Inequalities of DVT-type−the two-dimensional case

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Abstract. In this note, particular two-dimensional inequalities of Drápal-Valent type in integer numbers are investigated.

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In [3], A. Drápal and V. Valent proved that in a finite quasigroup Q of order n the number of associative triples $a(Q) \geq 2n - i(Q) + (\delta_1 + \delta_2)$, where $i(Q)$ is the number of idempotents in Q, i.e., $i(Q) = |\{\, x \in Q \,|\, xx = x \,\}|$, $\delta_1 = |\{\, z \in Q \,|\, zx \neq x \text{ for all } x \in Q \,\}|$ and $\delta_2 = |\{\, z \in Q \,|\, xz \neq x \text{ for all } x \in Q \,\}|$ [3,Theorem 2.5]. This important result is an easy consequence of the inequality

$$\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) - \sum_{i=1}^{k}(a_i + b_i) \geq 3n - 2k + (r + s),$$

where $n \geq k \geq 0$, $a_1, \dots, a_n, b_1 \dots, b_n$ are non-negative integers such that $\sum_{i=1}^{n} a_i = n = \sum_{i=1}^{n} b_i$, $a_i \geq 1$ and $b_i \geq 1$ for $1 \leq i \leq k$, r is the number of i with $a_i = 0$ and s is the number of i with $b_i = 0$ ([3, Proposition

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2.4(ii)]. The lengthy and complicated proof of this DVT-inequality (inequality of Drápal-Valent type) in [3] is based on highly semantically involved insight.

In [4], a very short elementary arithmetical proof of a more general inequality of this type was found. This inequality is two-dimensional in the sense that it works with two n-tuples of integers. The approach in [4] opens a road to investigation of similar inequalities of DVT-type which could be useful in further investigations of estimates in non-associative algebra and they are also of independent interest. Hence, they deserve a thorough examination, however the research is only at its beginning. In [1] and [2], the one-dimensional case working with one n-tuple of real numbers was investigated. In this note, the two-dimensional case of inequalities of Drápal's type is investigated. Among other results, it is shown that

$$2\sum_{i=1}^{n}(a_i^2+b_i^2+a_ib_i) \geq 3\sum_{i=1}^{n}(a_i+b_i)+2(r+s),$$

where $a_1,\dots,a_n,b_1,\dots,b_n$ are arbitrary integers with $\sum_{i=1}^n a_i \geq n$, $\sum_{i+1}^n b_i \geq n$, r is the number of i with $a_i=0$ and s is the number of i with $b_i=0$. The case when the equality holds is characterized and several other inequalities of this type are investigated.

1. First concepts

Let $n \geq 1$ and let $\alpha=(a_1,\dots,a_n)$, $\beta=(b_1,\dots,b_n)$ be an ordered n-tuples of integers. We put

1. $z(\alpha,a)=|\{\,i\,|\,1\leq i\leq n, a_i=a\}|$, for every $a\in\mathbb{R}$;
2. $z(\alpha)=z(\alpha,0)$;
3. $s(\alpha)=\sum_{i=1}^n a_i$;
4. $r(\alpha)=\sum_{i=1}^n a_i^2$;
5. $t(\alpha)=r(\alpha)-s(\alpha)-z(\alpha)$;
6. $p(\alpha,\beta)=\sum_{i=1}^n a_ib_i$;
7. $r(\alpha,\beta)=\sum_{i=1}^n a_i^2+\sum_{i=1}^n b_i^2+\sum_{i=1}^n a_ib_i$;
8. $t(\alpha,\beta) = 2\sum_{i=1}^n a_i^2+2\sum_{i=1}^n b_i^2+2\sum_{i=1}^n a_ib_i-3\sum_{i=1}^n a_i-3\sum_{i=1}^n b_i-2z(\alpha)-2z(\beta)$.

We thus have

(9) $r(\alpha,\beta)=\sum_{i=1}^n(a_i+b_i)^2-p(\alpha,\beta)$ $(=r(\alpha+\beta)-p(\alpha,\beta)\,)$;

(10) $t(\alpha,\beta)=2r(\alpha,\beta)-3s(\alpha)-3s(\beta)-2z(\alpha)-2z(\beta)$ $(=2(r(\alpha,\beta)-z(\alpha)-z(\beta))-3s(\alpha+\beta)=2(r(\alpha,\beta)-s(\alpha)-s(\beta)-z(\alpha)-z(\beta))-s(\alpha)-s(\beta)\,)$.

