On $IC\overline{s}$-subgroups of finite groups

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Abstract. A subgroup H of a group G is said to be an $IC\overline{s}$-subgroup of G if the intersection of H and $[H, G]$ is contained in $H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the maximal s-semipermutable subgroup of G contained in H. In this paper, we investigate the influence of $IC\overline{s}$-subgroups on the structure of finite groups. Some new results of p-nilpotency and supersolvability of finite groups are obtained.

Keywords: $IC\overline{s}$-subgroup, p-nilpotent group, supersoluble group.
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1. Introduction

In this paper, all groups are finite, G always denotes a finite group. $\pi(G)$ denotes the set of all primes dividing $|G|$. $Syl_p(G)$ denotes the set of Sylow p-subgroups of G. $\mathfrak{U}$ is the class of all supersolvable groups. $Z_{\mathfrak{U}}(G)$ denotes the product of all normal subgroups N of G such that every chief factor of G below N has prime order. The $\mathfrak{F}$-residual of G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G with the quotient in $\mathfrak{F}$. We use standard notation as in [1] and [2].

Let H be a subgroup of a group G. It is well known that the normal closure H^G of H in G is the smallest normal subgroup of G containing H and $H^G = H[H, G]$, where $[H, G]$ is the commutator subgroup of H and G. From [3], we know that a subgroup H of a group G has IC-property if the intersection of H and $[H, G]$ satisfies some conditions. It is an interesting question to research the relationship between IC-property of some subgroups and the structure of G. Many results have been obtained. For example, in [3], Gao and Li introduced the concept of $IC\Phi$-subgroup. A subgroup H of G is said to be an $IC\Phi$-subgroup of G if the intersection of H and $[H, G]$ is contained in $\Phi(H)$. They proved that a

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group G is p-nilpotent if every cyclic subgroup of G with order p and 4(if $p = 2$) or every maximal subgroup of P is an $IC\Phi$-subgroup of G, where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. Kaspczyk in [16] generalized these results and proved that a group G is p-nilpotent if there is a subgroup D of P with $1 < |D| \leq |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and $|P| \geq 8$) is an $IC\Phi$-subgroup of G. In [17], Gao and Li introduced the concept of $IC\Phi s$-subgroup. A subgroup H of G is said to be an $IC\Phi s$-subgroup of G if $(H \cap [H, G])H_G/H_G \leq \Phi(H/H_G)H_{sG}/H_G$, where H_{sG} is the maximal s-permutable subgroup of G contained in H. They proved that a group G is p-nilpotent if there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi s$-subgroup of G, where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. Zhang and Xu in [18] generalized this result and proved the following result. Let N be a normal subgroup of G and P a Sylow p-subgroup of N. Assume that there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an $IC\Phi s$-subgroup of G, then $N \leq Z_{p\mathfrak{U}}(G)$. Later, in [19], Gao and Li also introduced the concept of ICC-subgroup. A subgroup H of G is said to be an ICC-subgroup of G if the intersection of H and $[H, G]$ is contained in H_{cG}, where H_{cG} is a CAP-subgroup of G contained in H. They proved that a group G is p-nilpotent if there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ and 4 (if $|D| = 2$ and P is a non-abelian 2-group) is an ICC-subgroup of G, where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. In this paper, we continue to study the IC-property of some subgroups.

Recall that a subgroup H is s-semipermutable in G if H permutes with every Sylow q-subgroup of G for prime q not dividing $|H|$. Once the notion of s-semipermutable subgroup was introduced, many authors have been interested in it and have applied it to investigate the structure of groups. For example, Li et al. in [15] proved that a group G is p-nilpotent if P satisfies the following: P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ are s-semipermutable in G. When P is a non-abelian 2-group and $|P : D| > 2$, in addition, H is s-semipermutable in G if there exists $H_1 \trianglelefteq H \leq P$ with $2|H_1| = |D|$ and H/H_1 is cyclic of order 4, where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. Here, combining IC-property and s-semipermutable subgroup we give a new concept which covers properly s-semipermutable subgroup of G.

Definition 1.1. *Let H be a subgroup of a group G, then H is called $IC\overline{s}$-subgroup of G if $H \cap [H, G] \leq H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the maximal s-semipermutable subgroup of G contained in H.*

Obviously, a s-semipermutable subgroup of G is an $IC\overline{s}$-subgroup of G. The converse does not hold in general.

