SEP elements and the solution to constructed equations

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Abstract. For the last five years, SEP elements in a ring with involution have been discussed by many authors. In this paper, we obtain many new characterizations of SEP elements by using group inverse and Moore-Penrose inverse, also by constructing a lot of equations and discussing the expression forms of solution to these equations in certain given set.

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1. Introduction

The EP elements in a ring with involution have been investigated by a lot of authors [7, 13, 14, 16, 20, 21, 23, 25, 26, 28, 31, 32, 33, 34], which originates from the study of EP matrices and EP linear operators on Banach or Hilbert

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spaces [1, 2, 3, 5, 6, 8, 11, 9, 27]. The study of SEP elements originates from [20] and for further research, the readers can refer to [4, 29, 30, 32]. However, the excellent and significant fact is that the people can use generalized inverses of matrices such as EP matrices and SEP matrices, in the case when ordinary inverses do not exist, in order to solve some matrix equations, operator equations or differential equations.

Let R be a ring. An element $a \in R$ is called the group invertible [10, 12, 17] if there exists $a^{\#} \in R$ such that

$$a = aa^{\#}a,\ a^{\#} = a^{\#}aa^{\#},\ aa^{\#} = a^{\#}a.$$

According to [17], $a^{\#}$ is called the group inverse of a, which is unique if it exists. Clearly, $a \in R^{\#}$ if and only if $a \in a^2R \cap Ra^2$.

We usually write $R^{\#}$ to denote the set of all group invertible elements in R.

An involution $* : a \to a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*\ for\ a,\ b \in R.$$

$a \in R$ is called the Moore-Penrose invertible [13, 15, 19, 22] if there exists $a^+ \in R$ such that

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a.$$

We always use R^+ to denote the set of all Moore-Penrose invertible elements in R.

Let $a \in R^{\#} \cap R^+$. if $a^{\#} = a^+$, then a is called an EP element. We denote the set of all EP elements in R by R^{EP}.

If $a = aa^*a$, then a is said to be a partial isometry (or PI) element [20, 24, 25] and we use R^{PI} to denote the set of all PI elements in R. It is known that $a \in R^+$ is partial isometry if and only if $a^+ = a^*$.

If $a \in R^{EP} \cap R^{PI}$, then a is said to be a strong EP element. We used to write R^{SEP} to represent the set of all strong EP elements in R.

EP elements in rings with involution have been characterized by conditions involving their group inverse and Moore-Penrose inverse, the solution to related constructed equations. Motived by these ways, we discuss the properties of SEP elements.

Throughout this paper R is a $*-$ring with 1.

2. Several properties of SEP elements

In [20], the following equation and theorem are given.

$$axa^* = xa^{\#}a. \tag{2.1}$$

[20, Theorem 2.2]: Let $a \in R^{\#} \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (2.1) has at least one solution in $\chi_a = \{a, a^{\#}, a^+, a^*, (a^{\#})^*, (a^+)^*\}$.

In this section, inspired by [20], we discuss some properties of SEP elements using the expression of core inverse or dual inverse of certain generalized inverse element.

Let $a \in R^{\#} \cap R^{+}$. Then, we have the following formulas on core inverse and dual core inverse

$$a^{\textcircled{\#}} = a^{\#}aa^{+};,$$
$$a_{\textcircled{\#}} = a^{+}aa^{\#};,$$
$$(a^{\#})^{+} = a^{+}a^{3}a^{+};$$

and

$$(a^{+})^{\#} = (aa^{\#})^{*}a(aa^{\#})^{*}.$$

Hence, we have the following theorem.

Theorem 2.1. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if Eq.* (2.1) *has at least one solution in $\sigma_a = \left\{a^{\textcircled{\#}}, a_{\textcircled{\#}}, (a^{\#})^{+}, (a^{+})^{\#}\right\}$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then, $a^{\#} = a^{+} = a^{*}$, this gives

$$a^{2}a^{*} = a^{2}a^{\#} = a = aa^{\#}a.$$

Hence, $x = a$ is a solution.