If $s(\alpha) = n = s(\beta)$ then

(11) $t(\alpha, \beta) = 2(r(\alpha, \beta) - 3n - z(\alpha) - z(\beta))$.

Lemma 1.1. $t(\alpha, \beta) = 2t(\alpha) + 2t(\beta) + 2p(\alpha, \beta) - s(\alpha) - s(\beta)$.

Proof. Easy to check directly. □

Lemma 1.2. *Put* $\gamma = \alpha + \beta = (a_1 + b_1, \dots, a_n + b_n)$. *Then,* $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + t(\gamma) + z(\gamma) - s(\gamma) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\gamma) - 2s(\gamma)$.

Proof. By 1.1, $t(\alpha, \beta) = 2t(\alpha) + 2t(\beta) + 2p(\alpha, \beta) - s(\alpha) - s(\beta) = t(\alpha) + t(\beta) + r(\alpha) + r(\beta) + 2p(\alpha, \beta) - 2s(\alpha) - 2s(\beta) - z(\alpha) - z(\beta) = t(\alpha) + t(\beta) + r(\gamma) - 2s(\gamma) - z(\alpha) - z(\beta)$ and the rest is clear. □

Lemma 1.3. *Put* $\delta = \alpha + \beta - 1 = (a_1 + b_1 - 1, \dots, a_n + b_n - 1)$. *Then,* $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\delta) - n$.

Proof. We have $r(\alpha + \beta) - 2s(\alpha + \beta) = \sum_{i=1}^{n}(a_i + b_i)^2 - 2\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i - 2\sum_{i=1}^{n} a_i - 2\sum_{i=1}^{n} b_i = \sum_{i=1}^{n}(a_i + b_i - 1)^2 - n = r(\delta) - n$ and it remains to use 1.2. □

Lemma 1.4. *Put* $\varepsilon = \alpha + \beta - 2 = (a_1 + b_1 - 2, \dots, a_n + b_n - 2)$. *Then,* $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\varepsilon) + 2s(\varepsilon)$.

Proof. We have $r(\varepsilon) + 2s(\varepsilon) = r(\alpha + \beta) - 4s(\alpha + \beta) + 4n + 2s(\alpha + \beta) - 4n = r(\alpha + \beta) - 2s(\alpha + \beta)$ and it remains to use 1.2. □

Lemma 1.5. *Put* $\alpha_1 = \alpha - 1$ *and* $\beta_1 = \beta - 1$. *Then,* $t(\alpha, \beta) = 2r(\alpha_1) + 2r(\beta_1) + 2p(\alpha_1, \beta_1) + 3s(\alpha_1) + 3s(\beta_1) - 2z(\alpha_1, -1) - 2z(\beta_1, -1)$.

Proof. Easy to check directly. □

Lemma 1.6. (i) $r(\alpha) \geq \sum_{i=1}^{n} |a_i| \geq |s(\alpha)|$.
(ii) $r(\alpha) + s(\alpha) \geq 0$.
(iii) $r(\alpha) = \sum_{i=1}^{n} |a_i|$ *if and only if* $a_i \in \{0, 1, -1\}$, *for every* i.
(iv) $r(\alpha) = s(\alpha)$ *if and only if* $a_i \in \{0, 1\}$, *for every* i.

Proof. Easy to see. □

Lemma 1.7. (i) *If* $s(\alpha) \geq 0$ *then* $r(\alpha) + 2s(\alpha) \geq r(\alpha) \geq 0$.
(ii) *If* $\sum_{i=1}^{n} |a_i + 1| \geq n$ *then* $r(\alpha) + 2s(\alpha) \geq 0$.

Proof. (i) This is obvious.
(ii) We have $r(\alpha) + 2s(\alpha) = r(\alpha + 1) - n \geq (\sum_{i=1}^{n} |a_i + 1|) - n \geq 0$. □

Lemma 1.8. *If* $s(\alpha) \geq 2n$ *or* $\sum_{i=1}^{n} |a_i - 1| \geq n$ *then* $r(\alpha) - 2s(\alpha) \geq 0$.