Example 1.1. Let $G = S_4$. Take $H = \langle(34)\rangle$, then $[H,G] \leq A_4$ and $H \cap [H,G] = 1$. Hence, H is an $IC\overline{s}$-subgroup of G. But, clearly, H is not s-semipermutable subgroup of G because of $\langle(123)\rangle H \neq H\langle(123)\rangle$.

In this paper, we study the structure of a group G under the assumption that some subgroups of G are $IC\overline{s}$-subgroups and get some characterzations of p-nilpotency and supersolvability of finite groups.

2. Preliminary results

Lemma 2.1 ([4, Lemma 2.2])**.** *Let G be a group. Suppose that H is an s-semipermutable subgroup of G. Then*

(1) *If $H \leq K \leq G$, then H is s-semipermutable in K.*

(2) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N.*

(3) *If $H \leq O_p(G)$, then H is s-permutable in G.*

(4) *Suppose that H is a p-group for some prime $p \in \pi(G)$ and N is normal in G. Then, $H \cap N$ is also an s-semipermutable subgroup of G.*

Lemma 2.2 ([5, Lemma A])**.** *If H is an s-permutable subgroup of G and H is a p-group. Then, $O^p(G) \leq N_G(H)$.*

Lemma 2.3. *Let G be a group, $H \leq G$, $N \trianglelefteq G$. Suppose that H is an $IC\overline{s}$-subgroup of G. Then*

(1) *If $H \leq K \leq G$, then H is an $IC\overline{s}$-subgroup of K.*

(2) *Let $N \leq H$. If H is a p-group for some prime $p \in \pi(G)$, then H/N is an $IC\overline{s}$-subgroup of G/N.*

(3) *If H is a p-group and N is a p'-group for some prime $p \in \pi(G)$, then HN/N is an $IC\overline{s}$-subgroup of G/N.*

Proof. (1) By the hypothesis, $H \cap [H,G] \leq H_{\overline{s}G}$. Then, $H \cap [H,K] \leq H \cap [H,G] \leq H_{\overline{s}G} \leq H_{\overline{s}K}$ by Lemma 2.1(1). Hence, H is an $IC\overline{s}$-subgroup of K.

(2) By Lemma 2.1(2) and the hypothesis, we have $H/N \cap [H/N, G/N] = (H \cap [H,G]N)/N = (H \cap [H,G])N/N \leq H_{\overline{s}G}/N \leq (H/N)_{\overline{s}(G/N)}$. Hence, H/N is an $IC\overline{s}$-subgroup of G/N.

(3) By the hypothesis, we have $H \cap [H,G] = H \cap [H,G]N$. Hence, $HN/N \cap [HN/N, G/N] = (HN \cap [H,G]N)/N = (H \cap [H,G]N)N/N = (H \cap [H,G])N/N \leq H_{\overline{s}G}N/N \leq (HN/N)_{\overline{s}(G/N)}$ by Lemma 2.1(2). This shows that HN/N is an $IC\overline{s}$-subgroup of G/N.

□

Lemma 2.4 ([6, Theorem A]). *Let H be an s-semipermutable π-subgroup of G. Then, H^G contains a nilpotent π-complement and all π-complements in H^G are conjugate. Also, if π consists of a single prime, then H^G is solvable.*

Lemma 2.5 ([2, Lemma 1.2]). *Let U, V and W be subgroups of a group G. Then, the following statements are equivalent:*

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.6 ([7, Lemma 2.4]). *Let N be a normal subgroup of a group G such that G/N is p-nilpotent and let P be a Sylow p-subgroup of N, where p is a prime divisor of $|G|$. If $|P| \leq p^2$ and one of the following conditions holds, then G is p-nilpotent.*

(1) $(|G|, p-1) = 1$ *and* $|P| \leq p$.

(2) G *is* A_4*-free if* $p = min\pi(G)$.

(3) $(|G|, p^2 - 1) = 1$.

Lemma 2.7 ([8, Theorem 10.4.1]). *Let G be a finite group, p be an odd prime and P be a Sylow p-subgroup of G. Then, G is p-nilpotent if and only if $N_G(J(P))$ and $C_G(Z(P))$ are p-nilpotent.*

Lemma 2.8 ([9, Theorem B]). *Let $\mathfrak{F}$ be a formation and let E be a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