" $\Leftarrow$ " (1) If $x = a^{\textcircled{\#}} = a^{\#}aa^{+}$ is a solution, then

$$a(a^{\#}aa^{+})a^{*} = (a^{\#}aa^{+})a^{\#}a,$$

e.g., $aa^{+}a^{*} = a^{\#}$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(2) If $x = a_{\textcircled{\#}} = a^{+}aa^{\#}$, then

$$a(a^{+}aa^{\#})a^{*} = (a^{+}aa^{\#})a^{\#}a.$$

e.g., $aa^{\#}a^{*} = a^{+}aa^{\#}$. Multiplying the equality on the left by a, one has $aa^{*} = aa^{\#}$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(3) If $x = (a^{\#})^{+} = a^{+}a^{3}a^{+}$, then $a(a^{+}a^{3}a^{+})a^{*} = \left(a^{+}a^{3}a^{+}\right)a^{\#}a$, e.g.,

$$a^{3}a^{+}a^{*} = a^{+}a^{2}.$$

Multiplying the equality on the left by $a^{\#}a^{\#}$, one gets $aa^{+}a^{*} = a^{\#}$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(4) If $x = (a^{+})^{\#} = (aa^{\#})^{*}a(aa^{\#})^{*}$, then

$$a(aa^{\#})^{*}a(aa^{\#})^{*}a^{*} = (aa^{\#})^{*}a(aa^{\#})^{*}a^{\#}a.$$

Multiplying the equality on the left by $a^{+}a^{+}$, one yields

$$a^{*} = a^{+}a^{\#}a.$$

It follows that $a^{*}a = (a^{+}a^{\#}a)a = a^{+}a$. Hence, $a \in R^{PI}$ by [20, Theorem 1.5.2], this infers $a^{+} = a^{*} = a^{+}a^{\#}a$. Thus, $a \in R^{EP}$ by [20, Theorem 1.2.1]. Therefore, $a \in R^{SEP}$. $\square$

Observing the proof of Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $a^* = a^+ a^{\#} a$.*

Noting that if $a \in R^{\#} \cap R^{+}$, then $a_{\textcircled{\#}} = a^+ a^{\#} a$. Thus, by Corollary 2.2, we have

Corollary 2.3. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $a^* = a_{\textcircled{\#}}$.*

Since $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$, use a^* to replace a in Corollary 2.2, we have

Corollary 2.4. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $a = (a^+)^*(a^{\#})^* a^*$.*

Applying the involution on the equality of Corollary 2.4, we obtain.

Corollary 2.5. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $a^* = a^{\textcircled{\#}}$.*

Theorem 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $aa^+ = (a^{\#})^* a^{\#}$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then, $a^* = a^+ a^{\#} a$ by Corollary 2.2. This gives $aa^+ = (a^+)^* a^* = (a^+)^* a^+ a^{\#} a = (a^+)^* a^{\#} = (a^{\#})^* a^{\#}$.
" $\Leftarrow$ " From the equality $aa^+ = (a^{\#})^* a^{\#}$, one has

$$a^* = a^* a a^+ = a^*((a^{\#})^* a^{\#}) = (aa^{\#})^* a^{\#},$$

so

$$a^+ a^* = a^+ a^{\#}.$$

Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3]. □

Noting that $a^+ a^{\#} a = a^+ a^3 a^+ a^{\#} a^{\#} = (a^{\#})^+ a^{\#} a^{\#}$. Then, by Corollary 2.2, we have the following corollary.

Corollary 2.7. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $a^* = (a^{\#})^+ a^{\#} a^{\#}$.*

Since $a \in R^{SEP}$ if and only if $a^{\#} \in R^{SEP}$, use $a^{\#}$ to replace a in Corollary 2.7, we have

Corollary 2.8. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $(a^{\#})^* = a^+ a^2$.*

Noting that $a_{\textcircled{\#}} \in R^{EP}$ and $a^+_{\textcircled{\#}} = a^+ a^2$. Hence, we have.

Corollary 2.9. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $(a^{\#})^* = a^+_{\textcircled{\#}}$.*

Since $((a^{\#})^*)^+ = ((a^{\#})^+)^* = aa^+a^*a^+a$, Corollary 2.9 implies the following corollary.

Corollary 2.10. *Let $a \in R^{\#} \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+a^* = a_{\#\!\!\!\bigcirc}$.*

Remark 2.11. It is well known that $a \in R^{EP}$ if and only if $a \in R^+$ and $aa^+ = a^+a$. However, if $a^+a = (a^{\#})^*a^{\#}$, we can't obtain $a \in R^{SEP}$.

For example, let $R = M_3(Z_2)$ and choose $a = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, $a^{\#} = a$ and $a^+ = a^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Clearly, $a^+a = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; $(a^{\#})^*a^{\#} = a^*a = a^+a$. While $aa^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq (a^{\#})^*a^{\#}$. Hence, by Theorem 2.6, a is not SEP. In fact, even $a^{\#} \neq a^+$

3. Using group inverse and Moore-Penrose inverse to characterize SEP elements

In [20], the group inverse and the Moore Penrose inverse of the product of two generalize inverse elements of an element is represented. Inspired by this, we study in this section SEP elements by constructing the expresentation forms of the group inverse and the Moore Penrose inverse of the product of several generalized inverse elements of an element.

