A compression of finite topological spaces based on homomorphism

Liqin He
Department of General Education
Zhangzhou Institute of Technology
Zhangzhou, Fujian 363000
China
184499750@qq.com

Yidong Lin
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, 363000
China
yidong_lin@yeah.net

Jinjin Li*
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, 363000
China
jinjinlimnu@126.com

Abstract. The task of compressing large topologies into more manageable, smaller ones while leveraging the power of homomorphisms is a pivotal concern. Therefore, this paper focuses on the compression of topologies through the utilization of homomorphisms. In this paper, we devise a subbase-consistent function tailored specifically for topological spaces. To preserve certain properties of the original topological spaces, the concept of topological homomorphism is introduced. Additionally, we delve into the subbase reduction under homomorphism and subsequently construct the corresponding discernibility matrix. An experiment is conducted to validate the feasibility and effectiveness of our approach.

Keywords: discernibility matrix, homomorphism, subbase reduction, topology.
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1. Introduction

As a fundamental mathematical discipline, topology theory has many practical applications, such as computer science [15, 19, 20], computational geometry [16], graph theory [4, 23], structural biology [12], chemistry [21], physics [7], data mining [1, 9] and rough sets [8, 13, 14, 22, 27, 28]. For a finite topological space, the base and subbase contribute to improve the efficiency of data analysis.

*. Correspond author

Communication between data spaces is a very important topic in the field of artificial intelligence [18, 6, 3, 17]. The primary motivation for examining the communication between data spaces lies in the pursuit of identifying a relatively smaller database, a quest that often leads to techniques like data fusion and compression. These can be explained as a mapping between data spaces in mathematics. Grzymala-Busse first investigated the methodology of homomorphism between information systems [5], and he then considered the conditions that make the data space be selective under endomorphism [6]. A homomorphism serves as a valuable tool for aggregating sets of objects, attributes and descriptors from the original space, an it can be perceived as a distinctive form of communication between various data spaces. According to [3, 17], communications can be elucidated as the process of translating the information embedded in one granular realm into another, bridging the gap between disparate granular worlds. Li et al. [10] investigated homomorphism between information systems. Wang et al. [24, 25, 26] delved into the study of generalized information systems, fuzzy information systems and decision systems, employing the concept of homomorphism as a key analytical tool. Therefore, we will utilize the concept of homomorphism to delve into the comprehensive study of topological spaces.

The discernibility matrix holds a crucial role in the realm of knowledge reductions. Each element within this matrix represents the ensemble of attributes that uniquely differentiate a pair of corresponding objects. In simpler terms, it comprises the set of attributes on which the two objects exhibit contrasting values. Li and Zhang [11] first introduced discernibility matrix into topological space. Each entry within this matrix comprises a collection of topologies, uniquely identifying a pair of points. Furthermore, they delved into the subject of subbase reduction in families of subbases, emphasizing its role in preserving the topological rough membership function.

In this paper, our attention lies in exploring the to the reduction of subbases derived from homomorphism. The remainder of this paper is organized as follows. In Section 2, we provide a review the relevant concepts of topology theory, serving as a foundation for the subsequent discussions. In Section 3, the definition of subbase consistent function in topological space is proposed and its properties are delved into. In Section 4, a subbase discernibility matrix is constructed based on homomorphism to find reduction of subbases. Simultaneously, judgement theorem of reducts is put forward. we introduce a graph approach that facilitates the identification of a reduct of $f-$induced topological space in Section 5. Section 6 concludes this paper with a summary and discussion of our further work.

2. Preliminaries

For each i=1, . . . , n, let τ_i be a topology on X with subbase γ_i, $\mathbf{A}$=$\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, and $\gamma_{\mathbf{A}} = \bigwedge_{i=1}^{n} \gamma_i = \{\bigcap_{i=1}^{n} K_i : K_i \in \gamma_i, i = 1, \ldots, n\}$. Then, $\gamma_{\mathbf{A}}$ is a subbase for a topology on X, and $\bigcap(\gamma_{\mathbf{A}})_x = \bigcap_{i=1}^{n}(\bigcap(\gamma_i)_x)$, for $x \in X$. For $\mathbf{B} \subseteq \mathbf{A}$ and

$\gamma \in \mathbf{A}$, denote $(x)_{\mathbf{B}} = \bigcap(\gamma_{\mathbf{B}})_x$, simply denote $(x)_{\{\gamma\}}$ by $(x)_\gamma$. For any $x, y \in X$, we have the following properties[11]:

1. If $y \in (x)_{\mathbf{B}}$, then $(y)_{\mathbf{B}} \subseteq (x)_{\mathbf{B}}$.
2. If $y \notin (x)_{\mathbf{B}}$, there is $\gamma \in \mathbf{B}$ satisfying $y \notin (x)_\gamma$.
3. If $(y)_{\mathbf{B}} = (x)_{\mathbf{B}}$, then $(y)_\gamma = (x)_\gamma$, for each $\gamma \in \mathbf{B}$.

Let γ be a subbase for topology τ on X. $K \in \gamma$ is deemed reducible of γ if the set $\gamma - \{K\}$ is a subbase for τ. In essence, a reducible element K satisfies the condition that there exists $\gamma' \subseteq \gamma - \{K\}$ such that, for any $x \in K$, we have $x \in \bigcap \gamma' \subseteq K$. If $\alpha \subseteq \gamma$ is a subbase for τ and any proper subset of α is not a subbase for τ, then α is said to be a reduct of γ, which is sometimes also called a minimal subbase for τ. Alternatively, if $\alpha \subseteq \gamma$ is a reduct of γ, then $(x)_\alpha = (x)_\gamma$, for any $x \in X$. As is well-known, a subbase uniquely determines a topology on X and induces a basis for that topology. If an element $K \in \gamma$ is the union of some elements from γ, then K is called a union reducible element of γ; Conversely, if $K \in \gamma$ is the intersection of some members of γ, then K is called an intersection reducible element of γ. To get a reduct of γ, one can simply eliminate the reducible elements from γ in a prescribed order. (see [11])

Let $\mathbf{A}$ denote a family of subbases representing distinct topologies on X. $\mathbf{B} \subseteq \mathbf{A}$ is called a subbase consistent set if $(x)_{\mathbf{A}} = (x)_{\mathbf{B}}$, for every $x \in X$. If $\mathbf{B}$ is a subbase consistent set and no proper subset of $\mathbf{B}$ satisfies this consistency property, then $\mathbf{B}$ is called a reduct of $\mathbf{A}$. To facilitate the identification of all reducts of $\mathbf{A}$, we introduce the concept of a subbase discernibility matrix, which is defined as follows:[11]:

$$d((x)_{\mathbf{A}}, (y)_{\mathbf{A}}) = \begin{cases} \{\gamma \in \mathbf{A} : y \notin (x)_\gamma\}, & y \notin (x)_{\mathbf{A}}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

is called subbase discernibility set for $(x)_{\mathbf{A}}$ and $(y)_{\mathbf{A}}$ and $D = (d((x)_{\mathbf{A}}, (y)_{\mathbf{A}}) : x, y \in X)$ is called subbase discernibility matrix.