Proof. We have $r(\alpha) - 2s(\alpha) = r(\alpha - 1) - n \geq \sum_{i=1}^{n} |a_i - 1| - n \geq \sum_{i=1}^{n}(a_i - 1) - n = s(\alpha) - 2n$. □

Remark 1.9. First of all, if $a_i \leq -2$ then $a_i^2 - 1 \geq 3$. If $a_i = -1$ then $a_i^2 - 1 = 0$. If $a_i = 0$ then $a_i^2 - 1 = -1$. If $a_i = 1$ then $a_i^2 - 1 = 0$. If $a_i \geq 2$ then $a_i^2 - 1 \geq 3$. Now, if $g = |\{\, i \,|\, |a_i| \geq 2 \,\}|$ then $r(\alpha) - n \geq 3g - z(\alpha)$.

Remark 1.10. First of all, if $a_i \leq -1$ then $a_i^2 - 2a_i \geq 3$. If $a_i = 0$ then $a_i^2 - 2a_i = 0$. If $a_i = 1$ then $a_i^2 - 2a_i = -1$. If $a_i = 2$ then $a_i^2 - 2a_i = 0$. If $a_i \geq 3$ then $a_i^2 - 2a_i \geq 3$. Now, if $h = |\{\, i \,|\, a_i < 0 \,\}| + |\{\, i \,|\, a_i \geq 3 \,\}|$ then $r(\alpha) - 2s(a) \geq 3h - z(\alpha, 1)$.

Remark 1.11. First of all, if $a_i \leq -3$ then $a_i^2 + 2a_i \geq 3$. If $a_i = -2$ then $a_i^2 + 2a_i = 0$. If $a_i = -1$ then $a_i^2 + 2a_i = -1$. If $a_i = 0$ then $a_i^2 + 2a_i = 0$. If $a_i \geq 1$ then $a_i^2 + 2a_i \geq 3$. Now, if $k = |\{\, i \,|\, a_i \leq -3 \,\}| + |\{\, i \,|\, a_i > 0 \,\}|$ then $r(\alpha) + 2s(\alpha) \geq 3k - z(\alpha, -1)$.

Lemma 1.12. (i) $r(\alpha) \geq 2z(\alpha) - 2n + 3\sum_{i=1}^{n} |a_i| \geq 2z(\alpha) - 2n + 3s(\alpha)$.
(ii) $r(\alpha) = 2z(\alpha) - 2n + 3s(\alpha)$ *if and only if* $a_i \in \{0, 1, 2\}$, *for every* $i = 1, \dots, n$.

Proof. This result was proved in [1,6.1] in a more general setting. □

2. Technical results (a)

Let $n \geq 2$ and let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ be ordered n-tuples of integers. Choose $1 \leq j, k \leq n$, $j \neq k$, and define $\gamma = (c_1, \dots, c_n)$ as $c_j = a_j - 1$, $c_k = a_k + 1$ and $c_i = a_i$ for $i \neq j, k$ (see [1, Section 2]). Clearly, $s(\alpha) = s(\gamma)$.

The following assertions are easily seen:

Lemma 2.1. $p(\alpha, \beta) = p(\gamma, \beta) + (b_j - b_k)$.

Lemma 2.2. $r(\alpha, \beta) = r(\gamma, \beta) + 2(a_j - a_k - 1) + b_j - b_k$.

Lemma 2.3. $t(\alpha, \beta) = t(\gamma, \beta) + 4(a_j - a_k - 1) + 2(b_j - b_k) + 2z(\gamma) - 2z(\alpha)$.

Lemma 2.4. (i) *If* $a_j = 0$ *and* $a_k = -1$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k)$.
(ii) *If* $a_j = 1$ *and* $a_k = 0$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k)$.
(iii) *If* $a_j = 1$ *and* $a_k = -1$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k + 4)$.
(iv) *If* $a_j = 0 = a_k$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 2(b_j - b_k - 4)$.

