,K_3) = 1$.

Since by Table 7 we have $\Delta_G(5A,5A,7A) = 54320$, we then obtain that $\Delta^*_G(5A,5A,7A) \geq \Delta_G(5A,5A,7A) - 2\cdot\sum_{K_1}(5a,5a,7b) - \sum_{K_2}(5a,5a,7b) - \sum_{K_3}(5a,5a,7a) - 2\cdot\sum_{K_4}(5a,5a,7b) + 2\cdot\sum_{PSL_3(4)}(5x,5y,7b) + 2\cdot 2\cdot\sum_{A_7}(5a,5a,7b) = 54320 - 2(17920) - 3276 - 1344 - 2(91) + 2(2457) + 2(2)(84) = 18928 > 0$. Therefore, the group G is $(5A,5A,7A)$-generated. Since the same holds for $(5A,5A,7B)$, it follows that the group G is $(5A,5A,7X)$-generated for $X \in \{A,B\}$. $\square$

Proposition 4.26. *The group G is $(5A,5A,11X)$-generated for $X \in \{A,B\}$.*

Proof. By Proposition 4.3 we proved that the group G is $(2A, 5A, 11X)$-generated for $X \in \{A, B\}$. It follows by Theorem 2.4 that G is $(5A, 5A, (11A)^2)$- and $(5A, 5A, (11B)^2)$-generated. By GAP, we see that $(11A)^2 = 11B$ and $(11B)^2 = 11A$ and the results follow. $\square$

Proposition 4.27. *The group G is $(5A, 5A, 23X)$-generated for $X \in \{A, B\}$.*

Proof. By Proposition 4.4 we proved that G is $(2A, 5A, 23X)$-generated for $X \in \{A, B\}$. It follows by Theorem 2.4 that the group G is $(5A, 5A, (23A)^2)$- and $(5A, 5A, (23B)^2)$-generated. Since by GAP we have $(23A)^2 = 23A$ and $(23B)^2 = 23B$, then the results follow. $\square$

4.3.2 $(5, 7, r)$-generations

In this subsection we discuss the case $(5, 7, r)$-generations.

It follows that we will end up with 11 cases, namely $(5A, 7A, 7A)$ -, $(5A, 7A, 7B)$ -, $(5A, 7A, 11A)$ - , $(5A, 7A, 11B)$-, $(5A, 7A, 23A)$-, $(5A, 7A, 23B)$-, $(5A, 7B, 7B)$-, $(5A, 7B, 11A)$-, $(5A, 7B, 11B)$-, $(5A, 7B, 23A)$-, $(5A, 7B, 23B)$-generation.

Proposition 4.28. *The group G is $(5A, 7X, 7Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. As in Proposition 4.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies), $2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ may have contributions here.The groups $2^3{:}PLS_3(2)$ and $PSL_3(2)$ will not have any contributions because they both do not have elements of order 5. We obtained that $\sum_{PSL_3(4)}(5x, 7b, 7b) = \Delta_{PSL_3(4)}(5a, 7b, 7b) + \Delta_{PSL_3(4)}(5b, 7b, 7b) = 567 + 567 = 1134$, $\sum_{A_7}(5a, 7b, 7b) = 84$ and $h(7A, PSL_3(4)) = h(7A, A_7) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 meet the $5A$, $7A$ classes of G. We obtained that $\sum_{K_1}(5a, 7b, 7b) = 12544$, $\sum_{K_2}(5a, 7b, 7b) = 1134$, $\sum_{K_3}(5a, 7a, 7a) = 672$, $\sum_{K_4}(5a, 7b, 7b) = 189$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 7 we have $\Delta_G(5A, 7A, 7A) = 52584$, we then obtain that $\Delta^*_G(5A, 7A, 7A) \geq \Delta_G(5A, 7A, 7A) - 2 \cdot \sum_{K_1}(5a, 7b, 7b) - \sum_{K_2}(5a, 7b, 7b) - \sum_{K_3}(5a, 7a, 7a) - 2 \cdot \sum_{K_4}(5a, 7b, 7b) + 2 \cdot \sum_{PSL_3(4)}(5x, 7b, 7b) + 2 \cdot 2 \cdot \sum_{A_7}(5a, 7b, 7b) = 52584 - 2(12544) - 1134 - 672 - 2(189) + 2(1134) + 2(2)(84) = 27916 > 0$. Therefore, the group G is $(5A, 7A, 7A)$-generated. Since the same holds for $(5A, 7B, 7B)$, it follows that the group G is $(5A, 7X, 7X)$-generated for $X \in \{A, B\}$.

