Some criteria for solvability in finite groups

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Abstract. Let G be a finite group and $m(G) = \sum_{g\in G} \frac{1}{o(g)}$, where $o(g)$ is the order of the element $g \in G$. In this paper, we show that if G is a finite nonsolvable group with $m(G) = m(A_5)$, then $G \cong A_5$. Furthermore, we show that if $m(G) < m(\mathrm{SL}(2, 5))$ and $m(G) \neq m(A_5)$, then G is solvable.
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1. Introduction

Throughout let G be a finite group. There is a wide body of recent literature exploring the link between structural properties of G and arithmetic functions constructed on the orders of the elements of G. We denote by $o(g)$ the order of an element $g \in G$, and set $\psi(G) = \sum_{g\in G} o(g)$, that is, $\psi(G)$ denote the sum of the orders of the elements of G. H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs in [2] proved if $|G| = n$, then $\psi(G) < \psi(C_n)$, where C_n be the cyclic group of order n. Hence, the maximum value of ψ on the set of groups of order n will occur at the cyclic group C_n. In [1], the minimum value of ψ on the set of groups of the same order n is investigated. Specifically, it is shown that for nilpotent groups, the minimum value will be obtained for groups where each Sylow subgroup has prime exponent. It is also proved that if there exist nonnilpotent groups of order n, then the minimum value will be obtained for a nonnilpotent group.

Since then, the investigation of the influence of the value of $\psi(G)$ on the structure of the finite group G became one of the main topics in finite group theory. M. Herzog, P. Longobardi and M. Maj [10] proved that if $|G| = n$ and $\psi(G) \geq \psi(C_n)/6.68$, then G is a solvable group, and posed the following conjecture: If $\psi(G) > 211\psi(C_n)/1617$, then G is a solvable group. In [3], M. B. Azad and B. Khosravi prove the validity of this conjecture. Recently, they proved in [4] that If $|G| = n$ and $\psi(G) > 31\psi(C_n)/77$, then G is a supersolvable group. Moreover, they prove that if G is a group of odd order n and $\psi(G) > 271\psi(C_n)/3674$, then G is a supersolvable group.

Let $o(G) = \psi(G)/|G|$, that is, $o(G)$ is the average order of the elements of G. E. I. Khukhro, A. Moretó and M. Zarrin in [14] showed that there is no polynomial lower bound for $o(G)$ in terms of $o(N)$, where $N \trianglelefteq G$, even when G is a prime-power order group and N is abelian. This gives a negative answer to a question of A. Jaikin-Zapira in [13]. M. Herzog, P. Longobardi and M. Maj [11] proved that if $o(G) < o(A_5)$, then G is solvable. This confirm a conjecture posed by E. I. Khukhro, A. Moretó and M. Zarrin in [14]. M. Tărnăuceanu proved in [16] a sufficient condition for the supersolvability of finite groups in terms of $o(G)$. Specifically, if $o(G) < 211/60 = o(A_4)$, then G is supersolvable. The author in fact classifies all finite groups G such that $o(G) < 211/60$.

M. Garonzi and M. Patassini in [7] introduced the following fuction:

$$R_G(r, s) = \sum_{g \in G} \frac{o\,(g)^s}{\varphi\,(o\,(g))^r},$$

where r, s are two real numbers and φ denote Euler's totient function. The authors of [7] proved several results detecting cyclicity or nilpotency of G in terms of inequalities involving the fuction $R_G(r, s)$.

Note that, $R_G(0, 1) = \sum_{g \in G} o(g) = \psi(G)$ is the sum of the order of elements in G. On the other hand, $R_G(0, -1) = \sum_{g \in G} \frac{1}{o(g)}$ is the sum of the inverses of the element orders in finite groups.

Set $m(G) = \sum_{g \in G} \frac{1}{o(g)}$. Observe that $m(A_5) = 599/30$, and $m(A_4) = 31/6$, where A_n is the alternating group of degree n. B. Azad, B. Khosravi and H. Rashidi in [5] proved that if $m(G) < m(A_5)$, then G is solvable and if $m(G) < m(A_4)$, then G is supersolvable. Furthermore, many interesting results have been given, for example [9, 15].