Lemma 3.1. *Let $a \in R^{\#} \cap R^+$. Then, $aa^+a^* \in R^{EP}$ and $(aa^+a^*)^+ = aa^+(a^{\#})^*$.*

Proof. It is routine. □

Theorem 3.2. *Let $a \in R^{\#} \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+(a^{\#})^* = a^+a^2$.*

Proof. It is an immediate result of Corollary 2.10 and Lemma 3.1. □

Corollary 3.3. *Let $a \in R^{\#} \cap R^+$. Then, $a \in R^{SEP}$ if and only if $aa^+ = a^+a^2a^*$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$. Then, $a \in R^{EP}$ and $aa^+(a^{\#})^* = a^+a^2$ by Theorem 3.2. Noting that $a^+ = a^+(aa^{\#})^*$ and $(aa^{\#})^* = aa^{\#}$ by [20, Theorem 1.1.3]. Then, $aa^+ = aa^+(a^{\#})^*a^* = a^+a^2a^*$.

" $\Longleftarrow$ " From the equality $aa^+ = a^+a^2a^*$, one gets $aa^+ = a^+a^2a^+$. Hence, $a \in R^{EP}$, this induces $(aa^{\#})^* = aa^{\#}$ by [20, Theorem 1.1.3]. It follows that $aa^+(a^{\#})^* = a^+a^2a^*(a^{\#})^* = a^+a^2(aa^{\#}) = a^+a^2$. Hence, $a \in R^{SEP}$ by Theorem 3.2. □

Now, we give the following lemma which proof is routine.

Lemma 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then*

(1) $(a^{+}a^{2}a^{*})^{+} = (a^{+})^{*}a^{\#}$;

(2) $(a^{+}a^{2}a^{*})^{\#} = (a^{\#})^{*}a^{+}aa^{\#}(aa^{\#})^{*}$.

Noting that $(aa^{+})^{+} = (aa^{+})^{\#} = aa^{+}$. Hence, Corollary 3.3 and Lemma 3.4 induce the following two theorems.

Theorem 3.5. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $aa^{+} = (a^{+})^{*}a^{\#}$.*

Theorem 3.6. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if $aa^{+} = (a^{\#})^{*}a^{+}aa^{\#}(aa^{\#})^{*}$.*

Noting that $(a_{\textcircled{\#}})^{*} = (a^{\#})^{*}a^{+}a$. Then, Theorem 3.6 implies the following corollary.

Corollary 3.7. *Let $a \in R^{\#} \cap R^{+}$. Then, the followings are equivalent:*

(1) $a \in R^{SEP}$;

(2) $aa^{+} = (a_{\textcircled{\#}})^{*}a^{\#}(aa^{\#})^{*}$;

(3) $aa^{+} = (a^{\#})^{*}a_{\textcircled{\#}}(aa^{\#})^{*}$.

Remark 3.8. Though $a \in R^{EP}$ if and only if $aa^{+} = aa^{\#}$, but Theorem 3.5 can't draw conclusion $a \in R^{SEP}$ under condition $aa^{\#} = (a^{+})^{*}a^{\#}$.

In the example of Remark 2.11, $(a^{+})^{*}a^{\#} = (a^{*})^{*}a = a^{2} = aa^{\#}$. But $a \notin R^{SEP}$.

4. Generalized inverse equations and SEP elements

In this section, we characterize SEP elements through the solvability of several constructed generalized inverse equations in a given set.

By observing Theorem 3.5, when $a \in R^{SEP}$, we have

$$aa^{+} = (a^{+})^{*}aa^{+}a^{\#}.$$

So, we can construct the equation

$$xa^{+} = (a^{+})^{*}xa^{+}a^{\#}. \tag{4.1}$$

Theorem 4.1. *Let $a \in R^{\#} \cap R^{+}$. Then, $a \in R^{SEP}$ if and only if Eq. (4.1) has at least one solution in $\zeta_a = \chi_a \cup \sigma_a$.*

Proof. " $\Rightarrow$ " If $a \in R^{SEP}$, then $aa^{+} = (a^{+})^{*}a^{\#} = (a^{+})^{*}aa^{+}a^{\#}$ by Theorem 3.5. Hence, $x = a$ is a solution in ζ_a.
" $\Leftarrow$ " (1) If $x = a$, then $aa^{+} = (a^{+})^{*}aa^{+}a^{\#} = (a^{+})^{*}a^{\#}$. Hence, $a \in R^{SEP}$ by Theorem 3.5.

(2) If $x = a^{\#}$, then $a^{\#}a^{+} = (a^{+})^{*}a^{\#}a^{+}a^{\#}$, this gives

$$a = a^{\#}a^{+}a^{3} = (a^{+})^{*}a^{\#}a^{+}a^{\#}a^{3} = (a^{+})^{*}a^{\#}a^{+}a^{2} = (a^{+})^{*}.$$

Hence, $a^{\#}a^{+} = aa^{\#}a^{+}a^{\#} = a^{\#}a^{\#}$, this infers $a \in R^{EP}$. Thus, $a \in R^{SEP}$.