3. Subbase by consistent function

Definition 3.1. *Let X be a finite topological space with subbase $\gamma=\{K_1, \ldots, K_t\}$. A surjective map $f : X \to Y$ satisfies the following property: If for any $x, z \in X$, the condition $f(x) = f(z)$ implies that $(x)_\gamma = (z)_\gamma$, then f is called a subbase consistent function with respect to γ; Furthermore, given a subset $U \subseteq X$, if, for any $x, z \in X$ such that $f(x) = f(z)$ and $x \in U$, it follows that $z \in U$, then f is called a subbase consistent function with respect to U.*

Generally, when f is a subbase consistent function, for any $x \in X$, there exists $y \in Y$ satisfying $y = f(x')$, for each $x' \in (x)_\gamma \setminus \bigcup_{z \in (x)_\gamma \wedge (z)_\gamma \subset (x)_\gamma} (z)_\gamma$. For brevity, we denote this as $y = f((x)_\gamma \setminus \bigcup_{z \in (x)_\gamma \wedge (z)_\gamma \subset (x)_\gamma} (z)_\gamma)$. From Definition 3.1, it is easy to see that f is a subbase consistent function with respect to the subbase γ if and only if f is a subbase consistent function with respect to each $K \in \gamma$. A subbase consistent function is a specific type of mapping between two topological spaces, and this class of functions has the potential to compress a topological space into a relatively smaller one.

Example 1. Let $X = \{1, 2, \ldots, 9\}$ be a finite topological space with subbase $\gamma = \{\{1, 2, 4, 8\}, \{3, 4, 6, 7, 8, 9\}, \{3, 5, 7\}\}$. Then, $(1)_\gamma = (2)_\gamma = \{1, 2, 4, 8\}$, $(3)_\gamma = (7)_\gamma = \{3, 7\}$, $(4)_\gamma = (8)_\gamma = \{4, 8\}$, $(5)_\gamma = \{3, 5, 7\}$, $(6)_\gamma = (9)_\gamma = \{3, 4, 6, 7, 8, 9\}$. By Definition 3.1, we define a subbase consistent function $f : X \to Y$ as follows:

$$y_1{=}f(1){=}f(2), y_2{=}f(3){=}f(7), y_3{=}f(4){=}f(8), y_4{=}f(6){=}f(9), y_5{=}f(5).$$

Then, $f(\gamma) = \{\{y_1, y_3\}, \{y_2, y_3, y_4\}, \{y_2, y_5\}\}$.

In the following, we will delve into the examination of several crucial properties exhibited by the subbase consistent function.

Theorem 1. *Let γ be a subbase for X, f is a subbase consistent function with respect to γ iff $f^{-1}(f(K)) = K$, for any $K \in \gamma$.*

Proof. $\Rightarrow$ Since the inclusion $K \subseteq f^{-1}(f(K))$ is always valid, we only have to prove the reverse inclusion. For any $x \in f^{-1}(f(K))$, it follows that $f(x) \in f(K)$. Consequently, there exists $z \in K$ such that $f(x) = f(z)$. By the definition of $(z)_\gamma$, we know that $(z)_\gamma \subseteq K$. Since f is subbase consistent with respect to γ, it follows that $(x)_\gamma = (z)_\gamma$. This, in turn, implies that $(x)_\gamma \subseteq K$, thereby establishing that $x \in K$.

$\Leftarrow$ To arrive at a contradiction, suppose there exist $x, z \in X$ with $f(x) = f(z)$ satisfying $(x)_\gamma \neq (z)_\gamma$, then $z \in f^{-1}(f((x)_\gamma))$. Without loss of generality, suppose that there exists $\Delta \subseteq \{1, 2, \ldots, t\}$ such that $(x)_\gamma = \bigcap_{j\in\Delta} K_j$. Let $K_j = (x)_\gamma \cup (K_j \backslash (x)_\gamma)$, then $K_j = f^{-1}(f(K_j)) = f^{-1}(f((x)_\gamma \cup (K_j \backslash (x)_\gamma))) = f^{-1}(f((x)_\gamma)) \cup f^{-1}(f((K_j \backslash (x)_\gamma)))$, hence $f^{-1}(f((x)_\gamma)) \subseteq K_j$. This implies $z \in f^{-1}(f((x)_\gamma)) \subseteq \bigcap_{j\in\Delta} K_j = (x)_\gamma$, that is, $(z)_\gamma \subseteq (x)_\gamma$. Similarly, we can deduce $(x)_\gamma \subseteq (z)_\gamma$. This forces that $(x)_\gamma = (z)_\gamma$, a contradiction. Therefore, f is a subbase consistent function with respect to γ. □

Theorem 2. *Let γ be a subbase, for X. For any $K \in \gamma$, if f is subbase consistent with respect to γ, then $f((x)_\gamma) = \bigcap\{f(K) : K \in \gamma \wedge x \in K\}$.*

Proof. We only need to show that $f(K_i \cap K_j) = f(K_i) \cap f(K_j)$, for any $K_i, K_j \in \gamma$. It is straightforward to verify that $f(K_i \cap K_j) \subseteq f(K_i) \cap f(K_j)$. We now proceed to prove the converse, starting with the case where $K_i \cap K_j = \emptyset$. For contradiction, suppose $f(K_i) \cap f(K_j) = \emptyset$. Let y be an element in this intersection, i.e., $y \in f(K_i) \cap f(K_j)$. Then, there are $x \in K_i$ and $z \in K_j$ such that $f(x) = y = f(z)$, which implies $z \in K_i$. Thus, $z \in K_i \cap K_j$, a contradiction. Next, we consider the case where $K_i \cap K_j \neq \emptyset$. To show

$f(K_i) \cap f(K_j) \subseteq f(K_i \cap K_j)$, let $y \in f(K_i) \cap f(K_j)$. There exist $x \in K_i$ and $z \in K_j$ such that $f(x) = f(z) = y$. By the properties of f and the fact that $K_i \cap K_j$ is non-empty, we can infer that $\{x, z\} \subseteq K_i \cap K_j$. This, in turn, implies $y \in f(K_i \cap K_j)$. □

Corollary 3.1. *For any $K_i, K_j \in \gamma$, if f is subbase consistent with respect to K_i or K_j, then $f(K_i \cap K_j) = f(K_i) \cap f(K_j)$.*

Corollary 3.2. *f is a subbase consistent function with respect to γ iff*

$$f^{-1}(f((x)_\gamma)) = (x)_\gamma,$$

for each $x \in X$.