Lemma 2.9 ([10, Lemma 3.3]). *Let $\mathfrak{F}$ be a solubly saturated formation containing all supersoluble groups. Suppose that E is a normal subgroup of G such that $G/E \in \mathfrak{F}$. If $E \leq Z_{\mathfrak{F}}(G)$, then $G \in \mathfrak{F}$. In particular, if E is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.10 ([11, Lemma 2.12]). *Let $\mathfrak{F}$ be a solubly saturated formation. Suppose that P is a normal p-subgroup of G and C is a Thompson critical subgroup of P. If either $P/\Phi(P) \leq Z_{\mathfrak{F}}(G/\Phi(P))$ or $\Omega(C) \leq Z_{\mathfrak{F}}(G)$, then $P \leq Z_{\mathfrak{F}}(G)$.*

3. New characterization of p-nilpotentcy of groups

Theorem 3.1. *Let $N \trianglelefteq G$ such that G/N is p-nilpotent, where p is the smallest prime divisor of $|G|$. Suppose that every cyclic subgroup of N with order p and 4 (if $p = 2$) is an $IC\overline{s}$-subgroup of G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let (G, N) be a minimal counterexample with minimal order $|G| + |N|$.

(1) G is a minimal nonnilpotent group.

Let L be a proper subgroup of G. Since $L/(L \cap N) \cong LN/N \leq G/N$ and G/N is p-nilpotent, we have $L/(L \cap N)$ is p-nilpotent. It is clear that every cyclic subgroup of $L \cap N$ with order p and 4 (if $p = 2$) is an $IC\overline{s}$-subgroup of L by hypothesis and Lemma 2.3(1). Therefore, $(L, L \cap N)$ satisfies the hypothesis of the theorem. The minimal choice of (G, N) implies that L is p-nilpotent. This shows that G is a minimal non-p-nilpotent group. In view of ([1], Proposition 5.4; [2], Theorem 6.18), G is a minimal nonnilpotent group, $G = P \rtimes Q$, where P is a Sylow p-subgroup of G, Q is a Sylow q-subgroup of G and $p \neq q$; $P = G^{\mathfrak{N}}$, that is, P is the intersection of all normal subgroups K of G satisfying G/K is nilpotent; $P/\Phi(P)$ is a chief factor of G; the exponent of P is p or 4(when P is a nonabelian 2-group).

(2) Let $x \in P$ such that $x \notin \Phi(P)$, then $\langle x\rangle^G = \langle x\rangle[\langle x\rangle, G] = P$.

Since $P \trianglelefteq G$, we have $\langle x\rangle[\langle x\rangle, G] = \langle x\rangle^G \leq P$ and $\Phi(P)$ are normal subgroups of G, so $\langle x\rangle[\langle x\rangle, G]\Phi(P) \trianglelefteq G$, obviously, $\Phi(P) < \langle x\rangle[\langle x\rangle, G]\Phi(P) \leq P$. Since $P/\Phi(P)$ is a chief factor of G, we have $P = \langle x\rangle[\langle x\rangle, G]\Phi(P) = \langle x\rangle[\langle x\rangle, G]$.

(3) $p = 2$ and P has an element of order 4.

Obviously, $P \leq N$. If P has not an element of order 4, let $x \in P$ such that $x \notin \Phi(P)$, then $|\langle x\rangle| = p$ and $P = \langle x\rangle[\langle x\rangle, G]$. By hypothesis, $\langle x\rangle \cap [\langle x\rangle, G] \leq \langle x\rangle_{\overline{s}G}$. If $\langle x\rangle_{\overline{s}G} = 1$, then $\Phi(P) \leqslant [\langle x\rangle, G] < P$, Since $P/\Phi(P)$ is a chief factor of G, so $P = \langle x\rangle[\langle x\rangle, G] = \langle x\rangle\Phi(P) = \langle x\rangle$. Since p is the smallest prime divisor of $|G|$, we have G is p-nilpotent, a contradiction. If $\langle x\rangle = \langle x\rangle_{\overline{s}G}$, then $\langle x\rangle Q$ is a subgroup of G. Since $P \neq \langle x\rangle$, we have $\langle x\rangle Q < G$ is nilpotent, so $\langle x\rangle \leq N_G(Q)$, Therefore, $P \leq N_G(Q)$ and so $Q \trianglelefteq G$. This implies that G is nilpotent, a contradiction.

(4) The final contradiction.