(3) If $x = a^{+}$, then $a^{+}a^{+} = (a^{+})^{*}a^{+}a^{+}a^{\#}$. Multiplying the equality on the right by $a^{+}a$, one has

$$a^{+}a^{+} = a^{+}a^{+}a^{+}a.$$

By [30, Lemma 2.11], we have $a^{+} = a^{+}a^{+}a$. Hence, $a \in R^{EP}$, this gives $x = a^{+} = a^{\#}$. Thus, $a \in R^{SEP}$ by (2).

(4) If $x = a^{*}$, then $a^{*}a^{+} = (a^{+})^{*}a^{*}a^{+}a^{\#} = aa^{+}a^{+}a^{\#}$. Multiplying the equality on the right by $a^{+}a$, one gets

$$a^{*}a^{+} = a^{*}a^{+}a^{+}a$$

and

$$a^{+} = (a^{\#})^{*}a^{*}a^{+} = (a^{\#})^{*}a^{*}a^{+}a^{+}a = a^{+}a^{+}a.$$

Hence, $a \in R^{EP}$, this gives

$$a^{*}a^{\#} = a^{*}a^{+} = aa^{+}a^{+}a^{\#} = a^{+}a^{\#}.$$

By [20, Theorem 1.5.2], $a \in R^{PI}$. Therefore, $a \in R^{SEP}$.

(5) If $x = (a^{+})^{*}$, then $(a^{+})^{*}a^{+} = (a^{+})^{*}(a^{+})^{*}a^{+}a^{\#}$. Multiplying the equality on the left by a^{*}, one gets

$$a^{+} = a^{+}a(a^{+})^{*}a^{+}a^{\#} = (a^{+}a(a^{+})^{*}a^{+}a^{\#})a^{+}a = a^{+}a^{+}a.$$

Hence, $a \in R^{EP}$, this infers

$$a^{+} = a^{+}a(a^{+})^{*}a^{+}a^{\#} = a^{\#}a(a^{+})^{*}a^{+}a^{\#} = (a^{+})^{*}a^{+}a^{\#},$$
$$a^{*}a^{+} = a^{*}(a^{+})^{*}a^{+}a^{\#} = a^{+}a^{\#} = a^{\#}a^{+}.$$

Thus, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(6) If $x = (a^{\#})^{*}$, then $(a^{\#})^{*}a^{+} = (a^{+})^{*}(a^{\#})^{*}a^{+}a^{\#}$, it follows that

$$a^{+} = a^{*}(a^{\#})^{*}a^{+} = a^{*}(a^{+})^{*}(a^{\#})^{*}a^{+}a^{\#} = (a^{\#})^{*}a^{+}a^{\#}$$

and

$$a^{*}a^{+} = a^{*}(a^{\#})^{*}a^{+}a^{\#} = a^{+}a^{\#}.$$

Thus, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(7) If $x = (a^{\#})^{+} = a^{+}a^{3}a^{+}$, then $a^{+}a^{3}a^{+}a^{+} = (a^{+})^{*}a^{+}a^{3}a^{+}a^{+}a^{\#}$. Multiplying the equality on the right by $a^{+}a$, one has

$$a^{+}a^{3}a^{+}a^{+} = a^{+}a^{3}a^{+}a^{+}a^{+}a$$

and

$$a^+a^+ = a^+a^\#a^+a^3a^+a^+ = a^+a^\#a^+a^3a^+a^+a^+a = a^+a^+a^+a.$$

By [30, Lemma 2.11], $a^+ = a^+a^+a$. Hence, $a \in R^{EP}$, it follows that

$$x = \left(a^\#\right)^+ = \left(a^+\right)^+ = a.$$

Thus, $a \in R^{SEP}$ by (1).

(8) If $x = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then

$$(aa^\#)^*a(aa^\#)^*a^+ = (a^+)^*(aa^\#)^*a(aa^\#)^*a^+a^\#,$$

e.g.,

$$(aa^\#)^* = (a^+)^*(aa^\#)^*a^\#.$$

This gives

$$a^* = a^*(aa^\#)^* = a^*(a^+)^*(aa^\#)^*a^\# = (aa^\#)^*a^\#$$

and

$$a^+a^* = a^+(aa^\#)^*a^\# = a^+a^\#.$$

Thus, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(9) If $x = a^{\oplus\!\!\!\!\#} = a^\#aa^+$, then

$$a^\#aa^+a^+ = (a^+)^*a^\#aa^+a^+a^\# = (a^+)^*a^+a^+a^\#.$$

Multiplying the equality on the right by a^+a, one yields

$$a^\#aa^+a^+ = a^\#aa^+a^+a^+a$$

and

$$a^+a^+ = a^+a(a^\#aa^+a^+) = a^+a(a^\#aa^+a^+a^+a) = a^+a^+a^+a.$$

This implies $a^+ = a^+a^+a$ by [30, Lemma 2.11]. Hence, $a \in R^{EP}$, it follows that $x = a^{\oplus\!\!\!\!\#} = a^\#aa^+ = a^\#$. Thus, $a \in R^{SEP}$ by (2).

