Relation between ultra matroid and linear decomposition

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Abstract. The notion of linear tangle was initially introduced as an obstacle to mixed searching number and Linear decomposition, both of which have significant connections to game theory and graph theory. In this concise paper, we introduce the concept of ultra matroid on a connectivity system, which combines the matroid concept defined on a set with the idea of an ultrafilter. Finally, we establish the equivalence between linear tangle and ultra matroid under certain conditions.

Keywords: ultra matroid, linear tangle, linear decomposition, matroid, ultrafilter.

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1. Introduction

The width parameter is a well-established concept in the field of graph theory. Linear-width, which is one of its variations, has been extensively researched in graph theory as well as in other fields such as network theory, topology, geometry, and combinatorial mathematics. Scholars have conducted extensive research on linear-width and other width parameters (e.g., [1], [2], [3], [4], [12], [13], [14], [17], [18], [19], [20], [22], [23] [24] [25]).

Linear tangle, which obstructs Linear decomposition, was initially introduced in the literature as an obstacle to mixed searching number, a concept deeply related to both graph theory and game theory[1]. Given the significance of studying linear-width as mentioned earlier, studying linear tangles is also crucial.

Filter is widely used in various areas, such as set theory, combinatorial mathematics, and topology. When a filter is ultra, it is well-known that it includes either the set A or its complement. Numerous studies have been conducted in this area(e.g., [5], [6], [7]).

Matroids are concepts that find applications in optimization theory, combinatorial mathematics, topology, algebra, graph algorithm, game theory, geometry, network theory, artificial intelligence, fuzzy theory, and coding theory. Due to their broad range of applications, matroids have gained considerable attention, and several studies have been conducted on this subject(e.g., [8], [9], [10], [11]). Matroids are characterized by the well-known axiom of augmentation property or independent set exchange property.

In this brief paper, we define a connectivity system as a pair (X, f), where X is a finite set, and f is a symmetric submodular function. We then introduce the new concept of an ultra matroid on the connectivity system (X, f), which combines the ultrafilter idea with the matroid concept defined on the set X. Finally, we establish the equivalence between linear tangle and ultra matroid under certain conditions. The concept of matroids, which may seem unrelated to graph theory at first glance, can be shown to be very interesting and novel when combined with the properties of ultrafilters. This relationship adds a significant level of relevance, making it a fascinating area of study.

2. Definition and preparation

This section provides mathematical definitions of each concept.

2.1 Symmetric submodular function and connectivity system

The definition of a symmetric submodular function is given below.

Definition 2.1. Let X be a finite set. A function $f : X \to \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:

- For all $A \subseteq X$, $f(A) = f(X \setminus A)$.
- For all $A, B \subseteq X$, $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

Lemma 2.2 ([13])**.** A symmetric submodular function f satisfies

1. For all $A \subseteq X$, $f(A) \geq f(\emptyset) = f(X)$,
2. For all $A, B \subseteq X$, $f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A)$.

In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system. And we use the notation f for a symmetric submodular function, a finite set X, and a natural number k.

Example 2.3. Consider X as the finite set $\{1, 2, 3\}$. For a symmetric submodular function $f : X \to \mathbb{N}$, with $A = \{1, 2\}$ and $B = \{2, 3\}$, we find that $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$, demonstrating the concept of symmetric submodularity.

2.2 Matroid on boolean algebra $(X, \cup, \cap)$

The definition of Matroid on boolean algebra $(X, \cup, \cap)$ is shown below.

Definition 2.4. In Boolean algebra $(X, \cup, \cap)$, the set family $M \subseteq 2^X$ is called a matroid if the following axioms hold true:

(MB1) $\emptyset \in M$.

(MB2) If $A \in M$ and $B \subseteq A$ then $B \in M$.

(MB3) If $A, B \in M$ and $|A| < |B|$ then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

Axiom (MB2) is commonly known as the hereditary property, while axiom (MB3) is often referred to as the augmentation property or independent set exchange property. In addition, the combination of axiom (MB1) and axiom (MB2) defines a combinatorial concept called an independence system, which is also known as an abstract simplicial complex.