Lemma 3.1. *Let (X, τ) be a finite topological space endowed with subbase $\gamma = \{K_1, \ldots, K_t\}$. Consider a function $f : X \to Y$ that is subbase-consistent with respect to γ. We define the set $f(\gamma) = \{f(K_i) : i \leq t\}$ and $f(\tau) = \{f(K) : K \in \tau\}$. Then, $f(\gamma)$ constitutes a subbase, for $f(\tau)$ on Y.*

Corollary 3.3. (1) *$f((x)_\gamma) = (f(x))_{f(\gamma)}$, for any $x \in X$;* (2) *$f((\gamma)) = (f(\gamma))$.*

We may pose a question regarding the interplay between the reducible elements of γ and $f(\gamma)$. Specifically, can the reduct of γ be converted into the reduct of $f(\gamma)$? We shall proceed with our investigation to address these questions.

Theorem 3. *Let γ be a subbase, for X and f be a subbase consistent function. If $K \in \gamma$ is a reducible element of γ, then $f(K)$ is a reducible element of $f(\gamma)$. If $\alpha \subseteq \gamma$ is a reduct of γ, then $f(\alpha)$ is a reduct of $f(\gamma)$.*

Proof. We will initially establish that, for any reducible element K of γ, its image $f(K)$ under the function f is also a reducible element of $f(\gamma)$. Specifically, if K is a union-reducible element of γ, there exists a collection $\{K_i : i \in \Delta\}$ with $\Delta \subset \{1, 2, \ldots, t\}$ and $K \notin \{K_i : i \in \Delta\}$ such that $K = \bigcup_{i \in \Delta} K_i$. On the other hand, if K is an intersection-reducible element, then exists a distinct collection $\{K_j : j \in \Delta'\}$ with $\Delta' \subset \{1, 2, \ldots, t\}$ and $K \notin \{K_j : j \in \Delta'\}$ such that $K = \bigcap_{j \in \Delta'} K_j$. It follows from Corollary 3.5 that $f(K) = \bigcap_{j \in \Delta'} f(K_j)$.

Next, we aim to demonstrate that if $\alpha \subseteq \gamma$ is a reduct of γ, then $f(\alpha)$ is also a reduct of $f(\gamma)$. Obviously, $(x)_\alpha = (x)_\gamma$, for any $x \in X$. And there is $y \in Y$ such that $f(x) = y$. Then, $(y)_{f(\alpha)} = \bigcap\{f(K) : f(x) \in f(K) \wedge f(K) \in f(\alpha)\} = f((x)_\alpha) = f((x)_\gamma) = (f(x))_{f(\gamma)} = (y)_{f(\gamma)}$. This shows that $f(\alpha)$ forms a subbase, for $f(\tau)$ on Y. As a matter of fact, any proper subset of α is not a subbase, for τ, indicating that none of the members of α are reducible elements of α. Consequently, the members of $f(\alpha)$ are also not reducible elements of $f(\alpha)$. Therefore, we conclude that $f(\alpha)$ is indeed a reduct of $f(\gamma)$. □

Theorem 4. *Let (X, τ) be a finite topological space with subbase $\gamma=\{K_1, \ldots, K_t\}$ and $f : X \to Y$ is a subbase consistent function with respect to γ. Then, the following statements are equivalent:*

1. *K is a reducible element of γ.*
2. *For $y \in f(K)$, there is $\gamma' \subseteq \gamma \backslash \{K\}$ such that $y \in \bigcap f(\gamma') \subseteq f(K)$.*

Proof. (1) $\Rightarrow$ (2) If K is a reducible element of γ, then $(y)_{f(\gamma)\backslash\{f(K)\}} = (y)_{f(\gamma)}$, for any $y \in Y$.

(2) $\Rightarrow$ (1) It is easy to verify that $f(\gamma)\backslash\{f(K)\}$ constitutes a covering of Y. For any $y \in Y$, $(y)_{f(\gamma)\backslash\{f(K)\}} = (y)_{f(\gamma)}$ if $y \notin f(K)$; However, if $y \in f(K)$, there exists a subset $\gamma' \subseteq \gamma\backslash\{K\}$ such that $y \in \bigcap f(\gamma') \subseteq f(K)$. Consequently, $(y)_{f(\gamma)} = (y)_{f(\gamma)\backslash\{f(K)\}} \cap f(K) \supseteq (y)_{f(\gamma)\backslash\{f(K)\}} \cap (\bigcap f(\gamma')) \supseteq (y)_{f(\gamma)\backslash\{f(K)\}}$. This equality ensures that $(y)_{f(\gamma)} = (y)_{f(\gamma)\backslash\{f(K)\}}$, demonstrating that $f(\gamma)\backslash\{f(K)\}$ forms a subbase for $f(\tau)$ on Y, then $f(K)$ is a reducible element of $f(\gamma)$. Next, we shall demonstrate that K is a reducible element of γ. There exists a subset $\Delta \subseteq \{1, 2, \ldots, t\}$ satisfying $f(K) = \bigcap_{i\in\Delta} f(K_i)$ or $f(K) = \bigcup_{i\in\Delta} f(K_i)$. Since f is subbase consistent with respect to γ, we have $f^{-1}(f(K_i)) = K_i$, for any $K_i \in \gamma$ by Theorem 3.1. Hence, $K = f^{-1}(f(K)) = f^{-1}(\bigcap_{i\in\Delta} f(K_i)) = \bigcap_{i\in\Delta} K_i$ if $f(K)$ is an intersection reducible element. Conversely, $K = \bigcup_{i\in\Delta} K_i$ if $f(K)$ is a union reducible element. These deductions indicate that K is indeed a reducible element of γ. □

From the preceding discussion, we observe that the subbase consistent function possesses numerous advantageous properties that lend themselves to defining the concepts of homomorphisms in the following.

Definition 3.2. *Let (X, τ) be a finite topological space equipped with subbase $\gamma = \{K_1, \ldots, K_t\}$. If a function $f : X \to Y$ is a subbase consistent with respect to γ, then f is designated as a homomorphism from the triplet (X, τ, γ) to the $f-$induced topological space $(Y, f(\tau), f(\gamma))$ of (X, τ, γ), and $f(\gamma)$ is called an $f-$induced subbase.*

Utilizing the aforementioned results, the identification of a many-to-one homomorphism allows for the compression of a space into a relatively smaller size. Addressing the challenges posed by mathematical models and theories in data spaces, this compression process simplifies and abstracts the data space while preserving the topological relationships between its objects.

