Diameter estimate for generalized m-quasi-Einstein manifolds

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Abstract. In this paper, we study the diameter estimate for generalized m-quasi-Einstein manifold. Using the Bochner formula and the Hopf maximum principle, we get a gradient estimate for the potential function of the generalized quasi-Einstein manifold. Based on the gradient estimate, we get a diameter estimates for generalized m-quasi-Einstein manifolds under suitable conditions.

Keywords: Generalized m-quasi-Einstein manifold, diameter estimate, gradient estimate, the Hopf maximum principle.

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1. Introduction

The diameter estimate is an attractive topic in Riemannian geometry. The upper diameter estimate is mainly focus on the extension of the Bonnet-Myers theorem. Since the diameter of a manifold is related to the volume of this manifold, then the lower diameter estimate can help us to estimate the isoperimetric constant $\inf_\Omega \frac{|\partial\Omega|^n}{|\Omega|^{n-1}}$, where Ω ranges over all open submanifolds of M, with compact closure in M, and smooth boundary. The lower diameter estimate can also help us to study the Fraenkel asymmetry $\inf\{\frac{|\Omega\Delta B|}{|\Omega|} : B\ is\ ball, |B| = |\Omega|\}$, and the Hardy-Littlewood maximal operator $Mf(x) = \sup_{r>0} \frac{\int_{B(x,r)} |f(x)|dx}{|B(x,r)|}$.

In [13], [14], [16], [17], [19], the authors studied the upper diameter estimate and extended the Bonnet-Myers theorem to the Riemannian manifold with Bakry-Emery curvature bounded from below. Futaki and Sano [6] obtained a lower diameter bound for compact shrinking Ricci soliton. Futaki and Li [5] improved the diameter estimate in [6]. Wang [18] got a lower diameter bound for compact τ-quasi-Einstein manifold. Hu, Mao and Wang [8] obtained a lower diameter estimate for compact generalized τ-quasi-Einstein manifold under the strong conditions $\lambda = \lambda(f)$, $\lambda'(t) \geq 0$ and $[t^{\frac{2}{\alpha_0(n-2)}}\lambda(t)]' \geq 0$ (for more details, please see [8]). In [4], the author of this paper got a lower diameter estimate for a class of generalized quasi-Einstein manifolds.

The concept of generalized quasi-Einstein manifold was firstly introduced by Catino [3]. Barros and Ribeiro [2] introduced the concept of generalized

m-quasi-Einstein manifold and obtained some structural equations. Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 3$. If there exist smooth functions f and λ on (M, g) such that the Ricci tensor satisfies

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{1.1}$$

then (M, g) is called generalized m-quasi-Einstein manifold, where m is a positive integer. The function f in (1.1) is usually called potential function. Suppose that (M, g) is a generalized m-quasi-Einstein manifold with scalar curvature R. If ρ, C are real constants, $\lambda = \rho R + C$, then (M, g) is called (m, ρ)-quasi-Einstein manifold (see [10] for more details).

The classification of generalized m-quasi-Einstein manifolds has been extensively studied. Huang [10] discussed the classification of (m, ρ)-quasi-Einstein manifolds. Barros and Gomes [1] proved that compact generalized m-quasi-Einstein with constant scalar curvature must be isometric to a standard Euclidean sphere. Hu, Li and Zhai [9] proved that some generalized m-quasi-Einstein manifolds with constant Ricci curvature are Einstein manifolds. Jauregui and Wylie [12] discussed the conformal diffeomorphisms of generalized m-quasi-Einstein manifolds. Neto [15] proved that a 4-dimensional generalized m-quasi-Einstein manifold with harmonic anti-self dual Weyl tensor is locally a warped product with 3-dimensional Einstein fibers provided an additional condition holds. Huang and Zeng [11] discussed the classification for generalized m-quasi-Einstein manifolds under the assumption that the Bach tensor is flat.

As far as we know, the study of the diameter estimate for generalized m-quasi-Einstein manifolds is very few up to now. Motivated by [18] and [8], we study the diameter estimate for generalized m-quasi-Einstein manifolds in this paper. Since λ is a function, the lower diameter estimate of generalized m-quasi-Einstein manifolds is much more difficult than that of τ-quasi-Einstein manifolds. To overcome this difficult, we need to use some new skills. For example, we need to construct some proper auxiliary functions and get proper estimate for these functions.

2. Gradient estimate and some useful Lemmas

To consider the lower diameter estimate for compact generalized m-quasi-Einstein manifolds, we need to get a gradient estimate for $h = e^{-\frac{f}{m}}$. Firstly, we introduce some useful Lemmas.