Lemma 2.5. (i) *If* $a_j \neq 0, 1$ *and* $a_k \neq -1, 0$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 4(a_j - a_k - 1) + 2(b_j - b_k)$.
(ii) *If* $a_j \neq 0, 1$ *and* $a_k = -1$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 4a_j + 2(b_j - b_k + 1)$.
(iii) *If* $a_j = 1$ *and* $a_k \neq -1, 0$ *then* $t(\alpha, \beta) = t(\gamma, \beta) - 4a_k + 2(b_j - b_k + 1)$.
(iv) *If* $a_j \neq 0, 1$ *and* $a_k = 0$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 4a_j + 2(b_j - b_k - 3)$.
(v) *If* $a_j = 0$ *and* $a_k \neq -1, 0$ *then* $t(\alpha, \beta) = t(\gamma, \beta) - 4a_k + 2(b_j - b_k - 3)$.

Lemma 2.6. *If* $a_j \geq 2$ *and* $a_k = 0$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 2(2a_j - 3 + b_j - b_k) \geq t(\gamma, \beta) + 2(b_j - b_k + 1)$.

Lemma 2.7. *If* $a_j \geq 2$ *and* $a_k = -1$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 2(2a_j + 1 + b_j - b_k) \geq t(\gamma, \beta) + 2(b_j - b_k + 5)$.

Lemma 2.8. *If* $a_j \geq 2$ *and* $a_k \leq -2$ *then* $t(\alpha, \beta) = t(\gamma, \beta) + 2(2a_j - 2a_k - 2 + b_j - b_k) \geq t(\gamma, \beta) + 2(b_j - b_k + 6)$.

3. Technical results (b)

Let a, b be integers and put $z(0) = 1$, $z(a) = 0$ for $a \neq 0$ and $t(a, b) = 2a^2 + 2b^2 + 2ab - 3a - 3b - 2z(a) - 2z(b)$ $(= t(b, a))$.

Lemma 3.1. *Let $a \neq 0$ and $b \neq 0$. Then:*
(i) *$z(a) = 0 = z(b)$ and $t(a, b) \geq 0$.*
(ii) *$t(a, b) = 0$ if and only if $a = 1 = b$.*
(iii) *$t(a, b) \neq 1$.*
(iv) *$t(a, b) = 2$ if and only if either $a = 1$, $b = -1$ or $a = -1$, $b = 1$.*
(v) *$t(a, b) = 3$ if and only if either $a = 2$, $b = -1$ or $a = -1$, $b = 2$.*

Proof. Taking into account that $t(a, b) = t(b, a)$, the proof is divided into seven parts:
(1) Assume $a \geq 2$, $b \geq 2$. Then, $2a^2 - 3a \geq 2$, $2b^2 - 3b \geq 2$ and $2ab \geq 8$. Consequently, $t(a, b) \geq 12$.
(2) Assume $a \geq 2$, $b = 1$. Then, $2a^2 - 3a \geq 2$, $2b^2 - 3b = -1$ and $2ab \geq 4$. Consequently, $t(a, b) \geq 5$.
(3) Assume $a = 1 = b$. Then, $t(a, b) = 0$.
(4) Assume $a + b = 1$. Then, $t(a, b) = 2a^2 - 2a - 1$. Since $a + b = 1$, we have $a \neq 1$. If $a \geq 3$ then $2a^2 - 2a - 1 \geq 11$. If $a = 2$ then $2a^2 - 2a - 1 = 3$ and $b = -1$. If $a = -1$ then $2a^2 - 2a - 1 = 3$ and $b = 2$. If $a \leq -2$ then $2a^2 - 2a - 1 \geq 11$.
(5) Assume $a + b = 2$.Then, $t(a, b) = 2a^2 - 4a + 2$. Since $a + b = 2$, we have $a \neq 2$. If $a \geq 3$ then $2a^2 - 4a + 2 \geq 8$. If $a = 1$ then $2a^2 - 4a + 2 = 0$ and $b = 1$. If $a \leq -1$ then $2a^2 - 4a + 2 \geq 8$.
(6) Assume $a < 0$, $b > 0$ and put $c = 2a^2 + 2b^2 + 4ab - 3a - 3b = 2(a+b)^2 - 3(a+b)$. Then, $t(a, b) > c$. If $a + b \leq -1$ then $c \geq 5$ and $t(a, b) \geq 6$. If $a + b = 0$ then $t(a, b) = 2a^2$. Hence, $t(a, b) \geq 8$ for $a \leq -2$ and $2a^2 = 2$ for $a = -1$ (then $b = 1$). If $a + b = 2$ then (4) applies. If $a + b = 2$ then 5) applies. Finally, if $a + b \geq 3$ then $t(a, b) = (a + b)(a + b - 3) + a^2 + b^2 \geq 17$.
(7) Assume $a < 0$, $b < 0$. Then, $t(a, b) \geq 12$. □