We now prove that the group G is $(5A, 7A, 7B)$-generated. We obtained that $\sum_{PSL_3(4)}(5x, 7b, 7a) = \Delta_{PSL_3(4)}(5a, 7b, 7a) + \Delta_{PSL_3(4)}(5b, 7b, 7a) = 567 + 567 = 1134$, $\sum_{A_7}(5a, 7b, 7b) = 84$, $\sum_{K_1}(5a, 7b, 7a) = 12544$, $\sum_{K_2}(5a, 7b, 7a) = 1134$, $\sum_{K_3}(5a, 7a, 7b) = 672$ and $\sum_{K_4}(5a, 7b, 7a) = 189$. Since by same Table 7 we have $\Delta_G(5A, 7A, 7B) = 52584$, we then obtain that $\Delta^*_G(5A, 7A, 7B) \geq \Delta_G(5A, 7A, 7B) - 2 \cdot \sum_{K_1}(5a, 7b, 7a) - \sum_{K_2}(5a, 7b, 7a) - \sum_{K_3}(5a, 7a, 7b) - 2 \cdot \sum_{K_4}(5a, 7b, 7a) + 2 \cdot \sum_{PSL_3(4)}(5x, 7b, 7a) + 2 \cdot 2 \cdot \sum_{A_7}(5a, 7b, 7a) = 52584 - 2(12544) - 1134 - 672 - 2(189) + 2(1134) + 2(2)(84) = 27916 > 0$, proving that the group G is $(5A, 7A, 7B)$-generated. $\square$

Proposition 4.29. *The group G is $(5A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5, K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(5a, 7x, 11y) = 12672$ and $h(11Z, K_1) = 1$ for $Z \in \{A, B\}$. By Table 7 we have $\Delta_G(5A, 7X, 11Y) = 48576$ for all $X, Y \in \{A, B\}$. We then obtain that $\Delta^*_G(5A, 7X, 11Y) \geq \Delta_G(5A, 7X, 11Y) - \sum_{K_1}(5a, 7x, 11y) = 48576 - 12672 = 35904 > 0$, so that the group G becomes $(5A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$. □

Proposition 4.30. *The group G is $(5A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of orders 5 and 7. By Table 7 we have $\Delta_G(5A, 7X, 23Y) = 44160$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*_G(5A, 7X, 23Y) = \Delta_G(5A, 7X, 23Y) = 44160 > 0$, so that the group G is $(5A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

4.3.3 $(5, 11, r)$-generations

In this subsection we discuss the case $(5, 11, r)$-generations. It follows that we will end up with 7 cases, namely $(5A, 11A, 11A)$-, $(5A, 11A, 11B)$-, $(5A, 11A, 23A)$-, $(5A, 11A, 23B)$, $(5A, 11B, 11B)$-, $(5A, 11B, 23A)$- and $(5A, 11B, 23B)$-generation.

Proposition 4.31. *The group G is $(5A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. Looking at Proposition 4.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The group K_7 will not have any contributions because it does not have elements of order 5. We obtained that $\sum_{K_1}(5a, 11b, 11b) = 8448$, $\sum_{K_5}(5a, 11b, 11b) = 198$ and $\sum_{PSL_2(11)}(5x, 11b, 11b) = \Delta_{PSL_2(11)}(5a, 11b, 11b) + \Delta_{PSL_2(11)}(5b, 11b, 11b) = 11 + 11 = 22$. By Table 7, we have $\Delta_G(5A, 11A, 11A) = 62238$. We already have $h(11A, K_1) = h(11A, K_5) = h(11A, PSL_2(11)) = 1$. We then have $\Delta^*_G(5A, 11X, 11X) \geq \Delta_G(5A, 11A, 11A) - \sum_{K_1}(5a, 11b, 11b) - \sum_{K_5}(5a, 11b, 11b) + \sum_{PSL_2(11)}(5a, 11b, 11b) = 62238 - 8448 - 198 + 22 = 53614 > 0$, showing that the group G is $(5A, 11A, 11A)$-generated.