Lemma 2.1 ([2])**.** *Let (M, g) be an n-dimensional generalized m-quasi-Einstein manifold satisfying* (1.1)*, then*

$$\frac{1}{2}\nabla R = \frac{m-1}{m} Ric(\nabla f) + \frac{1}{m}[R - (n-1)\lambda]\nabla f + (n-1)\nabla\lambda, \tag{2.1}$$

where R is the scalar curvature of (M, g).

Lemma 2.2. *Suppose that (M,g) is an n-dimensional compact generalized m-quasi-Einstein manifold with scalar curvature R. Let $\lambda_{\max} = \max_{x\in M}\lambda(x)$ and $R_{\min} = \min_{x\in M} R(x)$, then $n\lambda_{\max} - R_{\min} \geq 0$. If $n\lambda_{\max} - R_{\min} = 0$, then (M,g) is trivial.*

Proof. Taking trace in both sides of (1.1), we get

$$R + \triangle f - \frac{1}{m}|\nabla f|^2 = n\lambda. \tag{2.2}$$

Let x_1 be the minimum points of $f(x)$ on M. By the Maximum principle, we have

$$\triangle f(x_1) \geq 0, \nabla f(x_1) = 0. \tag{2.3}$$

According to (2.2) and (2.3), we get

$$n\lambda_{\max} - R_{\min} \geq n\lambda(x_1) - R(x_1) \geq 0.$$

If $n\lambda_{\max} - R_{\min} = 0$, then $\triangle f - \frac{1}{m}|\nabla f|^2 \leq 0$. According to the Hopf maximum principle in [7], we conclude that f is a constant. Therefore, (M,g) is trivial. The proof of Lemma 2.2 is complete. □

In the following, we always suppose that M is an n-dimensional compact generalized m-quasi-Einstein manifold with scalar curvature R, $R_{\min} = \min_{x\in M} R(x)$, $\lambda_{\max} = \max_{x\in M}\lambda(x)$, $h = e^{-\frac{f}{m}}$, $D = \{x \in M; \nabla h((x) \neq 0\}$ and

$$F(h) = |\nabla h|^2 + \frac{n}{m}\lambda_{\max}h^2 - \frac{1}{m}R_{\min}h^2. \tag{2.4}$$

Lemma 2.3. *Let $h = e^{-\frac{f}{m}}$. Then, the following equality*

$$\begin{aligned}\triangle|\nabla h|^2 &= 2|\nabla^2 h|^2 + (2-m)\frac{\nabla|\nabla h|^2\nabla h}{h} + (\frac{4}{m} - 2)\lambda|\nabla h|^2 - \frac{2n}{m}\lambda|\nabla h|^2\\ &+ \frac{2}{m}R|\nabla h|^2 - \frac{4}{m}[R-(n-1)\lambda]|\nabla h|^2 + \frac{2n-4}{m}h\nabla\lambda\nabla h\end{aligned} \tag{2.5}$$

holds on generalized m-quasi-Einstein manifolds.

Proof. Direct calculation shows that

$$\Delta h = -\frac{n}{m}\lambda h + \frac{1}{m}hR. \tag{2.6}$$

Therefore,

$$2\nabla\Delta h\cdot\nabla h = -\frac{2n}{m}\lambda|\nabla h|^2 + \frac{2}{m}R|\nabla h|^2 - \frac{2n}{m}h\nabla\lambda\nabla h + \frac{2}{m}h\nabla R\nabla h. \tag{2.7}$$

Since $\nabla h = -\frac{1}{m} h\nabla f$, by (2.1) we have

$$\frac{2}{m}h\nabla R\nabla h = (\frac{4}{m}-4)\mathrm{Ric}(\nabla h,\nabla h) - \frac{4}{m}[R-(n-1)\lambda]|\nabla h|^2 + \frac{4(n-1)}{m}h\nabla\lambda\nabla h. \tag{2.8}$$

According to (1.1), we get

$$(\frac{4}{m}-2)\mathrm{Ric}(\nabla h,\nabla h) = (\frac{4}{m}-2)\lambda|\nabla h|^2 + (2-m)\frac{\nabla|\nabla h|^2\nabla h}{h}. \tag{2.9}$$

By the Bochner formula, we have

$$\triangle|\nabla h|^2 = 2|\nabla^2 h|^2 + 2\nabla\Delta h\cdot\nabla h + 2\mathrm{Ric}(\nabla h,\nabla h). \tag{2.10}$$

Putting (2.7), (2.8) and (2.9) into (2.10), we conclude that (2.5) is true. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *If F is the function defined in (2.4), then there exists a smooth vector field X on M such that*

$$\Delta F \geq \nabla F\cdot X + \frac{2n-4}{m}h\nabla\lambda\nabla h + (\frac{2n\lambda}{m} - \frac{2R}{m} - 2\lambda)|\nabla h|^2 + (2-2m)\frac{|\nabla h|^2}{h}G(h) \tag{2.11}$$

holds on $D = \{x\in M; \nabla h((x)\neq 0\}$.

