On the SS-supplemented modules over Dedekind domains

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Abstract. A module M is called *ss-supplemented* if every submodule U of M has a supplement V in M such that $U \cap V$ is semisimple. In this paper, we completely determine the structure of (amply) ss-supplemented modules over Dedekind domains. In particular, we prove that an abelian group M is ss-supplemented (as a $\mathbb{Z}$-module) if and only if $M \cong (\bigoplus_{p\in I} \mathbb{Z}_p^{(\upsilon)}) \oplus (\bigoplus_{q\in J} \mathbb{Z}_{q^2}^{(\nu)})$, where $\mathbb{P}$ is the set of all prime integers, I, J are some subsets of $\mathbb{P}$ and υ, ν are any index sets.

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1. Introduction

In [5], the concept of ss-supplement submodules have been introduced as a type of supplement submodules. In the same paper authors have obtained detailed information about modules with the help of ss-supplement submodules.

A submodule V of M of an R-module M is called a *supplement* of a submodule U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$

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and $U \cap V \ll V$ (see [8]). It follows from [8] that a module M is called *supplemented* if every submodule of M has a supplement in M. A submodule U of M has *ample supplements* in M if every submodule L of M such that $M = U + L$ contains a supplement of U in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M. As a generalization of semisimple modules, supplemented modules play an important role in module and ring theory, and Abelian groups.

Recall from [3] that a submodule V of a module M is a *Rad-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq Rad(V)$. Since $Rad(V)$ is the sum of all small submodules of V, every supplement submodule of M is a *Rad*-supplement submodule of M. M is called *Rad-supplemented* if all submodules have a *Rad*-supplement in M. Every module M with $M = Rad(M)$ is *Rad*-supplemented.

There are generally two different ways of working in module classes. The first one is the characterization of some important rings. A ring R is (semi) perfect if and only if every left (finitely generated) R-module is (amply) supplemented (see [8, 42.6§43.9]). $\frac{R}{P(R)}$ is perfect, where $P(R)$ is the sum of all radical left ideals of R if and only if every left R-module is Rad-supplemented, that is, every submodule of a module has a Rad-supplement in the module ([3, Theorem 6.1]). The other one is determine the structure of given modules classes over a Dedekind domain. It follows from [9, Proposition A.7 and Proposition A.8] and [3, Theorem 7.2 and Theorem 7.4] that the structure of ($Rad-$) supplemented modules over a (*local*) Dedekind domain is given.

For a module M, we consider the following submodule of M as in [4]:

$$Soc_s(M) = \sum \{N \ll M \mid N \text{ is simple}\}$$

Since $Soc_s(V) \subseteq Rad(V)$, it is of interest to investigate the analogue of this notion by replacing "$Rad(V)$"with "$Soc_s(V)$". We will use the same definition and notation as in [5] to call a submodule V of a module M *ss-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. It is shown in [5] that a submodule V of M is *ss*-supplement of some submodule U in M if and only if V is a supplement of U in M and $U \cap V$ is semisimple. Following [5], a module M is said to be *ss-supplemented* if every submodule of M has an *ss*-supplement in M. A submodule U of a module M has *ample ss-supplements* in M if every submodule V of M such that $M = U + V$ contains a *ss*-supplement of U in M. Also, It is called that a module M *amply ss-supplemented* if every submodule of M has ample *ss*-supplements in M. It is clear that every *ss*-supplemented module is supplemented. The basic properties of these modules were obtained in [5]. Clearly, the class of *ss*-supplemented modules is between the class of semisimple modules and the class of supplemented modules. It is shown in [5, Theorem 41] that a ring R is semiperfect with semisimple radical if and only if every left R-module is (amply) *ss*-supplemented.

2. Preliminaries

Throughout this study, all rings are associative with identity and all modules are unitary left modules. Let R be a ring and M be an R-module. The notation $U \subseteq M$ will mean that U is a submodule of M.