By (3), we obtain that P is a nonabelian 2-group and the exponent of $\Phi(P)$ is 2. Let x be an element of P with order 4, then $P = \langle x\rangle[\langle x\rangle, G]$ by (2). If $[\langle x\rangle, G] < P$, then $P/[\langle x\rangle, G]$ is a nontrivial cyclic group. Let $M/[\langle x\rangle, G]$ be the unique maximal subgroup of $P/[\langle x\rangle, G]$. Then, $[\langle x\rangle, G] \leq M < P, M \trianglelefteq G$, obviously, $\Phi(P) \leq M$, hence $M = \Phi(P)$. So P is cyclic. It follows that G is 2-nilpotent, a contradiction. This contradiction shows that $P = [\langle x\rangle, G]$. By hypothesis, $\langle x\rangle = \langle x\rangle \cap [\langle x\rangle, G] \leq \langle x\rangle_{\overline{s}G}$, hence $\langle x\rangle = \langle x\rangle_{\overline{s}G}$. Notice that $\langle x\rangle Q$ is a proper subgroup of G, so $\langle x\rangle Q$ is nilpotent, then $\langle x\rangle \leq N_G(Q)$. Let y be an element of P with order 2. By hypothesis, $\langle y\rangle \cap [\langle y\rangle, G] \leq \langle y\rangle_{\overline{s}G}$. If $\langle y\rangle_{\overline{s}G} = 1$, then $\langle y\rangle \cap [\langle y\rangle, G] = 1$. Suppose that $\langle y\rangle[\langle y\rangle, G] = P$, then $\Phi(P) \leqslant [\langle y\rangle, G] < P$, Since $P/\Phi(P)$ is a chief factor of G, we get $\Phi(P) = [\langle y\rangle, G]$ and so $P = \langle y\rangle$, a contradiction. Hence, $\langle y\rangle[\langle y\rangle, G]Q$ is a proper subgroup of G and so $\langle y\rangle \leq N_G(Q)$. If $\langle y\rangle_{\overline{s}G} = \langle y\rangle$, then $\langle y\rangle \leq N_G(Q)$. Therefore, $P \leq N_G(Q)$ and so $Q \trianglelefteq G$. This implies that G is nilpotent, a contradiction.

This completes the proof. □

Corollary 3.1. *Let p be the smallest prime divisor of $|G|$. If every cyclic subgroup of G with order p and 4 (if $p = 2$) is an $IC\overline{s}$-subgroup of G, then G is p-nilpotent.*

Theorem 3.2. *Let G be a group and $P \in Syl_p(G)$, where p is the smallest prime divisor of $|G|$. Suppose that every maximal subgroup of P is an $IC\overline{s}$-subgroup of G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let G be a minimal counterexample with minimal order.

(1) G has a unique minimal normal subgroup N, G/N is p-nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G and M/N be a maximal subgroup of PN/N. Then, $M = M \cap PN = (M \cap P)N$. Let $M \cap P = P_1$, then $P_1 \cap N = P \cap N$ and $|P : P_1| = |PN/N : M/N| = p$, this shows that P_1 is maximal in P. Since $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N, hence

$$|N \cap P_1[P_1, G]|_p = |N|_p = |N \cap P_1| = |(N \cap P_1)(N \cap [P_1, G])|_p$$

and

$$\begin{aligned} |N \cap P_1[P_1, G]|_{p'} &= \frac{|N|_{p'} |P_1[P_1, G]|_{p'}}{|NP_1[P_1, G]|_{p'}} \\ &= \frac{|N|_{p'} |[P_1, G]|_{p'}}{|N[P_1, G]|_{p'}} = |N \cap [P_1, G]|_{p'} = |(N \cap P_1)(N \cap [P_1, G])|_{p'}. \end{aligned}$$

This implies that $N \cap P_1[P_1, G] = (N \cap P_1)(N \cap [P_1, G])$. By Lemma 2.5, we have $P_1N \cap [P_1, G]N = (P_1 \cap [P_1, G])N$, and because P_1 is an $IC\overline{s}$-subgroup of G, hence $M/N \cap [M/N, G/N] = P_1N/N \cap [P_1N/N, G/N] = (P_1N \cap [P_1, G]N)/N = (P_1 \cap [P_1, G])N/N \leq (P_1)_{\overline{s}G}N/N \leq (P_1N/N)_{\overline{s}(G/N)} = (M/N)_{\overline{s}(G/N)}$ by Lemma 2.1(2). This shows that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we have N is a unique minimal normal subgroup of G and $\Phi(G) = 1$.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and G/N is p-nilpotent by (1). This implies that G is p-nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.