Proof. Direct calculation shows that

$$\nabla F = \nabla|\nabla h|^2 + \frac{2n}{m}\lambda_{\max}h\nabla h - \frac{2}{m}R_{\min}h\nabla h. \tag{2.12}$$

Therefore, we have

$$\triangle F = \triangle|\nabla h|^2 + \frac{2n}{m}\lambda_{\max}|\nabla h|^2 + \frac{2n}{m}\lambda_{\max}h\triangle h - \frac{2}{m}R_{\min}|\nabla h|^2 - \frac{2}{m}R_{\min}h\triangle h. \tag{2.13}$$

For the purpose of convenience, we let

$$G(h) = -\frac{n}{m}\lambda_{\max}h + \frac{1}{m}R_{\min}h. \tag{2.14}$$

Putting (2.5) into (2.13), we obtain

$$\begin{aligned}\triangle F &= 2|\nabla^2 h|^2 + (2-m)\frac{\nabla|\nabla h|^2\nabla h}{h} \\ &+ (\frac{4}{m}-2)\lambda|\nabla h|^2 - \frac{2n}{m}\lambda|\nabla h|^2 + \frac{2}{m}R|\nabla h|^2 \\ &- \frac{4}{m}[R-(n-1)\lambda]|\nabla h|^2 + \frac{2n-4}{m}h\nabla\lambda\nabla h - 2G(h)\frac{|\nabla h|^2}{h} - 2G(h)\triangle h.\end{aligned} \tag{2.15}$$

Consider a point $O \in D$. Rotating the orthonormal frame at O so that $|\nabla h|(O) = h_1(O) \neq 0$. According to (2.10) in [18], we have

$$\begin{aligned} & 2|\nabla^2 h|^2 + (2-m)\frac{\nabla|\nabla h|^2 \nabla h}{h} \\ \geq & \frac{2n}{n-1}h_{11}^2 - \frac{4}{n-1}h_{11}\triangle h + \frac{2}{n-1}(\triangle h)^2 + (4-2m)\frac{h_{11}|\nabla h|^2}{h}. \end{aligned} \tag{2.16}$$

According to (2.12), we get

$$h_{11} = \frac{\nabla F \cdot \nabla h}{2|\nabla h|^2} + G(h). \tag{2.17}$$

Putting (2.17) into (2.16), we conclude that there exists a smooth vector field X on M such that

$$\begin{aligned} 2|\nabla^2 h|^2 + (4-2m)\frac{\nabla|\nabla h|^2 \nabla h}{h} &\geq \nabla F \cdot X + \frac{2n}{n-1}[G(h)]^2 \\ - \frac{4}{n-1}G(h)\triangle h + \frac{2}{n-1}&(\triangle h)^2 + (4-2m)\frac{|\nabla h|^2}{h}G(h). \end{aligned} \tag{2.18}$$

From (2.6) and (2.14), we can easily find that $G(h) - \triangle h \leq 0$. According to Lemma 2.2 and (2.14), we have $G(h) \leq 0$. Therefore

$$nG(h) - \triangle h \leq G(h) - \triangle h \leq 0. \tag{2.19}$$

By (2.19), we obtain

$$\begin{aligned} & -2G(h)\triangle h + \frac{2n}{n-1}[G(h)]^2 - \frac{4}{n-1}G(h)\triangle h + \frac{2}{n-1}(\triangle h)^2 \\ = & \frac{2n}{n-1}[G(h)]^2 - \frac{2n}{n-1}G(h)\triangle h + \frac{2}{n-1}[L(h)]^2 - \frac{2}{n-1}G(h)\triangle h \\ = & \frac{2n}{n-1}G(h)[G(h) - \triangle h] - \frac{2}{n-1}\triangle h[G(h) - \triangle h] \\ = & \frac{2}{n-1}[G(h) - \triangle h][nG(h) - \triangle h] \geq 0. \end{aligned} \tag{2.20}$$

According to (2.15), (2.18) and (2.20), we conclude that (2.11) holds. The proof of Lemma 2.4 is complete. □

3. Diameter estimate for compact (m, ρ)-quasi-Einstein manifold

In [9], Huang and Wei introduced the concept of (m, ρ)-quasi-Einstein manifold. Let (M, g) be an n-dimensional Riemannian manifold with scalar curvature R. If $n \geq 3$ and there exist constants constants ρ, C and $m > 0$ such that

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = (\rho R + C)g, \tag{3.1}$$

then (M, g) is called (m, ρ)-quasi-Einstein manifold.