Definition 2.1 ([8]). *A submodule N of an R-module M is said to be small in M, denoted by $N \ll M$, if $M \neq N + K$ for every proper submodule K of M.*

Definition 2.2 ([8]). *Let M be an R-module. The sum of simple submodules of M is called the socle of M and is denoted by $Soc(M)$. If there are no simple submodules in M we put $Soc(M) = 0$. Also, $Soc(M)$ is the largest semisimple submodule of M.*

Definition 2.3 ([8]). *Let M be an R-module. The intersection of all maximal submodules of M or the sum of all small submodules of M is called the radical of M and is denoted by $Rad(M)$. If M has no maximal submodules we set $Rad(M) = M$.*

Lemma 2.1 ([5, Lemma 2]). *Let M be a module. Then, $Soc_s(M) = Rad(M) \cap Soc(M)$.*

Reduced modules and coatomic modules have a very important place in the theory of supplemented modules. Now we recall:

Definition 2.4 ([6]). *A module M is called reduced if whenever $A \subseteq M$ with $A = Rad(A)$ implies $A = 0$.*

Definition 2.5 ([6]). *A module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M.*

Note that, a coatomic module has small radical. Semisimple modules or finitely generated modules are coatomic.

Definition 2.6 ([8]). *A non-zero module M is called hollow if every proper submodule of M is small in M and is called local if M has a largest submodule, i.e. a proper submodule which contains all other proper submodules.*

It is obvious that a largest submodule has to be equal to the radical of M and that in this case $Rad(M)$ is maximal and small in M. It follows from [8, 41.4 (2)] that a module M is hollow and cyclic if and only if it is local. Note that hollow modules are clearly amply supplemented. A ring R is called *local ring* if ${}_RR$ is a local module.

Proposition 2.1 ([9, Proposition A.7 and Proposition A.8]). *Let R be a local Dedekind domain with maximal ideal P, quotient field K, and $Q = \frac{K}{R}$. Let a, b, c and n be natural numbers, and let $B(n_1, ..., n_s)$ denote the direct sum of arbitrarily many copies of $\frac{R}{P^{n_1}}$, ... ,$\frac{R}{P_s^n}$. Then, an R-module M is supplemented if and only if M is isomorphic to the direct sum $R^a \oplus K^B \oplus Q^c \oplus B(1, 2, ..., n)$.*

If R is a non-local Dedekind domain, an R-module M is supplemented if and only if every P-primary component (viewed as module over the localization R_P) has the structure described in the above form.

Definition 2.7 ([5]). *A module M strongly local if it is local and $Rad(M) \subseteq Soc(M)$ and a ring R left strongly local ring if ${}_RR$ is a strongly local module.*

It is also state that the following implications on modules:

$$\text{simple} \Longrightarrow \text{strongly local} \Longrightarrow \text{local}.$$

The following lemma comes from [5, Lemma 13].

Lemma 2.2 ([5, Lemma 13]). *Let M be an ss-supplemented module and N be a small submodule of M. Then, $N \subseteq Soc_s(M)$.*

In this paper, we completely determine the structure of ss-supplemented and amply ss-supplemented modules over Dedekind domains. We show that M is ss-supplemented if and only if it is isomorphic to a direct sum of strongly local R-modules. We prove that over a local Dedekind domain a module M is ss-supplemented if and only if it is amply ss-supplemented if and only if M is a bounded module with semisimple radical. In particular, we show that an abelian group M is ss-supplemented (as a $\mathbb{Z}$-module) if and only if $M \cong (\bigoplus_{p\in I} \mathbb{Z}_p^{(\upsilon)}) \oplus (\bigoplus_{q\in J} \mathbb{Z}_{q^2}^{(\nu)})$, where $\mathbb{P}$ is the set of all prime integers, I, J are some subsets of $\mathbb{P}$ and υ, ν are any index sets.

3. SS-supplemented modules over Dedekind domains

In this section, we aim to determine the structure of ss-supplemented modules over Dedekind domains. Unless stated otherwise, here we assume that every ring is a Dedekind domain which is not a field.