Following these footsteps, in this paper, we continue to study the relations between $m(G)$ and the structure of G.

The notation is standard and follows that of Isaacs [12]. Let $\pi(G)$ denote the set of all primes of $|G|$, $SL(2,5)$ denote the special linear group of dimension 2 over the field with 5 elements, and C_n denote a cyclic group of order n.

Since A_5 is not solvable and A_4 is not supersolvable, the bound in the above results [5] are the best possible. Using the idea in [5], we first prove the following two results:

Theorem 1.1. *Let G be a nonsolvable group with $m(G) = m(A_5)$. Then, $G \cong A_5$.*

Theorem 1.2. *Let G be a nonsupersolvable group with $m(G) = m(A_4)$. Then, $G \cong A_4$.*

The following theorems give new criteria for surpersolvability and solvability of a finite group by the sum of the inverses of the element orders.

Theorem 1.3. *Let G be a finite group. If $m(G) \leq |\pi(G)|$, then G is supersolvable.*

Theorem 1.4. *Let G be a finite group. If $m(A_5) \neq m(G) < m(\mathrm{SL}(2,5))$, then G is solvable.*

2. Some known lemmas

In this section, we collect some known results which will be used in the proof of main theorems.

Lemma 2.1 ([5, Lemma 2.1])**.** *Let G be a finite group with $m(G) \leq t$ for some natural number t. Then, there exists a cyclic subgroup $\langle x \rangle$ such that $|G : \langle x \rangle| \leq t$.*

Lemma 2.2 ([5, Lemma 1.3])**.** *If G is a finite noncyclic group of order n, then $m(C_n) < m(G)$.*

Lemma 2.3 ([12, Theorem 2.20])**.** *Let A be a cyclic proper subgroup of a finite group G, and let $K = Core_G(A)$. Then, $|A : K| < |G : A|$, and in particular, if $|A| > |G : A|$, then $K > 1$.*

Lemma 2.4 ([5, Lemma 2.2])**.** *Let G be a finite group and $H \leq G$. Then, $m(H) \leq m(G)$, with equality if and only if $H = G$.*

Lemma 2.5 ([5, Lemma 2.3])**.** *Let G be a finite group and $N \trianglelefteq G$. Then, $m(G/N) \leq m(G)$, with equality if and only if $N = 1$.*

Lemma 2.6 ([5, Lemma 2.6])**.** *Let G and H be finite groups. Then, $m(G \times H) \geq m(G)m(H)$, with equality if and only if $gcd(|G|, |H|) = 1$.*

Lemma 2.7 ([12, Lemma 9.1])**.** *Let G be a finite group, and suppose $G/Z(G)$ is simple. Then, $G/Z(G)$ is nonabelian, and G' is perfect. Also, $G'/Z(G')$ is isomorphic to $G/Z(G)$.*

Lemma 2.8 ([5, Lemma 2.4])**.** *Let G be a finite group, $P \in Syl_p(G)$ be a normal and cyclic subgroup of G. Then, $m(Px) \geq m(P)/o(Px)$, where $Px \in G/P$, and the equality holds if and only if x centralizes P. Also, $m(G) \geq m(P)m(G/P)$ with equality if and only if P is central in G.*

Lemma 2.9 ([9, Proposition 2.5])**.** *Let G be a finite group and suppose that there exists $x \in G$ such that $|G : \langle x\rangle| < 2p$, where p is the maximal prime divisor of $|G|$. Then, one of the following holds:*

(1) *G has a normal cyclic Sylow p-subgroup,*

(2) *G is solvable and $\langle x\rangle$ is a maximal subgroup of G of index either p or $p+1$.*

3. Proofs of the main theorems

Proof of Theorem 1.1 Assume that G contains a non-trivial normal subgroup N. Then, by Lemmas 2.4 and 2.5, we have that $m(N) < m(G) = m(A_5)$, and $m(G/N) < m(G) = m(A_5)$. It follows from [5, Theorem 2.7] that N and G/N are solvable, and thus G is solvable, which is a contradiction. Thus, G is a simple group.