4. Reduction of a family of topologies based on homomorphism

In this section, we define $\mathbf{A} = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and $\Gamma = \{\tau_1, \ldots, \tau_n\}$, where each τ_i represents a topology on the finite point set X with subbase γ_i. We refer to the family $\mathbf{A}$ of topologies on X paired with Γ as $(X, \Gamma, \mathbf{A})$ and abbreviate it as FTS for brevity.

Definition 4.1. *Let $f : X \to Y$ be a surjective mapping. If f is subbase consistent with respect to each $\gamma_i \in \mathbf{A}$, then f is called a homomorphism on $(X, \Gamma, \mathbf{A})$.*

Denote $f(\mathbf{A}) = \{f(\gamma_1), \ldots, f(\gamma_n)\}$ and $f(\Gamma) = \{f(\tau_1), \ldots, f(\tau_n)\}$. Obviously, for any $y \in Y$, there exists $x \in X$ such that

$$y = f((x)_{\mathbf{A}} \setminus \bigcup_{z \in (x)_{\mathbf{A}} \wedge (z)_{\mathbf{A}} \subset (x)_{\mathbf{A}}} (z)_{\mathbf{A}}).$$

Definition 4.2. *Let $f : X \to Y$ be a homomorphism on $(X, \Gamma, \mathbf{A})$, then $(Y, f(\Gamma), f(\mathbf{A}))$ is called the $f-$induced FTS of $(X, \Gamma, \mathbf{A})$.*

Example 2. Let $(X, \Gamma, \mathbf{A})$ be FTS with X=$\{1, 2, \ldots, 20\}$ and $\mathbf{A}$=$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\gamma_1 = \{\{2, 6, 7, 8, 10, 15, 16, 18\}, \{4, 6, 8, 9, 10, 13, 14, 18, 19\}, \{1, 3, 5, 11, 12, 17, 20\}\},$$
$$\gamma_2 = \{\{1, 2, 6, 7, 8, 10, 11, 12, 15, 16, 18, 20\}, \{2, 7, 15, 16\}, \{3, 4, 5, 9, 13, 14, 17, 19\}\},$$
$$\gamma_3 = \{\{1, 11, 12, 20\}, \{2, 3, 5, 7, 15, 16, 17\}, \{3, 4, 5, 9, 13, 14, 17, 19\}, \{2, 3, 5, 6, 7, 8, 10, 15, 16, 17, 18\}\},$$
$$\gamma_4 = \{\{2, 3, 5, 7, 15, 16, 17\}, \{3, 4, 5, 9, 13, 14, 17, 19\}, \{1, 3, 5, 6, 8, 10, 11, 12, 17, 18, 20\}\}.$$

Hence,

$(1)_{\mathbf{A}} = (11)_{\mathbf{A}} = (12)_{\mathbf{A}} = (20)_{\mathbf{A}} = \{1, 11, 12, 20\}$,
$(2)_{\mathbf{A}} = (7)_{\mathbf{A}} = (15)_{\mathbf{A}} = (16)_{\mathbf{A}} = \{2, 7, 15, 16\}$,
$(3)_{\mathbf{A}} = (5)_{\mathbf{A}} = (17)_{\mathbf{A}} = \{3, 5, 17\}$,
$(4)_{\mathbf{A}} = (9)_{\mathbf{A}} = (13)_{\mathbf{A}} = (14)_{\mathbf{A}} = \{4, 9, 13, 14\}$,
$(6)_{\mathbf{A}} = (8)_{\mathbf{A}} = (10)_{\mathbf{A}} = (18)_{\mathbf{A}} = \{6, 8, 10, 18\}$.

Next, we define a subbase consistent function $f : X \to Y$ as follows:

$$f((1)_{\mathbf{A}}) = y_1, f((2)_{\mathbf{A}}) = y_2, f((3)_{\mathbf{A}}) = y_3, f((4)_{\mathbf{A}}) = y_4, f((6)_{\mathbf{A}}) = y_5.$$

Then, $f(\mathbf{A}) = \{f(\gamma_1), f(\gamma_2), f(\gamma_3), f(\gamma_4)\}$, where

$$f(\gamma_1)=\{\{y_2, y_5\}, \{y_4, y_5\}, \{y_1, y_3\}\}, f(\gamma_3)=\{\{y_1\}, \{y_2, y_3\}, \{y_3, y_4\}, \{y_2, y_3, y_5\}\},$$
$$f(\gamma_2) = \{\{y_1, y_2, y_5\}, \{y_2\}, \{y_3, y_4\}\}, f(\gamma_4) = \{\{y_2, y_3\}, \{y_3, y_4\}, \{y_1, y_3, y_5\}\}.$$

The proposed definitions introduce a methodology for compressing a family of topologies into a smaller-sized representation. Subsequently, we shall delve into the equivalent formulations of subbase consistent functions.

Lemma 4.1. *Let $f : X \to Y$ be a surjective mapping and let γ_1, γ_2 be subbases for different topologies on X. If f is subbase consistent function with respect to γ_1 and γ_2 respectively, then*

1. *f is a subbase consistent with respect to $\gamma_1 \cup \gamma_2$.*
2. *f is a subbase consistent with respect to $\gamma_1 \cap \gamma_2$.*

Proof. We only prove (1). We observe that, for any $x \in X$ $(x)_{\gamma_1 \cup \gamma_2} = \bigcap\{K \in \gamma_1 \cup \gamma_2 : x \in K\}$. This can be expressed as $(x)_{\gamma_1 \cup \gamma_2} = (x)_{\gamma_1} \cap (x)_{\gamma_2}$. Since f is subbase consistent with respect to γ_1 and γ_2 respectively, it follows that $(x)_{\gamma_i} = (z)_{\gamma_i}$ $(i = 1, 2)$, for any $x, z \in X$ such that $f(x) = f(z)$. Therefore, $(x)_{\gamma_1} \cap (x)_{\gamma_2} = (z)_{\gamma_1} \cap (z)_{\gamma_2}$, which implies $(x)_{\gamma_1 \cup \gamma_2} = (z)_{\gamma_1 \cup \gamma_2}$. Consequently, f is a subbase consistent with respect to $\gamma_1 \cup \gamma_2$. □

Lemma 4.2. *Let $f : X \to Y$ and γ_1, γ_2 be subbases for topologies on X. If f is a subbase consistent with respect to γ_1 or γ_2, then $f((x)_{\gamma_1} \cap (x)_{\gamma_2}) = f((x)_{\gamma_1}) \cap f((x)_{\gamma_2})$, for any $x \in X$.*