Remark 3.2. Let a be a non-zero integer. Then, $t(a, 0) = 2a^2 - 3a - 2$, and hence $t(a, 0) \geq 7$ for $a \geq 3$, $t(a, 0) = 0$ for $a = 2$, $t(a, 0) = -3$ for $a = 1$, $t(-1, 0) = 3$ and $t(a, 0) \geq 12$ for $a \leq -2$. Further, $t(0, 0) = -4$.

Lemma 3.3. *Let $a \geq 2$ and $b \geq 1$. Then, $t(a, b) \geq t(a, -b) + 2$.*

Proof. We have $t(a, b) - t(a, -b) = 4ab - 6b \geq 2b \geq 2$. □

Lemma 3.4. *Let $a \geq 1$ and $b \geq 0$. Then:*
(i) *$t(a + 1, b) > t(a, b)$.*
(ii) *If $c \geq a$, $d \geq b$ and $c + d > a + b$ then $t(c, d) > t(a, b)$.*

Proof. We have $t(a + 1, b) - t(a, b) = 4a + 2b - 1 \geq 4a - 1 > 0$ and the rest is clear. □

4. The inequalities

Throughout this section, let $n \geq 1$, a, b be integers, $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ be ordered n-tuples of integers. Put $I = \{1, \dots, n\}$, $A = \{ i \in I \mid a_i \geq 0, b_i \geq 0, a_i + b_i \geq 3\}$, $B_1 = \{ i \in I \mid (a_i, b_i) = (2,0)\}$, $B_2 = \{ i \in I \mid (a_i, b_i) = (0,2)\}$, $B_3 = \{ i \in I \mid (a_i, b_i) = (1,1)\}$, $B = B_1 \cup B_2 \cup B_3$, $C_1 = \{ i \in I \mid (a_i, b_i) = (2,-1)\}$, $C_2 = \{ i \in I \mid (a_i, b_i) = (-1,2)\}$, $C = C_1 \cup C_2$, $D_1 = \{ i \in I \mid (a_i, b_i) = (0,1)\}$, $D_2 = \{ i \in I \mid (a_i, b_i) = (1,0)\}$, $D = D_1 \cup D_2$ and $E = \{ i \in I \mid (a_i, b_i) = (0,0)\}$. For $X = A, B_1, \dots, E$, denote $\mathrm{x} = |X|$. Clearly, $t(\alpha, \beta) = \sum_{i=1}^n t(a_i, b_i)$.

Example 4.1. Taking into account 3.1 and 3.2, it is easy to see that if $I = B \cup C \cup D \cup E$ and $3\mathrm{c} = 3\mathrm{d} + 4\mathrm{e}$ then $t(\alpha, \beta) = 0$.

Theorem 4.2. *Let* $\sum_{i=1}^n a_i \geq n$ *and* $\sum_{i=1}^n b_i \geq n$. *Put* $z_1 = |\{ i \in I \mid a_i = 0\}| = z(\alpha)$ *and* $z_2 = |\{ i \in I \mid b_i = 0\}| = z(\beta)$. *Then:*
(i) $2\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3\sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
(ii) $2\sum_{i=1}^n (a_i + b_i)^2 \geq 2\sum_{i=1}^n a_i b_i + 3\sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
(iii) *The equalities hold if and only if* $I = B$, $\mathrm{b}_1 = \mathrm{b}_2$, $2\mathrm{b}_1 \leq n$ *and* $\mathrm{b}_3 = n - 2\mathrm{b}_1$. *In this case,* $\sum_{i=1}^n a_i = n = \sum_{i=1}^n b_i$.

Proof. Clearly, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ $t(\alpha, \beta) \geq 0$. By 1.2, $t(\alpha, \beta) = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + q$, where $q = r(\alpha + \beta) - 2s(\alpha + \beta)$. By [1,6.1(i)],$t(\alpha) - z(\alpha) = \sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - 2z_1 \geq 2(\sum_{i=1}^n a_i - n) \geq 0$ and $t(\beta) - z(\beta) \geq 0$. By 1.8, $q \geq 0$, and hence $t(\alpha, \beta) \geq 0$.