Remark 3.1. Let R be a Dedekind domain and K be the quotient field of R. Since $\frac{K}{R}$ is the hollow R-module which is not local, it follows from [5, Proposition 16] that $\frac{K}{R}$ is not ss-supplemented. Hence, again applying [5, Proposition 16], K is also not ss-supplemented.

The following proposition gives the classes of ss-supplemented modules.

Proposition 3.1. *Let R be a Dedekind domain with the quotient field K and M be a left R-module. If M is ss-supplemented, then M is reduced, coatomic and $Rad(M) \subseteq Soc(M)$.*

Proof. Let M be a ss-supplemented module and N be a radical submodule of M. Then, by [7, Lemma 4.4], N is injective and so there exists a submodule L of M such that $M = N \oplus L$. Therefore, N is ss-supplemented by [5, Proposition 26]. Note that $N \cong K^{(I)} \oplus (\frac{K}{R})^{(J)}$ for some index sets I and J. It follows from

[5, Proposition 26] and Remark 3.1 that $I = J = \emptyset$. It means that $N = 0$, that is, M is reduced.

Let U be a proper submodule of M. Then, U has a ss-supplement, say V, in M. Since M is reduced, V has a maximal submodule P. It is easy to see that $U + P$ is a maximal submodule of M and so M is coatomic. Moreover, $Rad(M) \subseteq Soc(M)$ by Lemma 2.2. □

Note that a reduced hollow module is local.

Proposition 3.2. *Let R be a Dedekind domain and M be a non-zero left R-module. Then, M is ss-supplemented if and only if it is isomorphic to a direct sum of strongly local R-modules.*

Proof. Let M be a ss-supplemented module. By Proposition 3.1, we have M is reduced. Therefore, it is supplemented and so we can write $M \cong \bigoplus_{i\in I} L_i$, where each L_i is local by Proposition 2.1. Since M is ss-supplemented, it follows from [5, Proposition 26] that each L_i is ss-supplemented. By [5, Proposition 16], each L_i is a strongly local module. This is desired conclusion. The converse follows from [5, Theorem 27]. □

Example 3.1. Consider the left $\mathbb{Z}$-module $M = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{p^2}$. It follows from [5, Theorem 27] that M is ss-supplemented.

A module M over a commutative domain R is said to be *bounded* if $rM = 0$ for some non-zero $r \in R$.

Proposition 3.3. *Let R be a local Dedekind domain and M be a left R-module. Then, M is ss-supplemented if and only if it is a bounded module with semisimple radical.*

Proof. If M is ss-supplemented, then it is coatomic and $Rad(M) \subseteq Soc(M)$ by Proposition 3.1. By [6, Lemma 2.1], there exists a nonnegative integer and a bounded submodule N of M such that $M \cong R^{(n)} \oplus N$. By [5, Proposition 26], R is ss-supplemented. It follows from [5, Proposition 16] that R is strongly local and so $n = 0$ by [5, Proposition 11]. It means that M is bounded.

Conversely, let M be a bounded module with $Rad(M) \subseteq Soc(M)$. Then, there exists an ideal I of R such that $IM = 0$. So, M can be considered as a $\frac{R}{I}$-module. Since $\frac{R}{I}$ is an artinian ring, it follows from [8, 43.9] that M is a supplemented $\frac{R}{I}$-module. Therefore, M is a supplemented R-module. Hence it is ss-supplemented by [5, Lemma 19]. □

Lemma 3.1. *Let R be a local Dedekind domain and M be a left R-module. Then, M is a strongly local module if and only if $M \cong \frac{R}{pR}$ or $M \cong \frac{R}{p^2R}$, where p is the prime element of R.*

Proof. Since M is a strongly local module, it is local and so $M \cong \frac{R}{p^nR}$ for some $n \geq 1$. Therefore, we can write $Rad(\frac{R}{p^nR}) = \frac{pR}{p^nR} \subseteq Soc(\frac{R}{p^nR}) \cong \frac{R}{pR}$, and so $n = 1$ or $n = 2$. The converse is clear. □