(10) If $x = a_{\oplus\!\!\!\!\#} = a^+aa^\#$, then

$$a^+aa^\#a^+ = (a^+)^*a^+aa^\#a^+a^\# = (a^+)^*a^\#a^+a^\#.$$

Multiplying the equality on the left by $aa^\#$, one obtains

$$a^+aa^\#a^+ = aa^\#a^+aa^\#a^+ = a^\#a^+$$

and

$$a^+a^2 = a^+aa^\#a^+a^3 = a^\#a^+a^3 = a.$$

It follows that $a \in R^{EP}$, this shows $x = a_{\oplus\!\!\!\!\#} = a^+aa^\# = a^\#$. Thus, $a \in R^{SEP}$ by (2). $\square$

Now, we revise Eq. (4.1) as follows

$$xa^+ = (a^+)^* a^+ x a^\# . \tag{4.2}$$

Similar to the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.2) has at least one solution in ζ_a.*

Multiplying Eq. (4.2) on left by a^*, one gets

$$a^* x a^+ = a^+ x a^\# . \tag{4.3}$$

Theorem 4.3. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.3) has at least one solution in ζ_a.*

Now, we revise Eq. (4.2) as follow

$$xa^* = aa^+ x a^\# . \tag{4.4}$$

Theorem 4.4. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.4) has at least one solution in ζ_a.*

Clearly, Eq. (4.3) has the same solution as the following equation

$$a^* a^+ a x a^+ = a^+ x a^\# .$$

Hence, we can construct the following equation

$$a^* y a x a^+ = y x a^\# . \tag{4.5}$$

Theorem 4.5. *Let $a \in R^\# \cap R^+$. Then, $a \in R^{SEP}$ if and only if Eq. (4.5) has at least one solution in $\zeta_a^2 =: \{(x, y) | x, y \in \zeta_a\}$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then, $a^* = a^\# = a^+$, it follows that

$$a^* a^3 a^+ = a^\# a^3 a^\# = a = a^2 a^\# .$$

Thus, $(x, y) = (a, a)$ is a solution in ζ_a^2.
" $\Leftarrow$ " (I) If $y = a$, then $a^* a^2 x a^+ = a x a^\#$ has at least one solution x_0 in ζ_a, one gets

$$a^* a^2 x_0 a^+ = a x_0 a^\# . \tag{4.6}$$

Multiplying the equality on the right by aa^+, one has $ax_0a^\# = ax_0a^\# aa^+$, this gives $x_0^+ a^\# a x_0 a^\# = x_0^+ a^\# a x_0 a^\# a a^+$.

Noting that $x_0^+ a^\# a x_0 = \begin{cases} a^+ a, x_0 \in \left\{a, a^\#, (a^+)^*, a^{\textcircled{\#}}\right\} \\ aa^+, x_0 \in \left\{a^+, a^*, \left(a^\#\right)^*, a^{\textcircled{\#}}, \left(a^\#\right)^+, (a^+)^\#\right\} \end{cases}.$

Hence, we have

if $x_0 \in \left\{a, a^{\#}, (a^{+})^{*}, a_{\textcircled{\#}}\right\}$, then $a^{+}aa^{\#} = a^{+}aa^{\#}aa^{+} = a^{+}$;

if $x_0 \in \left\{a^{+}, a^{*}, \left(a^{\#}\right)^{*}, a^{\textcircled{\#}}, \left(a^{\#}\right)^{+}, (a^{+})^{\#}\right\}$, then $aa^{+}a^{\#} = aa^{+}a^{\#}aa^{+}$, that is, $a^{\#} = a^{\#}aa^{+}$.

