Intuitionistic fuzzy threshold hypergraphs

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Abstract. A hypergraph is a graph that allows any number of vertices to be connected by an edge. A threshold hypergraph is a hypergraph $\mathbb{H}$ for which there exists a function $w : U(\mathbb{H}) \to \mathbb{N} \cup \{0\}$ and a non-negative integer s such that for all $V \subseteq U(H)$, V is independent if and only if $\sum_{v \in V} w(v) \leq s$. We introduce Intuitionistic Fuzzy Threshold Hypergraph(IFTHG) along with some of the multiple forms of them such as core, simple, elementary, sectionally elementary IFTHGs and (μ, ν)- tempered IFTHGs, with few of its properties. It is further proven that if an IFTHG $\mathbb{H}_{\mathbb{G}}$ is elementary, support simple and simply ordered, then $\mathbb{H}_{\mathbb{G}}$ is a (μ, ν)-tempered IFTHG. Further, we discuss how this method can be used to recognize water wastage using IFTHG and identify regions where it can be reduced, demonstrating that it is more effective for controlling water management systems compared to other methodologies.

Keywords: Intuitionistic fuzzy threshold hypergraph(IFTHG), Core, Simple, Support Simple, Elementary and Sectionally elementary IFTHGs, (μ, ν)-tempered IFTHG.

MSC 2020: 05C65, 05C72.

1. Introduction

This research introduces the Inituitionistic Fuzzy Threshold Graphs(IFTHGs) and some types to address the limitations of traditional water management systems, which struggle to model complex real-world water distribution networks. The objective is to enhance the accuracy of detecting water leakage and wastage, leading to more efficient water management. The gap in existing research is the inadequacy of conventional graph-based methods to effectively manage intricate water systems. The novelty of this study is the application of IFTHG, which provides a more precise and effective solution for optimizing water usage.

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In set theory, Zadeh[21] created fuzzy sets as a technique of conveying ambiguity and vagueness. Fuzzy set theory has sparked attention to variety of fields. Atanassov[4, 7] came up with the concept of Intuitionistic Fuzzy Sets(IFS) as a generalisation of fuzzy sets, and Atanassov added an additional module to the concept of fuzzy set (which specifies the degree of non-membership). The concept behind intuitionistic fuzzy graphs were discussed in [5, 6].

Euler introduced the notion of graph theory in 1736. The graph theory is a powerful tool for addressing combinatorial issues in a various fields. To deal with the complexities of application base, the concept of graph was expanded to provide a hypergraph, which is a set of V vertices including a collection of V subsets. The notions of graph and hypergraph were introduced by Berge [8]. Hypergraphs, an extension of classical graphs, are used to model complex relationships among multiple elements by Lee-Kwang & Lee [11]. Muhammad Akram have explored the integration of fuzzy logic into hypergraph structures, known as fuzzy hypergraphs (Akram & Dubek, 2013)[1]. Akram and Luqman (2020)[12] conducted a comprehensive study on fuzzy hypergraphs, making significant contributions to the field. The study of hypergraphs, particularly in fuzzy and soft computing, plays a crucial role in advancing computational intelligence and decision-making methods (Akram et al.,)[2]. These structures are extensively studied in decision-making and computational applications due to their versatility by Akram et al.(2022)[3]. Sarwar et al. (2023)[20] introduced a new group decision-making approach using rough soft approximations of hypergraphs, demonstrating their practical applications. Akram and Nawaz (2023)[13] applied single-valued neutrosophic soft hypergraphs to model the human nervous system, showing the interdisciplinary nature of hypergraph theory. Nawaz, Akram, and Alcantud (2022)[16] developed an algorithm to compute the strength of competing interactions in the Bering Sea using Pythagorean fuzzy hypergraphs.

Intuitionistic fuzzy graphs and Intuitionistic fuzzy hypergraphs and intuitionistic fuzzy directed hypergraphs have been introduced in [17, 18, 19]. Some types of intuitionistic fuzzy directed hypergraphs are discussed in [14]. Lanzhen Yang and Hua Mao [10] introduced intuitionistic fuzzy threshold graph and explained its applications. Chvatal and Hammer were the first to introduce threshold graphs. Historical work by Chvatal and Hammer (1973)[9] on set-packing problems and threshold graphs underscores the longstanding interest in hypergraph theory. Myithili and Nandhini (2024)[15] reviewed intuitionistic fuzzy threshold hypergraphs, contributing to the theoretical expansion of fuzzy hypergraphs.

The new features of the proposed method in the water management system include its ability to accurately model complex structures using IFTHG, which surpasses traditional graph-based methods that often fall short in capturing real-life complexities. The main advantages of the results are the precise detection of water leakage or wastage in specific regions, facilitated by threshold values that maintain optimal water levels, leading to more efficient and targeted water management. This approach helps prioritize maintenance resources, such as manpower and materials, ensuring they are used where they are most needed, reducing overall costs.

2. Preliminaries

We stepped over some fundamental definitions in this section that are related to our main concept.

Definition 2.1 ([7])**.** Let a set E be fixed. An *intuitionistic fuzzy set (IFS)* V in E is an object of the form $V = \{\langle v_i, \mu_i(v_i), \nu_i(v_i)\rangle / v_i \in E\}$, where the function $\mu_i : E \to [0,1]$ and $\nu_i : E \to [0,1]$ determine the degree of membership and the degree of non-membership of the element $v_i \in E$, respectively and for every $v_i \in E$, $0 \leq \mu_i(v_i) + \nu_i(v_i) \leq 1$.

Definition 2.2 ([5])**.** The six Cartesian products of two IFSs V_1, V_2 of V over E are defined as
$V_1 \times_1 V_2 = \{\langle (v_1, v_2), \mu_1.\mu_2, \nu_1.\nu_2\rangle \,| v_1 \in V_1, v_2 \in V_2\}$,
$V_1 \times_2 V_2 = \{\langle (v_1, v_2), \mu_1 + \mu_2 - \mu_1.\mu_2, \nu_1.\nu_2\rangle \,| v_1 \in V_1, v_2 \in V_2\}$,
$V_1 \times_3 V_2 = \{\langle (v_1, v_2), \mu_1.\mu_2, \nu_1 + \nu_2 - \nu_1.\nu_2\rangle \,| v_1 \in V_1, v_2 \in V_2\}$,
$V_1 \times_4 V_2 = \{\langle (v_1, v_2), \min(\mu_1, \mu_2), \max(\nu_1, \nu_2)\rangle \,| v_1 \in V_1, v_2 \in V_2\}$,
$V_1 \times_5 V_2 = \{\langle (v_1, v_2), \max(\mu_1, \mu_2), \min(\nu_1, \nu_2)\rangle \,| v_1 \in V_1, v_2 \in V_2\}$.
$V_1 \times_6 V_2 = \{\left\langle (v_1, v_2), \frac{\mu_i(v_1)+\mu_i(v_2)}{2}, \frac{\nu_i(v_1)+\nu_i(v_2)}{2}, \right\rangle |v_1 \in V_1, v_2 \in V_2\}$.
It must be noted that $v_i \times_s v_j$ is an IFS, where $s = 1, 2, 3, 4, 5, 6$.