Since the same holds for $(5A, 11B, 11B)$ implies that the group G is $(5A, 11A, 11A)$-generated. For the $(5A, 11A, 11B)$-generations, we obtained that $\sum_{K_1}(5a, 11b, 11a) = 8448$, $\sum_{K_5}(5a, 11b, 11a) = 99$ and $\sum_{PSL_2(11)}(5a, 11x, 11y) = \Delta_{PSL_2(11)}(5a, 11b, 11b) + \Delta_{PSL_2(11)}(5b, 11b, 11b) = 11 + 11 = 22$. Since by Table 7 we have $\Delta_G(5A, 11A, 11B) = 61479$, we obtain that $\Delta^*_G(5A, 11A, 11B) \geq \Delta_G(5A, 11A, 11B) - \sum_{K_1}(5a, 11b, 11a) - \sum_{K_5}(5a, 11b, 11a) + \sum_{PSL_2(11)}(5a, 11b,$

$11a) = 61479 - 8448 - 99 + 22 = 52954 > 0$, proving that the group G is $(5A, 11A, 11B)$-generated. □

Proposition 4.32. *The group G is $(5A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table 7 we have $\Delta_G(5A, 11X, 23Y) = 61893$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta_G^*(5A, 11X, 23Y) = \Delta_G(5A, 11X, 23Y) = 61893 > 0$, so that G is $(5A, 11X, 23Y)$-generated group for all $X, Y \in \{A, B\}$. □

4.3.4 $(5, 23, r)$-generations

In this subsection we discuss the case $(5, 23, r)$-generations. It follows that we will end up with 3 cases, namely $(5A, 23A, 23A)$-, $(5A, 23A, 23B)$- and $(5A, 23B, 23B)$-generation.

Proposition 4.33. *The group G is $(5A, 23X, 23Y)$-generated for all $X, Y \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 5. By Table 7 we have $\Delta_G(5A, 23X, 23Y) = 32706$ for all $X, Y \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta_G^*(5A, 23X, 23Y) = \Delta_G(5A, 23X, 23Y) = 32706 > 0$, so that the group G is $(5A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. □

4.3.5 $(7, 7, r)$-generations

In this subsection we discuss the case $(7, 7, r)$-generations.

It follows that we will end up with 16 cases, namely $(7A, 7A, 7A)$-, $(7A, 7A, 7B)$-, $(7A, 7A, 11A)$-, $(7A, 7A, 11B)$, $(7A, 7A, 23A)$-, $(7A, 7A, 23B)$-, $(7A, 7B, 7B)$-, $(7A, 7B, 11A)$-, $(7A, 7B, 11B)$, $(7A, 7B, 23A)$-, $(7A, 7B, 23B)$, $(7B, 7B, 7B)$-, $(7B, 7B, 11A)$-, $(7B, 7B, 11B)$-, $(7B, 7B, 23A)$- and $(7B, 7B, 23B)$-generation.

Proposition 4.34. *The group G is $(7X, 7Y, 7Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.5 we proved that G is $(2A, 7A, 7X)$-generated for $X \in \{A, B\}$. It follows by Theorem 2.4 that the group G is $(7A, 7A, (7A)^2)$- and $(7A, 7A, (B)^2)$-generated. Since by the power maps, we have $(7A)^2 = 7A$ and $(7B)^2 = 7B$, the group G becomes $(7A, 7A, 7A)$- and $(7A, 7A, 7B)$-generated. Since G is $(7A, 7A, 7A)$-generated, the same will hold for $(7B, 7B, 7B)$.

We are left only to investigate of the $(7A, 7B, 7B)$ generation for the group G. As in Proposition 4.5, we observe that the groups $PSL_3(4)$, A_7 (2-copies),

$2^3{:}PLS_3(2)$ (2-copies) and $PSL_3(2)$ have contributions. We obtained that $\sum_{PSL_3(4)}(7b, 7a, 7a) = 357$, $\sum_{A_7}(7b, 7a, 7a) = 36$, $\sum_{2^3:PSL_3(2)}(7b, 7a, 7a) = 8$, $\sum_{PSL_3(2)}(7a, 7b, 7b) = 1$ and $h(7A, PSL_3(4)) = h(7A, A_7) = h(7A, 2^3{:}PSL_3(2)) = h(7A, PSL_3(2)) = 2$.