Proof. We only need to prove $f((x)_{\gamma_1}) \cap f((x)_{\gamma_2}) \subseteq f((x)_{\gamma_1} \cap (x)_{\gamma_2})$. Without loss of generality, we assume that the function f is subbase consistent with respect to γ_1. Let $y \in f((x)_{\gamma_1}) \cap f((x)_{\gamma_2})$. This means that there exist $z_1 \in (x)_{\gamma_1}$ and $z_2 \in (x)_{\gamma_2}$ such that $f(z_1) = f(z_2) = y$, which implies that $(z_1)_{\gamma_1} \subseteq (x)_{\gamma_1}$ and $(z_1)_{\gamma_1} = (z_2)_{\gamma_1}$. Therefore, $(z_2)_{\gamma_1} \subseteq (x)_{\gamma_1}$. It follows that $z_2 \in (x)_{\gamma_1} \cap (x)_{\gamma_2}$. This implies $y \in f((x)_{\gamma_1} \cap (x)_{\gamma_2})$. Consequently, $f((x)_{\gamma_1}) \cap f((x)_{\gamma_2}) \subseteq f((x)_{\gamma_1} \cap (x)_{\gamma_2})$. □

Thus, if f is homomorphism on $(X, \Gamma, \mathbf{A})$, then $f((x)_{\mathbf{B}}) = \bigcap_{\gamma \in \mathbf{B}} f((x)_\gamma)$ with each $x \in X$, for any $\mathbf{B} \subseteq \mathbf{A}$. Moreover, $f(\bigcap_{\gamma \in \mathbf{A}} \gamma) = \bigcap_{\gamma \in \mathbf{A}} f(\gamma)$.

Theorem 5. *If $f : X \to Y$ is a homomorphism on $(X, \Gamma, \mathbf{A})$, then $\mathbf{B} \subseteq \mathbf{A}$ is a subbase consistent set of $\mathbf{A}$ if and only if $f(\mathbf{B})$ is a subbase consistent set of $f(\mathbf{A})$.*

Proof. ⇒ For each $x \in X$, $f((x)_{\mathbf{B}}) = f((x)_{\mathbf{A}})$ since $(x)_{\mathbf{B}} = (x)_{\mathbf{A}}$. That is, $(y)_{f(\mathbf{B})} = (y)_{f(\mathbf{A})}$, for any $y \in Y$, hence $f(\mathbf{B})$ is a subbase consistent of $f(\mathbf{A})$.

⇐ For any $y \in Y$, $f^{-1}((y)_{f(\mathbf{A})}) = \bigcap_{\gamma \in \mathbf{A}} f^{-1}((y)_{f(\gamma)}) = \bigcap_{\gamma \in \mathbf{A}} \bigcap_{y \in f(K) \in f(\gamma)} K$. By applying this to the definition of $(x)_{\mathbf{A}}$, for any $x \in X$, we obtain $(x)_{\mathbf{A}} = \bigcap_{y \in f((x)_{\mathbf{A}})} (\bigcap_{\gamma \in \mathbf{A}} \bigcap_{y \in f(K) \in f(\gamma)} K) = \bigcap_{y \in f((x)_{\mathbf{A}})} f^{-1}((y)_{f(\mathbf{A})})$.

Analogously, $(x)_{\mathbf{B}} = \bigcap_{y \in f((x)_{\mathbf{B}})} f^{-1}((y)_{f(\mathbf{B})})$, for each $x \in X$. Since $(y)_{f(\mathbf{B})} = (y)_{f(\mathbf{A})}$ holds, for each $y \in Y$, it follows that $(x)_{\mathbf{B}} = (x)_{\mathbf{A}}$, for all $x \in X$. This demonstrates that $\mathbf{B}$ is a subbase consistent set of $\mathbf{A}$. □

Theorem 6. *Let $f : X \to Y$ be a homomorphism on $(X, \Gamma, \mathbf{A})$ and $(Y, f(\Gamma), f(\mathbf{A}))$ be the $f-$induced FTS. For any $\mathbf{B} \subseteq \mathbf{A}$, $\mathbf{B}$ is a reduct of $\mathbf{A}$ if and only if $f(\mathbf{B})$ is a reduct of $f(\mathbf{A})$.*

Proof. ⇒ Assume that there is a $\gamma \in \mathbf{B}$ such that $(y)_{f(\mathbf{B}) \setminus \{f(\gamma)\}} = (y)_{f(\mathbf{A})}$, for every $y \in Y$, i.e., $(y)_{f(\mathbf{B} \setminus \{\gamma\})} = (y)_{f(\mathbf{A})}$. This implies that $f(\mathbf{B} \setminus \{\gamma\})$ is a subbase consistent of $f(\mathbf{A})$. Invoking Theorem 4.6, we can see that $\mathbf{B} \setminus \{\gamma\}$ is a subbase consistent with $\mathbf{A}$. However, this conclusion contradicts the fact that $\mathbf{B}$ is a reduct of $\mathbf{A}$. This implies $f(\mathbf{B})$ is a reduct of $f(\mathbf{A})$.

⇐ Conversely, if $f(\mathbf{B})$ is a subbase consistent set of $f(\mathbf{A})$, then $\mathbf{B}$ must be also a subbase consistent set of $\mathbf{A}$. To illustrate this, suppose there exists $\gamma \in \mathbf{B}$

such that $(x)_{\mathbf{B}\backslash\{\gamma\}} = (x)_{\mathbf{A}}$, for any $x \in X$. Then, $\mathbf{B}\backslash\{\gamma\}$ is a subbase consistent set of $\mathbf{A}$. That is, $f(\mathbf{B}\backslash\{\gamma\})$ is a subbase consistent set of $f(\mathbf{A})$, this creates a contradiction. □

Definition 4.3. *Let $f : X \to Y$ be a homomorphism on $(X, \Gamma, \mathbf{A})$ and $(Y, f(\Gamma), f(\mathbf{A}))$ be the $f-$induced FTS. For any $y_i, y_j \in Y$, the subbase discernibility set related to y_i and y_j is defined by*

$$d(y_i, y_j) = \begin{cases} \{f(\gamma) : y_j \notin (y_i)_{f(\gamma)}, \gamma \in \mathbf{A}\}, & y_j \notin (y_i)_{f(\mathbf{A})}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

And $D = (d_{ij})_{|Y|\times|Y|}$ is called a subbase discernibility matrix, where $d_{ij} = d(y_i, y_j)$.