Definition 2.3 ([17])**.** An *intuitionistic fuzzy graph (IFG)* is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \ldots, v_n\}$ such that $\mu_i : V \to [0,1]$ and $\nu_i : V \to [0,1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \mu_i(v_i) + \nu_i(v_i) \leq 1$ for every $v_i \in V$, $i = 1, 2, \ldots, n$.

(ii) $E \subseteq V \times V$ and $\mu_{ij} : V \times V \to [0,1]$ and $\nu_{ij} : V \times V \to [0,1]$ are such that $\mu_{ij} \leq \mu_i \wedge \mu_j$, $\nu_{ij} \leq \nu_i \vee \nu_j$ and $0 \leq \mu_i(v_i) + \nu_i(v_i) \leq 1$, where μ_{ij} and ν_{ij} are the membership and non-membership values of the edge (ν_i, ν_j). The values of $\mu_i \wedge \mu_j$ and $\nu_i \vee \nu_j$ can be determined by one of the cartesian products $\times_s$, $s = 1, 2, \ldots, 6$ for all i and j given in Definition 2.2.

Note: Throughout this paper, it is assumed that the fourth Cartesian product

$$V_{i_1} \times V_{i_2} \times V_{i_3} \times \ldots \times V_{i_n} = \{\langle (v_1, v_2, \ldots, v_n), \min(\mu_1, \mu_2, \ldots, \mu_n), \max(\nu_1, \nu_2, \ldots, \nu_n)\rangle | v_1 \in V_1, v_2 \in V_2, \ldots, v_n \in V_n\},$$

is used to determine the edge membership μ_{ij} and the edge non-membership ν_{ij}.

Definition 2.4 ([19])**.** An intuitionistic fuzzy hypergraph(IFHG) is an ordered pair $H = (V, E)$ where

(i) $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of intuitionistic fuzzy vertices;

(ii) $E = \{E_1, E_2, \ldots, E_m\}$ is a family of crisp subsets of V;

(iii) $E_j = \{(v_i, \mu_j(v_i), \nu_j(v_i)) : \mu_j(v_i), \nu_j(v_i) \geq 0$ and $\mu_j(v_i) + \nu_j(v_i) \leq 1\}$, $j = 1, 2, \ldots, m$;

(iv) $E_j \neq \emptyset$, $j = 1, 2, \ldots, m$;

(v) $\bigcup_j \operatorname{supp}(E_j) = V$, $j = 1, 2, \ldots, m$.

Here, the hyperedges E_j are crisp sets of intuitionistic fuzzy vertices. Moreover, $\mu_j(v_i)$ and $\nu_j(v_i)$ denote the degrees of membership and non-membership of vertex v_i to edge E_j. Thus, the elements of the incidence matrix of IFHG are of the form $(v_{ij}, \mu_j(v_i), \nu_j(v_i))$. The sets (V, E) are crisp sets.

Notation list[14]

- $\langle \mu(u_i), \nu(u_i) \rangle$ or simply $\langle \mu_i, \nu_i \rangle$ denote the degrees of membership and non-membership of the vertex $v_i \in V$, such that $0 \leq \mu_i + \nu_i \leq 1$.
- $\langle \mu(u_{ij}), \nu(u_{ij}) \rangle$ or simply $\langle \mu_{ij}, \nu_{ij} \rangle$ denote the degrees of membership and non-membership of the hyperedge $(u_i, u_j) \in V \times V$, such that $0 \leq \mu_{ij} + \nu_{ij} \leq 1$.
- μ_{ij} and ν_{ij} are the membership and non-membership value of i^{th} vertex in j^{th} hyperedge.
- The support of an IFS V in $\mathcal{E}$ is denoted by $supp(\mathcal{E}_j) = \{u_i | \mu_{ij}(u_i) > 0$ and $\nu_{ij}(u_i) > 0\}$, where $\mathcal{E}_j$ is a hyperedge in intuitionistic fuzzy threshold hypergraph.

3. Intuitionistic fuzzy threshold hypergraph

Definition 3.1 ([15])**.** The *Intuitionistic Fuzzy Threshold Hypergraph*(IFTHG) is defined as $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ where

(i) $U = \{u_1, u_2, ..., u_n\}$ is a finite set of intuitionistic fuzzy vertices;

(ii) $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_m\}$ is a family of crisp subsets of U;

(iii) $\mathcal{E}_j = \{u_i, \mu_j(u_i), \nu_j(u_i)/0 \leq \mu_j(u_i) + \nu_j(u_i) \leq 1\}, j = 1, 2, ...m$;

(iv) $\mathcal{E}_j \neq \emptyset$, $j = 1, 2, \ldots, m$;

(v) $\bigcup_j \operatorname{supp}(\mathcal{E}_j) = U$, $j = 1, 2, \ldots, m$;

(vi) an independent set $V \subseteq U$ has a set of all distinct combinations of a non-adjacent vertices in $\mathbb{H}_{\mathbb{G}}$ if and only if there exists a threshold values $s_1, s_2 > 0$ such that $\sum_{u_i \in V} \mu_j(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_j(u_i)) \leq s_2$.

Example. Consider an IFTHG $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5, \mathcal{E}_6\}$, where $\mathcal{E}_1 = \{u_1, u_6\}$, $\mathcal{E}_2 = \{u_2, u_6\}$, $\mathcal{E}_3 = \{u_2, u_3\}$, $\mathcal{E}_4 = \{u_3, u_6\}$, $\mathcal{E}_5 = \{u_3, u_4\}$ and $\mathcal{E}_6 = \{u_4, u_5, u_6\}$ with threshold value $\langle 0.4, 0.5\rangle$ is represented in the following incidence matrix.

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \quad\mathcal{E}_1\quad & \quad\mathcal{E}_2\quad & \quad\mathcal{E}_3\quad & \quad\mathcal{E}_4\quad & \quad\mathcal{E}_5\quad & \quad\mathcal{E}_6\quad \end{array} \\ \begin{pmatrix} \langle 0.3, 0.6\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.2, 0.6\rangle & \langle 0.2, 0.6\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.5\rangle & \langle 0.3, 0.5\rangle & \langle 0.3, 0.5\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.7\rangle & \langle 0.3, 0.7\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.5\rangle \\ \langle 0.4, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 0.4, 0.5\rangle \end{pmatrix} \end{array}$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.4, 0.5)$ has been given below.