Theorem 3.1. *Let R be a local Dedekind domain and M be a left R-module. The following statements are equivalent:*

(1) *M is ss-supplemented.*

(2) *M is bounded and $Rad(M) \subseteq Soc(M)$.*

(3) *For every submodule N of M, N is bounded and $Rad(N) \subseteq Soc(N)$.*

(4) *Every submodule of M is ss-supplemented.*

(5) *M is amply ss-supplemented.*

(6) *M is isomorphic to a direct sum of $\frac{R}{pR}$'s and $\frac{R}{p^2R}$'s, where p is the prime element of R.*

Proof. (1) $\Longleftrightarrow$ (2) By Proposition 3.3.

(2) $\Longrightarrow$ (3) Let $N \subseteq M$. Therefore, N is bounded as a submodule of the bounded module M. By [8, 21.2 (2)], we obtain that $Rad(N) \subseteq Rad(M) \cap N \subseteq Soc(M) \cap N = Soc(N)$.

(3) $\Longrightarrow$ (4) It follows from Proposition 3.3

(4) $\Longrightarrow$ (5) By [5, Corollary 36].

(5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (1) By Proposition 3.2 and Lemma 3.1. □

Theorem 3.2. *Let R be a non-local Dedekind domain and M be a left R-module. The following statements are equivalent:*

(1) *M is ss-supplemented.*

(2) *M is supplemented, coatomic and $Rad(M) \subseteq Soc(M)$.*

(3) *M is isomorphic to a direct sum of strongly local R-modules.*

Proof. (1) $\Longleftrightarrow$ (3) By Proposition 3.2.

(1) $\Longleftrightarrow$ (2) It follows from By Proposition 3.1 and [5, Lemma 19]. □

Lemma 3.2. *An abelian group M is strongly local (as a $\mathbb{Z}$-module) if and only if $M \cong \mathbb{Z}_p$ or $M \cong \mathbb{Z}_{p^2}$, where p is a prime integer in $\mathbb{Z}$.*

Proof. Let M be a strongly local group. Then, M is local and so there exists a prime integer p and $1 \leq n \in \mathbb{Z}^+$ such that $M \cong \mathbb{Z}_{p^n}$. Note that $Rad(M) \cong \mathbb{Z}_{p^{n-1}}$ and $Soc(M) \cong \mathbb{Z}_p$. Since M is strongly local, we have $Rad(M) \subseteq Soc(M)$ and so $n = 1$ or $n = 2$. The converse is clear. □

Combining Proposition 3.2 and Lemma 3.2, we give the structure of *ss*-supplemented abelian groups.

Corollary 3.1. *An abelian group M is ss-supplemented (as a $\mathbb{Z}$-module) if and only if $M \cong (\bigoplus_{p \in I} \mathbb{Z}_p^{(\upsilon)}) \oplus (\bigoplus_{q \in J} \mathbb{Z}_{q^2}^{(\nu)})$, where $\mathbb{P}$ is the set of all prime integers, I, J are some subsets of $\mathbb{P}$ and υ, ν are any index sets.*

Example 3.2. Given the Abelian group $G = \mathbb{Z}_4^{(\mathbb{N})}$. Following Corollary 3.1, G is ss-supplemented as a $\mathbb{Z}$-module.

The following lemma is an analogous of [6, Lemma 2.3] for ss-supplement submodules.

Lemma 3.3. *Let R be a local Dedekind domain and M be an R-module. A submodule U of $Rad(M)$ has a ss-supplement in M if and only if $U = Rad(U) \oplus S$, where S is a semisimple submodule of U.*

Proof. Let V be a ss-supplement of U in M. Then, we can write $M = U + V$, $U \cap V << V$ and $U \cap V$ is semisimple. By [8, 41.1.(5)], $U \cap V + Rad(U) = (U \cap V) \cap Rad(M) + Rad(U) = U \cap (V \cap Rad(M)) + Rad(U) = U \cap (Rad(V) + Rad(U)) = U \cap (pV + pU) = U \cap pM = U \cap Rad(M) = U$, where p is the prime element of R. Thus, $U = Rad(U) \oplus S$ for some $S \subseteq U \cap V$.