The maximal subgroups K_1, K_2, K_3 and K_4 have elements of order 7. We obtained that $\sum_{K_1}(7b, 7a, 7a) = 8576$, $\sum_{K_2}(7b, 7a, 7a) = 379$, $\sum_{K_3}(7a, 7b, 7b) = 148$, $\sum_{K_4}(7b, 7a, 7a) = 379$. We found that $h(7A, K_1) = h(7A, K_4) = 2$ and $h(7A, K_2) = h(7A, K_3) = 1$.

Since by Table 7 we have $\Delta_G(7A, 7B, 7B) = 51948$, we then obtain that $\Delta^*_G(7A, 7B, 7B) \geq \Delta_G(7A, 7B, 7B) - 2 \cdot \sum_{K_1}(7b, 7a, 7a) - \sum_{K_2}(7b, 7a, 7a) - \sum_{K_3}(7a, 7b, 7b) - 2\cdot\sum_{K_4}(7b, 7a, 7a) + 2\cdot\sum_{PSL_3(4)}(7b, 7a, 7a) + 2\cdot 2\cdot\sum_{A_7}(7b, 7a, 7a) + 2 \cdot 2 \cdot \sum_{2^3:PSL_3(2)}(7b, 7a, 7a) + 2 \cdot \sum_{PSL_3(2)}(7a, 7b, 7b) = 51948 - 2(8576) - 379 - 148 - 2(379) + 2(379) + 2(2)(36) + 2(2)(8) + 2(1) = 34447 > 0$. Therefore, the group G is $(7A, 7B, 7B)$-generated. □

Proposition 4.35. *The group G is $(7X, 7Y, 11Z)$-generated for $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.6 we have proved that G is $(2A, 7X, 11Y)$-generated for all $X, Y \in \{A, B\}$. It follows by Theorem 2.4 that G is $(7X, 7X, (11Y)^2)$-generated. It follows that G is $(7X, 7X, (11A)^2)$- and $(7X, 7X, (11B)^2)$-generated for $X \in \{A, B\}$. Since by the power maps we have $(11A)^2 = 11B$ and $(11B)^2 = 11A$, it then follows that G is $(7X, 7X, 11B)$- and $(7X, 7X, 11A)$-generated group for $X \in \{A, B\}$.

We investigate the $(7A, 7B, 11X)$ generations of G, where $X \in \{A, B\}$. Looking at Proposition 4.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5, K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(7b, 7a, 11x) = 9856$ for $x \in \{a, b\}$. We already have $h(11X, K_1) = 1$ for $X \in \{A, B\}$. Since by Table 7 we have $\Delta_G(7A, 7B, 11X) = 56496$ for $X \in \{A, B\}$, we then obtain that $\Delta^*_G(7A, 7B, 11X) \geq \Delta_G(7A, 7B, 11X) - \sum_{K_1}(7b, 7a, 11x) = 56496 - 9856 = 46640 > 0$, proving G is $(7A, 7B, 11X)$-generated for $X \in \{A, B\}$. □

Proposition 4.36. *The group G is $(7X, 7Y, 23Z)$-generated for $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.7 we have proved that G is $(2A, 7X, 23Y)$-generated for all $X, Y \in \{A, B\}$. It follows by Theorem 2.4 that G is $(7X, 7X, (23Y)^2)$-generated. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ then it follows that G is $(7X, 7X, 23A)$- and $(7X, 7X, 23B)$-generated for $X \in \{A, B\}$.

We prove that G is $(7A, 7B, 23X)$-generated for $X \in \{A, B\}$. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table 7 we have $\Delta_G(7A, 7B, 23X) = 45264$ for $X \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7A, 7B, 23X) = \Delta_G(7A, 7B, 23X) = 45264 > 0$, so that G is $(7A, 7B, 23X)$-generated group for $X \in \{A, B\}$. □

4.3.6 $(7,11,r)$- and $(7,23,23)$-generations

In this subsection we discuss the cases $(7,11,r)$- and $(7,23,r)$-generations.

It follows that we will end up with 20 cases, namely $(7A,11A,11A)$-, $(7A,11A,11B)$-, $(7A,11A,23A)$-, $(7A,11A,23B)$-, $(7A,11B,11B)$-, $(7A,11B,23A)$-, $(7A,11B,23B)$-, $(7B,11A,11A)$-, $(7B,11A,11B)$-, $(7B,11A,23A)$-, $(7B,11A,23B)$-, $(7B,11B,11B)$-, $(7B,11B,23A)$-, $(7B,11B,23B)$-, $(7A,23A,23A)$-, $(7A,23A,23B)$-, $(7A,23B,23B)$-$(7B,23A,23A)$-, $(7B,23A,23B)$- and $(7B,23B,23B)$-generation.