(m, ρ)-quasi-Einstein manifolds are closely related to the ρ-Einstein flows (see page 844 in [9] or page 270 in [10] for more details). The classification of compact (m, ρ)-quasi-Einstein manifolds was extensively studied by Huang and Wei in [10]. One of the main results in [10] can be stated as

Theorem A. *Let (M, g) be an n-dimensional compact (m, ρ)-quasi-Einstein manifold satisfying* (3.1) *and $n \geq 3$. Then:*

(1) *If $\rho < \frac{1}{2(n-1)}$, then either $C > 0$ and $R \geq \frac{n(n-1)C}{m+(n-1)(1-n\rho)}$, or (M, g) is trivial.*

(2) *If $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$, then either $C < 0$ and $R \leq \frac{n(n-1)C}{m+(n-1)(1-n\rho)}$, or (M, g) is trivial.*

(3) *If $\rho = \frac{1}{2(n-1)}$, then (M, g) is trivial.*

(4) *If $\rho \geq \frac{1}{n}$ and $m > 1$, then (M, g) is trivial.*

According to Theorem A, we can easily get the following proposition.

Proposition 3.1. *Let (M, g) be an n-dimensional nontrivial compact (m, ρ)-quasi-Einstein manifold satisfying (3.1) and $n \geq 3$. If $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$, then $C < 0$ and $R \leq 0$.*

In this section, we only consider the lower diameter estimate for n-dimensional nontrivial compact (m, ρ)-quasi-Einstein manifolds satisfying $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$. Using Lemma 2.4, we get

Theorem 3.1. *Suppose that M is an n-dimensional nontrivial compact (m, ρ)-quasi-Einstein manifold satisfying (3.1) and $n \geq 3$. Let $\omega_f = \max_{x \in M} f(x) - \min_{x \in M} f(x)$. If $m > n$, $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$ and $\nabla \lambda \nabla f \leq 0$, then the diameter of M* satisfies

$$\operatorname{diam} M \geq \frac{\sqrt{m}}{\sqrt{n\rho R_{\max} + nC - R_{\min}}}\left(\frac{\pi}{2} - \arcsin e^{-\frac{w_f}{m}}\right). \tag{3.2}$$

Proof. Since $m > n \geq 3$, $G(h) \leq 0$, then

$$(2 - 2m)\frac{|\nabla h|^2}{h} G(h) \geq 0. \tag{3.3}$$

Since $\nabla \lambda \nabla f \leq 0$, by (2.11) and (3.3) we have

$$\Delta F \geq \nabla F \cdot X + \left(\frac{2n\lambda}{m} - \frac{2R}{m} - 2\lambda\right)|\nabla h|^2. \tag{3.4}$$

It follows from Proposition 3.1 and $\frac{1}{2(n-1)} \leq \rho < \frac{1}{n}$ that $\lambda = \rho R + C < 0$ and $R \leq 0$. Since $m > n$, by (3.4) we conclude that $\Delta F \geq \nabla F \cdot X$ holds on D.

Let x_0 be the maximum point of $F(h(x))$ on M. If $x_0 \in D$, then there exists a neighborhood $\mathcal{U}$ of x_0 so that $\mathcal{U} \subset D$. Moreover, x_0 is the maximum point of $F(h(x))$ on $\mathcal{U}$. It is obvious that $\Delta F \geq \nabla F \cdot X$ holds on $\mathcal{U}$. Therefore, by the Hopf maximum principle in [7] we conclude that F is constant on $\mathcal{U}$. Since $\Delta F(x_0) \leq 0$ and $\nabla F(x_0) = 0$, by (3.4) we conclude that $\nabla h(x_0) = 0$, which is a contradiction with $x_0 \in D$. Therefore, x_0 is not in D, which means that $\nabla h(x_0) = 0$. Thus, we arrive at

$$|\nabla h|^2(x) + \mathcal{G}(h(x)) \leq \mathcal{G}(h(x_0)), \tag{3.5}$$

where $\mathcal{G}(h) = \frac{n}{m}\lambda_{\max}h^2 - \frac{1}{m}R_{\min}h^2$.