Conversely, since $U = Rad(U) \oplus S$, we get $Rad(U) = Rad(Rad(U))$ and so $Rad(U)$ is injective by [7, Lemma 4.1]. It follows that $M = Rad(U) \oplus V$ for some submodule V of M. By the module law, we can write that $U = U \cap M = U \cap (Rad(U) \oplus V) = Rad(U) \oplus (U \cap V)$. Since R is a local ring, it follows from [2, Theorem 3.5] that $U \cap V \cong \frac{U}{Rad(U)}$ is semisimple. Since $M = U + V$ and $U \cap V \subseteq U \subseteq Rad(M)$, it follows from [8, 19.3 (5)] and [5, Lemma 1] that $U \cap V$ is small in V. It means that V is a ss-supplement of U in M. □

Let R be an arbitrary ring and M be an R-module. Recall from [8] that an injective module E together with essential monomorphism $\Phi : M \longrightarrow E$ *injective hull* of M. It is known in [8, 17.9] that every R-module has only one (up to isomorphism) injective hull. We denote by $E(M)$ the injective hull of a module M.

Next, we obtain the following result which determines the structure of a module has a ss-supplement in its injective hull over local Dedekind domains.

Theorem 3.3. *Let R be a local Dedekind domain and M be a left R-module. The following statements are equivalent:*

(1) *M has a ss-supplement in $E(M)$, where $E(M)$ is the injective hull of M,*

(2) *$M \cong K^{(I)} \oplus (\frac{K}{R})^{(J)} \oplus S$, where K is the quotient field of R, I and J denote some index sets and S is a semisimple R-module,*

(3) *$Rad(M)$ is a direct summand of M.*

Proof. (1) $\Longrightarrow$ (2) By [7, Lemma 4.1], $E(M)$ is a radical module and so $M \subseteq E(M) = Rad(E(M))$. Suppose that M has a ss-supplement in $E(M)$. Applying Lemma 3.3, we obtain that $M = Rad(M) \oplus S$, where S is semisimple. Therefore, $Rad(M) = P(M) \cong K^{(I)} \oplus (\frac{K}{R})^{(J)}$ for some index sets I and J. Hence $M \cong K^{(I)} \oplus (\frac{K}{R})^{(J)} \oplus S$, where S is a semisimple R-module.

(2) $\Longrightarrow$ (1) It is clear by Lemma 3.3.

(1) $\Longleftrightarrow$ (3) This can be proved by taking $U = Rad(M)$ in the Lemma 3.3. □

Example 3.3. Put $M = \mathbb{Q} \oplus \mathbb{Z}_3$, where $\mathbb{Q}$ is the fractional field of the domain $\mathbb{Z}$. Therefore, $Rad(M) = \mathbb{Q} \oplus \{\overline{0}\}$ and thus $M = Rad(M) \oplus \mathbb{Z}_3$. So, $Rad(M)$ is a direct summand of M. Hence, by Theorem 3.3, M has a ss-supplement in $E(M) = \mathbb{Q} \oplus \mathbb{Z}_3^{\infty}$.

In [1], a module M is said to be *$\oplus$-radical supplemented* if $Rad(M)$ has a supplement that is a direct summand of M. It is shown in [1, Theorem 3.1] that over a local Dedekind domain a module M is $\oplus$-radical supplemented if and only if $M \cong K^{(I)} \oplus (\frac{K}{R})^{(J)} \oplus R^{(n)} \oplus N$, where K is the quotient field of R, I and J denote some index sets, n is a non-negative integer and N is a bounded R-module. Note that over a local Dedekind domain (which is not field) every semisimple module is bounded. Using this fact, Theorem 3.3 and [1, Theorem 3.1], we get the following result:

Corollary 3.2. *Let R be a local Dedekind domain and M be a left R-module. If $Rad(M)$ has a ss-supplement in M, then it is $\oplus$-radical supplemented.*

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