Proposition 4.37. *The group G is $(7X,11Y,11Z)$-generated for $X,Y,Z \in \{A,B\}$.*

Proof. Looking at Proposition 4.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The groups K_5, K_7 and $PSL_2(11)$ will not have any contributions because they both do not have elements of order 7. We obtained that $\sum_{K_1}(7X,11Y,11Z) = 5632$ and $h(11Z,K_1) = 1$ for all $X,Y,Z \in \{A,B\}$. By Table 7 we have $\Delta_G(7X,11Y,11Z) = 64416$. We then obtained that $\Delta_G^*(7X,11Y,11Z) \geq \Delta_G(7X,11Y,11Z) - \sum_{K_1}(7X,11Y,11Z) = 64416 - 5632 = 58784 > 0$ and so G is $(7X,11Y,11Z)$-generated group for all $X,Y,Z \in \{A,B\}$. $\square$

Proposition 4.38. *The group G is $(7X,11Y,23Z)$-generated for all $X,Y,Z \in \{A,B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table 7 we have $\Delta_G(7X,11Y,23Z) = 67712$ for all $X,Y,Z \in \{A,B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7X,11Y,23Z) = \Delta_G(7X,11Y,23Z) = 67712 > 0$, so that G is $(7X,11Y,23Z)$-generated group for all $X,Y,Z \in \{A,B\}$. $\square$

Proposition 4.39. *The group G is $(7X,23Y,23Z)$-generated for all $X,Y,Z \in \{A,B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of order 23. This maximal subgroup will not have any contributions because it does not have elements of order 7. By Table 7 we have $\Delta_G(7X,23Y,23Z) = 32384$ for all $X,Y,Z \in \{A,B\}$. Since there is no contributing group, we then obtain that $\Delta^*(7X,23Y,23Z) = \Delta_G(7X,23Y,23Z) = 32384 > 0$, so that the group G is $(7X,23Y,23Z)$-generated for all $X,Y,Z \in \{A,B\}$. $\square$

4.3.7 $(11,11,r)$-generations

In this subsection we discuss the case $(11,11,r)$-generations.

It follows that we will end up with 10 cases, namely $(11A,11A,11A)$-, $(11A,11A,11B)$-, $(11A,11A,23A)$-, $(11A,11A,23B)$-, $(11A,11B,11B)$-, $(11A,11B,$

$23A$)-, $(11A, 11B, 23B)$-, $(11B, 11B, 11B)$-, $(11B, 11B, 23A)$- and $(11B, 11B, 23B)$-generation.

Proposition 4.40. *The group G is $(11X, 11Y, 11Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.8 we have proved that G is $(2A, 11X, 11Y)$-generated for all $X, Y \in \{A, B\}$. Then, by Theorem 2.4 it follows that G is $(11X, 11X, (11Y)^2)$-generated for all $X, Y, Z \in \{A, B\}$. Since $(11A)^2 = 11B$ and $(11B)^2) = 11A$ then it follows that G is $(11X, 11X, 11Y)$-generated for all $X, Y, Z \in \{A, B\}$.

We prove that G is $(11A, 11B, 11B)$-generated. Looking at Proposition 4.3, we see that K_1, K_5, K_7 and $PSL_2(11)$ are the only groups having elements of order 11. The maximal subgroup K_7 have its relevant structure constant zero, so it will not have any contributions. We obtained that $\sum_{K_1}(11b, 11a, 11a) = 3632$, $\sum_{K_5}(11b, 11a, 11a) = 35$ and $\sum_{PSL_2(11)}(11b, 11a, 11a) = 2$. We have found that $h(11B, K_1) = h(11B, K_5) = h(11B, PSL_2(11)) = 1$. By Table 8 we have $\Delta_G(11A, 11B, 11B) = 87485$, we then obtain $\Delta_G^*(11A, 11B, 11B) \geq \Delta_G(11A, 11B, 11B) - \sum_{K_1}(11b, 11a, 11a) - \sum_{K_5}(11b, 11a, 11a) + \sum_{PSL_2(11)}(11b, 11a, 11a) = 87485 - 3632 - 35 + 2 = 83820 > 0$, proving that the group G is $(11A, 11B, 11B)$-generated. □

Proposition 4.41. *The group G is $(11X, 11Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.9 we have proved that G is $(2A, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$. Then, by Theorem 2.4 it follows that G is $(11X, 11X, (23Y)^2)$-generated for all $X, Y \in \{A, B\}$. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ we then obtained that G is $(11X, 11X, 23Y)$-generated for all $X, Y \in \{A, B\}$.