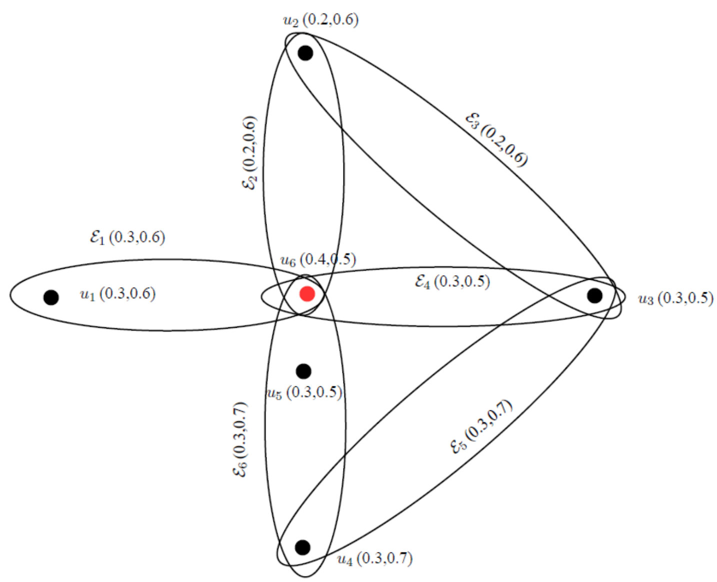

Figure 1: IFTHG

Proposition 3.1. *An IFTHG is an expansion of intuitionistic fuzzy threshold graph(IFTG).*

Proof. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an intuitionistic fuzzy threshold hypergraph. Then by its definition, an independent set $V \subseteq U$ has a set of all distinct combinations of non-adjacent vertices in $\mathbb{H}_{\mathbb{G}}$ if and only if there exists threshold values $s_1, s_2 > 0$ such that $\sum_{u_i \in V} \mu_j(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_j(u_i)) \leq s_2$.

We know that, in an IFTG, an edge has only two vertices. As a result, we consider a hyperedge instead of an edge, which contains two or more vertices which is an intuitionistic fuzzy threshold hypergraph. Therefore, an IFTHG is an expansion of IFTG. $\square$

Definition 3.2. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG. Then the *height of* $\mathbb{H}_{\mathbb{G}}$ is defined by $h(\mathbb{H}_{\mathbb{G}}) = \{\max(\min(\mu_{ij})), \max(\max(\nu_{ij}))\}$, for which $\sum_{u_i \in V} \mu_j(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_j(u_i)) \leq s_2$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, where μ_{ij} and ν_{ij} is the membership and non-membership value of i^{th} vertex in j^{th} hyperedge respectively.

Definition 3.3. An IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ is *simple* if there exists an independent set $V \subseteq U$ such that $\mathcal{E}$ has no repeated intuitionistic fuzzy hyperedges and whenever $\mathcal{E}_j, \mathcal{E}_k \in \mathcal{E}$ and $\mathcal{E}_j \subseteq \mathcal{E}_k$ then $\mathcal{E}_j = \mathcal{E}_k$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_{ij})(u_i) \leq s_2$ for all j and k.

Example. Consider an IFTHG $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ according to the incidence matrix below with threshold value $\langle 0.3, 0.6 \rangle$.

$$
\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}
\begin{array}{c}
\begin{array}{ccc} \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 \end{array} \\
\begin{pmatrix}
\langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\
\langle 0, 1 \rangle & \langle 0.2, 0.6 \rangle & \langle 0, 1 \rangle \\
\langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.5 \rangle \\
\langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.1, 0.7 \rangle \\
\langle 0.3, 0.6 \rangle & \langle 0.3, 0.6 \rangle & \langle 0.3, 0.6 \rangle
\end{pmatrix}
\end{array}
$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.3, 0.6)$ has been shown below.

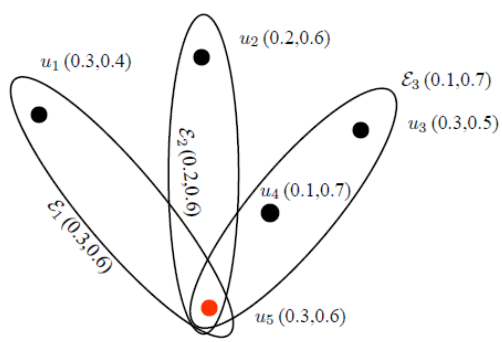

Figure 2: Simple IFTHG

Definition 3.4. An IFTHG, $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ is *support simple*, if there is an independent set $V \subseteq U$ such that $\mathcal{E}_j, \mathcal{E}_k \in \mathcal{E}$, $\mathcal{E}_j \subseteq \mathcal{E}_k$ and $\text{supp}(\mathcal{E}_j) = \text{supp}(\mathcal{E}_k)$ then $\mathcal{E}_j = \mathcal{E}_k$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_{ij})(u_i) \leq s_2$, for all j and k. Here, the hyperedges $\mathcal{E}_j$ and $\mathcal{E}_k$ are called supporting hyperedges.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ according to the incidence matrix below with threshold value $\langle 0.3, 0.7\rangle$.

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 \end{array} \\ \begin{pmatrix} \langle 0.1, 0.7\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.2, 0.5\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.2, 0.6\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.5\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.5\rangle & \langle 0, 1\rangle \\ \langle 0.3, 0.7\rangle & \langle 0.3, 0.7\rangle & \langle 0.3, 0.7\rangle & \langle 0.3, 0.7\rangle \end{pmatrix} \end{array}$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.3, 0.7)$ has been illustrated below.

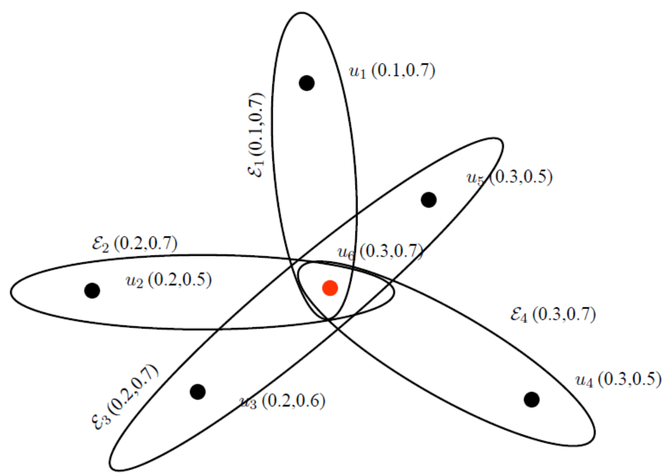

Figure 3: Support simple IFTHG

Here, $\mathcal{E}_2$ and $\mathcal{E}_4$ are supporting hyperedges.

Definition 3.5. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG and an independent set $V \subseteq U$ exists, if $\mathcal{E}_j, \mathcal{E}_k \in \mathcal{E}$ and $0 < \alpha, \beta \leq 1$. Then the (α, β)-*level* is defined by $(\mathcal{E}_j, \mathcal{E}_k)^{(\alpha,\beta)} = \{u_i \in U | \min(\mu^{\alpha}_{ij}(u_i)) \geq \alpha, \max(\nu^{\beta}_{ij}(u_i)) \leq \beta\}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_{ij})(u_i) \leq s_2$.