2.3 Matroid on a connectivity system (X, f)

In this manuscript, we present a definition of matroids on connectivity systems (X, f) that mimics the concept of matroids on Boolean algebra. By imposing restrictions on submodular functions for each axiom, we ensure a coherent and natural definition.

Definition 2.5. Let X be a finite set and f be a symmetric submodular function. In a connectivity system (X, f), the set family $M \subseteq 2^X$ is called a matroid of order $k+1$ on (X, f) if the following axioms hold true:

(M0) For every $A \in M$, $f(A) \leq k$,

(M1) $\emptyset \in M$,

(M2) If $A \in M$, $B \subseteq A$, and $f(B) \leq k$ then $B \in M$,

(M3) If $A, B \in M$, $|A| < |B|$, $e \in X$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$, then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

Furthermore, let us define an order $k+1$ matroid $M \subseteq 2^X$ on a connectivity system (X, f) as an Ultra Matroid if it satisfies the following Axiom (M4):

(M4) For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$.

Axiom (M4) imitates the idea of Ultrafilters, which states that "either a set A or its complement belongs to the (Ultra)filter." Therefore, ultra matroids can be interpreted as matroids that include only one side.

2.4 Linear tangle: deep relation to linear width and Linear decomposition

The definition of a linear tangle on a connectivity system (X, f) is given below. Linear tangle, which obstructs Linear decomposition, was first introduced in the literature [1].

Definition 2.6 ([1])**.** Let X be a finite set and f be a symmetric submodular function. A linear tangle of order $k+1$ on a connectivity system (X, f) is a family $L \subseteq 2^X$, satisfying the following axioms:

(L1) $\emptyset \in L$,

(L2) For $A \subseteq X$, if $f(A) \leq k$ then either $A \in M$ or $X \setminus A \in M$,

(L3) If $A, B \in L$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$.

We propose to use the augmentation property of matroids by considering the following axiom $L3'$ instead of $L3$. We refer to a linear tangle of order $k+1$ on a connectivity system (X, f) that satisfies the axioms $L1$, $L2$, and $L3'$ as a restricted linear tangle of order $k+1$.

(L3') If $A, B \in L$, $|A| \neq |B|$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$.

3. Main result: equivalence between restricted linear tangle and ultra matroid

We show equivalence between a restricted linear tangle on a connectivity system (X, f) and an ultra matroid on a connectivity system (X, f) under the assumption that $f(\{e\}) \leq k$ for every $e \in X$.

Theorem 3.1. Let X be a finite set and f be a symmetric submodular function. Under the assumption that $f(\{e\}) \leq k$ for every $e \in X$, the family $W \subseteq 2^X$ being an order $k+1$ restricted linear tangle and $W \subseteq 2^X$ being an order $k+1$ ultra matroid are equivalent necessary and sufficient conditions.

Proof. Suppose that W is an ultra matroid of order $k+1$ on the connectivity system (X, f). We need to show that W is a restricted linear tangle of order $k+1$ on the connectivity system (X, f).

First, we show that W satisfies the axioms (L1) and (L2) of a restricted linear tangle. Since W is an ultra matroid of order $k+1$, it satisfies Axiom (M4), which implies that for any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$. In particular, for the empty set $\emptyset$, we have $f(\emptyset) = f(A) \leq k$, and hence $\emptyset \in W$. Moreover, if $A \subseteq X$ and $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$ by Axiom (M4), which shows that W satisfies Axiom (L2) as well.

Next, we prove that W satisfies the axiom (L3') of a restricted linear tangle. Let $A, B \in W$ and $e \in X$ be such that $f(\{e\}) \leq k$. We need to show that $A \cup B \cup \{e\} \neq X$. Suppose, on the contrary, that $A \cup B \cup \{e\} = X$. We choose a triple $(A, B, \{e\})$ that minimizes $|A \cap B|$ among such triples. First, we claim that $A \cap B = \emptyset$.