Assume that $|G| > 361$. By Lemma 2.1, There exists $x \in G$ such that $|G : \langle x\rangle| \leq 599/30$. Now using Lemma 2.3, we get that $|\langle x\rangle : Core_G(\langle x\rangle)| < |G : \langle x\rangle| \leq 19$. Thus, $|\langle x\rangle| > \frac{361}{|G:\langle x\rangle|} \geq |G : \langle x\rangle|$. Again by Lemma 2.3, $Core_G(\langle x\rangle)$ is a non-trivial normal subgroup of G, which is a contradiction. Therefore, $|G| \leq 361$.

By [6], the simple nonabelian groups of order less than 362 are A_5, A_6 and PSL(2,7). If $G \cong A_6$, then the number of Sylow 5-subgroup of G is $n_5(G) = 36$. Therefore, G has 144 elements of order 5. By the definition of $m(G)$, we get that $m(G) > m(A_5)$ which is a contradiction. Similarly, if $G \cong \mathrm{PSL}(2, 7)$, then we get a contradiction. Thus, $G \cong A_5$, as desired. □

Proof of Theorem 1.2. Assume that G contains a nontrivial normal cyclic subgroup N. Then, by Lemma 2.5, we have that $m(G/N) < m(G) = m(A_4)$. It follows from [5, Theorem 2.8] that G/N is supersolvable. Therefore, G is supersolvable, which is a contradiction. Thus, G has no a non-trivial normal cyclic subgroup.

Assume that $|G| > 25$. By Lemma 2.1, there exists $x \in G$ such that $|G : \langle x\rangle| \leq 31/6$. By Lemma 2.3, we get that $|\langle x\rangle : Core_G(\langle x\rangle)| < |G : \langle x\rangle| \leq 5$. Thus, $|\langle x\rangle| > \frac{25}{|G:\langle x\rangle|} \geq |G : \langle x\rangle|$.

Again by Lemma 2.3, $Core_G(\langle x\rangle)$ is a nontrivial normal cyclic subgroup of G, which is a contradiction. Thus, $|G| \leq 25$.

Since G is a non-supersolvable group of order less than 26, we have that $G \cong A_4$, SL(2,3), S_4 or $C_2 \times A_4$. If $G \cong \mathrm{SL}(2, 3)$, S_4 or $C_2 \times A_4$, then the number of Sylow 3-subgroup of G is 4, and so G has 8 elements order of 3 and at least one element order of 2. Thus, $m(G) > m(A_4)$, which is a contradiction. Therefore, we obtain that $G \cong A_4$, as desired. □

Proof of Theorem 1.3. We prove by induction on $|\pi(G)|$.

Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r = p$ are primes, and $\alpha_i > 0$, for each $1 \leq i \leq r$.

If $|\pi(G)| = 1$, then G is a p-group, therefore G is supersolvable. Assume that $|\pi(G)| \geq 2$. We consider the following two cases:

Assume that G has no normal cyclic Sylow subgroups. By Lemma 2.1, there exists $x \in G$ such that $|G : \langle x \rangle| \leq |\pi(G)| < p$ and by Lemma 2.9, we get a contradiction.