If $O_p(G) = 1$, then by (2), G is not solvable, so $p = 2$. If $PN < G$, then PN satisfies the hypothesis of the theorem by Lemma 2.3(1), hence PN is 2-nilpotent, so N is also 2-nilpotent. By (1), G/N is 2-nilpotent, hence G is solvable, a contradiction. If $PN = G$, let P_1 be a maximal subgroup of P, by the hypothesis, $P_1 \cap [P_1, G] \leq (P_1)_{\overline{s}G}$. If $[P_1, G] = 1$, then $P_1 \trianglelefteq G$, so $|P| = p$, thus G is 2-nilpotent, a contradiction. Hence, $[P_1, G] \neq 1$. Then, $P_1 \cap N \leq P_1 \cap [P_1, G] \leq (P_1)_{\overline{s}G}$, therefore $P_1 \cap N = (P_1)_{\overline{s}G} \cap N$, obviously, $P_1 \cap N \neq 1$, so $(P_1 \cap N)^G = N$ is soluble by Lemma 2.1(4) and Lemma 2.4, so is G, a contradiction.

(4) The final contradiction.

By (1), $\Phi(O_p(G)) \leq \Phi(G) = 1$, so $O_p(G)$ is an elementary abelian p-subgroup. By (3), $N \leq O_p(G)$. If $N \leq \Phi(P)$, then $N \leq \Phi(G) = 1$, a contradiction. If $N \nleq \Phi(P)$, let P_1 be a maximal subgroup of P such that $N \nleq P_1$, then $P = P_1N$. If $N \cap P_1 = 1$, then $|N| = p$, by (1), G/N is p-nilpotent, hence G is p-nilpotent by Lemma 2.6, a contradiction. If $N \cap P_1 \neq 1$, obviously, $[P_1, G] \neq 1$, then $1 \neq P_1 \cap N \leq P_1 \cap [P_1, G] \leq (P_1)_{\overline{s}G}$, hence $P_1 \cap N = (P_1)_{\overline{s}G} \cap N$, so $O^p(G) \leq N_G(P_1 \cap N)$ by Lemma 2.1(3)(4) and Lemma 2.2. Clearly, $P_1 \cap N \trianglelefteq P$, so $P_1 \cap N \trianglelefteq G$, we obtain $P_1 \cap N = N$ by the minimality and uniqueness of N, hence $N \leq P_1$, a contradiction.

This completes the proof. □

Corollary 3.2. *If every maximal subgroup of every Sylow subgroup of G is an $IC\overline{s}$-subgroup of G, then G is a Sylow tower group of supersolvable type.*

Theorem 3.3. *Let G be a group and $P \in Syl_p(G)$, where p is a prime divisor of $|G|$. Suppose that $N_G(P)$ is p-nilpotent and every maximal subgroup of P is an $IC\overline{s}$-subgroup of G, then G is p-nilpotent.*

Proof. If $p = min\pi(G)$, then G is p-nilpotent by Theorem 3.2. Now, we consider the case when $p \neq min\pi(G)$, that is, p is an odd prime. Assume that the result is false. Let G be a minimal counterexample with minimal order.

(1) G has a unique minimal normal subgroup N, G/N is p-nilpotent and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G, then PN/N is a Sylow p-subgroup of G/N . If $N_G(P)N = G$, then $G/N = N_G(P)N/N \cong N_G(P)/N_G(P) \cap N$ is p-nilpotent. If $N_G(P)N \neq G$, then $N_G(P)N$ satisfies the hypothesis of the theorem by Lemma 2.3(1), so $N_G(P)N$ is p-nilpotent by the minimal choice of G, hence $N_{G/N}(PN/N) \cong N_G(P)N/N$ is p-nilpotent. We may obtain that every maximal subgroup of PN/N is an $IC\overline{s}$-subgroup of G/N by a similar discussion as in Theorem 3.2(1). This shows that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we have N is a unique minimal normal subgroup of G and $\Phi(G) = 1$.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ and G/N is p-nilpotent by (1). This implies that G is p-nilpotent, a contradiction.

(3) G is solvable.

Since G is not p-nilpotent and p is an odd prime, by Lemma 2.7, there exists a nontrivial characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, obviously, $P \leq N_G(H)$. If $N_G(H) < G$, then clearly $N_G(H)$ satisfies the hypothesis of the theorem, hence $N_G(H)$ is p-nilpotent, a contradiction. This contradiction shows that $N_G(H) = G$. Therefore,$1 \neq H \trianglelefteq G, H \leq O_p(G)$ and so $N \leq O_p(G)$. By (1), $G/O_p(G)$ is p-nilpotent and so G is p-solvable. Then, by [14,Theorem3.5], there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G, where $q||G|$ and $q \neq p$. If $PQ < G$, obviously, PQ satisfies the hypothesis of the theorem, hence PQ is p-nilpotent. So $O_p(G)Q = O_p(G) \times Q$ and $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [14,Theorem3.2], a contradiction. Hence, $G = PQ$ is solvable.