Definition 3.6. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, there exists an independent set $V \subseteq U$ such that $\mathbb{H}_{\mathbb{G}}{}^{y_i,z_i} = \langle U^{y_i,z_i}, \mathcal{E}^{y_i,z_i}\rangle$ be the (y_i, z_i)-level of $\mathbb{H}_{\mathbb{G}}$. The sequence of real numbers $\{y_1, y_2, \ldots, y_n; z_1, z_2, \ldots, z_n\}$ such that $0 \leq y_i \leq h_\mu(\mathbb{H}_{\mathbb{G}})$ and $0 \leq z_i \leq h_\nu(\mathbb{H}_{\mathbb{G}})$, satisfying the properties:

(i) If $y_1 < \alpha \leq 1$ and $0 \leq \beta < z_1$ then $\mathcal{E}^{\alpha,\beta} = \emptyset$;

(ii) If $y_{i+1} \leq \alpha \leq y_i$; $z_i \leq \beta \leq z_{i+1}$ then $\mathcal{E}^{\alpha,\beta} = \mathcal{E}^{y_i,z_i}$;

(iii) $\mathcal{E}^{y_i,z_i} \sqsubset \mathcal{E}^{y_{i+1},z_{i+1}}$;

for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1-\nu_{ij})(u_i) \leq s_2$ is called the *fundamental sequence* of $\mathbb{H}_{\mathbb{G}}$ and is denoted by $\mathscr{F}(\mathbb{H}_{\mathbb{G}})$.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ according to the incidence matrix below with threshold value $\langle 0.4, 0.6\rangle$.

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array}\begin{array}{c} \begin{array}{ccc} \quad\mathcal{E}_1 \quad & \quad\mathcal{E}_2\quad & \quad\mathcal{E}_3\quad \end{array} \\ \begin{pmatrix} \langle 0.2, 0.7\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.3, 0.6\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.4, 0.5\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.4, 0.4\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle \end{pmatrix} \end{array}$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.4, 0.6)$ has been given below.

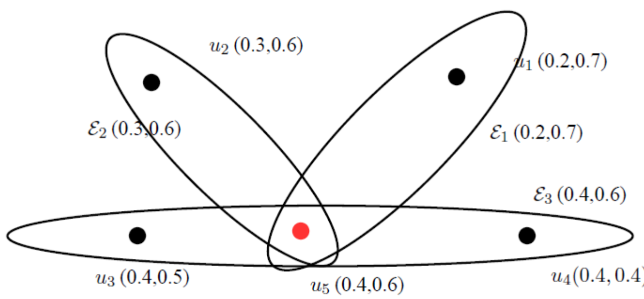

Figure 4: ${\mathbb{H}_{\mathbb{G}}}^{(\mu_j(y,z),\nu_j(y,z))}$-level IFTHG

By examining the (y_n, z_n)-level IFTHG of $\mathbb{H}_{\mathbb{G}}$, we see that ${\mathbb{H}_{\mathbb{G}}}^{\langle 0.2,0.7\rangle} = {\mathbb{H}_{\mathbb{G}}}^{\langle 0.3,0.6\rangle} = {\mathbb{H}_{\mathbb{G}}}^{\langle 0.4,0.6\rangle}$ and $\mathscr{F}(\mathbb{H}_{\mathbb{G}}) = \{\langle 0.2, 0.7\rangle, \langle 0.3, 0.6\rangle, \langle 0.4, 0.6\rangle\}$.

Definition 3.7. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, also an independent set $V \subseteq U$ exists. Then, the *core set* of $\mathbb{H}_{\mathbb{G}}$ is denoted by $\mathscr{C}(\mathbb{H}_{\mathbb{G}})$ for $0 < (y_i, z_i) \leq h(\mathbb{H}_{\mathbb{G}})$, and is defined by $\mathscr{C}(\mathbb{H}_{\mathbb{G}}) = \{{\mathbb{H}_{\mathbb{G}}}^{y_1,z_1}, {\mathbb{H}_{\mathbb{G}}}^{y_2,z_2}, \ldots, {\mathbb{H}_{\mathbb{G}}}^{y_n,z_n}\}$. The corresponding set of (y_i, z_i)-level hypergraphs is ${\mathbb{H}_{\mathbb{G}}}^{y_1,z_1} \subset {\mathbb{H}_{\mathbb{G}}}^{y_2,z_2} \subset \ldots \subset {\mathbb{H}_{\mathbb{G}}}^{y_n,z_n}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1-\nu_{ij})(u_i) \leq s_2$ is called the $\mathbb{H}_{\mathbb{G}}$-induced fundamental sequence and is denoted by $\mathscr{I}(\mathbb{H}_{\mathbb{G}})$. The (y_n, z_n)-level is called the support level of $\mathbb{H}_{\mathbb{G}}$ and the ${\mathbb{H}_{\mathbb{G}}}^{y_n,z_n}$ is called the support of $\mathbb{H}_{\mathbb{G}}$.

Definition 3.8. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG is said to be *elementary*, if an independent set $V \subseteq U$ exists. Then $\mu_{ij} : U \to [0, 1]$ and $\nu_{ij} : U \to [0, 1]$ are constant functions or has a range $\{0, r\}$, $r \neq 0$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1-\nu_{ij})(u_i) \leq s_2$. If $|\operatorname{supp}(\mu_{ij}, \nu_{ij})| = 1$, then it is called a *spike*.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4\}$ according to the incidence matrix below with threshold value $\langle 0.4, 0.5\rangle$.

$$\begin{array}{c c} & \begin{array}{cccc} \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \mathcal{E}_4 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} \langle 0.4, 0.6\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.4, 0.6\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.4, 0.6\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.4, 0.6\rangle & \langle 0, 1\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.4, 0.5\rangle \end{pmatrix} \end{array}$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.4, 0.5)$ has been shown below.

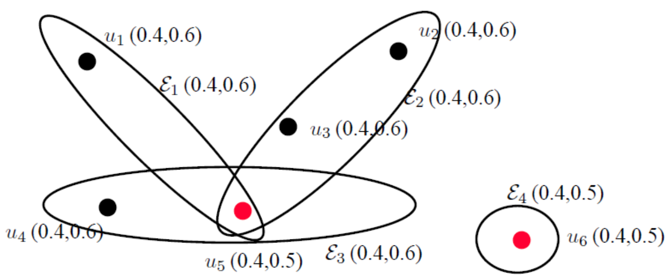

Figure 5: Elementary IFTHG

Theorem 3.1. *The intuitionistic fuzzy threshold hyperedges of an IFTHG are elementary.*

Definition 3.9. Consider $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, if there exists an independent set $V \subseteq U$ such that $\mathbb{H}_{\mathbb{G}}$ be an IFTHG and $\mathscr{F}(\mathbb{H}_{\mathbb{G}}) = \{y_n, y_{n-1}, \ldots, y_1, z_1, z_2, \ldots, z_n\}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_{ij})(u_i) \leq s_2$. Then $\mathbb{H}_{\mathbb{G}}$ is called *sectionally elementary* if for each μ_{ij}, $\nu_{ij} \in \mathcal{E}$ and $y_n, z_n \in \mathscr{F}(\mathbb{H}_{\mathbb{G}})$, $\mu_{ij}^{\alpha} = \mu_{ij}^{y_i}$ and $\nu_{ij}^{\beta} = \nu_{ij}^{z_i}$ for all $\alpha, \beta \in (y_{i+1}, z_i]$.(we assume $y_{i+1} = 0$).