We still have to prove the $(11A, 11B, 23X)$-generations where $X \in \{A, B\}$. By Table 4 we see that K_7 is the only maximal subgroup having elements of orders 11 and 23. We then obtain that $\sum_{K_7}(11x, 11y, 23z) = \Delta_{K_7}(11a, 11j, 23z) + \Delta_{K_7}(11c, 11h, 23z) + \Delta_{K_7}(11d, 11g, 23z) + \Delta_{K_7}(11e, 11f, 23z) + \Delta_{K_7}(11i, 11b, 23x) = 23 + 23 + 23 + 23 + 23 = 115$ for $z \in \{a, b\}$. We have found that $h(23X, K_7) = 1$ for $X \in \{A, B\}$. Since by Table 8 we have $\Delta_G(11A, 11B, 23X) = 79994$, then we obtained that $\Delta_G^*(11A, 11B, 23X) \geq \Delta_G(11A, 11B, 23Z) - \sum_{K_7}(11x, 11y, 23z) = 79994 - 115 = 79879 > 0$ for all $Z \in \{A, B\}$. Hence G is $(11A, 11B, 23X)$-generated group for $X \in \{A, B\}$. □

4.3.8 $(11, 23, r)$-generations

We will be looking at the cases $(11A, 23A, 23A)$-, $(11A, 23A, 23B)$-, $(11A, 23B, 23B)$-, $(11B, 23A, 23A)$-, $(11B, 23A, 23B)$- and $(11B, 23B, 23B)$-generation.

Proposition 4.42. *The group G is $(11X, 23Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Table 4, K_7 is the only maximal subgroup having elements of orders 11 and 23. This maximal subgroup K_7 will not have any contributions because its relevant structure constants are all zero. By Table 8 we have $\Delta_G(11X, 23Y, 23Z) = 42067$ for all $X, Y, Z \in \{A, B\}$. Since there is no contributing group, we then obtain that $\Delta^*_G(11X, 23Y, 23Z) = \Delta_G(11X, 23Y, 23Z) = 42067 > 0$, showing that G is $(11X, 23Y, 23Z)$-generated group for all $X, Y, Z \in \{A, B\}$. □

4.3.9 $(23, 23, r)$-generations

We conclude our investigation on the (p, q, r)-generations of the Mathieu sporadic simple group G by considering the $(23, 23, 23)$-generations. We will be looking at the cases $(23A, 23A, 23A)$-, $(23A, 23A, 23B)$-, $(23A, 23B, 23B)$- and $(23B, 23B, 23B)$-generation.

Proposition 4.43. *The group G is $(23X, 23Y, 23Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. By Proposition 4.10 we have proved that G is $(2A, 23X, 23Y)$-generated for all $X, Y \in \{A, B\}$. Then, by Theorem 2.4 it follows that G is $(23X, 23X, (23Y)^2)$-generated for all $X, Y \in \{A, B\}$. Since $(23A)^2 = 23A$ and $(23B)^2 = 23B$ then it follows that G is $(23X, 23X, 23Y)$-generated for all $X, Y \in \{A, B\}$. We now check the $(23A, 23B, 23B)$-generation of G. By Table 4, K_7 is the only maximal subgroup having elements of order 11. We obtained that $\sum_{K_7}(23a, 23b, 23b) = 5$ and $h(23B, K_7) = 1$. Since by Table 8 we have $\Delta_G(23A, 23B, 23B) = 17646$, then we obtained that $\Delta^*_G(23A, 23B, 23B) \geq \Delta_G(23A, 23B, 23B) - \sum_{K_7}(23a, 23b, 23b) = 17646 - 5 = 17641 > 0$. Hence the group G is $(23A, 23B, 23B)$-generated. □