In any case, we have $a \in R^{EP}$. Hence, $\zeta_a = \{a, a^{\#}, a^{*}, (a^{+})^{*}\}$

(1) If $x_0 = a$, we have $a^{*}a^{3}a^{+} = a^{2}a^{\#}$, e.g., $a^{*}a^{2} = a$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(2) If $x_0 = a^{\#}$, then we have $a^{*}a^{2}a^{\#}a^{+} = aa^{\#}a^{\#}$, e.g., $a^{*} = a^{\#}$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(3) If $x = a^{*}$, then $a^{*}a^{2}a^{*}a^{+} = aa^{*}a^{\#}$. Multiplying the equality by $a(a^{+})^{*}$ on the right and remind in heart that $a \in R^{EP}$. We have $a^{*}a^{2} = a$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(4) If $x = (a^{+})^{*}$, then $a^{*}a^{2}(a^{+})^{*}a^{+} = a(a^{+})^{*}a^{\#}$. Multiplying the equality on the right by aa^{*}, one has $a^{*}a^{2} = a$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(II) If $y = a^{\#}$, then the equation $a^{*}a^{\#}axa^{+} = a^{\#}xa^{\#}$ has at least one solution x_0 in ζ_a, so one gets

$$a^{*}a^{\#}ax_0a^{+} = a^{\#}x_0a^{\#}. \tag{4.7}$$

Multiplying Eq. (4.7) on the right by aa^{+}, one has

$$a^{\#}x_0a^{\#} = a^{\#}x_0a^{\#}aa^{+},$$

this gives $ax_0a^{\#} = ax_0a^{\#}aa^{+}$. Hence, $a \in R^{EP}$ by (I) and so

$$\zeta_a = \left\{a, a^{\#}, a^{*}, (a^{+})^{*}\right\}.$$

(5) If $x_0 = a$, then $a^{*}a^{\#}a^{2}a^{+} = a^{\#}aa^{\#}$, e.g., $a^{*} = a^{\#}$. Hence, $a \in R^{SEP}$;

(6) If $x_0 = a^{\#}$, then $a^{*}a^{\#}aa^{\#}a^{+} = a^{\#}a^{\#}a^{\#}$, that is, $a^{*}a^{\#}a^{\#} = a^{\#}a^{\#}a^{\#}$ because $a \in R^{EP}$. Hence, $a \in R^{SEP}$;

(7) If $x_0 = a^{*}$, then $a^{*}a^{\#}aa^{*}a^{+} = a^{\#}a^{*}a^{\#}$, that is, $a^{*}a^{*}a^{+} = a^{\#}a^{*}a^{\#}$. Multiplying the equality on the right by $a(a^{+})^{*}$, one gets $a^{*} = a^{\#}$. Hence, $a \in R^{SEP}$;

(8) If $x_0 = (a^{+})^{*}$, then $a^{*}a^{\#}a(a^{+})^{*}a^{+} = a^{\#}(a^{+})^{*}a^{\#}$, e.g., $a^{+} = a^{\#}(a^{+})^{*}a^{\#}$ and $a = aa^{+}a = aa^{\#}(a^{+})^{*}a^{\#}a = (a^{+})^{*}$. Hence, $a \in R^{SEP}$.

(III)If $y = a^{+}$, then the equation $a^{*}a^{+}axa^{+} = a^{+}xa^{\#}$ has at least one solution x_0 in ζ_a, so we have

$$a^{*}a^{+}ax_0a^{+} = a^{+}x_0a^{\#}. \tag{4.8}$$

Multiplying Eq. (4.8) on the right by aa^{+}, one has

$$a^{+}x_0a^{\#} = a^{+}x_0a^{\#}aa^{+}. \tag{4.9}$$

Mutiplying the last equation on the left by $x_0^+(aa^\#)^*a$, one gets

$$x_0^+(aa^\#)^*x_0a^\# = x_0^+(aa^\#)^*x_0a^\#aa^+.$$

Noting that

$$x_0^+(aa^\#)^*x_0 = \begin{cases} a^+a, x_0 \in \left\{a, a^\#, (a^+)^*, a^{\text{\textcircled{\#}}}\right\} \\ aa^+, x_0 \in \left\{a^+, a^*, \left(a^\#\right)^*, a^{\text{\textcircled{\#}}}, \left(a^\#\right)^+, (a^+)^\#\right\} \end{cases}.$$

It follows that $aa^+a^\# = aa^+a^\#aa^+$ or $a^+aa^\# = a^+aa^\#aa^+$, e.g., $a^\# = a^\#aa^+$ or $a^+aa^\# = a^+$.

In any case, we have $a \in R^{EP}$ and so $\zeta_a = \{a, a^\#, a^*, (a^+)^*\}$.