Now, assume that $t(\alpha, \beta) = 0$. Then, $t(\alpha) = z(\alpha)$, $t(\beta) = z(\beta)$ and $q = 0$. By 1.12(i), $0 = t(\alpha) - z(\alpha) = r(\alpha) - s(\alpha) - 2z(\alpha) \geq 2s(\alpha) - 2n \geq 0$, and hence $s(\alpha) = n$. Then, $\sum_{i=1}^n a_i = n$, $\sum_{i=1}^n b_i = n$ and $a_i, b_i \in \{0, 1, 2\}$, for every $i \in I$ by 1.12(ii). Further, by 1.4, $q = r(\alpha + \beta - 2) + 2s(\alpha + \beta - 2) = 0$. Since $s(\alpha + \beta - 2) = s(\alpha) + s(\beta) - 2n = 0$, we get $\alpha + \beta - 2 = 0$. Thus $a_i + b_i = 2$, for every $i \in I$ and $(a_i, b_i) \in B$. Finally, $s(\alpha) = 2\mathrm{b}_1 + \mathrm{b}_3 = n = s(\beta) = 2\mathrm{b}_2 + \mathrm{b}_3$. Hence, $\mathrm{b}_1 = \mathrm{b}_2$, $2\mathrm{b}_1 \leq n$ and $\mathrm{b}_3 = n - 2\mathrm{b}_1$. Conversely, if $2\mathrm{b}_1 \leq n$, $\mathrm{b}_2 = \mathrm{b}_1$ and $\mathrm{b}_3 = n - 2\mathrm{b}_1$ then $\sum_{i=1}^n a_i = 2\mathrm{b}_1 + \mathrm{b}_3 = n = 2\mathrm{b}_2 + \mathrm{b}_3 = \sum_{i=1}^n b_i$ and $t(\alpha, \beta) = 0$ by 3.1. □

Remark 4.3. By 4.2(iii), the situation $\sum_{i=1}^n a_i \geq n$, $\sum_{i=1}^n b_i \geq n$, $t(\alpha, \beta) = 0$ is completely described. In order to find all such pairs α, β, choose $p \geq 0$ such that $2p \leq n$ and take p pairs (2,0), p pairs (0,2) and $n - 2p$ pairs (1,1).

Remark 4.4. Consider the situation from 4.2. The following inequalities follow from 4.2(i),(ii): $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq \sum_{i=1}^n (a_i + b_i) + n + z_1 + z_2$, $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3n + z_1 + z_2$, $\sum_{i=1}^n (a_i + b_i)^2 \geq \sum_{i=1}^n (a_i + b_i + a_i b_i) + n + z_1 + z_2$, $\sum_{i=1}^n (a_i + b_i)^2 \geq \sum_{i=1}^n a_i b_i + 3n + z_1 + z_2$ and the equalities hold if and only if the conditions from 4.2(iii) are satisfied.

Remark 4.5. Consider the situation from 4.2 and its proof. Now, by 1.6(i), $r(\alpha + \beta - 1) \geq s(\alpha + \beta - 1) = s(\alpha) + s(\beta) - n$, so that $q \geq s(\alpha) + s(\beta) - 2n$.

Consequently, $2r(\alpha) + 2r(\beta) + 2p(\alpha, \beta) - 4s(a) - 4s(\beta) + 2n - 2z(\alpha) - 2z(\beta) = t(\alpha, \beta) - s(\alpha) - s(\beta) + 2n = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + q - s(\alpha) - s(\beta) + 2n \geq 0$. From this, $t(\alpha, \beta) \geq s(\alpha) + s(\beta) - 2n$ and $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 2\sum_{i=1}^n (a_i + b_i) - n + z_1 + z_2$. This inequality is a slight improvement of 4.2(i), since $4(s(\alpha) + s(\beta)) - 2n \geq 3(s(\alpha) + s(\beta))$. Of course, if $s(\alpha) = n = s(\beta)$ then we get $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3n + z_1 + z_2$.