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$, and it is denoted by the upcoming incidence matrix with threshold value $\langle 0.4, 0.6\rangle$.

$$\begin{array}{c c} & \begin{array}{ccc} \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} & \begin{pmatrix} \langle 0.4, 0.5\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.4, 0.5\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.3, 0.5\rangle \\ \langle 0, 1\rangle & \langle 0.3, 0.5\rangle & \langle 0.3, 0.5\rangle \\ \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle & \langle 0, 1\rangle \end{pmatrix} \end{array}$$

The IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.4, 0.6)$ has been shown below.

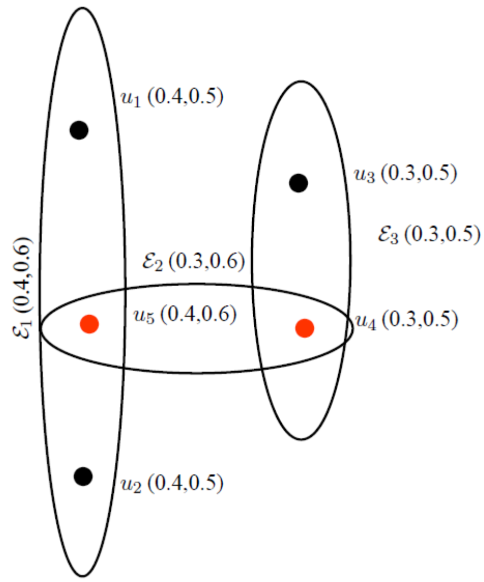

Figure 6: Sectionally Elementary IFTHG

Definition 3.10. Suppose $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ and $\mathbb{H}_{\mathbb{G}}' = (U', \mathcal{E}'; s_1', s_2')$ are IFTHGs, $\mathbb{H}_{\mathbb{G}}$ is called a *partial IFTHG* of $\mathbb{H}_{\mathbb{G}}'$ if there exists an independent set $V \subseteq U$ such that

$$u' = \begin{cases} \min(\operatorname{supp}(\mu_{ij})) \;\&\; \sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1 \,|\, \mu_{ij} \in \mathcal{E}' \\ \max(\operatorname{supp}(\nu_{ij})) \;\&\; \sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2 \,|\, \nu_{ij} \in \mathcal{E}' \end{cases}$$

The partial IFTHG generated by $\mathcal{E}'$ and is denoted by $\mathbb{H}_{\mathbb{G}} \subseteq \mathbb{H}_{\mathbb{G}}'$. Also, $\mathbb{H}_{\mathbb{G}} \subset \mathbb{H}_{\mathbb{G}}'$ if $\mathbb{H}_{\mathbb{G}} \subseteq \mathbb{H}_{\mathbb{G}}'$ and $\mathbb{H}_{\mathbb{G}} \neq \mathbb{H}_{\mathbb{G}}'$.

Definition 3.11. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an IFTHG, there exists an independent set $V \subseteq U$ such that $\mathscr{C}(\mathbb{H}_{\mathbb{G}}) = \{\mathbb{H}_{\mathbb{G}}^{y_1, z_1}, \mathbb{H}_{\mathbb{G}}^{y_2, z_2}, \ldots, \mathbb{H}_{\mathbb{G}}^{y_n, z_n}\}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$. $\mathbb{H}_{\mathbb{G}}$ is said to be *ordered* if $\mathscr{C}(\mathbb{H}_{\mathbb{G}})$ is ordered. That is $\mathbb{H}_{\mathbb{G}}^{y_1, z_1} \subset \ldots \subset \mathbb{H}_{\mathbb{G}}^{y_2, z_2} \subset \mathbb{H}_{\mathbb{G}}^{y_n, z_n}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$. The IFTHG is said to be *simply ordered* if the sequence $\{\mathbb{H}_{\mathbb{G}}^{y_i, z_i} | i = 1, 2, \ldots, n\}$ is simply ordered, that is if it is ordered and if whenever $\mathcal{E} \in \mathbb{H}_{\mathbb{G}}^{y_{i+1}, z_{i+1}} \backslash \mathbb{H}_{\mathbb{G}}^{y_i, z_i}$ then $\mathcal{E} \nsubseteq \mathbb{H}_{\mathbb{G}}^{y_i, z_i}$.

Theorem 3.2. *If $\mathbb{H}_{\mathbb{G}}$ is an elementary intuitionistic fuzzy threshold hypergraph, then $\mathbb{H}_{\mathbb{G}}$ is ordered. Also, if $\mathbb{H}_{\mathbb{G}}$ is ordered IFTHG with $\mathscr{C}(\mathbb{H}_{\mathbb{G}}) = \{\mathbb{H}_{\mathbb{G}}^{y_1, z_1}, \mathbb{H}_{\mathbb{G}}^{y_2, z_2}, \ldots, \mathbb{H}_{\mathbb{G}}^{y_n, z_n}\}$ and if $\mathbb{H}_{\mathbb{G}}^{y_n, z_n}$ is simple, then $\mathbb{H}_{\mathbb{G}}$ is elementary.*

Definition 3.12. Let an IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ and an independent set $V \subseteq U$. Then $\mathbb{H}_{\mathbb{G}}$ is called (μ, ν)-*tempered intuitionistic fuzzy threshold hypergraph*(tIFTHG), if there exists an intuitionistic fuzzy subset, $\mu_{ij} : U \to [0, 1]$ and $\nu_{ij} : U \to [0, 1]$ such that $\mathcal{E} = \{(\mu_{ij}(u_i), \nu_{ij}(u_i) | u_i \in \mathcal{E}_i\}$ where

$$\mu_{ij} = \begin{cases} \wedge\mu_j(a) \text{ and } \sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{ij} = \begin{cases} \vee\nu_j(a) \text{ and } \sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

where $\mu_{ij}(u_i)$ and $\nu_{ij}(u_i)$ are degrees of membership and non-membership functions of the element $u_i \in \mathcal{E}$ respectively for every $u_i \in \mathcal{E}$, $0 < \mu_{ij}(u_i) + \nu_{ij}(u_i) \leq 1$.

Example. Consider an IFTHG, $\mathbb{H}_{\mathbb{G}}$ such that $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, , \mathcal{E}_4\}$ according to the incidence matrix below with threshold value $\langle 0.3, 0.4 \rangle$.

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{E}_1\quad & \quad\mathcal{E}_2\quad & \quad\mathcal{E}_3\quad & \quad\mathcal{E}_4\quad \end{array} \\ \begin{pmatrix} \langle 0.2, 0.5 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.2, 0.5 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle \\ \langle 0, 1 \rangle & \langle 0.3, 0.4 \rangle & \langle 0.3, 0.4 \rangle & \langle 0, 1 \rangle \\ \langle 0.2, 0.5 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.2, 0.5 \rangle \end{pmatrix} \end{array}$$

The following hypergraph depicts the IFTHG $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; 0.3, 0.4)$.