Corollary 4.1. *For any $x_i, x_j \in X$, $d((x_i)_{\mathbf{A}}, (x_j)_{\mathbf{A}})$ is the subbase discernibility set for $(x_i)_{\mathbf{A}}$ and $(x_j)_{\mathbf{A}}$, then*

1. *$d((x_i)_{\mathbf{A}}, (x_j)_{\mathbf{A}}) = \emptyset$ if and only if $d(f(x_i), f(x_j)) = \emptyset$.*
2. *$\gamma \in d((x_i)_{\mathbf{A}}, (x_j)_{\mathbf{A}})$ if and only if $f(\gamma) \in d(f(x_i), f(x_j))$.*

Theorem 7. *For $\mathbf{B} \subseteq \mathbf{A}$, $f(\mathbf{B})$ is a subbase consistent set iff for $d(y_i, y_j) \neq \emptyset$, $f(\mathbf{B}) \cap d(y_i, y_j) \neq \emptyset$.*

Proof. $\Rightarrow$. Suppose $f(\mathbf{B})$ is subbase consistent set for $\mathbf{B} \subseteq \mathbf{A}$. If $d(y_i, y_j) \neq \emptyset$, then $y_j \notin (y_i)_{f(\mathbf{A})}$. If there exists no $\gamma \in \mathbf{B}$ such that $y_j \notin (y_i)_{f(\gamma)}$, that is, $y_j \in (y_i)_{f(\gamma)}$ for any $\gamma \in \mathbf{B}$, then $y_j \in (y_i)_{f(\mathbf{B})} = (y_i)_{f(\mathbf{A})}$, a contradiction. Thus, there is $\gamma \in \mathbf{B}$ satisfying $y_j \notin (y_i)_{f(\gamma)}$, then $f(\gamma) \in f(\mathbf{B}) \cap d(y_i, y_j) \neq \emptyset$.

$\Leftarrow$. For each $y_i \in Y$, it holds that $y_i \in (y_i)_{f(\mathbf{B})}$. Suppose $(y_i)_{f(\mathbf{B})} \neq (y_i)_{f(\mathbf{A})}$. Then, there must exist some $y_j \in (y_i)_{f(\mathbf{B})}$ but $y_j \notin (y_i)_{f(\mathbf{A})}$. Since $d(y_i, y_j) \neq \emptyset$, it follows that $f(\mathbf{B}) \cap d(y_i, y_j) \neq \emptyset$. This means there is some $\gamma \in \mathbf{B}$ such that $f(\gamma) \in f(\mathbf{B}) \cap d(y_i, y_j)$, this implies $y_j \notin (y_i)_{f(\gamma)}$ and consequently, $y_j \notin (y_i)_{f(\mathbf{B})} \subseteq (y_i)_{f(\gamma)}$. This creates a contradiction. Therefore, we must have $(y_i)_{f(\mathbf{B})} = (y_i)_{f(\mathbf{A})}$, indicating that $f(\mathbf{B})$ is a subbase consistent set. □

Definition 4.4. *A subbase discernibility function F_Y of $(Y, f(\Gamma), f(\mathbf{A}))$ is defined by $F_Y(f(\gamma_1), \ldots, f(\gamma_n)) = \wedge\{\vee d(y_i, y_j) : y_i, y_j \in Y,\ d(y_i, y_j) \neq \emptyset\}$ with Boolean variables $f(\gamma_1), \ldots, f(\gamma_n)$, where $\wedge$ is conjunction operation and $\vee$ is disjunction operation. Each conjunction operator of the minimal disjunctive form is called a reduct of $f(\mathbf{A})$.*

Example 3. Let $X = \{1, 2, \ldots, 15\}$, $\mathbf{A} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where

$$\gamma_1 = \{\{1, 2, 3, 4, 5, 8, 10, 15\}, \{3, 5, 7, 11, 12\}, \{4, 6, 8, 9, 10, 13, 14\}\},$$
$$\gamma_2 = \{\{1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 15\}, \{3, 4, 5, 8, 10\}, \{3, 5, 6, 9, 13, 14\}\},$$
$$\gamma_3 = \{\{1, 2, 3, 4, 5, 8, 10, 15\}, \{6, 9, 13, 14\}, \{4, 7, 8, 10, 11, 12\}\},$$
$$\gamma_4 = \{\{1, 2, 3, 5, 15\}, \{6, 7, 9, 11, 12, 13, 14\}, \{3, 4, 5, 8, 10\}\}.$$

Then, we have $(1)_{\mathbf{A}} = (2)_{\mathbf{A}} = (15)_{\mathbf{A}} = \{1,2,3,5,15\}$, $(4)_{\mathbf{A}} = (8)_{\mathbf{A}} = (10)_{\mathbf{A}} = \{4,8,10\}$, $(6)_{\mathbf{A}} = (9)_{\mathbf{A}} = (13)_{\mathbf{A}} = (14)_{\mathbf{A}} = \{6,9,13,14\}$, $(7)_{\mathbf{A}} = (11)_{\mathbf{A}} = (12)_{\mathbf{A}} = \{7,11,12\}$, $(3)_{\mathbf{A}} = (5)_{\mathbf{A}} = \{3,5\}$. Let $Y = \{y_1,y_2,y_3,y_4,y_5\}$, define a homomorphism $f : X \to Y$ as follows: $f((1)_{\mathbf{A}}) = \{y_1,y_2\}, f((3)_{\mathbf{A}}) = \{y_2\}, f((4)_{\mathbf{A}}) = \{y_3\}, f((6)_{\mathbf{A}}) = \{y_4\}, f((7)_{\mathbf{A}}) = \{y_5\}$. Hence,

$$
\begin{aligned}
f(\gamma_1) &= \{\{y_1,y_2,y_3\},\{y_2,y_5\},\{y_3,y_4\}\},\\
f(\gamma_2) &= \{\{y_1,y_2,y_3,y_5\},\{y_2,y_3\},\{y_3,y_4\}\},\\
f(\gamma_3) &= \{\{y_1,y_2,y_3\},\{y_4\},\{y_3,y_5\}\},\\
f(\gamma_4) &= \{\{y_1,y_2\},\{y_2,y_3\},\{y_4,y_5\}\}.
\end{aligned}
$$

Then, $(y_1)_{f(\mathbf{A})} = \{y_1,y_2\}$, $(y_2)_{f(\mathbf{A})} = \{y_2\}$, $(y_3)_{f(\mathbf{A})} = \{y_3\}$, $(y_4)_{f(\mathbf{A})} = \{y_4\}$, $(y_5)_{f(\mathbf{A})} = \{y_5\}$. Thus, we can easy to obtain the discernibility matrix $D =$

$$
\begin{pmatrix}
\emptyset & \emptyset & \{f(\gamma_4)\} & f(\mathbf{A}) & \{f(\gamma_1),f(\gamma_3),f(\gamma_4)\}\\
\{f(\gamma_1),f(\gamma_2),f(\gamma_4)\} & \emptyset & \{f(\gamma_1),f(\gamma_4)\} & f(\mathbf{A}) & f(\mathbf{A})\\
f(\mathbf{A}) & \{f(\gamma_1),f(\gamma_2),f(\gamma_3)\} & \emptyset & f(\mathbf{A}) & f(\mathbf{A})\\
f(\mathbf{A}) & f(\mathbf{A}) & \{f(\gamma_3),f(\gamma_4)\} & \emptyset & \{f(\gamma_1),f(\gamma_2),f(\gamma_3)\}\\
\{f(\gamma_1),f(\gamma_3),f(\gamma_4)\} & \{f(\gamma_3),f(\gamma_4)\} & \{f(\gamma_1),f(\gamma_4)\} & \{f(\gamma_1),f(\gamma_2),f(\gamma_3)\} & \emptyset
\end{pmatrix},
$$