(9) If $x_0 = a$, then $a^*a^+a^2a^+ = a^+aa^\#$, that is, $a^* = a^\#$. Hence, $a \in R^{SEP}$;

(10) If $x_0 = a^\#$, then $a^*a^+aa^\#a^+ = a^+a^\#a^\#$, that is, $a^*a^\#a^\# = a^\#a^\#a^\#$. Hence, $a \in R^{SEP}$;

(11) If $x_0 = a^*$, then $a^*a^+aa^*a^+ = a^+a^*a^\#$, that is, $a^*a^*a^+ = a^+a^*a^\#$, $a^*a^* = a^*a^*a^+a = a^+a^*a^\#a = a^+a^*$. Hence, $a \in R^{PI}$ by [30, Corollary 2.10]. Thus, $a \in R^{SEP}$;

(12) If $x_0{=}(a^+)^*$, then $a^*a^+a(a^+)^*a^+{=}a^+(a^+)^*a^\#$, that is, $a^+{=}a^+(a^+)^*a^\#$ and $a = aa^+a = aa^+(a^+)^*a^\#a = (a^+)^*$. Hence, $a \in R^{SEP}$.

(IV) If $y = a^*$, then the equation $a^*a^*axa^+ = a^*xa^\#$ has at least one solution x_0 in ζ_a. So, we have the following equition.

$$a^*a^*ax_0a^+ = a^*x_0a^\#. \tag{4.10}$$

Clearly, we can obtain $a^*x_0a^\# = a^*x_0a^\#aa^+$, this infers

$$x_0^+(aa^\#)^*x_0a^\# = x_0^+(aa^\#)^*x_0a^\#aa^+.$$

Thus, $a \in R^{EP}$ by (III), so $\zeta_a{=}\{a, a^\#, a^*, (a^+)^*\}$.

Multiplying Eq. (4.10) on the left by $(a^+)^*$, one yeilds

$$a^*ax_0a^+ = a^+ax_0a^\#. \tag{4.11}$$

(13) If $x_0 = a$, then $a^*a^2a^+ = a^+a^2a^\#$, e.g., $a^*a = a^+a$. Hence, $a \in R^{SEP}$;

(14) If $x_0 = a^\#$, then $a^*aa^\#a^+ = a^+aa^\#a^\#$, e.g., $a^*a^+ = a^+a^\#$. Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3];

(15) If $x_0 = a^*$, then

$$a^*aa^*a^+ = a^+aa^*a^\# = a^*a^\#,$$

and

$$a^*aa^* = a^*aa^*a^+a = a^*a^\#a = a^*.$$

Hence, $a \in R^{SEP}$;

(16) If $x = (a^+)^*$, then

$$a^*a(a^+)^*a^+ = a^+a(a^+)^*a^\# = (a^+)^*a^\#$$

and

$$a^*a(a^+)^* = a^*a(a^+)^*a^+a = (a^+)^*a^\#a = (a^+)^*,$$

this gives

$$a^+ = a^+a^*a.$$

Hence, $a \in R^{SEP}$ by [20, Theorem 1.5.3].

(V) $y = (a^+)^*$, then the equation $a^*(a^+)^*axa^+ = (a^+)^*xa^\#$ has at least one solution x_0 in ζ_a. So, we obtain

$$a^+a^2x_0a^+ = (a^+)^*x_0a^\#. \tag{4.12}$$

We can obtain $(a^+)^*x_0a^\# = a^+a(a^+)^*x_0a^\#$, so

$$(a^+)^*x_0a^\#ax_0^+ = a^+a(a^+)^*x_0a^\#ax_0^+.$$

Noting that

$$x_0a^\#ax_0^+ = \begin{cases} aa^+, & x_0 \in \left\{a, a^\#, a^*, (a^+)^*, a_{\textcircled{\#}}\right\} \\ a^+a, & x_0 \in \left\{a^+, a^*, (a^\#)^*, a^{\textcircled{\#}}, (a^\#)^+, (a^+)^\#\right\} \end{cases}.$$

Then, $(a^+)^*aa^+ = a^+a(a^+)^*aa^+$ or $(a^+)^*a^+a = a^+a(a^+)^*a^+a$.

In the first case, we have

$$(a^+)^* = (a^+)^*aa^\# = (a^+)^*aa^+aa^\# = a^+a(a^+)^*aa^+aa^\# = a^+a(a^+)^*.$$

In the second case, we also have $(a^+)^* = a^+a(a^+)^*$.

Hence, $a \in R^{EP}$. Now, we have $\zeta_a = \left\{a, a^\#, a^*, (a^+)^*\right\}$ and Eq. (4.12) changes into $ax_0a^+ = (a^+)^*x_0a^\#$, that is, $ax_0a^\# = (a^+)^*x_0a^\#$. Hence

$$ax_0a^\#ax_0^+ = (a^+)^\#x_0a^\#ax_0^+,$$

this infers

$$a^2a^+ = (a^+)^*aa^+$$

or

$$aa^+a = (a^+)^*a^+a.$$

Noting that $a \in R^{EP}$, then, in any case, we have $a = (a^+)^*$. Hence, $a \in R^{SEP}$.