(4) The final contradiction.

By (2) and (3), $O_p(G) \neq 1$, then $N \leq O_p(G)$ and $O_p(G)$ is an elementary abelian p-group by (1). Hence, there exists a maximal subgroup M of G such that $G = N \rtimes M$. Obviously, $O_p(G) \cap M \trianglelefteq M$, $O_p(G) \cap M \trianglelefteq O_p(G)$, so $O_p(G) \cap M \trianglelefteq G$. If $O_p(G) \cap M \neq 1$, then $N \leq O_p(G) \cap M$ and so $N \leq M$, a contradiction. Hence, $O_p(G) \cap M = 1$. This induces that $N = O_p(G) = C_G(N)$. Let $M_p = P \cap M$ be a Sylow p-subgroup of M, then $P = NM_p$. It is easy to see that $M_p \neq 1$. Let P_1 be a maximal subgroup of P containing M_p, by the hypothesis, $P_1 \cap [P_1, G] \leq (P_1)_{\overline{s}G}$, obviously, $N \nleq P_1$ and $N \leq [P_1, G]$, then $P_1 \cap N \leq P_1 \cap [P_1, G] \leq (P_1)_{\overline{s}G}$, so $P_1 \cap N = (P_1)_{\overline{s}G} \cap N$. If $P_1 \cap N \neq 1$, We may obtain $P_1 \cap N \trianglelefteq G$ by a similar discussion as in Theorem 3.2(4), hence $P_1 \cap N = N$ and so $N \leq P_1$, a contradiction. If $P_1 \cap N = 1$, it follows that $|N| = p$ and $P = N \times P_1$. Hence, $P \leq C_G(N) = N$, therefore, $P = N$ and $G = N_G(N) = N_G(P)$ is p-nilpotent, a contradiction.

This completes the proof. □

Remark 3.1. In Theorem 3.1, 3.2, 3.3, the assumptions that p is the smallest prime divisor of $|G|$ and $N_G(P)$ is p-nilpotent are necessary. To illustrate the situation, we consider the semidirect product $G = H \rtimes \langle c \rangle$, where $H = \langle a \rangle \times \langle b \rangle$ with $o(a) = o(b) = 5$ and $o(c) = 2$ defined by $a^c = a^{-1}$ and $b^c = b^{-1}$. Then, the subgroups of G of order 5 are normal in G, but G is not 5-nilpotent.

4. New characterization of supersolubility of groups

Lemma 4.1. *Let P be a normal p-subgroup of G. If every cyclic subgroup of P with order p and 4 (if $p = 2$) is an $IC\overline{s}$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Assume that the result is false. Let (G, P) be a minimal counterexample with $|G| + |P|$ minimal. Let K be a normal subgroup of G such that P/K is a cheif factor of G. Obviously, (G, K) satisfies the hypothesis of the theorem and so $K \leq Z_{\mathfrak{U}}(G)$. Suppose that P/L is another cheif factor of G which is different from P/K, then $L \leq Z_{\mathfrak{U}}(G)$ and so $P = KL \leq Z_{\mathfrak{U}}(G)$, a contradiction. This contradiction shows that K is the unique normal subgroup of G such that P/K is a cheif factor of G. Obviously, $|P/K| \neq p$. Let C be a Thompson critical subgroup of P. If $\Omega(C) < P$, then $\Omega(C) \leq K \leq Z_{\mathfrak{U}}(G)$, so $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.10, a contradiction. If $\Omega(C) = P$, then $exp(P) = p$ or 4. Let G_p be a Sylow p-subgroup of G containing P, obviously, $P/K \cap Z(G_p/K) \neq 1$. Suppose that $M/K \leq P/K \cap Z(G_p/K)$ and $|M/K| = p$. Let $x \in M \backslash K$, then $M = \langle x \rangle K$ and $\langle x \rangle \cap [\langle x \rangle, G] \leq \langle x \rangle_{\overline{s}G}$. If $\langle x \rangle = \langle x \rangle_{\overline{s}G}$, then M/K is s-semipermutable in G/K by Lemma 2.1(2), clearly, $M/K \trianglelefteq G_p/K$, and $O^p(G/K) \leq N_{G/K}(M/K)$ by Lemma 2.1(3) and Lemma 2.2. So $M/K \trianglelefteq G/K$. Hence, $|M/K| = |P/K| = p$, a contradiction. If $\langle x \rangle \neq \langle x \rangle_{\overline{s}G}$, then $1 < [\langle x \rangle, G] < P$, so $[\langle x \rangle, G] \leq K$. It follows that $K < M = \langle x\rangle K = \langle x\rangle[\langle x\rangle, G]K = \langle x\rangle^G K \le P$, since K is the unique normal subgroup of G such that P/K is a cheif factor of G, we have $M = P$, a contradiction.