The discernibility subbase function

$$
\begin{aligned}
&F_Y(f(\gamma_1),f(\gamma_2),f(\gamma_3),f(\gamma_4)) = f(\mathbf{A}) \wedge f(\gamma_4) \wedge (f(\gamma_1) \vee f(\gamma_3) \vee f(\gamma_4)) \wedge (f(\gamma_1)\\
&\vee f(\gamma_2) \vee f(\gamma_4)) \wedge (f(\gamma_1) \vee f(\gamma_4)) \wedge (f(\gamma_1)\\
&\vee f(\gamma_2) \vee f(\gamma_3)) \wedge (f(\gamma_3) \vee f(\gamma_4))\\
&= f(\gamma_4) \wedge (f(\gamma_1) \vee f(\gamma_2) \vee f(\gamma_3))\\
&= (f(\gamma_1) \wedge f(\gamma_4)) \vee (f(\gamma_2) \wedge f(\gamma_4)) \vee (f(\gamma_3) \wedge f(\gamma_4)),
\end{aligned}
$$

then $\{f(\gamma_1),f(\gamma_4)\}$, $\{f(\gamma_2),f(\gamma_4)\}$ and $\{f(\gamma_3),f(\gamma_4)\}$ are the reducts of $f(\mathbf{A})$. In other words, $\{\gamma_1,\gamma_4\}$, $\{\gamma_2,\gamma_4\}$ and $\{\gamma_3,\gamma_4\}$ are the reducts of $\mathbf{A}$.

Obviously, if $(X,\Gamma,\mathbf{A})$ is a quasi-discrete FTS, then $(Y,f(\Gamma),f(\mathbf{A}))$ is $f-$induced discrete FTS. And for any $y_i,y_j \in Y$, $d(y_i,y_j) = \{f(\gamma) : \gamma \in \mathbf{A} \wedge y_j \notin (y_i)_{f(\gamma)}\}$.

Example 4. From Example 4.3, we can get the subbase discernibility matrix $D =$

$$
\begin{pmatrix}
\emptyset & \{f(\gamma_1),f(\gamma_3),f(\gamma_4)\} & \{f(\gamma_2),f(\gamma_3)\} & f(\mathbf{A}) & \{f(\gamma_1),f(\gamma_3)\}\\
f(\mathbf{A}) & \emptyset & \{f(\gamma_1),f(\gamma_2)\} & f(\mathbf{A}) & \{f(\gamma_2),f(\gamma_3),f(\gamma_4)\}\\
\{f(\gamma_2),f(\gamma_3),f(\gamma_4)\} & f(\mathbf{A}) & \emptyset & \{f(\gamma_1),f(\gamma_3),f(\gamma_4)\} & f(\mathbf{A})\\
f(\mathbf{A}) & f(\mathbf{A}) & \{f(\gamma_1)\} & \emptyset & \{f(\gamma_2),f(\gamma_3),f(\gamma_4)\}\\
\{f(\gamma_1),f(\gamma_3)\} & \{f(\gamma_1),f(\gamma_4)\} & \{f(\gamma_1),f(\gamma_2)\} & f(\mathbf{A}) & \emptyset
\end{pmatrix},
$$

then $F_Y(f(\gamma_1),f(\gamma_2),f(\gamma_3),f(\gamma_4)) = f(\gamma_1) \wedge (f(\gamma_2) \vee f(\gamma_3))$ and so $\{f(\gamma_1,f(\gamma_2))\}$ and $\{f(\gamma_1,f(\gamma_3))\}$ are all the subbase reducts of $f(\mathbf{A})$. Namely, $\{\gamma_1,\gamma_2\}$, $\{\gamma_1,\gamma_3\}$ are the subbase reducts of $\mathbf{A}$.

To summarize, when dealing with a given FTS, we can utilize the homomorphism technique to efficiently identify a relatively compact representation. Specifically, this approach allows us to swiftly perform equivalent reductions of subbases in a compressed image of the FTS, all without altering the topological structure of the original space.

5. Vertex cover method for the reduction of FTS

As is widely known, computing all possible reducts of subbases based on discernibility matrices or discernibility subbase functions is an NP-hard problem. However, it is often unnecessary to obtain all possible reducts.

Note that the discernibility matrix D can also be represented in set notation, denoted as $D = \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\}$. Recall that each subset Y' of $Y = \{y_1, y_2, \ldots, y_m\}$ can be expressed through a row vector $g(Y')$, where g is the characteristic function. For example, $Y' = \{y_1, y_3\}$ of $Y = \{y_1, y_2, y_3, y_4\}$ can be represented as $g(Y') = (1, 0, 1, 0)$. Consequently, any non-empty subbase discernibility set $d(y_i, y_j) \subseteq f(\mathbf{A})$ can be rewritten as $g(d(y_i, y_j))$. By selecting an ordering for $D = \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\}$, the subbase discernibility matrix can be represented as a Boolean matrix. We refer to this as the subbase discernibility Boolean matrix, denoted by $M(d(y_i, y_j) : (y_i, y_j) \in Y \times Y)$. In simpler terms, this Boolean matrix corresponds uniquely to the subbase discernibility function $F_Y(f(\gamma_1), \ldots, f(\gamma_t))$.

Example 5. According to Example 4.12, $\{f(\gamma_1), f(\gamma_3), f(\gamma_4)\}$, $\{f(\gamma_1), f(\gamma_2), f(\gamma_4)\}$, $\{f(\gamma_1), f(\gamma_4)\}$, $\{f(\gamma_1), f(\gamma_2), f(\gamma_3)\}$, $\{f(\gamma_3), f(\gamma_4)\}$, $\{f(\gamma_4)\}$, $f(\mathbf{A})$ are all the subbase discernibility sets. From the subbase discernibility matrix D, we can get M shown as follows:

$$M^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

In the following, we propose a method to derive a reduct.

Firstly, it has been suggested by [2] that the matrix M can be viewed as the incidence matrix of a hypergraph $\mathcal{G} = (f(\mathbf{A}), \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\})$, in which $d(y_i, y_j)$ is considered as the hyperedge, and $f(\mathbf{A})$ is the collection of vertices. Subsequently, the task of obtaining reduction of subbases can be reframed as the problem of identifying minimal vertex covers in this hypergraph.