(VI) If $y = (a^\#)^*$, then the equation $a^*(a^\#)^*axa^+ = (a^\#)^*xa^\#$ has at least one solution x_0 in ζ_a. So, we have

$$(aa^\#)^*ax_0a^+ = (a^\#)^*x_0a^\#. \tag{4.13}$$

Multiplying Eq. (4.13) on the left by a^*a^*, one has

$$a^*a^*ax_0a^+ = a^*x_0a^\#. \tag{4.14}$$

Thus, $a \in R^{SEP}$ by (IV).

(VII) If $y = a^{\textcircled{\#}} = a^\#aa^+$, then we have the following equation

$$a^*a^\#axa^+ = a^\#aa^+xa^\#.$$

Multiplying the equation on the right by ax^+, one obtains

$$\begin{aligned} a^*a^\#axa^+ax^+ &= a^\#aa^+xa^\#ax^+ \\ &= \begin{cases} a^\#aa^+aa^+, & x \in \left\{a, a^\#, a^*, (a^+)^*, a_{\textcircled{\#}}\right\} \\ a^\#aa^+a^+a, & x \in \left\{a^+, a^*, (a^\#)^*, a^{\textcircled{\#}}, (a^\#)^+, (a^+)^\#\right\} \end{cases}. \end{aligned}$$

Multiplying the last equation on the left by a^+a, one gets

$$a^\#aa^+aa^+ = a^+aa^\#aa^+aa^+,$$

or

$$a^\#aa^+a^+a = a^+aa^\#aa^+a^+a.$$

In the first case, we have $a^\#aa^+ = a^+$.

In the second case, we have $a^\#aa^+a^+a = a^+a^+a$, this gives $a^\#aa^+ = a^\#aa^+a^*(a^\#)^* = a^\#aa^+a^+aa^*(a^\#)^* = a^+a^+aa^*(a^\#)^* = a^+$.

Hence, in any case, we have $a^+ = a^\#aa^+$, so $a \in R^{EP}$. Thus, $y = a^{\textcircled{\#}} = a^\#$. This shows $a \in R^{SEP}$ by (II).

(VIII) If $y = a_{\textcircled{\#}} = a^+aa^\#$, then

$$a^*a^+axa^+ = a^+aa^\#xa^\#.$$

Multiplying the equation on the right by aa^+, one has

$$a^+aa^\#xa^\# = a^+aa^\#xa^\#aa^+.$$

Multiplying the last equation on the left by x^+a, one gets

$$x^+aa^\#xa^\# = x^+aa^\#xa^\#aa^+.$$

This infers

$$a^+aa^\# = a^+aa^\#aa^+$$

or

$$aa^+a^\# = aa^+a^\#aa^+.$$

Hence, $a^+ = a^+aa^\#$ or $a^\# = aa^\#a^+$. In any case, we have $a \in R^{EP}$, this infers $y = a_{\text{\textcircled{\#}}} = a^\#$. Thus, $a \in R^{SEP}$ by (II).

(IX) If $y = (a^\#)^+ = a^+a^3a^+$, then we have

$$a^*a^+a^3xa^+ = a^+a^3a^+xa^\#.$$

Multiplying the equation on the right by aa^+, one yields

$$a^+a^3a^+xa^\# = a^+a^3a^+xa^\#aa^+.$$

Multiplying the last equation on the left by $a^+a^\#$, one obtains

$$a^+xa^\# = a^+xa^\#aa^+. \tag{4.15}$$

By the proof of (III), we have $a \in R^{EP}$. It follows that $y = (a^\#)^+ = (a^+)^+ = a$. Hence, $a \in R^{SEP}$ by (I).

(X) If $y = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then

$$a^*a(aa^\#)^*axa^+ = (aa^\#)^*a(aa^\#)^*xa^\#.$$

Multiplying the equation on the right by aa^+, one obtains

$$(aa^\#)^*a(aa^\#)^*xa^\# = (aa^\#)^*a(aa^\#)^*xa^\#aa^+.$$

Multiplying the last equation on the left by a^+a^+, one yields

$$a^+xa^\# = a^+xa^\#aa^+. \tag{4.16}$$

Hence, $a \in R^{EP}$ by (III). It follows that $y = (a^+)^\# = (a^\#)^\# = a$. Then, by (I), we have $a \in R^{SEP}$. □

5. Conclusions

In this paper, we mainly portray SEP elements with the help of representations of nuclear inverse, group inverse and Moore Penrose inverse of the product of several generalised inverse elements of an element, and also construct the corresponding generalised inverse equations through the SEP properties of the generalised inverse elements under discussion, and then in turn discuss the solvability of these equations in a given set in order to explore the SEP properties of the element.

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