5. Conclusion

As mentioned at the introduction of this paper that it is natural to ask whether a finite simple group G is (l, m, n)-generated or not. The motivation for this question came from the calculation of the genus of finite simple groups [34]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations (see [33] for further details). Our aim in this paper was to establish all the (p, q, r)-generations of the Mathieu group M_{23}, where p, q and r are prime numbers dividing $|M_{23}|$. The main result of the paper was to prove Theorem 1.1 and this was done though sequence of propositions. We found that the Mathieu group M_{23} is generated by the following:

- $(2A, 5A, 11X)$, $(2A, 5A, 23X)$, $(2A, 7X, 7Y)$, $(2A, 7X, 11Y)$, $(2A, 7X, 23Y)$, $(2A, 11X, 11Y)$, $(2A, 11X, 23Y)$ and $(2A, 23X, 23Y)$ for all $X, Y \in \{A, B\}$.
- $(3A, 3A, 11X)$, $(3A, 3A, 23X)$, $(3A, 5A, 5A)$, $(3A, 5A, 7X)$, $(3A, 5A, 11X)$, $(3A, 5A, 23X)$, $(3A, 7X, 7Y)$, $(3A, 7X, 11Y)$, $(3A, 7X, 23Y)$, $(3A, 11X, 11Y)$, $(3A, 11X, 23Y)$ and $(3A, 23X, 23Y)$ for all $X, Y \in \{A, B\}$.

- $(5A,5A,5A)$, $(5A,5A,7X)$, $(5A,5A,11X)$, $(5A,5A,23X)$, $(5A,7X,7Y)$, $(5A,7X,11Y)$, $(5A,7X,23Y)$, $(5A,11X,11Y)$, $(5A,11X,23Y)$ and $(5A, 23X,23Y)$ for all $X,Y \in \{A,B\}$.
- $(7X,7Y,7Z)$, $(7X,7Y,11Z)$, $(7X,7Y,23Z)$, $(7X,11Y,11Z)$, $(7X,11Y,23Z)$ and $(7X,23Y,23Z)$ for all $X,Y,Z \in \{A,B\}$.
- $(11X,11Y,11Z)$, $(11X,11Y,23Z)$ and $(11X,23Y,23Z)$ for all $X,Y,Z \in \{A,B\}$.
- $(23X,23Y,23Z)$ for all $X,Y,Z \in \{A,B\}$.

Theorem 1.1 tells us also that the Mathieu group M_{23} is not generated by the following: $(p,q,r) \in \{(2,3,r),(2,5,5),(2,5,7),(3,3,3),(3,3,5),(3,3,7)\}$, for all primes r that divides $|M_{23}|$.

Tables : Structure constants of M_{23}

Tables 5 to 8 following here below give the partial structure constants of M_{23} computed using GAP that were used in the calculations above.

Table 5:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(2A,2A,pX)$	98	30	5	0	0	0	0	0	0
$\Delta_{M_{23}}(2A,3A,pX)$	448	180	65	35	35	11	11	0	0
$\Delta_{M_{23}}(2A,5A,pX)$	896	780	605	364	364	253	253	138	138
$\Delta_{M_{23}}(2A,7A,pX)$	0	450	390	301	462	308	308	184	184
$\Delta_{M_{23}}(2A,7B,pX)$	0	450	390	462	301	308	308	184	184
$\Delta_{M_{23}}(2A,11A,pX)$	0	180	345	392	392	341	341	391	391
$\Delta_{M_{23}}(2A,11B,pX)$	0	180	345	392	392	341	341	391	391
$\Delta_{M_{23}}(2A,23A,pX)$	0	0	90	112	112	187	187	161	230
$\Delta_{M_{23}}(2A,23B,pX)$	0	0	90	112	112	187	187	230	161
$\lvert C_{M_{23}}(pX)\rvert$	688	180	15	14	14	11	11	23	23

Table 6:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(3A,3A,pX)$	2688	1681	855	511	511	275	275	138	138
$\Delta_{M_{23}}(3A,5A,pX)$	11648	10260	5490	4886	4886	4136	4136	2438	2438
$\Delta_{M_{23}}(3A,7A,pX)$	6720	6570	5490	4886	4886	4136	4136	3312	3312
$\Delta_{M_{23}}(3A,7B,pX)$	6720	6570	5490	4886	4886	4136	4136	3312	3312
$\Delta_{M_{23}}(3A,11A,pX)$	2688	4500	5175	5264	5264	5126	5379	5129	5129
$\Delta_{M_{23}}(3A,11B,pX)$	2688	4500	5175	5264	5264	5379	5126	5129	5129
$\Delta_{M_{23}}(3A,23A,pX)$	0	1080	1590	2016	2016	2453	2453	3082	2714
$\Delta_{M_{23}}(3A,23B,pX)$	0	1080	1590	2016	2016	2453	2453	2714	3082
$\lvert C_{M_{23}}(pX)\rvert$	688	180	15	14	14	11	11	23	23