Definition 5.1. *Let $(Y, f(\Gamma), f(\mathbf{A}))$ be $f-$induced FTS with subbase discernibility matrix D. For $d(y_i, y_j) \neq \emptyset$ and $d(y_i, y_j) \neq f(\mathbf{A})$, let $w(d(y_i, y_j))$ be equal to the repeated times of $d(y_i, y_j)$ in D. Then, for any $f(\gamma) \in f(\mathbf{A})$, the importance of $f(\gamma)$ is define by:*

$$sig(f(\gamma)) = \sum_{f(\gamma)\in d(y_i,y_j)\wedge(y_i,y_j)\in Y\times Y} w(d(y_i, y_j)).$$

Sometimes, we call $sig(f(\gamma))$ the weight of $f(\gamma)$.

In fact, $sig(f(\gamma))$ can be seen as the degree of vertex $f(\gamma)$ in the graph $\mathcal{G}$.

Secondly, the conjunction operation $\wedge$ satisfies absorption law, which states that $a \wedge (a \vee b) = a$. If there exist core elements within $f(\mathbf{A})$, meaning that at least one subbase discernibility set is a singleton, then any other discernibility sets that contain this core will be absorbed by the conjunction operation. In the graph representation, the edges corresponding to these core elements form cycles.

Thirdly, if there exists no circles in the graph, to get the minimal vertex covers, a greedy strategy can be employed to identify minimal vertex covers. This strategy involves selecting the vertex with the currently maximum degree.

Given a hypergraph $\mathcal{G} = (E, V)$ with vertices V and edges E, let M be its incidence matrix. For any $v \in V$ and $T \subset E$, we define $N(v, T) = \{e \in T : v \in e\}$ and $d(e, V) = \{v \in V : v \in e\}$. Let $M(v)$ be the column vector of M associated with vertex v and $M(N(v, T))$ represent the rows of M corresponding to the edges in $N(v, T)$. Based on the above analysis, we now outline an algorithm to achieve a reduction of subbases using the hypergraph representation:

Algorithm 1 reduct base weighted hypergraph(RWH)

RWH(S, A, δ) \\ $S = (X, A)$ is an information system, δ is threshold
Output: One reduct red
1: Generating the $f-$induced topology Y and $base_Y$
2: Generateing the subbase discernibility matrix D
3: Important vector $\mathbf{s} \leftarrow \emptyset$
4: **for** $a \in A$
5: $sig(f(\gamma_a)) \leftarrow \sum_{f(\gamma_a) \in d(y_i, y_j) \wedge (y_i, y_j) \in Y \times Y} w(d(y_i, y_j))$, $\mathbf{s} \leftarrow \mathbf{s} \cup \{sig(f(\gamma_a))\}$
6: **end for**
7: Generating hypergraph $\mathcal{G} = (V, E)$ and its incident matrix M, where $V = \{f(\gamma_a) : a \in A\}$ and $E = \{d(y_i, y_j) : (y_i, y_j) \in Y \times Y\}$
8: $red \leftarrow \emptyset$
9: **while** $M \neq \emptyset$
10: **for** $e \in E$
11: **if** $|d(e, V)| = 1$
12: $red \leftarrow red \cup \{f(\gamma_a) : f(\gamma_a) \in d(e, V)\}$
13: $M \leftarrow M - M(f(\gamma_a)) - M(N(f(\gamma_a), E))$ \\here M is incident matrix of sub-graph
14: $E \leftarrow E - N(f(\gamma_a), E)$, $A \leftarrow A - \{a\}$, $\mathbf{s} \leftarrow \mathbf{s} - sig(f(\gamma_a))$
15: **end if**
16: **end for**
17: **end while**
18: **while** $M \neq \emptyset$
19: Computing $\mathbf{s}' = \{sig'(f(\gamma_a)) : a \in A\}$
20: $B \leftarrow \{f(\gamma_a) : sig'(f(\gamma_a)) = max(\mathbf{s}')\}$
21: $red \leftarrow red \cup \{f(\gamma_a)\}$, where $f(\gamma_a) \in B$ such that $sig(f(\gamma_a)) = max(\mathbf{s})$
22: $M \leftarrow M - M(f(\gamma_a)) - M(N(f(\gamma_a), E))$, $E \leftarrow E - N(f(\gamma_a), E)$, $A \leftarrow A - \{a\}$ and $\mathbf{s} \leftarrow \mathbf{s} - sig(f(\gamma_a))$
23: **end while**

In Algorithm 1, both steps 1 and 2 can be done in $\mathbf{O}(|X|^2|A|)$, steps $4-6$ also require less than $\mathbf{O}(|X|^2|A|)$, while step 7, step $9-23$ can be finished in less than $\mathbf{O}(|X|^2)$. Therefore, the overall complexity of Algorithm 1 is dominated by steps 1, 2, and 4 through 6, resulting in a time complexity of is $\mathbf{O}(|X|^2|A|)$.

Example 6. From Example 5.1, it is easy to see that

$$\mathbf{s} = (sig(f(\gamma_1)), sig(f(\gamma_2)), sig(f(\gamma_3)), sig(f(\gamma_4))) = (16, 13, 14, 16).$$

By steps 9 to 16, $f(\gamma_4) \in red$ and $M \leftarrow M - M(f(\gamma_a)) - M(N(f(\gamma_a), E)) = (1, 1, 1)$ and $f(\mathbf{A}) = \{f(\gamma_1), f(\gamma_2), f(\gamma_3)\}$. Then, from steps 18 to 23, since $sig'(f(\gamma_1)) = sig'(f(\gamma_3)) = sig'(f(\gamma_3))$, furthermore, $sig(f(\gamma_1)) > sig(f(\gamma_2))$ and $sig(f(\gamma_1)) > sig(f(\gamma_3))$, then $f(\gamma_1) \in red$ and so $M = \emptyset$ by step 22 and the loop end. Hence, $red = \{f(\gamma_1), f(\gamma_4)\}$, then $\{\gamma_1, \gamma_4\}$ is a reduct of $\mathbf{A}$.

6. Conclusion and future work

In this paper, we have introduced a theoretical framework grounded in homomorphism to investigate the reduction of subbases. We demonstrate that the reduct of a subbase for a topological space is equivalent to the reduct of the subbase for the $f-$induced topological space, effectively transforming a complex topology into a more concise one. Additionally, we show that the reductions of subbases are equivalent to those of the $f-$induced FTS, which may be of a reduced size compared to the original. To facilitate the search for all possible reductions, we propose the subbase discernibility matrix for the $f-$induced FTS. However, it is worth noting that this is an NP-hard problem. As a result, we turn to graph theory to devise a heuristic approach to find a minimal subbase reduction. While we have made significant progress, there remain intriguing problems worthy of further consideration and discussion.

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