Remark 4.6. In view of 4.4, put $t'(\alpha, \beta) = r(\alpha, \beta) - 2s(\alpha) - 2s(\beta) + n - z(\alpha) - z(\beta)$. Now, $t(\alpha, \beta) - 2t'(\alpha, \beta) = s(\alpha) + s(\beta) - 2n$. Thus $t(\alpha, \beta) \geq 2t'(\alpha, \beta)$ if and only if $s(\alpha + \beta) \geq 2n$. Notice also that $t'(\alpha, \beta) = r(\alpha) - 2s(\alpha) - z(\alpha) + r(\beta) - 2s(\beta) - z(\beta) + p(\alpha, \beta) + n = t(\alpha) - s(\alpha) + t(\beta) - s(\beta) + p(\alpha, \beta) + n = r(\alpha + \beta) - 2s(\alpha + \beta) - p(\alpha, \beta) + n - z(\alpha) - z(\beta) = r(\alpha - 1) - z(\alpha) + r(\beta - 1) - z(\beta) + p(\alpha, \beta) - n = r(\alpha + \beta - 2) + 2s(\alpha + \beta + 2) - p(\alpha, \beta) + n - z(\alpha) - z(\beta) = r(\alpha + \beta - 1) - p(\alpha, \beta) - z(\alpha) - z(\beta)$.

Using 1.4, we have $2t'(\alpha, \beta) = t(\alpha, \beta) - s(\alpha) - s(\beta) + 2n = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(a + \beta - 2) + 2s(\alpha + \beta - 2) - s(\alpha) - s(\beta) + 2n = t(\alpha) - z(\alpha) + t(\beta) - z(\beta) + r(\alpha + \beta - 2) + s(\alpha + \beta - 2) \geq t(\alpha) - z(\alpha) + t(\beta) - z(\beta)$ by 1.6(ii).

Theorem 4.7. *Let* $\sum_{i=1}^n |a_i| \geq n$ *and* $\sum_{i=1}^n |b_i| \geq n$. *Put* $z_1 = z(\alpha)$ *and* $z_2 = z(\beta)$. *Then:*
(i) $\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 2\sum_{i=1}^n (a_i + b_i) - n + z_1 + z_2$.
(ii) $\sum_{i=1}^n (a_i + b_i)^2 \geq \sum_{i=1}^n a_i b_i + 2\sum_{i=1}^n (a_i + b_i) - n + z_1 + z_2$.
(iii) *The equalities hold if and only if the conditions from* 4.2(iii) *are satisfied.*

Proof. Clearly, the inequalities are equivalent to $t'(\alpha, \beta) \geq 0$. By 1.12(i), $t(\alpha) - z(\alpha) = r(\alpha) - \sum_{i=1}^n (\alpha) - 2z(\alpha) \geq r(\alpha) - \sum_{i=1}^n |a_i| - 2z(\alpha) \geq r(\alpha) - 3\sum_{i=1}^n |a)i| + 2n - 2z(\alpha) \geq 0$. Similarly, $t(\beta) - z(\beta) \geq 0$ and $t'(\alpha, \beta) \geq 0$ by 4.6.

If $t'(\alpha, \beta) = 0$ then $t(\alpha) - z(\alpha) = 0 = t(\beta) - z(\beta)$, hence (see the proof of 4.2) $s(\alpha) = n = s(\beta)$ and from 4.6 follows that $t(\alpha, \beta) = 2t'(\alpha, \beta) + s(\alpha) + s(b) - 2n = 0$. The rest follows from 4.2. □

Remark 4.8. It follows from 4.6 and 4.7 that $t(\alpha, \beta) \geq 0$, provided that $\sum_{i=1}^n |a_i| \geq n$, $\sum_{i=1}^n |b_i| \geq n$ and $s(\alpha) + s(\beta) \geq 2n$.

Proposition 4.9. *Let* $\alpha = (a_1, \dots, a_n)$ *and* $\beta = (b_1, \dots, b_n)$ *be* n*-tuples of non-zero integers. Then:*
(i) $2\sum_{i=1}^n (a_i^2 + b_i^2 + a_i b_i) \geq 3\sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
(ii) $2\sum_{i=1}^n (a_i + b_i)^2 \geq 2\sum_{i=1}^n a_i b_i + 3\sum_{i=1}^n (a_i + b_i) + 2(z_1 + z_2)$.
(iii) The equalities hold if and only if $a_1 = \dots = a_n = b_1 = \dots = b_n = 1$.

Proof. Use 4.1(i),(ii). □

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