Thus, G has some normal cyclic Sylow p-subgroup, say Q. If Q is a prime order cyclic group and $|\pi(G)| = 2$, then G is supersolvable. If not, then by Lemma 2.8, we have that

$$m(G/Q)m(Q) \leq m(G) \leq |\pi(G)|.$$

And thus

$$m(G/Q) \leq m(G)/m(Q) \leq m(G)/2 \leq |\pi(G)|/2 \leq |\pi(G)| - 1 = |\pi(G/Q)|.$$

By the inductive hypothesis, G/Q is a supersolvable group. Therefore, G is supersolvable. □

Proof of Theorem 1.4. Assume that G is nonsolvable. By Lemma 2.1, there exists $x \in G$ such that $|G : \langle x \rangle| \leq 131/5 < 27$. Let $H = Core_G(\langle x \rangle)$, then H is a normal cyclic subgroup of G. By Lemma 2.3, we get that $|\langle x \rangle : H| < |G : \langle x \rangle| \leq 26$. Hence,

$$|G : H| = |G : \langle x \rangle||\langle x \rangle : H| \leq 25 \times 26 = 650.$$

Since G is nonsolvable, it follows that G/H is nonsolvable. Let $\overline{G} = G/H$. Then, we have that $|\overline{G}| = |G/H| \leq 650$. By Lemma 2.5,

$$m(\overline{G}) \leq m(G) < m(\mathrm{SL}(2,5)) = \frac{131}{5}.$$

By GAP [8], we find that the only possibility for $\overline{G}$ of order up to 650 is $\overline{G} \cong A_5$. Therefore, H is a maximal normal subgroup of G and $H \leq C_G(H)$. On the other hand, $C_G(H) \trianglelefteq N_G(H) = G$. Thus, $C_G(H) = H$ or $C_G(H) = G$.

In the case $C_G(H) = H$, then $N_G(H)/C_G(H) = G/H \cong A_5 \lesssim Aut(H)$, which is a contradiction, since H is cyclic.

In the case $C_G(H) = G$, then $H \leq Z(G)$. Therefore, $H = Z(G)$ and $G/Z(G) = G/H \cong A_5$. Thus, $G' \nleq Z(G)$ and $G'Z(G) \trianglelefteq G$, by the definition of maximal normal subgroup, we have $G = G'Z(G)$ and

$$G'/(G' \cap Z(G)) \cong G'Z(G)/Z(G) = G/Z(G) \cong A_5.$$

By Lemma 2.8, G' is perfect and $G'/Z(G') \cong G/Z(G) \cong A_5$. Hence G' is a central extension of $Z(G')$ by A_5. Since the Schur multiplier of A_5 has order 2 and G' is perfect, it follows that $G' \cong \mathrm{SL}(2,5)$ or $G' \cong A_5$.

If $G' \cong \mathrm{SL}(2,5)$, then $|G' \cap Z(G)| = 2$ and

$$G/(G' \cap Z(G)) = G'Z(G)/(G' \cap Z(G)) \cong G'/(G' \cap Z(G)) \\ \times Z(G)/(G' \cap Z(G)) \cong A_5 \times C_n,$$

where $n = |G|/120$. By Lemmas 2.2 and 2.6,

$$\frac{599}{30} m(C_n) = m(A_5)m(C_n) \leq m(A_5 \times C_n) = m(G/G' \cap Z(G)) \leq m(G) < \frac{131}{5}.$$

Therefore, $m(C_n) < 786/599$. This implies that $n = 1$ and $G \cong \mathrm{SL}(2,5)$, a contradiction.

If $G' \cong A_5$, then $|G' \cap Z(G)| = 1$ and $G = G' \times Z(G) \cong A_5 \times Z(G)$. Set $|Z(G)| = n$. By Lemmas 2.2 and 2.6,

$$\frac{599}{30} m(C_n) = m(A_5)m(Z(G)) \leq m(A_5 \times Z(G)) = m(G) < \frac{131}{5}.$$

By the above discussion, we have that $n = 1$. Thus $Z(G) = 1$ and $G \cong A_5$, a contradiction. □

4. Conclusions

All the previous conclusions indicate that the sum of the inverses of the order of elements can effectively characterize the solvability and supersolvability of finite groups, and provide us with ideas and directions for future research on similar problems, such as the extension of known important conclusions and the necessary and sufficient conditions for A_4 and A_5. These will surely be the subject of some further research.

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Conflict of interest

The authors declare that they have no conflict of interest.

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