Table 7:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(5A,5A,pX)$	108416	78600	61058	54320	54320	45287	45287	37582	37582
$\Delta_{M_{23}}(5A,7A,pX)$	69888	65880	58200	52584	52584	48576	48576	44160	44160
$\Delta_{M_{23}}(5A,7B,pX)$	69888	65880	58200	52584	52584	48576	48576	44160	44160
$\Delta_{M_{23}}(5A,11A,pX)$	61824	62100	61755	61824	61824	62238	61479	61893	61893
$\Delta_{M_{23}}(5A,11B,pX)$	61824	62100	61755	61824	61824	61479	62238	61893	61893
$\Delta_{M_{23}}(5A,23A,pX)$	16128	19080	24510	26880	26880	29601	29601	32706	32706
$\Delta_{M_{23}}(5A,23B,pX)$	16128	19080	24510	26880	26880	29601	29601	32706	32706
$\Delta_{M_{23}}(7A,7A,pX)$	88704	62820	56340	51948	60412	48400	48400	52992	52992
$\Delta_{M_{23}}(7A,7B,pX)$	57792	62820	56340	51948	51948	56496	56496	45264	45264
$\Delta_{M_{23}}(7A,11A,pX)$	75264	67680	66240	71904	61600	64416	64416	67712	67712
$\Delta_{M_{23}}(7A,11B,pX)$	75264	67680	66240	71904	61600	64416	64416	67712	67712
$\Delta_{M_{23}}(7A,23A,pX)$	21504	25920	28800	27552	32256	32384	32384	32384	32384
$\Delta_{M_{23}}(7A,23B,pX)$	21504	25920	28800	27552	32256	32384	32384	32384	32384
$\Delta_{M_{23}}(7B,7B,pX)$	88704	62820	56340	60412	51948	48400	48400	52992	52992
$\Delta_{M_{23}}(7B,11A,pX)$	75264	67680	66240	61600	71904	64416	64416	67712	67712
$\Delta_{M_{23}}(7B,11B,pX)$	75264	67680	66240	61600	71904	64416	64416	67712	67712
$\Delta_{M_{23}}(7B,23A,pX)$	21504	25920	28800	32256	27552	32384	32384	32384	32384
$\Delta_{M_{23}}(7B,23B,pX)$	21504	25920	28800	32256	27552	32384	32384	32384	32384
$\|C_{M_{23}}(pX)\|$	688	180	15	14	14	11	11	23	23

Table 8:

pX	2A	3A	5A	7A	7B	11A	11B	23A	23B
$\Delta_{M_{23}}(11A,11A,pX)$	83328	88020	83835	81984	81984	87485	88520	81029	81029
$\Delta_{M_{23}}(11A,11B,pX)$	83328	83880	84870	81984	81984	87485	87485	79994	79994
$\Delta_{M_{23}}(11A,23A,pX)$	45696	40140	40365	41216	41216	38258	38753	42067	42067
$\Delta_{M_{23}}(11A,23B,pX)$	45696	40140	40365	41216	41216	38258	38753	42067	42067
$\Delta_{M_{23}}(11B,11B,pX)$	83328	88020	83835	81984	81984	88520	87485	81029	81029
$\Delta_{M_{23}}(11B,23A,pX)$	45696	40140	40365	41216	41216	38753	38258	42067	42067
$\Delta_{M_{23}}(11B,23B,pX)$	45696	40140	40365	41216	41216	38753	38258	42067	42067
$\Delta_{M_{23}}(23A,23A,pX)$	26880	21240	21330	19712	19712	20119	20119	17646	18222
$\Delta_{M_{23}}(23A,23B,pX)$	18816	24120	21330	19712	19712	20119	20119	17646	17646
$\Delta_{M_{23}}(23B,23B,pX)$	26880	21240	21330	19712	19712	20119	20119	18222	17646
$\|C_{M_{23}}(pX)\|$	688	180	15	14	14	11	11	23	23

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