According to (3.5), for all $x \in M$, we have

$$\mathcal{G}(h(x)) \leq |\nabla h|^2(x) + \mathcal{G}(h(x)) \leq \mathcal{G}(h(x_0)).$$

Therefore, x_0 is the maximum point of $\mathcal{G}(h(x))$, which means that x_0 is the maximum point of $h(x)$. Let x_1 and x_2 be the maximum and minimum points of $h(x)$ on M respectively. Without loss of generality, we can suppose that $x_1 = x_0$. Similar to the proof of Theorem 1.3 in [18], we choose a minimizing geodesic γ jointing x_1 and x_2. Let $h_1 = h(x_1)$, $h_2 = h(x_2)$. Since M is a nontrivial generalized quasi-Einstein manifold, by Lemma 2.2 we conclude that $n\lambda_{\max} - R_{\min} > 0$. Therefore, according to (3.5) we have

$$\begin{aligned} \mathrm{diam}M &\geq \int_{h_2}^{h_1} \frac{dh}{\sqrt{\mathcal{G}(h(x_1)) - \mathcal{G}(h(x))}} \\ &= \int_{h_2}^{h_1} \frac{dh}{\sqrt{\frac{1}{m}(n\lambda_{\max} - R_{\min})(h_1^2 - h^2)}} \\ &= \frac{1}{\sqrt{\frac{1}{m}(n\lambda_{\max} - R_{\min})}} \int_{\frac{h_2}{h_1}}^{1} \frac{d\sigma}{\sqrt{1-\sigma^2}} \\ &= \frac{\sqrt{m}}{\sqrt{n\lambda_{\max} - R_{\min}}}\left(\frac{\pi}{2} - \arcsin e^{-\frac{w_f}{m}}\right). \end{aligned} \tag{3.6}$$

Since $\lambda_{\max} = \rho R_{\max} + C$, by (3.6) we conclude that (3.2) holds. The proof of Theorem 3.1 is complete. □

4. Diameter estimate for generalized m-quasi-Einstein manifold

In this section, we study the lower diameter estimate for compact generalized m-quasi-Einstein manifold satisfying (1.1). This manifold is more general than (m, ρ)-quasi-Einstein manifold. Using Lemma 2.4, we get

Theorem 4.1. *Suppose that M is an n-dimensional nontrivial compact m-quasi-Einstein manifold satisfying (1.1) and $m \geq 1$. Let $\omega_f = \max_{x\in M} f(x) -$*

$\min_{x\in M} f(x)$. *If* $\nabla\lambda\nabla f \leq 0$ *and* $(n-m)\lambda > R$ *on* M, *then the diameter of* M *satisfies*

$$\mathrm{diam}M \geq \frac{\sqrt{m}}{\sqrt{n\lambda_{\max} - R_{\min}}}\left(\frac{\pi}{2} - \arcsin e^{-\frac{w_f}{m}}\right). \tag{4.1}$$

Proof. Since $m \geq 1$, $\nabla\lambda\nabla f \leq 0$ and $G(h) \leq 0$, by (2.11) we conclude that (3.4) holds. It follows from $(n-m)\lambda > R$ that $\frac{2n\lambda}{m} - \frac{2R}{m} - 2\lambda > 0$. Therefore, we have $\Delta F \geq \nabla F \cdot X$. Similar to the proof of Theorem 3.1, we conclude that (3.5) is true. Since M is a nontrivial generalized quasi-Einstein manifold, then $n\lambda_{\max} - R_{\min} > 0$. Let x_1 and x_2 be the maximum and minimum points of $h(x)$ on M, respectively. Set $h_1 = h(x_1)$, $h_2 = h(x_2)$. Similar to the proof of Theorem 3.1, we conclude that (4.1) is true. The proof of Theorem 4.1 is complete. □

Example 4.1. Suppose that $u(x)$ is a smooth function on n-dimensional Riemannian manifold (M, g) with $n \geq 3$, $\widetilde{\lambda}$ is a real increasing smooth positive function and $\widetilde{f}$ is a real decreasing smooth function. Then, $\lambda = \widetilde{\lambda} \circ u$ and $f = \widetilde{f} \circ u$ are smooth functions on (M, g). Obviously, $\nabla\lambda\nabla f \leq 0$. Choose a proper metric g, the Ricci tensor can satisfy (1.1) and the scalar curvature can satisfy $R \leq 0$. If $n > m$, then $(n-m)\lambda > R$. Therefore, the diameter of M satisfies (4.1).

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