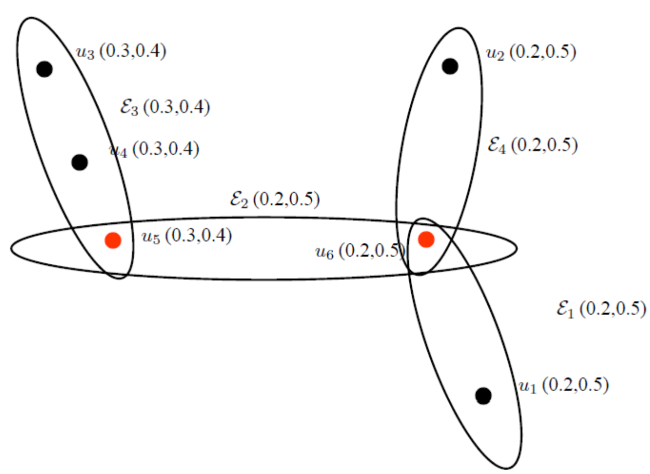

Figure 7: (μ, ν)-tIFTHG

$\mathcal{E}^{(0.3,0.4)} = \{\{u_3, u_5\}, \{u_4, u_5\}\}$
$\mathcal{E}^{(0.2,0.5)} = \{\{u_3, u_5\}, \{u_4, u_5\}, \{u_2, u_6\}, \{u_1, u_6\}\}$
Define $\mu_{ij} : U \to [0,1]$ and $\nu_{ij} : U \to [0,1]$ by
$\mu_{1j}(u_1) = 0.2,\ \mu_{2j}(u_2) = 0.2,\ \mu_{3j}(u_3) = 0.3,\ \mu_{4j}(u_4) = 0.3,\ \mu_{5j}(u_5) = 0.3,$ $\mu_{6j}(u_6) = 0.2$ and
$\nu_{1j}(u_1) = 0.5,\ \nu_{2j}(u_2) = 0.5,\ \nu_{3j}(u_3) = 0.4,\ \nu_{4j}(u_4) = 0.4,\ \nu_{5j}(u_5) = 0.4,$ $\nu_{6j}(u_6) = 0.5$
Note that
$\mu_{ij}(\mathcal{E}_1) = \mu_{1j}(u_1) \wedge \mu_{6j}(u_6) = 0.2$
$\mu_{ij}(\mathcal{E}_2) = \mu_{5j}(u_5) \wedge \mu_{6j}(u_6) = 0.2$
$\mu_{ij}(\mathcal{E}_3) = \mu_{3j}(u_3) \wedge \mu_{4j}(u_4) \wedge \mu_{5j}(u_5) = 0.3$
$\mu_{ij}(\mathcal{E}_4) = \mu_{2j}(u_2) \wedge \mu_{6j}(u_6) = 0.2$ and
$\nu_{ij}(\mathcal{E}_1) = \nu_{1j}(u_1) \vee \nu_{6j}(u_6) = 0.5$
$\nu_{ij}(\mathcal{E}_2) = \nu_{5j}(u_5) \vee \nu_{6j}(u_6) = 0.5$
$\nu_{ij}(\mathcal{E}_3) = \nu_{3j}(u_3) \vee \nu_{4j}(u_4) \vee \nu_{5j}(u_5) = 0.4$
$\nu_{ij}(\mathcal{E}_4) = \nu_{2j}(u_2) \vee \nu_{6j}(u_6) = 0.5$
Thus, $\mathbb{H}_{\mathbb{G}}$ is (μ, ν) - tIFTHG.

Theorem 3.3. *An IFTHG is (μ, ν)-tIFTHG of some crisp hypergraph $\mathbb{H}_{\mathbb{G}}$, if and only if $\mathbb{H}_{\mathbb{G}}$ is elementary, support simple and simply ordered.*

Proof. Let $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ be an (μ, ν)-tIFTHG of some crisp hypergraph $\mathbb{H}_{\mathbb{G}}$. There exists an independent set $V \subseteq U$ in $\mathbb{H}_{\mathbb{G}}$. Since, (μ, ν)-tempered the membership values and non-membership values of intuitionistic fuzzy hyperedges with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_{ij})(u_i) \leq s_2$ of $\mathbb{H}_{\mathbb{G}}$ are constant, hence it is elementary.

Clearly if support of two intuitionistic fuzzy hyperedges of the (μ, ν)-tIFTHG are equal then the intuitionistic fuzzy hyperedges with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_{ij})(u_i) \leq s_2$ are also equal. Therefore, it is support simple.

Let $\mathscr{C}(\mathbb{H}_{\mathbb{G}}) = \{\mathbb{H}_{\mathbb{G}}{}^{y_1,z_1}, \mathbb{H}_{\mathbb{G}}{}^{y_2,z_2}, \ldots, \mathbb{H}_{\mathbb{G}}{}^{y_n,z_n}\}$. Since $\mathbb{H}_{\mathbb{G}}$ is elementary, it is ordered.

To prove: $\mathbb{H}_{\mathbb{G}}$ is simply ordered.

Let $\mathcal{E} \in \mathbb{H}_{\mathbb{G}}{}^{y_{i+1},z_{i+1}} \backslash \mathbb{H}_{\mathbb{G}}{}^{y_i,z_i}$, then there exists $u_i \in \mathcal{E}$ such that $\mu_{ij}(u_i) = y_{i+1}$ and $\nu_{ij}(u_i) = z_{i+1}$. Since $y_{i+1} > y_i$ and $z_{i+1} > z_i$, it follows that $u_i \notin \mathbb{H}_{\mathbb{G}}{}^{y_i,z_i}$ and $\mathcal{E} \nsubseteq \mathbb{H}_{\mathbb{G}}{}^{y_i,z_i}$ for which $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V}(1 - \nu_{ij})(u_i) \leq s_2$. Thus, $\mathbb{H}_{\mathbb{G}}$ is simply ordered.

Conversely, suppose $\mathbb{H}_{\mathbb{G}} = (U, \mathcal{E}; s_1, s_2)$ is elementary, support simple and simply ordered.

To prove: (μ, ν) is tIFTHG.

We know that $\mathbb{H}_{\mathbb{G}}{}^{y_i,z_i} = \mathbb{H}_{\mathbb{G}i} = (U_i, \mathcal{E}_i; s_1, s_2)$. Also define $\mu_{ij} : U_i \to [0,1]$ and $\nu_{ij} : U_i \to [0,1]$ by

$$\mu_{ij}(u_i) = \begin{cases} y_1, & \text{if } u_i \in \mathcal{E}_1 \\ y_i, & \text{if } u_i \in \mathcal{E}_i \backslash \mathcal{E}_{i-1},\ i = 1, 2, \ldots, n \end{cases}$$

and

$$\nu_{ij}(u_i) = \begin{cases} z_1, & \text{if } u_i \in \mathcal{E}_1 \\ z_i, & \text{if } u_i \in \mathcal{E}_i \backslash \mathcal{E}_{i-1},\ i = 1, 2, \ldots, n \end{cases}$$

To prove: $\mathcal{E} = \{(\mu_{ij}(u_i), \nu_{ij}(u_i)) | u_i \in \mathcal{E}_i\}$ where

$$\mu_{ij}(u_i) = \begin{cases} \wedge \mu_i(a) \text{ and } \sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{ij}(u_i) = \begin{cases} \vee \nu_i(a) \text{ and } \sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2 | a \in \mathcal{E}, & \text{if } u_i \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

Let $\mathcal{E}' \in \mathcal{E}_i$.