Since $2k \geq f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A)$, at least one of $f(A \setminus B)$ or $f(B \setminus A)$ is at most k. Without loss of generality, assume that $f(A \setminus B)$ is at most k. Hence, by axiom (M2), $A \setminus B \in M$. If $A \cap B \neq \emptyset$, then we have $|A \cap B| > |(A \setminus B) \cap B|$, which contradicts the choice of the triple. Thus, we have shown that $A \cap B = \emptyset$.

Next, we claim that $e \notin A$ and $e \notin B$. Suppose, on the contrary, that $e \in A$ or $e \in B$. If $e \in A$, then $A \cup B = X$, which implies that $X \setminus A = B \in M$, but this contradicts the axiom (M4). Similarly, we know that $e \notin B$ holds. Now, we know that the triple $(A, B, \{e\})$ consists of a partition of X. Hence, we have $f(A \cup \{e\}) = f(X \setminus B) = f(B) \leq k$.

If $|A| < |B|$, by axiom (M3) there exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$. This contradicts that there not exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$ because the triple $(A, B, \{e\})$ consists of a partition of X. If $|B| < |A|$, by axiom (M3) there exists $e'' \in A \setminus B$ such that $B \cup \{e''\} \in M$. This contradicts that the triple $(A, B, \{e\})$ consists of a partition of X. So W satisfies the axiom (L3').

Therefore, we have shown that W satisfies all the axioms (L1), (L2), and (L3') of a restricted linear tangle. Hence, W is a restricted linear tangle of order $k + 1$ on the connectivity system (X, f).

Assume that $f(\{e\}) \leq k$ for every $e \in X$ and let W be a restricted linear tangle of order $k + 1$ on the connectivity system (X, f). We will show that W satisfies the axioms of an ultra matroid of order $k+1$ on the connectivity system (X, f).

(M0) For any $A \in W$, we have by definition of a restricted linear tangle that $f(A) \leq k$, hence (M0) is satisfied.

(M1) Since $\emptyset$ is an element of W by definition of a restricted linear tangle, it is also an element of W, hence the axiom (M1) is satisfied.

(M2) Let $A \in W$, $B \subseteq A$, and $f(B) \leq k$. We need to show that $B \in W$. Suppose, to the contrary, that there exist subsets A and B such that $A \subseteq B$, $f(A) \leq k$, $B \in W$, and $A \notin W$. Then, we have $X \setminus A \in W$ by the axiom (L2), and for any $e \in X$, $(X \setminus A) \cup \{e\} \cup B = X$ holds, but this contradicts the axiom (L3') of a restricted linear tangle. So the axiom (M2) is satisfied.

(M3) Let $A, B \in W$, $|A| < |B|$, $e \in X$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$. We need to show that there exists $e \in B \setminus A$ such that $A \cup \{e\} \in W$. Suppose, to the contrary, that there exists a subset A, $B \in W$ and an element $e \in B \setminus A$ such that $f(A \cup \{e\}) \leq k$ and $A \cup \{e\} \notin W$ hold. Then, we have $(X \setminus A) \cap (X \setminus \{e\}) \in W$ by the axiom (L2). Hence A, $(X \setminus A) \cap (X \setminus \{e\}) \in W$ and $A \cup (X \setminus A) \cap (X \setminus \{e\}) \cup \{e\} = X$ hold. However, this contradicts the axiom (L3'). So the axiom (M3) is satisfied.

Therefore, W satisfies all the axioms of an ultra matroid of order $k + 1$ on the connectivity system (X, f). This proof is completed. □

4. Conclusion

In this brief manuscript, we have established the equivalence between linear tangles and ultra matroids, subject to certain conditions. Going forward, we intend to further explore the connections between not only matroids, but also antimatroids and greedoids.

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Conflict of Interest Statement

The author declares no conflicts of interest.

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