This completes the proof. □

Theorem 4.1. *Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If every cyclic subgroup of every noncyclic Sylow subgroup of X with order p and 4 (if $p = 2$) is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$.*

Proof. We first prove that the theorem is true if $X = E$. Assume that the result is false. Let (G, E) be a minimal counterexample with $|G| + |E|$ minimal.

(1) E is soluble.

If E is not soluble, then $2||E|$. Suppose that E has a cyclic Sylow 2-subgroup, then E is 2-nilpotent and so E is soluble, a contradiction. This contradiction shows that every Sylow 2-subgroup of E is noncyclic. By the hypothesis and Corollary 3.1, we have E is 2-nilpotent and so E is soluble, a contradiction again. Thus, (1) holds.

(2) $G^{\mathfrak{F}} \le E$ and $G^{\mathfrak{F}}$ is a p-group.

Since $G/E \in \mathfrak{F}$, we have $G^{\mathfrak{F}} \le E$ and so $G^{\mathfrak{F}}$ is soluble by (1) . If $G^{\mathfrak{F}} \le \Phi(G)$, then $G^{\mathfrak{F}} \le M_G$ for every maximal subgroup M of G and so $G/M_G \in \mathfrak{F}$. Hence, $G^{\mathfrak{F}}$ is a p-group by [13, Theorem 3.4.2]. If $G^{\mathfrak{F}} \not\le \Phi(G)$, let K be a maximal subgroup of G such that $G^{\mathfrak{F}} \not\le K$. Then, $K/K \cap E \cong KE/E = G/E \in \mathfrak{F}$ and every cyclic subgroup of every noncyclic Sylow subgroup of $K \cap E$ with order p and 4 is an $IC\overline{s}$-subgroup of K by Lemma 2.3(1). Thus, $K \in \mathfrak{F}$. So $G^{\mathfrak{F}}$ is a p-group by [13, Theorem 3.4.2].

(3) The final contradiction.

If $G^{\mathfrak{F}}$ is noncyclic, then $G^{\mathfrak{F}} \le Z_{\mathfrak{U}}(G)$ by the hypothesis and Lemma 4.1. If $G^{\mathfrak{F}}$ is cyclic, obviously, $G^{\mathfrak{F}} \le Z_{\mathfrak{U}}(G)$. Hence, $G \in \mathfrak{F}$ by Lemma 2.9, a contradiction. This contradiction shows that the theorem holds if $X = E$.

Now, we prove that the theorem holds for $X = F^*(E)$.

We may obtain $F^*(E)$ is soluble by a similar discussion as in (1) and so $F(E) = F^*(E)$. Let P be a Sylow p-subgroup of $F(E)$, then $P \trianglelefteq G$. If P is noncyclic, then $P \le Z_{\mathfrak{U}}(G)$ by the hypothesis and Lemma 4.1. If P is cyclic, obviously, $P \le Z_{\mathfrak{U}}(G)$. It follows that $F(E) \le Z_{\mathfrak{U}}(G)$. Thus, we have $G \in \mathfrak{F}$ by Lemma 2.8 and Lemma 2.9.

This completes the proof. □

Lemma 4.2. *Let P be a normal p-subgroup of G. If every maximal subgroup of P is an $IC\overline{s}$-subgroup of G, then $P \le Z_{\mathfrak{U}}(G)$.*