Since $\mathbb{H}_{\mathbb{G}}$ is elementary and support simple, the intuitionistic fuzzy hyperedge (c_{ij}, d_{ij}) is unique in $\mathcal{E}$ which has support $\mathcal{E}'$. In fact, distinct hyperedges in $\mathcal{E}$ should have distinct supports which lie in $\mathcal{E}_i$ with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$.

To prove: For each $\mathcal{E}' \in \mathcal{E}_i$, $\mu_{ij}(u_i) = c_{ij}$, $\nu_{ij}(u_i) = d_{ij}$.

Since, all the hyperedges are elementary and different hyperedges have different supports, it follows from the definition of fundamental sequence, that $h(c_{ij}, d_{ij})$ is equal to some number of (y_i, z_i) in $\mathscr{F}(\mathbb{H}_{\mathbb{G}})$.

Consequently, $\mathcal{E}' \subseteq U_i$. Moreover if $i > 1$, then $\mathcal{E}' \in \mathcal{E}' \backslash \mathcal{E}_{i-1}$. Since $\mathcal{E} \subseteq U_i$, it follows from the definition of (μ, ν)-tempered that for each $u_i \in \mathcal{E}_i$, $\mu_{ij}(u_i) \geq y_i$ and $\mu_{ij}(u_i) \leq z_i$ with $\sum_{u_i \in V} \mu_{ij}(u_i) \leq s_1$ and $\sum_{u_i \in V} (1 - \nu_{ij})(u_i) \leq s_2$.

Claim: $\mu_{ij}(u_i) = y_i$ and $\nu_{ij}(u_i) = z_i$ for some $u_i \in \mathcal{E}_i$. By definition of (μ, ν)-tempered $\mu_{ij}(u_i) \geq y_{i-1}$ and $\nu_{ij}(u_i) \leq z_{i-1}$ for all $U_i \in \mathcal{E}_i \Rightarrow \mathcal{E} \subseteq U_{i-1}$ and so $\mathcal{E} \in \mathcal{E}_i \backslash \mathcal{E}_{i-1}$ and since $\mathbb{H}_{\mathbb{G}}$ is simply ordered $\mathcal{E} \nsubseteq U_{i-1}$, which is a contradiction. Hence, it follows from the definition that $\mu_{ij} = c_{ij}$, $\nu_{ij} = d_{ij}$. $\square$

4. Application of IFTHG

Water is required for the survival of all living organisms throughout the universe. Water is, without a doubt, the reason why the earth is still the only planet on which life can survive. This universal solvent is one of the most essential resources we have on this world. Life would be difficult to survive without water. Furthermore, it feeds nearly 70% of the world's population. A water distribution system is a segment of a water supply network that distributes drinking water from a centralised treatment plant or wells to people to meet the needs of residential, commercial, industrial and firefighting users.

The value of other commodities rises as a result of wasting water. Water shortages makes it more difficult to manufacture certain goods that require it. As a result, certain commodities would become scarce or may be priced higher. It's worth noting that there's a critical challenge with leakages in underground pipelines, which results in water waste in each area. Clearly, we need to know the quantity of actual water usage, estimated water leakage and unexpected water consumption in order to manage and control water supplies in each area. The pictorial representation of water distribution system is shown in Figure 8. The houses in the particular areas and two major suppliers are denoted as vertices and the supply of water to certain areas are considered as hyperedges. Furthermore, the two reservoirs must deliver the appropriate quantity of water to ensure the basic water usage of nine areas, as well as the minimum quantity of water storage required to maintain the suppliers normal water level.

Consider nine areas $e_1, e_2, e_3, ..., e_9$ and there are two suppliers p and q. During that particular period, water was distributed one by one from major suppliers to all areas of town.

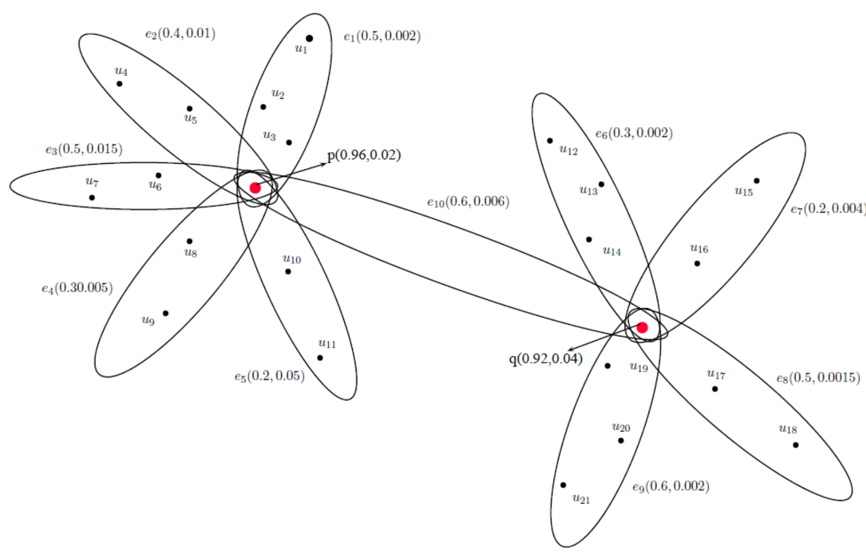

Figure 8: IFTHG linked to water management system

Now, consider a vertex(house) u_1 in the hyperedge(area) e_1 and the supplier p

- $\mu_i(u_1)$-denotes the actual water usage of the vertex u_1;
- $\nu_i(u_1)$-denotes the quantity of water leakage of the vertex u_1;
- $1 - \mu_i(u_1) - \nu_i(u_1)$-(the index of u_1)- denotes the quantity of unexpected water usage of u_1;
- $\mu_i(p)$-denotes the quantity of water supply of the supplier p;
- $\nu_i(p)$-denotes the minimal quantity of water storage of supplier p;
- $1 - \mu_i(p) - \nu_i(p)$-(the index of v_1)- denotes the quantity of unexpected water usage of p;

- $\mu_{ij}(e_1)$-denotes the total water distributed to the houses u_1, u_2, u_3;
- $\nu_{ij}(e_1)$-denotes the quantity of water leakage of the area e_1.

The values of vertices are tabulated below:

$u_1 - (0.6, 0.002)$	$u_{12} - (0.5, 0.001)$
$u_2 - (0.5, 0.0015)$	$u_{13} - (0.3, 0.002)$
$u_3 - (0.55, 0.002)$	$u_{14} - (0.4, 0.0015)$
$u_4 - (0.5, 0.001)$	$u_{15} - (0.6, 0.003)$
$u_5 - (0.4, 0.001)$	$u_{16} - (0.2, 0.004)$
$u_6 - (0.5, 0.001)$	$u_{17} - (0.5, 0.001)$
$u_7 - (0.6, 0.0015)$	$u_{18} - (0.7, 0.0015)$
$u_8 - (0.3, 0.004)$	$u_{19} - (0.6, 0.001)$
$u_9 - (0.5, 0.005)$	$u_{20} - (0.7, 0.002)$
$u_{10} - (0.3, 0.05)$	$u_{21} - (0.65, 0.0015)$
$u_{11} - (0.2, 0.045)$	

4.1 Working mechanism

The water distribution system begins by emphasizing the necessity of a raw water source such as a lake, river or groundwater. Water is stored in reservoirs to ensure a consistent supply for the system, with pumps facilitating its transportation from the source to storage facilities. Prior to entering the system, water undergoes filtration to prevent corrosion, followed by purification to meet quality standards.

Main pumping stations near storage facilities supply water directly into the piping system. Reservoirs play a crucial role in system capacity, supporting both consumer consumption and fire protection needs during peak demand periods. Ground-level and raised storage options are common.

Water is then distributed to users through pipes, with each area typically receiving water from reservoirs within a two-hour timeframe. If unexpected water needs arise, interconnected reservoirs can provide additional water via hyperedges. Once all areas are serviced, reservoirs replenish tanks for the next distribution cycle. This process ensures consistent water supply and addresses fluctuating demand effectively.

From the above comparison that using IFTHG in the water distribution system is essential. It can also be used to regulate water resources. Furthermore, in a water distribution system, IFTHGs are more appropriate than fuzzy threshold graph and intuitionistic fuzzy threshold graphs.

4.2 Analysis

Our water management system has two major suppliers, each with a capacity of 1000 litres. And each vertices (houses/industries) has a regular water allocation. Suppose that house 1 takes 60 litres of water per day, which indicates that the membership value of that vertex is 0.6. Similarly, intuitionistic fuzzy data from a collected data has been generated.

Because reservoirs influence the quantity of water required in each area, also, known that the threshold dimension can simply calculated by the number of reservoirs. As seen in the figures 9 and 10, in which the threshold dimension is 2 and the intuitionistic fuzzy threshold partition number is 2 as considered in [10]. As a result, we could generate two intuitionistic fuzzy threshold subhypergraphs(IFTsHGs). It must be observed that as thresholds s_1 and $m - s_2$ refer to the maximum quantity of water replenishment and unexpected water consumption respectively, where m is the number of elements in the maximum independent set of subhypergraphs. For e.g., consider $s_1 = 0.96$ and $s_2 = 4.98$ for first IFTsHG(Fig:9). Here, number of elements in the maximum independent set is $\{u_1, u_4, u_7, u_9, u_{10}\}$ and

$$\mu_i(u_1) + \mu_i(u_4) + \mu_i(u_7) + \mu_i(u_9) + \mu_i(u_{10}) = 2.5.$$

As a conclusion, it is clear that the reservior p can supply 0.96 quantity of water for the five areas, which requires at least 2.5 quantity of water for actual consumption and 0.02 (5-4.96=0.02) quantity of water goes under unforeseen water consumption/leakage. We can conclude from this that the p supplier can supply 960 litres for five areas, with at least 250 litres for actual consumption and 2 litres for unforeseen water consumption/leakage.

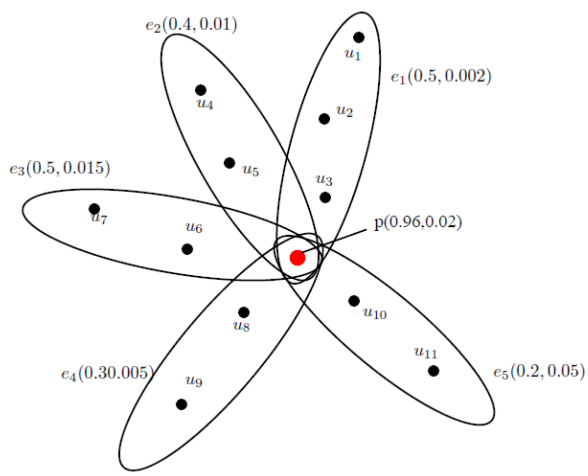

Figure 9: The first IFTHsG linked to water management system

Similarly, consider $s_1 = 0.92$ and $s_2 = 3.96$ for second IFTsHG(Fig:10) and the number of elements in the maximum independent set is $\{u_{12}, u_{15}, u_{18}, u_{20}\}$ which results in

$$\mu_i(u_{12}) + \mu_i(u_{15}) + \mu_i(u_{18}) + \mu_i(u_{20}) = 2.5.$$

Finally, the reservior q can supply 0.92 quantity of water for the four areas, with at least 2.5 quantity of water for actual consumption and 0.04 (4-3.96=0.04) quantity of water goes under unforeseen water consumption/leakage. Now, conclude from this that the q supplier can supply 920 litres for four areas, with at least 250 litres for actual consumption and 4 litres for unforeseen water consumption/leakage.

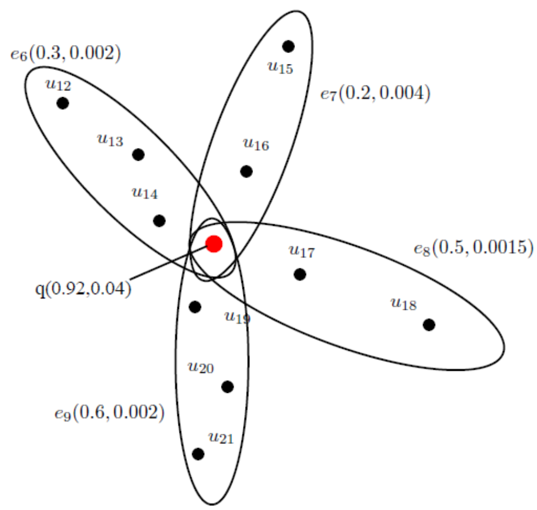

Figure 10: The second IFTsHG linked to water management system

In the first area, there are significant water leakages near houses u_1, u_4, u_7, u_9, and u_{10}. Similarly, in the second area, leakages are observed around houses u_{12}, u_{15}, u_{18}, and u_{20}, or in the pipes near these locations. By identifying these problem areas, we can take targeted actions to reduce water waste effectively. As a result of above analysis, it is easily seen that the importance of IFTHG as an application part to manage water resources. Furthermore, it is analyzed that IFTHGs are better than fuzzy threshold graphs for controlling water management systems through threshold values. In this application, threshold values are most important to maintain the certain limit of each vertices. Additionally, this method excels in identifying water wastage and, while illustrated here with a smaller example, it can be applied to larger and more complex structures with equal effectiveness.

5. Conclusion

In many real-world problems, data exhibit uncertain behavior and can vary significantly based on different parameters, highlighting the need for IFTHG. The application of fuzzy graph theory in addressing real-time situations is expanding, with IF models offering greater precision and compatibility than traditional methods. IFTHGs, in particular, are better equipped to handle uncertainty and ambiguity, making them a powerful tool in various applications. In this paper, we introduced the concept and definitions of IFTHGs, including core, fundamental sequence, elementary, and sectionally elementary types, and discussed several key results. Our IFTHG model, designed to reduce water wastage, demonstrates the practical significance of this approach. Future research in this area holds promise for a wide range of applications.

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