Proof. Assume that the result is false. Let (G, P) be a minimal counterexample with $|G| + |P|$ minimal. Let N be a minimal normal subgroup of G such that $N \le P$, by Lemma 2.3(2), $(G/N, P/N)$ satisfies the hypothesis of the theorem and so $P/N \le Z_{\mathfrak{U}}(G/N)$. Obviously, $|N| \ne p$. Suppose that G has another minimal normal subgroup $K \leq P$, then $P/K \leq Z_{\mathfrak{U}}(G/K)$. It follows that $NK/K \leq Z_{\mathfrak{U}}(G/K)$ and so $|N| = p$, a contradiction. This contradiction shows that N is the unique minimal normal subgroup of G contained in P. By Lemma 2.10, we have $N \nleq \Phi(P)$, so $\Phi(P) = 1$, then there exists a subgroup T of P such that $P = N \rtimes T$. Let N_1 be a maximal subgroup of N and $N_1 \trianglelefteq P$. It is clear that $P_1 = N_1T$ is maximal in P, so $P_1 \cap [P_1, G] \leq (P_1)_{\overline{s}G}$. Since $P_1[P_1, G] = P_1^G \leq P$, we have $[P_1, G] \leq P$. If $[P_1, G] = 1$, then $P_1 \leq Z(G)$ and $P_1 \cap N \trianglelefteq G$. If $[P_1, G] \neq 1$, then $N \leq [P_1, G]$, we may obtain $P_1 \cap N \trianglelefteq G$ by a similar discussion as in Theorem 3.2(4), hence $N_1 = P_1 \cap N = 1$, that is, $|N| = p$, a contradiction.

This completes the proof. □

Theorem 4.2. *Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If every maximal subgroup of every Sylow subgroup of X is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$.*

Proof. We first prove that the theorem is true if $X = E$. Assume that the result is false. Let G be a minimal counterexample with minimal order.

(1) G has a minimal normal subgroup $N \leq P$ and N is an elementary abelian p-group, where $P \in Syl_p(E)$ and p is the largest prime divisor of $|E|$.

By the hypothesis and Lemma 2.3(1), every maximal subgroup of every Sylow subgroup of E is an $IC\overline{s}$-subgroup of E, so E is a Sylow tower group of supersolvable type by Corollary 3.2. Thus, P is a normal Sylow p-subgroup of E, where p is the largest prime divisor of E, and so $P \trianglelefteq G$. Hence, we have (1).

(2) $G/N \in \mathfrak{F}$ and $N = P$.

Since $(G/N)/(E/N) \cong G/E \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of E/N is an $IC\overline{s}$-subgroup of G/N by Lemma 2.3(2)(3), we have $G/N \in \mathfrak{F}$. The solvability of E implies that $F(E) = N$ and $C_E(N) \leq F(E)$ by [13, Theorem 1.8.17 and Theorem 1.8.18], so $C_E(N) = N = F(E)$ as N is elementary abelian. Clearly, $P \leq F(E)$, hence $N = P$.

(3) The final contradiction.

By the hypothesis and Lemma 4.2, $P \leq Z_{\mathfrak{U}}(G)$. By (2), $G/P \in \mathfrak{F}$. Hence, $G \in \mathfrak{F}$ by Lemma 2.9, a contradiction. This contradiction shows that the theorem holds if $X = E$.

Now, we prove that the theorem holds for $X = F^*(E)$.

Assume that $F^*(E)$ is not soluble, then $2||F^*(E)|$. By the hypothesis and Theorem 3.2, we have $F^*(E)$ is 2-nilpotent and so $F^*(E)$ is soluble, a contradiction. This contradiction shows that $F^*(E)$ is soluble. Hence, $F(E) = F^*(E)$. Let Q be a Sylow q-subgroup of $F(E)$, then $Q \trianglelefteq G$. By the hypothesis and Lemma 4.2, $Q \leq Z_{\mathfrak{U}}(G)$. It follows that $F(E) \leq Z_{\mathfrak{U}}(G)$. Thus, we have $G \in \mathfrak{F}$ by Lemma 2.8 and Lemma 2.9.

This completes the proof. □

5. Conclusion

An interesting question in finite group theory is to determine the influence of the embedding properties of members of some distinguished families of subgroups on the structure of the group. The present paper adds some results to this line of research. We proved that a group G is p-nilpotent if every cyclic subgroup of G with order p and 4 (if $p = 2$) or every maximal subgroup of P is an $IC\overline{s}$-subgroup of G, where $P \in Syl_p(G)$ and p is the smallest prime divisor of $|G|$. We also proved the following results: Let $\mathfrak{F}$ be a solubly saturated formation containing $\mathfrak{U}$ and let E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that $X = E$ or $X = F^*(E)$. If every cyclic subgroup of every noncyclic Sylow subgroup of X with order p and 4 (if $p = 2$) or every maximal subgroup of every Sylow subgroup of X is an $IC\overline{s}$-subgroup of G, then $G \in \mathfrak{F}$. The $IC\overline{s}$-subgroup in Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 4.1 and Theorem 4.2 can be replaced by s-semipermutable subgroup and the results are also true. Hence, we generalize some results on the basis of s-semipermutability.

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